Math 421 Notes

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Contents

1 Introduction

Definition 1.1. A topological space $(S, \mathcal{T})$ is a nonempty set with a family of subsets $\mathcal{T}$ such that

1. $\emptyset \in \mathcal{T}$
2. $S \in \mathcal{T}$
3. $\mathcal{T}$ is closed under finite intersections and arbitrary unions

Examples: $\{\emptyset, S\}$ (indiscrete topology), 2^S (discrete topology).
A metric on a metric space defines a topology. Not all topologies have a corresponding metric. A topology is called metrizable if we can define a metric such that "open" has the same meaning.
Topologies can be partially ordered. $\mathcal{T}_1 \prec \mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$ as sets. Denote $\mathcal{T}(\mathcal{E})$ to be the topology generated by $\mathcal{E} \subset 2^S$.

Definition 1.2. A base of $\mathcal{T}$ is a family $\mathcal{B} \subset \mathcal{T}$ such that for any nonempty open set $O \in \mathcal{T}$, there exists a colletion $\{B_\alpha : B_\alpha \in \mathcal{B}\}$ such that $O = \bigcup \alpha B_\alpha$.

Definition 1.3. Let $(S, \mathcal{T})$ be a topological space and $X \subset S$. Then, $\mathcal{T}_x = \{O \cap X : O \subset \mathcal{T}\}$ is the relative topology $(X, \mathcal{T}_x)$.

Definition 1.4. A set X is closed if $\exists Y \in \mathcal{T}$ such that $X = Y^c$.

Definition 1.5. The interior of X is the largest open set $X^o \subset X$.

Definition 1.6. The closure of X is the smallest closed set $\overline{X} \supset X$.

Definition 1.7. The boundary of X is $\overline{X} \setminus X^o$.

Definition 1.8. A neighbourhood of $x \in S$ is a set $N_x \subset S$ such that $x \in N_x^o$

Definition 1.9. A neighbourhood base of x is a family $\mathcal{N}_x$ such that each $N \in \mathcal{N}_x$ is a neighbourhood of x and for any neighbourhood M_x, there exists some $N \in \mathcal{N}_x$ such that $N \subset M_x$.

Definition 1.10 (Classification of topological spaces)**.** A topological space is called T_2 or Hausdorff if $\forall x, y \in S$, $x \neq y$, there exists $O_x, O_y \in \mathcal{T}$ such that $x \in O_x$, $y \in O_y$, and $O_x \cap O_y = \emptyset$.

Definition 1.11. A topological space $(S, \mathcal{T})$ is

- separable if there exists a countable dense set
- first countable if $\forall x \in S$, there exists a countable neighbourhood base
- second countable if there exists a countable base

Proposition 1.12. Second countable implies both first countable and separable.

Proof. (Second countable implies first countable) Let $x \in S$, and let $M_x \subset \mathcal{T}$ be a neighbourhood of x. Since $\mathcal{B}$ is a base, there exists open sets $N_\alpha \in \mathcal{B}$ such that $\bigcup_\alpha N_\alpha = M_x^o$. Observe that there exists some N_α such that $x \in N_\alpha$, whence second countable.
(Second countable implies separable) For each $B \in \mathcal{B}$, choose some $x_B \in B$, and let $D = \bigcup_B x_B$. Suppose that $\overline{D} \neq S$, then $\overline{D}^c$ is open. Since $\mathcal{B}$ is a base, there exists some $B \in \mathcal{B}$ such that $B \subset \overline{D}^c$. Contradiction. $\square$

Definition 1.13. A sequence $\{x_n\}_{x \in \mathbb{N}}$ in $(S, \mathcal{T})$ is convergent if $\exists x \in S$ such that for any neighbourhood of x, there exists some $N \in \mathbb{N}$ such that $x_n \in N_x$ for all $n > N$.

Proposition 1.14. Let $(S, \mathcal{T})$ be a first countable topological space, and $X \subset S$. Then $x \in \overline{X}$ if and only if x is the limit point of a convergent sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$.

Proof. Let $\mathcal{N}_x = \{O_n : n \in \mathbb{N}\}$ be a countable neighbourhood base of x such that $O_n \subset O_{n-1}$ for all $n \in \mathbb{N}$. If $x \in \overline{X}$, then $O_n \cap X \neq \emptyset$ for all $n \in \mathbb{N}$. Then we can pick $x_n \in O_n \cap X$, whence $x_n \to x$. Converse is similar. $\square$

Definition 1.15. Let $(S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)$ be topological spaces. A function $f : S_1 \to S_2$ is continuous if $f^{-1}(O) \in \mathcal{T}_1$ for any $O \in \mathcal{T}_2$. Ie. the preimage of any open set is open.

Definition 1.16. Let $(S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)$ be topological spaces. A function $f : S_1 \to S_2$ is open if $f(O) \in \mathcal{T}_2$ for any $O \in \mathcal{T}_1$.

Definition 1.17. A homeomorphism is an invertible function that is open and continuous.

Definition 1.18. Let S_1 be a set and let $(S_2, \mathcal{T}_2)$ be a topological space. Let $\mathcal{F}$ be a family of functions from S_1 to S_2. Then, the topology on S_1 generated by $\left\{f^{-1}(O) : O \in \mathcal{T}_2\right\}$ is called the $\mathcal{F}$-weak topology.

Remark 1.19. By definition, all functions $f \in \mathcal{F}$ are continuous with respect to the above topology on S_1.

Example 1.20. Let $S_1 = C([a, b]; \mathbb{R})$ be the set of continuous functions, and let $S_2 = \mathbb{R}$ with the usual metric topology. Let $E_x : S_1 \to S_2$ where $E_x(f) = f(x)$ be the evaluation functions, and let $\mathcal{F} = \{E_x : x \in [a, b]\}$. The $\mathcal{F}$-weak topology on $C([a, b]; \mathbb{R})$ is the topology of pointwise convergence.

Definition 1.21. A topological space $(S, \mathcal{T})$ is compact if any open cover has a finite subcover.

Definition 1.22. A subset $X \subset S$ is compact if it is compact in the relative topology.

Definition 1.23. A subset $X \subset S$ is precompact if its closure is compact.

Definition 1.24. We say that $(S, \mathcal{T})$ has the finite intersection property if for any family of closed sets $\mathcal{C}$ such that $\bigcap_{i=1}^n C_i \neq \emptyset$ for any finite subfamily $\{C_1, ..., C_n\}$ also satisfies $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

Exercise 1.25. S is compact if and only if it has the finite intersection property.

Proposition 1.26. Let $X \subset S$ be a subset of a compact topological space $(S, \mathcal{T})$. Then X is compact if X is closed.

Proof. Let $\mathcal{C}$ be an open cover of X. Let $\mathcal{C}' = \mathcal{C} \cup \{X^c\}$ be an open cover of S. There exists a finite subcover of $\mathcal{C}'$, so there exists a finite subcover of X (we can safely remove X^c from the finite subcover of S as $X \cap X^c = \emptyset$). $\square$

Proposition 1.27. Let $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$, and let $f : S_1 \to S_2$ be continuous. If S_1 is compact, then $f(S_1) \subset S_2$ is compact.

Proof. Let $\mathcal{C}$ be an open cover of $f(S_1)$. Let $\mathcal{C}' = \left\{f^{-1}(C) : C \in \mathcal{C}\right\}$ be an open cover of S_1 (preimages of open sets are open by continuity). Hence, there exists a finite subcover of S_1, from which we get a finite subcover of $f(S_1)$. $\square$

Proposition 1.28 (Bolzano-Weierstrass property)**.** A second countable topological space is compact if and only if every sequence has a convergent subsequence.

Proof. Suppose that S is compact, and suppose, for contradiction, that $\{z_n\}_{n\in\mathbb{N}}$ does not have a convergent subsequence. Since S is first countable, this means that for any $x \in S$, there exists some neighbourhood O_x of x and some $N_x \in \mathbb{N}$ such that $z_n \notin O_x$ for all $n > N_x$. Let $\mathcal{C} = \{O_x^o : x \in S\}$ be an open cover of S. Since S is compact, there exists some finite subcover $\mathcal{C}' = \left\{O_{x_1}^o, ..., O_{x_m}^o\right\}$. Then, let $N = \max\{n_{x_1}, ..., n_{x_m}\}$, whence $z_n \notin \bigcup_i O_{x_i} = S$ for all $n > N$, which is a contradiction. Suppose that every sequence of S has a convergent subsequence. Since S is second countable, there exists a countable open cover $\mathcal{C} = \{O_i : i \in \mathbb{N}\}$. Suppose, for contradiction, that $\mathcal{C}$ has no finite subcover. Then, for any $i \in \mathbb{N}$, there exists some $x_i \notin \bigcup_{j=1}^{i} O_j$. Let $\{x_{n_i}\}_{i\in\mathbb{N}}$ be a convergent subsequence and let x be its limit. Since $\mathcal{C}$ is a cover, there exists j such that $x \in O_j$. It follows that there exists some $N \in \mathbb{N}$ such that $x_{n_k} \in O_j$ for all $k > N$, which is a contradiction. $\square$