

# Math 420 Notes

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## 1 Introduction

**Definition 1.1.** A  $\sigma$ -algebra on  $X$  is a collection of subsets of  $2^X$  that is closed under complement and countable union.

**Definition 1.2.** Let  $\mathcal{M} \subset 2^X$  be the measurable subsets of  $X$ . A measure  $\mu$  on  $(X, \mathcal{M})$  is a function  $\mu : \mathcal{M} \rightarrow [0, \infty]$  satisfying the following.

- (i)  $\mu(\emptyset) = 0$
- (ii)  $\mu(\bigcup_j^\infty E_j) = \sum_j^\infty \mu(E_j)$

Note that  $\mathcal{M}$  is a  $\sigma$ -algebra.

**Example 1.1.** The counting measure.  $\mu(E) = \# \{X : X \in E\}$

**Example 1.2.** The Dirac measure. Fix  $x_0 \in X$ .  $\mu(E) = 1$  if  $x_0 \in E$ , and  $\mu(E) = 0$  otherwise.

**Example 1.3.** An unmeasurable set. (Folland p.20).

Let  $E_r = E + r \bmod 1$ . There exists a set  $E \subset [0, 1)$  such that

- $\{E_r\}_{r \in \mathbb{Q} \cap [0,1]}$  are disjoint
- $\bigcup_{r \in \mathbb{Q} \cap [0,1]} E_r = [0, 1)$

This set  $E$  is inconsistent with (ii) of the definition when  $\mu([0, 1)) = 1$  and  $\mu(E_r) = \mu(E)$ .

**Definition 1.3.** Let non-empty  $\mathcal{E} \subset 2^X$ . The  $\sigma$ -algebra generated by  $\mathcal{E}$  is  $\mathcal{M}(\mathcal{E})$ , that is the smallest  $\sigma$ -algebra containing  $\mathcal{E}$ . (We can get this by taking the intersection of all the  $\sigma$ -algebras containing  $\mathcal{E}$ )

**Example 1.4.** Let  $X$  be a topological space. The Borel  $\sigma$ -algebra  $B_X$  is a  $\sigma$ -algebra generated by open sets. This contains open sets, closed sets, countable union of closed sets ( $F\sigma$ -sets), countable intersection of open sets ( $G\delta$ -sets).

$B_{\mathbb{R}}$  can be generated by any of

- open intervals.  $\{(a, b)\}$
- closed intervals.  $\{[a, b]\}$
- half open intervals.  $\{(a, b]\}$
- semi-infinite intervals.  $\{(a, \infty)\}$

## 2 The Lebesgue Measure

### 2.1 Premeasures

Let  $\mathcal{A}$  be the set of finite disjoint unions of h-intervals, where h-intervals are of the following form:  $(a, b]$ ,  $(a, \infty)$ ,  $\emptyset$ , where  $-\infty \leq a < b < \infty$ .

**Proposition 2.1.**  $\mathcal{A}$  is an algebra.

*Proof.* The intersection of two h-intervals is also an h-interval. The complement of an h-interval is the union of at most two disjoint h-intervals. Refer to text (Folland Prop 1.7).  $\square$

Define the “Length” of sets in  $\mathcal{A}$  to be a function  $m_0 : \mathcal{A} \rightarrow [0, \infty]$  with finite additivity and  $m_0(\emptyset) = 0$ .

**Definition 2.1.** A premeasure is a function  $m : \mathcal{A} \rightarrow [0, \infty]$  such that

- (i)  $m(\emptyset) = 0$
- (ii) For countably many disjoint  $A_j \in \mathcal{A}$  whose union  $A = \bigcup A_j$  is also in  $\mathcal{A}$ , we have  $m(\bigcup A_j) = \sum m(A_j)$ .

**Theorem 2.1.** *The following is true*

1.  $m_0$  is well defined.
2.  $m_0$  is a premeasure.

*Proof of 1.* This is just bookkeeping. See text.  $\square$

*Proof of 2.* Let  $A = (a, b] \in \mathcal{A}$  be a countable disjoint union of  $A_j = (a_j, b_j] \in \mathcal{A}$ . We can assume that  $A_j$ , because each  $A_j$  would otherwise be the finite union of some disjoint set of intervals in  $\mathcal{A}$ . We can also assume that  $A$  is an interval by the same argument. Consider  $A = \bigcup_{j=1}^n A_j \cup (A \setminus \bigcup_{j=1}^n A_j)$ . Then we have

$$m_0(A) = m_0\left(\bigcup_{j=1}^n A_j\right) + m_0\left(A \setminus \bigcup_{j=1}^n A_j\right) \geq m_0\left(\bigcup_{j=1}^n A_j\right).$$

Taking the limit gives  $m_0(A) \geq m_0(\bigcup_{j=1}^\infty A_j)$ .

Now let  $\epsilon > 0$ . Consider the compact interval  $[a + \epsilon, b]$  covered by  $\bigcup_{j=1}^\infty (a_j, b_j + \frac{\epsilon}{2^j})$ . There must be a finite subcover. Now,  $(a + \epsilon, b]$  is also covered by this finite subcover, and we can relabel the finite subcover so that  $a_j < a_{j+1}$ . Then

$$\begin{aligned} m_0(A) &= m_0((a, a + \epsilon]) + m_0((a + \epsilon, b]) \\ &\leq \epsilon + m_0((a_1, b_n + \frac{\epsilon}{2^n})) = \epsilon + b_n + \frac{\epsilon}{2^n} - a_n + \sum_{j=2}^n (a_j - a_{j-1}) \\ &\leq \epsilon + (b_n - a_n) + \sum_{j=1}^n \left(b_j + \frac{\epsilon}{2^j} - a_{j-1}\right) \leq \epsilon + \sum_{j=1}^n \frac{\epsilon}{2^j} + \sum_{j=1}^n m_0(A_j) \\ &\leq 7\epsilon + \sum m_0(A_j), \end{aligned}$$

and countable additivity follows. □

## 2.2 Lebesgue Outer Measure

**Definition 2.2.** The Lebesgue outer measure  $m^*$  of a set  $E \subset \mathbb{R}$  is defined as follows.

$$m^*(E) = \inf \left\{ \sum_{j=1}^\infty m_0(I_j) : E \subset \bigcup_{j=1}^\infty I_j \right\}.$$

**Definition 2.3.** In general, given  $\mathcal{E} \subset 2^X$  with  $\emptyset, X \in \mathcal{E}$  and  $\mu_0 : \mathcal{E} \rightarrow [0, \infty]$  with  $\mu_0(\emptyset) = 0$ , we can define  $\mu^* : 2^X \rightarrow [0, \infty]$  as follows.

$$\mu^*(E) = \inf \left\{ \sum_{j=1}^\infty \mu_0(E_j) : E \subset \bigcup_{j=1}^\infty E_j \text{ and } E_j \in \mathcal{E} \right\}.$$

**Proposition 2.2.**  $\mu^*$  is an outer measure, where an outer measure satisfies three properties.

1.  $\mu^*(\emptyset) = 0$
2.  $A \subset B \implies \mu^*(A) \leq \mu^*(B)$
3.  $\mu^*\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty \mu^*(E_j)$

*Proof of 1.*  $\emptyset \subset \bigcup_{j=1}^\infty \emptyset \implies \mu^*(\emptyset) \leq \sum_{j=1}^\infty \mu_0(\emptyset) = 0$ . □

*Proof of 2.* Let  $A \subset B$ . Then  $\left\{ \{E_j\}_j \subset \mathcal{E} : B \subset \bigcup_j E_j \right\} \subset \left\{ \{E_j\}_j \subset \mathcal{E} : A \subset \bigcup_j E_j \right\}$ . Hence  $\mu^*(A) \leq \mu^*(B)$ . □

*Proof of 3.* Let  $\{A_{j,k}\}_k \subset \mathcal{E}$  such that  $E_j \subset \bigcup_{k=1}^{\infty} A_{j,k}$ . Observe that  $\bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j,k=1}^{\infty} A_{j,k}$ . Let  $\epsilon > 0$ , then

$$\mu^* \left( \bigcup_{j=1}^{\infty} E_j \right) \leq \sum_{j,k=1}^{\infty} \mu_0(A_{j,k}) \leq \sum_{j=1}^{\infty} \left( \frac{\epsilon}{2^j} + \mu^*(E_j) \right) = \epsilon + \sum_{j=1}^{\infty} \mu^*(E_j)$$

Since  $\epsilon$  is arbitrary, we get  $\mu^*(\bigcup_j E_j) \leq \sum_j \mu^*(E_j)$ .  $\square$

Observe that  $\mu^*$  is defined for every set in  $2^X$ , but it is not additive. To fix this, we will remove some “bad” sets.

**Definition 2.4.** Let  $\mu^*$  be an outer measure on  $X$ . A set  $A \subset X$  is  $\mu^*$ -measurable if for every  $E \subset X$ , we have  $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$ .

### 2.3 Carathéodory's Extension Theorem

**Theorem 2.2.** Let  $\mathcal{M}$  be the set of  $\mu^*$ -measurable sets, and  $\mu^* \upharpoonright_{\mathcal{M}}$  is a complete measure.

*Proof.* 1. We show that  $\mathcal{M}$  is an algebra. Clearly,  $\emptyset \in \mathcal{M}$ , and  $\mathcal{M}$  is closed under complement. Now let  $A, B \in \mathcal{M}$ . Then

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c). \end{aligned}$$

The other inequality is automatic by monotonicity. Hence  $A \cup B \in \mathcal{M}$ .

2. We show that  $\mu^*$  is finitely additive. Let  $A, B \in \mathcal{M}$  be disjoint. Then

$$\mu^*(A \cup B) = \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c) = \mu^*(A) + \mu^*(B).$$

3. We show that  $\mathcal{M}$  is closed under countable union and  $\mu^*$  is countably additive.

Let  $\{A_j\} \subset \mathcal{M}$ ,  $B_n = \bigcup_{j=1}^n A_j$ , and  $B = \bigcup_{j=1}^{\infty} A_j$ . Let  $E \subset X$ , then

$$\begin{aligned} \mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) = \sum_{j=1}^n \mu^*(E \cap A_j). \end{aligned}$$

By the definition, we get

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*(E \cap B^c).$$

Take  $n \rightarrow \infty$ , then we get closure under countable union.

$$\mu^*(E) \geq \sum_{j=1}^{\infty} \mu^*(E \cap A_j) + \mu^*(E \cap B^c) \geq \mu^*(E \cap B) + \mu^*(E \cap B^c).$$

Take  $E = B$ , then we get countable additivity.

$$\mu^*(B) = \sum_{j=1}^{\infty} \mu^*(A_j) + \mu^*(\emptyset) = \sum_{j=1}^{\infty} \mu^*(A_j).$$

We can easily check that  $\mathcal{M}$  is complete. This theorem is complete.  $\square$

**Proposition 2.3.** If  $A \in \mathcal{A}$ , then  $A$  is  $\mu^*$ -measurable.

*Proof.* Let  $A \in \mathcal{A}$  and  $E \subset X$ . Let  $\epsilon > 0$ . There exists  $\{A_j\} \subset \mathcal{A}$  with  $E \subset \bigcup_{j=1}^{\infty} A_j$  such that  $\mu^*(E) + \epsilon \geq \sum_{j=1}^{\infty} \mu_0(A_j)$  by the definition of  $\mu^*$ . Then

$$\mu^*(E) + \epsilon \geq \sum \mu_0(A_j \cap A) + \sum \mu_0(A_j \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Take  $\epsilon \rightarrow 0$ , and we see that  $A$  is  $\mu^*$ -measurable. □

**Proposition 2.4.**  $\mu^* \upharpoonright_{\mathcal{A}} = \mu_0 \upharpoonright_{\mathcal{A}}$ .

*Proof.* See text. □

## 2.4 Lebesgue Measure on $\mathbb{R}$

Let  $X = \mathbb{R}$  and define  $m_0$  to be the length of h-intervals.

1.  $m^*(E) = \inf \{ \sum m_0(I_j) : E \subset \bigcup I_j \}$ , where  $I_j$  are h-intervals.
2.  $\mathcal{L}$  is the  $m^*$ -measurable sets (Lebesgue measurable).
3.  $m = m^* \upharpoonright_{\mathcal{L}}$ .

**Remark 2.1.** The measure  $m$  is a Borel measure, that is it is defined for all Borel sets. Also,  $m$  is the unique Borel measure with  $m((a, b]) = b - a$ .

*Proof.* See text. Basically if  $\mu_0$  is  $\sigma$ -finite on  $\mathcal{A}$ , then Carathéodory gives uniqueness. □

**Remark 2.2.** We can also construct a measure with any non-decreasing right-continuous  $F : \mathbb{R} \rightarrow \mathbb{R}$  with  $m_F((a, b]) = F(b) - F(a)$ . This is the Lebesgue-Stieltjes measure. Observe that the Lebesgue measure simply has  $F(x) = x$ .

**Proposition 2.5.** Any Borel measure  $\mu$  that is finite on bounded sets defines a non-decreasing right-continuous function  $F : \mathbb{R} \rightarrow \mathbb{R}$  as follows

$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \mu((-x, 0]) & \text{if } x < 0 \end{cases}.$$

**Proposition 2.6.** The Lebesgue measure is **translation invariant**  $m(E+s) = m(E)$  and **dilation invariant**  $m(rE) = |r| m(E)$ .

**Remark 2.3.** open sets, closed sets, etc.  $\subsetneq \mathcal{B}_{\mathbb{R}} \subsetneq \mathcal{L} \subsetneq 2^{\mathbb{R}}$ .

**Lemma 2.3.**  $m^*(E) = \inf \left\{ \sum_{j=1}^{\infty} (b_j - a_j) : E \subset \bigcup_{j=1}^{\infty} (a_j, b_j) \right\}$ .

*Proof.* See text. □

**Theorem 2.4.** Let  $E \subset \mathbb{R}$ . All of the following imply one another.

- (a)  $E \in \mathcal{L}$ .
- (b) There exists  $U_{\text{open}} \supset E$  such that  $m^*(U \setminus E) \leq \epsilon$ .
- (c) There exists  $F_{\text{closed}} \subset E$  such that  $m^*(E \setminus F) \leq \epsilon$ .

(d) There exists a  $G\delta$  set  $V \supset E$  such that  $E = V \setminus N_1$  with  $N_1$  null.

(e) There exists a  $F\sigma$  set  $H \supset E$  such that  $E = H \cup N_2$  with  $N_2$  null.

*Proof.* .

(a  $\implies$  b). Let  $\epsilon > 0$ . There exists  $U = \bigcup_{j=1}^{\infty} I_j \supset E$  (where each  $I_j$  is an open interval) such that  $m(E) + \epsilon \geq \sum_{j=1}^{\infty} m(I_j) \geq m(U)$ . Then, using the definition of a measurable set,

$$m(U) = m(U \cap E) + m(U \cap E^c) = m(E) + m(U \cap E^c) \leq m(E) + \epsilon.$$

Hence  $m(U \setminus E) < \epsilon$  holds for  $m(E)$  finite.

If  $m(E)$  is infinite, then let  $E_j = E \cap (j, j+1]$  and  $U_j = U \cap (j, j+1]$ . Then  $m(U_j \setminus E_j) \leq \epsilon 2^{-|j|}$  from the finite case, and countable additivity gives the desired result.

(a  $\implies$  c). Use  $E^c$  and (a) implies (b).

(b  $\implies$  d). There exists open  $U_j \supset E$  such that  $m^*(U_j \setminus E) \leq \frac{1}{j}$ . Then  $V = \bigcap_{j=1}^{\infty} U_j \supset E$  is a  $G\delta$  set. Let  $N_1 = V \setminus E$ , then  $E = V \setminus N_1$ . It follows that  $N_1 \subset U_j \setminus E$  for all  $j$ , so  $m^*(N_1) \leq m^*(U_j \setminus E) \leq \frac{1}{j}$ . Hence  $m^*(N_1) = 0$ , so  $N_1$  is a null set.

(c  $\implies$  e). Similar to above.

(d  $\implies$  a) and (e  $\implies$  a).  $G\delta$  and  $F\sigma$  sets are Borel, so they are measurable. Null sets are also measurable by completeness. Hence  $E$  is measurable.

□

### 3 Integrals and Convergence

**Definition 3.1.** Let  $(X, \mathcal{M})$  and  $Y, \mathcal{N})$  be measurable spaces. A function  $f : X \rightarrow Y$  is  $(\mathcal{M}, \mathcal{N})$  measurable if  $f^{-1}(E) \in \mathcal{M}$  for all  $E \in \mathcal{N}$ .

**Remark 3.1.** If  $\mathcal{N}$  is generated by  $\mathcal{E} \subset \mathcal{N}$ , then  $f : X \rightarrow Y$  is measurable if and only if  $f^{-1}(E) \in \mathcal{M}$  for all  $E \in \mathcal{E}$ .

**Remark 3.2.** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is Borel measurable if it is continuous.

**Remark 3.3.** Composition of measurable functions is measurable.

**Remark 3.4.**  $f : X \rightarrow \bar{\mathbb{R}}$  is measurable if it is  $(\mathcal{M}, \mathcal{B}_{\bar{\mathbb{R}}})$ -measurable, where

$$\mathcal{B}_{\bar{\mathbb{R}}} = \{E \subset \bar{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\mathbb{R}}\}.$$

**Proposition 3.1.** Let  $(X, \mathcal{M})$  be a measurable space. Then

1.  $f : X \rightarrow \mathbb{C}$  is measurable if and only if the real and imaginary parts of  $f$  are measurable.
2.  $f, g : X \rightarrow \mathbb{C}$  are measurable implies  $f + g$  and  $f \cdot g$  are measurable.
3.  $f_j : X \rightarrow \bar{\mathbb{R}}$  is measurable implies  $\sup_j f_j$ ,  $\inf_j f_j$ ,  $\limsup_{j \rightarrow \infty} f_j$  and  $\liminf_{j \rightarrow \infty} f_j$  are measurable.
4.  $f_j : X \rightarrow \mathbb{C}$  is measurable implies  $\lim_{j \rightarrow \infty} f_j$  is measurable if the limit exists.

**Definition 3.2.** A simple function on  $(X, \mathcal{M})$  is of the form  $f(x) = \sum_{j=1}^n z_j \chi_{E_j}(x)$  for  $z_j \in \mathbb{C}$  and  $E_j \in \mathcal{M}$ .

**Remark 3.5.**  $f$  is in “standard form” if  $E_j = f^{-1}(\{z_j\})$ .

**Definition 3.3.** Let  $(X, \mathcal{M}, \mu)$  be a measure space and let  $f = \sum_{j=1}^n z_j \chi_{E_j}$  be a simple function. Then

$$\int f = \sum_{j=1}^n z_j \mu(E_j).$$

**Proposition 3.2.** Let  $\phi, \psi$  be simple functions.

- (a)  $c \in \mathbb{C}$  implies  $\int c\psi = c \int \psi$ . (linearity)
- (b)  $\int \phi + \psi = \int \phi + \int \psi$ .
- (c) If  $\phi, \psi \in \mathbb{R}$ , then  $\phi \leq \psi \implies \int \phi \leq \int \psi$ .
- (d) If  $\phi \geq 0$ , then  $\mathcal{M} \ni A \mapsto \int_A \phi := \int \chi_A \phi$  is a measure.

*Proof.* .

(a), (b), and (d). See text / exercise.

(c). Let  $\phi = \sum_{j=1}^n z_j \chi_{E_j}$  and  $\psi = \sum_{k=1}^m w_k \chi_{F_k}$  in standard form. Then

$$\begin{aligned} \int \phi &= \sum_j z_j \mu(E_j) = \sum_j z_j \sum_k \mu(E_j \cap F_k) = \sum_j \sum_k z_j \mu(E_j \cap F_k) \\ &\leq \sum_k \sum_j w_k \mu(E_j \cap F_k) = \sum_k w_k \sum_j \mu(E_j \cap F_k) = \sum_k w_k \mu(F_k) = \int \psi. \end{aligned}$$

□

**Definition 3.4.** Define  $L^+ = \{f : X \rightarrow [0, \infty), \text{measurable}\}$ . Then for  $f \in L^+$ , define

$$\int f = \sup \left\{ \int \phi : 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

**Remark 3.6.** We have monotonicity and linearity for  $f \in L^+$ .

### 3.1 Approximation by Simple Functions and Monotone Convergence

**Theorem 3.1** (Approximation Theorem). (a) Let measurable  $f : X \rightarrow [0, \infty]$ . There exists simple  $0 \leq \phi_1 \leq \phi_2 \leq \dots \leq f$  such that  $\phi_n \rightarrow f$  pointwise, and  $\phi_n \rightarrow f$  uniformly on sets where  $f$  is bounded.

(b) Let measurable  $f : X \rightarrow \mathbb{C}$ . There exists simple  $\{\phi_n\}$  with  $0 \leq |\phi_1| \leq |\phi_2| \leq \dots \leq |f|$  such that  $\phi_n \rightarrow f$  pointwise, and  $\phi_n \rightarrow f$  uniformly on sets where  $f$  is bounded.

*Proof.* Proof by construction with powers of 2. □

**Theorem 3.2** (Monotone Convergence Theorem). Let  $\{f_n\} \subset L^+$  with  $0 \leq f_1 \leq f_2 \leq \dots$ . Then

$$\int \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int f_n.$$

*Proof.* Let  $f(x) = \sup_n f_n(x) \in L^+$ . Then  $\{\int f_n\}$  is increasing, so  $\lim_{n \rightarrow \infty} \int f_n = \sup_n \int f_n$  (which exists). Since  $f_n \leq f$ , we have  $\int f_n \leq \int f$ , so  $\lim_{n \rightarrow \infty} \int f_n \leq \int f$ . Let  $\phi$  be a simple function such that  $0 \leq \phi \leq f$ . Fix  $\alpha \in (0, 1)$ . Let  $E_n = \{x : f_n(x) \geq \alpha \phi(x)\}$ . Observe that  $E_n$  is measurable and  $E_1 \subset E_2 \subset \dots \cup_{n=1}^{\infty} E_n = X$ . Since  $E \mapsto \int_E \phi$  is a measure, we get  $\int_{E_n} \phi \mapsto \int \phi$  by continuity from below. Then

$$\int f_n \geq \int_{E_n} f_n \geq \alpha \int_{E_n} \phi \implies \lim_{n \rightarrow \infty} \int f_n \geq \alpha \int \phi.$$

If we take  $\alpha \rightarrow 1$ , then  $\lim \int f_n \geq \int \phi$ . Then the Monotone Convergence Theorem follows by simple function approximation.  $\square$

**Proposition 3.3.** Let  $\{f_n\} \subset L^+$ , then  $\int \sum_n f_n = \sum_n \int f_n$ .

*Proof.* Let  $f_1, f_2 \in L^+$ . By approximation, we have  $\phi_n \uparrow f_1$  and  $\psi_n \uparrow f_2$ . Then  $\phi_n + \psi_n \uparrow f_1 + f_2$ . Then by Monotone Convergence,

$$\int f_1 + f_2 = \lim_{n \rightarrow \infty} \int (\phi_n + \psi_n) = \lim_{n \rightarrow \infty} \left( \int \phi_n + \int \psi_n \right) = \lim_{n \rightarrow \infty} \int \phi_n + \lim_{n \rightarrow \infty} \int \psi_n = \int f_1 + \int f_2.$$

Now let  $\{f_n\}_{n=1}^{\infty}$ . Then using MCT on  $\sum_{n=1}^N f_n \uparrow \sum_{n=1}^{\infty} f_n$  implies

$$\int \sum_{n=1}^{\infty} f_n = \int \lim_{N \rightarrow \infty} \sum_{n=1}^N f_n = \lim_{N \rightarrow \infty} \int \sum_{n=1}^N f_n = \lim_{N \rightarrow \infty} \sum_{n=1}^N \int f_n = \sum_{n=1}^{\infty} \int f_n.$$

$\square$

**Proposition 3.4.** If  $f \in L^+$ , then  $\int f = 0 \iff f = 0$  almost everywhere.

*Proof.* For simple  $f = \sum_{k=1}^n a_k \chi_{E_k}$ , then  $\mu(E_k) = 0$  or  $a_k = 0$ . The result follows because the finite union of null sets is still a null set.

Now we prove this for  $f \in L^+$ . If  $f = 0$  almost everywhere, then any simple  $\phi$  satisfying  $0 \leq \phi \leq f$  is also 0 almost everywhere. Then  $\int \phi = 0$ , implying that  $\int f = 0$ . If  $f$  is not 0 almost everywhere, then  $\mu(\{f(x) > 0\}) > 0$ . Let  $E_n = \{f(x) > \frac{1}{n}\}$  for  $n \in \mathbb{N}$ . Then  $\{f(x) > 0\} = \bigcup E_n$ . It follows that there exists some  $k$  such that  $\mu(E_k) > 0$ . Hence  $f \geq \frac{1}{k} \chi_{E_k}$ , so  $\int f \geq \frac{1}{k} \mu(E_k) > 0$ .  $\square$

**Remark 3.7.** We don't care about null sets. If  $f_n \in L^+$  and  $f_n \uparrow f$  almost everywhere, then  $\int f = \lim_{n \rightarrow \infty} \int f_n$ .

*Proof.* Apply MCT to  $f_n \chi_{N^c}$  where  $N$  is the null set on which  $f_n$  does not converge to  $f$ .  $\square$

**Theorem 3.3** (Fatou's Lemma). Let  $\{f_n\}_{n=1}^{\infty} \subset L^+$ . Then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n.$$

**Corollary 3.3.1.** If  $f_n \rightarrow f$  almost everywhere, then  $\int f \leq \liminf_{n \rightarrow \infty} \int f_n$ .

*Proof of Fatou's Lemma.* Let  $g_k(x) = \inf_{n \geq k} f_n(x)$  be an increasing sequence of functions. Then for each  $j \geq k$ , we have

$$\inf_{n \geq k} f_n \leq f_j \implies \int \inf_{n \geq k} f_n \leq \int f_j \implies \int \inf_{n \geq k} f_n \leq \inf_{j \geq k} \int f_j$$



It follows by MCT and the above that

$$\int \sup_k g_k = \int \liminf f_n = \lim_{k \rightarrow \infty} \int \inf_{n \geq k} f_n \leq \lim_{k \rightarrow \infty} \inf_{j \geq k} \int f_j = \liminf_{n \rightarrow \infty} \int f_n.$$

□

### 3.2 Integration of Complex Functions

Let  $f : X \rightarrow \mathbb{R}$  be measurable, then  $f^+ = \max(f, 0)$  and  $f^- = \max(-f, 0)$ . Observe that  $f^+, f^- \in L^+$  and we can write  $f = f^+ - f^-$ . Also observe that if  $f : X \rightarrow \mathbb{C}$ , then we write  $f = \Re(f) + i\Im(f)$ , and hence  $\int f = \int \Re(f) + i \int \Im(f)$ .

**Definition 3.5.** We say  $f : X \rightarrow \mathbb{C}$  is integrable if  $\int |f| < \infty$ . Define

$$L^1(\mu) = \left\{ f : X \rightarrow \mathbb{C} : \int |f| < \infty \right\}.$$

**Proposition 3.5.** (a)  $L^1$  is a vector space.

(b)  $\int$  is a linear map on  $L^1$ .

(c)  $f \in L^1$  implies  $|\int f| \leq \int |f|$ .

(d) If  $f, g \in L^1$ , then  $\int |f - g| = 0 \iff f = g \text{ a.e.} \iff \int_E f = \int_E g \text{ for all } E \in \mathcal{M}$ .

*Proof.* See text. □

**Remark 3.8.** If we define  $L^1$  to be the equivalence class of almost everywhere defined integrable functions under  $f \sim g \iff f = g \text{ a.e.}$ , then  $L^1$  is a Banach space under  $|f - g|$ .

**Remark 3.9.** If  $f \in L^+$  with  $\int f < \infty$ , then  $\mu(\{f = \infty\}) = 0$ .

*Proof.* **Exercise.** □

**Theorem 3.4** (Dominated Convergence Theorem). *Let  $L^1 \ni f_n \rightarrow f$  almost everywhere and  $|f_n| \leq g \in L^1$  for all  $n$ . Then  $f \in L^1$  and  $\int f = \lim_{n \rightarrow \infty} \int f_n$ .*

*Proof.* First we show that  $f \in L^1$ . Observe that  $|f_n| \leq g$  implies  $|f| \leq g$  almost everywhere (**why a.e.?**), so  $f \in L^1$ .

We take  $f_n \in \mathbb{R}$ . Otherwise, consider  $\Re(f_n)$  and  $\Im(f_n)$ . Observe that  $g \pm f_n \geq 0$ , so Fatou's Lemma implies

$$\begin{aligned} \int g + \int f &= \int g + f \leq \liminf \int g + f_n = \int g + \liminf \int f_n \\ \int g - \int f &= \int g - f \leq \liminf \int g - f_n = \int g - \limsup \int f_n \end{aligned}$$

Since  $\int g < \infty$ , we have  $\limsup \int f_n \leq \int f \leq \liminf \int f_n$ . It follows that  $\lim \int f_n = \int f$ . □

**Proposition 3.6.** Let  $\{f_j\}_{j=1}^\infty \subset L^1$  with  $\sum_{j=1}^\infty \int |f_j| < \infty$ . Then  $\sum_{j=1}^\infty f_j$  converges almost everywhere and  $\int \sum_{j=1}^\infty f_j = \sum_{j=1}^\infty \int f_j$ .

*Proof.* Each  $|f_j| \in L^+$ , so MCT gives  $\int \sum_{j=1}^{\infty} |f_j| = \sum_{j=1}^{\infty} \int |f_j| < \infty$ . Hence,  $\sum_{j=1}^{\infty} |f_j| \in L^1$ . It follows that  $\sum_{j=1}^{\infty} |f_j(x)| < \infty$  almost everywhere, so  $\sum_{j=1}^{\infty} f_j$  converges almost everywhere. Since  $\left| \sum_{j=1}^N f_j \right| \leq \sum_{j=1}^N |f_j| \leq \sum_{j=1}^{\infty} |f_j| = g \in L^1$ , we can apply DCT to the partial sums to get the result.  $\square$

**Definition 3.6.** The support of a function  $f : X \rightarrow \mathbb{C}$  is the set  $\{x : f(x) \neq 0\}$ .

**Theorem 3.5** ( $L^1$  Approximation of Functions). *Let  $f \in L^1(\mu)$ . For any  $\epsilon > 0$ , there exists a simple function  $\phi = \sum_{j=1}^n a_j \chi_{E_j}$  such that  $\int |f - \phi| < \epsilon$ . If  $(X, \mu) = (\mathbb{R}, m)$ , then we can take each  $E_j$  to be a finite union of open intervals. Also, there exists a continuous function  $g$  with compact support such that  $\int |f - g| < \epsilon$ .*

## 4 Modes of Convergence

Let  $f_n : X \rightarrow \mathbb{C}$  and  $f : X \rightarrow \mathbb{C}$ .

**Definition 4.1.**  $f_n \rightarrow f$  pointwise if  $f_n(x) \rightarrow f(x)$  for all  $x \in X$ .

**Definition 4.2.**  $f_n \rightarrow f$  uniformly if  $\sum_{x \in X} |f_n(x) - f(x)| \rightarrow 0$ .

**Definition 4.3.**  $f_n \rightarrow f$  almost everywhere if  $f_n(x) \rightarrow f(x)$  for all  $x \in N^c$  with  $\mu(N) = 0$ .

**Definition 4.4.**  $f_n \rightarrow f$  in  $L^1$  if  $\int_X |f_n - f| d\mu \rightarrow 0$ .

**Definition 4.5.**  $f_n \rightarrow f$  in measure if for all  $\epsilon > 0$ ,  $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \rightarrow 0$ .

We have the following implications

- uniform convergence implies pointwise convergence
- pointwise convergence implies almost everywhere convergence
- convergence in  $L^1$  implies convergence in measure
- uniform convergence implies convergence in measure
- uniform convergence implies convergence in  $L^1$  on a finite measure space
- almost everywhere convergence implies convergence in measure on a finite measure space
- almost everywhere convergence implies convergence in  $L^1$  if we can apply DCT
- convergence in measure implies almost everywhere convergence if we allow subsequences

**Theorem 4.1** (Egoroff). *If  $\mu(X) < \infty$ , and  $\{f_n\}_{n=1}^{\infty}$  are measurable, with  $f_n \rightarrow f$  almost everywhere, then  $f_n \rightarrow f$  almost uniformly. That is, for any  $\epsilon > 0$ , there exists  $E \subset X$  with  $\mu(E) < \epsilon$  such that  $f_n \rightarrow f$  uniformly on  $E^c$ .*

**Remark 4.1.** Almost uniform convergence implies convergence in measure.

*Proof.* For  $k \in \mathbb{N}$  let  $E_n(k) = \bigcup_{m=n}^{\infty} \{x \in X : |f_m(x) - f(x)| \geq \frac{1}{k}\}$ . These sets are decreasing with  $\mu(\bigcap_{n=1}^{\infty} E_n(k)) = 0$  by almost everywhere convergence. By continuity from above, we have  $\lim_{n \rightarrow \infty} \mu(E_n(k)) = 0$ . Hence, for any  $k$  and  $\epsilon$ , there is some  $n_k$  such that  $\mu(E_{n_k}(k)) < \epsilon 2^{-k}$ , so

$$\mu\left(E := \bigcup_{k=1}^{\infty} E_{n_k}(k)\right) \leq \sum_{k=1}^{\infty} \mu(E_{n_k}(k)) < \epsilon.$$

If  $x \notin E$ , then  $|f_n(x) - f(x)| < \frac{1}{k}$  for sufficiently large  $n$ , so  $f_n \rightarrow f$  uniformly on  $E^c$ .  $\square$

**Definition 4.6.** A sequence of functions  $f_n$  is Cauchy in measure if for any  $\epsilon > 0$ ,

$$\lim_{m,n \rightarrow \infty} \mu(\{|f_n - f_m| \geq \epsilon\}) = 0.$$

**Theorem 4.2.** Let  $\{f_n\}_{n=1}^{\infty}$  be Cauchy in measure. Then

- $f_n \rightarrow f$  in measure for some  $f$ .
- There exists a subsequence  $f_{n_j}$  that converges to  $f$  almost everywhere.
- If  $f_n \rightarrow g$  in measure, then  $f = g$  almost everywhere.

*Proof.* See text.  $\square$

## 5 Product Measures

Consider measure spaces  $(X, \mathcal{M}, \mu)$  and  $(Y, \mathcal{N}, \nu)$ .

**Definition 5.1.** Define  $\mathcal{M} \otimes \mathcal{N}$  to be the  $\sigma$ -algebra generated by rectangles of the form  $A \times B = \{(x, y) : x \in A, y \in B\}$  where  $A \in \mathcal{M}$  and  $B \in \mathcal{N}$ .

Let  $\mathcal{A}$  be the set of finite disjoint unions of rectangles. Then  $\pi : \mathcal{A} \rightarrow [0, \infty]$  with  $\bigcup (A_j \times B_j) \mapsto \sum \mu(A_j) \nu(B_j)$  is a well-defined premeasure.

**Definition 5.2.** The product measure  $\mu \times \nu$  is the extension of  $\pi$  to  $\mathcal{M} \otimes \mathcal{N}$ .

**Definition 5.3.** Let  $E \in \mathcal{M} \otimes \mathcal{N}$ . Define  $E_x = \{y \in Y : (x, y) \in E\}$ , and  $E^y = \{x \in X : (x, y) \in E\}$ . Let  $f : X \times Y \rightarrow \mathbb{C}$ . Define  $f_x : y \mapsto f(x, y)$  and  $f^y : x \mapsto f(x, y)$ .

**Proposition 5.1.** Let  $E \in \mathcal{M} \otimes \mathcal{N}$ . Then  $E_x \in \mathcal{N}$  and  $E^y \in \mathcal{M}$ .

*Proof.* Let  $R = \{E \subset X \times Y : E_x \in \mathcal{N}, E^y \in \mathcal{M}\}$ . Then  $R$  contains all the rectangles. Furthermore,  $R$  is a  $\sigma$ -algebra (exercise). Hence  $\mathcal{M} \otimes \mathcal{N} \subset R$ .  $\square$

**Proposition 5.2.** If  $f$  is  $\mathcal{M} \otimes \mathcal{N}$ -measurable, then  $f_x$  is  $\mathcal{N}$ -measurable and  $f^y$  is  $\mathcal{M}$ -measurable.

*Proof.*  $f_x^{-1}(B) = (f^{-1}(B))_x \in \mathcal{N}$  for any Borel  $B$ .  $\square$

**Theorem 5.1** (Slicing Theorem). Let  $E \in \mathcal{M} \otimes \mathcal{N}$ . Then  $x \mapsto \nu(E_x)$  and  $y \mapsto \mu(E^y)$  are measurable, and  $(\mu \times \nu)(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$ .

*Proof.* Suppose that  $\mu$  and  $\nu$  are finite.

1. We check that the conclusion holds for rectangles  $E = A \times B$ . Observe that

$$\nu(E_x) = \chi_A(x) \nu(B) \implies \int_X \nu(E_x) d\mu = \mu(A) \nu(B) = (\mu \times \nu)(E).$$

2. The conclusion also holds for finite disjoint unions of rectangles by the additivity of measures and integrals.
3. Let  $\mathcal{C} = \{E \in \mathcal{M} \otimes \mathcal{N} : \text{conclusion holds}\}$ . Then  $\mathcal{A} \subset \mathcal{C}$  where  $\mathcal{A}$  is the finite disjoint union of rectangles.

We show that  $\mathcal{C}$  is a monotone class. That is it is closed under increasing union and decreasing intersection (ie. monotone union and intersection).

$\cup$ : Let increasing  $\{E_n\} \subset \mathcal{C}$  with  $E = \bigcup E_n$ . Then  $f_n(y) = \mu(E_n^y) \uparrow f(y) = \mu(E^y)$ . Hence by MCT and continuity from below,

$$\int_Y \mu(E^y) d\nu = \lim_{n \rightarrow \infty} \int_Y \mu(E_n^y) d\nu = \lim_{n \rightarrow \infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E).$$

Therefore,  $E \in \mathcal{C}$ .

$\cap$ : Let decreasing  $\{E_n\} \subset \mathcal{C}$  with  $E = \bigcap E_n$ . Then  $f_n(y) = \mu(E_n^y) \leq f_1(y) < \infty$  by monotonicity and finiteness of  $\mu$  and  $\nu$ . Hence by DCT and continuity from above,

$$\int_Y \mu(E^y) d\nu = \lim_{n \rightarrow \infty} \int_Y \mu(E_n^y) d\nu = \lim_{n \rightarrow \infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E).$$

Therefore,  $E \in \mathcal{C}$ .

Since  $\mathcal{C}$  is a monotone class that contains  $\mathcal{A}$ , it contains the  $\sigma$ -algebra generated by  $\mathcal{A}$  (See text for proof). Hence  $\mathcal{C} = \mathcal{M} \otimes \mathcal{N}$ .

4. If  $\mu$  and  $\nu$  are  $\sigma$ -finite, then consider  $X \times Y = \bigcup_{j=1}^{\infty} A_j \times B_j$  as the union of rectangles with finite measure. Then apply the above to each rectangle. Details omitted.

□

**Theorem 5.2** (Tonelli). *Let  $f \in L^+(\mu \times \nu)$  with  $\sigma$ -finite  $\mu$  and  $\nu$ . Then  $x \mapsto \int_Y f_x d\nu \in L^+(\mu)$  and  $y \mapsto \int_X f^y d\mu \in L^+(\nu)$ , and*

$$\int_X \int_Y f d\nu d\mu = \int_Y \int_X f d\mu d\nu = \int_{X \times Y} f d(\mu \times \nu).$$

*Proof.* • If  $f = \chi_E$ , use the slicing theorem.

- If  $f \in L^+$  is simple, follows from above by additivity.
- If  $f \in L^+$ , let  $0 \leq f_n^{\text{simple}} \uparrow f$ . Let  $g_n(x) = \int_Y (f_n)_x d\nu$  and  $g(x) = \int_Y f_x d\nu$ . Then  $g_n \uparrow g$  and  $\int g_n \rightarrow \int g$  by MCT, and we get the result

$$\int_{X \times Y} f d(\mu \times \nu) = \lim_{n \rightarrow \infty} \int_{X \times Y} f_n d(\mu \times \nu) = \lim_{n \rightarrow \infty} \int_X g_n d\mu = \int_X g d\mu.$$

□

**Theorem 5.3** (Fubini). *Let  $f \in L^1(\mu \times \nu)$ . ... TLDR: We can also change the order of iterated integrals.*

*Proof.* Let  $f : X \times Y \rightarrow \mathbb{C}$ . Write  $f = (\Re(f)^+ - \Re(f)^-) + i(\Im(f)^+ - \Im(f)^-)$  and apply Tonelli. □

### 5.1 Lebesgue Measure on $\mathbb{R}^n$

All the nice properties for  $\mathbb{R}$  also happen in  $\mathbb{R}^n$  ( + invariance of rotation ).

## 6 Differentiation of Measures