Math 539 Notes

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Contents

1 Introduction

Motivating questions (some statistics):

- the "probability" that a random number has some property
- the "distribution" of some given multiplicative/additive function

Idea: we can answer the question for $\{1, ..., \lfloor x \rfloor\}$ for some parameter x. Then, take the limit $x \to \infty$ for all natural numbers.

1.1 Notation

Let $g(x) \geq 0$.

Definition 1.1.1. $O(g(x))$ means some unspecified function $u(x)$ such that $|u(x)| \leq cg(x)$ for some constant $c > 0$.

Example 1.1.2. Show that $e^{2x} - 1 = 2x + O(x^2)$ for $x = [-1, 1]$.

Proof. Observe that $f(z) = e^{2z} - 1 - 2z$ is analytic (and entire) and has a double zero at $z = 0$ (one can check that $f(z) = f'(z) = 0$. Hence, $g(z) = (e^{2z} - 1 - 2z)/z^2$ has a removable singularity at $z = 0$, whence g is analytic and entire. Let $C = \max\{|g(z)| : |z| \leq 1\}$. Then

$$|g(z)| \leq C \implies \left|e^{2z} - 1 - 2z\right| \leq C\left|z^2\right| \implies e^{2z} - 1 - 2z = O(|z|^2).$$

$\square$

Exercise 1.1.3. Show that $\sqrt{x+1} = \sqrt{x} + O(1/\sqrt{x})$ for $x \in [1, \infty)$.

Definition 1.1.4. $f(x) \ll g(x)$ means $f(x) = O(g(x))$.

Exercise 1.1.5. Suppose that $f_1 \ll g_1, f_2 \ll g_2$, then $f_1 + f_2 \ll \max\{g_1, g_2\}$. ✓

Exercise 1.1.6. Let f, g be continuous on $[0, \infty)$, and $f \ll g$ on $[123, \infty)$. Show that $f \ll g$ on $[0, \infty)$. ✓

Definition 1.1.7. $f(x) \sim g(x)$ means $\lim \frac{f(x)}{g(x)} = 1$.

Definition 1.1.8. $f(x) = o(g(x))$ means $\lim \frac{f(x)}{g(x)} = 0$.

Definition 1.1.9. $f(x) = O_y(g(x))$ means f, g depend on some parameter y, and the implicit constant depends on y.

Exercise 1.1.10. For any $A, \epsilon > 0$, show that $(\log x)^A \ll_{A,\epsilon} x^\epsilon$.

1.2 Riemann-Stieltjes Integral

Appendix A in the book.

Definition 1.2.1. Some definitions for partitions

1. Let $\underline{x} = \{x_0, ..., x_N\}$ be a partition of $[c, d]$ if $c = x_0 < \cdots < x_N = d$.
2. The mesh size $m(\underline{x}) = \max_{1 \le j \le N} x_j - x_{j-1}$.
3. Sample points $\xi_j \in [x_{j-1}, x_j]$.

Definition 1.2.2 (Riemann-Stieltjes Integral)**.** Given two functions $f(x)$ and $g(x)$, define the Riemann-Stieltjes integral as

$$\int_c^d f(x)\ dg(x) = \lim_{m(\underline{x}) \to 0} \sum_{j=1}^{N} f(\xi_j)(g(x_j) - g(x_{j-1})).$$

Remark 1.2.3. Setting $g(x) = x$ gives the Riemann integral.

Theorem 1.2.4. *Let $f(x)$ have bounded variation and let $g(x)$ be continuous on $[c, d]$, or vice versa. Then $\int_c^d f(x)\ dg(x)$ exists.*

Remark 1.2.5. If a function is piecewise monotone, then it has bounded variation.

Example 1.2.6. Given a sequence $a_{n n \in \mathbb{N}}$, define the summatory function $A(x) = \sum_{n \le x} a_n$. Then, on any $[c, d]$, $A(x)$ is bounded, piecewise continuous and piecewise monotone. Hence, the Riemann-Stieltjes integral exists when g is continuous.

Remark 1.2.7. We present 3 facts that we will use.

1. If $A(x)$ is the summatory function as above, and $f(x)$ is continuous, then

$$\int_c^d f(x)\ dA(x) = \sum_{c < n \le d} a_n f(n).$$

2. (Integration by parts). If the integrals exist, then

$$\int_c^d f(x)\ dg(x) = f(x)g(x)|_c^d - \int_c^d g(x)\ df(x).$$

3. If $f(x)$ is continuously differentiable, then

$$\int_c^d g(x)\ df(x) = \int_c^d g(x)f'(x)\ dx.$$

Example 1.2.8 (Summation by parts)**.** Consider $\sum_{n\leq y} \frac{a_n}{n}$. Let $f(x) = 1/x$, then we can write

$$\sum_{n\leq y} \frac{a_n}{n} = \sum_{n\leq y} a_n \cdot \frac{1}{n} = \int_0^y \frac{1}{x}\ dA(x) = \frac{1}{x}A(x)\bigg|_0^y - \int_0^y A(x)\ d\left(\frac{1}{x}\right) = \frac{A(y)}{y} - \int_0^y A(x)\frac{1}{x^2}\ dx.$$

The final manipulation that we want to get is

$$\sum_{n\leq y} a_n f(n) = A(y)f(y) - \int_0^y A(x)f'(x)\ dx. \tag{1}$$

2 Dirichlet Series

A Dirichlet series is $\sum_{n=1}^{\infty} a_n n^{-s}$.
Facts about Dirichlet series:

- converge in some right half-plane $\{s \in \mathbb{C} : \Re s > R\}$ for some R (possibly $R = \pm\infty$).
- Sometimes converge conditionally. Example: $\sum_{n=1}^{\infty}(-1)^n/n^{1/2}$.
- $\left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\left(\sum_{n=1}^{\infty} b_n n^{-s}\right) = \sum_{n=1}^{\infty} c_n n^{-s}$ where $c = \sum_{de=n} a_d b_e = \sum_{d|n} a_d b_{e/d}$. (multiplicative convolution)

Some notation: for $s \in \mathbb{C}$, we write $s = \sigma + it$, that is σ is the real part of s, and t is the imaginary part of s. Note that if $x > 0$, then $|x^s| = |x^\sigma|\left|x^{it}\right| = |x^\sigma|\left|e^{it\log x}\right| = |x^\sigma|$.

Theorem 2.0.1 (thm 1.1)**.** *Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. Suppose that $s_0 \in \mathbb{C}$ is such that $\alpha(s_0)$ converges. Then $\alpha(s)$ converges uniformly in the sector*

$$S = \{s \in \mathbb{C} : \sigma \geq \sigma_0, |t - t_0| \leq H\,|\sigma - \sigma_0|\}$$

for any $H > 0$.

Proof. WLOG, let $s_0 = 0$, otherwise we can do a change of variables.
Let $A(x) = \sum_{n\leq x} a_n = \alpha(0) - R(x)$. Then, for $\sigma > 0$,

$$\begin{aligned}\sum_{M<n\leq N} a_n n^s &= \int_M^N x^{-s}\ dA(x) = \int_M^N x^{-s}\ d(\alpha(0) - R(x)) \\ &= \int_M^N x^{-s}\ d\alpha(0) - \int_M^N x^{-s}\ dR(x) = -\int_M^N x^{-s}\ dR(x) \\ &= -x^{-s}R(x)\big|_M^N + \int_M^N R(x)\ d(x^{-s}) \\ &= R(M)M^{-s} - R(N)N^{-s} - s\int_M^N R(x)x^{-s-1}\ dx.\end{aligned}$$

Note that $R(N)N^{-s} \to 0$ as $N \to \infty$, and that $R(x)x^{-s-1} \ll x^{-\sigma-1}$. Hence, letting $N \to \infty$ gives

$$\sum_{M<n} a_n n^{-s} = R(M)M^{-s} - s\int_M^\infty R(x)x^{-s-1}\,dx \to 0 \text{ as } M \to \infty.$$

Now, choose M large such that $|R(x)| < \epsilon$ for all $x \geq M$. Then,

$$\begin{aligned}\left|\sum_{n>M} a_n n^{-s}\right| &\leq \epsilon M^{-\sigma} + |s|\int_M^\infty \epsilon x^{-\sigma-1}\,dx \\ &= \epsilon M^{-\sigma} + |s|\,\epsilon x^{-\sigma}\frac{1}{-\sigma}\Big|_M^\infty \\ &= \epsilon M^{-\sigma} + |s|\,\epsilon\frac{M^{-\sigma}}{\sigma} = \frac{\epsilon}{M^\sigma}\left(1 + \frac{|s|}{\sigma}\right).\end{aligned}$$

Since $s \in S$, we have

$$|s| = \sqrt{\sigma^2 + t^2} \leq \sqrt{\sigma^2 + |H\sigma|^2} = \sigma\sqrt{1 + H^2},$$

so $\left|\sum_{n>M} a_n n^{-s}\right| \leq \epsilon(1 + \sqrt{1 + H^2})$ as $M \to \infty$. Observe that the latter only depends on H, so the convergence is uniform. □

Corollary 2.0.2. *If $\alpha(s_0)$ converges, then $\alpha(s)$ converges for all s with $\sigma > \sigma_0$.*

Corollary 2.0.3. *If $\alpha(s_0)$ diverges, then $\alpha(s)$ diverges for all s with $\sigma < \sigma_0$.*

Remark 2.0.4. The Dirichlet series $\alpha(s)$ has an abscissa of convergence σ_c such that $\alpha(s)$ converges if $\sigma > \sigma_c$, and diverges if $\sigma < \sigma_c$. It is allowed to have $\sigma_c = \pm\infty$. Furthermore, $\alpha(s)$ converges locally uniformly right of σ_c and each $a_n n^{-s}$ is analytic, whence $\alpha(s)$ is analytic. (Conway; Theorem VII.2.1; p.147)

Remark 2.0.5. Observe that $\int_1^N x^{-s}\,dA(x) = \sum_{1<n\leq N} a_n n^{-s} = \sum_{n=2}^N a_n n^{-s}$. Sometimes we write $\int_{-1}^N$ to include the 1.

Theorem 2.0.6 (thm 1.3)**.** *Let $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$ have an abscissa of convergence $\sigma_c \geq 0$. Then for $\sigma > \sigma_c$, we have $\alpha(s) = s\int_1^\infty A(x)x^{-s-1}\,dx$. Moreover,*

$$\limsup_{x\to\infty} \frac{\log|A(x)|}{\log x} = \sigma_c.$$

Proof. Observe that

$$\begin{aligned}\sum_{n=1}^N a_n n^{-s} &= \int_{1^-}^N x^{-s}\,dA(x) = x^{-s}A(x)\big|_{1^-}^N - \int_{1^-}^N A(x)\,d(x^{-s}) \\ &= A(N)N^{-s} - \int_{1^-}^N A(x)(-sx^{-s-1}\,dx) = A(N)N^{-s} + s\int_1^N A(x)x^{-s-1}\,dx.\end{aligned}$$

Observe that in the last line, we can replace 1^- with 1 because the integrand is bounded.
Define $\phi = \limsup_{x\to\infty} \frac{\log|A(x)|}{\log x}$. We compare this to σ_c.
Let $\sigma = \phi + \epsilon$ for some $\varepsilon > 0$. Then $\frac{\log|A(x)|}{\log x} < \phi + \frac{\varepsilon}{2}$ for large x, so $A(x) \ll x^{\phi+\varepsilon/2}$. Then, $A(N)N^{-s} \ll N^{\phi+\varepsilon/2}N^{-(\phi+\varepsilon)} = N^{-\varepsilon/2}$. Hence,

$$\int_N^\infty A(x)x^{-\sigma-1}\,dx \ll \int_N^\infty x^{-\phi+\varepsilon/2}x^{-(\phi+\varepsilon+1)}\,dx = \int_N^\infty x^{-1-\varepsilon/2}\,dx \ll N^{-\varepsilon/2}.$$

It follows that

$$\sum_{n=1}^{N} a_n n^{-s} = O(N^{-\varepsilon/2}) + s\left(\int_1^\infty A(x)x{-}s-1\ dx + O(N^{-\varepsilon/2})\right).$$

Let $N \to \infty$ gives $s\int_1^\infty A(x)x^{-s-1}\ dx$ converges. Hence $\sigma_c \leq \phi$.
Conversely, let $\sigma_0 = \sigma_c + \varepsilon$, and let $R_0(x) = \sum_{n>x} a_n n^{-\sigma_0} = \alpha(\sigma_0) - \sum_{n\leq x} a_n n^{-\sigma_0}$. Observe that

$$A(N) = -R_0(N)N^{\sigma_0} + \sigma_0 \int_0^N R_0(x)x^{\sigma_0-1}\ dx.$$

Since $\alpha(0)$ converges, $R_0(x) = o(1)$ so $R_0(x) \ll 1$. Then

$$A(N) \ll 1 \cdot N^{\sigma_0} + \sigma_0 \int_0^N 1 \cdot x^{\sigma_0-1}\ dx = N^{\sigma_0} + N^{\sigma_c} \ll N^{\sigma_0} = N^{\sigma_c+\varepsilon}.$$

Hence $\frac{\log|A(x)|}{\log x} \ll \frac{(\sigma_c+\varepsilon)\log x}{\log x} = \sigma_c + \varepsilon$, so $\phi \leq \sigma_c + \varepsilon$. Take $\varepsilon \to 0$, so $\phi \leq \sigma_c$. □

Definition 2.0.7. The abscissa of absolute convergence is $\sigma_a = \inf\{\sigma \in \mathbb{R} : \sum_{n=1}^\infty |a_n|\, n^{-\sigma} \text{ converges}\}$.

Example 2.0.8. Let $\eta(s) = \sum_{n=1}^\infty (-1)^{n-1} n^{-s}$. Observe that $\sigma_c = 0$ by the alternating series test. However, we only have absolute convergence when $\sigma > 1$, so the abscissa of absolute convergence is $\sigma_a = 1$.

Remark 2.0.9. When $a_n \geq 0$ for all $n \in \mathbb{N}$, we have $\sigma_c = \sigma_a$.

Theorem 2.0.10 (thm 1.4)**.** *For any Dirichlet series $\alpha(s)$, we have $\sigma_c \leq \sigma_a \leq \sigma_c + 1$.*

Proof. The first inequality is trivial.
Let $\sigma = \sigma_c + 1 + \varepsilon$ where $\varepsilon > 0$. We show that $\alpha(\sigma)$ converges absolutely. Note that $\alpha(s)$ converges at $s = \sigma_c + \varepsilon/2$, that is

$$a_n n^{-(\sigma_c+\varepsilon/2)} = o(1) \implies a_n n^{-(\sigma_c+\varepsilon/2)} \ll 1.$$

Then,

$$\sum_{n=1}^\infty |a_n|\, n^{-(\sigma_c+1+\varepsilon)} = \sum_{n=1}^\infty \left|a_n n^{-(\sigma_c+\varepsilon/2)}\right| n^{-(1+\varepsilon/2)} \ll \sum_{n=1}^\infty n^{-(1+\varepsilon/2)} \ll 1.$$

It follows that $\alpha(\sigma)$ converges absolutely for all $\varepsilon > 0$. □

Theorem 2.0.11 (Landau's Theorem (thm 1.7))**.** *Let $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$ with $\sigma_c < \infty$. If $a_n \geq 0$ for each $n \in \mathbb{N}$. Then, $\alpha(s)$ has a singularity at $s = \sigma_c$.*

Proof. Suppose that there does not exist a singularity at $s = \sigma_c$. Then, there exists an analytic continuation of α to $C = \{s \in \mathbb{C} : |s - \sigma_c| < \delta\}$.
Let $z = \sigma_c - \frac{1}{4}\delta$ and let $w = \sigma_c + \frac{3}{4}\delta$. Let $D = \left\{s \in \mathbb{C} : |s - w| < \frac{5}{4}\delta\right\}$. Observe that $D \subset C \cup \{s \in \mathbb{C} : \sigma > 0\}$, so α has an analytic continuation to D. Let $P(s)$ be the power series of α

centered at w. Observe that $z \in D$, so it suffices to show that $P(z) = \alpha(z)$, whence we contradict the assumption that the abscissa of convergence is σ_c. Note that

$$\begin{aligned}
P(z) &= \sum_{k=0}^{\infty} \frac{\alpha^{(k)}(w)}{k!}(z-w)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}(z-w)^k \sum_{n=1}^{\infty} a_n(-\log n)^k n^{-w} && \text{we can differentiate termwise for } \alpha^{(k)}(w) \\
&= \sum_{k=1}^{\infty} \frac{1}{k!}(w-z)^k \sum_{n=1}^{\infty} a_n(\log n)^k n^{-w} && \text{where the terms are all nonnegative} \\
&= \sum_{n=1}^{\infty} a_n n^{-w} \sum_{k=1}^{\infty} \frac{1}{k!}(w-z)^k (\log n)^k \\
&= \sum_{n=1}^{\infty} a_n n^{-w} e^{(w-z)\log n} = \sum_{n=1}^{\infty} a_n n^{-z}.
\end{aligned}$$

It follows that $\alpha(z)$ converges left of σ_c, which is a contradiction. □

2.1 Dirichlet convolutions

Motivating question: are these calculations legitimate?

- $\zeta(s)^2 = \sum_{l,m=1}^{\infty}(lm)^{-s} = \sum_{n=1}^{\infty} d(n)n^{-s}$.
- $\zeta(s) = \prod_{p \text{ prime}}(1 + p^{-s} + p^{-2s} + \cdots) = \prod_{p \text{ prime}}(1 - p^{-s})^{-1}$.

Definition 2.1.1. Let $a = \{a_n\}$, $b = \{b_n\}$ be sequences. The Dirichlet/multiplicative convolution $a * b$ by $c = \{c_n\}$ where $c_n = \sum_{de=n} a_d b_e = \sum_{d|n} a_d b_{n/d}$.

Theorem 2.1.2 (thm 1.8)**.** *Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, let $\beta(s) = \sum_{n=1}^{\infty} b_n n^{-s}$, and let $\gamma(s) = \sum_{n=1}^{\infty} c_n n^{-s}$. If $s \in \mathbb{C}$ is such that $\alpha(s)$ and $\beta(s)$ converge absolutely, and if $c = a * b$, then $\gamma(s)$ converges absolutely and $\gamma(s) = \alpha(s)\beta(s)$.*

Example 2.1.3. Observe that $d(n) = (1 * 1)(n)$.

Example 2.1.4. Let $M(s) = \sum_{n=1}^{\infty} \mu(n)n^{-s}$, where μ is the Möbius function which defined as

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not square free} \\ 1 & \text{if } n \text{ has an even number of prime divisors} \\ -1 & \text{if } n \text{ has an odd number of prime divisors} \end{cases}.$$

Equivalently, we can define μ as the function that satisfies

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}.$$

Observe that $M(s)\zeta(s) = \sum_{n=1}^{\infty}(\mu * 1)(n)n^{-s} = 1$ for $\sigma > 1$, since $(\mu * 1)(n) = \sum_{d|n} \mu(d)$. It follows that $M(s) = 1/\zeta(s)$.
Since the abscissa of convergence of M is $\sigma_c = 1$, we get that $\zeta(s)$ has no zeroes when $\sigma > 1$.

Example 2.1.5 (Möbius Inversion Formula)**.** Write $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ and $G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$. Then

$$\begin{aligned} F(s)\zeta(s) = G(s) &\iff F(s) = \frac{G(s)}{\zeta(s)} = G(s)M(s) \\ (f * 1)(n) = g(n) &\iff f(n) = (g * \mu)(n) \\ \sum_{d|n} f(n) = g(n) &\iff f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right). \end{aligned} \tag{2}$$

Example 2.1.6. It is known that $\sum_{d|n} \phi(d) = n$. This gives $(\phi * 1)(n) = \sum_{d|n} \phi(d) = n$. Then, for $\sigma > 2$, we have

$$\left(\sum_{n=1}^{\infty} \phi(n)n^{-s}\right)\left(\sum_{n=1}^{\infty} n^{-s}\right) = \sum_{n=1}^{\infty} n \cdot n^{-s} = \zeta(s-1).$$

This gives $\sum_{n=1}^{\infty} \phi(n)n^{-s} = \zeta(s-1)/\zeta(s)$.

Exercise 2.1.7. Let $\sigma_1(n) = \sum_{d|n} d$. Show that $\sum_{n=1}^{\infty} \sigma_1(n)n^{-s} = \zeta(s-1)\zeta(s)$.

Definition 2.1.8. A function f is multiplicative if $f(m)f(n) = f(mn)$ if $\gcd(m, n) = 1$.

Definition 2.1.9. A number n is y-friable if $p \mid n \implies p \leq y$.

Theorem 2.1.10 (thm 1.9)**.** *Let f be a multiplicative function, and let $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$. If $\sum_{n=1}^{\infty} |f(n)|\, n^{-\sigma}$ converges, we have the Euler product*

$$F(s) = \prod_{p \text{ prime}} (1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots).$$

Proof. Let $\sigma > \sigma_a$. Then, for all p, we have

$$\left|1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right| \leq 1 + |f(p)|\, p^{-s} + |f(p)|\, p^{-2s} + \cdots \leq \sum_{n=1}^{\infty} |f(n)|\, n^{-s}.$$

Since the above converges, we can rearrange the finite product

$$\prod_{p \leq y}(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots) = \sum_{n\ y\text{-friable}} f(n)n^{-s}.$$

Now, we can compute

$$\begin{aligned} \left|F(s) - \prod_{p \leq y}(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots)\right| &= \left|F(s) - \sum_{n\ y\text{-friable}} f(n)n^{-s}\right| \\ &= \left|\sum_{n \text{ not } y\text{-friable}} f(n)n^{-s}\right| \\ &\leq \sum_{n > y} |f(n)|\, n^{-s} = o(1). \end{aligned}$$

The tail goes to 0, so the theorem is proved. ∎

Remark 2.1.11. Almost the same proof shows that the Euler product converges absolutely. In particular, it is nonzero (unless an individual factor is zero). Note that the convergence of a product is defined as the convergence of the sum of logs.

Example 2.1.12. Note that μ is multiplicative, so $M(s) = \prod_{p \text{ prime}}(1 - p^{-s})$.

Property 2.1.13. If f and g are multiplicative, then $f * g$ is also multiplicative (If $F(s)$ and $G(s)$ have Euler products, the $F(s)G(s)$ also has an Euler product).

Property 2.1.14. Dirichlet convolutions are associative, that is $(f * g) * h = f * (g * h)$.

Definition 2.1.15. Let $\omega(n)$ be the number of distinct prime factors of n.

Definition 2.1.16. Let $\Omega(n)$ be the number of prime factors of n counting with multiplicity.

Definition 2.1.17 (Liouville lambda function)**.** Let $\lambda(n) = (-1)^{\Omega(n)}$. Note that $\lambda(n) = \mu(n)$ if and only if n is squarefree.

Example 2.1.18. Find an Euler product for $L(s) = \sum_{n=1}^{\infty} \lambda(n) n^{-s}$.
Solution: First, note that $\lambda(n)$ is totally multiplicative, that is $\lambda(mn) = \lambda(m)\lambda(n)$ for all $m, n \in \mathbb{N}$ (not just when $(m, n) = 1$). Also, $\sum_{n=1}^{\infty} |\lambda(n) n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma}$ converges when $\sigma > 1$. So, for $\sigma > 1$, by theorem 2.1.10,

$$\begin{aligned} L(s) &= \prod_{p \text{ prime}} \left(1 + \lambda(p)p^{-s} + \lambda(p^2)p^{-2s} + \cdots\right) \\ &= \prod_{p \text{ prime}} \left(1 - p^{-s} + p^{-2s} - \cdots\right) \\ &= \prod_{p \text{ prime}} \left(1 + p^{-s}\right)^{-1}. \end{aligned}$$

Exercise 2.1.19. Show that $L(s) = \frac{\zeta(2s)}{\zeta(s)}$.

Remark 2.1.20. If $f(n)$ is totally multiplicative, then

$$\begin{aligned} \sum_{n=1}^{\infty} f(n) n^{-s} &= \prod_{p} \left(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right) \\ &= \prod_{p} \left(1 + f(p)p^{-s} + f(p)^2 p^{-2s} + \cdots\right) \\ &= \prod_{p} \left(1 - f(p)p^{-s}\right)^{-1}. \end{aligned}$$

Definition 2.1.21 (von Mangoldt Lambda function)**.** Define the von Mangoldt Lambda function as

$$\Lambda(n) = \begin{cases} \log p & n = p^r \text{ for some prime } p \\ 0 & \text{otherwise} \end{cases}.$$

Remark 2.1.22. Recall that $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$ for $\sigma > 1$, so we can take logarithms.

$$\log \zeta(s) = \sum_p \log(1 - p^{-s})^{-1} = \sum_p \left(p^{-s} + \frac{1}{2}p^{-2s} + \frac{1}{3}p^{-3s} + \cdots\right).$$

This is a Dirichlet series which we can differentiate term by term, hence

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} &= \frac{d}{ds}\sum_p \left(p^{-s} + \frac{1}{2}p^{-2s} + \frac{1}{3}p^{-3s} + \cdots\right) \\ &= \sum_p \left((-\log p)p^{-s} + \frac{1}{2}(-2\log p)p^{-2s} + \cdots\right) \\ &= -\sum_{n=1}^{\infty}\Lambda(n)n^{-s}.\end{aligned}$$

2.2 Meromorphic continuation of $\zeta(s)$

Example 2.2.1. We prove that $\eta(s) = (1-2^{1-s})\zeta(s)$ in two different ways (for $\sigma > 1$).

Proof 1. For $\sigma > 1$, $\eta(s)$ converges absolutely, so

$$\begin{aligned}\eta(s) &= 1^{-s} + (2^{-s} - 2\cdot 2^{-s}) + 3^{-s} + (4^{-s} - 2\cdot 4^{-s}) + \cdots \\ &= (1^{-s} + 2^{-s} + 3^{-s} + \cdots) - 2(2^{-s} + 4^{-s} + \cdots) \\ &= \zeta(s) - 2\cdot 2^{-s}\zeta(s) \\ &= (1-2^{1-s})\zeta(s).\end{aligned}$$

Proof 2. Note that $(-1)^{n-1}$ is multiplicative, and its value at p^r equals -1 if $p = 2$ and 1 if $p \geq 3$. Thus,

Note that $\zeta(s) = (1-2^{1-s})^{-1}\eta(s)$, and $\eta(s)$ converges when $\sigma > 0$. Hence, this is a meromorphic continuation of $\zeta(s)$ to $\sigma > 0$. Note that $(1-2^{1-s})^{-1}\eta(s)$ has singularities when $1 - 2^{1-s} = 0$, that is $s = 1 + \frac{2\pi i}{\log 2}$.

Exercise 2.2.2. Show that $\zeta(s)$ has a simple pole at $s = 1$ with residue 1.

Theorem 2.2.3 (thm 1.12)**.** *For $\sigma > 0$ and $s \neq 1$, we can write*

$$\zeta(s) = \sum_{n\leq x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^{\infty}\{u\}u^{-s-1}\ du. \tag{3}$$

Proof. For $\sigma > 1$, we have

$$\begin{aligned}\sum_{n>x} n^{-s} &= \int_x^{\infty} u^{-s}\ d(\lfloor u\rfloor) \\ &= \int_x^{\infty} u^{-s}\ du - \int_x^{\infty} u^{-s}\ d(\{u\}) \\ &= \left.\frac{u^{1-s}}{1-s}\right|_x^{\infty} - \left(\{u\}u^{-s}\big|_x^{\infty} - \int_x^{\infty}\{u\}\ d(u^{-s})\right) \\ &= \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s\int_x^{\infty}\{u\}u^{-s-1}\ du.\end{aligned}$$

Let $\varepsilon > 0$, then for $\sigma > \varepsilon$, we have

$$\left|\int_x^{\infty}\{u\}u^{-s-1}\ du\right| \leq \int_x^{\infty} 1\cdot u^{-\sigma-1}\ du = \frac{x^{-\sigma}}{\sigma}.$$

Note that this is uniform for $\sigma > \varepsilon$, so we have analyticity. Then, we conclude that the equation 3 holds for all $\sigma > 0$ by the uniqueness of analytic continuation. □

Remark 2.2.4. We can use a similar method to show that $\zeta(s)$ is defined for all $s \in \mathbb{C} \setminus \{1\}$.

Corollary 2.2.5. *When $x = 1$, we have*

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \{u\} u^{-s-1}\, du,$$

so $\zeta(s) - \frac{1}{s-1}$ has a removable singularity at $s = 1$ with value

$$C_0 = 1 - \int_1^\infty \{u\} u^{-2}\, du.$$

Then, by DCT, we get $\zeta(s) = \frac{1}{s-1} + C_0 + O(|s-1|)$.

Corollary 2.2.6. *We can rearrange to get*

$$\begin{aligned}
\sum_{n \le x} n^{-s} &= \zeta(s) - \frac{x^{1-s}}{s-1} - \frac{\{x\}}{x^s} + s \int_x^\infty \{u\} u^{-s-1}\, du \\
&= \zeta(s) + \frac{x^{1-s}}{1-s} + O\left(\frac{1}{x^\sigma} + |s| \int_x^\infty 1 \cdot u^{-\sigma-1}\, du\right) \\
&= \zeta(s) + \frac{x^{1-s}}{1-s} + O\left(\frac{1}{x^\sigma} + \frac{|s|}{\sigma} \frac{1}{x^\sigma}\right).
\end{aligned}$$

Hence, we get asymptotics for $\sum_{n \le x} n^{-\alpha}$.

$$\sum_{n \le x} \frac{1}{n^\alpha} = \begin{cases} O(x^{1-\alpha}) & \text{if } 0 < \alpha < 1 \\ \zeta(\alpha) + O(x^{1-\alpha}) & \text{if } \alpha > 1 \end{cases}.$$

Corollary 2.2.7. *Let $s = 1$, then*

$$\begin{aligned}
\sum_{n \le x} \frac{1}{n} &= \int_{1^-}^x \frac{1}{t}\, d\lfloor t \rfloor = \int_{1^-}^x \frac{1}{t}\, dt - \int_{1^-}^x \frac{1}{t}\, d\{t\} \\
&= \log x - \left.\frac{\{t\}}{t}\right|_{1^-}^x + \int_1^x \{t\} \frac{1}{t^2}\, dt \\
&= \log x - \frac{\{x\}}{x} + 1 - \int_1^x \{t\} \frac{1}{t^2}\, dt \\
&= \log x + 1 - \int_1^\infty \{t\} t^{-2}\, dt + \int_x^\infty \{t\} t^{-2}\, dt - \frac{\{x\}}{x} \\
&= \log x + C_0 + O\left(\frac{1}{x}\right).
\end{aligned}$$

Note that an error of $1/x$ is the best approximation with a smooth function (because we can't get better than the jumps).

Remark 2.2.8. Note that C_0 is Euler's constant, that is

$$C_0 = \lim_{x \to \infty} \left(\sum_{n \le x} \frac{1}{n} - \log x\right) \approx 0.577.$$

3 Elementary Estimates for Arithmetic Functions

Motivating question: what is the expectation of $\frac{\phi(n)}{n}$?
Note that $(\phi * 1)(n) = n$, so by Möbius inversion,

$$\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d} \implies \frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}.$$

Then, we get

$$\sum_{n\le x} \frac{\phi(n)}{n} = \sum_{n\le x}\sum_{d|n} \frac{\mu(d)}{d} = \sum_{d\le x} \frac{\mu(d)}{d}\left\lfloor \frac{x}{d} \right\rfloor = \sum_{d\le x} \frac{\mu(d)}{d}\left(\frac{x}{d} + O(1)\right).$$

Hence, dividing by x gives

$$\begin{aligned}
\frac{1}{x}\sum_{n\le x} \frac{\phi(n)}{n} &= \sum_{d\le x} \frac{\mu(d)}{d^2} + O\left(\frac{1}{x}\sum_{d\le x}\left|\frac{\mu(d)}{d}\right|\right) \\
&= \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O\left(\sum_{d>x}\left|\frac{\mu(d)}{d^2}\right| + \frac{1}{x}\sum_{d\le x}\left|\frac{\mu(d)}{d}\right|\right) \\
&= \frac{1}{\zeta(2)} + O\left(\sum_{d>x}\frac{1}{d^2} + \frac{1}{x}\sum_{d\le x}\frac{1}{d}\right) \\
&= \frac{1}{\zeta(2)} + O\left(\frac{\log x}{x}\right).
\end{aligned}$$

Example 3.0.1. Estimate the number of square-free numbers up to x. Let $Q(x) = \sum_{n\le x} \mu(n)^2$. Note that $\mu(n)^2$ is the indicator function for square-free numbers.

Lemma 3.0.2. *Define $g(d)$ as*

$$g(d) = \begin{cases} \mu(m) & \textit{if } d = m^2 \textit{ for some } m \in \mathbb{N} \\ 0 & \textit{if } d \textit{ is not a square} \end{cases}.$$

Then $\mu^2 = 1 * g$.

Proof 1. Let k^2 be the largest square divisor of n. Then

$$\begin{aligned}
\mu(n)^2 &= \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases} \\
&= \sum_{d|k} \mu(d) = \sum_{\substack{d|n \\ d=m^2}} \mu(m) = \sum_{d|n} g(d).
\end{aligned}$$

□

Exercise 3.0.3 (Proof 2)**.** Show that $\sum_{n=1}^{\infty} \mu(n)^2 n^{-s} = \frac{\zeta(s)}{\zeta(2s)}$ and $\sum_{n=1}^{\infty} g(n) n^{-s} = \frac{1}{\zeta(2s)}$.

Exercise 3.0.4 (Proof 3)**.** Show that $\mu(n)^2 = \sum_{d|n} g(d)$ by Möbius inversion, along with the fact that every $n \in \mathbb{N}$ can be uniquely written as $n = a^2 b$ where b is square-free.

We can approximate $Q(x)$ as follows.

$$
\begin{aligned}
Q(x) &= \sum_{n\le x}\mu(n)^2 = \sum_{n\le x}\sum_{d\mid n} g(d) = \sum_{d\le x} g(d)\left(\frac{x}{d}+O(1)\right) = x\sum_{d\le x}\frac{g(d)}{d} + O\left(\sum_{d\le x}\left|\frac{g(d)}{d}\right|\right)\\
&= x\sum_{m\le\sqrt{x}}\frac{\mu(m)}{m^2} + O\left(\sum_{m\le\sqrt{x}}|\mu(m)|\right) \qquad \text{by the definition of } g\\
&= x\left(\frac{1}{\zeta(2)}+O\left(\frac{1}{\sqrt{x}}\right)\right)+O(\sqrt{x})\\
&= \frac{x}{\zeta(2)}+O(\sqrt{x})
\end{aligned}
$$

Definition 3.0.5. The density of $A\subset\mathbb{N}$ is

$$\delta(A)=\lim_{x\to\infty}\frac{1}{x}\#\left\{n\le x : n\in A\right\}.$$

Exercise 3.0.6. Show $\forall y\ge 1$ that

$$\#\left\{n\le x : p\le y \implies p^2\nmid n\right\} = x\prod_{p\le y}\left(1-\frac{1}{p^2}\right)+O_y(1).$$

The general method we get from the above is as follows. Let $A(x)=\sum_{n\le x}a_n$ and $B(x)=\sum_{n\le x}b_n$. Define $c=a*b$ and $C(x)=\sum_{n\le x}c_n$. Then,

$$
\begin{aligned}
C(x) &= \sum_{n\le x}\sum_{d\mid n}a_d b_{n/d} = \sum_{d\le x}\sum_{\substack{n\le x\\ d\mid n}}a_d b_{n/d}\\
&= \sum_{d\le x}a_d\sum_{l\le x/d}b_l = \sum_{d\le x}a_d B\left(\frac{x}{d}\right).
\end{aligned}
$$

Example 3.0.7. Let $a_n=b_n=1$ and $c_n=d(n)$. Then,

$$
\begin{aligned}
\sum_{n\le x}d(n) = C(x) &= \sum_{d\le x}a_d B\left(\frac{x}{d}\right)\\
&= \sum_{n\le x}\left\lfloor\frac{x}{d}\right\rfloor = \sum_{d\le x}\left(\frac{x}{d}+O(1)\right) = x\sum_{d\le x}\frac{1}{d}+O(x)\\
&= x\left(\log x + C_0 + O\left(\frac{1}{x}\right)\right)+O(x) = x\log x + O(x).
\end{aligned}
$$

Remark 3.0.8. Note that

$$x\log x - x = \int_1^x \log t\ dt < \sum_{n\le x}\log n < \int_1^{x+1}\log t\ dt = (x+1)\log(x+1)-(x+1).$$

Exercise 3.0.9. For $x\ge 2$, we have $\sum_{n\le x}\log x = x\log x - x + O(\log x)$.

Then, $\sum_{n\leq x} d(n) \sim x\log x \sim \sum_{n\leq x}\log n$, so we say $d(n)$ has average order $\log n$.
Note that example 3.0.7 does not give a very good error term. This is because $B\left(\frac{x}{d}\right)$ gives a poor approximation when x/d is small. Instead, we can use Dirichlet's Hyperbola Method:

$$C(x) = \sum_{d\leq y} a_d B\left(\frac{x}{d}\right) + \sum_{l\leq x/y} b_l A\left(\frac{x}{l}\right) - A(y)B\left(\frac{x}{y}\right). \tag{4}$$

Now we can improve example 3.0.7 as follows.

$$\begin{aligned}
\sum_{n\leq x} d(n) &= \sum_{d\leq y}\left\lfloor\frac{x}{d}\right\rfloor + \sum_{l\leq x/y}\left\lfloor\frac{x}{l}\right\rfloor - \lfloor y\rfloor\left\lfloor\frac{x}{y}\right\rfloor \\
&= x\left(\log y + C_0 + O\left(\frac{1}{y}\right)\right) + O(y) + x\left(\log\frac{x}{y} + C_0 + O\left(\frac{y}{x}\right)\right) + O\left(\frac{x}{y}\right) \\
&\qquad - y\frac{x}{y} + O(y) + O\left(\frac{x}{y}\right) + O(1) \\
&= x\log x + (2C_0 - 1)x + O\left(y + \frac{x}{y}\right) \\
&= x\log x + (2C_0 - 1)x + O\left(\sqrt{x}\right).
\end{aligned}$$

3.1 Prime number estimates

We can think of the von Mangoldt Lambda function as a prime indicator function because proper prime powers are rare, so they should not influence the main term. Let $\Psi(x) = \sum_{n\leq x}\Lambda(n)$. Note that the Prime Number Theorem is equivalent to

$$\Psi(x) \sim x.$$

Recall that $\Psi(s) = -\frac{\zeta'(s)}{\zeta(s)}$. Since $-\frac{\zeta'}{\zeta}(s)\zeta(s) = -\zeta'(s)$, we get

$$\Lambda * 1 = \log \implies \Lambda = \log * \mu \implies \Lambda(n) = \sum_{d|n}\mu(d)\log\left(\frac{n}{d}\right).$$

Exercise 3.1.1. Show that $\Lambda(n) = -\sum_{d|n}\mu(d)\log(d)$.

Note that for $x\geq 2$, we get

$$\sum_{n\leq x}\log n = \sum_{n\leq x}\sum_{d|n}\Lambda(d) = \sum_{d\leq x}\Lambda(d)\left\lfloor\frac{x}{d}\right\rfloor \tag{5}$$

$$\sum_{n\leq x}\log n = x\log x - x + O(\log x). \tag{6}$$

Call these two equations $[x]$. Now, replace x with $x/2$ in $[x]$ and call it $[x/2]$. Let $E(t) = \lfloor t\rfloor - 2\lfloor t/2\rfloor$. Taking $[x] - 2[x/2]$ gives

LHS: $\quad (x\log x - x + O(\log x)) - 2\left(\frac{x}{2}\log\frac{x}{2} - \frac{x}{2} + O(\log x)\right) = (\log 2)x + O(\log x)$

RHS: $\quad \sum_{d\leq x}\Lambda(d)\lfloor x\rfloor d - 2\sum_{d\leq x}\Lambda(d)\left\lfloor\frac{x}{2d}\right\rfloor = \sum_{d\leq x}\Lambda(d)E\left(\frac{x}{d}\right).$

This gives the bounds

$$\Psi(x) - \Psi\left(\frac{x}{2}\right) = \sum_{\frac{x}{2} \leq d \leq x} \Lambda(d) \leq \sum_{d \leq x} \Lambda(d) E\left(\frac{x}{d}\right) \leq \sum_{d \leq x} \Lambda(d) = \Psi(x).$$

Immediately we get

$$\Psi(x) \geq \sum_{d \leq x} \Lambda(d) E\left(\frac{x}{d}\right) = (\log 2)x + O(\log x). \tag{7}$$

With some calculation, we get

$$\begin{aligned}
\Psi(x) &= \left(\Psi(x) - \Psi\left(\frac{x}{2}\right)\right) + \left(\Psi\left(\frac{x}{2}\right) - \Psi\left(\frac{x}{4}\right)\right) + \cdots \\
&\leq ((\log 2)x + O(\log x)) + \left((\log 2)\frac{x}{2} + O(\log x)\right) + \cdots \\
&= (2\log 2)x + O\left(\log^2(x)\right) \qquad (8)
\end{aligned}$$

Chebyshev took $[x] - [x/2] - [x/3] - [x/5] + [x/30]$ to get the bounds

$$0.9212x + O(\log x) \leq \Psi(x) \leq 1.1056x + O(\log^2 x).$$

Definition 3.1.2. We write $f \asymp g$ if $f \ll g$ and $g \ll f$. We say "f is of the same order of magnitude as g".

Therefore, $\Psi(x) \asymp x$.

Exercise 3.1.3 (a weak version of Bertrand's Postulate)**.** Note that $\sum_{x<n\leq 2x} \Lambda(n) = \Psi(2x) - \Psi(x) \gg x$. Then,

$$\#\{p : x < p \leq 2x\} \gg \frac{x}{\log x}.$$

Theorem 3.1.4 (Mertens)**.** *For $x \geq 2$,*

1. $\sum_{n\leq x} \frac{\Lambda(n)}{n} = \log x + O(1)$.
2. $\sum_{p\leq x} \frac{\log p}{p} = \log x + O(1)$.
3. $\sum_{p\leq x} \frac{1}{p} = \log\log x + b + O\left(\frac{1}{\log x}\right)$.
4. $\prod_{p\leq x} \left(1 - \frac{1}{p}\right)^{-1} = e^c \log x + O(1)$.

Proof of (a). Note that equations 5 and 6 give

$$\begin{aligned}
x\log x + O(x) &= \sum_{d\leq x} \Lambda(d) \left\lfloor \frac{x}{d} \right\rfloor = x \sum_{d\leq x} \frac{\Lambda(d)}{d} + O\left(\sum_{d\leq x} \Lambda(d)\right) \\
&= x \sum_{d\leq x} \frac{\Lambda(d)}{d} + O(x).
\end{aligned}$$

Dividing by x gives the desired result. □

Proof of (b). Note that

$$\sum_{n\le x}\frac{\Lambda(n)}{n}-\sum_{p\le x}\frac{\log p}{p}=\sum_{\substack{p^r\le x\\ r\ge 2}}\frac{\log p}{p^r}\le\sum_{p\le x}\log p\sum_{r=2}^{\infty}\frac{1}{p^r}$$
$$=\sum_{p\le x}(\log p)\cdot\frac{1}{p(p-1)}\ll\sum_{p=1}^{\infty}\frac{p^{\varepsilon}}{p^2}=O(1).$$

□

Proof of (d). Write $R(x)=\sum_{p\le x}\frac{\log p}{p}-\log x$, so $R(x)\ll 1$ and $R(2^-)=-\log 2$. Then,

$$\begin{aligned}\sum_{p\le x}\frac{1}{p}&=\int_{2^-}^{x}\frac{1}{\log u}\;d\left(\sum_{p\le u}\frac{\log p}{p}\right)\\
&=\int_{2^-}^{x}\frac{1}{\log u}\;d(\log u)+\int_{2^-}^{x}\frac{1}{\log u}\;dR(u)\\
&=\int_{2^-}^{x}\frac{1}{u\log u}\;du+\left.\frac{R(u)}{\log u}\right|_{2^-}^{x}-\int_{2^-}^{x}R(u)\;d\left(\frac{1}{\log u}\right)\\
&=\log\log u|_{2^-}^{x}+\frac{R(x)}{\log x}-\frac{R(2^-)}{\log 2}+\int_{2^-}^{x}R(u)\frac{1}{u\log^2 u}\;du\\
&=\log\log x-\log\log 2+1+O\left(\frac{1}{\log x}\right)+\int_{2}^{\infty}\frac{R(u)}{u\log^2 u}+O\left(\int_{x}^{\infty}\frac{|R(u)|}{u\log^2 u}\;du\right)\\
&=\log\log x+\left(1-\log\log 2+\int_{2}^{\infty}\frac{R(u)}{u\log^2 u}\;du\right)+O\left(\frac{1}{\log x}\right).\end{aligned}$$

□

Proof of (e). Note that $\log(1-t)^{-1}-t\ll|t|^2$ for $|t|\le\frac{1}{2}$ by power series. Hence

$$S=\sum_{p>x}\log\left(1-\frac{1}{p}\right)^{-1}-\frac{1}{p}\ll\sum_{p>x}\left(\frac{1}{p}\right)^2\le\sum_{n>x}\frac{1}{n^2}\ll\frac{1}{x}.$$

Note that S exists and

$$\begin{aligned}\sum_{p\le x}\log\left(1-\frac{1}{p}\right)^{-1}&=\sum_{p\le x}\frac{1}{p}+S+\sum_{p>x}\left(\log\left(1-\frac{1}{p}\right)^{-1}-\frac{1}{p}\right)\\
&=\log\log x+b+O\left(\frac{1}{\log x}\right)+S+O\left(\frac{1}{x}\right)\\
&=\log\log x+c+O\left(\frac{1}{\log x}\right).\end{aligned}$$

Note that $e^t=1+O(|t|)$ by power series, so

$$\begin{aligned}\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1}&=\exp\left(\sum_{p\le x}\log\left(1-\frac{1}{p}\right)^{-1}\right)=\exp\left(\log\log x+c+O\left(\frac{1}{\log x}\right)\right)\\
&=\log x\cdot e^c\cdot e^{O(1/\log x)}=e^c\log x\left(1+O\left(\frac{1}{\log x}\right)\right)\\
&=e^c\log x+O(1).\end{aligned}$$

$\square$

Remark 3.1.5. Note that in theorem 3.1.4, we have $c = C_0$ and $b = C_0 - \sum_p \sum_{k \geq 2} \frac{1}{kp^k}$.

Proposition 3.1.6. The average order of $\omega(n) = \# \{p : p|w\}$ is $\log \log n$.

Proof. We can compute

$$\sum_{n \leq x} \omega(n) = \sum_{n \leq x} \sum_{p|n} 1 = \sum_{p \leq x} \sum_{\substack{n \leq x \\ p|n}} 1 = \sum_{p \leq x} \left\lfloor \frac{x}{p} \right\rfloor = x \sum_{p \leq x} \frac{1}{p} + O\left(\sum_{p \leq x} 1 \right)$$
$$= x \log \log x + O(x).$$

$\square$

Exercise 3.1.7. Check that $\sum_{n \leq x} \log \log n \sim x \log \log x$.

Definition 3.1.8. The variance of $f(n)$ is

$$\lim_{x \to \infty} \sum_{n \leq x} (f(n) - g(x))^2$$

where $g(x)$ is $g(x) = \frac{1}{x} \sum_{n \leq x} f(n)$.

Now we compute the variance of $\omega(n)$.

Lemma 3.1.9. *We show that*

$$\sum_{n \leq x} \omega(n)^2 \leq x (\log \log x)^2 + O(x \log \log x).$$

Proof.

$$\sum_{n \leq x} \omega(n)^2 = \sum_{n \leq x} \left(\sum_{p|n} 1 \right) \left(\sum_{q|n} 1 \right) = \sum_{p \leq x} \sum_{q \leq x} \sum_{\substack{n \leq x \\ p|n \\ q|n}} 1$$
$$= \sum_{p \leq x} \sum_{\substack{q \leq x \\ q \neq p}} \left\lfloor \frac{x}{pq} \right\rfloor + \sum_{p \leq x} \left\lfloor \frac{p}{x} \right\rfloor$$
$$\leq \sum_{p \leq x} \sum_{q \leq x} \frac{x}{pq} + \sum_{p \leq x} \frac{x}{p} = x \left(\sum_{p \leq x} \frac{1}{p} \right)^2 + x \sum_{p \leq x} \frac{1}{p}$$
$$= x (\log \log x + O(1))^2 + O(x \log \log x).$$

$\square$

Proposition 3.1.10. The variance of $\omega(n)$ is $\ll \log \log x$.

Proof.

$$\begin{aligned}\sum_{n\le x}(\omega(n)-\log\log x)^2 &= \sum_{n\le x}\omega(n)^2 - 2(\log\log x)\sum_{n\le x}\omega(n) + (\log\log x)^2\sum_{n\le x}1 \\ &= \sum_{n\le x}\omega(n)^2 - 2(\log\log x)\left(x\log\log x + O(x)\right) + (x+O(1))\left(\log\log x\right)^2 \\ &\le \big(x(\log\log x)^2 + O(x\log\log x)\big) - (x\log\log x)^2 + O(x\log\log x).\end{aligned}$$

□

Corollary 3.1.11. *It follows that*

$$S = \#\left\{n\le x : |\omega(n)-\log\log x| > (\log\log x)^{\frac{1}{2}+\varepsilon}\right\} \ll \frac{x}{(\log\log x)^{2\varepsilon}}.$$

Proof.

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in S}}1 &\le \sum_{n\le x}\left(\frac{|\omega(n)-\log\log x|}{(\log\log x)^{\frac{1}{2}+\varepsilon}}\right)^2 \\ &= \frac{1}{(\log\log x)^{1+2\varepsilon}}\sum_{n\le x}(\omega(n)-\log\log x)^2 \\ &\ll \frac{x\log\log x}{(\log\log x)^{1+2\varepsilon}}.\end{aligned}$$

□

Exercise 3.1.12 (Hardy-Ramanujan)**.** Show that

$$\#\left\{n\le x : |\omega(n)-\log\log n| > (\log\log n)^{\frac{1}{2}+\varepsilon}\right\} \ll \frac{x}{(\log\log x)^{2\varepsilon}} = o(x).$$

It follows that the natural density is 0.

Exercise 3.1.13. Do the above computations for $\Omega(n)$.

Exercise 3.1.14. Prove that $2^{\omega(n)} \le d(n) \le 2^{\Omega(n)}$.

From Hardy-Ramanujan, for almost all $n\in\mathbb{N}$,

$$(1-\delta)\log\log n \le \omega(n) \le \Omega(n) \le (1+\delta)\log\log n,$$

$$(\log n)^{(1-\delta)\log 2} = 2^{(1-\delta)\log\log n} \le d(n) \le 2^{(1+\delta)\log\log n} = (\log n)^{(1+\delta)\log 2}.$$

So for most $n\in\mathbb{N}$, we have $d(n)\approx(\log n)^{\log 2}$, but the average order of $d(n)$ is $\log n$.