# Math 421 Notes

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## Contents



# 1 Topological Spaces

**Definition 1.0.1.** A topological space $(S, \mathcal{T})$ is a nonempty set with a family of subsets $\mathcal{T}$ such that

1. $\emptyset \in \mathcal{T}$
2. $S \in \mathcal{T}$
3. $\mathcal{T}$ is closed under finite intersections and arbitrary unions

Examples: $\{\emptyset, S\}$ (indiscrete topology), $2^S$ (discrete topology).
A metric on a metric space defines a topology. Not all topologies have a corresponding metric. A topology is called metrizable if we can define a metric such that "open" has the same meaning.
Topologies can be partially ordered. $\mathcal{T}_1 \prec \mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$ as sets. Denote $\mathcal{T}(\mathcal{E})$ to be the topology generated by $\mathcal{E} \subset 2^S$.

**Definition 1.0.2.** A base of $\mathcal{T}$ is a family $\mathcal{B} \subset \mathcal{T}$ such that for any nonempty open set $O \in \mathcal{T}$, there exists a colletion $\{B_\alpha : B_\alpha \in \mathcal{B}\}$ such that $O = \bigcup \alpha B_\alpha$.

**Definition 1.0.3.** Let $(S, \mathcal{T})$ be a topological space and $X \subset S$. Then, $\mathcal{T}_x = \{O \cap X : O \subset \mathcal{T}\}$ is the relative topology $(X, \mathcal{T}_x)$.

**Definition 1.0.4.** A set $X$ is closed if $\exists Y \in \mathcal{T}$ such that $X = Y^c$.

**Definition 1.0.5.** The interior of $X$ is the largest open set $X^o \subset X$.

**Definition 1.0.6.** The closure of $X$ is the smallest closed set $\overline{X} \supset X$.

**Definition 1.0.7.** The boundary of $X$ is $\overline{X} \setminus X^o$.

**Definition 1.0.8.** A neighbourhood of $x \in S$ is a set $N_x \subset S$ such that $x \in N_x^o$

**Definition 1.0.9.** A neighbourhood base of $x$ is a family $\mathcal{N}_x$ such that each $N \in \mathcal{N}_x$ is a neighbourhood of $x$ and for any neighbourhood $M_x$, there exists some $N \in \mathcal{N}_x$ such that $N \subset M_x$.

**Definition 1.0.10** (Classification of topological spaces)**.** A topological space is called $T_2$ or Hausdorff if $\forall x, y \in S$, $x \neq y$, there exists $O_x, O_y \in \mathcal{T}$ such that $x \in O_x$, $y \in O_y$, and $O_x \cap O_y = \emptyset$.

**Definition 1.0.11.** A topological space $(S, \mathcal{T})$ is

- separable if there exists a countable dense set
- first countable if $\forall x \in S$, there exists a countable neighbourhood base
- second countable if there exists a countable base

**Proposition 1.0.12.** Second countable implies both first countable and separable.

*Proof.* (Second countable implies first countable) Let $x \in S$, and let $M_x \subset \mathcal{T}$ be a neighbourhood of $x$. Since $\mathcal{B}$ is a base, there exists open sets $N_\alpha \in \mathcal{B}$ such that $\bigcup_\alpha N_\alpha = M_x^o$. Observe that there exists some $N_\alpha$ such that $x \in N_\alpha$, whence second countable.
(Second countable implies separable) For each $B \in \mathcal{B}$, choose some $x_B \in B$, and let $D = \bigcup_B x_B$. Suppose that $\overline{D} \neq S$, then $\overline{D}^c$ is open. Since $\mathcal{B}$ is a base, there exists some $B \in \mathcal{B}$ such that $B \subset \overline{D}^c$. Contradiction. □

**Definition 1.0.13.** A sequence $\{x_n\}_{x \in \mathbb{N}}$ in $(S, \mathcal{T})$ is convergent if $\exists x \in S$ such that for any neighbourhood of $x$, there exists some $N \in \mathbb{N}$ such that $x_n \in N_x$ for all $n > N$.

**Proposition 1.0.14.** Let $(S, \mathcal{T})$ be a first countable topological space, and $X \subset S$. Then $x \in \overline{X}$ if and only if $x$ is the limit point of a convergent sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$.

*Proof.* Let $\mathcal{N}_x = \{O_n : n \in \mathbb{N}\}$ be a countable neighbourhood base of $x$ such that $O_n \subset O_{n-1}$ for all $n \in \mathbb{N}$. If $x \in \overline{X}$, then $O_n \cap X \neq \emptyset$ for all $n \in \mathbb{N}$. Then we can pick $x_n \in O_n \cap X$, whence $x_n \to x$. Converse is similar. □

**Definition 1.0.15.** Let $(S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)$ be topological spaces. A function $f : S_1 \to S_2$ is continuous if $f^{-1}(O) \in \mathcal{T}_1$ for any $O \in \mathcal{T}_2$. Ie. the preimage of any open set is open.

**Definition 1.0.16.** Let $(S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)$ be topological spaces. A function $f : S_1 \to S_2$ is open if $f(O) \in \mathcal{T}_2$ for any $O \in \mathcal{T}_1$.

**Definition 1.0.17.** A homeomorphism is an invertible function that is open and continuous.

**Definition 1.0.18.** Let $S_1$ be a set and let $(S_2, \mathcal{T}_2)$ be a topological space. Let $\mathcal{F}$ be a family of functions from $S_1$ to $S_2$. Then, the topology on $S_1$ generated by $\left\{f^{-1}(O) : O \in \mathcal{T}_2\right\}$ is called the $\mathcal{F}$-weak topology.

**Remark 1.0.19.** By definition, all functions $f \in \mathcal{F}$ are continuous with respect to the above topology on $S_1$.

**Example 1.0.20.** Let $S_1 = C([a, b]; \mathbb{R})$ be the set of continuous functions, and let $S_2 = \mathbb{R}$ with the usual metric topology. Let $E_x : S_1 \to S_2$ where $E_x(f) = f(x)$ be the evaluation functions, and let $\mathcal{F} = \{E_x : x \in [a, b]\}$. The $\mathcal{F}$-weak topology on $C([a, b]; \mathbb{R})$ is the topology of pointwise convergence.

**Definition 1.0.21.** A topological space $(S, \mathcal{T})$ is compact if any open cover has a finite subcover.

**Definition 1.0.22.** A subset $X \subset S$ is compact if it is compact in the relative topology.

**Definition 1.0.23.** A subset $X \subset S$ is precompact if its closure is compact.

**Definition 1.0.24.** We say that $(S, \mathcal{T})$ has the finite intersection property if for any family of closed sets $\mathcal{C}$ such that $\bigcap_{i=1}^{n} C_i \neq \emptyset$ for any finite subfamily $\{C_1, ..., C_n\}$ also satisfies $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

**Exercise 1.0.25.** *$S$ is compact if and only if it has the finite intersection property.*

**Proposition 1.0.26.** Let $X \subset S$ be a subset of a compact topological space $(S, \mathcal{T})$. Then $X$ is compact if $X$ is closed.

*Proof.* Let $\mathcal{C}$ be an open cover of $X$. Let $\mathcal{C}' = \mathcal{C} \cup \{X^c\}$ be an open cover of $S$. There exists a finite subcover of $\mathcal{C}'$, so there exists a finite subcover of $X$ (we can safely remove $X^c$ from the finite subcover of $S$ as $X \cap X^c = \emptyset$). $\square$

**Proposition 1.0.27.** Let $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$, and let $f : S_1 \to S_2$ be continuous. If $S_1$ is compact, then $f(S_1) \subset S_2$ is compact.

*Proof.* Let $\mathcal{C}$ be an open cover of $f(S_1)$. Let $\mathcal{C}' = \left\{f^{-1}(C) : C \in \mathcal{C}\right\}$ be an open cover of $S_1$ (preimages of open sets are open by continuity). Hence, there exists a finite subcover of $S_1$, from which we get a finite subcover of $f(S_1)$. $\square$

**Proposition 1.0.28** (Bolzano-Weierstrass property)**.** A second countable topological space is compact if and only if every sequence has a convergent subsequence.

*Proof.* Suppose that $S$ is compact, and suppose, for contradiction, that $\{z_n\}_{n \in \mathbb{N}}$ does not have a convergent subsequence. Since $S$ is first countable, this means that for any $x \in S$, there exists some neighbourhood $O_x$ of $x$ and some $N_x \in \mathbb{N}$ such that $z_n \notin O_x$ for all $n > N_x$. Let $\mathcal{C} = \{O_x^o : x \in S\}$ be an open cover of $S$. Since $S$ is compact, there exists some finite subcover $\mathcal{C}' = \left\{O_{x_1}^o, ..., O_{x_m}^o\right\}$. Then, let $N = \max\{n_{x_1}, ..., n_{x_m}\}$, whence $z_n \notin \bigcup_i O_{x_i} = S$ for all $n > N$, which is a contradiction. Suppose that every sequence of $S$ has a convergent subsequence. Since $S$ is second countable, there exists a countable open cover $\mathcal{C} = \{O_i : i \in \mathbb{N}\}$. Suppose, for contradiction, that $\mathcal{C}$ has no finite subcover. Then, for any $i \in \mathbb{N}$, there exists some $x_i \notin \bigcup_{j=1}^{i} O_j$. Let $\{x_{n_i}\}_{i \in \mathbb{N}}$ be a convergent subsequence and let $x$ be its limit. Since $\mathcal{C}$ is a cover, there exists $j$ such that $x \in O_j$. It follows that there exists some $N \in \mathbb{N}$ such that $x_{n_k} \in O_j$ for all $k > N$, which is a contradiction. $\square$

## 1.1 Weierstrass theorems

**Theorem 1.1.1** ("Classical" Weierstrass)**.** *If $f$ is a continuous function on $[a, b]$, then there exists a sequence of polynomials $\{P_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} P_n = f$ uniformly.*

**Remark 1.1.2.** This theorem implies that the set of polynomials is dense in $C_{\mathbb{R}}([a, b])$ (real-valued continuous functions).

**Definition 1.1.3.** Let $X$ be compact and Hausdorff, and let $C_{\mathbb{R}}(X)$ be the set of real-valued continuous functions on $X$ equipped with pointwise multiplication: $(fg)(x) = f(x)g(x)$. This is an algebra.

**Definition 1.1.4.** An algebra $\mathcal{A} \subset C_{\mathbb{R}}(X)$ separates points if for any $x \neq y$ in $X$, there exists an $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

**Theorem 1.1.5** (Stone-Weierstrass)**.** *Let $X$ be a compact Hausdorff space. Let $\mathcal{A}$ be a closed subalgebra (wrt $\|\cdot\|_\infty$) of $C_{\mathbb{R}}(X)$ that seaprates points. Then, either $\mathcal{A} = C_{\mathbb{R}}(X)$ or $\exists x_0 \in X$ such that $\mathcal{A} = \{f \in C_{\mathbb{R}}(X) : f(x_0) = 0\}$.*

**Remark 1.1.6.** If $\mathcal{A}$ separates points and $1 \in \mathcal{A}$, we must have $\overline{\mathcal{A}} = C_{\mathbb{R}}(X)$. Hence, any unital subalgebra $\mathcal{A}$ of $C_{\mathbb{R}}(X)$ is dense.

**Definition 1.1.7.** Let $f$ and $g$ be functions on the same domain. Write $f \wedge g = \min\{f, g\}$ and $f \vee g = \max\{f, g\}$.

**Definition 1.1.8.** A family $\mathcal{F} \subset C_{\mathbb{R}}(X)$ is a lattice if any functions $f, g \in \mathcal{F}$, we have $f \wedge g \in \mathcal{F}$ and $f \vee g \in \mathcal{F}$.

**Lemma 1.1.9.** *Any closed unital subalgebra $\mathcal{A} \subset C_{\mathbb{R}}(X)$ is a lattice.*

*Proof.* Observe that

$$
\begin{aligned}
f \vee g &= \frac{1}{2}|f - g| + \frac{1}{2}(f + g) \\
f \wedge g &= -((-f) \vee (-g)),
\end{aligned}
$$

so it suffices to show that $f \in \mathcal{A}$ means $|f| \in \mathcal{A}$. By the classical Weierstrass theorem, there is a sequence of polynomials $\{P_n\}_{n\in\mathbb{N}}$ such that $|P_n(x) - |x|| < \frac{1}{n}$ for all $x \in [-1, 1]$. Hence,

$$
\|P_n(h) - |h|\|_\infty < \frac{1}{n} \text{ where } h = \frac{f}{\|f\|_\infty},
$$

that is $P_n(h) \to |h|$ uniformly. Note that $P_n(h) \in \mathcal{A}$, so we are done. $\square$

**Proposition 1.1.10** (Kakutani-Klein)**.** If $\mathcal{L}$ is a closed lattice that separates points such that $1 \in \mathcal{L}$. Then, $\mathcal{L} = C_{\mathbb{R}}(X)$.

*Proof.* Let $g \in C_{\mathbb{R}}(X)$, let $\varepsilon > 0$, and let $x \neq y \in X$. The map $\phi : \mathcal{L} \to \mathbb{R}^2$ such that $h \mapsto (h(x), h(y))$ is an algebra homomorphism. The image contains $(1, 1)$ since $1 \in \mathcal{L}$. The image also contains $(a, b)$ with $a \neq b$ since $\mathcal{L}$ separates points. It suffices to look at the subalgebras of $\mathbb{R}^2$, whence the image is all of $\mathbb{R}^2$. Now, there exists $f_{xy} \in \mathcal{L}$ such that $f_{xy}(x) = g(x)$ and $f_{xy}(y) = g(y)$. By continuity, there exists an open neighbourhood $N_y$ of $y$ such that $f_{xy}(z) + \varepsilon = g(z)$ for all $z \in N_y$. By compactness, there exists a finite subcover $\{N_{y_1}, ..., N_{y_m}\} \subset \{N_y : y \in X\}$. Let $f_x = \max\{f_{xy_1}, ..., f_x y_m\} \in \mathcal{L}$. Note that $f_x(x) = g(x)$ and $f_x(z) > g(z) - \varepsilon$ for all $z \in X$. This gives a lower bound. We can also find an upper bound whence there exists some $f \in \mathcal{L}$ such that $\|f - g\|_\infty < 2\varepsilon$. $\square$