

Math 421 Notes

Henry Xia

January 12, 2020

Contents

1 Introduction

1

1 Introduction

Definition 1.1. A topological space (S, \mathcal{T}) is a nonempty set with a family of subsets \mathcal{T} such that

1. $\emptyset \in \mathcal{T}$
2. $S \in \mathcal{T}$
3. \mathcal{T} is closed under finite intersections and arbitrary unions

Examples: $\{\emptyset, S\}$ (indiscrete topology), 2^S (discrete topology).

A metric on a metric space defines a topology. Not all topologies have a corresponding metric. A topology is called metrizable if we can define a metric such that “open” has the same meaning.

Topologies can be partially ordered. $\mathcal{T}_1 \prec \mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$ as sets. Denote $\mathcal{T}(\mathcal{E})$ to be the topology generated by $\mathcal{E} \subset 2^S$.

Definition 1.2. A base of \mathcal{T} is a family $\mathcal{B} \subset \mathcal{T}$ such that for any nonempty open set $O \in \mathcal{T}$, there exists a collection $\{B_\alpha : B_\alpha \in \mathcal{B}\}$ such that $O = \bigcup_\alpha B_\alpha$.

Definition 1.3. Let (S, \mathcal{T}) be a topological space and $X \subset S$. Then, $\mathcal{T}_x = \{O \cap X : O \in \mathcal{T}\}$ is the relative topology (X, \mathcal{T}_x) .

Definition 1.4. A set X is closed if $\exists Y \in \mathcal{T}$ such that $X = Y^c$.

Definition 1.5. The interior of X is the largest open set $X^\circ \subset X$.

Definition 1.6. The closure of X is the smallest closed set $\overline{X} \supset X$.

Definition 1.7. The boundary of X is $\overline{X} \setminus X^\circ$.

Definition 1.8. A neighbourhood of $x \in S$ is a set $N_x \subset S$ such that $x \in N_x^\circ$.

Definition 1.9. A neighbourhood base of x is a family \mathcal{N}_x such that each $N \in \mathcal{N}_x$ is a neighbourhood of x and for any neighbourhood M_x , there exists some $N \in \mathcal{N}_x$ such that $N \subset M_x$.

Definition 1.10 (Classification of topological spaces). A topological space is called T_2 or Hausdorff if $\forall x, y \in S, x \neq y$, there exists $O_x, O_y \in \mathcal{T}$ such that $x \in O_x, y \in O_y$, and $O_x \cap O_y = \emptyset$.

Definition 1.11. A topological space (S, \mathcal{T}) is

- separable if there exists a countable dense set
- first countable if $\forall x \in S$, there exists a countable neighbourhood base
- second countable if there exists a countable base

Proposition 1.12. Second countable implies both first countable and separable.

Proof. (Second countable implies first countable) Let $x \in S$, and let $M_x \subset \mathcal{T}$ be a neighbourhood of x . Since \mathcal{B} is a base, there exists open sets $N_\alpha \in \mathcal{B}$ such that $\bigcup_\alpha N_\alpha = M_x$. Observe that there exists some N_α such that $x \in N_\alpha$, whence second countable.

(Second countable implies separable) For each $B \in \mathcal{B}$, choose some $x_B \in B$, and let $D = \bigcup_B x_B$. Suppose that $\overline{D} \neq S$, then \overline{D}^c is open. Since \mathcal{B} is a base, there exists some $B \in \mathcal{B}$ such that $B \subset \overline{D}^c$. Contradiction. \square

Definition 1.13. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in (S, \mathcal{T}) is convergent if $\exists x \in S$ such that for any neighbourhood of x , there exists some $N \in \mathbb{N}$ such that $x_n \in N_x$ for all $n > N$.

Proposition 1.14. Let (S, \mathcal{T}) be a first countable topological space, and $X \subset S$. Then $x \in \overline{X}$ if and only if x is the limit point of a convergent sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$.

Proof. Let $\mathcal{N}_x = \{O_n : n \in \mathbb{N}\}$ be a countable neighbourhood base of x such that $O_n \subset O_{n-1}$ for all $n \in \mathbb{N}$. If $x \in \overline{X}$, then $O_n \cap X \neq \emptyset$ for all $n \in \mathbb{N}$. Then we can pick $x_n \in O_n \cap X$, whence $x_n \rightarrow x$. Converse is similar. \square

Definition 1.15. Let $(S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)$ be topological spaces. A function $f : S_1 \rightarrow S_2$ is continuous if $f^{-1}(O) \in \mathcal{T}_1$ for any $O \in \mathcal{T}_2$. I.e. the preimage of any open set is open.

Definition 1.16. Let $(S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)$ be topological spaces. A function $f : S_1 \rightarrow S_2$ is open if $f(O) \in \mathcal{T}_2$ for any $O \in \mathcal{T}_1$.

Definition 1.17. A homeomorphism is an invertible function that is open and continuous.

Definition 1.18. Let S_1 be a set and let (S_2, \mathcal{T}_2) be a topological space. Let \mathcal{F} be a family of functions from S_1 to S_2 . Then, the topology on S_1 generated by $\{f^{-1}(O) : O \in \mathcal{T}_2\}$ is called the \mathcal{F} -weak topology.

Remark 1.19. By definition, all functions $f \in \mathcal{F}$ are continuous with respect to the above topology on S_1 .

Example 1.20. Let $S_1 = C([a, b]; \mathbb{R})$ be the set of continuous functions, and let $S_2 = \mathbb{R}$ with the usual metric topology. Let $E_x : S_1 \rightarrow S_2$ where $E_x(f) = f(x)$ be the evaluation functions, and let $\mathcal{F} = \{E_x : x \in [a, b]\}$. The \mathcal{F} -weak topology on $C([a, b]; \mathbb{R})$ is the topology of pointwise convergence.