Math 539 Notes

Henry Xia

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1 Introduction

Motivating questions (some statistics):

- the "probability" that a random number has some property
- the "distribution" of some given multiplicative/additive function

Idea: we can answer the question for $\{1, ..., \lfloor x \rfloor\}$ for some parameter x. Then, take the limit $x \to \infty$ for all natural numbers.

1.1 Notation

Let $g(x) \geq 0$.

Definition 1.1. $O(g(x))$ means some unspecified function $u(x)$ such that $|u(x)| \leq cg(x)$ for some constant $c > 0$.

Example 1.1. Show that $e^{2x} - 1 = 2x + O(x^2)$ for $x = [-1, 1]$.

Proof. Observe that $f(z) = e^{2z} - 1 - 2z$ is analytic (and entire) and has a double zero at $z = 0$ (one can check that $f(z) = f'(z) = 0$. Hence, $g(z) = (e^{2z} - 1 - 2z)/z^2$ has a removable singularity at $z = 0$, whence g is analytic and entire. Let $C = \max\{|g(z)| : |z| \leq 1\}$. Then

$$|g(z)| \leq C \implies \left|e^{2z} - 1 - 2z\right| \leq C\left|z^2\right| \implies e^{2z} - 1 - 2z = O(|z|^2).$$

□

Exercise 1.1. Show that $\sqrt{x+1} = \sqrt{x} + O(1/\sqrt{x})$ for $x \in [1, \infty)$.

Definition 1.2. $f(x) \ll g(x)$ means $f(x) = O(g(x))$.

Exercise 1.2. Suppose that $f_1 \ll g_1, f_2 \ll g_2$, then $f_1 + f_2 \ll \max\{g_1, g_2\}$. ✓

Exercise 1.3. Let f, g be continuous on $[0, \infty)$, and $f \ll g$ on $[123, \infty)$. Show that $f \ll g$ on $[0, \infty)$. ✓

Definition 1.3. $f(x) \sim g(x)$ means $\lim \frac{f(x)}{g(x)} = 1$.

Definition 1.4. $f(x) = o(g(x))$ means $\lim \frac{f(x)}{g(x)} = 0$.

Definition 1.5. $f(x) = O_y(g(x))$ means f, g depend on some parameter y, and the implicit constant depends on y.

Exercise 1.4. For any $A, \epsilon > 0$, show that $(\log x)^A \ll_{A,\epsilon} x^\epsilon$.