Math 420 Notes

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Contents

1 Introduction

Definition 1.1. A σ-algebra on X is a collection of subsets of 2^X that is closed under complement and countable union.

Definition 1.2. Let $\mathcal{M} \subset 2^X$ be the measurable subsets of X. A measure μ on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ satisfying the following.

(i) $\mu(\emptyset) = 0$

(ii) $\mu(\bigcup_j^\infty E_j) = \sum_j^\infty \mu(E_j)$

Note that $\mathcal{M}$ is a σ-algebra.

Example 1.1. The counting measure. $\mu(E) = \#\{X : X \in E\}$

Example 1.2. The Dirac measure. Fix $x_0 \in X$. $\mu(E) = 1$ if $x_0 \in E$, and $\mu(E) = 0$ otherwise.

Example 1.3. An unmeasurable set. (Folland p.20).
Let $E_r = E + r \bmod 1$. There exists a set $E \subset [0, 1)$ such that

- $\{E_r\}_{r \in \mathbb{Q} \cap [0,1)}$ are disjoint
- $\bigcup_{r \in \mathbb{Q} \cap [0,1)} E_r = [0, 1)$

This set E is inconsistent with (ii) of the definition when $\mu([0, 1)) = 1$ and $\mu(E_r) = \mu(E)$.

Definition 1.3. Let non-empty $\mathcal{E} \subset 2^X$. The σ-algebra generated by $\mathcal{E}$ is $\mathcal{M}(\mathcal{E})$, that is the smallest σ-algebra containing $\mathcal{E}$. (We can get this by taking the intersection of all the σ-algebras containing $\mathcal{E}$)

Example 1.4. Let X be a topological space. The Borel σ-algebra B_X is a σ-algebra generated by open sets. This contains open sets, closed sets, countable union of closed sets ($F\sigma$-sets), countable intersection of open sets ($G\delta$-sets).
$B_{\mathbb{R}}$ can be generated by any of

- open intervals. $\{(a, b)\}$
- closed intervals. $\{[a, b]\}$
- half open intervals. $\{(a, b]\}$
- semi-infinite intervals. $\{(a, \infty)\}$

2 The Lebesgue Measure

2.1 Premeasures

Let $\mathcal{A}$ be the set of finite disjoint unions of h-intervals, where h-intervals are of the following form: $(a, b]$, (a, ∞), $\emptyset$, where $-\infty \leq a < b < \infty$.

Proposition 2.1. $\mathcal{A}$ is an algebra.

Proof. The intersection of two h-intervals is also an h-interval. The complement of an h-interval is the union of at most two disjoint h-intervals. Refer to text (Folland Prop 1.7). $\square$

Define the "Length" of sets in $\mathcal{A}$ to be a function $m_0 : \mathcal{A} \to [0, \infty]$ with finite additivity and $m_0(\emptyset) = 0$.

Definition 2.1. A premeasure is a function $m : \mathcal{A} \to [0, \infty]$ such that

(i) $m(\emptyset) = 0$

(ii) For countably many disjoint $A_j \in \mathcal{A}$ whose union $A = \bigcup A_j$ is also in $\mathcal{A}$, we have $m(\bigcup A_j) = \sum m(A_j)$.

Theorem 2.1. *The following is true*

1. *m_0 is well defined.*
2. *m_0 is a premeasure.*

Proof of 1. This is just bookkeeping. See text. $\square$

Proof of 2. Let $A = (a, b] \in \mathcal{A}$ be a countable disjoint union of $A_j = (a_j, b_j] \in \mathcal{A}$. We can assume that A_j, because each A_j would otherwise be the finite union of some disjoint set of intervals in $\mathcal{A}$. We can also assume that A is an interval by the same argument.
Consider $A = \bigcup_{j=1}^{n} A_j \cup (A \setminus \bigcup_{j=1}^{n})$. Then we have

$$m_0(A) = m_0\left(\bigcup_{j=1}^{n} A_j\right) + m_0\left(A \setminus \bigcup_{j=1}^{n}\right) \geq m_0\left(\bigcup_{j=1}^{n} A_j\right).$$

Taking the limit gives $m_0(A) \geq m_0(\bigcup_{j=1}^{\infty} A_j)$.

Now let $\epsilon > 0$. Consider the compact interval $[a+\epsilon, b]$ covered by $\bigcup_{j=1}^{\infty}(a_j, b_j + \frac{\epsilon}{2^j})$. There must be a finite subcover. Now, $(a+\epsilon, b]$ is also covered by this finite subcover, and we can relabel the finite subcover so that $a_j < a_{j+1}$. Then

$$\begin{aligned}
m_0(A) &= m_0((a, a+\epsilon]) + m_0((a+\epsilon, b]) \\
&\leq \epsilon + m_0((a_1, b_n + \frac{\epsilon}{2^n})) = \epsilon + b_n + \frac{\epsilon}{2^n} - a_n + \sum_{j=2}^{n}(a_j - a_{j-1}) \\
&\leq \epsilon + (b_n - a_n) + \sum_{j}^{n}\left(b_j + \frac{\epsilon}{2^j} - a_{j-1}\right) \leq \epsilon + \sum_{j}^{n}\frac{\epsilon}{2^j} + \sum_{j}^{n} m_0(A_j) \\
&\leq 7\epsilon + \sum m_0(A_j),
\end{aligned}$$

and countable additivity follows. □

2.2 Lebesgue Outer Measure

Definition 2.2. The Lebesgue outer measure m^* of a set $E \subset \mathbb{R}$ is defined as follows.

$$m^*(E) = \inf\left\{\sum_{j=1}^{\infty} m_0(I_j) : E \subset \bigcup_{j=1}^{\infty} I_j\right\}.$$

Definition 2.3. In general, given $\mathcal{E} \subset 2^X$ with $\emptyset, X \in \mathcal{E}$ and $\mu_0 : \mathcal{E} \to [0, \infty]$ with $\mu_0(\emptyset) = 0$, we can define $\mu^* : 2^X \to [0\infty]$ as follows.

$$\mu^*(E) = \inf\left\{\sum_{j=1}^{\infty} \mu_0(E_j) : E \subset \bigcup_{j=1}^{\infty} E_j \text{ and } E_j \in \mathcal{E}\right\}.$$

Proposition 2.2. μ^* is an outer measure, where an outer measure satisfies three properties.

1. $\mu^*(\emptyset) = 0$
2. $A \subset B \implies \mu^*(A) \leq \mu^*(B)$
3. $\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \mu^*(E_j)$

Proof of 1. $\emptyset \subset \bigcup_{j=1}^{\infty} \emptyset \implies \mu^*(\emptyset) \leq \sum_{j=1}^{\infty} \mu_0(\emptyset) = 0$. □

Proof of 2. Let $A \subset B$. Then $\left\{\{E_j\}_j \subset \mathcal{E} : B \subset \bigcup_j E_j\right\} \subset \left\{\{E_j\}_j \subset \mathcal{E} : A \subset \bigcup_j E_j\right\}$. Hence $\mu^*(A) \leq \mu^*(B)$. □

Proof of 3. Let $\{A_{j,k}\}_k \subset \mathcal{E}$ such that $E_j \subset \bigcup_{k=1}^{\infty} A_{j,k}$. Observe that $\bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j,k=1}^{\infty} A_{j,k}$. Let $\epsilon > 0$, then

$$\mu^*\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j,k=1}^{\infty} \mu_0(A_{j,k}) \leq \sum_{j=1}^{\infty}\left(\frac{\epsilon}{2^j} + \mu^*(E_j)\right) = \epsilon + \sum_{j=1}^{\infty} \mu^*(E_j)$$

Since ϵ is arbitrary, we get $\mu^*(\bigcup_j E_j) \leq \sum_j \mu^*(E_j)$. □

Observe that μ^* is defined for every set in 2^X, but it is not additive. To fix this, we will remove some "bad" sets.

Definition 2.4. Let μ^* be an outer measure on X. A set $A \subset X$ is μ^*-measurable if for every $E \subset X$, we have $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

2.3 Carathéodory's Extension Theorem

Theorem 2.2. *Let $\mathcal{M}$ be the set of μ^*-measurable sets, and $\mu^*|_{\mathcal{M}}$ is a complete measure.*

Proof.

1. We show that $\mathcal{M}$ is an algebra. Clearly, $\emptyset \in \mathcal{M}$, and $\mathcal{M}$ is closed under complement. Now let $A, B \in \mathcal{M}$. Then

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).\end{aligned}$$

   The other inequality is automatic by monoticity. Hence $A \cup B \in \mathcal{M}$.

2. We show that μ^* is finitely additive. Let $A, B \in \mathcal{M}$ be disjoint. Then

$$\mu^*(A \cup B) = \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c) = \mu^*(A) + \mu^*(B).$$

3. We show that $\mathcal{M}$ is closed under countable union and μ^* is countably additive. Let $\{A_j\} \subset \mathcal{M}$, $B_n = \bigcup_{j=1}^n A_j$, and $B = \bigcup_{j=1}^\infty A_j$. Let $E \subset X$, then

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) = \sum_{j=1}^n \mu^*(E \cap A_j).\end{aligned}$$

   By the definition, we get

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*(E \cap B^c).$$

   Take $n \to \infty$, then we get closure under countable union.

$$\mu^*(E) \geq \sum_{j=1}^\infty \mu^*(E \cap A_j) + \mu^*(E \cap B^c) \geq \mu^*(E \cap B) + \mu^*(E \cap B^c).$$

   Take $E = B$, then we get countable additivity.

$$\mu^*(B) = \sum_{j=1}^\infty \mu^*(A_j) + \mu^*(\emptyset) = \sum_{j=1}^\infty \mu^*(A_j).$$

We can easily check that $\mathcal{M}$ is complete. This theorem is complete. □