Math 421 Notes

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Contents

1 Topological Spaces

Definition 1.0.1. A topological space $(S, \mathcal{T})$ is a nonempty set with a family of subsets $\mathcal{T}$ such that

1. $\emptyset \in \mathcal{T}$
2. $S \in \mathcal{T}$
3. $\mathcal{T}$ is closed under finite intersections and arbitrary unions

Examples: $\{\emptyset, S\}$ (indiscrete topology), 2^S (discrete topology).
A metric on a metric space defines a topology. Not all topologies have a corresponding metric. A topology is called metrizable if we can define a metric such that "open" has the same meaning.
Topologies can be partially ordered. $\mathcal{T}_1 \prec \mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$ as sets. Denote $\mathcal{T}(\mathcal{E})$ to be the topology generated by $\mathcal{E} \subset 2^S$.

Definition 1.0.2. A base of $\mathcal{T}$ is a family $\mathcal{B} \subset \mathcal{T}$ such that for any nonempty open set $O \in \mathcal{T}$, there exists a colletion $\{B_\alpha : B_\alpha \in \mathcal{B}\}$ such that $O = \bigcup \alpha B_\alpha$.

Definition 1.0.3. Let $(S, \mathcal{T})$ be a topological space and $X \subset S$. Then, $\mathcal{T}_x = \{O \cap X : O \subset \mathcal{T}\}$ is the relative topology $(X, \mathcal{T}_x)$.

Definition 1.0.4. A set X is closed if $\exists Y \in \mathcal{T}$ such that $X = Y^c$.

Definition 1.0.5. The interior of X is the largest open set $X^o \subset X$.

Definition 1.0.6. The closure of X is the smallest closed set $\overline{X} \supset X$.

Definition 1.0.7. The boundary of X is $\overline{X} \setminus X^o$.

Definition 1.0.8. A neighbourhood of $x \in S$ is a set $N_x \subset S$ such that $x \in N_x^o$

Definition 1.0.9. A neighbourhood base of x is a family $\mathcal{N}_x$ such that each $N \in \mathcal{N}_x$ is a neighbourhood of x and for any neighbourhood M_x, there exists some $N \in \mathcal{N}_x$ such that $N \subset M_x$.

Definition 1.0.10 (Classification of topological spaces)**.** A topological space is called T_2 or Hausdorff if $\forall x, y \in S$, $x \neq y$, there exists $O_x, O_y \in \mathcal{T}$ such that $x \in O_x$, $y \in O_y$, and $O_x \cap O_y = \emptyset$.

Definition 1.0.11. A topological space $(S, \mathcal{T})$ is

- separable if there exists a countable dense set
- first countable if $\forall x \in S$, there exists a countable neighbourhood base
- second countable if there exists a countable base

Proposition 1.0.12. Second countable implies both first countable and separable.

Proof. (Second countable implies first countable) Let $x \in S$, and let $M_x \subset \mathcal{T}$ be a neighbourhood of x. Since $\mathcal{B}$ is a base, there exists open sets $N_\alpha \in \mathcal{B}$ such that $\bigcup_\alpha N_\alpha = M_x^o$. Observe that there exists some N_α such that $x \in N_\alpha$, whence second countable.
(Second countable implies separable) For each $B \in \mathcal{B}$, choose some $x_B \in B$, and let $D = \bigcup_B x_B$. Suppose that $\overline{D} \neq S$, then $\overline{D}^c$ is open. Since $\mathcal{B}$ is a base, there exists some $B \in \mathcal{B}$ such that $B \subset \overline{D}^c$. Contradiction. □

Definition 1.0.13. A sequence $\{x_n\}_{x \in \mathbb{N}}$ in $(S, \mathcal{T})$ is convergent if $\exists x \in S$ such that for any neighbourhood of x, there exists some $N \in \mathbb{N}$ such that $x_n \in N_x$ for all $n > N$.

Proposition 1.0.14. Let $(S, \mathcal{T})$ be a first countable topological space, and $X \subset S$. Then $x \in \overline{X}$ if and only if x is the limit point of a convergent sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$.

Proof. Let $\mathcal{N}_x = \{O_n : n \in \mathbb{N}\}$ be a countable neighbourhood base of x such that $O_n \subset O_{n-1}$ for all $n \in \mathbb{N}$. If $x \in \overline{X}$, then $O_n \cap X \neq \emptyset$ for all $n \in \mathbb{N}$. Then we can pick $x_n \in O_n \cap X$, whence $x_n \to x$. Converse is similar. □

Definition 1.0.15. Let $(S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)$ be topological spaces. A function $f : S_1 \to S_2$ is continuous if $f^{-1}(O) \in \mathcal{T}_1$ for any $O \in \mathcal{T}_2$. Ie. the preimage of any open set is open.

Definition 1.0.16. Let $(S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)$ be topological spaces. A function $f : S_1 \to S_2$ is open if $f(O) \in \mathcal{T}_2$ for any $O \in \mathcal{T}_1$.

Definition 1.0.17. A homeomorphism is an invertible function that is open and continuous.

Definition 1.0.18. Let S_1 be a set and let $(S_2, \mathcal{T}_2)$ be a topological space. Let $\mathcal{F}$ be a family of functions from S_1 to S_2. Then, the topology on S_1 generated by $\left\{f^{-1}(O) : O \in \mathcal{T}_2\right\}$ is called the $\mathcal{F}$-weak topology.

Remark 1.0.19. By definition, all functions $f \in \mathcal{F}$ are continuous with respect to the above topology on S_1.

Example 1.0.20. Let $S_1 = C([a, b]; \mathbb{R})$ be the set of continuous functions, and let $S_2 = \mathbb{R}$ with the usual metric topology. Let $E_x : S_1 \to S_2$ where $E_x(f) = f(x)$ be the evaluation functions, and let $\mathcal{F} = \{E_x : x \in [a, b]\}$. The $\mathcal{F}$-weak topology on $C([a, b]; \mathbb{R})$ is the topology of pointwise convergence.

Definition 1.0.21. A topological space $(S, \mathcal{T})$ is compact if any open cover has a finite subcover.

Definition 1.0.22. A subset $X \subset S$ is compact if it is compact in the relative topology.

Definition 1.0.23. A subset $X \subset S$ is precompact if its closure is compact.

Definition 1.0.24. We say that $(S, \mathcal{T})$ has the finite intersection property if for any family of closed sets $\mathcal{C}$ such that $\bigcap_{i=1}^{n} C_i \neq \emptyset$ for any finite subfamily $\{C_1, ..., C_n\}$ also satisfies $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

Exercise 1.0.25. *S is compact if and only if it has the finite intersection property.*

Proposition 1.0.26. Let $X \subset S$ be a subset of a compact topological space $(S, \mathcal{T})$. Then X is compact if X is closed.

Proof. Let $\mathcal{C}$ be an open cover of X. Let $\mathcal{C}' = \mathcal{C} \cup \{X^c\}$ be an open cover of S. There exists a finite subcover of $\mathcal{C}'$, so there exists a finite subcover of X (we can safely remove X^c from the finite subcover of S as $X \cap X^c = \emptyset$). $\square$

Proposition 1.0.27. Let $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$, and let $f : S_1 \to S_2$ be continuous. If S_1 is compact, then $f(S_1) \subset S_2$ is compact.

Proof. Let $\mathcal{C}$ be an open cover of $f(S_1)$. Let $\mathcal{C}' = \left\{f^{-1}(C) : C \in \mathcal{C}\right\}$ be an open cover of S_1 (preimages of open sets are open by continuity). Hence, there exists a finite subcover of S_1, from which we get a finite subcover of $f(S_1)$. $\square$

Proposition 1.0.28 (Bolzano-Weierstrass property)**.** A second countable topological space is compact if and only if every sequence has a convergent subsequence.

Proof. Suppose that S is compact, and suppose, for contradiction, that $\{z_n\}_{n \in \mathbb{N}}$ does not have a convergent subsequence. Since S is first countable, this means that for any $x \in S$, there exists some neighbourhood O_x of x and some $N_x \in \mathbb{N}$ such that $z_n \notin O_x$ for all $n > N_x$. Let $\mathcal{C} = \{O_x^o : x \in S\}$ be an open cover of S. Since S is compact, there exists some finite subcover $\mathcal{C}' = \left\{O_{x_1}^o, ..., O_{x_m}^o\right\}$. Then, let $N = \max\{n_{x_1}, ..., n_{x_m}\}$, whence $z_n \notin \bigcup_i O_{x_i} = S$ for all $n > N$, which is a contradiction. Suppose that every sequence of S has a convergent subsequence. Since S is second countable, there exists a countable open cover $\mathcal{C} = \{O_i : i \in \mathbb{N}\}$. Suppose, for contradiction, that $\mathcal{C}$ has no finite subcover. Then, for any $i \in \mathbb{N}$, there exists some $x_i \notin \bigcup_{j=1}^{i} O_j$. Let $\{x_{n_i}\}_{i \in \mathbb{N}}$ be a convergent subsequence and let x be its limit. Since $\mathcal{C}$ is a cover, there exists j such that $x \in O_j$. It follows that there exists some $N \in \mathbb{N}$ such that $x_{n_k} \in O_j$ for all $k > N$, which is a contradiction. $\square$

1.1 Weierstrass Theorems

Theorem 1.1.1 ("Classical" Weierstrass)**.** *If f is a continuous function on $[a, b]$, then there exists a sequence of polynomials $\{P_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} P_n = f$ uniformly.*

Remark 1.1.2. This theorem implies that the set of polynomials is dense in $C_{\mathbb{R}}([a, b])$ (real-valued continuous functions).

Definition 1.1.3. Let X be compact and Hausdorff, and let $C_{\mathbb{R}}(X)$ be the set of real-valued continuous functions on X equipped with pointwise multiplication: $(fg)(x) = f(x)g(x)$. This is an algebra.

Definition 1.1.4. An algebra $\mathcal{A} \subset C_{\mathbb{R}}(X)$ separates points if for any $x \neq y$ in X, there exists an $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Theorem 1.1.5 (Stone-Weierstrass)**.** *Let X be a compact Hausdorff space. Let $\mathcal{A}$ be a closed subalgebra (wrt $\|\cdot\|_\infty$) of $C_{\mathbb{R}}(X)$ that seaprates points. Then, either $\mathcal{A} = C_{\mathbb{R}}(X)$ or $\exists x_0 \in X$ such that $\mathcal{A} = \{f \in C_{\mathbb{R}}(X) : f(x_0) = 0\}$.*

Remark 1.1.6. If $\mathcal{A}$ separates points and $1 \in \mathcal{A}$, we must have $\overline{\mathcal{A}} = C_{\mathbb{R}}(X)$. Hence, any unital subalgebra $\mathcal{A}$ of $C_{\mathbb{R}}(X)$ is dense.

Definition 1.1.7. Let f and g be functions on the same domain. Write $f \wedge g = \min\{f, g\}$ and $f \vee g = \max\{f, g\}$.

Definition 1.1.8. A family $\mathcal{F} \subset C_{\mathbb{R}}(X)$ is a lattice if any functions $f, g \in \mathcal{F}$, we have $f \wedge g \in \mathcal{F}$ and $f \vee g \in \mathcal{F}$.

Lemma 1.1.9. *Any closed unital subalgebra $\mathcal{A} \subset C_{\mathbb{R}}(X)$ is a lattice.*

Proof. Observe that

$$f \vee g = \frac{1}{2}|f - g| + \frac{1}{2}(f + g)$$
$$f \wedge g = -((-f) \vee (-g)),$$

so it suffices to show that $f \in \mathcal{A}$ means $|f| \in \mathcal{A}$. By the classical Weierstrass theorem, there is a sequence of polynomials $\{P_n\}_{n \in \mathbb{N}}$ such that $|P_n(x) - |x|| < \frac{1}{n}$ for all $x \in [-1, 1]$. Hence,

$$\|P_n(h) - |h|\|_\infty < \frac{1}{n} \text{ where } h = \frac{f}{\|f\|_\infty},$$

that is $P_n(h) \to |h|$ uniformly. Note that $P_n(h) \in \mathcal{A}$, so we are done. □

Proposition 1.1.10 (Kakutani-Klein)**.** If $\mathcal{L}$ is a closed lattice that separates points such that $1 \in \mathcal{L}$. Then, $\mathcal{L} = C_{\mathbb{R}}(X)$.

Proof. Let $g \in C_{\mathbb{R}}(X)$, let $\varepsilon > 0$, and let $x \neq y \in X$. The map $\phi : \mathcal{L} \to \mathbb{R}^2$ such that $h \mapsto (h(x), h(y))$ is an algebra homomorphism. The image contains $(1, 1)$ since $1 \in \mathcal{L}$. The image also contains (a, b) with $a \neq b$ since $\mathcal{L}$ separates points. It suffices to look at the subalgebras of $\mathbb{R}^2$, whence the image is all of $\mathbb{R}^2$. Now, there exists $f_{xy} \in \mathcal{L}$ such that $f_{xy}(x) = g(x)$ and $f_{xy}(y) = g(y)$. By continuity, there exists an open neighbourhood N_y of y such that $f_{xy}(z) + \varepsilon = g(z)$ for all $z \in N_y$. By compactness, there exists a finite subcover $\{N_{y_1}, ..., N_{y_m}\} \subset \{N_y : y \in X\}$. Let $f_x = \max\{f_{xy_1}, ..., f_x y_m\} \in \mathcal{L}$. Note that $f_x(x) = g(x)$ and $f_x(z) > g(z) - \varepsilon$ for all $z \in X$. This gives a lower bound. We can also find an upper bound whence there exists some $f \in \mathcal{L}$ such that $\|f - g\|_\infty < 2\varepsilon$. □

The Stone-Weierstrass Theorem has two generalizations.

1. It extends to complex-valued functions, provided the subalgebras are also closed under conjugation. Any $f \in C_{\mathbb{C}}(X)$ can be written as $f = (f + \overline{f})/2 - i(f - \overline{f})/2$.

2. It extends to locally compact Hausdorff (LCH) spaces, that is topological spaces S such that every $x \in S$ has a compact neighbourhood. Here, the relevant algebra is the set of functions that vanish at infinity, that is $f \in C_{\mathbb{R}}(S)$ such that $\forall \epsilon > 0$, the set $\{x \in S : |f(x)| \geq \epsilon\}$ is compact. Hence, consider $S \cup \{\infty\}$.

1.2 Hausdorff Spaces

Definition 1.2.1. A space S is locally compact if each $x \in S$ has a compact neighbourhood.

Lemma 1.2.2. *Let $(S, \mathcal{T})$ be Hausdorff. Let $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence in S. Then the limit $x = \lim_{n\to\infty} x_n$ is unique.*

Proof. Let $y \neq x$. By Hausdorff, there exists disjoint open sets O_x and O_y such that $x \in O_x$ and $y \in O_y$. Since $x_n \to x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in O_x$ for all $n > n_0$. Hence $x_n \notin O_y$, so $\{x_n\}_{n\in\mathbb{N}}$ does not converge to y. $\square$

Lemma 1.2.3. *Let $(S, \mathcal{T})$ be Hausdorff, and let $K \subset S$ be compact. For any $x \notin K$, there are open disjoint sets U, V such that $x \in U$ and $K \subset V$.*

Proposition 1.2.4. Let $(S, \mathcal{T})$ be Hausdorff, and let $K \subset S$ be compact. Then K is closed.

Theorem 1.2.5. *Let $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ be two compact Hausdorff spaces, and let $f : S_1 \to S_2$. If f is continuous and bijective, then f is a homeomorphism.*

Proposition 1.2.6. Let S be a LCH space. Let $K \subset U \subset S$, where K is compact and U is open. Then, there is an open set O with compact closure such that

$$K \subset O \subset \overline{O} \subset U.$$

Proof. Since S is LCH, every point of K has an open neighbourhood with compact closure. Since K is compact, there is a finite subcover of such neighbourhoods. Hence K is a subset of their union V which has a compact closure. If $U = S$, then $O = V$. Otherwise, U^c nonempty. By Hausdorff, for any $x \in U^c \subset K^c$, there exists an open set O_x such that $K \subset O_x$ and $x \notin \overline{O_x}$. It follows that

$$\bigcap_{x\in U^c} U^c \cap \overline{V O_x} = \emptyset.$$

By the finite intersection property, there are finitely many $\{x_1, ..., x_n\}$ such that

$$U^c \cap \overline{V} \cap \overline{O_{x_1}} \cap \cdots \cap \overline{O_{x_n}} = \emptyset.$$

Let $O = V \cap O_{x_1} \cap \cdots \cap O_{x_n}$. $\square$

Definition 1.2.7. The support of a complex-valued function f is $\overline{\{x \in S : f(x) \neq 0\}}$.

Definition 1.2.8. Let $C_c(S)$ be the set of compactly supported continuous functions on S. Write $K \prec f$ for a compact set K and a function $f \in C_c(S)$ such that $0 \leq f(x) \leq 1$ for all $x \in S$ and $f(x) = 1$ for all $x \in K$. Write $f \prec U$ for an open set U and a function $f \in C_c(S)$ such that $0 \leq f(x) \leq 1$ for all $x \in S$ and the support of f is a subset of U.

Lemma 1.2.9 (Urysohn's Lemma)**.** *Let S be a LCH space. Let $K \subset U \subset S$, where K is compact and U is open. There exists a function $f \in C_c(S)$ such that $K \prec f \prec U$.*

Proof. We have a family of open sets $\{O_r : r \in \mathbb{Q} \cap [0, 1]\}$ with compact closures such that $K \subset O_1$, $\overline{O_0} \subset U$, and $\overline{O_s} \subset O_r$ for $s > r$. Define

$$f_r(x) = r\chi_{O_r} = \begin{cases} r & \text{if } x \in O_r \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g_s(x) = s + (1-s)\chi_{\overline{O_s}} = \begin{cases} 1 & \text{if } x \in \overline{O_s} \\ s & \text{otherwise} \end{cases}.$$

Also define

$$f(x) = \sup\{f_r(x) : r \in \mathbb{Q} \cap [0,1]\} \quad \text{and} \quad g(x) = \inf\{g_s(x) : s \in \mathbb{Q} \cap [0,1]\}.$$

Observe that $\{x : f(x) > a\}$ is open for all $a \in \mathbb{R}$ (lower semicontinuous), and $\{x : g(x) < a\}$ is open for all $a \in \mathbb{R}$ (upper semicontinuous). Moreover, $0 \le f \le 1$ and $f(x) = 1$ for all $x \in K \subset O_1$. Also, $\text{supp}(f) \subset \overline{O_0} \subset U$. It suffices to show that $f = g$. Note that $f_r(x) > g_s(x)$ if $r > s$ and $x \in O_r \cap \overline{O_s}^c$. But $r > s$ means $O_r \subset O_s$, so $f_r \le g_s$ for all $r, s \in \mathbb{Q}$, so $f \le g$. Suppose, for contradiction, that there exists an x such that $f(x) < g(x)$. Then, $\exists r, s \in \mathbb{Q}$ such that $f(x) < r < s < g(x)$. The first inequality implies that $x \notin O_r$, while the last implies that $x \in \overline{O_s}$, which is a contradiction since $r < s$. Therefore $f = g$, and we are done. □

Proposition 1.2.10. Let $(S, \mathcal{T})$ be LCH, let K be compact, and let $\{O_i : 1 \le i \le n\}$ be a finite open cover of K. Then there are functions $\{f_i \in C_c(S) : 1 \le i \le n\}$ such that

1. $f_i \prec O_i$ for $i \in [n]$.
2. $\sum_{i=1}^n f_i(x) = 1$ for all $x \in K$.

The set $\{f_i\}$ is called a partition of unity subordinate to the cover $\{O_i\}$.

Proof. Let $x \in K$, there exists i such that $x \in O_i$. There is a compact neighbourhood N_x of x such that $N_x \subset O_i$. By compactness, there exists $\{x_1, ..., x_m\}$ such that $K \subset \bigcup_{j=1}^m N_x^o \subset \bigcup_{j=1}^m N_x$. For $i \in [n]$, define $K_i = \bigcup_{N_{x_j} \subset O_i} N_{x_j}$. By Urysohn's lemma, there exists continuous functions g_i such that $K_i \prec g_i \prec O_i$. Since $K \subset \bigcup_{i=1}^n K_i$, we have $\sum_{i=1}^n g_i \ge 1$ on K.
Let $W = \{x : \sum_{i=1}^n g_i(x) > 0\}$ is open because W is the preimage of an open set, so by Urysohn's lemma, there exists f such that $K \prec f \prec W$. Let $g_{n+1} = 1 - f$. By construction, $G(x) = \sum_{i=1}^{n+1} g_i(x) > 0$ everywhere. Now, let $f_i(x) = g_i(x)/G(x)$ for $i \in [n]$. Therefore, $f_i \prec O_i$ (ie. $\text{supp}(f_i) \subset O_i$), and $\sum_{i=1}^n f_i = 1$ on K. □

Proposition 1.2.11 (Tietze's Extension)**.** Let $(S, \mathcal{T})$ be LCH, let K be compact, and let $f \in C(K)$. Then there exists $F \in C_c(S)$ such that $F(x) = f(x)$ for all $x \in K$.

Proof. Since f is continuous on a compact set, it is bounded. We assume WLOG that $|f| \le 1$ on K. Let V be an open set with compact closure such that $K \subset V$. Let $K^+ = \left\{x \in K : f(x) \ge \frac{1}{3}\right\}$, $K^- = \left\{x \in K : f(x) \le -\frac{1}{3}\right\}$. Note that K^+ and K^- are disjoint closed (compact) subsets of K. Applying Urysohn's Lemma to K^+ and $V \setminus K^-$, and to K^- and $V \setminus K^+$, and rescaling gives $f_1 \in C_c(S)$ such that $f_1 = \frac{1}{3}$ on K^+, $f_1 = -\frac{1}{3}$ on K^-, and $|f_1| \le \frac{1}{3}$ everywhere. Moreover, $\text{supp}(f) \subset V$ and $|f - f_1| \le \frac{2}{3}$ on K. Repeat this procedure for $f - f_1$ to get f_2 such that $|f - f_1 - f_2| \le \left(\frac{2}{3}\right)^2$ on K. Hence, we obtain a sequence $\{f_n\}_{n \in \mathbb{N}}$ such that $f_n \in C_c(S)$ where $|f_n| \le \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$ and $|f - \sum_{i=1}^n| \le \left(\frac{2}{3}\right)^n$ on K. Now, let $F = \sum_{i=1}^\infty f_i$. Note that F is convergent uniformly everywhere so F is continuous, and $F = f$ on K. Moreover, $\text{supp}(F) \subset \overline{V}$. □

2 Normed Vector Spaces

Definition 2.0.1. A normed vector space $(V, \|\cdot\|)$ is a vector space V over $\mathbb{C}$ or $\mathbb{R}$ equipped with a norm $\|\cdot\| : V \to [0, \infty)$ such that

1. $\|v\| \ge 0$ for all $v \in V$ and $\|v\| = 0 \iff v = 0$
2. $\|\lambda v\| = |\lambda| \, \|v\|$ for all $v \in V, \lambda \in \mathbb{C}$

3. $\|v+w\| \leq \|v\| + \|w\|$ for all $v, w \in V$ (Minkowski's inequality)

Definition 2.0.2. Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be two normed linear spaces. A bounded linear transformation is a function $T : V_1 \to V_2$ such that

1. $T(\lambda v + w) = \lambda T(v) + T(w)$ for all $v, w \in V_1, \lambda \in \mathbb{C}$
2. There exists some $C \geq 0$ such that $\|Tv\|_2 \leq C\|v\|_1$ for all $v \in V_1$

The norm of T is

$$\|T\| = \sup\left\{\frac{\|Tv\|_2}{\|v\|_1} : v \in V_1, v \neq 0\right\}.$$

The set of all bounded linear transformations is a vector space denoted $\mathcal{L}(V_1, V_2)$, and the norm is referred to as the operator norm.

Proposition 2.0.3. Let $T : V_1 \to V_2$ be a linear transformation between two normed linear spaces $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$. The following are equivalent:

1. T is continuous at $v_0 \in V_1$
2. T is continuous everywhere
3. T is bounded

Proof. $(2 \implies 1)$ is trivial.
If (1) holds, then there is some $r > 0$ such that

$$\|v - v_0\|_1 < 2r^{-1} \implies \|Tv - Tv_0\|_2 < 1.$$

For any $w \in V_1$, the vector $v = \frac{w}{r\|w\|_1} + v_0$ satisfies $\|v - v_0\|_1 = r^{-1}$, so

$$\|Tw\|_2 = r\|w\|_1 \|T(v - v_0)\|_2 = r\|w\|_1 \|Tv - Tv_0\|_2 \leq r\|w\|_1.$$

Therefore $(1 \implies 3)$.
If (3) holds, then

$$\|Tv_1 - Tv_2\|_2 = \|T(v_1 - v_2)\|_2 \leq C\|v_1 - v_2\|_1.$$

Therefore $(3 \implies 2)$. $\square$

Theorem 2.0.4. *Let V be an infinite-dimensional normed linear space. Then the set $\mathcal{B}_1 = \{v \in V : \|v\| \leq 1\}$ is not compact.*

Proof. We recursively construct a sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\mathcal{B}_1$ as follows. Let $w_1 \in \mathcal{B}_1$. Given $\{w_1, ..., w_n\}$, let W_n be their span, which is finite dimensional and hence closed. Since V is infinite-dimensional, $V \setminus W_n \neq \emptyset$. Then, there exists some $w_{n+1} \in V$ such that

$$\|w_{n+1}\| = 1, \qquad \text{and} \qquad \|w_{n+1} - w\| > \frac{1}{2} \quad (w \in W_n).$$

Indeed, let $x \in V \setminus W_n$. Since W_n is closed, $\delta_0 = \inf\{\|x - w\| : w \in W_n\} > 0$. In particular, there exists some $z \in W_n$ such that $\|x - z\| < 2\delta_0$. Let $w_{n+1} = \frac{x-z}{\|x-z\|}$. Then, $\|w_{n+1}\| = 1$ and

$$\inf_{w\in W_n} \|w_{n+1} - w\| = \inf_{w\in W_n} \frac{\|x - z - w\|}{\|x - z\|} = \frac{\inf \|x - w\|}{\|x - z\|} > \frac{1}{2}.$$

Now, it follows that $\|w_i - w_j\| > \frac{1}{2}$ for all $i, j \in \mathbb{N}$, so the sequence $\{w_n\}_{n\in\mathbb{N}}$ in $\mathcal{B}_1$ has no convergent subsequence. Since the norm induces a metric topology which is first countable, we conclude that $\mathcal{B}_1$ is not sequentially compact, and hence not compact. $\square$

Definition 2.0.5. A Banach space is a complete normed linear space.

Theorem 2.0.6.