Math 420 Notes

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Contents

1 Introduction

Definition 1.1. A σ-algebra on X is a collection of subsets of 2^X that is closed under complement and countable union.

Definition 1.2. Let $\mathcal{M} \subset 2^X$ be the measurable subsets of X. A measure μ on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ satisfying the following.

(i) $\mu(\emptyset) = 0$

(ii) $\mu(\dot{\bigcup}_j^\infty E_j) = \sum_j^\infty \mu(E_j)$

Note that $\mathcal{M}$ is a σ-algebra.

Example 1.1. The counting measure. $\mu(E) = \#\{X : X \in E\}$

Example 1.2. The Dirac measure. Fix $x_0 \in X$. $\mu(E) = 1$ if $x_0 \in E$, and $\mu(E) = 0$ otherwise.

Example 1.3. An unmeasurable set. (Folland p.20).
Let $E_r = E + r$ mod 1. There exists a set $E \subset [0, 1)$ such that

- $\{E_r\}_{r\in\mathbb{Q}\cap[0,1)}$ are disjoint
- $\bigcup_{r\in\mathbb{Q}\cap[0,1)} E_r = [0, 1)$

This set E is inconsistent with (ii) of the definition when $\mu([0, 1)) = 1$ and $\mu(E_r) = \mu(E)$.

Definition 1.3. Let non-empty $\mathcal{E} \subset 2^X$. The σ-algebra generated by $\mathcal{E}$ is $\mathcal{M}(\mathcal{E})$, that is the smallest σ-algebra containing $\mathcal{E}$. (We can get this by taking the intersection of all the σ-algebras containing $\mathcal{E}$)

Example 1.4. Let X be a topological space. The Borel σ-algebra B_X is a σ-algebra generated by open sets. This contains open sets, closed sets, countable union of closed sets ($F\sigma$-sets), countable intersection of open sets ($G\delta$-sets).
$B_{\mathbb{R}}$ can be generated by any of

- open intervals. $\{(a, b)\}$
- closed intervals. $\{[a, b]\}$
- half open intervals. $\{(a, b]\}$
- semi-infinite intervals. $\{(a, \infty)\}$

2 The Lebesgue Measure

2.1 Premeasures

Let $\mathcal{A}$ be the set of finite disjoint unions of h-intervals, where h-intervals are of the following form: $(a, b]$, (a, ∞), $\emptyset$, where $-\infty \le a < b < \infty$.

Proposition 2.1. $\mathcal{A}$ is an algebra.

Proof. The intersection of two h-intervals is also an h-interval. The complement of an h-interval is the union of at most two disjoint h-intervals. Refer to text (Folland Prop 1.7). □
Define the "Length" of sets in $\mathcal{A}$ to be a function $m_0 : \mathcal{A} \to [0, \infty]$ with finite additivity and $m_0(\emptyset) = 0$.

Definition 2.1. A premeasure is a function $m : \mathcal{A} \to [0, \infty]$ such that

(i) $m(\emptyset) = 0$

(ii) For countably many disjoint $A_j \in \mathcal{A}$ whose union $A = \bigcup A_j$ is also in $\mathcal{A}$, we have $m(\bigcup A_j) = \sum m(A_j)$.

Theorem 2.1. *The following is true*

1. *m_0 is well defined.*
2. *m_0 is a premeasure.*

Proof of 1. This is just bookkeeping. See text. □

Proof of 2. Let $A = (a, b] \in \mathcal{A}$ be a countable disjoint union of $A_j = (a_j, b_j] \in \mathcal{A}$. We can assume that A_j, because each A_j would otherwise be the finite union of some disjoint set of intervals in $\mathcal{A}$. We can also assume that A is an interval by the same argument.
Consider $A = \bigcup_{j=1}^n A_j \cup (A \setminus \bigcup_{j=1}^n)$. Then we have

$$m_0(A) = m_0\left(\bigcup_{j=1}^n A_j\right) + m_0\left(A \setminus \bigcup_{j=1}^n\right) \geq m_0\left(\bigcup_{j=1}^n A_j\right).$$

Taking the limit gives $m_0(A) \geq m_0(\bigcup_{j=1}^\infty A_j)$.
Now let $\epsilon > 0$. Consider the compact interval $[a+\epsilon, b]$ covered by $\bigcup_{j=1}^\infty (a_j, b_j + \frac{\epsilon}{2^j})$. There must be a finite subcover. Now, $(a+\epsilon, b]$ is also covered by this finite subcover, and we can relabel the finite subcover so that $a_j < a_{j+1}$. Then

$$\begin{aligned}
m_0(A) &= m_0((a, a+\epsilon]) + m_0((a+\epsilon, b]) \\
&\leq \epsilon + m_0((a_1, b_n + \frac{\epsilon}{2^n})) = \epsilon + b_n + \frac{\epsilon}{2^n} - a_n + \sum_{j=2}^n (a_j - a_{j-1}) \\
&\leq \epsilon + (b_n - a_n) + \sum_j^n \left(b_j + \frac{\epsilon}{2^j} - a_{j-1}\right) \leq \epsilon + \sum_j^n \frac{\epsilon}{2^j} + \sum_j^n m_0(A_j) \\
&\leq 7\epsilon + \sum m_0(A_j),
\end{aligned}$$

and countable additivity follows. □

2.2 Lebesgue Outer Measure

Definition 2.2. The Lebesgue outer measure m^* of a set $E \subset \mathbb{R}$ is defined as follows.

$$m^*(E) = \inf\left\{\sum_{j=1}^\infty m_0(I_j) : E \subset \bigcup_{j=1}^\infty I_j\right\}.$$

Definition 2.3. In general, given $\mathcal{E} \subset 2^X$ with $\emptyset, X \in \mathcal{E}$ and $\mu_0 : \mathcal{E} \to [0, \infty]$ with $\mu_0(\emptyset) = 0$, we can define $\mu^* : 2^X \to [0\infty]$ as follows.

$$\mu^*(E) = \inf\left\{\sum_{j=1}^\infty \mu_0(E_j) : E \subset \bigcup_{j=1}^\infty E_j \text{ and } E_j \in \mathcal{E}\right\}.$$

Proposition 2.2. μ^* is an outer measure, where an outer measure satisfies three properties.

1. $\mu^*(\emptyset) = 0$
2. $A \subset B \implies \mu^*(A) \leq \mu^*(B)$
3. $\mu^*\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j=1}^\infty \mu^*(E_j)$

Proof of 1. $\emptyset \subset \bigcup_{j=1}^\infty \emptyset \implies \mu^*(\emptyset) \leq \sum_{j=1}^\infty \mu_0(\emptyset) = 0$. □

Proof of 2. Let $A \subset B$. Then $\left\{\{E_j\}_j \subset \mathcal{E} : B \subset \bigcup_j E_j\right\} \subset \left\{\{E_j\}_j \subset \mathcal{E} : A \subset \bigcup_j E_j\right\}$. Hence $\mu^*(A) \leq \mu^*(B)$. □

Proof of 3. Let $\{A_{j,k}\}_k \subset \mathcal{E}$ such that $E_j \subset \bigcup_{k=1}^\infty A_{j,k}$. Observe that $\bigcup_{j=1}^\infty E_j \subset \bigcup_{j,k=1}^\infty A_{j,k}$. Let $\epsilon > 0$, then

$$\mu^*\left(\bigcup_{j=1}^\infty E_j\right) \leq \sum_{j,k=1}^\infty \mu_0(A_{j,k}) \leq \sum_{j=1}^\infty \left(\frac{\epsilon}{2^j} + \mu^*(E_j)\right) = \epsilon + \sum_{j=1}^\infty \mu^*(E_j)$$

Since ϵ is arbitrary, we get $\mu^*(\bigcup_j E_j) \leq \sum_j \mu^*(E_j)$. □

Observe that μ^* is defined for every set in 2^X, but it is not additive. To fix this, we will remove some "bad" sets.

Definition 2.4. Let μ^* be an outer measure on X. A set $A \subset X$ is μ^*-measurable if for every $E \subset X$, we have $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

2.3 Carathéodory's Extension Theorem

Theorem 2.2. *Let $\mathcal{M}$ be the set of μ^*-measurable sets, and $\mu^* \restriction_{\mathcal{M}}$ is a complete measure.*

Proof.

1. We show that $\mathcal{M}$ is an algebra. Clearly, $\emptyset \in \mathcal{M}$, and $\mathcal{M}$ is closed under complement. Now let $A, B \in \mathcal{M}$. Then

$$\begin{aligned}\mu^*(E) &= \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &= \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c) \\ &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).\end{aligned}$$

   The other inequality is automatic by monoticity. Hence $A \cup B \in \mathcal{M}$.

2. We show that μ^* is finitely additive. Let $A, B \in \mathcal{M}$ be disjoint. Then

$$\mu^*(A \cup B) = \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c) = \mu^*(A) + \mu^*(B).$$

3. We show that $\mathcal{M}$ is closed under countable union and μ^* is countably additive.
   Let $\{A_j\} \subset \mathcal{M}$, $B_n = \bigcup_{j=1}^n A_j$, and $B = \bigcup_{j=1}^\infty A_j$. Let $E \subset X$, then

$$\begin{aligned}\mu^*(E \cap B_n) &= \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c) \\ &= \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1}) = \sum_{j=1}^n \mu^*(E \cap A_j).\end{aligned}$$

   By the definition, we get

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*(E \cap B^c).$$

   Take $n \to \infty$, then we get closure under countable union.

$$\mu^*(E) \geq \sum_{j=1}^\infty \mu^*(E \cap A_j) + \mu^*(E \cap B^c) \geq \mu^*(E \cap B) + \mu^*(E \cap B^c).$$

   Take $E = B$, then we get countable additivity.

$$\mu^*(B) = \sum_{j=1}^\infty \mu^*(A_j) + \mu^*(\emptyset) = \sum_{j=1}^\infty \mu^*(A_j).$$

We can easily check that $\mathcal{M}$ is complete. This theorem is complete. □

Proposition 2.3. If $A \in \mathcal{A}$, then A is μ^*-measurable.

Proof. Let $A \in \mathcal{A}$ and $E \subset X$. Let $\epsilon > 0$. There exists $\{A_j\} \subset \mathcal{A}$ with $E \subset \bigcup_{j=1}^{\infty} A_j$ such that $\mu^*(E) + \epsilon \geq \sum_{j=1}^{\infty} \mu_0(A_j)$ by the definition of μ^*. Then

$$\mu^*(E) + \epsilon \geq \sum \mu_0(A_j \cap A) + \sum \mu_0(A_j \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Take $\epsilon \to 0$, and we see that A is μ^*-measurable. $\square$

Proposition 2.4. $\mu^* \restriction_{\mathcal{A}} = \mu_0 \restriction_{\mathcal{A}}$.

Proof. See text. $\square$

2.4 Lebesgue Measure on $\mathbb{R}$

Let $X = \mathbb{R}$ and define m_0 to be the length of h-intervals.

1. $m^*(E) = \inf\{\sum m_0(I_j) : E \subset \bigcup I_j\}$, where I_j are h-intervals.
2. $\mathcal{L}$ is the m^*-measurable sets (Lebesgue measurable).
3. $m = m^* \restriction_{\mathcal{L}}$.

Remark 2.1. The measure m is a Borel measrue, that is it is defined for all Borel sets. Also, m is the unique Borel measure with $m((a, b]) = b - a$.

Proof. See text. Basically if μ_0 is σ-finite on $\mathcal{A}$, then Carathéodory gives uniqueness. $\square$

Remark 2.2. We can also construct a measure with any non-decreasing right-continuous $F : \mathbb{R} \to \mathbb{R}$ with $m_F((A, b]) = F(b) - F(a)$. This is the Lebesgue-Stieltjes measure. Observe that the Lebesgue measure simply has $F(x) = x$.

Proposition 2.5. Any Boren measure μ that is finite on bounded sets defines a non-decreasing right-continuous function $F : \mathbb{R} \to \mathbb{R}$ as follows

$$F(x) = \begin{cases} \mu((0, x]) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \mu((-x, 0]) & \text{if } x < 0 \end{cases}.$$

Proposition 2.6. The Lebesgue measure is **translation invariant** $m(E+s) = m(E)$ and **dilation invariant** $m(rE) = |r|\, m(E)$.

Remark 2.3. open sets, closed sets, etc. $\subsetneq \mathcal{B}_{\mathbb{R}} \subsetneq \mathcal{L} \subsetneq 2^{\mathbb{R}}$.

Lemma 2.3. $m^*(E) = \inf\left\{\sum_{j=1}^{\infty}(b_j - a_j) : E \subset \bigcup_{j=1}^{\infty}(a_j, b_j)\right\}$.

Proof. See text. $\square$

Theorem 2.4. *Let $E \subset \mathbb{R}$. All of the following imply one another.*

(a) $E \in \mathcal{L}$.

(b) There exists $U_{open} \supset E$ such that $m^*(U \setminus E) \leq \epsilon$.

(c) There exists $F_{closed} \subset E$ such that $m^*(E \setminus F) \leq \epsilon$.

(d) *There exists a $G\delta$ set $V \supset E$ such that $E = V \setminus N_1$ with N_1 null.*

(e) *There exists a $F\sigma$ set $H \supset E$ such that $E = H \cup N_2$ with N_2 null.*

Proof. .

$(a \implies b)$. Let $\epsilon > 0$. There exists $U = \bigcup_{j=1}^{\infty} I_j \supset E$ (where each I_j is an open interval) such that $m(E) + \epsilon \geq \sum_{j=1}^{\infty} m(I_j) \geq m(U)$. Then, using the definition of a measurable set,

$$m(U) = m(U \cap E) + m(U \cap E^c) = m(E) + m(U \cap E^c) \leq m(E) + \epsilon.$$

Hence $(U \setminus E) < \epsilon$ holds for $m(E)$ finite.

If $m(E)$ is infinite, then let $E_j = E \cap (j, j+1]$ and $U_j = U \cap (j, j+1]$. Then $m(U_j \setminus E_j) \leq \epsilon 2^{-|j|}$ from the finite case, and countable additivity gives the desired result.

$(a \implies c)$. Use E^c and (a) implies (b).

$(b \implies d)$. There exists open $U_j \supset E$ such that $m^*(U_j \setminus E) \leq \frac{1}{j}$. Then $V = \bigcap_{j=1}^{\infty} U_j \supset E$ is a $G\delta$ set. Let $N_1 = V \setminus E$, then $E = V \setminus N_1$. It follows that $N_1 \subset U_j \setminus E$ for all j, so $m^*(N_1) \leq m^*(U_j \setminus E) \leq \frac{1}{j}$. Hence $m^*(N_1) = 0$, so N_1 is a null set.

$(c \implies e)$. Similar to above.

$(d \implies a)$ and $(e \implies a)$. $G\delta$ and $F\sigma$ sets are Borel, so they are measurable. Null sets are also measurable by completeness. Hence E is measurable.

□

3 Integrals and Convergence

Definition 3.1. Let $(X, \mathcal{M})$ and $Y, \mathcal{N})$ be measurable spaces. A functoin $f : X \to Y$ is $(\mathcal{M}, \mathcal{N})$ measurable if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{N}$.

Remark 3.1. If $\mathcal{N}$ is generated by $\mathcal{E} \subset \mathcal{N}$, then $f : X \to Y$ is measurable if and only if $f^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$.

Remark 3.2. A function $f : \mathbb{R} \to \mathbb{R}$ is Borel measurable if it is continuous.

Remark 3.3. Composition of measurable functions is measurable.

Remark 3.4. $f : X \to \bar{\mathbb{R}}$ is measurable if it is $(\mathcal{M}, \mathcal{B}_{\bar{\mathbb{R}}})$-measurable, where

$$\mathcal{B}_{\bar{\mathbb{R}}} = \left\{ E \subset \bar{\mathbb{R}} : E \cap \mathbb{R} \in \mathcal{B}_{\bar{\mathbb{R}}} \right\}.$$

Proposition 3.1. Let $(X, \mathcal{M})$ be a measurable space. Then

1. $f : X \to \mathbb{C}$ is measurable if and only if the real and imaginary parts of f are measurable.
2. $f, g : X \to \mathbb{C}$ are measurable implies $f + g$ and $f \cdot g$ are measurable.
3. $f_j : X \to \bar{\mathbb{R}}$ is measurable implies $\sup_j f_j$, $\inf_j f_j$, $\limsup_{j\to\infty} f_j$ and $\liminf_{j\to\infty} f_j$ are measurable.
4. $f_j : X \to \mathbb{C}$ is measurable implies $\lim_{j\to\infty} f_j$ is measurable if the limit exists.

Definition 3.2. A simple function on $(X, \mathcal{M})$ is of the form $f(x) = \sum_{j=1}^{n} z_j \chi_{E_j}(x)$ for $z_j \in \mathbb{C}$ and $E_j \in \mathcal{M}$.

Remark 3.5. f is in "standard form" if $E_j = f^{-1}(\{z_j\})$.

Definition 3.3. Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f = \sum_{j=1}^{n} z_j \chi_{E_j}$ be a simple function. Then

$$\int f = \sum_{j=1}^{n} z_j \mu(E_j).$$

Proposition 3.2. Let ϕ, ψ be simple functions.

(a) $c \in \mathbb{C}$ implies $\int c\psi = c \int \psi$. (linearity)

(b) $\int \phi + \psi = \int \phi + \int \psi$.

(c) If $\phi, \psi \in \mathbb{R}$, then $\phi \leq \psi \implies \int \phi \leq \int \psi$.

(d) If $\phi \geq 0$, then $\mathcal{M} \ni A \mapsto \int_A \phi := \int \chi_A \phi$ is a measure.

Proof. .

(a), (b), and (d). See text / exercise.

(c). Let $\phi = \sum_{j=1}^{n} z_j \chi_{E_j}$ and $\psi = \sum_{k=1}^{m} w_k X_{F_k}$ in standard form. Then

$$\begin{aligned} \int \phi &= \sum_j z_j \mu(E_j) = \sum_j z_j \sum_k \mu(E_j \cap F_k) = \sum_j \sum_k z_j \mu(E_j \cap F_k) \\ &\leq \sum_k \sum_j w_k \mu(E_j \cap F_k) = \sum_k w_k \sum_j \mu(E_i \cap F_k) = \sum_k w_k \mu(F_k) = \int \psi. \end{aligned}$$

□

Definition 3.4. Define $L^+ = \{f : X \to [0, \infty), \text{measurable}\}$. Then for $f \in L^+$, define

$$\int f = \sup \left\{ \int \phi : 0 \leq \phi \leq f,\ \phi \text{ simple} \right\}.$$

Remark 3.6. We have monotonicity and linearity for $f \in L^+$.

3.1 Approximation by Simple Functions and Monotone Convergence

Theorem 3.1 (Approximation Theorem). *(a) Let measurable $f : X \to [0, \infty]$. There exists simple $0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq f$ such that $\phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on sets where f is bounded.*

(b) Let measurable $f : X \to \mathbb{C}$. There exists simple $\{\phi_n\}$ with $0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |f|$ such that $\phi_n \to f$ pointwise, and $\phi_n \to f$ uniformly on sets where f is bounded.

Proof. Proof by construction with powers of 2. □

Theorem 3.2 (Monotone Convergence Theorem). *Let $\{f_n\} \subset L^+$ with $0 \leq f_1 \leq f_2 \cdots$. Then*

$$\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n.$$

Proof. Let $f(x) = \sup_n f_n(x) \in L^+$. Then $\left\{\int f_n\right\}$ is increasing, so $\lim_{n\to\infty} \int f_n = \sup_n \int f_n$ (which exists). Since $f_n \le f$, we have $\int f_n \le \int f$, so $\lim_{n\to\infty} \int f_n \le \int f$.
Let ϕ be a simple function such that $0 \le \phi \le f$. Fix $\alpha \in (0, 1)$. Let $E_n = \{x : f_n(x) \ge \alpha\phi(x)\}$. Observe that E_n is measurable and $E_1 \subset E_2 \subset \cdots \bigcup_{n=1}^{\infty} E_n = X$. Since $E \mapsto \int_E \phi$ is a measure, we get $\int_{E_n} \phi \mapsto \int \phi$ by continuity from below. Then

$$\int f_n \ge \int_{E_n} f_n \ge \alpha \int_{E_n} \phi \implies \lim_{n\to\infty} f_n \ge \alpha \int \phi.$$

If we take $\alpha \to 1$, then $\lim \int f_n \ge \int \phi$. Then the Monotone Convergence Theorem follows by simple function approximation. $\square$

Proposition 3.3. Let $\{f_n\} \subset L^+$, then $\int \sum_n f_n = \sum_n \int f_n$.

Proof. Let $f_1, f_2 \in L^+$. By approximation, we have $\phi_n \uparrow f_1$ and $\psi_n \uparrow f_2$. Then $\phi_n + \psi_n \uparrow f_1 + f_2$. Then by Monotone Convergence,

$$\int f_1 + f_2 = \lim_{n\to\infty} \int (\phi_n + \psi_n) = \lim_{n\to\infty} \left(\int \phi_n + \int \psi_n \right) = \lim_{n\to\infty} \int \phi_n + \lim_{n\to\infty} \int \psi_n = \int f_1 + \int f_2.$$

Now let $\{f_n\}_{n=1}^{\infty}$. Then using MCT on $\sum_{n=1}^{N} f_n \uparrow \sum_{n=1}^{\infty} f_n$ implies

$$\int \sum_{n=1}^{\infty} f_n = \int \lim_{N\to\infty} \sum_{n=1}^{N} f_n = \lim_{N\to\infty} \int \sum_{n=1}^{N} f_n = \lim_{N\to\infty} \sum_{n=1}^{N} \int f_n = \sum_{n=1}^{\infty} \int f_n.$$

$\square$

Proposition 3.4. If $f \in L^+$, then $\int f = 0 \iff f = 0$ almost everywhere.

Proof. For simple $f = \sum_{k=1}^{n} a_k \chi_{E_k}$, then $\mu(E_k) = 0$ or $a_k = 0$. The result follows because the finite union of null sets is still a null set.
Now we prove this for $f \in L^+$. If $f = 0$ almost everywhere, then any simple ϕ satisfying $0 \le \phi \le f$ is also 0 almost everywhere. Then $\int \phi = 0$, implying that $\int f = 0$. If f is not 0 almost everywhere, then $\mu(\{f(x) > 0\}) > 0$. Let $E_n = \left\{f(x) > \frac{1}{n}\right\}$ for $n =\in \mathbb{N}$. Then $\{f(x) > 0\} = \bigcup E_n$. It follows that there exists some k such that $\mu(E_k) > 0$. Hence $f \ge \frac{1}{k}\chi_{E_k}$, so $\int f \ge \frac{1}{k}\mu(E_k) > 0$. $\square$

Remark 3.7. We don't care about null sets. If $f_n \in L^+$ and $f_n \uparrow f$ almost everywhere, then $\int f = \lim_{n\to\infty} \int f_n$.

Proof. Apply MCT to $f_n \chi_{N^c}$ where N is the null set on which f_n does not converge to f. $\square$

Theorem 3.3 (Fatou's Lemma). *Let $\{f_n\}_{n=1}^{\infty} \subset L^+$. Then*

$$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n.$$

Corollary 3.3.1. *If $f_n \to f$ almost everywhere, then $\int f \le \liminf_{n\to\infty} \int f_n$.*

Proof of Fatou's Lemma. Let $g_k(x) = \inf_{n\ge k} f_n(x)$ be an increasing sequence of functions. Then for each $j \ge k$, we have

$$\inf_{n\ge k} f_n \le f_j \implies \int \inf_{n\ge k} f_n \le \int f_j \implies \int \inf_{n\ge k} f_n \le \inf_{j\ge k} \int f_j$$

It follows by MCT and the above that

$$\int \sup_k g_k = \int \liminf f_n = \lim_{k\to\infty} \int \inf_{n\geq k} f_n \leq \lim_{k\to\infty} \inf_{j\geq k} \int f_j = \liminf_{n\to\infty} \int f_n.$$

□

3.2 Integration of Complex Functions

Let $f : X \to \mathbb{R}$ be measurable, then $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Observe that $f^+, f^- \in L^+$ and we can write $f = f^+ - f^-$. Also observe that if $f : X \to \mathbb{C}$, then we write $f = \Re(f) + i\Im(f)$, and hence $\int f = \int \Re(f) + i \int \Im(f)$.

Definition 3.5. We say $f : X \to \mathbb{C}$ is integrable if $\int |f| < \infty$. Define

$$L^1(\mu) = \left\{ f : X \to \mathbb{C} : \int |f| < \infty \right\}.$$

Proposition 3.5. (a) L^1 is a vector space.

(b) $\int$ is a linear map on L^1.

(c) $f \in L^1$ implies $\left|\int f\right| \leq \int |f|$.

(d) If $f, g \in L^1$, then $\int |f - g| = 0 \iff f = g$ a.e. $\iff \int_E f = \int_E g$ for all $E \in \mathcal{M}$.

Proof. See text. □

Remark 3.8. If we define L^1 to be the equivalence class of almost everywhere defined integrable functions under $f \sim g \iff f = g$ a.e., then L^1 is a Banach space under $|f - g|$.

Remark 3.9. If $f \in L^+$ with $\int f < \infty$, then $\mu(\{f = \infty\}) = 0$.

Proof. Exercise. □

Theorem 3.4 (Dominated Convergence Theorem)**.** *Let $L^1 \ni f_n \to f$ almost everywhere and $|f_n| \leq g \in L^1$ for all n. Then $f \in L^1$ and $\int f = \lim_{n\to\infty} \int f_n$.*

Proof. First we show that $f \in L^1$. Observe that $|f_n| \leq g$ implies $|f| \leq g$ almost everywhere, so $f \in L^1$.
We take $f_n \in \mathbb{R}$. Otherwise, consider $\Re(f_n)$ and $\Im(f_n)$. Observe that $g \pm f_n \geq 0$, so Fatou's Lemma implies

$$\int g + \int f = \int g + f \leq \liminf \int g + f_n = \int g + \liminf \int f_n$$
$$\int g - \int f = \int g - f \leq \liminf \int g - f_n = \int g - \limsup \int f_n$$

Since $\int g < \infty$, we have $\limsup \int f_n \leq \int f \leq \liminf \int f_n$. It follows that $\lim \int f_n = \int f$. □

Proposition 3.6. Let $\{f_j\}_{j=1}^\infty \subset L^1$ with $\sum_{j=1}^\infty \int |f_j| < \infty$. Then $\sum_{j=1}^\infty f_j$ converges almost everywhere and $\int \sum_{j=1}^\infty f_j = \sum_{j=1}^\infty \int f_j$.

Proof. Each $|f_j| \in L^+$, so MCT gives $\int \sum_{j=1}^{\infty} |f_j| = \sum_{j=1}^{\infty} \int |f_j| < \infty$. Hence, $\sum_{j=1}^{\infty} |f_j| \in L^1$. It follows that $\sum_{j=1}^{\infty} |f_j(x)| < \infty$ almost everywhere, so $\sum_{j=1}^{\infty} f_j$ converges almost everywhere. Since $\left|\sum_{j=1}^{N} f_j\right| \leq \sum_{j=1}^{N} |f_j| \leq \sum_{j=1}^{\infty} |f_j| = g \in L^1$, we can apply DCT to the partial sums to get the result. $\square$

Definition 3.6. The support of a function $f : X \to \mathbb{C}$ is the set $\{x : f(x) \neq 0\}$.

Theorem 3.5 (L^1 Approximation of Functions)**.** *Let $f \in L^1(\mu)$. For any $\epsilon > 0$, there exists a simple functoin $\phi = \sum_{j=1}^{n} a_j \chi_{E_j}$ such that $\int |f - \phi| < \epsilon$.*
If $(X, \mu) = (\mathbb{R}, m)$, then we can take each E_j to be a finite union of open intervals. Also, there exists a continuous function g with compact support such that $\int |f - g| < \epsilon$.

4 Modes of Convergence

Let $f_n : X \to \mathbb{C}$ and $f : X \to \mathbb{C}$.

Definition 4.1. $f_n \to f$ pointwise if $f_n(x) \to f(x)$ for all $x \in X$.

Definition 4.2. $f_n \to f$ uniformly if $\sum_{x \in X} |f_n(x) - f(x)| \to 0$.

Definition 4.3. $f_n \to f$ almost everywhere if $f_n(x) \to f(x)$ for all $x \in N^c$ with $\mu(N) = 0$.

Definition 4.4. $f_n \to f$ in L^1 if $\int_X |f_n - f| \, d\mu \to 0$.

Definition 4.5. $f_n \to f$ in measure if for all $\epsilon > 0$, $\mu(\{x : |f_n(x) - f(x)| \geq \epsilon\}) \to 0$.

We have the following implications

- uniform convergence implies pointwise convergence
- pointwise convergence implies almost everywhere convergence
- convergence in L^1 implies convergence in measure
- uniform convergence implies convergence in measure
- uniform convergence implies convergence in L^1 on a finite measure space
- almost everywhere convergence implies convergence in measure on a finite measure space
- almost everywhere convergence implies convergence in L^1 if we can apply DCT
- convergence in measure implies almost everywhere convergence if we allow subsequences

Theorem 4.1 (Egoroff)**.** *If $\mu(X) < \infty$, and $\{f_n\}_{n=1}^{\infty}$ are measurable, with $f_n \to f$ almost everywhere, then $f_n \to f$ almost uniformly. That is, for any $\epsilon > 0$, there exists $E \subset X$ with $\mu(E) < \epsilon$ such that $f_n \to f$ uniformly on E^c.*

Remark 4.1. Almost uniform convergence implies convergence in measure.

Proof. For $k \in \mathbb{N}$ let $E_n(k) = \bigcup_{m=n}^{\infty} \left\{x \in X : |f_m(x) - f(x)| \geq \frac{1}{k}\right\}$. These sets are decreasing with $\mu\left(\bigcap_{n=1}^{\infty} E_n(k)\right) = 0$ by almost everywhere convergence. By continuity from above, we have $\lim_{n\to\infty} \mu(E_n(k)) = 0$. Hence, for any k and ϵ, there is some n_k such that $\mu(E_{n_k}(k)) < \epsilon 2^{-k}$, so

$$\mu\left(E := \bigcup_{k=1}^{\infty} E_{n_k}(k)\right) \leq \sum_{k=1}^{\infty} \mu(E_{n_k}(k)) < \epsilon.$$

If $x \notin E$, then $|f_n(x) - f(x)| < \frac{1}{k}$ for sufficiently large n, so $f_n \to f$ uniformly on E^c. □

Definition 4.6. A sequence of functions f_n is Cauchy in measure if for any $\epsilon > 0$,

$$\lim_{m,n\to\infty} \mu(\{|f_n - f_m| \geq \epsilon\}) = 0.$$

Theorem 4.2. *Let $\{f_n\}_{n=1}^{\infty}$ be Cauchy in measure. Then*

- *$f_n \to f$ in measure for some f.*
- *There exists a subsequence f_{n_j} that converges to f almost everywhere.*
- *If $f_n \to g$ in measure, then $f = g$ almost everywhere.*

Proof. See text. □

5 Product Measures

Consider measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$.

Definition 5.1. Define $\mathcal{M} \otimes \mathcal{N}$ to be the σ-algebra generated by rectangles of the form $A \times B = \{(x, y) : x \in A, y \in B\}$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$.

Let $\mathcal{A}$ be the set of finite disjoint unions of rectangles. Then $\pi : \mathcal{A} \to [0, \infty]$ with $\bigcup(A_j \times B_j) \mapsto \sum \mu(A_j)\nu(B_j)$ is a well-defined premeasure.

Definition 5.2. The product measure $\mu \times \nu$ is the extension of π to $\mathcal{M} \otimes \mathcal{N}$.

Definition 5.3. Let $E \in X \times Y$. Define $E_x = \{y \in Y : (x, y) \in E\}$, and $E^y = \{x \in X : (x, y) \in E\}$. Let $f : X \times Y \to \mathbb{C}$. Define $f_x : y \mapsto f(x, y)$ and $f^y : x \mapsto f(x, y)$.

Proposition 5.1. Let $E \in \mathcal{M} \otimes \mathcal{N}$. Then $E_x \in \mathcal{N}$ and $E^y \in \mathcal{M}$.

Proof. Let $R = \{E \subset X \times Y : E_x \in \mathcal{N},\ E^y \in \mathcal{M}\}$. Then R contains all the rectangles. Furthermore, R is a σ-algebra (exercise). Hence $\mathcal{M} \otimes \mathcal{N} \subset R$. □

Proposition 5.2. If f is $\mathcal{M} \otimes \mathcal{N}$-measurable, then f_x is $\mathcal{N}$-measurable and f^y is $\mathcal{M}$-measurable.

Proof. $f_x^{-1}(B) = (f^{-1}(B))_x \in \mathcal{N}$ for any Borel B. □

Theorem 5.1 (Slicing Theorem). *Let $E \in \mathcal{M} \otimes \mathcal{N}$. Then $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable, and $(\mu \times \nu)(E) = \int_X \nu(E_x) d\mu(x) = \int_Y \mu(E^y) d\nu(y)$.*

Proof. Suppose that μ and ν are finite.

1. We check that the conclusion holds for rectangles $E = A \times B$. Observe that

$$\nu(E_x) = \chi_A(x)\nu(B) \implies \int_X \nu(E_x) d\mu = \mu(A)\nu(B) = (\mu \times \nu)(E).$$

2. The conclusion also holds for finite disjoint unions of rectangles by the additivity of measures and integrals.

3. Let $\mathcal{C} = \{E \in \mathcal{M} \otimes \mathcal{N} : \text{ conclusion holds}\}$. Then $\mathcal{A} \subset \mathcal{C}$ where $\mathcal{A}$ is the finite disjoint union of rectangles.

   We show that $\mathcal{C}$ is a monotone class. That is it is closed under increasing union and decreasing intersection (ie. monotone union and intersection).

   $\bigcup$: Let increasing $\{E_n\} \subset \mathcal{C}$ with $E = \bigcup E_n$. Then $f_n(y) = \mu(E_n^y) \uparrow f(y) = \mu(E^y)$. Hence by MCT and continuity from below,

   $$\int_Y \mu(E^y) d\nu = \lim_{n\to\infty} \int_Y \mu(E_n^y) d\nu = \lim_{n\to\infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E).$$

   Therefore, $E \in \mathcal{C}$.

   $\bigcap$: Let decreasing $\{E_n\} \subset \mathcal{C}$ with $E = \bigcap E_n$. Then $f_n(y) = \mu(E_n^y) \le f_1(y) < \infty$ by monotonicity and finiteness of μ and ν. Hence by DCT and continuity from above,

   $$\int_Y \mu(E^y) d\nu = \lim_{n\to\infty} \int_Y \mu(E_n^y) d\nu = \lim_{n\to\infty} (\mu \times \nu)(E_n) = (\mu \times \nu)(E).$$

   Therefore, $E \in \mathcal{C}$.

   Since $\mathcal{C}$ is a monotone class that contains $\mathcal{A}$, it contains the σ-algebra generated by $\mathcal{A}$ (See text for proof). Hence $\mathcal{C} = \mathcal{M} \otimes \mathcal{N}$.

4. If μ and ν are σ-finite, then consider $X \times Y = \bigcup_{j=1}^{\infty} A_j \times B_j$ as the union of rectangles with finite measure. Then apply the above to each rectangle. Details omitted.

□

Theorem 5.2 (Tonelli)**.** *Let $f \in L^+(\mu \times \nu)$ with σ-finite μ and ν. Then $x \mapsto \int_Y f_x \, d\nu \in L^+(\mu)$ and $y \mapsto \int_X f^y \, d\mu \in L^+(\nu)$, and*

$$\int_X \int_Y f \, d\nu d\mu = \int_Y \int_X f \, d\mu d\nu = \int_{X \times Y} f \, d(\mu \times \nu).$$

Proof.

- If $f = \chi_E$, use the slicing theorem.
- If $f \in L^+$ is simple, follows from above by additivity.
- If $f \in L^+$, let $0 \le f_n^{\text{simple}} \uparrow f$. Let $g_n(x) = \int_Y (f_n)_x \, d\nu$ and $g(x) = \int_Y f_x \, d\nu$. Then $g_n \uparrow g$ and $\int g_n \to \int g$ by MCT, and we get the result

  $$\int_{X \times Y} f \, d(\mu \times \nu) = \lim_{n\to\infty} \int_{X \times Y} f_n \, d(\mu \times \nu) = \lim_{n\to\infty} \int_X g_n \, d\mu = \int_X g \, d\mu.$$

□

Theorem 5.3 (Fubini)**.** *Let $f \in L^1(\mu \times \nu)$. Then $f_x \in L^1(\nu)$ a.e.x, $f^y \in L^1(\mu)$ a.e.y, $x \mapsto \int_Y f_x \, d\nu \in L^1(\mu)$, and $y \mapsto \int_X f^y \, d\mu \in L^1(\nu)$.*
... TLDR: We can also change the order of iterated integrals.

Proof. Let $f : X \times Y \to \mathbb{C}$. Write $f = (\Re(f)^+ - \Re(f)^-) + i(\Im(f)^+ - \Im(f)^-)$ and apply Tonelli. □

5.1 Lebesgue Measure on $\mathbb{R}^n$

All the nice properties for $\mathbb{R}$ also happen in $\mathbb{R}^n$ (+ invariance of rotation).

6 Differentiation of Measures

Example 6.1. Let $(X, \mathcal{M}, \mu)$ be a measurable space, and $f \in L^+$. Let $\nu : \mathcal{M} \to [0, \infty]$ such that $\nu : E \mapsto \int_E f \; d\mu$. Observe that ν is a measure, and we write $d\nu = f \; d\mu$.

Definition 6.1. Define an extended μ-integrable function as follows. Let $g : X \to [-\infty, \infty]$ be measurable. We can write $g = g^+ - g^-$ with $g^+, g^- \in L+$ such that we have $\int_X g^+ \; d\mu < \infty$ or $\int_X g^- \; d\mu < \infty$.

Observe that if g is extended μ-integrable, and we have a function $\nu : E \mapsto \int_E g \; d\mu$, then at most one of ∞ and $-\infty$ can be in the range of ν.

Definition 6.2. A signed measure on $(X, \mathcal{M})$ is a function $\nu : \mathcal{M} \to [-\infty, \infty]$ such that

- $\nu(\emptyset) = 0$
- ν assumes at most one of ∞ and $-\infty$
- If $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ are disjoint, then $\nu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \nu(E_j)$

Definition 6.3. The set $E \in \mathcal{M}$ is positive, negative, or null for ν if respectively $\nu(F) \geq 0$, $\nu(F) \leq 0$, or $\nu(F) = 0$ for all measurable $F \subset E$.

Example 6.2. Let $\nu : E \mapsto \int_E g \; d\mu$. Then E is positive for ν if and only if $g \geq 0$ μ almost everywhere (μ-a.e.) on E.

Theorem 6.1 (Hahn Decomposition)**.** *Let ν be a signed measure on $(X, \mathcal{M})$. There exists sets P positive for ν and N negative for ν such that $X = P \cup N$ with $P \cap N = \emptyset$.*
If there exists another pair of such sets P' and N', then $P \triangle P'$ and $N \triangle N$ are null for ν.

Proof. Without loss of generality, assume that $\nu : \mathcal{M} \to [-\infty, \infty)$. Otherwise, we can take $-\nu$. Let $m = \sup\{\nu(E) : E \text{ positive for } \nu\}$, so we have positive sets P_j such that $\lim_{j\to\infty} \nu(P_j) = m$. Let $P = \bigcup_{j=1}^{\infty} P_j$. Observe that P is also positive for ν. It follows from continuity that $m = \nu(P) < \infty$.
Exercise: show that continuity from above/below holds for signed measures.
Let $N = X \setminus P$. We show by contradiction that N does not contain any positive subsets. Suppose that N contains some positive E with $\nu(E) > 0$, then $\nu(P \cup E) = \nu(P) + \nu(E) > m$, which contradicts the maximality of m.
Suppose that N contains some set A with $\nu(A) > 0$. Since A is not positive, there exists some $C \subset A$ with $\nu(C) < 0$. Then if $B = A \setminus C$, we have $B \subset A$ and $\nu(B) = \nu(A) - \nu(C) > \nu(A)$.
Now we show that we can construct some problematic set. We can define a sequence $\{A_j\}_{j=1}^{\infty}$ as follows.

- Choose $A_1 \subset N$ with $\nu(A_1) > \frac{1}{n_1}$ where n_1 is the smallest positive integer for the inequality to hold.
- For $j > 1$, choose $A_j \subset A_{j-1}$ with $\nu(A_j) > \nu(A_{j-1}) + \frac{1}{n_j}$ where n_j is the smallest positive integer for the inquality to hold.

Let $A = \bigcap_{j=1}^{\infty} A_j$. Note that $\nu(A)$ needs to be finite, so the sum $\sum_{j=1}^{\infty} \frac{1}{n_j}$ must be summable, which means $n_j \to \infty$. But, since $\nu(A) > 0$, there is some $B \subset A$ with $\nu(B) > \nu(A)$, that is $\nu(B) > \nu(A) + \frac{1}{n}$ for some sufficiently large n. So we contradict the minimality of n_j when $n_j > n$. This means that N is negative, so we are done.
Now we consider the uniqueness of P and N. Let $X = P' \cup N'$ where P' is positive and N' is negative, and these two sets are disjoint. Then the symmetric difference is

$$P \triangle P' = (P \cap N') \cup (N \cap P').$$

These two intersections are both ν-null, so we are done. □

Definition 6.4. Signed measures μ and ν on $(X, \mathcal{M})$ are mutually singular if there exists some disjoint $A, B \in \mathcal{M}$ such that $X = A \cup B$ where B is μ-null and A is ν-null. We write $\mu \perp \nu$.

Example 6.3. Consider $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Then δ_{x_0} is mutually singular to the Lebesgue measure, but not the counting measure.

Theorem 6.2 (Jordan Decomposition)**.** *Let ν be a signed measure on $(X, \mathcal{M})$. Then there exist unique positive measures ν^+ and ν^- such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.*

Proof. Let $X = P \cup N$ be the Hahn decomposition. Let $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = -\nu(E \cap N)$. Observe the following.

- $(\nu^+ - \nu^-)(E) = \nu(E \cap P) + \nu(E \cap P^c) = \nu(E)$
- P is ν^--null and N is ν^+-null, so $\nu^+ \perp \nu^-$.

For uniqueness, suppose $\nu = \mu^+ - \mu^-$ with $\mu^+ \perp \mu^-$. Then $X = A \cup B$ with $\mu^+(B) = \mu^-(A) = 0$ for some disjoint $A, B \in \mathcal{M}$. Observe that this implies that $X = A \cup B$ is Hahn, so $A \triangle P$ is null. Then

$$\mu^+(E) = \mu^+(E \cap A) = \nu(E \cap A) = \nu(E \cap P) = \nu^+(E \cap P) = \nu^+(E),$$

so $\mu^+ = \nu^+$. We can do the same for $\mu^- = \nu^-$.
Exercise: show that $A \triangle P \implies \nu(E \cap A) = \nu(E \cap P)$. □

Definition 6.5. Let $|\nu| = \nu^+ + \nu^-$ be the total variation measure of ν.

Exercise: E ν-null $\iff |\nu|(E) = 0$.
Exercise: $\nu \perp \mu \iff |\nu| \perp \mu \iff \nu^+ \perp \mu$ and $\nu^- \perp \mu$.

6.1 The Radon-Nikodym Derivative

Let f be extended μ-integrable and $\nu : E \mapsto \int_E f \, d\mu$ where μ is a positive measure. We write $d\nu = f \, d\mu$.

Remark 6.1. Here are some remarks.

- $X = P \cup N = \{f > 0\} \cup \{f \leq 0\}$
- $f = f^+ - f^-$, so $\int_E f^+ \, d\mu - \int_E f^- \, d\mu = \nu^+ - \nu^-$
- $|f| = f^+ + f^-$, so $(\nu^+ + \nu^-)(E) = |\nu|(E) = \int_E (f^+ + f^-) \, d\mu = \int_E |f| \, d\mu$. Hence $d|\nu| = d|\mu|$
- $\nu \perp \mu$ only if $\nu = 0$. That is $f = 0$ μ-a.e.
- $\nu(E) = \int_E (\chi_P - \chi_N) \, d|\nu|$

- Integration with respect to signed ν: $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-) = L^1(|\nu|)$. Hence $\int f \, d\nu = \int f \, d\nu^+ - \int f \, d\nu^-$
- $|\nu(E)| \le |\nu|(E)$ (by the triangle inequality)

Definition 6.6. let ν be signed and $\mu \ge 0$ on $(X, \mathcal{M})$. We say ν is absolutely continuous with respect to μ if $\nu(E) = 0$ for all $E \in \mathcal{M}$ where $\mu(E) = 0$. We write $\nu \ll \mu$.

Exercise: $\nu \ll \mu \iff \nu^+ \ll \mu$ and $\nu^- \ll \mu$.
Exercise: $\nu \ll \mu$ and $\nu \perp \mu \implies \nu = 0$.

Remark 6.2. If $d\nu = f \, d\mu$, then $\nu \ll \mu$.

Theorem 6.3. *If ν is finite, that is $|\nu|(X) < \infty$, then $\nu \ll \mu$ if and only if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $E \in \mathcal{M}$ with $\mu(E) < \delta$ implies $|\nu(E)| < \epsilon$.*

Proof. If the latter holds, then the former must hold. (Eg. take a sequence of decreasing sets and use continuity from above).
Let $\nu \ll \mu$. Without loss of generality, assume that $\nu \ge 0$. Otherwise, $|\nu(E)| \le |\nu|(E)$, so we consider $|\nu|$ instead. We proceed by contradiction. There exists some $\epsilon > 0$ and for each j, we have $E_j \in \mathcal{M}$ such that $\mu(E_j) < 2^{-j}$ and $\nu(E_j) > \epsilon$. Set $F_k = \bigcup_{j=k}^{\infty} E_j$. Observe that $F_k \searrow F = \bigcap_{k=1}^{\infty} F_k$. Then

$$\mu(F_k) \le \sum_{j=k}^{\infty} \mu(E_j) \le \sum_{j=k}^{\infty} 2^{-j} = \frac{1}{2^{k-1}}.$$

Now by continuity from above, $\mu(F) = \lim_{k \to \infty} \mu(F_k) = 0$. Since $E_k \subset F_k$, we get

$$\nu(F) = \lim_{k \to \infty} \nu(F_k) \ge \lim_{k \to \infty} \nu(E_k) \ge \epsilon.$$

However, this contradicts $\nu \ll \mu$, so we are done. ∎

Definition 6.7. Let ν be signed and $\mu \ge 0$ on $(X, \mathcal{M})$. Then we can write $\nu = \lambda + \rho$ with $\lambda \perp \mu$ and $\rho \ll \mu$. This is the Lebesgue Decomposition.

Theorem 6.4 (Radon-Nikodym Theorem)**.** *Let ν be signed and $\mu \ge 0$ on $(X, \mathcal{M})$ where both are σ-finite. Then there exists a unique Lebesgue Decomposition $\nu = \lambda + \rho$, and there exists a unique extended μ-integrable $f : X \to \mathbb{R}$ such that $d\rho = f \, d\mu$ (up to μ-null sets). Then $\rho(E) = \int_E f \, d\mu$, and $f = \frac{d\rho}{d\mu}$ is the Radon-Nikodym derivative of ρ with respect to μ.*

Proof. We can assume $\nu \ge 0$, because we can take $\nu = \nu^+ - \nu^-$. We will assume that ν and μ are finite (see text for an extension to σ-finite).
Let $\mathcal{F} = \left\{ f \in L^+ : \int_E f \, d\mu \le \nu(E) \quad \forall E \in \mathcal{M} \right\}$. Observe that $0 \in \mathcal{F}$, so $\mathcal{F}$ is non-empty. Observe that if $f, g \in \mathcal{F}$, then

$$\int_E \max(f, g) \, d\mu = \int_{E \cap \{f > g\}} f \, d\mu + \int_{E \cap \{f \le g\}} g \, d\mu \le \nu(E \cap \{f > g\}) + \nu(E \cap \{f \le g\}) = \nu(E),$$

so $\max(f, g) \in \mathcal{F}$.

Let $a = \sup\left\{\int f \, d\mu : f \in \mathcal{F}\right\}$. Note that $a \le \nu(X) < \infty$ (by assumption of finiteness), and $a = \lim_{j\to\infty} \int f_j \, d\mu$ (by definition of supremum). Set $g_j = \max(f_1, f_2, ..., f_j) \in \mathcal{F}$. Now $\{g_j\}_{j=1}^{\infty}$ is monotone, so using MCT, we get

$$a = \lim_{j\to\infty} \int f_j \, d\mu \le \lim_{j\to\infty} \int g_j \, d\mu \le a,$$

so $\int f \, d\mu = \lim_{j\to\infty} \int g_j \, d\mu = a$. More MCT gives

$$\int_E f \, d\mu = \lim_{j\to\infty} \int_E g_j \, d\mu \le \nu(E),$$

so $f \in \mathcal{F}$.
Now set $d\lambda = d\nu - f \, d\mu$.

$\square$