

Math 539 Notes

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1 Introduction

Motivating questions (some statistics):

- the “probability” that a random number has some property
- the “distribution” of some given multiplicative/additive function

Idea: we can answer the question for $\{1, \dots, \lfloor x \rfloor\}$ for some parameter x . Then, take the limit $x \rightarrow \infty$ for all natural numbers.

1.1 Notation

Let $g(x) \geq 0$.

Definition 1.1.1. $O(g(x))$ means some unspecified function $u(x)$ such that $|u(x)| \leq cg(x)$ for some constant $c > 0$.

Example 1.1.2. Show that $e^{2x} - 1 = 2x + O(x^2)$ for $x = [-1, 1]$.

Proof. Observe that $f(z) = e^{2z} - 1 - 2z$ is analytic (and entire) and has a double zero at $z = 0$ (one can check that $f(z) = f'(z) = 0$). Hence, $g(z) = (e^{2z} - 1 - 2z)/z^2$ has a removable singularity at $z = 0$, whence g is analytic and entire. Let $C = \max\{|g(z)| : |z| \leq 1\}$. Then

$$|g(z)| \leq C \implies |e^{2z} - 1 - 2z| \leq C|z^2| \implies e^{2z} - 1 - 2z = O(|z|^2).$$

□

Exercise 1.1.3. Show that $\sqrt{x+1} = \sqrt{x} + O(1/\sqrt{x})$ for $x \in [1, \infty)$.

Definition 1.1.4. $f(x) \ll g(x)$ means $f(x) = O(g(x))$.

Exercise 1.1.5. Suppose that $f_1 \ll g_1, f_2 \ll g_2$, then $f_1 + f_2 \ll \max\{g_1, g_2\}$. ✓

Exercise 1.1.6. Let f, g be continuous on $[0, \infty)$, and $f \ll g$ on $[123, \infty)$. Show that $f \ll g$ on $[0, \infty)$. ✓

Definition 1.1.7. $f(x) \sim g(x)$ means $\lim \frac{f(x)}{g(x)} = 1$.

Definition 1.1.8. $f(x) = o(g(x))$ means $\lim \frac{f(x)}{g(x)} = 0$.

Definition 1.1.9. $f(x) = O_y(g(x))$ means f, g depend on some parameter y , and the implicit constant depends on y .

Exercise 1.1.10. For any $A, \epsilon > 0$, show that $(\log x)^A \ll_{A, \epsilon} x^\epsilon$.

1.2 Riemann-Stieltjes Integral

Appendix A in the book.

Definition 1.2.1. Some definitions for partitions

1. Let $\underline{x} = \{x_0, \dots, x_N\}$ be a partition of $[c, d]$ if $c = x_0 < \dots < x_N = d$.
2. The mesh size $m(\underline{x}) = \max_{1 \leq j \leq N} x_j - x_{j-1}$.
3. Sample points $\xi_j \in [x_{j-1}, x_j]$.

Definition 1.2.2 (Riemann-Stieltjes Integral). Given two functions $f(x)$ and $g(x)$, define the Riemann-Stieltjes integral as

$$\int_c^d f(x) dg(x) = \lim_{m(\underline{x}) \rightarrow 0} \sum_{j=1}^N f(\xi_j)(g(x_j) - g(x_{j-1})).$$

Remark 1.2.3. Setting $g(x) = x$ gives the Riemann integral.

Theorem 1.2.4. Let $f(x)$ have bounded variation and let $g(x)$ be continuous on $[c, d]$, or vice versa. Then $\int_c^d f(x) dg(x)$ exists.

Remark 1.2.5. If a function is piecewise monotone, then it has bounded variation.

Example 1.2.6. Given a sequence $a_{nn \in \mathbb{N}}$, define the summatory function $A(x) = \sum_{n \leq x} a_n$. Then, on any $[c, d]$, $A(x)$ is bounded, piecewise continuous and piecewise monotone. Hence, the Riemann-Stieltjes integral exists when g is continuous.

Remark 1.2.7. We present 3 facts that we will use.

1. If $A(x)$ is the summatory function as above, and $f(x)$ is continuous, then

$$\int_c^d f(x) dA(x) = \sum_{c < n \leq d} a_n f(n).$$

2. (Integration by parts). If the integrals exist, then

$$\int_c^d f(x) dg(x) = f(x)g(x)|_c^d - \int_c^d g(x) df(x).$$

3. If $f(x)$ is continuously differentiable, then

$$\int_c^d g(x) df(x) = \int_c^d g(x) f'(x) dx.$$

Example 1.2.8 (Summation by parts). Consider $\sum_{n \leq y} \frac{a_n}{n}$. Let $f(x) = 1/x$, then we can write

$$\sum_{n \leq y} \frac{a_n}{n} = \sum_{n \leq y} a_n \cdot \frac{1}{n} = \int_0^y \frac{1}{x} dA(x) = \frac{1}{x} A(x) \Big|_0^y - \int_0^y A(x) d\left(\frac{1}{x}\right) = \frac{A(y)}{y} - \int_0^y A(x) \frac{1}{x^2} dx.$$

The final manipulation that we want to get is

$$\sum_{n \leq y} a_n f(n) = A(y) f(y) - \int_0^y A(x) f'(x) dx. \quad (1)$$

2 Dirichlet Series

A Dirichlet series is $\sum_{n=1}^{\infty} n^{-s}$.

Facts about Dirichlet series:

- converge in some right half-plane $\{s \in \mathbb{C} : \Re s > R\}$ for some R (possibly $R = \pm\infty$).
- Sometimes converge conditionally. Example: $\sum_{n=1}^{\infty} (-1)^n / n^{1/2}$.
- $(\sum_{n=1}^{\infty} a_n n^{-s})(\sum_{n=1}^{\infty} b_n n^{-s}) = \sum_{n=1}^{\infty} c_n n^{-s}$ where $c = \sum_{de=n} a_d b_e = \sum_{d|n} a_d b_{e/d}$. (multiplicative convolution)

Some notation: for $s \in \mathbb{C}$, we write $s = \sigma + it$, that is σ is the real part of s , and t is the imaginary part of s . Note that if $x > 0$, then $|x^s| = |x^\sigma| |x^{it}| = |x^\sigma| |e^{it \log x}| = |x^\sigma|$.

Theorem 2.0.1. Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. Suppose that $s_0 \in \mathbb{C}$ is such that $\alpha(s_0)$ converges. Then $\alpha(s)$ converges uniformly in the sector $S = \{s \in \mathbb{C} : \sigma \geq \sigma_0, |t - t_0| \leq H |\sigma - \sigma_0|\}$ for any $H > 0$.

Proof. WLOG, let $s_0 = 0$, otherwise we can do a change of variables.

Let $A(x) = \sum_{n \leq x} a_n = \alpha(0) - R(x)$. Then, for $\sigma > 0$,

$$\begin{aligned} \sum_{M < n \leq N} a_n n^s &= \int_M^N x^{-s} dA(x) = \int_M^N x^{-s} d(\alpha(0) - R(x)) \\ &= \int_M^N x^{-s} d\alpha(0) - \int_M^N x^{-s} dR(x) = - \int_M^N x^{-s} dR(x) \\ &= -x^{-s} R(x) \Big|_M^N + \int_M^N R(x) d(x^{-s}) \\ &= R(M) M^{-s} - R(N) N^{-s} - s \int_M^N R(x) x^{-s-1} dx. \end{aligned}$$

Note that $R(N) N^{-s} \rightarrow 0$ as $N \rightarrow \infty$, and that $R(x) x^{-s-1} \ll x^{-\sigma-1}$. Hence, letting $N \rightarrow \infty$ gives

$$\sum_{M < n} a_n n^{-s} = R(M) M^{-s} - s \int_M^{\infty} R(x) x^{-s-1} dx \rightarrow 0 \text{ as } M \rightarrow \infty.$$

Now, choose M large such that $|R(x)| < \epsilon$ for all $x \geq M$. Then,

$$\begin{aligned} \left| \sum_{n>M} a_n n^{-s} \right| &\leq \epsilon M^{-\sigma} + |s| \int_M^\infty \epsilon x^{-\sigma-1} dx \\ &= \epsilon M^{-\sigma} + |s| \epsilon x^{-\sigma} \frac{1}{-\sigma} \Big|_M^\infty \\ &= \epsilon M^{-\sigma} + |s| \epsilon \frac{M^{-\sigma}}{\sigma} = \frac{\epsilon}{M^\sigma} \left(1 + \frac{|s|}{\sigma} \right). \end{aligned}$$

Since $s \in S$, we have

$$|s| = \sqrt{\sigma^2 + t^2} \leq \sqrt{\sigma^2 + |H\sigma|^2} = \sigma \sqrt{1 + H^2},$$

so $|\sum_{n>M} a_n n^{-s}| \leq \epsilon(1 + \sqrt{1 + H^2})$ as $M \rightarrow \infty$. Observe that the latter only depends on H , so the convergence is uniform. \square

Corollary 2.0.2. *If $\alpha(s_0)$ converges, then $\alpha(s)$ converges for all s with $\sigma > \sigma_0$.*

Corollary 2.0.3. *If $\alpha(s_0)$ diverges, then $\alpha(s)$ diverges for all s with $\sigma < \sigma_0$.*

Remark 2.0.4. The Dirichlet series $\alpha(s)$ has an abscissa of convergence σ_c such that $\alpha(s)$ converges if $\sigma > \sigma_c$, and diverges if $\sigma < \sigma_c$. It is allowed to have $\sigma_c = \pm\infty$. Furthermore, $\alpha(s)$ converges locally uniformly right of σ_c , whence $\alpha(s)$ is analytic.

Remark 2.0.5. Observe that $\int_1^N x^{-s} dA(x) = \sum_{1 < n \leq N} a_n n^{-s} = \sum_{n=2}^N a_n n^{-s}$. Sometimes we write \int_{-1}^N to include the 1.

Theorem 2.0.6. *Let $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$ have an abscissa of convergence $\sigma_c \geq 0$. Then for $\sigma > \sigma_c$, we have $\alpha(s) = s \int_1^\infty A(x) x^{-s-1} dx$. Moreover,*

$$\limsup_{x \rightarrow \infty} \frac{\log |A(x)|}{\log x} = \sigma_c.$$

Proof. Observe that

$$\begin{aligned} \sum_{n=1}^N a_n n^{-s} &= \int_{1^-}^N x^{-s} dA(x) = x^{-s} A(x) \Big|_{1^-}^N - \int_{1^-}^N A(x) d(x^{-s}) \\ &= A(N)N^{-s} - \int_{1^-}^N A(x)(-sx^{-s-1} dx) = A(N)N^{-s} + s \int_1^N A(x)x^{-s-1} dx. \end{aligned}$$

Observe that in the last line, we can replace 1^- with 1 because the integrand is bounded.

Define $\phi = \limsup_{x \rightarrow \infty} \frac{\log |A(x)|}{\log x}$. We compare this to σ_c .

Let $\sigma = \phi + \epsilon$ for some $\epsilon > 0$. Then $\frac{\log |A(x)|}{\log x} < \phi + \frac{\epsilon}{2}$ for large x , so $A(x) \ll x^{\phi+\epsilon/2}$. Then, $A(N)N^{-s} \ll N^{\phi+\epsilon/2} N^{-(\phi+\epsilon)} = N^{-\epsilon/2}$. Hence,

$$\int_N^\infty A(x)x^{-\sigma-1} dx \ll \int_N^\infty x^{-\phi+\epsilon/2} x^{-(\phi+\epsilon+1)} dx = \int_N^\infty x^{-1-\epsilon/2} dx \ll N^{-\epsilon/2}.$$

It follows that

$$\sum_{n=1}^N a_n n^{-s} = O(N^{-\epsilon/2}) + s \left(\int_1^\infty A(x)x^{-s} - 1 dx + O(N^{-\epsilon/2}) \right).$$

Let $N \rightarrow \infty$ gives $s \int_1^\infty A(x)x^{-s-1} dx$ converges. Hence $\sigma_c \leq \phi$.

Conversely, let $\sigma_0 = \sigma_c + \varepsilon$, and let $R_0(x) = \sum_{n>x} a_n n^{-\sigma_0} = \alpha(\sigma_0) - \sum_{n \leq x} a_n n^{-\sigma_0}$. Observe that

$$A(N) = -R_0(N)N^{\sigma_0} + \sigma_0 \int_0^N R_0(x)x^{\sigma_0-1} dx.$$

Since $\alpha(0)$ converges, $R_0(x) = o(1)$ so $R_0(x) \ll 1$. Then

$$A(N) \ll 1 \cdot N^{\sigma_0} + \sigma_0 \int_0^N 1 \cdot x^{\sigma_0-1} dx = N^{\sigma_0} + N^{\sigma_c} \ll N^{\sigma_0} = N^{\sigma_c+\varepsilon}.$$

Hence $\frac{\log|A(x)|}{\log x} \ll \frac{(\sigma_c+\varepsilon)\log x}{\log x} = \sigma_c + \varepsilon$, so $\phi \leq \sigma_c + \varepsilon$. Take $\varepsilon \rightarrow 0$, so $\phi \leq \sigma_c$. \square

Definition 2.0.7. The abscissa of absolute convergence is $\sigma_a = \inf \{ \sigma \in \mathbb{R} : \sum_{n=1}^\infty |a_n| n^{-\sigma} \text{ converges} \}$.

Example 2.0.8. Let $\eta(s) = \sum_{n=1}^\infty (-1)^{n-1} n^{-s}$. Observe that $\sigma_c = 0$ by the alternating series test. However, we only have absolute convergence when $\sigma > 1$, so the abscissa of absolute convergence is $\sigma_a = 1$.

Remark 2.0.9. When $a_n \geq 0$ for all $n \in \mathbb{N}$, we have $\sigma_c = \sigma_a$.

Theorem 2.0.10. For any Dirichlet series $\alpha(s)$, we have $\sigma_c \leq \sigma_a \leq \sigma_c + 1$.

Proof. The first inequality is trivial.

Let $\sigma = \sigma_c + 1 + \varepsilon$ where $\varepsilon > 0$. We show that $\alpha(\sigma)$ converges absolutely. Note that $\alpha(s)$ converges at $s = \sigma_c + \varepsilon/2$, that is

$$a_n n^{-(\sigma_c+\varepsilon/2)} = o(1) \implies a_n n^{-(\sigma_c+\varepsilon/2)} \ll 1.$$

Then,

$$\sum_{n=1}^\infty |a_n| n^{-(\sigma_c+1+\varepsilon)} = \sum_{n=1}^\infty \left| a_n n^{-(\sigma_c+\varepsilon/2)} \right| n^{-(1+\varepsilon/2)} \ll \sum_{n=1}^\infty n^{-(1+\varepsilon/2)} \ll 1.$$

It follows that $\alpha(\sigma)$ converges absolutely for all $\varepsilon > 0$. \square

Theorem 2.0.11 (Landau's Theorem). Let $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$ with $\sigma_c < \infty$. If $a_n \geq 0$ for each $n \in \mathbb{N}$. Then, $\alpha(s)$ has a singularity at $s = \sigma_c$.

Proof. Suppose that there does not exist a singularity at $s = \sigma_c$. Then, there exists an analytic continuation of α to $C = \{s \in \mathbb{C} : |s - \sigma_c| < \delta\}$.

Let $z = \sigma_c - \frac{1}{4}\delta$ and let $w = \sigma_c + \frac{3}{4}\delta$. Let $D = \{s \in \mathbb{C} : |s - w| < \frac{5}{4}\delta\}$. Observe that $D \subset C \cup \{s \in \mathbb{C} : \sigma > 0\}$, so α has an analytic continuation to D . Let $P(s)$ be the power series of α centered at w . Observe that $z \in D$, so it suffices to show that $P(z) = \alpha(z)$, whence we contradict

the assumption that the abscissa of convergence is σ_c . Note that

$$\begin{aligned}
P(z) &= \sum_{k=0}^{\infty} \frac{\alpha^{(k)}(w)}{k!} (z-w)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} (z-w)^k \sum_{n=1}^{\infty} a_n (-\log n)^k n^{-w} && \text{we can differentiate termwise for } \alpha^{(k)}(w) \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} (w-z)^k \sum_{n=1}^{\infty} a_n (\log n)^k n^{-w} && \text{where the terms are all nonnegative} \\
&= \sum_{n=1}^{\infty} a_n n^{-w} \sum_{k=1}^{\infty} \frac{1}{k!} (w-z)^k (\log n)^k \\
&= \sum_{n=1}^{\infty} a_n n^{-w} e^{(w-z) \log n} = \sum_{n=1}^{\infty} a_n n^{-z}.
\end{aligned}$$

It follows that $\alpha(z)$ converges left of σ_c , which is a contradiction. \square

2.1 Dirichlet convolutions

Motivating question: are these calculations legitimate?

- $\zeta(s)^2 = \sum_{l,m=1}^{\infty} (lm)^{-s} = \sum_{n=1}^{\infty} d(n) n^{-s}$.
- $\zeta(s) = \prod_p (1 + p^{-s} + p^{-2s} + \dots) = \prod_p (1 - p^{-s})^{-1}$.

Definition 2.1.1. Let $a = \{a_n\}$, $b = \{b_n\}$ be sequences. The Dirichlet/multiplicative convolution $a * b$ by $c = \{c_n\}$ where $c_n = \sum_{de=n} a_d b_e = \sum_{d|n} a_d b_{n/d}$.

Theorem 2.1.2. Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, let $\beta(s) = \sum_{n=1}^{\infty} b_n n^{-s}$, and let $\gamma(s) = \sum_{n=1}^{\infty} c_n n^{-s}$. If $s \in \mathbb{C}$ is such that $\alpha(s)$ and $\beta(s)$ converge absolutely, and if $c = a * b$, then $\gamma(s)$ converges absolutely and $\gamma(s) = \alpha(s)\beta(s)$.

Example 2.1.3. Observe that $d(n) = (1 * 1)(n)$.

Example 2.1.4. Let $M(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s}$, where μ is the Möbius function which defined as

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not square free} \\ 1 & \text{if } n \text{ has an even number of prime divisors} \\ -1 & \text{if } n \text{ has an odd number of prime divisors} \end{cases}$$

Equivalently, we can define μ as the function that satisfies

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

Observe that $M(s)\zeta(s) = \sum_{n=1}^{\infty} (\mu * 1)(n) n^{-s} = 1$ for $\sigma > 1$, since $(\mu * 1)(n) = \sum_{d|n} \mu(d)$. It follows that $M(s) = 1/\zeta(s)$.

Since the abscissa of convergence of M is $\sigma_c = 1$, we get that $\zeta(s)$ has no zeroes when $\sigma > 1$.

Example 2.1.5 (Möbius Inversion Formula). Write $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ and $G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$. Then

$$\begin{aligned} F(s)\zeta(s) = G(s) &\iff F(s) = \frac{G(s)}{\zeta(s)} = G(s)M(s) \\ (f * 1)(n) = g(n) &\iff f(n) = (g * \mu)(n) \\ \sum_{d|n} f(d) = g(n) &\iff f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right). \end{aligned}$$

Example 2.1.6. It is known that $\sum_{d|n} \phi(d) = n$. This gives $(\phi * 1)(n) = \sum_{d|n} \phi(d) = n$. Then, for $\sigma > 2$, we have

$$\left(\sum_{n=1}^{\infty} \phi(n)n^{-s}\right) \left(\sum_{n=1}^{\infty} n^{-s}\right) = \sum_{n=1}^{\infty} n \cdot n^{-s} = \zeta(s-1).$$

This gives $\sum_{n=1}^{\infty} \phi(n)n^{-s} = \zeta(s-1)/\zeta(s)$.

Exercise 2.1.7. Let $\sigma_1(n) = \sum_{d|n} d$. Show that $\sum_{n=1}^{\infty} \sigma_1(n)n^{-s} = \zeta(s-1)\zeta(s)$.

Definition 2.1.8. A function f is multiplicative if $f(m)f(n) = f(mn)$ if $\gcd(m, n) = 1$.

Definition 2.1.9. A number n is y -friable if $p \mid n \implies p \leq y$.

Theorem 2.1.10. Let f be a multiplicative function, and let $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$. If $\sum_{n=1}^{\infty} |f(n)|n^{-\sigma}$ converges, we have the Euler product

$$F(s) = \prod_{p \text{ prime}} (1 + f(p)p^{-s} + f(p^2)p^{-2s} + \dots).$$

Proof. Let $\sigma > \sigma_a$. Then, for all p , we have

$$|1 + f(p)p^{-s} + f(p^2)p^{-2s} + \dots| \leq 1 + |f(p)|p^{-s} + |f(p)|p^{-2s} + \dots \leq \sum_{n=1}^{\infty} |f(n)|n^{-s}.$$

Since the above converges, we can rearrange the finite product

$$\prod_{p \leq y} (1 + f(p)p^{-s} + f(p^2)p^{-2s} + \dots) = \sum_{n \text{ } y\text{-friable}} f(n)n^{-s}.$$

Now, we can compute

$$\begin{aligned} \left| F(s) - \prod_{p \leq y} (1 + f(p)p^{-s} + f(p^2)p^{-2s} + \dots) \right| &= \left| F(s) - \sum_{n \text{ } y\text{-friable}} f(n)n^{-s} \right| \\ &= \left| \sum_{n \text{ not } y\text{-friable}} f(n)n^{-s} \right| \\ &\leq \sum_{n > y} |f(n)|n^{-s} = o(1). \end{aligned}$$

The tail goes to 0, so the theorem is proved. \square

Remark 2.1.11. Almost the same proof shows that the Euler product converges absolutely. In particular, it is nonzero (unless an individual factor is zero). Note that the convergence of a product is defined as the convergence of the sum of logs.

Example 2.1.12. Note that μ is multiplicative, so $M(s) = \prod_{p \text{ prime}} (1 - p^{-s})$.