

# Math 539 Notes

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## 1 Introduction

Motivating questions (some statistics):

- the “probability” that a random number has some property
- the “distribution” of some given multiplicative/additive function

Idea: we can answer the question for  $\{1, \dots, \lfloor x \rfloor\}$  for some parameter  $x$ . Then, take the limit  $x \rightarrow \infty$  for all natural numbers.

### 1.1 Notation

Let  $g(x) \geq 0$ .

**Definition 1.1.1.**  $O(g(x))$  means some unspecified function  $u(x)$  such that  $|u(x)| \leq cg(x)$  for some constant  $c > 0$ .

**Example 1.1.2.** Show that  $e^{2x} - 1 = 2x + O(x^2)$  for  $x = [-1, 1]$ .

*Proof.* Observe that  $f(z) = e^{2z} - 1 - 2z$  is analytic (and entire) and has a double zero at  $z = 0$  (one can check that  $f(z) = f'(z) = 0$ ). Hence,  $g(z) = (e^{2z} - 1 - 2z)/z^2$  has a removable singularity at  $z = 0$ , whence  $g$  is analytic and entire. Let  $C = \max\{|g(z)| : |z| \leq 1\}$ . Then

$$|g(z)| \leq C \implies |e^{2z} - 1 - 2z| \leq C|z^2| \implies e^{2z} - 1 - 2z = O(|z|^2).$$

□

**Exercise 1.1.3.** Show that  $\sqrt{x+1} = \sqrt{x} + O(1/\sqrt{x})$  for  $x \in [1, \infty)$ .

**Definition 1.1.4.**  $f(x) \ll g(x)$  means  $f(x) = O(g(x))$ .

**Exercise 1.1.5.** Suppose that  $f_1 \ll g_1, f_2 \ll g_2$ , then  $f_1 + f_2 \ll \max\{g_1, g_2\}$ . ✓

**Exercise 1.1.6.** Let  $f, g$  be continuous on  $[0, \infty)$ , and  $f \ll g$  on  $[123, \infty)$ . Show that  $f \ll g$  on  $[0, \infty)$ . ✓

**Definition 1.1.7.**  $f(x) \sim g(x)$  means  $\lim \frac{f(x)}{g(x)} = 1$ .

**Definition 1.1.8.**  $f(x) = o(g(x))$  means  $\lim \frac{f(x)}{g(x)} = 0$ .

**Definition 1.1.9.**  $f(x) = O_y(g(x))$  means  $f, g$  depend on some parameter  $y$ , and the implicit constant depends on  $y$ .

**Exercise 1.1.10.** For any  $A, \epsilon > 0$ , show that  $(\log x)^A \ll_{A, \epsilon} x^\epsilon$ .

## 1.2 Riemann-Stieltjes Integral

Appendix A in the book.

**Definition 1.2.1.** Some definitions for partitions

1. Let  $\underline{x} = \{x_0, \dots, x_N\}$  be a partition of  $[c, d]$  if  $c = x_0 < \dots < x_N = d$ .
2. The mesh size  $m(\underline{x}) = \max_{1 \leq j \leq N} x_j - x_{j-1}$ .
3. Sample points  $\xi_j \in [x_{j-1}, x_j]$ .

**Definition 1.2.2** (Riemann-Stieltjes Integral). Given two functions  $f(x)$  and  $g(x)$ , define the Riemann-Stieltjes integral as

$$\int_c^d f(x) dg(x) = \lim_{m(\underline{x}) \rightarrow 0} \sum_{j=1}^N f(\xi_j)(g(x_j) - g(x_{j-1})).$$

**Remark 1.2.3.** Setting  $g(x) = x$  gives the Riemann integral.

**Theorem 1.2.4.** Let  $f(x)$  have bounded variation and let  $g(x)$  be continuous on  $[c, d]$ , or vice versa. Then  $\int_c^d f(x) dg(x)$  exists.

**Remark 1.2.5.** If a function is piecewise monotone, then it has bounded variation.

**Example 1.2.6.** Given a sequence  $a_{nn \in \mathbb{N}}$ , define the summatory function  $A(x) = \sum_{n \leq x} a_n$ . Then, on any  $[c, d]$ ,  $A(x)$  is bounded, piecewise continuous and piecewise monotone. Hence, the Riemann-Stieltjes integral exists when  $g$  is continuous.

**Remark 1.2.7.** We present 3 facts that we will use.

1. If  $A(x)$  is the summatory function as above, and  $f(x)$  is continuous, then

$$\int_c^d f(x) dA(x) = \sum_{c < n \leq d} a_n f(n).$$

2. (Integration by parts). If the integrals exist, then

$$\int_c^d f(x) dg(x) = f(x)g(x)|_c^d - \int_c^d g(x) df(x).$$

3. If  $f(x)$  is continuously differentiable, then

$$\int_c^d g(x) df(x) = \int_c^d g(x) f'(x) dx.$$

**Example 1.2.8** (Summation by parts). Consider  $\sum_{n \leq y} \frac{a_n}{n}$ . Let  $f(x) = 1/x$ , then we can write

$$\sum_{n \leq y} \frac{a_n}{n} = \sum_{n \leq y} a_n \cdot \frac{1}{n} = \int_0^y \frac{1}{x} dA(x) = \frac{1}{x} A(x) \Big|_0^y - \int_0^y A(x) d\left(\frac{1}{x}\right) = \frac{A(y)}{y} - \int_0^y A(x) \frac{1}{x^2} dx.$$

The final manipulation that we want to get is

$$\sum_{n \leq y} a_n f(n) = A(y) f(y) - \int_0^y A(x) f'(x) dx. \quad (1)$$

## 2 Dirichlet Series

A Dirichlet series is  $\sum_{n=1}^{\infty} a_n n^{-s}$ .

Facts about Dirichlet series:

- converge in some right half-plane  $\{s \in \mathbb{C} : \Re s > R\}$  for some  $R$  (possibly  $R = \pm\infty$ ).
- Sometimes converge conditionally. Example:  $\sum_{n=1}^{\infty} (-1)^n / n^{1/2}$ .
- $(\sum_{n=1}^{\infty} a_n n^{-s})(\sum_{n=1}^{\infty} b_n n^{-s}) = \sum_{n=1}^{\infty} c_n n^{-s}$  where  $c = \sum_{de=n} a_d b_e = \sum_{d|n} a_d b_{e/d}$ . (multiplicative convolution)

Some notation: for  $s \in \mathbb{C}$ , we write  $s = \sigma + it$ , that is  $\sigma$  is the real part of  $s$ , and  $t$  is the imaginary part of  $s$ . Note that if  $x > 0$ , then  $|x^s| = |x^\sigma| |x^{it}| = |x^\sigma| |e^{it \log x}| = |x^\sigma|$ .

**Theorem 2.0.1** (thm 1.1). *Let  $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$  be a Dirichlet series. Suppose that  $s_0 \in \mathbb{C}$  is such that  $\alpha(s_0)$  converges. Then  $\alpha(s)$  converges uniformly in the sector*

$$S = \{s \in \mathbb{C} : \sigma \geq \sigma_0, |t - t_0| \leq H |\sigma - \sigma_0|\}$$

for any  $H > 0$ .

*Proof.* WLOG, let  $s_0 = 0$ , otherwise we can do a change of variables.

Let  $A(x) = \sum_{n \leq x} a_n = \alpha(0) - R(x)$ . Then, for  $\sigma > 0$ ,

$$\begin{aligned} \sum_{M < n \leq N} a_n n^s &= \int_M^N x^{-s} dA(x) = \int_M^N x^{-s} d(\alpha(0) - R(x)) \\ &= \int_M^N x^{-s} d\alpha(0) - \int_M^N x^{-s} dR(x) = - \int_M^N x^{-s} dR(x) \\ &= -x^{-s} R(x) \Big|_M^N + \int_M^N R(x) d(x^{-s}) \\ &= R(M) M^{-s} - R(N) N^{-s} - s \int_M^N R(x) x^{-s-1} dx. \end{aligned}$$

Note that  $R(N)N^{-s} \rightarrow 0$  as  $N \rightarrow \infty$ , and that  $R(x)x^{-s-1} \ll x^{-\sigma-1}$ . Hence, letting  $N \rightarrow \infty$  gives

$$\sum_{M < n} a_n n^{-s} = R(M)M^{-s} - s \int_M^\infty R(x)x^{-s-1} dx \rightarrow 0 \text{ as } M \rightarrow \infty.$$

Now, choose  $M$  large such that  $|R(x)| < \epsilon$  for all  $x \geq M$ . Then,

$$\begin{aligned} \left| \sum_{n > M} a_n n^{-s} \right| &\leq \epsilon M^{-\sigma} + |s| \int_M^\infty \epsilon x^{-\sigma-1} dx \\ &= \epsilon M^{-\sigma} + |s| \epsilon x^{-\sigma} \frac{1}{-\sigma} \Big|_M^\infty \\ &= \epsilon M^{-\sigma} + |s| \epsilon \frac{M^{-\sigma}}{\sigma} = \frac{\epsilon}{M^\sigma} \left( 1 + \frac{|s|}{\sigma} \right). \end{aligned}$$

Since  $s \in S$ , we have

$$|s| = \sqrt{\sigma^2 + t^2} \leq \sqrt{\sigma^2 + |H\sigma|^2} = \sigma \sqrt{1 + H^2},$$

so  $\left| \sum_{n > M} a_n n^{-s} \right| \leq \epsilon(1 + \sqrt{1 + H^2})$  as  $M \rightarrow \infty$ . Observe that the latter only depends on  $H$ , so the convergence is uniform.  $\square$

**Corollary 2.0.2.** *If  $\alpha(s_0)$  converges, then  $\alpha(s)$  converges for all  $s$  with  $\sigma > \sigma_0$ .*

**Corollary 2.0.3.** *If  $\alpha(s_0)$  diverges, then  $\alpha(s)$  diverges for all  $s$  with  $\sigma < \sigma_0$ .*

**Remark 2.0.4.** The Dirichlet series  $\alpha(s)$  has an abscissa of convergence  $\sigma_c$  such that  $\alpha(s)$  converges if  $\sigma > \sigma_c$ , and diverges if  $\sigma < \sigma_c$ . It is allowed to have  $\sigma_c = \pm\infty$ . Furthermore,  $\alpha(s)$  converges locally uniformly right of  $\sigma_c$  and each  $a_n n^{-s}$  is analytic, whence  $\alpha(s)$  is analytic. (Conway; Theorem VII.2.1; p.147)

**Remark 2.0.5.** Observe that  $\int_1^N x^{-s} dA(x) = \sum_{1 < n \leq N} a_n n^{-s} = \sum_{n=2}^N a_n n^{-s}$ . Sometimes we write  $\int_{-1}^N$  to include the 1.

**Theorem 2.0.6** (thm 1.3). *Let  $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$  have an abscissa of convergence  $\sigma_c \geq 0$ . Then for  $\sigma > \sigma_c$ , we have  $\alpha(s) = s \int_1^\infty A(x)x^{-s-1} dx$ . Moreover,*

$$\limsup_{x \rightarrow \infty} \frac{\log |A(x)|}{\log x} = \sigma_c.$$

*Proof.* Observe that

$$\begin{aligned} \sum_{n=1}^N a_n n^{-s} &= \int_{1^-}^N x^{-s} dA(x) = x^{-s} A(x) \Big|_{1^-}^N - \int_{1^-}^N A(x) d(x^{-s}) \\ &= A(N)N^{-s} - \int_{1^-}^N A(x)(-sx^{-s-1} dx) = A(N)N^{-s} + s \int_1^N A(x)x^{-s-1} dx. \end{aligned}$$

Observe that in the last line, we can replace  $1^-$  with 1 because the integrand is bounded.

Define  $\phi = \limsup_{x \rightarrow \infty} \frac{\log |A(x)|}{\log x}$ . We compare this to  $\sigma_c$ .

Let  $\sigma = \phi + \epsilon$  for some  $\epsilon > 0$ . Then  $\frac{\log |A(x)|}{\log x} < \phi + \frac{\epsilon}{2}$  for large  $x$ , so  $A(x) \ll x^{\phi+\epsilon/2}$ . Then,  $A(N)N^{-s} \ll N^{\phi+\epsilon/2} N^{-(\phi+\epsilon)} = N^{-\epsilon/2}$ . Hence,

$$\int_N^\infty A(x)x^{-\sigma-1} dx \ll \int_N^\infty x^{-\phi+\epsilon/2} x^{-(\phi+\epsilon+1)} dx = \int_N^\infty x^{-1-\epsilon/2} dx \ll N^{-\epsilon/2}.$$

It follows that

$$\sum_{n=1}^N a_n n^{-s} = O(N^{-\varepsilon/2}) + s \left( \int_1^\infty A(x) x^{-s-1} dx + O(N^{-\varepsilon/2}) \right).$$

Let  $N \rightarrow \infty$  gives  $s \int_1^\infty A(x) x^{-s-1} dx$  converges. Hence  $\sigma_c \leq \phi$ .

Conversely, let  $\sigma_0 = \sigma_c + \varepsilon$ , and let  $R_0(x) = \sum_{n>x} a_n n^{-\sigma_0} = \alpha(\sigma_0) - \sum_{n \leq x} a_n n^{-\sigma_0}$ . Observe that

$$A(N) = -R_0(N)N^{\sigma_0} + \sigma_0 \int_0^N R_0(x) x^{\sigma_0-1} dx.$$

Since  $\alpha(0)$  converges,  $R_0(x) = o(1)$  so  $R_0(x) \ll 1$ . Then

$$A(N) \ll 1 \cdot N^{\sigma_0} + \sigma_0 \int_0^N 1 \cdot x^{\sigma_0-1} dx = N^{\sigma_0} + N^{\sigma_c} \ll N^{\sigma_0} = N^{\sigma_c+\varepsilon}.$$

Hence  $\frac{\log|A(x)|}{\log x} \ll \frac{(\sigma_c+\varepsilon)\log x}{\log x} = \sigma_c + \varepsilon$ , so  $\phi \leq \sigma_c + \varepsilon$ . Take  $\varepsilon \rightarrow 0$ , so  $\phi \leq \sigma_c$ .  $\square$

**Definition 2.0.7.** The abscissa of absolute convergence is  $\sigma_a = \inf \{ \sigma \in \mathbb{R} : \sum_{n=1}^\infty |a_n| n^{-\sigma} \text{ converges} \}$ .

**Example 2.0.8.** Let  $\eta(s) = \sum_{n=1}^\infty (-1)^{n-1} n^{-s}$ . Observe that  $\sigma_c = 0$  by the alternating series test. However, we only have absolute convergence when  $\sigma > 1$ , so the abscissa of absolute convergence is  $\sigma_a = 1$ .

**Remark 2.0.9.** When  $a_n \geq 0$  for all  $n \in \mathbb{N}$ , we have  $\sigma_c = \sigma_a$ .

**Theorem 2.0.10** (thm 1.4). *For any Dirichlet series  $\alpha(s)$ , we have  $\sigma_c \leq \sigma_a \leq \sigma_c + 1$ .*

*Proof.* The first inequality is trivial.

Let  $\sigma = \sigma_c + 1 + \varepsilon$  where  $\varepsilon > 0$ . We show that  $\alpha(\sigma)$  converges absolutely. Note that  $\alpha(s)$  converges at  $s = \sigma_c + \varepsilon/2$ , that is

$$a_n n^{-(\sigma_c+\varepsilon/2)} = o(1) \implies a_n n^{-(\sigma_c+\varepsilon/2)} \ll 1.$$

Then,

$$\sum_{n=1}^\infty |a_n| n^{-(\sigma_c+1+\varepsilon)} = \sum_{n=1}^\infty \left| a_n n^{-(\sigma_c+\varepsilon/2)} \right| n^{-(1+\varepsilon/2)} \ll \sum_{n=1}^\infty n^{-(1+\varepsilon/2)} \ll 1.$$

It follows that  $\alpha(\sigma)$  converges absolutely for all  $\varepsilon > 0$ .  $\square$

**Theorem 2.0.11** (Landau's Theorem (thm 1.7)). *Let  $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$  with  $\sigma_c < \infty$ . If  $a_n \geq 0$  for each  $n \in \mathbb{N}$ . Then,  $\alpha(s)$  has a singularity at  $s = \sigma_c$ .*

*Proof.* Suppose that there does not exist a singularity at  $s = \sigma_c$ . Then, there exists an analytic continuation of  $\alpha$  to  $C = \{s \in \mathbb{C} : |s - \sigma_c| < \delta\}$ .

Let  $z = \sigma_c - \frac{1}{4}\delta$  and let  $w = \sigma_c + \frac{3}{4}\delta$ . Let  $D = \{s \in \mathbb{C} : |s - w| < \frac{5}{4}\delta\}$ . Observe that  $D \subset C \cup \{s \in \mathbb{C} : \sigma > 0\}$ , so  $\alpha$  has an analytic continuation to  $D$ . Let  $P(s)$  be the power series of  $\alpha$

centered at  $w$ . Observe that  $z \in D$ , so it suffices to show that  $P(z) = \alpha(z)$ , whence we contradict the assumption that the abscissa of convergence is  $\sigma_c$ . Note that

$$\begin{aligned}
P(z) &= \sum_{k=0}^{\infty} \frac{\alpha^{(k)}(w)}{k!} (z-w)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} (z-w)^k \sum_{n=1}^{\infty} a_n (-\log n)^k n^{-w} && \text{we can differentiate termwise for } \alpha^{(k)}(w) \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} (w-z)^k \sum_{n=1}^{\infty} a_n (\log n)^k n^{-w} && \text{where the terms are all nonnegative} \\
&= \sum_{n=1}^{\infty} a_n n^{-w} \sum_{k=1}^{\infty} \frac{1}{k!} (w-z)^k (\log n)^k \\
&= \sum_{n=1}^{\infty} a_n n^{-w} e^{(w-z) \log n} = \sum_{n=1}^{\infty} a_n n^{-z}.
\end{aligned}$$

It follows that  $\alpha(z)$  converges left of  $\sigma_c$ , which is a contradiction.  $\square$

## 2.1 Dirichlet convolutions

Motivating question: are these calculations legitimate?

- $\zeta(s)^2 = \sum_{l,m=1}^{\infty} (lm)^{-s} = \sum_{n=1}^{\infty} d(n) n^{-s}$ .
- $\zeta(s) = \prod_{p \text{ prime}} (1 + p^{-s} + p^{-2s} + \dots) = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$ .

**Definition 2.1.1.** Let  $a = \{a_n\}$ ,  $b = \{b_n\}$  be sequences. The Dirichlet/multiplicative convolution  $a * b$  by  $c = \{c_n\}$  where  $c_n = \sum_{de=n} a_d b_e = \sum_{d|n} a_d b_{n/d}$ .

**Theorem 2.1.2** (thm 1.8). Let  $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ , let  $\beta(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ , and let  $\gamma(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ . If  $s \in \mathbb{C}$  is such that  $\alpha(s)$  and  $\beta(s)$  converge absolutely, and if  $c = a * b$ , then  $\gamma(s)$  converges absolutely and  $\gamma(s) = \alpha(s)\beta(s)$ .

**Example 2.1.3.** Observe that  $d(n) = (1 * 1)(n)$ .

**Example 2.1.4.** Let  $M(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s}$ , where  $\mu$  is the Möbius function which defined as

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not square free} \\ 1 & \text{if } n \text{ has an even number of prime divisors} \\ -1 & \text{if } n \text{ has an odd number of prime divisors} \end{cases}$$

Equivalently, we can define  $\mu$  as the function that satisfies

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}.$$

Observe that  $M(s)\zeta(s) = \sum_{n=1}^{\infty} (\mu * 1)(n) n^{-s} = 1$  for  $\sigma > 1$ , since  $(\mu * 1)(n) = \sum_{d|n} \mu(d)$ . It follows that  $M(s) = 1/\zeta(s)$ .

Since the abscissa of convergence of  $M$  is  $\sigma_c = 1$ , we get that  $\zeta(s)$  has no zeroes when  $\sigma > 1$ .

**Example 2.1.5** (Möbius Inversion Formula). Write  $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$  and  $G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$ . Then

$$\begin{aligned} F(s)\zeta(s) = G(s) &\iff F(s) = \frac{G(s)}{\zeta(s)} = G(s)M(s) \\ (f * 1)(n) = g(n) &\iff f(n) = (g * \mu)(n) \\ \sum_{d|n} f(n) = g(n) &\iff f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right). \end{aligned} \tag{2}$$

**Example 2.1.6.** It is known that  $\sum_{d|n} \phi(d) = n$ . This gives  $(\phi * 1)(n) = \sum_{d|n} \phi(d) = n$ . Then, for  $\sigma > 2$ , we have

$$\left(\sum_{n=1}^{\infty} \phi(n)n^{-s}\right) \left(\sum_{n=1}^{\infty} n^{-s}\right) = \sum_{n=1}^{\infty} n \cdot n^{-s} = \zeta(s-1).$$

This gives  $\sum_{n=1}^{\infty} \phi(n)n^{-s} = \zeta(s-1)/\zeta(s)$ .

**Exercise 2.1.7.** Let  $\sigma_1(n) = \sum_{d|n} d$ . Show that  $\sum_{n=1}^{\infty} \sigma_1(n)n^{-s} = \zeta(s-1)\zeta(s)$ .

**Definition 2.1.8.** A function  $f$  is multiplicative if  $f(m)f(n) = f(mn)$  if  $\gcd(m, n) = 1$ .

**Definition 2.1.9.** A number  $n$  is  $y$ -friable if  $p \mid n \implies p \leq y$ .

**Theorem 2.1.10** (thm 1.9). Let  $f$  be a multiplicative function, and let  $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ . If  $\sum_{n=1}^{\infty} |f(n)| n^{-\sigma}$  converges, we have the Euler product

$$F(s) = \prod_{p \text{ prime}} (1 + f(p)p^{-s} + f(p^2)p^{-2s} + \dots).$$

*Proof.* Let  $\sigma > \sigma_a$ . Then, for all  $p$ , we have

$$|1 + f(p)p^{-s} + f(p^2)p^{-2s} + \dots| \leq 1 + |f(p)|p^{-s} + |f(p)|p^{-2s} + \dots \leq \sum_{n=1}^{\infty} |f(n)| n^{-s}.$$

Since the above converges, we can rearrange the finite product

$$\prod_{p \leq y} (1 + f(p)p^{-s} + f(p^2)p^{-2s} + \dots) = \sum_{n \text{ } y\text{-friable}} f(n)n^{-s}.$$

Now, we can compute

$$\begin{aligned} \left| F(s) - \prod_{p \leq y} (1 + f(p)p^{-s} + f(p^2)p^{-2s} + \dots) \right| &= \left| F(s) - \sum_{n \text{ } y\text{-friable}} f(n)n^{-s} \right| \\ &= \left| \sum_{n \text{ not } y\text{-friable}} f(n)n^{-s} \right| \\ &\leq \sum_{n > y} |f(n)| n^{-s} = o(1). \end{aligned}$$

The tail goes to 0, so the theorem is proved. □

**Remark 2.1.11.** Almost the same proof shows that the Euler product converges absolutely. In particular, it is nonzero (unless an individual factor is zero). Note that the convergence of a product is defined as the convergence of the sum of logs.

**Example 2.1.12.** Note that  $\mu$  is multiplicative, so  $M(s) = \prod_{p \text{ prime}} (1 - p^{-s})$ .

**Property 2.1.13.** If  $f$  and  $g$  are multiplicative, then  $f * g$  is also multiplicative (If  $F(s)$  and  $G(s)$  have Euler products, the  $F(s)G(s)$  also has an Euler product).

**Property 2.1.14.** Dirichlet convolutions are associative, that is  $(f * g) * h = f * (g * h)$ .

**Definition 2.1.15.** Let  $\omega(n)$  be the number of distinct prime factors of  $n$ .

**Definition 2.1.16.** Let  $\Omega(n)$  be the number of prime factors of  $n$  counting with multiplicity.

**Definition 2.1.17** (Liouville lambda function). Let  $\lambda(n) = (-1)^{\Omega(n)}$ . Note that  $\lambda(n) = \mu(n)$  if and only if  $n$  is squarefree.

**Example 2.1.18.** Find an Euler product for  $L(s) = \sum_{n=1}^{\infty} \lambda(n)n^{-s}$ .

*Solution:* First, note that  $\lambda(n)$  is totally multiplicative, that is  $\lambda(mn) = \lambda(m)\lambda(n)$  for all  $m, n \in \mathbb{N}$  (not just when  $(m, n) = 1$ ). Also,  $\sum_{n=1}^{\infty} |\lambda(n)n^{-s}| = \sum_{n=1}^{\infty} n^{-\sigma}$  converges when  $\sigma > 1$ . So, for  $\sigma > 1$ , by theorem 2.1.10,

$$\begin{aligned} L(s) &= \prod_{p \text{ prime}} (1 + \lambda(p)p^{-s} + \lambda(p^2)p^{-2s} + \cdots) \\ &= \prod_{p \text{ prime}} (1 - p^{-s} + p^{-2s} - \cdots) \\ &= \prod_{p \text{ prime}} (1 + p^{-s})^{-1}. \end{aligned}$$

**Exercise 2.1.19.** Show that  $L(s) = \frac{\zeta(2s)}{\zeta(s)}$ .

**Remark 2.1.20.** If  $f(n)$  is totally multiplicative, then

$$\begin{aligned} \sum_{n=1}^{\infty} f(n)n^{-s} &= \prod_p (1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots) \\ &= \prod_p (1 + f(p)p^{-s} + f(p)^2p^{-2s} + \cdots) \\ &= \prod_p (1 - f(p)p^{-s})^{-1}. \end{aligned}$$

**Definition 2.1.21** (von Mangoldt Lambda function). Define the von Mangoldt Lambda function as

$$\Lambda(n) = \begin{cases} \log p & n = p^r \text{ for some prime } p \\ 0 & \text{otherwise} \end{cases}.$$

**Remark 2.1.22.** Recall that  $\zeta(s) = \prod_p (1 - p^{-s})^{-1}$  for  $\sigma > 1$ , so we can take logarithms.

$$\log \zeta(s) = \sum_p \log(1 - p^{-s})^{-1} = \sum_p \left( p^{-s} + \frac{1}{2}p^{-2s} + \frac{1}{3}p^{-3s} + \cdots \right).$$



This is a Dirichlet series which we can differentiate term by term, hence

$$\begin{aligned}\frac{\zeta'(s)}{\zeta(s)} &= \frac{d}{ds} \sum_p \left( p^{-s} + \frac{1}{2} p^{-2s} + \frac{1}{3} p^{-3s} + \dots \right) \\ &= \sum_p \left( (-\log p) p^{-s} + \frac{1}{2} (-2 \log p) p^{-2s} + \dots \right) \\ &= - \sum_{n=1}^{\infty} \Lambda(n) n^{-s}.\end{aligned}$$

## 2.2 Meromorphic continuation of $\zeta(s)$

**Example 2.2.1.** We prove that  $\eta(s) = (1 - 2^{1-s})\zeta(s)$  in two different ways (for  $\sigma > 1$ ).

*Proof 1.* For  $\sigma > 1$ ,  $\eta(s)$  converges absolutely, so

$$\begin{aligned}\eta(s) &= 1^{-s} + (2^{-s} - 2 \cdot 2^{-s}) + 3^{-s} + (4^{-s} - 2 \cdot 4^{-s}) + \dots \\ &= (1^{-s} + 2^{-s} + 3^{-s} + \dots) - 2(2^{-s} + 4^{-s} + \dots) \\ &= \zeta(s) - 2 \cdot 2^{-s} \zeta(s) \\ &= (1 - 2^{1-s})\zeta(s).\end{aligned}$$

*Proof 2.* Note that  $(-1)^{n-1}$  is multiplicative, and its value at  $p^r$  equals  $-1$  if  $p = 2$  and  $1$  if  $p \geq 3$ . Thus,

Note that  $\zeta(s) = (1 - 2^{1-s})^{-1} \eta(s)$ , and  $\eta(s)$  converges when  $\sigma > 0$ . Hence, this is a meromorphic continuation of  $\zeta(s)$  to  $\sigma > 0$ . Note that  $(1 - 2^{1-s})^{-1} \eta(s)$  has singularities when  $1 - 2^{1-s} = 0$ , that is  $s = 1 + \frac{2\pi i}{\log 2}$ .

**Exercise 2.2.2.** Show that  $\zeta(s)$  has a simple pole at  $s = 1$  with residue 1.

**Theorem 2.2.3** (thm 1.12). *For  $\sigma > 0$  and  $s \neq 1$ , we can write*

$$\zeta(s) = \sum_{n \leq x} n^{-s} + \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s \int_x^\infty \{u\} u^{-s-1} du. \quad (3)$$

*Proof.* For  $\sigma > 1$ , we have

$$\begin{aligned}\sum_{n > x} n^{-s} &= \int_x^\infty u^{-s} d(\lfloor u \rfloor) \\ &= \int_x^\infty u^{-s} du - \int_x^\infty u^{-s} d(\{u\}) \\ &= \frac{u^{1-s}}{1-s} \Big|_x^\infty - \left( \{u\} u^{-s} \Big|_x^\infty - \int_x^\infty \{u\} d(u^{-s}) \right) \\ &= \frac{x^{1-s}}{s-1} + \frac{\{x\}}{x^s} - s \int_x^\infty \{u\} u^{-s-1} du.\end{aligned}$$

Let  $\varepsilon > 0$ , then for  $\sigma > \varepsilon$ , we have

$$\left| \int_x^\infty \{u\} u^{-s-1} du \right| \leq \int_x^\infty 1 \cdot u^{-\sigma-1} du = \frac{x^{-\sigma}}{\sigma}.$$

Note that this is uniform for  $\sigma > \varepsilon$ , so we have analyticity. Then, we conclude that the equation 3 holds for all  $\sigma > 0$  by the uniqueness of analytic continuation.  $\square$

**Remark 2.2.4.** We can use a similar method to show that  $\zeta(s)$  is defined for all  $s \in \mathbb{C} \setminus \{1\}$ .

Consequences of Theorem 2.2.3:

- When  $x = 1$ , we have

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^\infty \{u\} u^{-s-1} du,$$

so  $\zeta(s) - \frac{1}{s-1}$  has a removable singularity at  $s = 1$  with value

$$C_0 = 1 - \int_1^\infty \{u\} u^{-2} du.$$

Then, by DCT, we get  $\zeta(s) = \frac{1}{s-1} + C_0 + O(|s-1|)$ .

- We can rearrange to get

$$\begin{aligned} \sum_{n \leq x} n^{-s} &= \zeta(s) - \frac{x^{1-s}}{s-1} - \frac{\{x\}}{x^s} + s \int_x^\infty \{u\} u^{-s-1} du \\ &= \zeta(s) + \frac{x^{1-s}}{1-s} + O\left(\frac{1}{x^\sigma} + |s| \int_x^\infty 1 \cdot u^{-\sigma-1} du\right) \\ &= \zeta(s) + \frac{x^{1-s}}{1-s} + O\left(\frac{1}{x^\sigma} + \frac{|s|}{\sigma} \frac{1}{x^\sigma}\right). \end{aligned}$$

Hence, we get asymptotics for  $\sum_{n \leq x} n^{-\alpha}$ .

$$\sum_{n \leq x} \frac{1}{n^\alpha} = \begin{cases} O(x^{1-\alpha}) & \text{if } 0 < \alpha < 1 \\ \zeta(\alpha) + O(x^{1-\alpha}) & \text{if } \alpha > 1 \end{cases}.$$

- Let  $s = 1$ , then

$$\begin{aligned} \sum_{n \leq x} \frac{1}{n} &= \int_{1^-}^x \frac{1}{t} d[t] = \int_{1^-}^x \frac{1}{t} dt - \int_{1^-}^x \frac{1}{t} d\{t\} \\ &= \log x - \frac{\{t\}}{t} \Big|_{1^-}^x + \int_1^x \{t\} \frac{1}{t^2} dt \\ &= \log x - \frac{\{x\}}{x} + 1 - \int_1^x \{t\} \frac{1}{t^2} dt \\ &= \log x + 1 - \int_1^\infty \{t\} t^{-2} dt + \int_x^\infty \{t\} t^{-2} dt - \frac{\{x\}}{x} \\ &= \log x + C_0 + O\left(\frac{1}{x}\right). \end{aligned}$$

Note that an error of  $1/x$  is the best approximation with a smooth function (because we can't get better than the jumps).

**Remark 2.2.5.** Note that  $C_0$  is Euler's constant, that is

$$C_0 = \lim_{x \rightarrow \infty} \left( \sum_{n \leq x} \frac{1}{n} - \log x \right) \approx 0.577.$$

### 3 Elementary Estimates for Arithmetic Functions

Motivating question: what is the expectation of  $\frac{\phi(n)}{n}$ ?

Note that  $(\phi * 1)(n) = n$ , so by Möbius inversion,

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d} \implies \frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}.$$

Then, we get

$$\sum_{n \leq x} \frac{\phi(n)}{n} = \sum_{n \leq x} \sum_{d|n} \frac{\mu(d)}{d} = \sum_{d \leq x} \frac{\mu(d)}{d} \left\lfloor \frac{x}{d} \right\rfloor = \sum_{d \leq x} \frac{\mu(d)}{d} \left( \frac{x}{d} + O(1) \right).$$

Hence, dividing by  $x$  gives

$$\begin{aligned} \frac{1}{x} \sum_{n \leq x} \frac{\phi(n)}{n} &= \sum_{d \leq x} \frac{\mu(d)}{d^2} + O \left( \frac{1}{x} \sum_{d \leq x} \left| \frac{\mu(d)}{d} \right| \right) \\ &= \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} + O \left( \sum_{d > x} \left| \frac{\mu(d)}{d^2} \right| + \frac{1}{x} \sum_{d \leq x} \left| \frac{\mu(d)}{d} \right| \right) \\ &= \frac{1}{\zeta(2)} + O \left( \sum_{d > x} \frac{1}{d^2} + \frac{1}{x} \sum_{d \leq x} \frac{1}{d} \right) \\ &= \frac{1}{\zeta(2)} + O \left( \frac{\log x}{x} \right). \end{aligned}$$

**Example 3.0.1.** Estimate the number of square-free numbers up to  $x$ . Let  $Q(x) = \sum_{n \leq x} \mu(n)^2$ . Note that  $\mu(n)^2$  is the indicator function for square-free numbers.

**Lemma 3.0.2.** Define  $g(d)$  as

$$g(d) = \begin{cases} \mu(m) & \text{if } d = m^2 \text{ for some } m \in \mathbb{N} \\ 0 & \text{if } d \text{ is not a square} \end{cases}.$$

Then  $\mu^2 = 1 * g$ .

*Proof 1.* Let  $k^2$  be the largest square divisor of  $n$ . Then

$$\begin{aligned} \mu(n)^2 &= \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1 \end{cases} \\ &= \sum_{d|k} \mu(d) = \sum_{\substack{d|n \\ d=m^2}} \mu(m) = \sum_{d|n} g(d). \end{aligned}$$

□

**Exercise 3.0.3** (Proof 2). Show that  $\sum_{n=1}^{\infty} \mu(n)^2 n^{-s} = \frac{\zeta(s)}{\zeta(2s)}$  and  $\sum_{n=1}^{\infty} g(n) n^{-s} = \frac{1}{\zeta(2s)}$ .

**Exercise 3.0.4** (Proof 3). Show that  $\mu(n)^2 = \sum_{d|n} g(d)$  by Möbius inversion, along with the fact that every  $n \in \mathbb{N}$  can be uniquely written as  $n = a^2 b$  where  $b$  is square-free.

We can approximate  $Q(x)$  as follows.

$$\begin{aligned}
Q(x) &= \sum_{n \leq x} \mu(n)^2 = \sum_{n \leq x} \sum_{d|n} g(d) = \sum_{d \leq x} g(d) \left( \frac{x}{d} + O(1) \right) = x \sum_{d \leq x} \frac{g(d)}{d} + O \left( \sum_{d \leq x} \left| \frac{g(d)}{d} \right| \right) \\
&= x \sum_{m \leq \sqrt{x}} \frac{\mu(m)}{m^2} + O \left( \sum_{m \leq \sqrt{x}} |\mu(m)| \right) \quad \text{by the definition of } g \\
&= x \left( \frac{1}{\zeta(2)} + O \left( \frac{1}{\sqrt{x}} \right) \right) + O(\sqrt{x}) \\
&= \frac{x}{\zeta(2)} + O(\sqrt{x})
\end{aligned}$$