# Math 539 Notes

Henry Xia

January 16, 2020

## Contents



# 1 Introduction

Motivating questions (some statistics):

- the "probability" that a random number has some property
- the "distribution" of some given multiplicative/additive function

Idea: we can answer the question for $\{1, ..., \lfloor x \rfloor\}$ for some parameter $x$. Then, take the limit $x \to \infty$ for all natural numbers.

## 1.1 Notation

Let $g(x) \geq 0$.

**Definition 1.1.** $O(g(x))$ means some unspecified function $u(x)$ such that $|u(x)| \leq cg(x)$ for some constant $c > 0$.

**Example 1.2.** Show that $e^{2x} - 1 = 2x + O(x^2)$ for $x = [-1, 1]$.

*Proof.* Observe that $f(z) = e^{2z} - 1 - 2z$ is analytic (and entire) and has a double zero at $z = 0$ (one can check that $f(z) = f'(z) = 0$. Hence, $g(z) = (e^{2z} - 1 - 2z)/z^2$ has a removable singularity at $z = 0$, whence $g$ is analytic and entire. Let $C = \max\{|g(z)| : |z| \leq 1\}$. Then

$$|g(z)| \leq C \implies \left|e^{2z} - 1 - 2z\right| \leq C \left|z^2\right| \implies e^{2z} - 1 - 2z = O(|z|^2).$$

$\square$

**Exercise 1.3.** Show that $\sqrt{x+1} = \sqrt{x} + O(1/\sqrt{x})$ for $x \in [1, \infty)$.

**Definition 1.4.** $f(x) \ll g(x)$ means $f(x) = O(g(x))$.

**Exercise 1.5.** Suppose that $f_1 \ll g_1, f_2 \ll g_2$, then $f_1 + f_2 \ll \max\{g_1, g_2\}$. ✓

**Exercise 1.6.** Let $f, g$ be continuous on $[0, \infty)$, and $f \ll g$ on $[123, \infty)$. Show that $f \ll g$ on $[0, \infty)$. ✓

**Definition 1.7.** $f(x) \sim g(x)$ means $\lim \frac{f(x)}{g(x)} = 1$.

**Definition 1.8.** $f(x) = o(g(x))$ means $\lim \frac{f(x)}{g(x)} = 0$.

**Definition 1.9.** $f(x) = O_y(g(x))$ means $f, g$ depend on some parameter $y$, and the implicit constant depends on $y$.

**Exercise 1.10.** For any $A, \epsilon > 0$, show that $(\log x)^A \ll_{A,\epsilon} x^\epsilon$.

## 1.2 Riemann-Stieltjes Integral

Appendix A in the book.

**Definition 1.11.** Some definitions for partitions

1. Let $\underline{x} = \{x_0, ..., x_N\}$ be a partition of $[c, d]$ if $c = x_0 < \cdots < x_N = d$.
2. The mesh size $m(\underline{x}) = \max_{1 \leq j \leq N} x_j - x_{j-1}$.
3. Sample points $\xi_j \in [x_{j-1}, x_j]$.

**Definition 1.12** (Riemann-Stieltjes Integral)**.** Given two functions $f(x)$ and $g(x)$, define the Riemann-Stieltjes integral as

$$\int_c^d f(x)\ dg(x) = \lim_{m(\underline{x}) \to 0} \sum_{j=1}^N f(\xi_j)(g(x_j) - g(x_{j-1})).$$

**Remark 1.13.** Setting $g(x) = x$ gives the Riemann integral.

**Theorem 1.14.** *Let $f(x)$ have bounded variation and let $g(x)$ be continuous on $[c, d]$, or vice versa. Then $\int_c^d f(x)\ dg(x)$ exists.*

**Remark 1.15.** If a function is piecewise monotone, then it has bounded variation.

**Example 1.16.** Given a sequence $a_{n n \in \mathbb{N}}$, define the summatory function $A(x) = \sum_{n \leq x} a_n$. Then, on any $[c, d]$, $A(x)$ is bounded, piecewise continuous and piecewise monotone. Hence, the Riemann-Stieltjes integral exists when $g$ is continuous.

**Remark 1.17.** We present 3 facts that we will use.

1. If $A(x)$ is the summatory function as above, and $f(x)$ is continuous, then

$$\int_c^d f(x)\ dA(x) = \sum_{c < n \leq d} a_n f(n).$$

2. (Integration by parts). If the integrals exist, then

$$\int_c^d f(x)\ dg(x) = f(x)g(x)|_c^d - \int_c^d g(x)\ df(x).$$

3. If $f(x)$ is continuously differentiable, then

$$\int_c^d g(x)\ df(x) = \int_c^d g(x)f'(x)\ dx.$$

**Example 1.18** (Summation by parts)**.** Consider $\sum_{n\le y} \frac{a_n}{n}$. Let $f(x) = 1/x$, then we can write

$$\sum_{n\le y} \frac{a_n}{n} = \sum_{n\le y} a_n \cdot \frac{1}{n} = \int_0^y \frac{1}{x}\ dA(x) = \frac{1}{x}A(x)\Big|_0^y - \int_0^y A(x)\ d\left(\frac{1}{x}\right) = \frac{A(y)}{y} - \int_0^y A(x)\frac{1}{x^2}\ dx.$$

The final manipulation that we want to get is

$$\sum_{n\le y} a_n f(n) = A(y)f(y) - \int_0^y A(x)f'(x)\ dx. \tag{1}$$

# 2 Dirichlet Series

A Dirichlet series is $\sum_{n=1}^{\infty} n^{-s}$.
Facts about Dirichlet series:

- converge in some right half-plane $\{s \in \mathbb{C} : \Re s > R\}$ for some $R$ (possibly $R = \pm\infty$).
- Sometimes converge conditionally. Example: $\sum_{n=1}^{\infty}(-1)^n/n^{1/2}$.
- $\left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\left(\sum_{n=1}^{\infty} b_n n^{-s}\right) = \sum_{n=1}^{\infty} c_n n^{-s}$ where $c = \sum_{de=n} a_d b_e = \sum_{d\mid n} a_d b_{e/d}$. (multiplicative convolution)

Some notation: for $s \in \mathbb{C}$, we write $s = \sigma + it$, that is $\sigma$ is the real part of $s$, and $t$ is the imaginary part of $s$. Note that if $x > 0$, then $|x^s| = |x^\sigma|\left|x^{it}\right| = |x^\sigma|\left|e^{it\log x}\right| = |x^\sigma|$.

**Theorem 2.1.** *Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. Suppose that $s_0 \in \mathbb{C}$ is such that $\alpha(s_0)$ converges. Then $\alpha(s)$ converges uniformly in the sector $S = \{s \in \mathbb{C} : \sigma \ge \sigma_0, |t - t_0| \le H|\sigma - \sigma_0|\}$ for any $H > 0$.*

*Proof.* WLOG, let $s_0 = 0$, otherwise we can do a change of variables.
Let $A(x) = \sum_{n\le x} a_n = \alpha(0) - R(x)$. Then, for $\sigma > 0$,

$$\begin{aligned}
\sum_{M<n\le N} a_n n^s &= \int_M^N x^{-s}\ dA(x) = \int_M^N x^{-s}\ d(\alpha(0) - R(x)) \\
&= \int_M^N x^{-s}\ d\alpha(0) - \int_M^N x^{-s}\ dR(x) = -\int_M^N x^{-s}\ dR(x) \\
&= -x^{-s}R(x)\big|_M^N + \int_M^N R(x)\ d(x^{-s}) \\
&= R(M)M^{-s} - R(N)N^{-s} - s\int_M^N R(x)x^{-s-1}\ dx.
\end{aligned}$$

Note that $R(N)N^{-s} \to 0$ as $N \to \infty$, and that $R(x)x^{-s-1} \ll x^{-\sigma-1}$. Hence, letting $N \to \infty$ gives

$$\sum_{M<n} a_n n^{-s} = R(M)M^{-s} - s\int_M^{\infty} R(x)x^{-s-1}\ dx \to 0 \text{ as } M \to \infty.$$

Now, choose $M$ large such that $|R(x)| < \epsilon$ for all $x \geq M$. Then,

$$\begin{aligned}\left|\sum_{n>M} a_n n^{-s}\right| &\leq \epsilon M^{-\sigma} + |s| \int_M^\infty \epsilon x^{-\sigma-1}\ dx \\ &= \epsilon M^{-\sigma} + |s|\,\epsilon x^{-\sigma} \frac{1}{-\sigma}\Big|_M^\infty \\ &= \epsilon M^{-\sigma} + |s|\,\epsilon \frac{M^{-\sigma}}{\sigma} = \frac{\epsilon}{M^\sigma}\left(1 + \frac{|s|}{\sigma}\right).\end{aligned}$$

Since $s \in S$, we have

$$|s| = \sqrt{\sigma^2 + t^2} \leq \sqrt{\sigma^2 + |H\sigma|^2} = \sigma\sqrt{1 + H^2},$$

so $\left|\sum_{n>M} a_n n^{-s}\right| \leq \epsilon(1 + \sqrt{1 + H^2})$ as $M \to \infty$. Observe that the latter only depends on $H$, so the convergence is uniform. $\square$

**Corollary 2.2.** *If $\alpha(s_0)$ converges, then $\alpha(s)$ converges for all $s$ with $\sigma > \sigma_0$.*

**Corollary 2.3.** *If $\alpha(s_0)$ diverges, then $\alpha(s)$ diverges for all $s$ with $\sigma < \sigma_0$.*

**Remark 2.4.** The Dirichlet series $\alpha(s)$ has an abscissa of convergence $\sigma_c$ such that $\alpha(s)$ converges if $\sigma > \sigma_c$, and diverges if $\sigma < \sigma_c$. It is allowed to have $\sigma_c = \pm\infty$. Furthermore, $\alpha(s)$ converges locally uniformly right of $\sigma_c$, whence $\alpha(s)$ is analytic.

**Remark 2.5.** Observe that $\int_1^N x^{-s}\ dA(x) = \sum_{1<n\leq N} a_n n^{-s} = \sum_{n=2}^N a_n n^{-s}$. Sometimes we write $\int_{-1}^N$ to include the 1.

**Theorem 2.6.** *Let $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$ have an abscissa of convergence $\sigma_c \geq 0$. Then for $\sigma > \sigma_c$, we have $\alpha(s) = s\int_1^\infty A(x)x^{-s-1}\ dx$. Moreover,*

$$\limsup_{x\to\infty} \frac{\log|A(x)|}{\log x} = \sigma_c.$$

*Proof.* Observe that

$$\begin{aligned}\sum_{n=1}^N a_n n^{-s} &= \int_{1^-}^N x^{-s}\ dA(x) = x^{-s}A(x)\Big|_{1^-}^N - \int_{1^-}^N A(x)\ d(x^{-s}) \\ &= A(N)N^{-s} - \int_{1^-}^N A(x)(-sx^{-s-1}\ dx) = A(N)N^{-s} + s\int_1^N A(x)x^{-s-1}\ dx.\end{aligned}$$

Observe that in the last line, we can replace $1^-$ with 1 because the integrand is bounded.
Define $\phi = \limsup_{x\to\infty} \frac{\log|A(x)|}{\log x}$. We compare this to $\sigma_c$.
Let $\sigma = \phi + \epsilon$ for some $\varepsilon > 0$. Then $\frac{\log|A(x)|}{\log x} < \phi + \frac{\varepsilon}{2}$ for large $x$, so $A(x) \ll x^{\phi+\varepsilon/2}$. Then, $A(N)N^{-s} \ll N^{\phi+\varepsilon/2}N^{-(\phi+\varepsilon)} = N^{-\varepsilon/2}$. Hence,

$$\int_N^\infty A(x)x^{-\sigma-1}\ dx \ll \int_N^\infty x^{-\phi+\varepsilon/2}x^{-(\phi+\varepsilon+1)}\ dx = \int_N^\infty x^{-1-\varepsilon/2}\ dx \ll N^{-\varepsilon/2}.$$

It follows that

$$\sum_{n=1}^N a_n n^{-s} = O(N^{-\varepsilon/2}) + s\left(\int_1^\infty A(x)x{-s-1}\ dx + O(N^{-\varepsilon/2})\right).$$

Let $N \to \infty$ gives $s \int_1^\infty A(x) x^{-s-1}\, dx$ converges. Hence $\sigma_c \leq \phi$.
Conversely, let $\sigma_0 = \sigma_c + \varepsilon$, and let $R_0(x) = \sum_{n>x} a_n n^{-\sigma_0} = \alpha(\sigma_0) - \sum_{n \leq x} a_n n^{-\sigma_0}$. Observe that

$$A(N) = -R_0(N) N^{\sigma_0} + \sigma_0 \int_0^N R_0(x) x^{\sigma_0 - 1}\, dx.$$

Since $\alpha(0)$ converges, $R_0(x) = o(1)$ so $R_0(x) \ll 1$. Then

$$A(N) \ll 1 \cdot N^{\sigma_0} + \sigma_0 \int_0^N 1 \cdot x^{\sigma_0 - 1}\, dx = N^{\sigma_0} + N^{\sigma_c} \ll N^{\sigma_0} = N^{\sigma_c + \varepsilon}.$$

Hence $\frac{\log |A(x)|}{\log x} \ll \frac{(\sigma_c + \varepsilon) \log x}{\log x} = \sigma_c + \varepsilon$, so $\phi \leq \sigma_c + \varepsilon$. Take $\varepsilon \to 0$, so $\phi \leq \sigma_c$. ☐

**Definition 2.7.** The abscissa of absolute convergence is $\sigma_a = \inf \left\{ \sigma \in \mathbb{R} : \sum_{n=1}^\infty |a_n| n^{-\sigma} \text{ converges} \right\}$.

**Example 2.8.** Let $\eta(s) = \sum_{n=1}^\infty (-1)^{n-1} n^{-s}$. Observe that $\sigma_c = 0$ by the alternating series test. However, we only have absolute convergence when $\sigma > 1$, so the abscissa of absolute convergence is $\sigma_a = 1$.

**Remark 2.9.** When $a_n \geq 0$ for all $n \in \mathbb{N}$, we have $\sigma_c = \sigma_a$.

**Theorem 2.10.** *For any Dirichlet series $\alpha(s)$, we have $\sigma_c \leq \sigma_a \leq \sigma_c + 1$.*

*Proof.* The first inequality is trivial.
Let $\sigma = \sigma_c + 1 + \varepsilon$ where $\varepsilon > 0$. We show that $\alpha(\sigma)$ converges absolutely. Note that $\alpha(s)$ converges at $s = \sigma_c + \varepsilon/2$, that is

$$a_n n^{-(\sigma_c + \varepsilon/2)} = o(1) \implies a_n n^{-(\sigma_c + \varepsilon/2)} \ll 1.$$

Then,

$$\sum_{n=1}^\infty |a_n| n^{-(\sigma_c + 1 + \varepsilon)} = \sum_{n=1}^\infty \left| a_n n^{-(\sigma_c + \varepsilon/2)} \right| n^{-(1+\varepsilon/2)} \ll \sum_{n=1}^\infty n^{-(1+\varepsilon/2)} \ll 1.$$

It follows that $\alpha(\sigma)$ converges absolutely for all $\varepsilon > 0$. ☐

**Theorem 2.11** (Landau's Theorem)**.** *Let $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$ with $\sigma_c < \infty$. If $a_n \geq 0$ for each $n \in \mathbb{N}$. Then, $\alpha(s)$ has a singularity at $s = \sigma_c$.*

*Proof.* Suppose that there does not exist a singularity at $s = \sigma_c$. Then, there exists an analytic continuation of $\alpha$ to $C = \{s \in \mathbb{C} : |s - \sigma_c| < \delta\}$.
Let $z = \sigma_c - \frac{1}{4}\delta$ and let $w = \sigma_c + \frac{3}{4}\delta$. Let $D = \left\{ s \in \mathbb{C} : |s - w| < \frac{5}{4}\delta \right\}$. Observe that $D \subset C \cup \{s \in \mathbb{C} : \sigma > 0\}$, so $\alpha$ has an analytic continuation to $D$. Let $P(s)$ be the power series of $\alpha$ centered at $w$. Observe that $z \in D$, so it suffices to show that $P(z) = \alpha(z)$, whence we contradict

the assumption that the abscissa of convergence is $\sigma_c$. Note that

$$
\begin{aligned}
P(z) &= \sum_{k=0}^{\infty} \frac{\alpha^{(k)}(w)}{k!}(z-w)^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}(z-w)^k \sum_{n=1}^{\infty} a_n(-\log n)^k n^{-w} && \text{we can differentiate termwise for } \alpha^{(k)}(w) \\
&= \sum_{k=1}^{\infty} \frac{1}{k!}(w-z)^k \sum_{n=1}^{\infty} a_n(\log n)^k n^{-w} && \text{where the terms are all nonnegative} \\
&= \sum_{n=1}^{\infty} a_n n^{-w} \sum_{k=1}^{\infty} \frac{1}{k!}(w-z)^k (\log n)^k \\
&= \sum_{n=1}^{\infty} a_n n^{-w} e^{(w-z)\log n} = \sum_{n=1}^{\infty} a_n n^{-z}.
\end{aligned}
$$

It follows that $\alpha(z)$ converges left of $\sigma_c$, which is a contradiction. $\square$

### 2.1 Dirichlet convolutions

Motivating question: are these calculations legitimate?

- $\zeta(s)^2 = \sum_{l,m=1}^{\infty}(lm)^{-s} = \sum_{n=1}^{\infty} d(n)n^{-s}$.
- $\zeta(s) = \prod_{p \text{ prime}}(1 + p^{-s} + p^{-2s} + \cdots) = \prod_{p \text{ prime}}(1 - p^{-s})^{-1}$ .

**Definition 2.12.** Let $a = \{a_n\}$, $b = \{b_n\}$ be sequences. The Dirichlet/multiplicative convolution $a * b$ by $c = \{c_n\}$ where $c_n = \sum_{de=n} a_d b_e = \sum_{d|n} a_d b_{n/d}$.

**Theorem 2.13.** *Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, let $\beta(s) = \sum_{n=1}^{\infty} b_n n^{-s}$, and let $\gamma(s) = \sum_{n=1}^{\infty} c_n n^{-s}$. If $s \in \mathbb{C}$ is such that $\alpha(s)$ and $\beta(s)$ onvege absolutely, and if $c = a * b$, then $\gamma(s)$ converges absolutely and $\gamma(s) = \alpha(s)\beta(s)$.*

**Example 2.14.** Observe that $d(n) = (1 * 1)(n)$.

**Example 2.15.** Let $M(s) = \sum_{n=1}^{\infty} \mu(n)n^{-s}$, where $\mu$ is the Möbius function which defined as

$$
\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not square free} \\ 1 & \text{if } n \text{ has an even number of prime divisors} \\ -1 & \text{if } n \text{ has an odd number of prime divisors} \end{cases}.
$$

Equivalently, we can define $\mu$ as the function that satisfies

$$
\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}.
$$

Observe that $M(s)\zeta(s) = \sum_{n=1}^{\infty}(\mu * 1)(n)n^{-s} = 1$ for $\sigma > 1$, since $(\mu * 1)(n) = \sum_{d|n} \mu(d)$. It follows that $M(s) = 1/\zeta(s)$.
Since the abscissa of convergence of $M$ is $\sigma_c = 1$, we get that $\zeta(s)$ has no zeroes when $\sigma > 1$.

**Example 2.16** (Möbius Inversion Formula)**.** Write $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ and $G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$. Then

$$\begin{aligned} F(s)\zeta(s) = G(s) &\iff F(s) = \frac{G(s)}{\zeta(s)} = G(s)M(s) \\ (f * 1)(n) = g(n) &\iff f(n) = (g * \mu)(n) \\ \sum_{d|n} f(n) = g(n) &\iff f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right). \end{aligned}$$

**Example 2.17.** It is known that $\sum_{d|n} \phi(d) = n$. This gives $(\phi * 1)(n) = \sum_{d|n} \phi(d) = n$. Then, for $\sigma > 2$, we have

$$\left(\sum_{n=1}^{\infty} \phi(n)n^{-s}\right)\left(\sum_{n=1}^{\infty} n^{-s}\right) = \sum_{n=1}^{\infty} n \cdot n^{-s} = \zeta(s-1).$$

This gives $\sum_{n=1}^{\infty} \phi(n)n^{-s} = \zeta(s-1)/\zeta(s)$.

**Exercise 2.18.** Let $\sigma_1(n) = \sum_{d|n} d$. Show that $\sum_{n=1}^{\infty} \sigma_1(n)n^{-s} = \zeta(s-1)\zeta(s)$.

**Definition 2.19.** A function $f$ is multiplicative if $f(m)f(n) = f(mn)$ if $\gcd(m,n) = 1$.

**Definition 2.20.** A number $n$ is $y$-friable if $p \mid n \implies p \leq y$.

**Theorem 2.21.** *Let $f$ be a multiplicative function, and let $F(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$. If $\sum_{n=1}^{\infty} |f(n)|\, n^{-\sigma}$ converges, we have the Euler product*

$$F(s) = \prod_{p \text{ prime}} (1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots).$$

*Proof.* Let $\sigma > \sigma_a$. Then, for all $p$, we have

$$\left|1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots\right| \leq 1 + |f(p)|\, p^{-s} + |f(p)|\, p^{-2s} + \cdots \leq \sum_{n=1}^{\infty} |f(n)|\, n^{-s}.$$

Since the above converges, we can rearrange the finite product

$$\prod_{p \leq y}(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots) = \sum_{n\ y\text{-friable}} f(n)n^{-s}.$$

Now, we can compute

$$\begin{aligned} \left|F(s) - \prod_{p \leq y}(1 + f(p)p^{-s} + f(p^2)p^{-2s} + \cdots)\right| &= \left|F(s) - \sum_{n\ y\text{-friable}} f(n)n^{-s}\right| \\ &= \left|\sum_{n \text{ not } y\text{-friable}} f(n)n^{-s}\right| \\ &\leq \sum_{n > y} |f(n)|\, n^{-s} = o(1). \end{aligned}$$

The tail goes to 0, so the theorem is proved. □

**Remark 2.22.** Almost the same proof shows that the Euler product converges absolutely. In particular, it is nonzero (unless an individual factor is zero). Note that the convergence of a product is defined as the convergence of the sum of logs.

**Example 2.23.** Note that $\mu$ is multiplicative, so $M(s) = \prod_{p \text{ prime}}(1 - p^{-s})$.