Math 539 Notes

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Contents

1 Introduction

Motivating questions (some statistics):

- the "probability" that a random number has some property
- the "distribution" of some given multiplicative/additive function

Idea: we can answer the question for $\{1, ..., \lfloor x \rfloor\}$ for some parameter x. Then, take the limit $x \to \infty$ for all natural numbers.

1.1 Notation

Let $g(x) \geq 0$.

Definition 1.1. $O(g(x))$ means some unspecified function $u(x)$ such that $|u(x)| \leq cg(x)$ for some constant $c > 0$.

Example 1.2. Show that $e^{2x} - 1 = 2x + O(x^2)$ for $x = [-1, 1]$.

Proof. Observe that $f(z) = e^{2z} - 1 - 2z$ is analytic (and entire) and has a double zero at $z = 0$ (one can check that $f(z) = f'(z) = 0$. Hence, $g(z) = (e^{2z} - 1 - 2z)/z^2$ has a removable singularity at $z = 0$, whence g is analytic and entire. Let $C = \max\{|g(z)| : |z| \leq 1\}$. Then

$$|g(z)| \leq C \implies \left|e^{2z} - 1 - 2z\right| \leq C\left|z^2\right| \implies e^{2z} - 1 - 2z = O(|z|^2).$$

□

Exercise 1.3. Show that $\sqrt{x+1} = \sqrt{x} + O(1/\sqrt{x})$ for $x \in [1, \infty)$.

Definition 1.4. $f(x) \ll g(x)$ means $f(x) = O(g(x))$.

Exercise 1.5. Suppose that $f_1 \ll g_1, f_2 \ll g_2$, then $f_1 + f_2 \ll \max\{g_1, g_2\}$. ✓

Exercise 1.6. Let f, g be continuous on $[0, \infty)$, and $f \ll g$ on $[123, \infty)$. Show that $f \ll g$ on $[0, \infty)$. ✓

Definition 1.7. $f(x) \sim g(x)$ means $\lim \frac{f(x)}{g(x)} = 1$.

Definition 1.8. $f(x) = o(g(x))$ means $\lim \frac{f(x)}{g(x)} = 0$.

Definition 1.9. $f(x) = O_y(g(x))$ means f, g depend on some parameter y, and the implicit constant depends on y.

Exercise 1.10. For any $A, \epsilon > 0$, show that $(\log x)^A \ll_{A,\epsilon} x^\epsilon$.

1.2 Riemann-Stieltjes Integral

Appendix A in the book.

Definition 1.11. Some definitions for partitions

1. Let $\underline{x} = \{x_0, ..., x_N\}$ be a partition of $[c, d]$ if $c = x_0 < \cdots < x_N = d$.
2. The mesh size $m(\underline{x}) = \max_{1 \le j \le N} x_j - x_{j-1}$.
3. Sample points $\xi_j \in [x_{j-1}, x_j]$.

Definition 1.12 (Riemann-Stieltjes Integral)**.** Given two functions $f(x)$ and $g(x)$, define the Riemann-Stieltjes integral as

$$\int_c^d f(x)\ dg(x) = \lim_{m(\underline{x}) \to 0} \sum_{j=1}^N f(\xi_j)(g(x_j) - g(x_{j-1})).$$

Remark 1.13. Setting $g(x) = x$ gives the Riemann integral.

Theorem 1.14. *Let $f(x)$ have bounded variation and let $g(x)$ be continuous on $[c, d]$, or vice versa. Then $\int_c^d f(x)\ dg(x)$ exists.*

Remark 1.15. If a function is piecewise monotone, then it has bounded variation.

Example 1.16. Given a sequence $a_{n n\in\mathbb{N}}$, define the summatory function $A(x) = \sum_{n \le x} a_n$. Then, on any $[c, d]$, $A(x)$ is bounded, piecewise continuous and piecewise monotone. Hence, the Riemann-Stieltjes integral exists when g is continuous.

Remark 1.17. We present 3 facts that we will use.

1. If $A(x)$ is the summatory function as above, and $f(x)$ is continuous, then
$$\int_c^d f(x)\ dA(x) = \sum_{c<n\le d} a_n f(n).$$
2. (Integration by parts). If the integrals exist, then
$$\int_c^d f(x)\ dg(x) = f(x)g(x)|_c^d - \int_c^d g(x)\ df(x).$$
3. If $f(x)$ is continuously differentiable, then
$$\int_c^d g(x)\ df(x) = \int_c^d g(x)f'(x)\ dx.$$

Example 1.18 (Summation by parts)**.** Consider $\sum_{n\leq y}\frac{a_n}{n}$. Let $f(x)=1/x$, then we can write

$$\sum_{n\leq y}\frac{a_n}{n}=\sum_{n\leq y}a_n\cdot\frac{1}{n}=\int_0^y\frac{1}{x}\ dA(x)=\frac{1}{x}A(x)\bigg|_0^y-\int_0^y A(x)\ d\left(\frac{1}{x}\right)=\frac{A(y)}{y}-\int_0^y A(x)\frac{1}{x^2}\ dx.$$

The final manipulation that we want to get is

$$\sum_{n\leq y}a_nf(n)=A(y)f(y)-\int_0^y A(x)f'(x)\ dx. \tag{1}$$