Math 421 Notes

Henry Xia

January 8, 2020

Contents

1 Introduction **1**

1 Introduction

Definition 1.1. A topological space $(S, \mathcal{T})$ is a nonempty set with a family of subsets $\mathcal{T}$ such that

1. $\emptyset \in \mathcal{T}$
2. $S \in \mathcal{T}$
3. $\mathcal{T}$ is closed under finite intersections and arbitrary unions

Examples: $\{\emptyset, S\}$ (indiscrete topology), 2^S (discrete topology).
A metric on a metric space defines a topology. Not all topologies have a corresponding metric. A topology is called metrizable if we can define a metric such that "open" has the same meaning.
Topologies can be partially ordered. $\mathcal{T}_1 \prec \mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$ as sets. Denote $\mathcal{T}(\mathcal{E})$ to be the topology generated by $\mathcal{E} \subset 2^S$.

Definition 1.2. A base of $\mathcal{T}$ is a family $\mathcal{B} \subset \mathcal{T}$ such that for any nonempty open set $O \in \mathcal{T}$, there exists a colletion $\{B_\alpha : B_\alpha \in \mathcal{B}\}$ such that $O =\subset_\alpha B_\alpha$.

Definition 1.3. Let $(S, \mathcal{T})$ be a topological space and $X \subset S$. Then, $\mathcal{T}_x = \{O \cap X : O \subset \mathcal{T}\}$ is the relative topology $(X, \mathcal{T}_x)$.

Definition 1.4. A set X is closed if $\exists Y \in \mathcal{T}$ such that $X = Y^c$.

Definition 1.5. The interior of X is the largest open set $X^o \subset X$.

Definition 1.6. The closure of X is the smallest closed set $\overline{X} \supset X$.

Definition 1.7. The boundary of X is $\overline{X} \setminus X^o$.

Definition 1.8. A neighbourhood of $x \in S$ is a set $N_x \subset S$ such that $x \in N_x^o$

Definition 1.9. A neighbourhood base of x is a family $\mathcal{N}_x$ such that each $N \in \mathcal{N}_x$ is a neighbourhood of x and for any neighbourhood M_x, there exists some $N \in \mathcal{N}_x$ such that $N \subset M_x$.

Definition 1.10 (Classification of topological spaces)**.** A topological space is called T_2 or Hausdorff if $\forall x, y \in S$, $x \neq y$, there exists $O_x, O_y \in \mathcal{T}$ such that $x \in O_x$, $y \in O_y$, and $O_x \cap O_y = \emptyset$.

Definition 1.11. A topological space $(S, \mathcal{T})$ is

- separable if there exists a countable dense set
- first countable if $\forall x \in S$, there exists a countable neighbourhood base
- second countable if there exists a countable base

Proposition 1.1. Second countable implies both first countable and separable.

Proof. (Second countable implies first countable) Let $x \in S$, and let $M_x \subset \mathcal{T}$ be a neighbourhood of x. Since $\mathcal{B}$ is a base, there exists open sets $N_\alpha \in \mathcal{B}$ such that $\bigcup_\alpha N_\alpha = M_x^o$. Observe that there exists some N_α such that $x \in N_\alpha$, whence second countable.
(Second countable implies separable) For each $B \in \mathcal{B}$, choose some $x_B \in B$, and let $D = \bigcup_B x_B$. Suppose that $\overline{D} \neq S$, then $\overline{D}^c$ is open. Since $\mathcal{B}$ is a base, there exists some $B \in \mathcal{B}$ such that $B \subset \overline{D}^c$. Contradiction. □

Definition 1.12. A sequence $\{x_n\}_{x \in \mathbb{N}}$ in $(S, \mathcal{T})$ is convergent if $\exists x \in S$ such that for any neighbourhood of x, there exists some $N \in \mathbb{N}$ such that $x_n \in N_x$ for all $n > N$.