Math 539 Notes

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Contents

1 Introduction

Motivating questions (some statistics):

- the "probability" that a random number has some property
- the "distribution" of some given multiplicative/additive function

Idea: we can answer the question for $\{1, ..., \lfloor x \rfloor\}$ for some parameter x. Then, take the limit $x \to \infty$ for all natural numbers.

1.1 Notation

Let $g(x) \geq 0$.

Definition 1.1. $O(g(x))$ means some unspecified function $u(x)$ such that $|u(x)| \leq cg(x)$ for some constant $c > 0$.

Example 1.2. Show that $e^{2x} - 1 = 2x + O(x^2)$ for $x = [-1, 1]$.

Proof. Observe that $f(z) = e^{2z} - 1 - 2z$ is analytic (and entire) and has a double zero at $z = 0$ (one can check that $f(z) = f'(z) = 0$. Hence, $g(z) = (e^{2z} - 1 - 2z)/z^2$ has a removable singularity at $z = 0$, whence g is analytic and entire. Let $C = \max\{|g(z)| : |z| \leq 1\}$. Then

$$|g(z)| \leq C \implies \left|e^{2z} - 1 - 2z\right| \leq C\left|z^2\right| \implies e^{2z} - 1 - 2z = O(|z|^2).$$

□

Exercise 1.3. Show that $\sqrt{x+1} = \sqrt{x} + O(1/\sqrt{x})$ for $x \in [1, \infty)$.

Definition 1.4. $f(x) \ll g(x)$ means $f(x) = O(g(x))$.

Exercise 1.5. Suppose that $f_1 \ll g_1, f_2 \ll g_2$, then $f_1 + f_2 \ll \max\{g_1, g_2\}$. ✓

Exercise 1.6. Let f, g be continuous on $[0, \infty)$, and $f \ll g$ on $[123, \infty)$. Show that $f \ll g$ on $[0, \infty)$. ✓

Definition 1.7. $f(x) \sim g(x)$ means $\lim \frac{f(x)}{g(x)} = 1$.

Definition 1.8. $f(x) = o(g(x))$ means $\lim \frac{f(x)}{g(x)} = 0$.

Definition 1.9. $f(x) = O_y(g(x))$ means f, g depend on some parameter y, and the implicit constant depends on y.

Exercise 1.10. For any $A, \epsilon > 0$, show that $(\log x)^A \ll_{A,\epsilon} x^\epsilon$.

1.2 Riemann-Stieltjes Integral

Appendix A in the book.

Definition 1.11. Some definitions for partitions

1. Let $\underline{x} = \{x_0, ..., x_N\}$ be a partition of $[c, d]$ if $c = x_0 < \cdots < x_N = d$.

2. The mesh size $m(\underline{x}) = \max_{1 \le j \le N} x_j - x_{j-1}$.

3. Sample points $\xi_j \in [x_{j-1}, x_j]$.

Definition 1.12 (Riemann-Stieltjes Integral)**.** Given two functions $f(x)$ and $g(x)$, define the Riemann-Stieltjes integral as

$$\int_c^d f(x)\ dg(x) = \lim_{m(\underline{x}) \to 0} \sum_{j=1}^{N} f(\xi_j)(g(x_j) - g(x_{j-1})).$$

Remark 1.13. Setting $g(x) = x$ gives the Riemann integral.

Theorem 1.14. *Let $f(x)$ have bounded variation and let $g(x)$ be continuous on $[c, d]$, or vice versa. Then $\int_c^d f(x)\ dg(x)$ exists.*

Remark 1.15. If a function is piecewise monotone, then it has bounded variation.

Example 1.16. Given a sequence $a_{n n \in \mathbb{N}}$, define the summatory function $A(x) = \sum_{n \le x} a_n$. Then, on any $[c, d]$, $A(x)$ is bounded, piecewise continuous and piecewise monotone. Hence, the Riemann-Stieltjes integral exists when g is continuous.

Remark 1.17. We present 3 facts that we will use.

1. If $A(x)$ is the summatory function as above, and $f(x)$ is continuous, then

$$\int_c^d f(x)\ dA(x) = \sum_{c < n \le d} a_n f(n).$$

2. (Integration by parts). If the integrals exist, then

$$\int_c^d f(x)\ dg(x) = f(x)g(x)|_c^d - \int_c^d g(x)\ df(x).$$

3. If $f(x)$ is continuously differentiable, then

$$\int_c^d g(x)\ df(x) = \int_c^d g(x)f'(x)\ dx.$$

Example 1.18 (Summation by parts)**.** Consider $\sum_{n\leq y} \frac{a_n}{n}$. Let $f(x) = 1/x$, then we can write

$$\sum_{n\leq y} \frac{a_n}{n} = \sum_{n\leq y} a_n \cdot \frac{1}{n} = \int_0^y \frac{1}{x}\ dA(x) = \frac{1}{x}A(x)\bigg|_0^y - \int_0^y A(x)\ d\left(\frac{1}{x}\right) = \frac{A(y)}{y} - \int_0^y A(x)\frac{1}{x^2}\ dx.$$

The final manipulation that we want to get is

$$\sum_{n\leq y} a_n f(n) = A(y)f(y) - \int_0^y A(x)f'(x)\ dx. \tag{1}$$

2 Dirichlet Series

A Dirichlet series is $\sum_{n=1}^{\infty} n^{-s}$.
Facts about Dirichlet series:

- converge in some right half-plane $\{s \in \mathbb{C} : \Re s > R\}$ for some R (possibly $R = \pm\infty$).
- Sometimes converge conditionally. Example: $\sum_{n=1}^{\infty}(-1)^n/n^{1/2}$.
- $\left(\sum_{n=1}^{\infty} a_n n^{-s}\right)\left(\sum_{n=1}^{\infty} b_n n^{-s}\right) = \sum_{n=1}^{\infty} c_n n^{-s}$ where $c = \sum_{de=n} a_d b_e = \sum_{d|n} a_d b_{e/d}$. (multiplicative convolution)

Some notation: for $s \in \mathbb{C}$, we write $s = \sigma + it$, that is σ is the real part of s, and t is the imaginary part of s. Note that if $x > 0$, then $|x^s| = |x^\sigma|\left|x^{it}\right| = |x^\sigma|\left|e^{it\log x}\right| = |x^\sigma|$.

Theorem 2.1. *Let $\alpha(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series. Suppose that $s_0 \in \mathbb{C}$ is such that $\alpha(s_0)$ converges. Then $\alpha(s)$ converges uniformly in the sector $S = \{s \in \mathbb{C} : \sigma \geq \sigma_0, |t - t_0| \leq H|\sigma - \sigma_0|\}$ for any $H > 0$.*

Proof. WLOG, let $s_0 = 0$, otherwise we can do a change of variables.
Let $A(x) = \sum_{n\leq x} a_n = \alpha(0) - R(x)$. Then, for $\sigma > 0$,

$$\begin{aligned}
\sum_{M<n\leq N} a_n n^s &= \int_M^N x^{-s}\ dA(x) = \int_M^N x^{-s}\ d(\alpha(0) - R(x)) \\
&= \int_M^N x^{-s}\ d\alpha(0) - \int_M^N x^{-s}\ dR(x) = -\int_M^N x^{-s}\ dR(x) \\
&= -x^{-s}R(x)\big|_M^N + \int_M^N R(x)\ d(x^{-s}) \\
&= R(M)M^{-s} - R(N)N^{-s} - s\int_M^N R(x)x^{-s-1}\ dx.
\end{aligned}$$

Note that $R(N)N^{-s} \to 0$ as $N \to \infty$, and that $R(x)x^{-s-1} \ll x^{-\sigma-1}$. Hence, letting $N \to \infty$ gives

$$\sum_{M<n} a_n n^{-s} = R(M)M^{-s} - s\int_M^\infty R(x)x^{-s-1}\ dx \to 0 \text{ as } M \to \infty.$$

Now, choose M large such that $|R(x)| < \epsilon$ for all $x \geq M$. Then,

$$\begin{aligned}\left|\sum_{n>M} a_n n^{-s}\right| &\leq \epsilon M^{-\sigma} + |s| \int_M^\infty \epsilon x^{-\sigma-1}\, dx \\ &= \epsilon M^{-\sigma} + |s|\, \epsilon x^{-\sigma} \frac{1}{-\sigma}\Big|_M^\infty \\ &= \epsilon M^{-\sigma} + |s|\, \epsilon \frac{M^{-\sigma}}{\sigma} = \frac{\epsilon}{M^\sigma}\left(1 + \frac{|s|}{\sigma}\right).\end{aligned}$$

Since $s \in S$, we have

$$|s| = \sqrt{\sigma^2 + t^2} \leq \sqrt{\sigma^2 + |H\sigma|^2} = \sigma\sqrt{1+H^2},$$

so $\left|\sum_{n>M} a_n n^{-s}\right| \leq \epsilon(1 + \sqrt{1+H^2})$ as $M \to \infty$. Observe that the latter only depends on H, so the convergence is uniform. □

Corollary 2.2. *If $\alpha(s_0)$ converges, then $\alpha(s)$ converges for all s with $\sigma > \sigma_0$.*

Corollary 2.3. *If $\alpha(s_0)$ diverges, then $\alpha(s)$ diverges for all s with $\sigma < \sigma_0$.*

Remark 2.4. The Dirichlet series $\alpha(s)$ has an abscissa of convergence σ_c such that $\alpha(s)$ converges if $\sigma > \sigma_c$, and diverges if $\sigma < \sigma_c$. It is allowed to have $\sigma_c = \pm\infty$. Furthermore, $\alpha(s)$ converges locally uniformly right of σ_c, whence $\alpha(s)$ is analytic.

Remark 2.5. Observe that $\int_1^N x^{-s}\, dA(x) = \sum_{1<n\leq N} a_n n^{-s} = \sum_{n=2}^N a_n n^{-s}$. Sometimes we write $\int_{-1}^N$ to include the 1.

Theorem 2.6. *Let $\alpha(s) = \sum_{n=1}^\infty a_n n^{-s}$ have an abscissa of convergence $\sigma_c \geq 0$. Then for $\sigma > \sigma_c$, we have $\alpha(s) = s\int_1^\infty A(x)x^{-s-1}\, dx$. Moreover,*

$$\limsup_{x\to\infty} \frac{\log|A(x)|}{\log x} = \sigma_c.$$

Proof. Observe that

$$\begin{aligned}\sum_{n=1}^N a_n n^{-s} &= \int_{1^-}^N x^{-s}\, dA(x) = x^{-s}A(x)\big|_{1^-}^N - \int_{1^-}^N A(x)\, d(x^{-s}) \\ &= A(N)N^{-s} - \int_{1^-}^N A(x)(-sx^{-s-1}\, dx) = A(N)N^{-s} + s\int_1^N A(x)x^{-s-1}\, dx.\end{aligned}$$

Observe that in the last line, we can replace 1^- with 1 because the integrand is bounded.
Define $\phi = \limsup_{x\to\infty} \frac{\log|A(x)|}{\log x}$. We compare this to σ_c.
Let $\sigma = \phi + \epsilon$ for some $\epsilon > 0$. Then $\frac{\log|A(x)|}{\log x} < \phi + \frac{\epsilon}{2}$ for large x, so $A(x) \ll x^{\phi+\epsilon/2}$. Then, $A(N)N^{-s} \ll N^{\phi+\epsilon/2}N^{-(\phi+\epsilon)} = N^{-\epsilon/2}$. Hence,

$$\int_N^\infty A(x)x^{-\sigma-1}\, dx \ll \int_N^\infty x^{-\phi+\epsilon/2}x^{-(\phi+\epsilon+1)}\, dx = \int_N^\infty x^{-1-\epsilon/2}\, dx \ll N^{-\epsilon/2}.$$

It follows that

$$\sum_{n=1}^N a_n n^{-s} = O(N^{-\epsilon/2}) + s\left(\int_1^\infty A(x)x{-}s-1\, dx + O(N^{-\epsilon/2})\right).$$

Let $N \to \infty$ gives $s \int_1^\infty A(x)x^{-s-1}\, dx$ converges. Hence $\sigma_c \le \phi$.
Conversely, let $\sigma_0 = \sigma_c + \epsilon$, and let $R_0(x) = \sum_{n>x} a_n n^{-\sigma_0} = \alpha(\sigma_0) - \sum_{n \le x} a_n n^{-\sigma_0}$. Observe that

$$A(N) = -R_0(N)N^{\sigma_0} + \sigma_0 \int_0^N R_0(x)x^{\sigma_0 - 1}\, dx.$$

Since $\alpha(0)$ converges, $R_0(x) = o(1)$ so $R_0(x) \ll 1$. Then

$$A(N) \ll 1 \cdot N^{\sigma_0} + \sigma_0 \int_0^N 1 \cdot x^{\sigma_0 - 1}\, dx = N^{\sigma_0} + N^{\sigma_c} \ll N^{\sigma_0} = N^{\sigma_c + \epsilon}.$$

Hence $\frac{\log |A(x)|}{\log x} \ll \frac{(\sigma_c + \epsilon)\log x}{\log x} = \sigma_c + \epsilon$, so $\phi \le \sigma_c + \epsilon$. Take $\epsilon \to 0$, so $\phi \le \sigma_c$. $\square$