

Math 421 Notes

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1 Topological Spaces

Definition 1.0.1. A topological space (S, \mathcal{T}) is a nonempty set with a family of subsets \mathcal{T} such that

1. $\emptyset \in \mathcal{T}$
2. $S \in \mathcal{T}$
3. \mathcal{T} is closed under finite intersections and arbitrary unions

Examples: $\{\emptyset, S\}$ (indiscrete topology), 2^S (discrete topology).

A metric on a metric space defines a topology. Not all topologies have a corresponding metric. A topology is called metrizable if we can define a metric such that “open” has the same meaning. Topologies can be partially ordered. $\mathcal{T}_1 \prec \mathcal{T}_2$ if $\mathcal{T}_1 \subset \mathcal{T}_2$ as sets. Denote $\mathcal{T}(\mathcal{E})$ to be the topology generated by $\mathcal{E} \subset 2^S$.

Definition 1.0.2. A base of \mathcal{T} is a family $\mathcal{B} \subset \mathcal{T}$ such that for any nonempty open set $O \in \mathcal{T}$, there exists a collection $\{B_\alpha : B_\alpha \in \mathcal{B}\}$ such that $O = \bigcup \alpha B_\alpha$.

Definition 1.0.3. Let (S, \mathcal{T}) be a topological space and $X \subset S$. Then, $\mathcal{T}_x = \{O \cap X : O \in \mathcal{T}\}$ is the relative topology (X, \mathcal{T}_x) .

Definition 1.0.4. A set X is closed if $\exists Y \in \mathcal{T}$ such that $X = Y^c$.

Definition 1.0.5. The interior of X is the largest open set $X^\circ \subset X$.

Definition 1.0.6. The closure of X is the smallest closed set $\overline{X} \supset X$.

Definition 1.0.7. The boundary of X is $\overline{X} \setminus X^\circ$.

Definition 1.0.8. A neighbourhood of $x \in S$ is a set $N_x \subset S$ such that $x \in N_x^\circ$

Definition 1.0.9. A neighbourhood base of x is a family \mathcal{N}_x such that each $N \in \mathcal{N}_x$ is a neighbourhood of x and for any neighbourhood M_x , there exists some $N \in \mathcal{N}_x$ such that $N \subset M_x$.

Definition 1.0.10 (Classification of topological spaces). A topological space is called T_2 or Hausdorff if $\forall x, y \in S, x \neq y$, there exists $O_x, O_y \in \mathcal{T}$ such that $x \in O_x, y \in O_y$, and $O_x \cap O_y = \emptyset$.

Definition 1.0.11. A topological space (S, \mathcal{T}) is

- separable if there exists a countable dense set
- first countable if $\forall x \in S$, there exists a countable neighbourhood base
- second countable if there exists a countable base

Proposition 1.0.12. Second countable implies both first countable and separable.

Proof. (Second countable implies first countable) Let $x \in S$, and let $M_x \subset \mathcal{T}$ be a neighbourhood of x . Since \mathcal{B} is a base, there exists open sets $N_\alpha \in \mathcal{B}$ such that $\bigcup_\alpha N_\alpha = M_x$. Observe that there exists some N_α such that $x \in N_\alpha$, whence second countable.

(Second countable implies separable) For each $B \in \mathcal{B}$, choose some $x_B \in B$, and let $D = \bigcup_B x_B$. Suppose that $\overline{D} \neq S$, then \overline{D}^c is open. Since \mathcal{B} is a base, there exists some $B \in \mathcal{B}$ such that $B \subset \overline{D}^c$. Contradiction. \square

Definition 1.0.13. A sequence $\{x_n\}_{n \in \mathbb{N}}$ in (S, \mathcal{T}) is convergent if $\exists x \in S$ such that for any neighbourhood of x , there exists some $N \in \mathbb{N}$ such that $x_n \in N_x$ for all $n > N$.

Proposition 1.0.14. Let (S, \mathcal{T}) be a first countable topological space, and $X \subset S$. Then $x \in \overline{X}$ if and only if x is the limit point of a convergent sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$.

Proof. Let $\mathcal{N}_x = \{O_n : n \in \mathbb{N}\}$ be a countable neighbourhood base of x such that $O_n \subset O_{n-1}$ for all $n \in \mathbb{N}$. If $x \in \overline{X}$, then $O_n \cap X \neq \emptyset$ for all $n \in \mathbb{N}$. Then we can pick $x_n \in O_n \cap X$, whence $x_n \rightarrow x$. Converse is similar. \square

Definition 1.0.15. Let $(S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)$ be topological spaces. A function $f : S_1 \rightarrow S_2$ is continuous if $f^{-1}(O) \in \mathcal{T}_1$ for any $O \in \mathcal{T}_2$. I.e. the preimage of any open set is open.

Definition 1.0.16. Let $(S_1, \mathcal{T}_1), (S_2, \mathcal{T}_2)$ be topological spaces. A function $f : S_1 \rightarrow S_2$ is open if $f(O) \in \mathcal{T}_2$ for any $O \in \mathcal{T}_1$.

Definition 1.0.17. A homeomorphism is an invertible function that is open and continuous.

Definition 1.0.18. Let S_1 be a set and let (S_2, \mathcal{T}_2) be a topological space. Let \mathcal{F} be a family of functions from S_1 to S_2 . Then, the topology on S_1 generated by $\{f^{-1}(O) : O \in \mathcal{T}_2\}$ is called the \mathcal{F} -weak topology.

Remark 1.0.19. By definition, all functions $f \in \mathcal{F}$ are continuous with respect to the above topology on S_1 .

Example 1.0.20. Let $S_1 = C([a, b]; \mathbb{R})$ be the set of continuous functions, and let $S_2 = \mathbb{R}$ with the usual metric topology. Let $E_x : S_1 \rightarrow S_2$ where $E_x(f) = f(x)$ be the evaluation functions, and let $\mathcal{F} = \{E_x : x \in [a, b]\}$. The \mathcal{F} -weak topology on $C([a, b]; \mathbb{R})$ is the topology of pointwise convergence.

Definition 1.0.21. A topological space (S, \mathcal{T}) is compact if any open cover has a finite subcover.

Definition 1.0.22. A subset $X \subset S$ is compact if it is compact in the relative topology.

Definition 1.0.23. A subset $X \subset S$ is precompact if its closure is compact.

Definition 1.0.24. We say that (S, \mathcal{T}) has the finite intersection property if for any family of closed sets \mathcal{C} such that $\bigcap_{i=1}^n C_i \neq \emptyset$ for any finite subfamily $\{C_1, \dots, C_n\}$ also satisfies $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

Exercise 1.0.25. S is compact if and only if it has the finite intersection property.

Proposition 1.0.26. Let $X \subset S$ be a subset of a compact topological space (S, \mathcal{T}) . Then X is compact if X is closed.

Proof. Let \mathcal{C} be an open cover of X . Let $\mathcal{C}' = \mathcal{C} \cup \{X^c\}$ be an open cover of S . There exists a finite subcover of \mathcal{C}' , so there exists a finite subcover of X (we can safely remove X^c from the finite subcover of S as $X \cap X^c = \emptyset$). \square

Proposition 1.0.27. Let (S_1, \mathcal{T}_1) and (S_2, \mathcal{T}_2) , and let $f : S_1 \rightarrow S_2$ be continuous. If S_1 is compact, then $f(S_1) \subset S_2$ is compact.

Proof. Let \mathcal{C} be an open cover of $f(S_1)$. Let $\mathcal{C}' = \{f^{-1}(C) : C \in \mathcal{C}\}$ be an open cover of S_1 (preimages of open sets are open by continuity). Hence, there exists a finite subcover of S_1 , from which we get a finite subcover of $f(S_1)$. \square

Proposition 1.0.28 (Bolzano-Weierstrass property). A second countable topological space is compact if and only if every sequence has a convergent subsequence.

Proof. Suppose that S is compact, and suppose, for contradiction, that $\{z_n\}_{n \in \mathbb{N}}$ does not have a convergent subsequence. Since S is first countable, this means that for any $x \in S$, there exists some neighbourhood O_x of x and some $N_x \in \mathbb{N}$ such that $z_n \notin O_x$ for all $n > N_x$. Let $\mathcal{C} = \{O_x^o : x \in S\}$ be an open cover of S . Since S is compact, there exists some finite subcover $\mathcal{C}' = \{O_{x_1}^o, \dots, O_{x_m}^o\}$. Then, let $N = \max\{n_{x_1}, \dots, n_{x_m}\}$, whence $z_n \notin \bigcup_i O_{x_i} = S$ for all $n > N$, which is a contradiction. Suppose that every sequence of S has a convergent subsequence. Since S is second countable, there exists a countable open cover $\mathcal{C} = \{O_i : i \in \mathbb{N}\}$. Suppose, for contradiction, that \mathcal{C} has no finite subcover. Then, for any $i \in \mathbb{N}$, there exists some $x_i \notin \bigcup_{j=1}^i O_j$. Let $\{x_{n_i}\}_{i \in \mathbb{N}}$ be a convergent subsequence and let x be its limit. Since \mathcal{C} is a cover, there exists j such that $x \in O_j$. It follows that there exists some $N \in \mathbb{N}$ such that $x_{n_k} \in O_j$ for all $k > N$, which is a contradiction. \square

1.1 Weierstrass Theorems

Theorem 1.1.1 (“Classical” Weierstrass). *If f is a continuous function on $[a, b]$, then there exists a sequence of polynomials $\{P_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} P_n = f$ uniformly.*

Remark 1.1.2. This theorem implies that the set of polynomials is dense in $C_{\mathbb{R}}([a, b])$ (real-valued continuous functions).

Definition 1.1.3. Let X be compact and Hausdorff, and let $C_{\mathbb{R}}(X)$ be the set of real-valued continuous functions on X equipped with pointwise multiplication: $(fg)(x) = f(x)g(x)$. This is an algebra.

Definition 1.1.4. An algebra $\mathcal{A} \subset C_{\mathbb{R}}(X)$ separates points if for any $x \neq y$ in X , there exists an $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Theorem 1.1.5 (Stone-Weierstrass). *Let X be a compact Hausdorff space. Let \mathcal{A} be a closed subalgebra (wrt $\|\cdot\|_{\infty}$) of $C_{\mathbb{R}}(X)$ that separates points. Then, either $\mathcal{A} = C_{\mathbb{R}}(X)$ or $\exists x_0 \in X$ such that $\mathcal{A} = \{f \in C_{\mathbb{R}}(X) : f(x_0) = 0\}$.*

Remark 1.1.6. If \mathcal{A} separates points and $1 \in \mathcal{A}$, we must have $\overline{\mathcal{A}} = C_{\mathbb{R}}(X)$. Hence, any unital subalgebra \mathcal{A} of $C_{\mathbb{R}}(X)$ is dense.

Definition 1.1.7. Let f and g be functions on the same domain. Write $f \wedge g = \min\{f, g\}$ and $f \vee g = \max\{f, g\}$.

Definition 1.1.8. A family $\mathcal{F} \subset C_{\mathbb{R}}(X)$ is a lattice if any functions $f, g \in \mathcal{F}$, we have $f \wedge g \in \mathcal{F}$ and $f \vee g \in \mathcal{F}$.

Lemma 1.1.9. Any closed unital subalgebra $\mathcal{A} \subset C_{\mathbb{R}}(X)$ is a lattice.

Proof. Observe that

$$\begin{aligned} f \vee g &= \frac{1}{2}|f - g| + \frac{1}{2}(f + g) \\ f \wedge g &= -((-f) \vee (-g)), \end{aligned}$$

so it suffices to show that $f \in \mathcal{A}$ means $|f| \in \mathcal{A}$. By the classical Weierstrass theorem, there is a sequence of polynomials $\{P_n\}_{n \in \mathbb{N}}$ such that $|P_n(x) - |x|| < \frac{1}{n}$ for all $x \in [-1, 1]$. Hence,

$$\|P_n(h) - |h|\|_{\infty} < \frac{1}{n} \text{ where } h = \frac{f}{\|f\|_{\infty}},$$

that is $P_n(h) \rightarrow |h|$ uniformly. Note that $P_n(h) \in \mathcal{A}$, so we are done. \square

Proposition 1.1.10 (Kakutani-Klein). If \mathcal{L} is a closed lattice that separates points such that $1 \in \mathcal{L}$. Then, $\mathcal{L} = C_{\mathbb{R}}(X)$.

Proof. Let $g \in C_{\mathbb{R}}(X)$, let $\varepsilon > 0$, and let $x \neq y \in X$. The map $\phi : \mathcal{L} \rightarrow \mathbb{R}^2$ such that $h \mapsto (h(x), h(y))$ is an algebra homomorphism. The image contains $(1, 1)$ since $1 \in \mathcal{L}$. The image also contains (a, b) with $a \neq b$ since \mathcal{L} separates points. It suffices to look at the subalgebras of \mathbb{R}^2 , whence the image is all of \mathbb{R}^2 . Now, there exists $f_{xy} \in \mathcal{L}$ such that $f_{xy}(x) = g(x)$ and $f_{xy}(y) = g(y)$. By continuity, there exists an open neighbourhood N_y of y such that $f_{xy}(z) + \varepsilon = g(z)$ for all $z \in N_y$. By compactness, there exists a finite subcover $\{N_{y_1}, \dots, N_{y_m}\} \subset \{N_y : y \in X\}$. Let $f_x = \max\{f_{xy_1}, \dots, f_{xy_m}\} \in \mathcal{L}$. Note that $f_x(x) = g(x)$ and $f_x(z) > g(z) - \varepsilon$ for all $z \in X$. This gives a lower bound. We can also find an upper bound whence there exists some $f \in \mathcal{L}$ such that $\|f - g\|_{\infty} < 2\varepsilon$. \square

The Stone-Weierstrass Theorem has two generalizations.

1. It extends to complex-valued functions, provided the subalgebras are also closed under conjugation. Any $f \in C_{\mathbb{C}}(X)$ can be written as $f = (f + \bar{f})/2 - i(f - \bar{f})/2$.
2. It extends to locally compact Hausdorff (LCH) spaces, that is topological spaces S such that every $x \in S$ has a compact neighbourhood. Here, the relevant algebra is the set of functions that vanish at infinity, that is $f \in C_{\mathbb{R}}(S)$ such that $\forall \epsilon > 0$, the set $\{x \in S : |f(x)| \geq \epsilon\}$ is compact. Hence, consider $S \cup \{\infty\}$.

1.2 Hausdorff Spaces

Definition 1.2.1. A space S is locally compact if each $x \in S$ has a compact neighbourhood.

Lemma 1.2.2. Let (S, \mathcal{T}) be Hausdorff. Let $\{x_n\}_{n \in \mathbb{N}}$ be a convergent sequence in S . Then the limit $x = \lim_{n \rightarrow \infty} x_n$ is unique.

Proof. Let $y \neq x$. By Hausdorff, there exists disjoint open sets O_x and O_y such that $x \in O_x$ and $y \in O_y$. Since $x_n \rightarrow x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in O_x$ for all $n > n_0$. Hence $x_n \notin O_y$, so $\{x_n\}_{n \in \mathbb{N}}$ does not converge to y . \square

Lemma 1.2.3. Let (S, \mathcal{T}) be Hausdorff, and let $K \subset S$ be compact. For any $x \notin K$, there are open disjoint sets U, V such that $x \in U$ and $K \subset V$.

Proposition 1.2.4. Let (S, \mathcal{T}) be Hausdorff, and let $K \subset S$ be compact. Then K is closed.

Theorem 1.2.5. Let (S_1, \mathcal{T}_1) and (S_2, \mathcal{T}_2) be two compact Hausdorff spaces, and let $f : S_1 \rightarrow S_2$. If f is continuous and bijective, then f is a homeomorphism.

Proposition 1.2.6. Let S be a LCH space. Let $K \subset U \subset S$, where K is compact and U is open. Then, there is an open set O with compact closure such that

$$K \subset O \subset \overline{O} \subset U.$$

Proof. Since S is LCH, every point of K has an open neighbourhood with compact closure. Since K is compact, there is a finite subcover of such neighbourhoods. Hence K is a subset of their union V which has a compact closure. If $U = S$, then $O = V$. Otherwise, U^c nonempty. By Hausdorff, for any $x \in U^c \subset K^c$, there exists an open set O_x such that $K \subset O_x$ and $x \notin \overline{O_x}$. It follows that

$$\bigcap_{x \in U^c} U^c \cap \overline{V O_x} = \emptyset.$$

By the finite intersection property, there are finitely many $\{x_1, \dots, x_n\}$ such that

$$U^c \cap \overline{V} \cap \overline{O_{x_1}} \cap \dots \cap \overline{O_{x_n}} = \emptyset.$$

Let $O = V \cap O_{x_1} \cap \dots \cap O_{x_n}$. □

Definition 1.2.7. The support of a complex-valued function f is $\overline{\{x \in S : f(x) \neq 0\}}$.

Definition 1.2.8. Let $C_c(S)$ be the set of compactly supported continuous functions on S . Write $K \prec f$ for a compact set K and a function $f \in C_c(S)$ such that $0 \leq f(x) \leq 1$ for all $x \in S$ and $f(x) = 1$ for all $x \in K$. Write $f \prec U$ for an open set U and a function $f \in C_c(S)$ such that $0 \leq f(x) \leq 1$ for all $x \in S$ and the support of f is a subset of U .

Lemma 1.2.9 (Urysohn's Lemma). Let S be a LCH space. Let $K \subset U \subset S$, where K is compact and U is open. There exists a function $f \in C_c(S)$ such that $K \prec f \prec U$.

Proof. We have a family of open sets $\{O_r : r \in \mathbb{Q} \cap [0, 1]\}$ with compact closures such that $K \subset O_1$, $\overline{O_0} \subset U$, and $\overline{O_s} \subset O_r$ for $s > r$. Define

$$f_r(x) = r\chi_{O_r} = \begin{cases} r & \text{if } x \in O_r \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g_s(x) = s + (1-s)\chi_{\overline{O_s}} = \begin{cases} 1 & \text{if } x \in \overline{O_s} \\ s & \text{otherwise} \end{cases}.$$

Also define

$$f(x) = \sup \{f_r(x) : r \in \mathbb{Q} \cap [0, 1]\} \quad \text{and} \quad g(x) = \inf \{g_s(x) : s \in \mathbb{Q} \cap [0, 1]\}.$$

Observe that $\{x : f(x) > a\}$ is open for all $a \in \mathbb{R}$ (lower semicontinuous), and $\{x : g(x) < a\}$ is open for all $a \in \mathbb{R}$ (upper semicontinuous). Moreover, $0 \leq f \leq 1$ and $f(x) = 1$ for all $x \in K \subset O_1$. Also, $\text{supp}(f) \subset \overline{O_0} \subset U$. It suffices to show that $f = g$. Note that $f_r(x) > g_s(x)$ if $r > s$ and $x \in O_r \cap \overline{O_s}^c$. But $r > s$ means $O_r \subset O_s$, so $f_r \leq g_s$ for all $r, s \in \mathbb{Q}$, so $f \leq g$. Suppose, for contradiction, that there exists an x such that $f(x) < g(x)$. Then, $\exists r, s \in \mathbb{Q}$ such that $f(x) < r < s < g(x)$. The first inequality implies that $x \notin O_r$, while the last implies that $x \in \overline{O_s}$, which is a contradiction since $r < s$. Therefore $f = g$, and we are done. □

Proposition 1.2.10. Let (S, \mathcal{T}) be LCH, let K be compact, and let $\{O_i : 1 \leq i \leq n\}$ be a finite open cover of K . Then there are functions $\{f_i \in C_c(S) : 1 \leq i \leq n\}$ such that

1. $f_i \prec O_i$ for $i \in [n]$.
2. $\sum_{i=1}^n f_i(x) = 1$ for all $x \in K$.

The set $\{f_i\}$ is called a partition of unity subordinate to the cover $\{O_i\}$.

Proof. Let $x \in K$, there exists i such that $x \in O_i$. There is a compact neighbourhood N_x of x such that $N_x \subset O_i$. By compactness, there exists $\{x_1, \dots, x_m\}$ such that $K \subset \bigcup_{j=1}^m N_{x_j}^o \subset \bigcup_{j=1}^m N_{x_j}$. For $i \in [n]$, define $K_i = \bigcup_{N_{x_j} \subset O_i} N_{x_j}$. By Urysohn's lemma, there exists continuous functions g_i such that $K_i \prec g_i \prec O_i$. Since $K \subset \bigcup_{i=1}^n K_i$, we have $\sum_{i=1}^n g_i \geq 1$ on K . Let $W = \{x : \sum_{i=1}^n g_i(x) > 0\}$ is open because W is the preimage of an open set, so by Urysohn's lemma, there exists f such that $K \prec f \prec W$. Let $g_{n+1} = 1 - f$. By construction, $G(x) = \sum_{i=1}^{n+1} g_i(x) > 0$ everywhere. Now, let $f_i(x) = g_i(x)/G(x)$ for $i \in [n]$. Therefore, $f_i \prec O_i$ (ie. $\text{supp}(f_i) \subset O_i$), and $\sum_{i=1}^n f_i = 1$ on K . \square

Proposition 1.2.11 (Tietze's Extension). Let (S, \mathcal{T}) be LCH, let K be compact, and let $f \in C(K)$. Then there exists $F \in C_c(S)$ such that $F(x) = f(x)$ for all $x \in K$.

Proof. Since f is continuous on a compact set, it is bounded. We assume WLOG that $|f| \leq 1$ on K . Let V be an open set with compact closure such that $K \subset V$. Let $K^+ = \{x \in K : f(x) \geq \frac{1}{3}\}$, $K^- = \{x \in K : f(x) \leq -\frac{1}{3}\}$. Note that K^+ and K^- are disjoint closed (compact) subsets of K . Applying Urysohn's Lemma to K^+ and $V \setminus K^-$, and to K^- and $V \setminus K^+$, and rescaling gives $f_1 \in C_c(S)$ such that $f_1 = \frac{1}{3}$ on K^+ , $f_1 = -\frac{1}{3}$ on K^- , and $|f_1| \leq \frac{1}{3}$ everywhere. Moreover, $\text{supp}(f) \subset V$ and $|f - f_1| \leq \frac{2}{3}$ on K . Repeat this procedure for $f - f_1$ to get f_2 such that $|f - f_1 - f_2| \leq (\frac{2}{3})^2$ on K . Hence, we obtain a sequence $\{f_n\}_{n \in \mathbb{N}}$ such that $f_n \in C_c(S)$ where $|f_n| \leq \frac{1}{3} (\frac{2}{3})^{n-1}$ and $|f - \sum_{i=1}^n f_i| \leq (\frac{2}{3})^n$ on K . Now, let $F = \sum_{i=1}^{\infty} f_i$. Note that F is convergent uniformly everywhere so F is continuous, and $F = f$ on K . Moreover, $\text{supp}(F) \subset \bar{V}$. \square