Vague Preferences and Contracts

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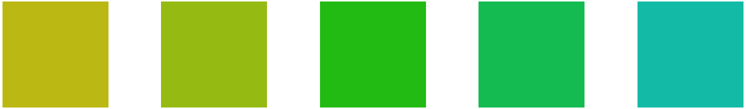
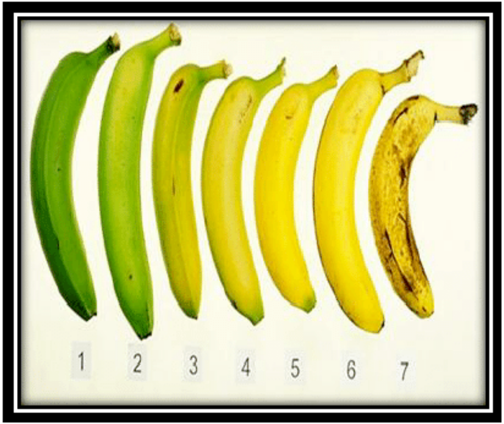
Which of the following are green:



(a) (b) (c) (d) (e)

Which of the following are ripe:



Vagueness

The language we use is less precise than the reality it describes.

- Descriptions that appear semantically crisp are in fact nebulous
- A statement that is neither (absolutely) true nor false is **vague**
- Contrasted with ‘uncertainty’: when the truth of statements is not known

Vagueness is economically relevant—a principal function of the (tort) legal system is to determine *the degree of truth*. Consider:

> *“Gilead Science’s Hepatitis-C treatment, sofosbuvir, infringes on Idenix Pharmaceutical’s patent.”*

- Essentially: Idenix claimed that the structure of sofosbuvir is based on already known molecules
- The truth of which was eventually settled by a jury to the tune of 2.5 billion dollars

This paper...

- Constructs a framework for modeling vagueness
- Provides a methodology for eliciting decision maker’s subjective beliefs
- Examines how vagueness affects strategic contracting (principal-agent style)

Language is Important

- Vagueness arises from the gap between descriptions of reality and reality itself
- It’s not clear *what* a state-space should look like, and if it would be describable in the actual language
- So, begin with a formal language, $\mathscr{L}$

Language is Important

The language $\mathscr{L}$ represents payoff relevant parameters, constructed by

- $\mathscr{P}$ a set of atomic propositions
- For any $\varphi, \psi \in \mathscr{L}$ we have
  - $\neg\varphi$: not φ
  - $\varphi \to \psi$: φ implies ψ

Language is Important

It will be notationally convenient to define several other operations from implication and negation:

- $\varphi \vee \psi =_{def} (\varphi \rightarrow \psi) \rightarrow \psi$
- $\varphi \wedge \psi =_{def} \neg(\neg\varphi \vee \neg\psi)$
- $\varphi \oplus \psi =_{def} \neg\varphi \rightarrow \psi$
- $\varphi \odot \psi =_{def} \neg(\varphi \rightarrow \neg\psi)$

Valuations

To allow for partial-truth, vagueness, fuzziness, what-have-you, we value statements via $v : \mathscr{L} \to [0, 1]$

- $v(\varphi) = 0$ indicates φ is **absolutely false**
- $v(\varphi) = 1$ indicates φ is **absolutely true**
- $v(\varphi) > v(\psi)$ indicates φ is **more true** than ψ

Valuations

A function $v : \mathscr{L} \to [0,1]$ is a valuation if

$$[\![\neg]\!] \quad v(\neg\varphi) = 1 - v(\varphi)$$

$$[\![\to]\!] \quad v(\varphi \to \psi) = \min\{1, 1 - v(\varphi) + v(\psi)\}$$

which implies:

$$[\![\vee]\!] \quad v(\varphi \vee \psi) = \max\{v(\varphi), v(\psi)\}$$

$$[\![\wedge]\!] \quad v(\varphi \wedge \psi) = \min\{v(\varphi), v(\psi)\}$$

$$[\![\oplus]\!] \quad v(\varphi \oplus \psi) = \min\{1, v(\varphi) + v(\psi)\}$$

$$[\![\odot]\!] \quad v(\varphi \odot \psi) = \max\{0, v(\varphi) + v(\psi) - 1\}$$

Valuations

- If we interpret “or” as:
  - $\vee$: maximum — “Man is evil or man is not evil”
  - $\oplus$: (truncated) summation — “The rectangle is green or its yellow”
- If v sends $\mathscr{P}$ to $\{0,1\}$ then all statements are $\{0,1\}$-valued; this is classic logic
- For classical logic $\vee$ and $\oplus$ coincide
- $\wedge$ and $\odot$ are dual

Models

A *vague model of uncertainty* is a tuple (Ω, V, μ):

- Ω is a topological state space,
- $V = \{v_\omega\}_{\omega \in \Omega}$ measurably assigns each state ω a valuation $v_\omega : \mathscr{L} \to [0, 1]$.
- μ a regular Borel probability measure over Ω.

A model allows for both vagueness (V) and uncertainty (μ).

Decision Theory

- An **act** is a function $f : \mathscr{L} \to \mathbb{R}_+$, finite support, $\sum_{\varphi \in \mathscr{L}} f(\varphi) \leq 1$.
- A **bet** x_φ is the act that maps φ to x and all other statements to 0.
- Bets are the extreme points of the set of acts
- The primitive is $\succcurlyeq$ over acts.

Decision Theory

Interpretation: Payoffs depend on truth values but contracts can only be written using the language $\mathscr{L}$

- A bet x_φ is less valuable the less true x is

  E.g., x_φ is an investment in a project, φ is the statement that the project does not infringe on intellectual property

Representation

Given a model (Ω, V, μ) and and act f define $f^V : \Omega \to [0,1]$ as

$$f^V : \omega \mapsto \sum_{\varphi \in \mathscr{L}} f(\varphi) v_\omega(\varphi).$$

The map f^V yields the 'weighted' payoff of f. (Ω, V, μ) **represents** $\succcurlyeq$ if

$$f \succcurlyeq g \iff \int_\Omega f^V \, \mathrm{d}\mu \geq \int_\Omega g^V \, \mathrm{d}\mu \tag{$\star$}$$

Representation

This model assumes linearity in both probability (expected utility) and in truth value:

- This is necessary is we want $x_\varphi + x_{\neg\varphi} = x_{TRUE}$
- Somewhat arbitrary: we can *define* a truth of $\frac{1}{2}$ to be what provides the $\frac{1}{2}$ payoff

Standard Axioms

A1 **Order** $\succcurlyeq$ is a non-trivial, continuous weak order.

A2 **Payoff Monotonicity** if f point-wise dominates g then $f \succcurlyeq g$.

A3 **Independence** $f \succcurlyeq g$ if and only if $\alpha f + (1-\alpha)h \succcurlyeq \alpha g + (1-\alpha)h$.

Axiom: Łukasiewicz Consistency

Call φ and ψ **disjoint** if $v(\varphi \odot \psi) = 0$ for any valuation v:

- Disjointness is tantamount to $\varphi \to \neg\psi$ (and vice-versa)
  - In classical logic, φ and ψ can never be true at the same time
  - Allowing for vagueness: the more true φ is, the less true ψ must be

Axiom: Łukasiewicz Consistency

A4 **Łukasiewicz Consistency** If φ and ψ are disjoint then:

$$\frac{1}{2}_{\varphi} + \frac{1}{2}_{\psi} \sim \frac{1}{2}_{\varphi \oplus \psi},$$

Theorem

$\succcurlyeq$ satisfies A1-4 if and only if it is represented by some vague model of uncertainty. Moreover this model is unique up-to isomorphism.

Contracting — Principal Agent

Principal Agent Model

- Let Ω denote a state-space, the states of which are associated with the various outcomes of the project.
- Agent chooses unobservable $e \in E$;
  - agent pays a utility cost $c(e) \in \mathbb{R}$,
  - induces μ_e, distribution over Ω
- The agent's continuously differentiable and strictly monotone utility index over money is $u : \mathbb{R} \to \mathbb{R}$: her ex-post utility is $u(x) - c(e)$.
- Outside option is $\bar{u} \in \mathbb{R}$.

Principal Agent Model

Departure from the standard model: Ω is not directly contactable.

- There exists a language, $\mathscr{L}$ and a valuation $V = \{v_\omega\}_{w \in \Omega}$
- The Principal must write an actual (linguistic) contract
- Each contract induces a function $f : \Omega \to \mathbb{R}$,
- Not all such functions might be induceable — each ‘contract writing technology’ is associated with $C \subseteq \mathbb{R}^\Omega$

Principal Agent Model

Say that a contract $f \in \mathbb{R}^{\Omega}$ *implements* $e \in E$ if

$$e = \arg\max_{e' \in E} \int_{\Omega} u \circ f \, \mathsf{d}\mu_{e'} - c(e') \tag{IC}$$

$$\int_{\Omega} u \circ f \, \mathsf{d}\mu_{e} \geq \bar{u} \tag{IR}$$

Let $\mathscr{L}$ be constructed from $\mathscr{P} = \mathbf{p}_1 \dots \mathbf{p}_n$, and consider the set of contracts:

$$C^{\mathscr{P}} = \left\{ x v_\omega(\mathbf{p}) + y v_\omega(\neg\mathbf{p}) \mid x, y \in \mathbb{R}, \mathbf{p} \in \mathscr{P} \right\}.$$

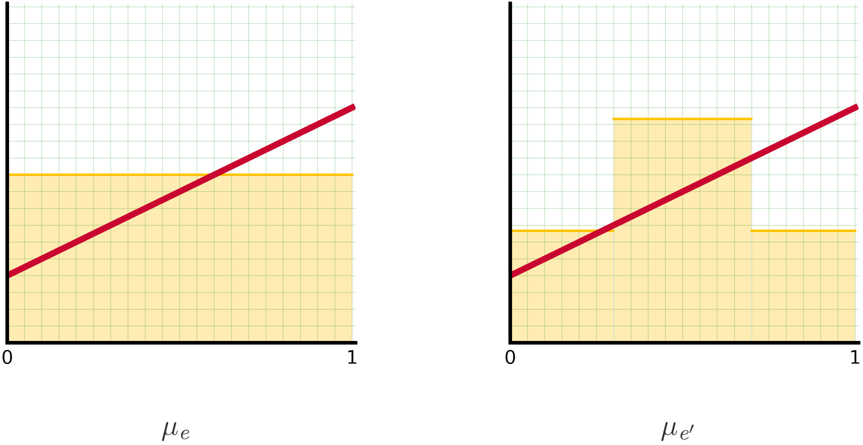
- $f \in C^{\mathscr{P}}$ is predicated directly on the truth of the propositions in $\mathscr{P}$.
- $x v_\omega(\mathbf{p}) + y v_\omega(\neg\mathbf{p})$ induces the affine function $f : \omega \mapsto (x - y) v_\omega(\mathbf{p}) + y$
- $C^{\mathscr{P}}$ is the set of all affine contracts

There are outcomes implementable by direct contracts (i.e., continuous functions over Ω) not implementable by C^P. This is obvious; for example:

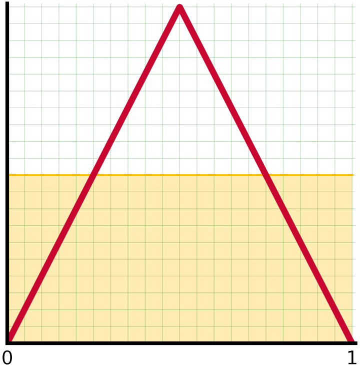
- Single $\mathbf{p}$, $\Omega = [0, 1]$ is the truth of $\mathbf{p}$.
- $E = \{e, e'\}$:
  - e induces uniform measure, less costly
  - e' concentrates probability symmetrically around $\frac{1}{2}$, more costly
- u is linear

Distribution of Outcomes

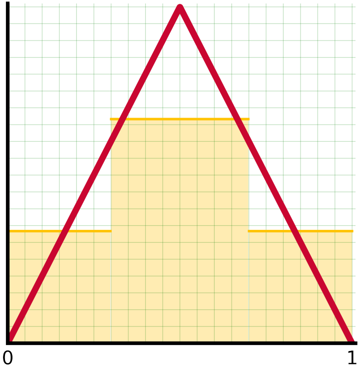
Contracts in $C^{\mathscr{P}}$ cannot implement e'



Contracts not in $C^{\mathscr{P}}$ can implement e'



μ_e



$\mu_{e'}$

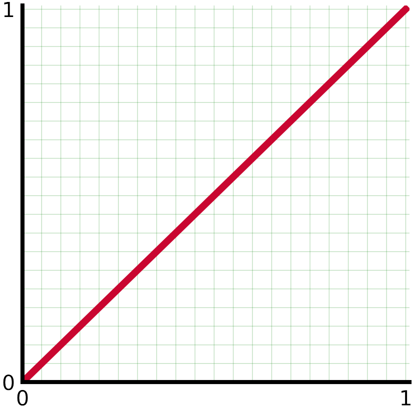
What about a richer set of linguistic contracts:

$$C^{\mathscr{L}} = \{ x v_\omega(\varphi) + y v_\omega(\neg\varphi) \mid x, y \in \mathbb{R}, \varphi \in \mathscr{L} \}.$$

- $f \in C^{\mathscr{L}}$ is predicated directly on compound statements.
- Still constructable contracts, but over more complex language

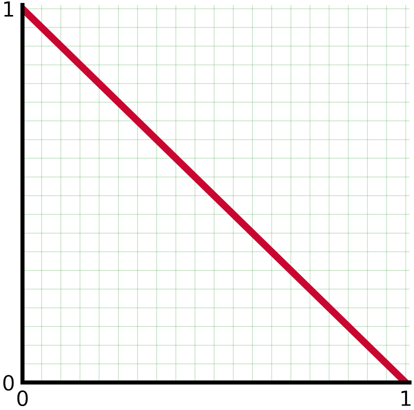
$$\mathbf{p} \mapsto 1$$

$$\omega \mapsto \omega$$



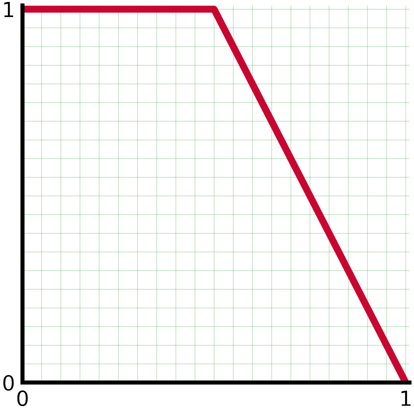
$$\neg\mathbf{p} \mapsto 1$$

$$\omega \mapsto 1 - \omega$$

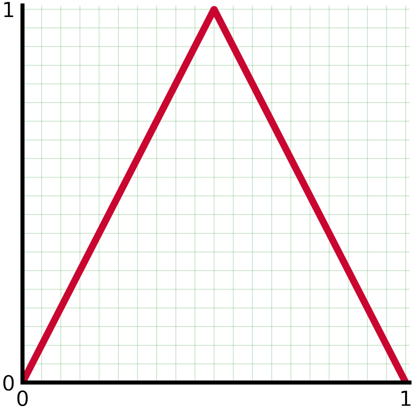


$$(\mathbf{p} \to \neg\mathbf{p}) \mapsto 1$$

$$\omega \mapsto \min\{1, 1 - \omega + (1 - \omega)\} = \min\{1, 2 - 2\omega\}.$$



$$
\begin{aligned}
(\mathbf{p} \to \neg\mathbf{p}) \wedge (\neg\mathbf{p} \to \mathbf{p}) &\mapsto 1 \\
\omega &\mapsto \min\left\{\min\{1, 1-\omega+(1-\omega)\}, \min\{1, 1-(1-\omega)+\omega\}\right\} \\
&= \min\{1, 2-2\omega, 2\omega\}.
\end{aligned}
$$



Theorem

$C^{\mathscr{L}}$ implements the same outcomes, at the same cost to the principal, as the set of all direct contracts (continuous maps from Ω).

- Subject to some mild regularity conditions, of course
- Even though the Principal cannot directly condition on Ω, the calculus of $\mathscr{L}$ is rich enough to approximate