ECON 1200: Game Theory

Evan Piermont

Summer 2015

Contents

What is game theory

- ▷ What is game theory?
  - ▷ The formal analysis of strategic interaction between players (note, we refer to the agents within the model as players, and the model itself as a game, but these references should not be taken too seriously). What makes the interactions *strategic* is that each player's actions affect not only her own outcome, but also the outcome of other players.
  - ▷ That is, we will analyze (and try to predict) which actions players will choose, keeping in mind that one player's action may affect the action that another person takes.
  - ▷ This simple idea leads to many moving parts: for instance, how does the anticipation of this effect change the action of the first player; does the acknowledgement that this change in behavior itself change how the other players react to the first players actions, etc. One of the main considerations in game theory is this feedback loop. From this observation, we will see that the main tools in game theory is the notion of an *equilbrium* where players' actions, and their actions' effect on other players' actions are in balance.
  - ▷ Game theory is a collection of models (and tools to analyze these models) that are applied to a variety of situations in which such strategic interactions are present.
- ▷ What will we use game theory for?
  - ▷ Form a simplified model that allows for analysis.
  - ▷ Highlight the key tensions that drive behavior.
  - ▷ Make predictions as to how agents will behave, and, in particular, the consequences of small changes to the model.
  - ▷ When the predictions are normatively undesirable, game theory provides tools to shape the situation to promote better outcomes.
- ▷ What are some examples of the situations of interest?
  - ▷ Firms, producing similar products, deciding when to enter the market, how much to produce, or at what price to sell. For example, the price of a competitors product clearly influences the demand a firm faces, and hence the optimal price to set. But by changing the price optimally, each firm might induce its competitors to change their prices, rendering the initial choice no longer optimal.
  - ▷ Politicians deciding what positions to take (under the cynical (although perhaps correct) view that politicians do not really have firm beliefs, but take stands so as to maximize their vote share). Each politician wants to distinguish herself from the others without alienating voters, but how to do so is a function of the position the opponents take.

- A group of students working on a joint project. Each wants to do the minimal work required, and so, will put in less effort the more the other group members contribute.

Simple Examples

We can consider some simple examples of games. Keep in mind that, even in the already stylized world of games, these examples are very synthetic. Because we do not yet have the formal apparatus needed to specify games, these games will be described in natural language and their analysis is informal. Once the formal notation is introduced, it may be worth considering how these games might be written and analyzed.

The following is a very simple, and well known game:

Game. (Rock Paper Scissors). There are two players. Each player has the same three actions: **Rock** (r), **Paper** (p), and **Scissors** (s). Both players play an action simultaneously. The rules are as follows: r wins against s, s wins against p, and p wins against r. Each actions ties against itself. Each player prefers to win than to tie than to lose. What can we say about this game?

- What is a good strategy? Might it be a good strategy to play a random combination of actions?
- If you knew for certain the action of the opponent, then what is a good strategy? What if you knew the other player was playing randomly? Then what is the best response?

A slightly less simple example:

Game. (Beauty Contest). There are n players, where n is a natural number (weakly) greater than 2. Each player has the same 101 actions: pick $x \in \{0, 1, 2, \ldots, 99, 100\}$. Each player plays an action (picks a number) simultaneously. The winner is the player whose number is closest to half of the average (i.e., if the players are labeled $i = 1 \ldots n$, with corresponding actions, x_i then the winner satisfies $i = \underset{j \in 1 \ldots n}{\operatorname{argmin}} \left| x_j - \frac{\sum_{k=1}^{n} x_k}{2n} \right|$). Ties are decided by a coin flip. Again, all players prefer to win than tie than lose.

- What is a good strategy? Clearly depends on the guesses of opponents.
- Perhaps its easier to think of bad strategies.
  - 51-100 is not good. Even if *everyone* guessed 100, half the average is no greater than 50, so 50 is a better guess.
  - But if we know this, and everyone else is just as smart, then no one guesses between 50-100. So, then, even if everyone guesses at 50 its better to guess 25 than anything in 26-50.
  - This logic can be iterated forever, where does it end? It seems that 0 would be the best guess. Is this reasonable?

Utility Representation

One of the key assumptions (read: requirements) of game theory models is that we, as modelers, understand the motivations of the players. It is impossible to make predictions about behavior in strategic settings if we do not first understand behavior in non-strategic settings (i.e., what would the player choose if she could select an outcome unencumbered by strategic considerations).

Sometimes it is easy to understand the motivations of players. For example, firms maximizing profit. It is more or less obvious that firms want always prefer more profit to less. But there are plenty of settings where motivations are less clear. For example, the decision to donate to a charity. It may be that I prefer to have more money to less (pushing me not to give), but prefer that the charity is effective to it not being effective (pushing me to give). These two forces counter one another. This is even more confounded by the fact that if other people give to charity then the charity will be effective regardless of my action (pushing me to not give when others do). But I also might have an intrinsic preference to help out, regardless of the actual effectiveness of the charity, because it make me feel like a good person (pushing me to give no matter the actions of others). None of these motivations are mutually exclusive. In other words, it might not be at all clear how to model players preferences over outcomes of the game.

To simplify matters considerably, we will assume throughout the class that the players' preferences (over outcomes of the game) will be described a by numerical representation. That is, we will associate, for each player, each possible outcome of the game with a real number, under the assumption that players prefer larger (numbered) outcomes to smaller ones. For example, in Rock Paper Scissors, we could say, win = 1, tie = 0, lose = -1. This respects the ordering we assumed in the game.

Notice, however, that when we described player' preferences, we did so through comparisons, and not by assigning numerical values. That is to say, from an intuitive, or empirical, standpoint, people have preferences (I like option a more than option b) without any reference to a numerical scale. Even more (potentially) damning to the whole exercise is that fact that preference between options is observable through choices (I choose option a over option b), but even if such a numerical scale existed, it would not be observable in any meaningful sense. The raises the fundamental question: **under what conditions is it okay to assign numerical values to outcomes, such that these values represent a players underlying preferences over the outcomes?** (When we say represent, we mean if a is preferred to b then the value assigned to a is larger than that assigned to b.)

Let us formalize this notion.

- We begin with a set of outcomes X.
  - X could be numbers: amounts of money, votes, etc,
  - or it could be an abstract set: $\{banana, apple, Toyota\}$
- Our fundamental assumption is that there exists an ordering (pairwise comparison) over these items.
- We call this comparison a preference relation: $\succcurlyeq$.
- We say that $a \succcurlyeq b$ if a is weakly preferred to b. From this we can define

- ▷ the strict prefernece relation, $\succ$. $a \succ b$ if $a \succcurlyeq b$ and **not** $b \succcurlyeq a$.
- ▷ the indifference relation, $\sim$. $a \sim b$ if $a \succcurlyeq b$ and $b \succcurlyeq a$.

For example, perhaps we could consider the set of grades one could get in this class. $X = \{A+, A, A-, B+, \ldots, F\}$. Then some preference relations are:

- ▷ A student who cares about everything:

$$A+ \succ A \succ A- \succ B + ...$$

- ▷ Only care about GPA. Since $A+$ and A both carry a GPA of 4, this induces the preference,

$$A+ \sim A \succ A- \succ B + ...$$

- ▷ A student taking the class pass fail, cares only if she passes or fails,

$$A+ \sim A \sim A- \sim B + \ldots D+ \succ D \sim D- \sim F$$

But *any* ordering is technically possible. This is an intrinsic and underlying trait of the decision maker, her *preference*. We assume that the preference relation is observable thorough choices, and therefore will take $\succcurlyeq$ as the underlying parameter of the problem (the primitive of the model).

As discussed above, in game theory models it is cumbersome to carry around all the elements of X. We would like to be able to use the convenient numerical representation. This will also allow us to find optimal actions by trying to maximize this number; we can use mathematical tools to do this. For the first point in the above example, $A+ = 100$, $A = 95$ etc., would represent the preference.

Unfortunately, not every $\succcurlyeq$ can be described numerically. For example if *apple* $\succ$ *toyota* $\succ$ *banana* $\succ$ *apple*, it is clearly impossible since that value given to *apple* must be both (strictly) bigger and smaller that that given to *toyota*. Therefore, we need to place restrictions of $\succcurlyeq$:[1]

A1 **Completeness** $\forall a, b \in X$ either $a \succcurlyeq b$ or $b \succcurlyeq a$ or both.

A2 **Transitivity** $\forall a, b, c \in X$, $a \succcurlyeq b, b \succcurlyeq c \Rightarrow a \succcurlyeq c$

Are these restrictions appealing? Which one makes more sense? Can we think of examples where they might not hold? These two restrictions guarantee the existence of a numerical representation called a **utility function**:

[1]The logical symbols $\forall, \exists, \Rightarrow$ will be used on occasion in this class. $\forall x \in X, \phi(x)$, is read "for all x in X, $\phi(x)$ is true", meaning that the succeeding sentence is true regarding all the elements in X. $\exists x \in X, \phi(x)$ is read "there exists an x in X, such that $\phi(x)$ is true", meaning that the succeeding sentence is true for at least some element of X. Lastly, $\phi \Rightarrow \psi$ is read "ϕ implies ψ", meaning if the sentence ϕ is true so is the sentence ψ.

Theorem (Utility Representation)**.** *Let X be a finite set. Then $\succcurlyeq$ satisfies A1-2 if and only if there exists a function $U : X \to \mathbb{R}$ such that*

$$a \succcurlyeq b \iff U(a) \geqslant U(b). \tag{1}$$

A few points:

- The theorem is an if and only if statement. Every numerical representation produces a $\succcurlyeq$ that satisfies A1-2 (so we say that A1-2 *characterize* such functions).
- We can relax the restriction that X is finite. We get countable X for free, but arbitrary X requires more structure.
- If we only want one direction of the implication of (1), we can drop completeness. Even if there are incomparable objects, we can find a a function U such that $a \succcurlyeq b \implies U(a) \geqslant U(b)$. However, the other direction clearly does not hold for the incomparable objects.

Ordinal v Cardinal Representations

There are two major shortcomings of the above model. They are closely related and both pertain to the fact that preferences are ordinal (comparisons, i.e., a is better that b) and not cardinal (magnitudes, i.e., a is twice as good as b). As shown below, this means that the utility function is far from unique.

- Any strictly increasing transformation of U will also represent $\succcurlyeq$.
  - A strictly increasing function, $f : \mathbb{R} \to \mathbb{R}$, is one such that $x \geqslant y \iff f(x) \geqslant f(y)$.
  - So, if $a \succcurlyeq b$ then $U(a) \geqslant U(b)$, so for a strictly increasing $f : \mathbb{R} \to \mathbb{R}$, $f(U(a)) \geqslant f(U(b)$. So $f \circ U$ represents $\succcurlyeq$.
  - There is a great deal of arbitrariness with respect to the utility function as there are many increasing functions: $log(x), x^3, 5x, e^x$, $x - 3$, etc.
- How do we take into account for things that only happen **sometimes**.
  - We have the players preference over outcomes, but we are interested in predicting behavior, so we need to examine the induced preference over actions. This can be problematic, however, since actions might induce outcomes randomly, depending on the realization of unknown actions by the other players.
  - For example, if the opponent is playing Rock Paper Scissors with random strategies, then given any action the outcomes are random. So, how do we rank actions that induce *random* outcomes (i.e., distributions over outcomes).
  - For example, say action a yields win half the time and lose half the time; action b produces a tie all of the time. Preferences are given as before by win = 1, tie = 0, lose = -1.
    - Is action a preferred to b? The ranking over outcomes does not give us enough information,

- Perhaps we could use statistics and take expectations. The expected value of a is 0 (since $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot -1 = 0$) and the expected value of b is 0, too. Does this mean $a \sim b$?
- Of course not, the numerical representation is only a metaphor. For example, win = 100, tie = 0, lose = -1, represents exactly the same preferences, but now taking expectations yields that $a > b$. With win = 1, tie = 0, lose = -100, we get the opposite.

- In this domain, taking expectations is meaningless, and this the consequence of the meaninglessness of the magnitudes given by the utility function (itself a consequence of the non uniqueness).

So we want an environment in which the magnitude of the utility (at least in relation to some reference point) has meaning. This is achievable, but it comes at a cost. (1) we must look at a more complex environment, and (2) we must impose further restrictions. Instead of considering preference over X we can consider preference over $\Delta(X)$: the set of all lotteries over X.

Lotteries and Expectations

We will begin with a finite set X, and let $\Delta(X)$ be the set of all lotteries over X.

- If $X = \{a_1 \dots a_N\}$ then $\pi \in \Delta(X)$ is a vector in $[0,1]^N$ that is $\pi = (\pi_1 \dots \pi_N)$ such that $\sum_{i=1}^N \pi_i = 1$ and all $\pi_i \geqslant 0$
- Each element of $\Delta(X)$, referred to as a lottery, assigns a probability to each element of X. The restrictions simply ensure that the probability is well defined.

In general, a *random variable* is a (real-valued) variable that takes values according to some underlying probabilities. Note that the elements of $\Delta(X)$ are not necessarily random variables, since the elements of X need not be real numbers.

Examples of random variables:

- Generally, we think $x = x_i$ with probability p_i for $i \in I$, where $\sum_{i \in I} p_i = 1$ and each p_i is weakly positive.
- Ex: A bet of a coin that pays 1 dollar on head and 0 on tails
  - $x_1 = 1$, $x_2 = 0$ and $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{2}$
- Ex: a bet on a die that pays 1 dollar on a 1, 2 on a 2, 3 on a 3, etc.
  - $y_1 = 1$, $y_2 = 2$, $y_3 = 3$, $y_4 = 4$, $y_5 = 5$, $y_6 = 6$ and $y_i = \frac{1}{6}$, for all i.
- Then, the expectation of a random variable x, denoted $\mathrm{E}(x)$, is

$$\sum_{i \in I} x_i p_i$$

  - So for the above examples $\mathrm{E}(x) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$, and $\mathrm{E}(y) = \sum_{i=1}^6 \frac{i}{6} = \frac{1}{6} \sum_{i=1}^6 i = \frac{21}{6}$.

Expected Utility

While we cannot take expectations of $\pi \in \Delta(X)$, (since these need not be random variables) we could (potentially) take expectations over the utility obtained from elements in X. That is, associate with each lottery, π, a random variable over **utilities**. Just as before, this raises the question: **when is it possible to describe a decision makers preferences over lotteries as expectations over the induced random variables of utilities?**

There is an immediate restriction on behavior: the decision maker's preferences must change proportionally to the change in probabilities of the lotteries, since expectations change in proportion to the probabilities. Mathematically: expectations are linear in probability so imposing that the utility function is an expectation imposes likewise that the decision maker treats probabilities in a linear fashion. For example, if $u(w) = 1$, $u(t) = 0$, and $u(l) = -1$ then it must be that the lottery $\frac{1}{2}w + \frac{1}{2}l \sim t$.

- ▷ Note that mixing together lotteries (with probabilistic weights) always results in a new lottery.
  - ▷ If π and ρ are lotteries (in $\Delta(X)$), then so is $\alpha\pi + (1-\alpha)\rho \sim \sigma$ for some $\alpha \in [0,1]$.
  - ▷ The new lottery is the point-wise combination π and ρ.
  - ▷ For each $a_i \in X$, $(\alpha\pi + (1-\alpha)\rho)_i = \alpha\pi_i + (1-\alpha)\rho_i$.
  - ▷ It is straightforward to show that the resulting weights are a lottery (non-negative and sum to 1).

A second restriction is the *continuity* of utilities. That is, if $u(w) > u(t) > u(l)$, there must be some lottery over w and l that is indifferent to t. This can be seen as a direct application of the intermediate value theorem, and the fact that linear functions (over finite domains) are continuous.

Of course, guaranteeing this behavior this will require a bit more structure than simple the utility representation.

A1 **Completeness** $\forall \pi, \sigma \in \Delta X$ either $\pi \succcurlyeq \sigma$ or $\sigma \succcurlyeq \pi$ or both.

A2 **Transitivity** $\forall \pi, \sigma, \rho \in \Delta X$, $\pi \succcurlyeq \sigma, \sigma \succcurlyeq \rho \Rightarrow \pi \succcurlyeq \rho$

A3 **Independence** $\forall \pi, \sigma, \rho \in \Delta X$ $\pi \succcurlyeq \sigma \Leftrightarrow \alpha\pi + (1-\alpha)\rho \succcurlyeq \alpha\sigma + (1-\alpha)\rho$ for all $\alpha \in [0,1]$

A4 **Continuity** $\forall \pi, \sigma, \rho \in \Delta X$ such that $\pi \succcurlyeq \sigma \succcurlyeq \rho$ then there exists a $\alpha \in [0,1]$ such that $\alpha\pi + (1-\alpha)\rho \sim \sigma$

Since we are after a specific utility function, we require more structure, but since the resulting functional is nonetheless a utility function, Theorem 2 implies that we will still need A1 and A2.

A3, Independence, provides the linearity of the utility function. It dictates that the decision maker only cares about the difference in the lotteries. (Hence the name, the decision maker's preferences are *independent* of the similarities in the lotteries. This is most easily seen via compound lotteries as shown in figure 1.

A3 says that if the DM prefers ρ to ν, then she must also prefer π to τ. The intuition is the following: she will end up with lottery σ $(1-\alpha)$ of the time with either π or τ. Therefore, she can ignore the right branch of the two lotteries completely, making her choice by focusing only on the left branch where the lotteries differ. Of course the left branches are exactly, ρ and ν. As we can see by the the example, the independence

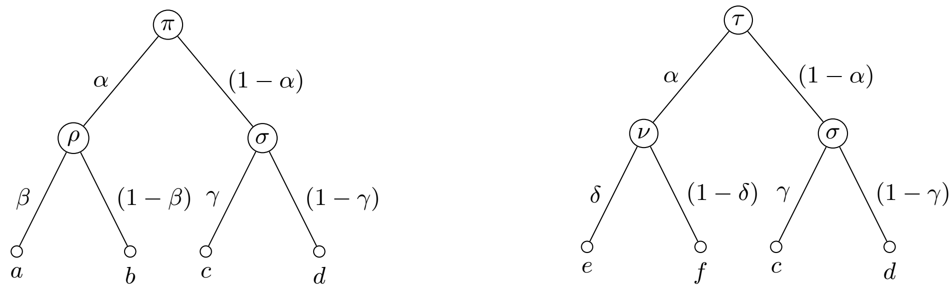

Figure 1: The compound lotteries $\pi = \alpha\rho + (1-\alpha)\sigma$ and $\tau = \alpha\nu + (1-\alpha)\sigma$.

axiom assumes that the decision maker reduces compound lotteries. It is easy to show that, π induces the lottery $(\alpha\beta, \alpha(1-\beta), (1-\alpha)\gamma, (1-\alpha)(1-\gamma)$ over the outcomes (a, b, c, d). The independence axioms states that it must be that the decision maker is indifferent between π and $(\alpha\beta, \alpha(1-\beta), (1-\alpha)\gamma, (1-\alpha)(1-\gamma)$. Hence, we are requiring a great deal of statistical sophistication from our decision maker. There is a deep literature on the empirical failings of the independence axiom and how it can be weakened appropriately.

A4, continuity, dictates you can always create indifferences –there are no jumps in preference (hence continuity). The change from a good outcome to a bad one is smooth with respect to the probabilities. Continuity is largely a technical axiom, but is nonetheless required. Like independence, there are intuitive examples where continuity fails.

These axioms allow us to obtain a utility function that respects probabilities, and allows us to take expectations.

Theorem (von Neumann Morgenstern)**.** *If $\succcurlyeq$ over ΔX satisfies A1-4 then there exists a $v : X \to \mathbb{R}$ such that $\pi \succcurlyeq \sigma$ if and only if*

$$U(\pi) = \sum_{i=1}^{N} \pi_i v(a_i) \succcurlyeq \sum_{i=1}^{N} \sigma_i v(a_i) = U(\sigma)$$

Moreover, v and u both represent preferences, then $v = x \cdot u + y$ with $x \in \mathbb{R}_+$ and $y \in \mathbb{R}$.

This functional form is called an expected utility representation; it is the cornerstone of much of economic/game theory modeling.

Have we solved our two problems?

- ▷ We can now accommodate the ranking of things that happen "sometimes". This is done naturally by looking at preferences over lotteries rather that just outcomes. In the process, we also require the decision maker to behave linearly with respect to the probabilities.
- ▷ On the other hand, the representation is still not unique (we can take affine translations, i.e., multiply by a positive number and add any number).
  - ▷ But is is close enough – the proportions between outcomes remain the same.

- The "absolute" magnitude is still meaningless (since we can add 10 million to everything), but the relative magnitude does matter. The proportional difference between utilities remains the same and is enough for us to use expectations.
- For example: $u(w) = 1$, $u(t) = 0$, and $u(l) = -1$.
  - So, $U(\frac{1}{2}w + \frac{1}{2}l) = 0 = U(t)$ implies $\frac{1}{2}w + \frac{1}{2}l \sim t$
- If we transform utilities by $v = 2u + 5$ we obtain:
  - $v(w) = 7$, $u(t) = 5$, and $u(l) = 3$
  - So $U(\frac{1}{2}w + \frac{1}{2}l) = 5 = U(t)$, we have the same preference $\frac{1}{2}w + \frac{1}{2}l \sim t$.
- Because of the ability to take affine transformations without distorting the underlying preference, we can always *normalize* the utility values in two places.
  - For example, we can set the utility of the best outcome, b, to $u(b) = 1$ and the worst outcome w to $u(w) = 0$.
  - To see this, we have $v = \frac{u-u(w)}{u(b)-u(w)}$.
  - This is clearly an affine transformation, and it satisfies the above properties.

We will assume, almost always, and unless otherwise noted, that the players in our models/games are expected utility maximizers. This mean we will represent the outcomes with numbers. We assume that these numbers correspond to the v in the above theorem. So players wish to maximize their expected payoff (in utility). This is an important assumption. There is a wide and active body of research on how to analyze games absent the expected utility assumption (related closely to the decision making paradigms without the independence assumption).

Example:

- An expected utility maximizer has preferences over lotteries of the 4 alternatives. In particular: $a_1 \succ a_2 \succ a_3 \succ a_4$
- We can normalize $u(a_1) = 1$ and $u(a_4) = 0$.
- Now assume the following facts
  1. $(.6, 0, 0, .4) \sim (0, 1, 0, 0)$
  2. $(0, .5, 0, .5) \succ (0, 0, 1, 0)$
- What does this tell us? What can we say about the utility value of a_2, a_3?
  - From fact 1 we know that $.6(1) + .4(0) = u(a_2) = .6$
  - From fact 2 we know that $.5(.6) + .5(0) > u(x_3)$
  - So $u(x_2) = .6$ and $u(x_3) \in (0, .3)$

The Game Theoretic Environment

With the above prerequisite ideas, we can now define formally what we mean by a game. At first glance, the notational burden imposed by such formalism might seem to outweigh any practical benefit. The truth is to the contrary; much of the subsequent analysis will require a specific environment, one in which we can speak directly and accurately about detailed concepts with no fear of ambiguity. A game, just like a board game, consists of three things:

1. A list of players: $i = 1 \dots N$.
2. A set of actions to be taken by each player A_i for each i.
3. A function that maps actions into outcomes, $\phi : A \to X$.

Preferences for each player over (lotteries of) outcomes. But, as we will see, with the help of ϕ, and the assumption that players adhere to the expected utility axioms, that preferences can be defined over strategy profiles. Again, because of our expected utility assumption, we can just talk about the **utilities** associated with the outcomes. Hence, under our various assumptions, we can think of ϕ as assigning to each action profile an associated utility. Then, random strategies map to lotteries over utilities, which, as previously discussed, will behave as expectations.

A word on notation. A_i refers to an actions set of each player. On the other hand, A refers to profiles of actions (one for each player). Formally, $A = \prod_i A_i$. An element in A, therefore, looks like $a_1 \dots a_N$, where each a_i is in A_i. Lastly, it will be very helpful to talk about a profile of strategies for every player except i. That is, specifying an action for all players but i. We will denote the set of all such action profiles by A_{-i}. Formally, $A_{-i} = \prod_{j \neq i} A_j$.

A word on strategies. A pure strategy (in a normal form game) is an action. However, we will allow players to take random strategies (called mixed strategies). A mixed strategy is an element of $\Delta(A_i)$ a lottery over actions. For example, a strategy is Rock Paper Scissors could be to play each action with $\frac{1}{3}$ probability. Just like with actions, we will use s_i to denote strategies and s_{-i} to denote strategies of opponents. Note that each action, a_i is also a strategy (called a *pure* strategy). For each player, the set of all strategies (both pure and mixed) will be denoted by $S_i \equiv \Delta(A_i)$ and the set of all strategy profiles, $S = \prod_i S_i$, and opponents profiles, $S_{-i} = \prod_{j \neq i} S_j$, are defined as with actions.

Using this formal language we can describe Rock Paper Scissors.

Game. (Rock Paper Scissors, Continued). The players are $P1$ and $P2$. And $i = 1, 2$ and $j \neq i$:

▷ $A_i = A_j = \{r, p, s\}$

▷

$$u_i(r_i, r_j) = 0 \qquad u_i(r_i, p_j) = -1 \qquad u_i(r_i, s_j) = 1$$
$$u_i(p_i, r_j) = 1 \qquad u_i(p_i, p_j) = 0 \qquad u_i(p_i, s_j) = -1$$
$$u_i(s_i, r_j) = -1 \qquad u_i(s_i, p_j) = 1 \qquad u_i(s_i, s_j) = 0$$

Notice how to describe the utilities, we made a kind of matrix. We could extend this thinking to represent the game in it's entirety, see Figure 2.

		P2: r	P2: p	P2: s
P1:	r	$0, 0$	$-1, 1$	$1, -1$
	p	$1, -1$	$0, 0$	$-1, 1$
	s	$-1, 1$	$1, -1$	$0, 0$

Figure 2: Rock Paper Scissors payoff matrix.

The first entry in each cell represents the payoff to the *row* player (here P1), and the second the payoff to the *column* player (P2), associated with action profile given by the corresponding row and column. This notation is very minimal and so we will use it often. However, it is not so good for games with many players or many actions (and of course would not work if there were infinite actions). For example, how would we create such a matrix for Beauty Contest? Many many rows and many dimensions. It is better to simply describe the game outright.

Moreover, it is worth pointing out that nowhere in the matrix version of the game did we describe *outcomes* of the game. Implicitly, we think that if $P1$ plays r and $P2$ plays s then $P1$ wins the game and $P2$ loses. But in our representation we have skipped this step entirely, talking only about the utility consequence of actions. Under our expected utility hypothesis this is okay, since players preference over actions (and therefore the predictions of the model) are going to be driven only by the utility consequence. However, this means that the correct assignment of utilities to outcomes is vital to making good predictions.

This is often overlooked when analyzing a model: for example, is it realistic that tying is just as preferred as winning and losing with equal chance, maybe $u(w) =$, $u(t) = 0$, and $u(l) = -2$ is better? The answer is of course subjective, and depends on the players, the game, and myriad other compilations. In this class, we will usually just take the utility values as given without worrying too much about from where they arise. Nonetheless, many empirical failings of game theory can be traced back to bad utility assignments (more on this latter).

Basic Examples

Lets construct another game:

$\mathscr{G}$**ame**. (coordination game). A husband and wife are trying to decide where to go to dinner and one of their cells phones have died. They previously talked about two restaurants. A tapas place (T)

and sushi bar (S). If they both go to the same restaurant then they will both be happy but if they mis-coordinate they will both be unhappy. In addition they both like the tapas place more. In particular: $u(T, T) = 10, u(T, S) = 2, u(S, T) = 1, u(S, S) = 6$ (there are no subscripts because the game is symmetric).

W:

H:	T	S
T	$10, 10$	$2, 1$
S	$1, 2$	$6, 6$

Figure 3: COORDINATION GAME payoff matrix.

We represent a game as a matrix as shown in Figure 10. Further, recall that the u is not unique. We only care about the relative positions of lotteries. Note that

$$.5u(T, T) + .5u(T, S) = u(S, S)$$

We can transform payoffs by any affine function. So $u(T, T) = 20, u(T, S) = 4, u(S, T) = 2, u(S, S) = 12$ represents the same game. The above still holds. As does $u(T, T) = 9, u(T, S) = 1, u(S, T) = 0, u(S, S) = 5$.

Now lets examine the game above.

- ▷ What would be a good strategy for both players? What makes sense?
  - ▷ The best outcome would be to both go to the Tapas restaurant as it achieves the highest payoff.
- ▷ Is there any other reasonable outcome?
  - ▷ What if the wife thinks the husband will go to Sushi place. Then what is the best thing to do?
  - ▷ If both decide to go to sushi, would either want to change?

Underlying these questions in the fact that in order to make a good choice, each player must think about what the other players a doing. This is the crux of what we mean when we say there are strategic concerns in the game. To more directly get at this, we will define a *best response.* Formally a best response is defined below, informally, a best response for player i is a strategy, s_i such that, **for a fixed profile of opponents strategies**, player i is maximizing her payoff by playing s_i. Keep in mind that we are fixing the other players choices, so a best response only cares about player i's payoff, ignoring any larger strategic complications.

Definition (Best Response)**.** *Given a strategy of other players, s_{-i} a strategy s_i^* is a **best response** for player i if there is no other strategy that can do better. That is,*

$$s_i \in argmax_{s_i \in \Delta(A_i)} u_i(s_i^*, s_{-i})$$

or equivalently,

$$u_i(s_i^*, s_{-i}) \geqslant u_i(s_i, s_{-i}) \quad \forall s_i \in \Delta(A_i)$$

If you **knew** what the other players were doing what would you choose to do? Hopefully, play a best response. If you were not playing a best response then you could change your action and obtain a higher payoff. But, since the payoffs in our games are in utility terms, this means this is an outcome that you like strictly better than the one induced by your action. The most basic assumption about players behavior that we make is that players are **rational**. Rationality has a very specific meaning here, namely, that players will best respond to their beliefs about other players actions. We will talk more about the philosophical implications of rationality in a bit.

For now, to get a better feel for best responses, lets look at another classic game, the prisoners dilemma.

$\mathscr{G}$**ame**. (PRISONERS DILEMMA). Two members of a criminal gang are arrested. Each prisoner is in solitary confinement with no means of speaking to or exchanging messages with the other. The police admit they don't have enough evidence to convict the pair on the principal charge. They plan to sentence both to a year in prison on a lesser charge. Simultaneously, the police offer each prisoner a bargain. Each prisoner is given the opportunity either to **defect** on the other, by testifying that the other committed the crime, or to **cooperate** with the other by remaining silent. Defecting gets the prisoner out of the lesser charges, only if the other does not also defect. If both defect then both will end up facing the full charges, with a relaxed sentence for admitting their guilt. If one remains silent while the other defects then he will face the full charges and a maximum sentence.

We need to put some payoffs with our story. The criminals prefer to go free, to doing lesser charges, to do a reduced sentence, to doing the maximum sentence. We are free to chose the relative weight in utility we put on these outcomes, different choices will tell different stories. For now lets choose the seemingly terrible:

$$\begin{array}{cccc} u_1(C_1, C_2) = 17 & u_1(C_1, D_2) = -43 & u_1(D_1, C_2) = 57 & u_1(D_1, D_2) = 7 \\ u_2(D_1, D_2) = 9 & u_2(C_1, D_2) = 21 & u_2(D_1, C_2) = -9 & u_2(D_1, D_2) = 6 \end{array}$$

We can construct the game matrix as in

		P2: C	P2: D
P1:	C	$17, 9$	$-43, 21$
	D	$57, -9$	$7, 6$

The PD game with weird payoffs

These values are terrible, they give us little intuition about the relative preference. But, as always, we can transform the payoffs as $\hat{v} = \alpha v + \beta$ where $\alpha > 0$. (Note also, we can change the two different players preferences by two different affine functions. Why? Because the two players preference over outcomes are independent of one another. As long as we keep *each* players expected utility ordering unchanged, we have not distorted the preferences, and, therefore, no distorted the predictions of the game). So, lets make it so that (C_1, C_2) has a payoff of $(1, 1)$ and (D_{1,D_2}) has a payoff of $(0, 0)$.

First we can subtract off the (D, D) payoff. So we subtract $(7, 6)$ from all cells:

	P2: C	P2: D
P1: C	$10, 3$	$-50, 15$
P1: D	$50, -15$	$0, 0$

The PD game subtracting $(7, 6)$

Now we can divide each cell by the (C, C) payoff. We divide each cell by $(10, 3)$:

	P2: C	P2: D
P1: C	$1, 1$	$-5, 5$
P1: D	$5, -5$	$0, 0$

The PD game dividing $(10, 3)$

- ▷ This game is MUCH easier to work with
- ▷ This is one of the reasons why our assumptions about EU are important. If the decision makers do not obey the EU axioms then we cannot renormalize games.
- ▷ The game is symmetric which means the matrix is its own transpose.
- ▷ We have chosen these particular values, but there are other that would fit the preferences.

We could tell a different story that fits the same model. Imagine a joint project between two co-workers. If they both work hard they get the best joint payoff, but if either slacks off and the other works hard then the slacker enjoys an even better payoff (at the expense of the hard worker, of course). Or, perhaps, there are two firms with the same product. They can either sell it at a high price or a low price. If they both choose the high price, they both get high profits, but if one undercuts the other it will make an even larger profit (at the expense of the high priced firm, of course). While these stories are all quite different, they are "equivalent" in some sense because they have the same tensions. This is another nice property of formalizing our ideas with game theory. Once we examine the tensions in one situation we can import our understanding to similar situations; the formalization gives un a nice way to interpret what "similar" means.

Back to the Prisoners Dilemma. What is your prediction for this game? We look at what is the best response to a particular action.

- ▷ If player 1 plays C: the best response for player 2 is to defect $(5 > 1)$.
- ▷ If player 1 plays D: the best response for player 2 is to defect $(0 > -5)$.
- ▷ By the symmetry of the game, the same is true for player 1.
- ▷ We can drawn in stars as in Figure 4.
    - ▷ In each row (a fixed strategy for player 1) we star the highest payoff for all columns (the best response for player 2). Likewise, for each column we star the highest payoff for each row.

		P2:	
		C	D
P1:	C	$1, 1$	$-5, 5*$
	D	$*5, -5$	$*0, 0*$

Figure 4: Prisoners Dilemma payoff matrix, with best responses starred.

So whatever the opponent does, the best response is to defect. In other words, regardless of the belief about opponents strategies, a "rational" player will never play C. The Prisoners Dilemma is the go to example of individual agents aggregating presences in an inefficient way. But, is this reasonable? We see cooperation in the real world (and in lab experiments of the Prisoners Dilemma). How do we rectify our theory with this evidence. Here are two points to keep in mind: (1) In lab experiments, the payoffs are monetary, and so, decidedly, not in utility terms. In other words, if players care about other players' monetary payoffs (and this seems reasonable), their preferences over outcomes will not be reflected by the associated dollar amount. This is what was meant by many empirical failings of game theory is just a bad mapping between outcomes and preferences. (2) In the real world, decisions are not made in isolation. As the adage goes, a thief that snitches might subsequently receive stitches. In other words, reputation and repeated interaction matter greatly. We will directly address this later in the course, but for now abstract away from it in order to create a baseline understanding from which to add more complicated interactions.

Dominance

As noted in the previous example, an expected utility maximizer will never play an action if there is another action which is always better than it (better for all possible actions the other players). We will call such action dominated. Formally

Definition (Dominance)**.** *Let s_i and t_i be two stategies that can be taken by player i. Then we say that s_i **strictly dominates** t_i if (or that t_i is **strictly dominated by** s_i)*

$$u_i(s_i, a_{-i}) > u_i(t_i, a_{-i})$$

*for **all** actions that can be taken by other players (that is for all $a_{-i} \in A_{-i}$).*

*Similarly, we say that s_i **weakly dominates** t_i (or that t_i is **weakly dominated by** s_i) if*

$$u_i(s_i, a_{-i}) \geqslant u_i(t_i, a_{-i})$$

*for **all** actions that can be taken by other players (that is for all a_{-i}) and strict for **at least one** action.*

Dominance gives us our first tool to analyze games:

- ▷ A strictly dominated action cannot be a best response to *any* profile of opponents actions.
- ▷ As hinted at above, a rational player will never play a strictly dominated strategy. Moreover, a strictly dominated action will never by part of a mixed strategy.

- ▷ This is a consequence of our assumption of expected utility maximization.
- ▷ No matter what the beliefs are that the player holds about his opponents actions, she could **always** improve her payoff by switching actions (or, more subtly, but placing more probability on better actions).

▷ Can we say the same about weakly dominated strategies?

- ▷ Consider the game in figure 5.
- ▷ D weakly dominates U, for player 1.
- ▷ But, we know that player 2 will not play R by the above claim (since it is strictly dominated by L).
- ▷ So we cannot rule out player 1 playing U.

P2:

P1:	L	R
U	$1, 3$	$2, 2$
D	$1, 5$	$3, 3$

Figure 5: A payoff matrix where we cannot eliminate weakly dominated strategies.

Iterated Deletion of Strictly Dominated Strategies

So no we know that a rational player will not play a strictly dominated strategy. If we begin with a game, there are a lot of possible action profiles – we can use strict domination as a way of reducing this set. Since rational player will not play strictly dominated strategies, we can get delete them (in the sense that we, as modelers, should never make a prediction of any action in the set of strictly dominated strategies). But, if player i knows that her opponents are all rational, then they will not be playing strictly dominated strategies, and so when she forms her belief about her opponents actions, she will not believe that they will play any strictly dominated strategies. But this means, when defining player i's strictly dominated strategies, we only need to concern ourselves with the outcomes that are consequences of non-strictly dominated strategies. In other words, if a strategy for player s_i is strictly worse than s_i' whenever her opponents are not playing strictly dominated strategies, then for all intents and purposes, s_i' dominates s_i. It is strictly better for any outcome that could be generated from rational behavior of her opponents. So even if s_i is not actually strictly dominated, we can delete that from our predictions as well.

This process is called **iterated deletion of strictly dominated strategies** (IDSDS). For the reasons outlined above we need to use strictly dominated strategies and **not** weakly dominated. We call it iterated deletion because when we delete a strategy, we then re-evaluate and find new strictly dominated strategies (they don't have to have existed before). Lets begin with the previous PD game. (**FIGURE** 4).

▷ First we delete C_1 for player 1 (it is dominated by D_1).

P2:

P1:	C	D
C	$1, 1$	$-5, 5$
D	$5, -5$	$0, 0$

Figure 6: Prisoners Dilemma payoff matrix, IDSDS.

▷ Then we delete C_2 for player 2 (Note, we only look at outcomes in the bottom row, since the top row was deleted.)

Lets, examine a larger game, shown in figure 7.

P2:

P1:	L	C	R
U	$5, 3$	$3, 2$	$0, 4$
M	$4, 6$	$6, 5$	$6, 4$
D	$6, 0$	$7, 3$	$1, 1$

Figure 7: A dominance solvable payoff matrix.

1. D dominates U for player 1: delete U

P2:

P1:	L	C	R
~~U~~	~~$5, 3$~~	~~$3, 2$~~	~~$0, 4$~~
M	$4, 6$	$6, 5$	$6, 4$
D	$6, 0$	$7, 3$	$1, 1$

2. C dominates R for player 2: delete R

P2:

P1:	L	C	~~R~~
~~U~~	~~$5, 3$~~	~~$3, 2$~~	~~$0, 4$~~
M	$4, 6$	$6, 5$	~~$6, 4$~~
D	$6, 0$	$7, 3$	~~$1, 1$~~

3. D dominates M for player 1: delete M

P2:

P1:	L	C	~~R~~
~~U~~	~~$5, 3$~~	~~$3, 2$~~	~~$0, 4$~~
~~M~~	~~$4, 6$~~	~~$6, 5$~~	~~$6, 4$~~
D	$6, 0$	$7, 3$	~~$1, 1$~~

4. C dominates L for player 2: delete L

P2:

P1:	~~L~~	C	~~R~~
~~U~~	~~$5, 3$~~	~~$3, 2$~~	~~$0, 4$~~
~~M~~	~~$4, 6$~~	~~$6, 5$~~	~~$6, 4$~~
D	~~$6, 0$~~	$7, 3$	~~$1, 1$~~

We are left with the unique pure strategy profile (D, C). Note how at each step (except 1) the strategy that gets deleted was not strictly dominated until the prior deletion. (For example, C did not dominate R

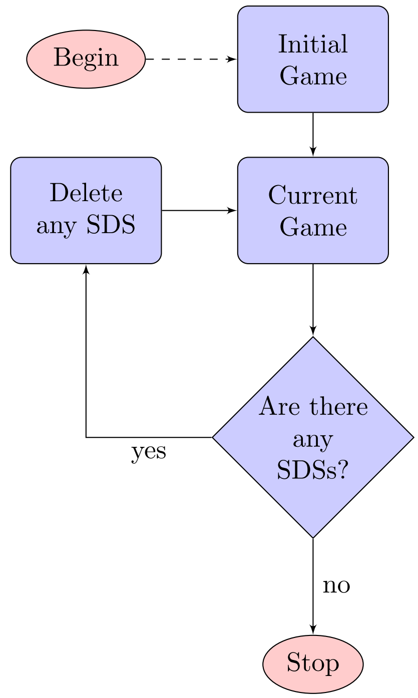

Figure 8: Flowchart for IDSDS.

until U was deleted, since $u_2(U, R) = 4 > 4 = u_2(U, C)$. Nonetheless, the order of deletion does not matter (so long as at each step, the strategy being deleted is strictly dominated)! The process of IDSDS can be represented graphically, via a flowchart, as in figure 8.

If we look back at the definition of a dominated strategy, it can be dominated by any other strategy, including a mixed strategy. So far we have only consider dominance by pure strategies. Consider the game figure 9. It does not initially look like anything is strictly dominated. However, not that the strategy for player two given by $[\frac{1}{2}, 0, \frac{1}{2}]$ has a payoff of $2\frac{1}{2}$ for regardless of the strategy chosen by player 1. Hence this strategy dominates C which has a constant payoff of 2. Once we delete C, U strictly dominates D for player 1, and finally, L dominates R for player 2, leaving use with the unique prediction (U, L).

P2:

P1:	L	C	R
U	$2,5$	$3,2$	$6,0$
D	$1,0$	$4,2$	$2,5$

Figure 9: Actions can be dominated by mixtures.

In the games so far analyzed there have been a unique prediction. This is not always the case. Lets look back at the COORDINATION GAME in Figure 10. There are no strictly dominated strategies in this game, so our analysis produces no predictions beyond that players will choose *some* action. So IDSDS can yield no deletions, but can it delete all strategies, leaving nothing to predict? The following result show this is not the case:

Theorem. *In a game with finite players and finite actions, there always exists a set of undominated strategies* *left after IDSDS. Moreover, the set that remains after deletion does not depend on the order we delete things.*

The first part of the theorem is obvious (since there would be no strategy left to strictly dominate the last strategy). The second is a bit harder to see. It follows to observations. First, by the definition of dominance, if s_i dominates t_i and t_i dominate r_i then s_i dominates r_i (i.e., the definition is transitive). Second, and again from the definition, if s_i garners a strictly larger payoff than t_i for every action in A_{-i} then s_i this is still the case for regarding any subset of A_{-i} (and so strict dominance is not lost after deletion). Note, this does not hold for weak domination (why not?), part of the reason why the result fails in that domain.

Assumptions on beliefs. Throughout this course, we will discuss in depth the assumptions regarding behavior (in particular, beliefs) associated with particular game theoretic tools. We start here with regards to IDSDS. The strategies that survive IDSDS are called *rationalizable*. What are the assumptions implicit in the predictions given by rationalization? Recall that we saw that a rational player will never play a strictly dominated strategy, no matter what her belief. So, one might assume that the answer is that the rationality of all players will give rise to strategies that survive IDSDS, and while this is required, it is not enough. For example, in the game presented in figure 7, if P2 did not believe P1 was rational she could not rule out that he might play U, but in that case, it is her best response to play R; the whole process breaks down. So rationality is not sufficient. What is needed is called *rationality and common belief in rationality*, or that each player is rational, believes her opponents are rational, believes her opponents believe their opponents are rational and so on. Each level of iteration of belief roughly corresponds to an iteration of the deletion mechanism. This last point suggests (correctly) that if at any level of belief (i.e., a belief about a belief about a belief, etc) a player does *not* believe an opponent is rational, then there exists a game, and a strategy thereof, that might be played, but does not survive the IDSDS process.

To see idea in action, lets recall the BEAUTY CONTEST game. If you look back to the argument made when the game was introduced, you will now recognize that we were iteratively deleting dominated strategies! The unique rationalizable strategy was for each player to guess 0. Let's examine a player, i's, decision.

1. If i is irrational, she can choose any action.
2. If i is rational, she will not guess above 50 (as all those guesses are strictly dominated). However, if i, while rational, believes her opponents are not rational. Then she might believe they will all guess 100, in which case she could *rationalize* guessing a number close to 50. Hence the predictions did not converge to 0.
3. If, in addition to assuming i is rational, also assume that she believes her opponents to be likewise rational, then she believes they will not play strictly dominated strategies. But now the logic of our initial argument begins to take hold –she believes all of her opponents will not guess above 50 (since they are rational), so she will not guess above 25. Of course, if i's opponents do not believe their opponents are rational (like the player i described in (2) above), they might play very close to 50, and hence i can rationalize a guess very close to 25. With two levels of belief in rationality, the predictions get closer to 0.
4. It is only if we assume the entire infinite hierarchy of beliefs of beliefs of beliefs in rationality that we get the final result, that everyone guesses 0.

While rationality seems like a relatively benign assumption, common belief in rationality is strikingly more involved. When, for instance, was the last time you thought about someone else's beliefs about someone else's beliefs about someone else's belief? These forms of knowledge and belief are indeed strange. The good news, however, is that for most simple games, fully common knowledge of rationality is not needed. For example, to obtain our prediction in the Prisoners Dilemma game, we only needed that players are rational; their beliefs about other players were irrelevant.

Nash Equilibrium

Rationalizability (or IDSDS) gives us a set of strategies that are best responses to the rationalizable strategies of each player's opponents. So each player believes that her opponents are playing rationalizable strategies, and best responds accordingly. However, the predictions are silent on the correctness of the beliefs of each player. For example in the coordination game, if the Husband thinks the Wife is going to the tapas restaurant, he will as well; if the Wife thinks the Husband is going to go to the sushi restaurant she will as well. Since all strategies are rationalizable, this is a perfectly permissible strategy profile. But, it does not take into account that each player, if she knew the other player's choices, would prefer to change her choice. In essence, we have not have not restricted, beyond regarding rationality, that the players' beliefs be correct.

Nash Equilibrium (NE), on the other hand, which will become the work horse of most of the rest of this class, dictates that player are best responding, not only to some rationalizable strategy, but to the strategy that other players are actually playing. Of course, this must hold for each player simultaneously. It is in this sense that we mean all the players actions are in *equilibrium*. We will discuss this more, and deal more directly with the behavioral assumptions later.

The concept of Nash Equilibrium predates John Nash, whom-after it was named. The oldest use of the concept (without formal characterization) is in regards to the Cournot duopoly model, introduced in the next section. It was also examined (again, in a specific context, this time zero sum games) by John von Neumann and Oskar Morgenstern in the 1940's. You might recognize these names from the expected utility theorem: they were the first to prove the characterization of expected utility as A1-4. It was John Nash who, in his *doctoral thesis*, proposed the concept of Nash Equilibrium in full generality. Much of its appeal comes from the result, due again to Nash's thesis, that under loose conditions, there always exists a mixed strategy NE.

First the formalities, then some examples. Recall the definition of a Best Response to a profile of opponents' strategies. From this, we can define a *best response correspondence* (a correspondence is a set valued function, or a mapping from A to B, where $f(a)$ is a subset of B (whereas for a function the image must be a point in B)). The best response correspondence for player i takes a profile of opponents strategies, s_{-i}, and returns *all* best responses for player i.

Definition (Best Response Correspondence)**.** *The* ***best response correspondence*** *for player i, or $BR_i : \Delta(A_{-i}) \rightrightarrows \Delta(A_i)$ is such that*

$$BR_i(s_{-i}) = \{s_i^* \in \Delta(A_i) | u_i(s_i^*, s_{-i}) \geqslant u_i(s_i, s_{-i}) \quad \forall s_i \in \Delta(A_i)\}$$

From here, the idea of a a Nash Equilibrium is easy: a set of strategies (one for each player) such that given everyone is playing according to the strategy, no one can do any better by changing his strategy (everyone is playing a best response, simultaneously).

- ▷ This means all the tensions are in balance, given what you are doing, I cannot do better by changing, and given what I'm doing you cannot do better by changing.
- ▷ So we will stay where we are. No player has an incentive to change.
- ▷ Every player is simultaneously playing a best response.

Before we get to a formal definition, lets look at the PRISONERS DILEMMA. We can see that (D, D) is a Nash Equilibrium. Why? If $P1$ plays D then $P2$'s best response is to play D, since $(0 > -5)$. The same is clearly true for $P1$ if $P2$ plays D. So we have met the conditions for equilibrium. Notice that we get the same prediction as we got from IDSDS in this case (more of the connection later).

We can formally define a Nash Equilibrium as follows:

Definition (Nash Equilibrium)**.** *Given a strategy profile $(s_1^* \dots s_N^*)$ is is a **Nash Equilibrium** if*

$$s_i^* \in argmax_{s_i \in \Delta(A_i)} u_i(s_i, s_{-i}^*)$$

or equivalently if

$$u_i(s_i^*, s_{-i}^*) \geqslant u_i(s_i, s_{-i}^*), \forall s_i \in \Delta(A_i)$$

for all $i \in N$.

Defining the Nash Eq. via the Best Response Correspondence: It might be useful to think of an abstract game in terms of the BR correspondences. Thus, the NE definition is equivalent to specifying that for a profile $(s_1^* \dots s_N^*)$: we have that $s_i^* \in BR_i(s_{-i}^*)$ for all i.

This is what we are doing by labeling the stars like in Figure 4. This will help us figure out what are the pure strategy NE. Notice that the only cell with stars for both players was the only NE. This is no-coincidence. Since the cell had a star for each player, it was a best response for both players, given that the other player was playing the corresponding action. If a cell does not have a star for each player, then it cannot be a NE (since by definition it cannot be a mutual best response, any player for whom there is no star could by definition do better by changing strategies). Equivalently, if there is a star for each player, then the corresponding actions constitute a NE.

To reiterate: in Figure 4, (D, D) satisfies the condition and is a NE as shown above. Conversely, (C, C) is not a NE (even though it is better for everyone). It is not a NE because

$$u_1(D, C) = 5 > 1 = u_1(C, C)$$
$$u_2(C, D) = 5 > 1 = u_2(C, C)$$

We can analyze the pure strategy NE of the COORDINATION GAME by adding BR stars to Figure 10:

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{\text{W:}} \\
 & & T & S \\ \cline{3-4}
\text{H:} & T & *10, 10* & 2, 1 \\ \cline{3-4}
 & S & 1, 2 & *6, 6* \\ \cline{3-4}
\end{array}$$

Figure 10: COORDINATION GAME payoff matrix, with BR stars.

- ▷ We star the best response in each column and row (fixing another players strategy)
- ▷ There are two NE, (T, T) and (S, S)
- ▷ We can check this manually, with numbers
  - ▷ $u_H(T, T) = 10 > 1 = u_H(S, T), u_W(T, T) = 10 > 1 = u_W(T, S)$ So (T, T) satisfies the criterion.
  - ▷ $u_H(S, S) = 6 > 2 = u_H(T, S), u_W(S, S) = 6 > 2 = u_W(S, T)$ So (S, S) satisfies the criterion.
  - ▷ $u_H(S, T) = 1 > 10 = u_H(T, T)$ So (S, T) fails.
  - ▷ $u_H(T, S) = 2 > 6 = u_H(S, S)$ So (T, S) fails.
- ▷ This example shows
  - ▷ There can be multiple equilibria.
  - ▷ Failure only needs one counter example (one deviation) but existence need exhaustiveness. We only had to show that the Husband wanted to deviate to demonstrate that (S, T) was not a NE. On the other hand, we had to show that every player was playing a best response to show that (T, T) was a NE.
  - ▷ The deviation does not have to be maximal, showing *any* deviation suffices.

$\mathscr{G}$**ame**. (MATCHING PENNIES). A boy is interested in a girl, but the feelings are not mutual. They are both going to go to the beach today, and there are two beaches to choose from: Atlantic Avenue beach and Indian Wells beach. If they both end up at either Atlantic or Indian Wells, then the boy gets 1 and the girl -1. If they they end up at different beaches, the girl gets 1 and the boy -1.

The corresponding matrix is:

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{\text{g:}} \\
 & & A & I \\ \cline{3-4}
\text{b:} & A & *1, -1 & -1, 1* \\ \cline{3-4}
 & I & -1, 1* & *1, -1 \\ \cline{3-4}
\end{array}$$

Figure 11: MATCHING PENNIES payoff matrix, with best response stars.

Notice that there are no cells that have two stars. No matter what actions are being played, one of the two players will want to deviate (either the boy moves to the beach with the girl, if they are apart, or the girl moves away from the beach with the boy, if they are together). This game has no pure strategy NE!

If we look back, it is easy to check that for each game, the NE strategies are within the set that survived IDSDS. This idea can be made formal through the following result:

Theorem. *Every strategy that is part of a Nash Equilibrium profile is rationalizable (survives IDSDS).*

This is left unproven. Nonetheless, this result will be useful once we show the existence of NE. Note that the converse is not true, there are profiles of rationalizable strategies that do no constitute a NE. This is all related to the following simpler result, which we will prove:

Theorem. *No Nash Equilibrium profile can contain a strictly dominated strategy.*

Proof. Assume this was not the case. Then there is some NE profile $(s_1^* \dots s_i^* \dots s_N^*)$ such that s_i^* is strictly dominated by t_i. So by definition of NE we have

$$u_i(s_i^*, s_{-i}^*) \geqslant u_i(s_i, s_{-i}^*), \forall s_i \in \Delta(A_i) \tag{1}$$

$$u_i(s_i^*, s_{-i}^*) \geqslant u_i(t_i, s_{-i}^*) \tag{2}$$

where (2) follows from (1) since $t_i \in \Delta(A_i)$. And by definition of strict domination we have

$$u_i(t_i, s_{-i}) > u_i(s_i^*, s_{-i}), \forall s_{-i} \in \Delta(A_{-i}) \tag{3}$$

$$u_i(t_i, s_{-i}) > u_i(s_i^*, s_{-i}^*) \tag{4}$$

where (3) follows from (4) since $s_{-i}^* \in \Delta(A_{-i})$. Combining (2) and (4) we have

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i^*, s_{-i}^*) \tag{5}$$

which is a contradiction. ★

We can define the notions of strict and weak Nash equilibria.

Definition. *Given a strategy profile $(s_1^* \dots s_N^*)$ is is a **strict Nash Equilibrium** if*

$$u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*), \forall s_i \in \Delta(A_i)$$

for all $i \in N$.

A *weak* NE is a NE that is not strict (so some player is indifferent to switching). For example in figure 12, (U, L) is strict (since $4 > 1$) and (D, R) is weak (since $1 = 1$).

P2:

		L	R
P1:	U	$*4, 4*$	$-1, 1$
	D	$1, -1$	$*1, 1*$

Figure 12: Strict and Weak Nash Equilibria.

Theorem. *No strict NE can contain a weakly dominated strategy.*

The proof is very similar to that of the previous result, and will be assigned as homework.

Assumptions on beliefs. As the results above show, there is a connection between rationalizable strategy profiles and NE strategy profiles, namely that every NE strategy profile is also rationalizable. This means that, if we are to believe the predictions of NE, at the very least we must believe that our players satisfy rationality and common knowledge of rationality. Why? Well, if they did not satisfy these conditions, then we could support strategy profiles that were not rationalizable, which we just said would not be a NE. But we also know that the converse is not true: not every rationalizable strategy is a NE. So we are assuming *strictly more* out of the beliefs of our players. NE gives us stronger predictions, but it also makes stronger assumptions!

So what is it that we are assuming about players beliefs and behavior in NE that we are not assuming in rationalizability? Well, recall that in the COORDINATION GAME all strategies were rationalizable, so a strategy like (T, S) would have been admissible. Nonetheless, (T, S) is clearly not a NE. Why? Because *given the action of P1*, P2 would like to switch to playing T. And this pin points the difference, in NE, players have to be playing a BR to what their opponents are *actually playing*. In other words, each player must have correct beliefs about other players strategies. A strategy is rationalizable if it is a BR to any rationalizable strategy of the opponent, whereas a strategy is part of a NE if it is a BR to the *true* strategy of the opponent.

While NE gives sharper predictions, and the notions of stability are well founded (if we find ourselves posed to play a NE, we will continue to do so), this behavioral assumption is somewhat stringent. In games of coordination, or games with multiple equilibria, it is unclear how players arrive at the correct beliefs. Notice that when rationality and common knowledge of rationality completely pin down the opponents strategies (for instance, in the PRISONERS DILEMMA), then no addition information is needed to characterize player beliefs about their opponents strategies, and so NE and rationalizability coincide.

Voting Game

Why are these concepts used? Sometimes we want to rule out unintuitive or downright silly equilibria as shown in the following example.

$\mathscr{G}$**ame**. (SIMPLE VOTING GAME). There are N (where N is an odd number) players of two types. k A-types and $N - k$ B-types. i.e., $N = 9, k = 5$ corresponds to $AAAAABBBB$. Each player has an action set of $\{a, b\}$. Which ever action gets more votes wins. Players of the winning type get a payoff of 1 and other players get a payoff of 0.

- There are multiple NE.
- The simplest is –everyone votes their type.
- Voting one's type is a weakly dominate strategy:
  - There are two scenarios we should worry about, because *only* the count matters we can boil down all strategy profiles into the number of votes:
    - Your vote changes the outcome from your type losing to winning (i.e., you are type A lose if vote b and win if vote a.) –then voting your type is strictly better that voting against your type.
    - Tour vote does not change the outcome. Then voting your type is weakly better (since the outcome is the same either way).

- ▷ If everyone is using a weakly dominate strategy, the profile must be a NE (can you show this?).

▷ This is an intuitive equilibrium (people vote for the policy the want).

▷ However, lets assume $N > 3$. Then everyone voting a is also a NE.

- ▷ No one person can change the outcome, so everyone is indifferent (weak NE).

But this is not intuitive. While it fits the bill of a NE, it goes against the spirit of the example, and the way we think the world works. Even believe it will not make a difference, people might still vote for their true preference, just incase it came out to matter. Therefore, we might want to consider NE in **undominated** strategies. This will rule out the silly eq. Such restrictions on the types of strategies that can be used in eq are called *refinements*.

Industrial Organization Games: Cournot and Bertrand

We will now examine the some topics in industrial organization and use the tools we just learned to analyze them. Industrial organization (IO) refers to how firms operate at the firm level. So when thinking about IO from a game theoretic perspective, the players in the games as firms, and the actions will be market level decisions. The first game is called the Cournot Duopoly model. Antoine Augustin Cournot, who first proposed the model, was an economist in the 1800 hundreds and proposed the following model in a 1838 book. Hence, this far predates the advent of Nash Equilibrium or formalized game theory. His model was described in isolation, and his proposed solution to the model was indeed a Nash Equilibrium.

We are interested in the behavior of firms when there is some competition (not a monopoly) but not so much competition that the individual firms actions has no effect on other firms (not perfect competition). This is the domain where strategic issues come into play. One firms market choices will have a direct effect on others –we want to make predictions on what will happen (in equilibrium). For simplicity we will only think about 2 firms for now.

$\mathscr{G}$**ame**. (Cournot Duopoly). There are two firms, 1 and 2. Each firm can produce a single type of good (they produce the same good). The cost of producing q_i units of the good is $C_i(q_i)$: an increasing function. The price at which the goods are sold is a function of the aggregate supply (i.e., $Q = q_1 + q_2$). Firms can choose any quantity to produce, and choose simultaneously. After production, the payoff function of firm i is

$$\pi_i(q_i, q_j) = q_i P(Q) - C_i(q_i)$$

Notice that the profit, $\pi_i(q_i, q_j)$ is a function of both firms' quantities, this is because both choices in part determine Q. We will make the following simplifying assumption:

▷ There is constant marginal cost: $C_i = cq_i$

- ▷ Linear aggregate demand: $P(Q) = \max\{0, a - Q\}$.
- ▷ The cost of producing one unit is not so high that no one will buy it: $c < a$.

The players are firms 1 and 2, the actions are $A_1 = A_2 = [0, \infty)$, and the payoffs are given by the profit function. We can therefore talk about this as a game. Using our simplifying assumptions can now write the profit function of firm 1 as

$$\pi_1(q_1, q_2) = \begin{cases} q_1[a - q_1 - q_2 - c] & \text{if } Q < a \\ -cq_1 & \text{if } Q \geqslant a \end{cases} \tag{6}$$

This is a function of firms 2's decision; we can find firm 1's best response given a strategy of firm 2.[2] The first order condition of the profit function with respect to q_1 is

$$a - c - q_2 - 2q_1 = 0 \tag{7}$$

$$q_1 = \frac{a - c - q_2}{2} \tag{8}$$

Notice that we only take the derivative with respect to the first line of (6). If $Q \geqslant a$, the best response is clearly to produce a negative amount, but since this is not allowed, the best possible option is to produce 0. Likewise, the q_1 described by in (8) cannot be negative however (as the strategy space does not allow that) so the real best response is

$$BR_1(q_2) = \begin{cases} \frac{a-c-q_2}{2} & \text{if } q_2 < a - c \\ 0 & \text{if } q_2 \geqslant a - c \end{cases} \tag{9}$$

Firm two is identical. So, utilizing the symmetry we are going to get

$$BR_2(q_1) = \begin{cases} \frac{a-c-q_1}{2} & \text{if } q_1 < a - c \\ 0 & \text{if } q_1 \geqslant a - c \end{cases} \tag{10}$$

We can graph these best responses, as shown in figure 13.

We are looking for a Nash equilibrium, or a mutual best response, so we need a q_1^*, q_2^* such that $q_1^* = BR_1(q_2^*)$ and $q_2^* = BR_2(q_1^*)$. We can do this by substitution. Plugging in firm 2's BR to firm 1's decision, we can look at firm 1's choice taking into account how its choice will effect firms 2's best response. That is, substituting (10) into (9):

$$q_1^* = \frac{a - c - \frac{a-c-q_1^*}{2}}{2} \tag{11}$$

$$= \frac{a - c + q_1^*}{4} \tag{12}$$

[2]Recall that to find the maximum of a differentiable function, f with respect to a variable, x, we can impose restrictions on its derivative. These are the so called first order condition (FOC) and second order condition (SOC). If at a point x, (FOC:) the first derivative is 0, i.e., $\frac{\partial f}{\partial x} = 0$, and (SOC:) the second derivative is negative $\frac{\partial^2 f}{\partial x^2} < 0$, the x is a local maximum of the function. Moreover, if $\frac{\partial^2 f}{\partial y^2} < 0, \forall y$, i.e., the function is concave, then every local maximum is a global maximum.

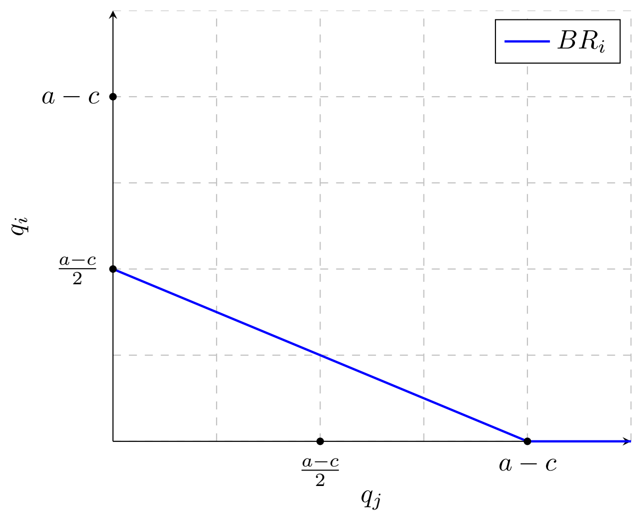

Figure 13: The best response correspondences for the COURNOT DUOPOLY game.

subtracting $\frac{q_1^*}{4}$,

$$\frac{3q_1^*}{4} = \frac{a-c}{4} \tag{13}$$

and solving for q_1^*,

$$q_1^* = \frac{a-c}{3} \tag{14}$$

By the symmetry of firm 2, we have that $q_1^* = q_2^* = \frac{a-c}{3}$. This is a NE, which can be verified by that fact that $q_1^* + q_2^* = Q = \frac{2}{3}(a-c) < (a-c) < a$, and so, payoffs are given by the first case in (6), so by the above derivation (q_1^*, q_2^*) are mutual best responses. The amount the each firm would produce, taking into account that the other firm is behaving similarly.

We can also see that this is a NE graphically, by using a neat trick that will come up again later in the course. We will graph both firms' BR correspondences on the same graph. However, we will rotate firm 2's graph by 90 degrees clockwise. Notice how on on axis of figure 13 is firm i's production and on the other is firm j's. So, when we rotate firm 2's graph, the y axis will be firm 1's fixed output, and the x axis to be the resulting BR for firm 2. This is presented in figure 14. Then, note that wherever the BR correspondences cross is a NE.

What properties does this solution have:

- ▷ What are the firms profit?
  - ▷ The price of each unit is $P(\frac{2(a-c)}{3}) = a - \frac{2(a-c)}{3} = \frac{1}{3}a + \frac{2}{3}c$
  - ▷ The profit for each firm is therefore $\frac{a-c}{3}[\frac{1}{3}a + \frac{2}{3}c - c] = \frac{(a-c)^2}{9}$
- ▷ What if the firms colluded? Could they increase their profit?
  - ▷ The joint profit function is: $\Pi = Q[a - Q - c]$.

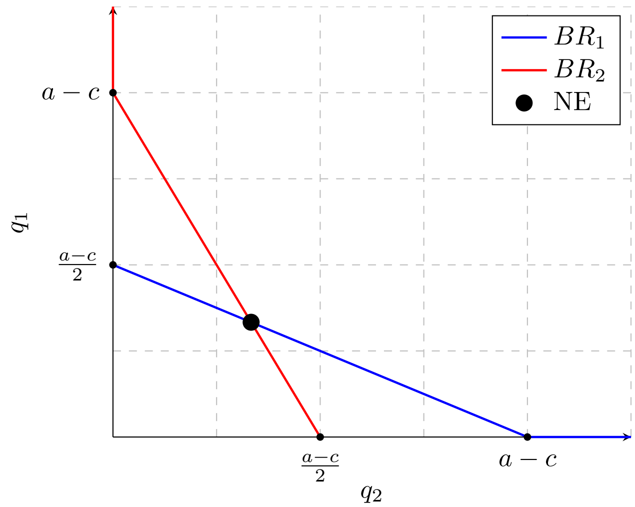

Figure 14: The best response correspondences for the Cournot Duopoly game, superimposed to show NE.

- ▷ The FOC is $a - c - 2Q = 0$ so the optimal $Q = \frac{a-c}{2}$
  - ▷ This is the amount a monopolist would produce. In some microeconomics class, you probably showed that this is socially inefficient; as the monopolist extracts rents from the consumers, it reduces the consumer surplus more than the corresponding gain in producer surplus.
- ▷ Then the price would be $P(\frac{(a-c)}{2}) = a - \frac{(a-c)}{2} = \frac{1}{2}a + \frac{1}{2}c$
- ▷ Assume they split evenly, so they are each producing $q_i = \frac{a-c}{4}$
- ▷ The profit function for each firm is therefore $\frac{a-c}{4}[\frac{1}{2}a + \frac{1}{2}c - c] = \frac{(a-c)^2}{8}$
- ▷ They make more profit by decreasing their production. But this is unsustainable, because, if one firm produced $\frac{a-c}{4}$, the best response of the competitor is *not* $\frac{a-c}{4}$. The other firm would deviate and produce more, taking advance of the smaller supply. This is like a continuous version of the prisoners dilemma.

What about the same game with N firms rather than 2? The demand and profit functions look the same, and the price is set in the same, only there are more firms producing to make the aggregate supply:

$$Q = \sum_{i=1}^{N} q_i$$

We can follow almost the same derivation as the two firm case. The profit of a firm, given a production vector $q = (q_1 \dots q_N)$ is

$$\pi_i(q) = \begin{cases} q_i[a - \sum_{j=1}^{N} q_j - c] & \text{if } Q < a \\ -cq_i & \text{if } Q \geqslant a \end{cases}$$

We can re-write the profit function as

$$\pi_i(q) = q_i[a - q_i - \sum_{j \neq i} q_j - c] \quad \text{if } Q < a$$

Again, we can take the FOC:

$$a - 2q_i - \sum_{j \neq i} q_j - c = 0$$

Here, we can exploit the symmetry of the firms. To so this, we will add together the N first order conditions:

$$\sum_{i=1}^{N} (a - 2q_i - \sum_{j \neq i} q_j - c) = 0$$

Then, by the symmetry, we know that in equilibrium each firm produces the same amount q^*, so

$$N(a - 2q^* - (N-1)q^* - c) = 0$$

or

$$N(N+1)q^* = N(a-c)$$

or finally, that

$$q^* = \frac{(a-c)}{N+1}$$

Setting, $N = 2$ produces the output we found in the 2 firm case, as it should. The total production in the economy is therefore $Q = \frac{N(a-c)}{N+1}$.

We can analyze the predictions of this model:

- ▷ The quantity the firms produce is increasing with demand (as a increases).
- ▷ The quantity is decreasing with the cost (as c increases).
- ▷ Recall that in perfect competition, firms produce until marginal profit is zero (MC=MR), so where $P(Q) = c$ or where $a - Q = c$, so $Q = a - c$.
  - ▷ The total amount produced tends to the perfectly competitive outcome as $N \to \infty$.
  - ▷ Competition alleviates the inefficiency of monopoly (closer to the competitive output), but we only reach the socially optimal production in the limit.
- ▷ Overall takeaway: small scale competition is good (for consumer/market efficiency) but not as good as perfect competition.

But these predictions rely on a key assumption: that firms compete in quantity. Is this reasonable? It seems that often firms set a price and then produce after facing demand (which of course, is a function of the price. Joseph Louis Franois Bertrand came up with this criticism of the model: firms actually compete in price. The following alternative model was formulated in the 1883 by Bertrand in a review of Cournot's (1838) book.

$\mathscr{G}$ame. (BERTRAND DUOPOLY). There are 2 firms producing the same good. The firms compete in price: they choose a price at which to sell the good. The amount they sell is a function of the demand function: $D(p)$, a decreasing function. Further, we assume that all demand goes to the cheaper firm. (All consumers buy at the lowest price). So, given prices the profit of each firm is

$$\pi_i(p_i, p_j) = \begin{cases} D(p_i)(p_i - c) & \text{if } p_i < p_j \\ \frac{D(p_i)(p_i - c)}{2} & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

Again, to simplify we are assuming a constant marginal cost. Further, we will assume linear demand again, so $D(p) = a - p$.

To begin our analysis, consider what happens when if the second firm set the price very high, so high that it is never binding. Then, the best response maximizes

$$(a - p_i)(p_i - c) = ap_i - ac - p_i^2 + cp_i$$

with the corresponding first order condition,

$$a - 2p_i + c = 0 \Rightarrow p_i^* = \frac{a + c}{2}$$

We will refer to this price, denoted p^M, as the monopoly price. It is the price that would be set by a monopolist.

The best response correspondence is given by:

$$BR_i(p_j) = \begin{cases} \{p_i | p_i > p_j\} & \text{if } p_j < c \\ \{p_i | p_i \geqslant p_j = c\} & \text{if } p_j = c \\ \varnothing & \text{if } p_j \in (c, p^M] \\ p^M & \text{if } p_j > p^M \end{cases}$$

We will examine each of the four cases, explaining why the BR is as given:

- ▷ If $p_j < c$:
  - ▷ Every unit sold loses money. Hence the best thing to do is to not sell any units, which is accomplished by any price higher than the other firm's.
- ▷ If $p_j = c$:

- ▷ Every unit sold make no profit at all. Undercutting the other firm is clearly a bad idea, as it means you lose money on each unit. So you are indifferent to selling and making no profit ($p_i = c$) or not selling at all $p_i > p_j = c$.

▷ If $p_j \in (c, p^M]$:

- ▷ There is profit to be made (so selling above p_j cannot be a best response). Matching the other firm (setting $p_i = p_j$) is no good either, as it splits the market; but lowering your price just a little bit would allow you to capture the whole market. Since you only lose a little bit of profit on each sale, but make twice as many sales, it is clearly better to undercut by just a bit. But that does work either, since for every price below p_j you would prefer to move up, just a little closer to p_j. There is no downside to this (no lost sales), but it makes more money per sale. And thats the whole strategy set, so there is no best response!, but hey, no one said that there was always a best response.[3]

▷ If $p_j > p^M$:

- ▷ If the other firm prices above the monopoly price, then, as argued above, choosing the monopoly price is optimal.

Since we are interested in the NE, we want a pair of prices such that each is in the other best response correspondence. The unique pair that satisfies this is $(p_1, p_2) = (c, c)$. That NE is to set price equal to marginal cost. Profit is $D(c)(p_i - c) = (a - c)(c - c) = 0$.

- ▷ So, firms make zero profit and sell at marginal cost: this is exactly the perfect competition outcome.
- ▷ If either firm tries to make profit, it will be undercut by the other firm.
- ▷ The Bertrand model a very different takeaway than the Cournot: any competition, even only 2 firms, will completely erode the monopoly rents.

One last game in this section:

$\mathscr{G}$**ame**. (PUBLIC GOODS GAME). There are N players. Each player has \$1 and can allocate it between a public and private good (thus the action space is $S_i = [0, 1]$. Private goods are consumed at a 1-to-1 ratio privately, whereas public goods are scaled (by some scale $1 > \alpha > \frac{1}{N}$ and then returned to *everyone*. Payoffs are therefore

$$\pi_i(s) = (1 - s_i) + \alpha \sum_{j=1}^{N} s_j$$

What we give gets used by everyone, what we keep only we get. This game could represent the tensions of

- ▷ A joint project

[3] This is a technical issue. Under mild conditions (finite strategies, or continuity of the payoff functions and compactness, best responses are guaranteed to exist.

- ▷ Charitable donations
- ▷ Legislative process

Given the strategy of all the other players s_{-i} the best response of player i is to maximize

$$\pi_i(s) = (1 - s_i) + \alpha s_i + \sum_{j \neq i} s_j$$

the derivative with respect to s_i is

$$\alpha - 1$$

Note two things:

- ▷ The derivative does not depend on s_{-i}. So it does not matter what the opponents do
- ▷ The derivative is negative always: so payoff is always decreasing when a player increases her donation. The best response (regardless of other players actions, and hence, the dominate strategy) is to set $s_i = 0$
- ▷ This is called the *free rider* problem. Each player want the other players to donate, but want to free ride on their contributions.

So every positive strategy is dominated by $s_i = 0$ the NE is therefore no donation, everyone's payoff is 1. However, if everyone donated everything, $s = 1$, then the payoff to everyone is $\alpha N > 1$ (where the inequality comes from our assumption on α.

Each of three games has similar tensions but with different twists. The are all slightly reminiscent of the Prisoners Dilemma game, in that cooperation, although it would be better for everyone, is unsustainable because of strategic deviations. In other words, each player would be best off if the her opponents acted cooperatively, but when they do, she is best of to act selfishly; so no one cooperates. This raises the interesting question as to why we see so much cooperation and collusion in the real world? As always a first reason why is that we just have the utilities wrong. In the Public Goods Game, people might care not only about their own payment but that of all the other players (i.e., altruists). But this is not reflected in our assignment of payoffs. A second reason is, again, that interactions are repeated. Firms compete not just at an instant but over a long period of time. Lastly, there are many tension that these games fail to capture, and the environment is very stylized. Nonetheless, these IO models provide a sense of what we can do with game theory, and how we can apply the tools of dominance, rationalizability, and Nash Equilibrium. Moreover, while their predictions should not be taken as exact, they give a good sense about the effect of the different tensions in a model: competition drives down the price, free riding keeps people from efficiently utilizing resources, etc.

Mixed Strategies and Mixed Eq

Recall that the Matching pennies game, given by the payoff matrix in Figure 11, has no pure strategy equilibrium (there was no box that has two stars in it). It seems reasonable that when playing Rock Paper

Scissors, or Matching pennies, a player should not play one action over and over. They the real life solution seems to make a (seemingly) random choice. Within our framework, this is perfectly admissible since we define strategies to be elements of $\Delta(A_i)$. But what does it mean?

Conceptually, mixed strategies, or randomizations over actions, are odd entities to interpret. From our intuitive point of view, we know that only one action will ultimately be taken, regardless of what the selection of that action involved. Because of this, it is unclear how a mixed strategy would ever be observed in real life, since we can only see the realization of the random process.[4] There are three resolutions to this problem of interpretation:

1. The use of mixed strategies is a mathematical convenience. That is to say, although players do not actually behave randomly, we can describe their aggregated play as if they did, and this leads to very nice results. Of course, this argument is one of utility, and is therefore not particularly convincing.

2. Player's base their strategies on some underlying randomness, and play accordingly. Imagine you could program a computer to to play Rock Paper Scissors for you. You would likely not want the computer to play in a deterministic manner. To this end, you could utilize a random number generator to choose each action with a specified probability (and hence, play a mixed strategy). Then, the jump from computers to people, requires only that we can preform some type of random operation within our brains, and carry out the action it intends. This argument is probably the most common, especially in introductions to the concept.

3. Lastly, and most subtly, is an argument that mixes together the first two. Player's do not need to literally randomize their actions, so long as before the game begins, the players opponents *believe* there uncertainty about which action will be taken. That is to say, whether or not my (one time) opponent in Rock Paper Scissors plays randomly, or chooses rock for sure, so long as I believe there is an equal chance that she plays each of his actions. This interpretation is not exactly standard, it relegates the randomness to the opponents beliefs, rather than the player herself. While this interpretation comes with its own difficulties, it has the nice feature that the predictions of games will be actions (which are observable) rather than randomizations (which are not), while still allowing us to work with tools mixed strategies provides.

There is no correct interpretation, and the list above is by no means exhaustive (nor are its contents mutually exclusive). The arguments for how we interpret our models are very important, but are often overlooked or pushed into the realm of philosophy; we need not only to consider how to mechanically solve models, but also what it is that the models represent. However, for the time being we will take a mechanical route (circling back from time to time to discuss the foundational philosophy). Specifically, we will treat mixed strategies as actual objects, things that can be played, and which result in random outcomes (over which, thanks to our expected utility hypothesis, players have well defined preference).

When defined Nash Equilibrium, we did so over lotteries of actions (i.e., mixed strategies), and so the definition still applies. We are searching for the same object as always, just in an expanded space:

[4] Even when looking at many observations of the same game, it is impossible to distinguish between players who act randomly, and players who act deterministically but whose strategies are very complex (i.e., play rock, then paper twice, then rock again, then scissors twice, etc.).

- ▷ In an NE each player is choosing a probability distribution over actions such that no one wants to change her **probability distribution**.
- ▷ Pure strategy NE are just a special case, where the probability distribution places probability 1 on a particular action.

Consider the MATCHING PENNIES game, shown (again) below

$$
\begin{array}{ccc|c|c|}
 & & \multicolumn{2}{c}{\text{g:}} \\
 & & \multicolumn{1}{c}{A} & \multicolumn{1}{c}{I} \\
\cline{3-4}
\text{b:} & A & 1,-1 & -1,1 \\
\cline{3-4}
 & I & -1,1 & 1,-1 \\
\cline{3-4}
\end{array}
$$

Figure 15: MATCHING PENNIES payoff matrix.

- ▷ We know there is no pure strategy NE. Lets look at mixed strategy eq.
- ▷ We will first examine the best response for b given that g plays a random strategy.
  - ▷ We will do this by denoting p as the probability that g plays A.
  - ▷ Then it must be that g plays I with probability $1-p$.
  - ▷ And correspondingly, q, $(1-q)$ are the probabilities that b plays A or I, respectively.
- ▷ What if g plays A with more than .5, i.e., $p > .5$.
  - ▷ Intuitively, it is obvious that b should play A all the time.
- ▷ Likewise, if $p < .5$ then always play I. If $p = .5$ anything goes.
- ▷ We can show this formally:
  - ▷ We start by finding the BR correspondence for b:
  - ▷ We will look at the best response for each of g possible strategies.
  - ▷ Fixing any p, then for a given q, b's payoff is

$$
\begin{aligned}
u_b((q,1-q),(p,1-p)) =& q[p+(-(1-p))]+(1-q)[1-p+(-p)] \\
=& q[2p-1]+(1-q)[1-2p]
\end{aligned}
$$

  - ▷ So if $p < .5$, then $1-2p > 2p-1$ (make q as small as possible, therefore 0 -always play I)
  - ▷ So if $p > .5$, then $2p-1 > 1-2p$ (make q as big as possible, therefore 1 -always play A)
  - ▷ If $p = .5$, then $1-2p = 2p-1$ therefore changing q does not change the payoff –anything goes.
- ▷ In the left half of figure 16, we plot this BR correspondence graphically.
- ▷ We can repeat this to find g's best response, maximizing her payoff given b's choice of q.

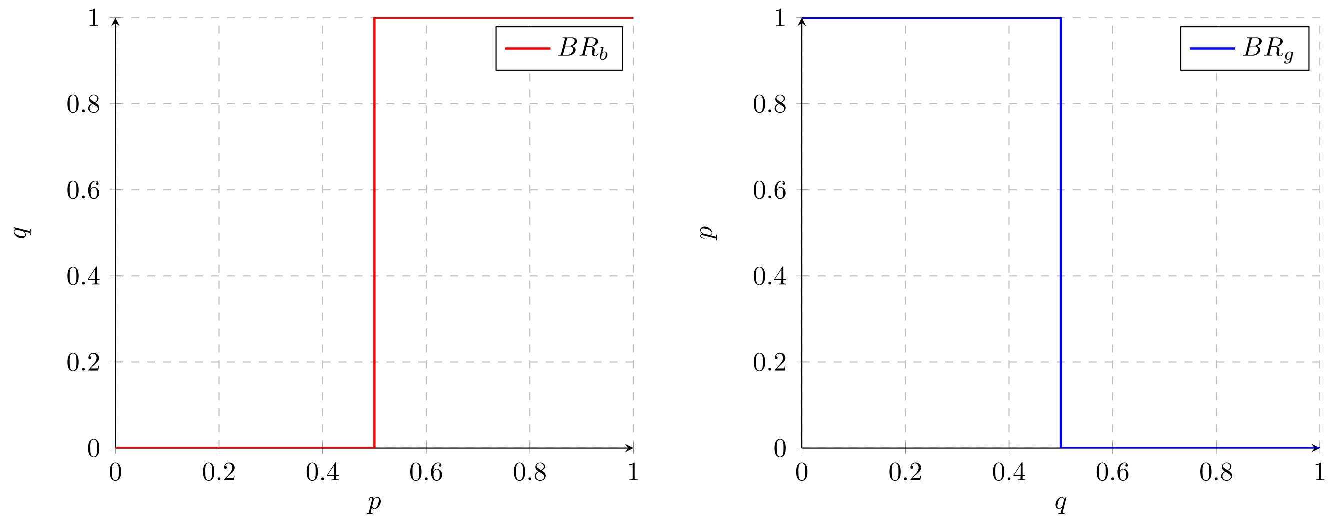

Figure 16: The best response correspondences for the MATCHING PENNIES game.

- ▷ We will get the exact opposite.
  - ▷ If player b plays A with more than .5, i.e., $q > .5$, it is obvious that g should play I all the time (she wants the mis-match),
  - ▷ Likewise, if $q < .5$ then always play A. If $q = .5$ anything goes.
- ▷ Again, we can graph this, as shown in the right half of figure 16.

Now that we have the BR correspondences, we just need to find a pair of strategies that are contained in each others BR correspondences. To help us with this, we will use the same trick from our analysis of the COURNOT DUOPOLY model. We will flip the axes of g's BR graph, and superimpose it ob b's BR graph, as shown in figure 17. Then, any intersection is a NE. So the unique NE in the MATCHING PENNIES game, is for play players to randomize evenly between the two actions, which we donate by $([\frac{1}{2}, \frac{1}{2}], [\frac{1}{2}, \frac{1}{2}])$. Each pair of outcomes occurs $\frac{1}{4}$ of the time.

$\mathscr{G}$**ame.** (BATTLE OF THE SEXES). Imagine a couple (who, after meeting by chance at the beach, and beginning a relationship based on their mutual love of Tapas) have agreed to meet at a concert this evening, but cannot recall if was Bach's Cello Suites or Stravinsky's The Rite of Spring on which they had agreed (and the fact that they forgot is common knowledge). The husband, of the quieter disposition, prefers most of all like to see the Cello Suites. On the other hand, the wife, a lifelong dabbler in dance, would like to go to The Rite of Spring. Both, however, would prefer to go to the same place rather than different ones. If they cannot communicate, where should they go?

Lets imagine the BATTLE OF THE SEXES game with the following payoff matrix, shown if figure 18. There are two obvious pure strategy NE, (B, B), and (S, S). However, as we shall see, there is also a mixed strategy NE.

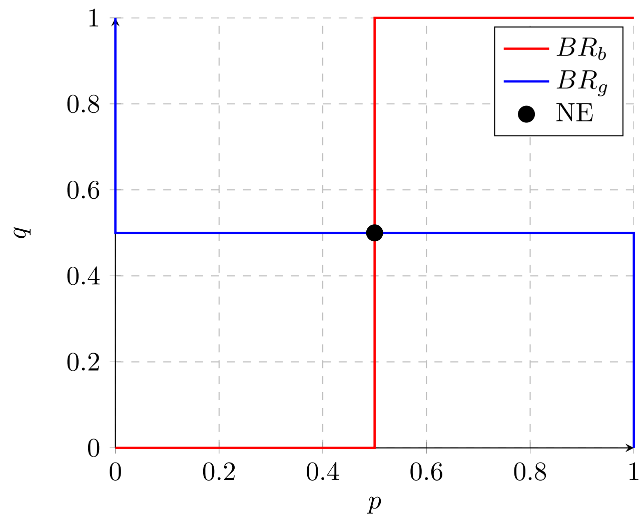

Figure 17: The best response correspondences for the MATCHING PENNIES game, superimposed.

W:

		B	S
H:	B	$*2,1*$	$0,0$
	S	$0,0$	$*1,2*$

Figure 18: BATTLE OF THE SEXES payoff matrix, with BR stars.

▷ We will start by analyzing the BATTLE OF THE SEXES game by looking at the best response for the H, given W's strategy. As before, we can let p denote the probability W plays B, and q the probability H plays B.

▷ So the utility of a given strategy profile (p,q) is

$$\begin{aligned} u_H(q,p) &= q[2p + 0(1-p)] + (1-q)[0p + 1(1-p)] \\ &= q[2p] + (1-q)[(1-p)] \end{aligned}$$

▷ So the best response correspondence is given by

$$BR_H(p) = \begin{cases} q = 1 & \text{if } 2p > 1-p \text{ if } p > \frac{1}{3} \\ q = 0 & \text{if } 2p < 1-p \text{ if } p < \frac{1}{3} \\ q \in [0,1] & \text{if } 2p = 1-p \text{ if } p = \frac{1}{3} \end{cases}$$

▷ Now, looking at the best response for the W, given hubby's q.

▷ So the utility of a given strategy profile (p,q) is

$$\begin{aligned} u_W(q,p) &= p[1q + 0(1-q)] + (1-p)[0q + 2(1-q)] \\ &= p[q] + (1-p)[2(1-q)] \end{aligned}$$

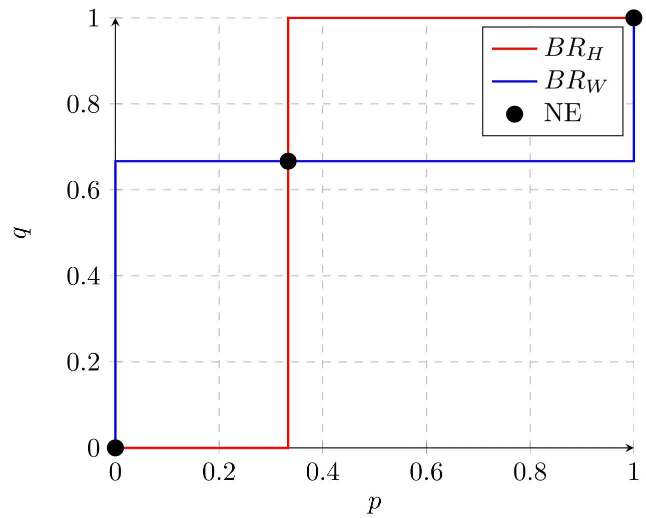

Figure 19: The best response correspondences for the Battle of the Sexes game, superimposed.

- ▷ So the best response correspondence is given by

$$BR_W(q) = \begin{cases} p = 1 & \text{if if } q > 2(1-q) \text{ if } q > \frac{2}{3} \\ p = 0 & \text{if if } q < 2(1-q) \text{ if } q < \frac{2}{3} \\ p \in [0,1] & \text{if } q = 2(1-q) \text{ if } q = \frac{2}{3} \end{cases}$$

- ▷ These BR correspondences are graphed (in standard superimposed fashion) in figure 19
- ▷ So there is a NE at $([\frac{1}{3}, \frac{2}{3}], [\frac{2}{3}, \frac{1}{3}])$.
- ▷ Also, note that the pure strategy NE are also found via this method, and labeled on the graph. As such, a search for mixed strategy NE finds all NE of the game.

The Indifference Principle

Look back at the previous two examples; notice that each player, when playing a mixed strategy, is indifferent to whatever she plays. In other words, in these equilibria, although the players must play specific proportions, they are indifferent to playing distribution over any of the strategies they are mixing between. Looking specifically at Matching pennies: both players play each strategy with probability $\frac{1}{2}$. Examining the BR correspondences, when either player randomizes in such a way, the opponent is just as happy to play A or I or any distribution thereover.

This is not a coincidence. Consider a player of a game choosing between the actions a and b, or a lottery of the two actions, given her opponents strategy. If action a is such that the expected payoff is higher than action b, she is always better off just playing that action a than mixing (or, obviously, than choosing b). We can see this mathematically by considering the payoff is given a mixed strategy. Recall that a mixed strategy is a lottery over actions so we can write the strategy as

$$s_i = (p(a_1)\dots p(a_k)\dots p(a_K)) \qquad a_n \in A_i$$

where the vector is weakly positive and sums to 1. Now, fix the other players strategy s_{-i} (we can do this because any equilibrium strategy must be a best response to the equilibrium profile of strategies, so assuming we are in an equilibrium, player i's strategy must be a best response to that specific strategy). The utility that player i receives when playing mixed strategy s_i is

$$u_i(s_i, s_{-i}) = \sum_{n=1}^{N} u_i(a_n, s_{-i})p(a_n)$$

As always, out ability to write the players' utility as an expectation over the outcomes they induce, is founded by our expected utility assumption. So clearly if there exists some a_n, a_m such that $u_i(a_n, s_{-i}) > u_i(a_m, s_{-i})$ and $p(a_m) \neq 0$ then the player can increase her utility by shifting probability form a_m to a_n. In particular, by playing the strategy $t_i = (q(a_1)\dots q(a_k)\dots q(a_K))$, where q is defined by

$$q(a_k) = \begin{cases} p(a_k) & \text{if } k \neq n, m \\ p(a_n) + p(a_m) & \text{if } k \neq n \\ 0 & \text{if } k \neq m \end{cases} \tag{15}$$

It is immediate that the vector q forms a probability distribution, and so t_i is a strategy. But the utility of t_i is higher than the utility of s_i, so s_i is not a best response, and therefore, (s_i, s_{-i}) cannot be a NE profile.

In fact we can say more:

Theorem (Indifference Principle)**.** *A mixed strategy profile is a NE if and only if*

1. *the expected payoff for player i, given s_{-i}, is the same for all a_i played with positive probability, and,*
2. *the expected payoff for player i, given s_{-i}, to all actions played with zero probably is at **most** the payoff to any action assigned positive probability.*

The proof of this theorem is very similar to the line of reasoning use above, albeit with a bit more care to obtain the bidirectional implication, and is so omitted. Since the theorem is an *if and only if* statement, it mean that it runs both ways. If a strategy profile is a NE is satisfies (1) and (2), and if a strategy profile satisfies (1) and (2), it is a NE. This is very useful, and will help us to find and verify NE in complicated games. We know that if a player is playing a mixed strategy, it must be that she is indifferent to all the actions on which he places positive weight. So her opponents must be playing a strategy so as to make this happen.

The indifference principle can be used to verify a NE. For example, we will verify that $([\frac{3}{4}, 0, \frac{1}{4}], [0, \frac{1}{3}, \frac{2}{3}])$ is a NE in the game shown in figure 20. All we need to do is calculate the payoffs for each player, given the other players strategy and then check that (1) and (2) of the Indifference Principle hold. This is much easier than comparing the strategy to every possible deviation.

▷ Given player 2s strategy, player 1's payoffs to each action are:

		P2:	
	L	C	R
P1: U	$2,2$	$3,3$	$1,1$
P1: M	$6,0$	$0,6$	$2,2$
P1: D	$6,4$	$5,1$	$0,7$

Figure 20: $([\frac{3}{4}, 0, \frac{1}{4}], [0, \frac{1}{3}, \frac{2}{3}])$ is a mixed strategy NE.

- ▷ $T : \frac{1}{3}3 + \frac{2}{3}1 = \frac{5}{3}$
- ▷ $M : \frac{1}{3}0 + \frac{2}{3}2 = \frac{4}{3}$
- ▷ $B\frac{1}{3}5 + \frac{2}{3}0 = \frac{5}{3}$

▷ So she is indifferent to T and B and prefers these to M.

▷ Likewise, we do the same for player 2.

- ▷ $L : \frac{3}{4}2 + \frac{1}{4}4 = \frac{10}{4}$
- ▷ $C : \frac{3}{4}3 + \frac{1}{4}1 = \frac{10}{4}$
- ▷ $R : \frac{3}{4}1 + \frac{1}{4}7 = \frac{10}{4}$

▷ So she is indifferent to all three actions. But then playing any lottery over them is a BR, including a lottery that puts no probability on L.

We can also use the Indifference Principle to find mixed strategy NE. This can be very useful.

		P2:
	L	R
P1: U	$10,9$	$0,8$
P1: D	$7,1$	$6,5$

Figure 21: A payoff matrix.

Consider the game presented in figure 21. There are two pure strategy NE, (U, L), (D, R). In addition there is a mixed strategy NE, which we will finding using the indifference principle. Since we know that if P1 is mixing, she must be indifferent between U and D. So, without worrying about her actions, we can find the strategy that P2 *must* be playing to induce indifference: letting q denote the probability that P2 plays L, it must be that:

$$\begin{aligned} 10q &= 7q + 6(1-q) \Rightarrow \\ 10p - 7p = 3q &= 6(1-q) \Rightarrow \\ 9q &= 6 \Rightarrow \\ q = \frac{6}{9} &= \frac{2}{3} \end{aligned}$$

so, P2 plays $(\frac{3}{2}, \frac{1}{3})$. Since this is a mixed strategy, it likewise must be that she is indifferent between L and

R. This means, given p as the probability that P1 plays U:

$$\begin{aligned}
9p + 1(1-p) &= 8p + 5(1-p) \Rightarrow \\
9p - 8p = p = 5(1-p) - (1-p) &= 4(1-p) \Rightarrow \\
5p &= 4 \Rightarrow \\
p &= \frac{4}{5}
\end{aligned}$$

So P1 plays $(\frac{4}{5}, \frac{1}{5})$. Since this is a mixed strategy, she must be indifferent between U and D, but, of course, we already verified that. So this pair of strategies satisfies the conditions of the Indifference Principle and is therefore a NE.

We ignored the constraints on the players utilities, only worrying about making the other player indifferent. But if the other player is indifferent (so all her actions) then she can play *any* strategy, and so the resulting profile is a NE.

There is a subtly here that is worth exposing. It seems counterintuitive that, rather than attempting to maximize payoff, we find a strategy that makes the *other* player indifferent between her actions. The story we use to motivate NE, about the non-stability of a profile if players can selfishly deviate, seems to have no place in this methodology. The resolution is that we are simply using the Indifference Principle as a tool to find NE (and it works, as you can see above!). That the Indifference Principle fully characterizes NE is a deep result, but the fact that it is easier to solve the for NE by finding the strategies that make player indifferent is not deep at all. It is simply makes the math easier, and we utilize the simplification.

Of course, it is not always possible to make an opponent indifferent to all actions. For instance, it is impossible induce a player to play a strictly dominated action. Consider the Prisoners Dilemma game. No weights that player 2 plays will ever induce player 1 to play anything but D. This can be formalized,

Theorem. *No player will ever play a strictly dominated action with positive probability a NE.*

Recall that even without the Indifference Principle machinery, we already had a theorem that stated a strictly dominated strategy will not be played in a NE. Therefore, even without appealing to indifference conditions, the above result follows directly from the following lemma.

Lemma. *A mixed strategy that places positive probability on strictly dominated action is itself strictly dominated.*

The proof of this lemma is straightforward, and so left as an exercise. Hint: to construct a dominating strategy, use a similar argument as in the construction of q as in (15). Then to see that the Lemma proves the Theorem, note that if s_i places positive probability on a strictly dominated action, it must be strictly dominated, and therefore, by the previous theorem, cannot be part of a NE.

Existence

We now have all the requisite ideas to express one of the nicest results in game theory. It asserts the existence of a Nash Equilibrium.

Theorem (Existence of Nash Equilibria)**.** *Every game with a finite number of expected utility maximizing players, each of which has a finite number of actions, has a mixed strategy NE.*

- ▷ There does not always exist a NE is pure strategies, so this result is not trivial –it assures we are not searching for something that doesn't exist.
- ▷ The proof of this is quite advanced.
  - ▷ First construct a giant BR correspondence that takes a *profile* of strategies, and returns each player's individual BR to the profile. This is itself a profile of strategies. So we have $BR : S \rightrightarrows S$.
  - ▷ Then, show that this correspondence satisfies certain properties to ensure the existence of a fixed point (where the functional value is the same as the argument).
  - ▷ A fixed point of the BR correspondence is a NE by construction: everyone is best responding simultaneously.
- ▷ While the proof is very elegant it is not constructive –it does not help up find the NE, only asserts their existence.
- ▷ This is unfortunate, finding NE is difficult, especially if the number of players or actions gets big (as an aside, this problem is thought to be live outside P, the class of polynomial time solvable problems.)
- ▷ Note, that finiteness is *not* a requirement, but a sufficient condition. For example:
  - ▷ Imagine the game: An infinite number of people choose an integer, if the all the integers are equal everyone wins, otherwise everyone loses.
  - ▷ This game has an infinite number of players, and an infinite action space for each player.
  - ▷ But, everyone choosing 1 is clearly a NE.

Extensive Form Games

We now have, more or less, a basic understanding of what we want to get out of a game (a prediction on players choices, given different behavioral assumptions). For most of the remainder of the class, we will be changing different aspects of the game environment, to see how our solution concepts, predictions, and tools, must change in response.

Until now, we have assumed that players take actions simultaneously. This is a very limiting assumption. Many, if note most, real world economic interactions happen over time, where each player gets to make decision contingent of previous events. For example of some environments that require sequential moves and strategies over time:

- ▷ A firm deciding to enter a market, and an incumbent deciding how to price.
- ▷ Contracts in firms, followed by worker effort.
- ▷ Board games, like chess, checkers, etc.

In order to represent this aspect of strategic interaction we need to include *timing* into out notion of a game. We will retain all of the previous elements: a set of players, a set of actions, and a set of payoffs, but we extend this to include timing: when each action is permitted, and in what order the players play.

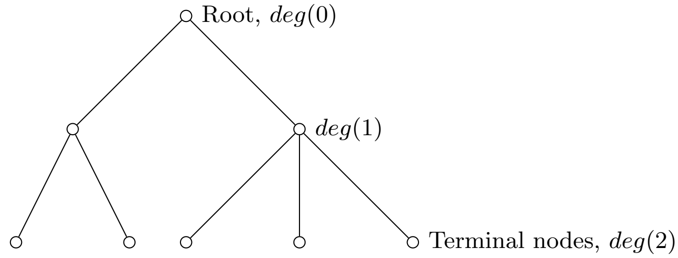

Figure 22: A rooted tree.

Just as it was notationally convenient to represent simultaneous move games as matrices (called a *normal form* game), it will be helpful to sequential move games using a type of graph called a *rooted tree* (and these games are called *extensive form* games). A rooted tree is a graph (a set of nodes and edges connecting them) with the following defining properties:

- ▷ There is a distinguished node, called the *root*.
- ▷ Every node is connected to the root via a unique path (called a branch).
- ▷ The number of edges between a node and the root, is the *degree* of the node. The root has degree 0, every other node has degree > 0.
- ▷ For two connected nodes, n, m, that are connected, if $deg(n) < deg(m)$, then n is called a parent of m, and m a child of n.

From these properties, we can show that:

- ▷ Every node, except the root, has a single parent.
- ▷ No two nodes can be connected and have the same degree.
- ▷ There are no cycles in the graph.

An example of a rooted tree is shown in figure 22.

Using the language of trees, we can easily define extensive form games. Before the formalities, the basic idea is that the root of the tree is the beginning of the interaction, and each action moves along the path of the tree. As time moves forward, the degree of the nodes increase as well. (We can see how the properties of the tree support this interpretation).

Definition. *An* ***extensive form game*** *is a set of players, $i = 1 \dots N$ and a rooted tree, such that*

1. *Each non-terminal node belongs to some player, and has a corresponding action set for that player.*
2. *The edges flowing out of each node correspond to this set of actions.*

3. *Each terminal node specifies a payoff for each player.*

For simplicity, we will often keep all nodes of the same degree belonging to the same player (this is always possible by adding dummy nodes with a single action). Lets construct the game tree for the following game (which is shown in figure 23).

$\mathscr{G}$**ame**. (SIMPLE ULTIMATUM GAME). There are 2 players. Player 1 has 10 dollars. She can choose to **give** (10 dollars to P2 and 0 to P1) **share** (5 dollars to each player) or **keep** (10 dollars to P1 and 0 to P2). After he makes his decision, P2 can **accept** or **reject**. After accepting payoffs are as specified, after rejecting and everyone gets 0.

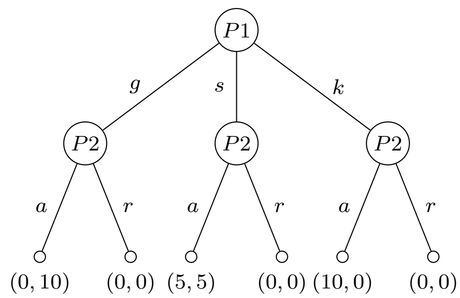

Figure 23: The game tree for the SIMPLE ULTIMATUM GAME.

Because there is timing involved, we need to be more careful about how we think about strategies. In the above game, player 2 can condition her action on the action of player 1. It is reasonable, for example, that player 2 may want to play reject if player 1 offers 0, but play accept in player 1 offers 10, and to randomize if player 1 offers 5. We need to specify what the player will do in **any possible situation**. A strategy is therefore no longer just an action (or lottery over actions) but an assignment of actions (or lotteries over actions) to **every node** of the game tree belonging to the player in question.

Definition. *In an extensive from game, a* ***strategy*** *for player i is a complete contingent plan: a lottery over the available actions at* every *node in the game tree belonging to player i.*

This is important! We need to specify what will happen even in hypothetical situations, (had a previous player played differently). By specifying actions at every point in the tree, we will be able to see the effects of tempting options, threats, etc., on players actions. Think about the PRISONERS DILEMMA: if players just choose between (C, C) and (D, D), they would pick (C, C). It is the presence of the other payoffs that create the motivation to play D. Because these considerations will dictate what solutions to games we find reasonable, it is necessary to define strategies as complete continent plans, rather than just actions along the path of play. But it will also make our lives a little more complicated: as we will now see, even simple games have very large strategy spaces.

- ▷ The game Tic-Tac-Toe has *lots* of (pure) strategies.
    - ▷ The first player has 9 possible actions for the first move.
    - ▷ The second player has 8 possible moves for *each* of the 9 first moves (so there are $8 \times 9 = 72$ 2 move configurations).

- ▷ So, iterating all the way down there are: $9! = 362880$ possible game outcomes.

▷ But there are a lot more strategies: we need to specify not only which outcome obtains, but also what the player would do at every node.

- ▷ It turns out that there are $5^{9\times8\times7\times6}$ strategies for player 1.
- ▷ This far exceeds the number of atoms in the observable universe!

▷ And, Tic-Tac-Toe is a simple game. In chess, there are 32 pieces, and many many board configurations. Each piece can go many places.

▷ Enumerating all the strategies in complex games is impossible.

To see why this methodology is necessary, and to get a sense of how to analyze extensive form games, lets look at the strategies for the Simple Ultimatum Game.

▷ P1 only has one node, so her actions are $\{G, S, K\}$.

▷ The situation for player 2 more complicated.

- ▷ A strategy specifies what he will play after each of player 1's actions.
- ▷ So her actions are $\{AAA, AAR, ARR, ARA, RRA, RRR\}$, where each entry in each pure strategy corresponds to the action to take at the corresponding node.

▷ (We only have to worry about pure actions, the result of any randomization will, as before, be the expectation over the pure strategies).

Using our different notion of a strategy does nothing to change the definition of a NE. The exact same definition, and reasoning, applies. Now we can analyze the game in the same way as before. There are many NE:

▷ Consider the action profile: (K, AAA).

- ▷ Player 2 will always accept, given this it is P1's best response to keep all the money.
- ▷ Player 2 also cannot do better, since no matter what she does P1 keeps all the money.

▷ What about (K, RRR).

- ▷ This is also a NE.
- ▷ Player 2 always rejects so it does not matter (to P1) what P1 offers, he cannot deviate.
- ▷ Since he offers nothing, P2 can do no better than by rejecting.

▷ What about (G, ARR).

- ▷ This is also a NE.
- ▷ Player 2 will reject unless he gets all the money. So P1 gets 0 after every action, and therefore G is a best response.

- ▷ Since P1 offers P2 a positive amount, her best response is to accept.

▷ Likewise we have (S, AAR) and (S, RAR).

- ▷ The second one seems funny: why wouldn't P2 play A after play 1 offers G but does after S?
- ▷ It doesn't matter: because P1 is playing S (and because it is a best response given P2's strategy) then P2's action after G has no bearing on her payoff.

So what isn't a NE? There are two things that could go wrong:

▷ If P1 offers something and P2 would have accepted less: (G, AAA), (G, AAR), (S, AAA), (S, RAA), (S, ARR).

▷ If P2 rejects when she would have gotten a higher payoff by accepting: (G, RRR), (G, RAA), (S, ARR), etc...

This is a lot of different equilibria. If one of our goals is to make predictions in strategic environments, we are not doing so well. Also there are lot of unintuitive things happening: why would P2 ever reject after G? We would like to kill two bunnies with one stone[5] and reduce the number of predictions while also ruling out unintuitive behavior. To get a better sense of this, lets look at another game.

$\mathscr{G}$**ame**. (ENTRANT GAME). There is an incumbent firm in a market. A new firm wants to enter. If the entrant does not enter, the incumbent firm remains a monopolist, profit for the two firms is (0,10). If the entrant firm enters then there are two possible actions for the incumbent. It can fight (lower prices to hurt the entering firm) or acquiesce (keep prices high and share profits). If the incumbent fights then the payoffs are $(-2, 2)$ and if it acquiesces then the payoffs are $(5, 5)$.

We can draw this game as in figure 24.

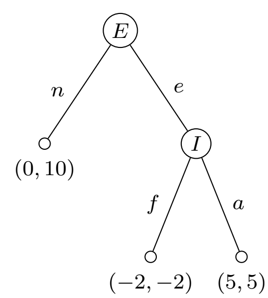

Figure 24: The game tree for the ENTRANT GAME.

What are the strategies in this game? The strategy profiles?

▷ For the entrant: $\{E, N\}$.

▷ For the incumbent $\{F, A\}$.

▷ And since each player only has one node, the strategy profiles are $(N, F), (N, A), (E, F), (E, A)$.

[5] You really realize how morbid that expression is when you change birds to bunnies

What are the NE?

- ▷ (N, F): since P1 is not entering, both or P2's actions are BR (we call this off path, since they do not occur along the path of play). Further, since P2 would fight, P1 is better off not entering.

- ▷ (E, A): Since P1 is entering, P2 gets a higher payoff by acquiescing. Since P1 is acquiescing, P2 should enter.

Now we can check that the other strategies are not NE:

- ▷ (N, A): Since P2 is playing A, the entrant could get a higher payoff by entering $(5 > 0)$ so N is not a BR.

- ▷ (E, F): Since P2 is playing F, the entrant could get a higher payoff by not entering $(0 > -2)$ so E is not a BR. Likewise, since P1 is entering, the incumbent would be better off playing A: $(5 > 2)$.

(N, F) is a strange equilibrium: if the entrant had entered, and *then* the incumbent made the its decision, we know that the incumbent would not want to fight. It is only because, in our definition of a strategy profile, we allow the incumbent to commit to fighting, that the entrant does want to enter. But this seems to assume away the most important aspect of extensive form games, that the timing matters! If these actions were actually taken in a sequence, then the incumbent would have to make its decision after seeing the entrants choice to enter or not. And if the entrant did choose to enter, fighting would not be the best response.

So, the incumbents threat to fight is not credible. The incumbent is able to make the threat (and so, it is technically a NE to do so) but when the time comes to carry out the action, it would want to change its action. The same is true in some of the equilibria we found in the ultimatum game: after being offered S, player 2 should never reject, however, (G, ARR) is an NE.

Subgame Perfect Equilibrium

In this section we will propose a *refinement* of NE, that will restrict players from making incredible threats. The motivation is that, if the game is being played sequentially, we know that when the time comes a rational player would change her strategy to maximize her forward looking payoff. By nature of time being one directional (or, in the leaguing of rooted trees, that actions always increase the degree of the node), player will care only about how their action affects future happenings, namely payoffs.

If a player is only looking forward, at what will happen further down the tree, then whatever happened previously is irrelevant; the actions taken so far in the game have no bearing. As such, we can think about a new game that begins at the current node. A rational player must play rationally in this *subgame* as well. We can make this formal:

Definition. *A* ***subgame*** *is any game tree whose root is a single node of the larger tree.*

Note: the game itself is also a subgame. To elucidate this definition figures 25 and 26 highlight the various subgames contained within the previous examples, the Simple Ultimatum Game and entrant game. Now the problem noted above with equilibria like (G, ARR) or (N, F), is that players are playing strategies that dictate non-maximal behavior in particular subgames (i.e., after the offerer offered some money, or

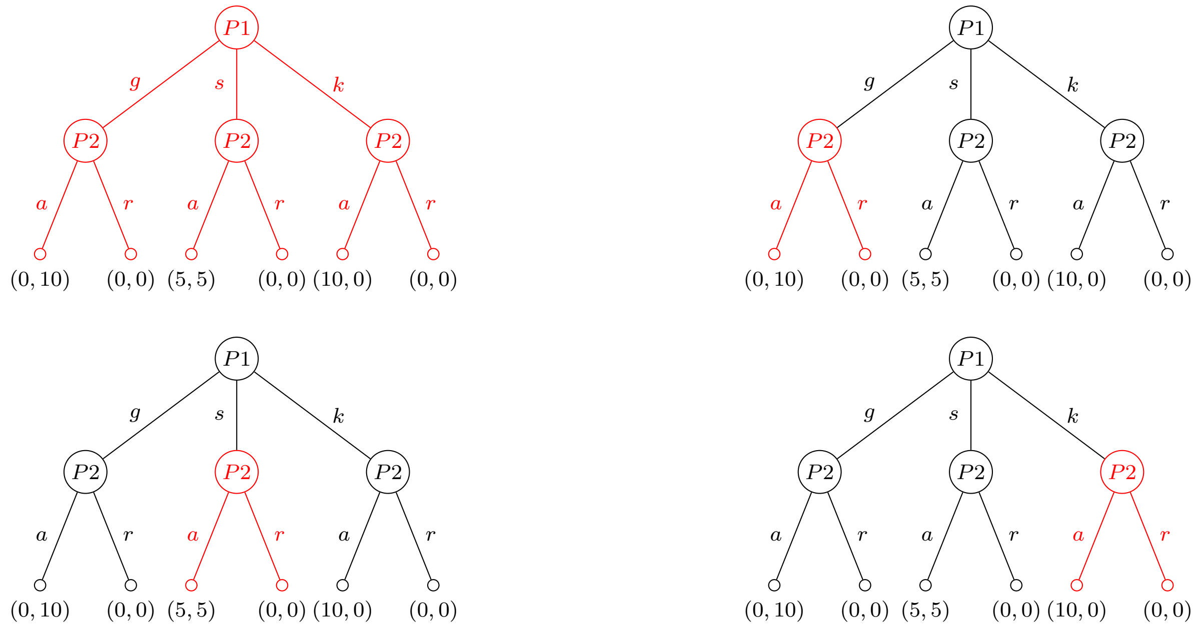

Figure 25: The four subgames of the SIMPLE ULTIMATUM GAME (in red).

Figure 26: The two subgames of the ENTRANT GAME (in red).

after the firm entered) –these strategies constituted NE only because the problematic subgames were never reached on path. But, we we discussed above, such behavior is, in a sense, incredible (i.e., *not* credible), since a rational player would know that her rational opponent would actually prefer to switch her strategy if the subgame was in fact reached.

We can get rid of this behavior, and the counterintuitive predictions, by insisting that player play a best response in *every* subgame, even the ones that might not get reached. If a profile of strategies satisfies this for all players we call it a subgame perfect equilibrium.

Definition. *A **subgame perfect NE** (SPNE) is strategy profile* $(s_1^* \dots s_N^*)$ *such that each strategy is a Nash Equilibrium given the profile, in every subgame.*

One immediate consequence: every SPNE is also NE. This follows immediately from that fact that the game itself is also a subgame.

Given the definition of SPNE, it may seem that finding and verifying such equilibria is substantially harder that with regards to regular NE, since it requires checking at every subgame. However, a nice property of SPNE is that they are easy find, because they are the result of backwards induction. Backwards induction is the idea that we should look to the end of the game first: when the consequences of actions are easy to distinguish. Then work our way back to the first stage.

The working our way back process works roughly like this. If the game is of degree n,[6] then at all (non-terminal) nodes of degree $n-1$, every action leads to a terminal node, and hence a payoff. In these subgames, it is easy to see which actions will be taken by the active player: we just maximize her payoff. But then, we can replace each node of degree $n-1$ with the corresponding payoff. Why? Because by the requirement of SPNE, if we get to this node, the active player must best respond, and so, the outcome is already known. But then, we have a new game of degree $n-1$, so we can repeat with all nodes of $n-2$. Eventually, this process terminates, and the resulting string of best responses is a SPNE.

This all may seem very ephemeral, so let's see how this works for the simple ultimatum game.

- ▷ Given an action of P1, P2's best response is
  - ▷ $BR_2(G) = \{A\}$
  - ▷ $BR_2(S) = \{A\}$
  - ▷ $BR_2(K) = \{A, R\}$
- ▷ Therefore we can replace the three proper subgames in figure 25 with the outcome that will follow these best responses. This is show in figure 27.
- ▷ Notice that, because there are two best responses to K, there are actually two cases to check. They were both placed on a single graph solely for readability.
- ▷ Now, the subgame prefect equilibria are clear: (K, AAA), or (S, AAR).

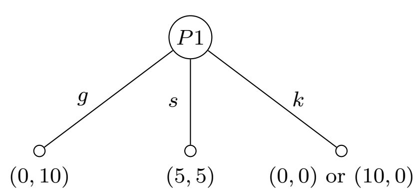

Figure 27: The game tree for the Simple Ultimatum Game, after one round of backwards induction reasoning.

We can do this again with the entrance game: The incumbent's best response correspondence, at each action set is given by

- ▷ $BR_2(E) = A$

Again, we can we can replace the subgames with the outcome that will follow it, as in figure 28. Again, SPNE predict (E, A) which does not fall victim to the problem of incredible threats. In both of these cases, we got rid of the equilibria that we did not want.

[6] The degree of the game is the highest degree of any node in the game.

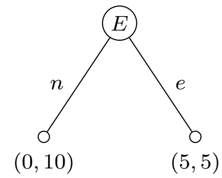

Figure 28: The game tree for the ENTRANT GAME game, after one round of backwards induction reasoning.

Next, we will look at a game that requires more than a single round of backwards induction reasoning. In fact, we will describe a game such that, for any $n \in \mathbb{N}$, can be made to require n iterations.

$\mathscr{G}$**ame**. (n-STAGE CENTIPEDE GAME). There are two players and n rounds. The two players sit across from each other at a table. Initially there are two stacks of coins, in front of player 1: one of size 4 and the other of 1. The active player alternates each round, beginning with P1. At each round (strictly less than n) the active player can choose between **take** –where she gets the larger pile of coins, and the inactive player gets the smaller pile of coins, and the game ends– or **pass** –where the active player pushes the coins across the table, (switching the roles of the players) and in the process, doubling *both* stacks of coins. In the final round, the actions and resulting payoffs are the same, but the game ends after either action.

The centipede game, originally named because it had 100 rounds, and the game tree sort of looked like a centipede, is show in figure 29, with 4 rounds. We draw the game tree, and show each round of backwards induction reasoning leading to the *unique* SPNE, (TT, TT). Each player takes the coins whenever she is active. This leads to the enormously inefficient outcome $(4, 1)$, and sixteen times smaller (for each player) than the outcome generated by (PP, PP). In the original game, with 100 rounds, the payoff would have been 2^{100} times larger after cooperation. Of course, this should not be surprising, as we have seen time and again, that large social payoffs do not induce cooperation in the face of strategic concerns pushing players to act selfishly. Notice that while the SPNE is unique, there are other NE. These take the form of changing off path play, which, by nature of not being reached, does not affect payoffs. For example, it is easy to check that (TT, TP) is a NE.

Arguments such as the one that solves for the SPNE for the n-STAGE CENTIPEDE GAME are often called *unraveling* arguments. The instability of the (PP, PP) action profile begins with P2's desire to deviate to (PP, PT), and from there P1 wants (PT, PT), and so on, unraveling the game back to the initial node, and creating a global instability for cooperation. Such arguments are common, and in many circumstances the defection of a single player causes the defection of another, causing the unsalvageable collapse of a particular strategy profile.

Lets take a look at one last game, before moving on to some more formal results. The game is the sequential version of the COURNOT DUOPOLY model.

$\mathscr{G}$**ame**. (STACKELBERG DUOPOLY). There are two firms, 1 and 2. Each firm can produce a single type of good (they produce the same good). The cost of producing q_i units of the good is cq_i. The price at which the goods are sold is a function of the aggregate supply $P(Q) = \max\{0, a - Q\}$, where $Q = q_1 + q_2$. Firms can choose any positive quantity to produce, but firm 1 chooses first, and then

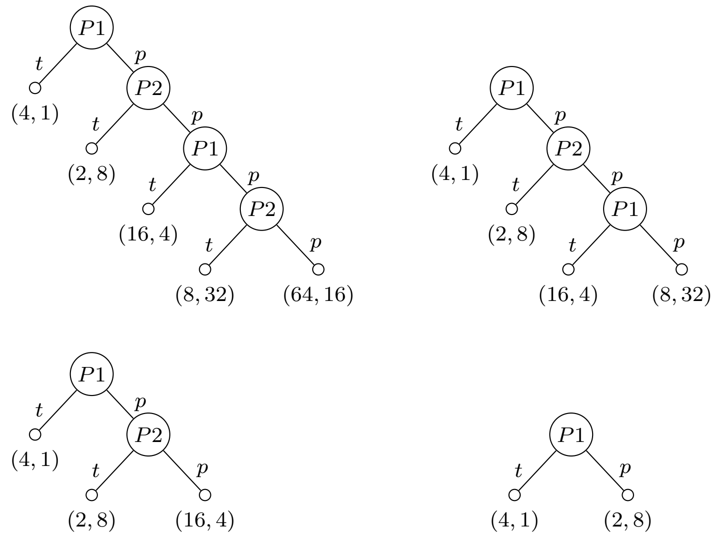

Figure 29: The game tree for the n-STAGE CENTIPEDE GAME game, with $n = 4$, after various iterations of backwards induction reasoning.

after seeing this decision, firm 2 chooses. After production, the payoff function of firm i is

$$\pi_1(q_1, q_2) = \begin{cases} q_1[a - q_1 - q_2 - c] & \text{if } Q < a \\ -cq_1 & \text{if } Q \geqslant a \end{cases}$$

We are going to look for the Subgame perfect equilibrium, and so, use backwards induction. We begin, therefore, by analyzing P2s best response given any move of P1. So lets take q_1 as fixed, and find the best response. The first order condition of the profit function with respect to q_2 is

$$a - c - q_1 - 2q_2 = 0$$
$$q_2 = \frac{a - c - q_1}{2}$$

Unsurprisingly, this is the same FOC as in the COURNOT DUOPOLY model. Except, that P1 can now take this as given, and not as an equilibrium condition. His profit is, given a choice of q_1,

$$\pi_1(q_1, q_2) = q_1[a - q_1 - \frac{a - c - q_1}{2} - c] = q_1[\frac{a - c - q_1}{2}]$$

Note the difference, in Cournot we plugged firm 2's choice into the FOC: choices were already made, we just needed the solution. Now, we plug firm 2's choice into the profit function and *then* optimize. P1 makes his choice knowing that firm two will best response accordingly.

Taking the derivative with respect to q_1, The FOC for firm 1 is:

$$\frac{a - c - q_1}{2} - \frac{q_1}{2} = 0$$

or that

$$q_1^* = \frac{a-c}{2}$$

- ▷ Plugging in to solve for firm 2's production: $q_2^* = \frac{a-c}{4}$.
- ▷ Compare to the Cournot case:
  - ▷ $q_1^* = \frac{a-c}{2} > \frac{a-c}{3} = q_1^{cd}$
  - ▷ $q_2^* = \frac{a-c}{4} < \frac{a-c}{3} = q_2^{cd}$
  - ▷ $P = a - 3\frac{a-c}{4} = \frac{a-3c}{4}$
  - ▷ $\pi_1 = q_1[\frac{a-3c}{4} - c] = \frac{a-c}{2}[\frac{a-c}{4}] = \frac{(a-c)^2}{8} > \frac{(a-c)^2}{9} = \pi_1^{cd}$
  - ▷ $\pi_2 = q_2[\frac{a-3c}{4} - c] = \frac{a-c}{4}[\frac{a-c}{4}] = \frac{(a-c)^2}{1}6 < \pi_2^{cd}$
- ▷ Total production increases.
- ▷ The first firm captures more of the profit.

Assumptions on beliefs. Just as the jump from rationalizability to NE required additional behavioral assumptions so does the jump from NE to SPNE. In keeping the analogy, since we are refining our solution concept, we require more structure. That is, since all SPNE profiles are NE (and so also rationalizable), it must be that we keep our initial assumptions on rationality, common belief thereof, and correct beliefs regarding other players' strategies. Since the converse is not true, we need something else.

The additional assumption comes from the fact that we now have different points of time modeled within the game. As modelers, we do not need to impose that players' beliefs stay the same throughout the game, we will allow player's to update their beliefs. This is only natural, for example in the ENTRANT GAME, if, at the beginning of the game, the incumbent believes with certainty (with probability 1) that the entrant will not enter, then surely it cannot still hold this belief after witnessing the entrant enter! For any coherent analysis we must allow player to revise their beliefs in the face of new information (i.e., moving along the game tree). The necessity of modeling beliefs at every node in the game makes matters a bit messy at times, and we will keep the discussion cursory.

Nonetheless, without being to formal about things, we can allude to the restriction on updating that leads to SPNE. Loosely speaking, we require that our player never loose the belief that all players are rational (so, even when changing their beliefs, will still believe players are rational), and that *this* is commonly believed at every history. So players believe their opponents are rational, and will continue to choose rationally at every point in the game, and that their opponents believe this of their own opponents, and so on. From this vantage point, just as always, the assumptions seem a bit more restrictive than initially thought. In particular, no matter how irrational a opponent seems for any length of time, you must believe she will act rationally for every point onward.

Normal Form Equivalence

Let's look at the even more simple ultimatum game where player 1 only has 2 actions: share $(5, 5)$ or keep $(10, 0)$. This is represented by the game tree in figure 30.

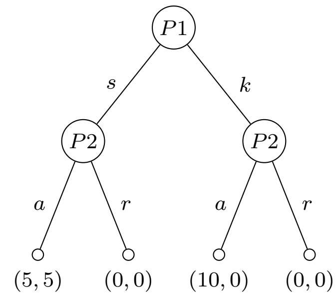

Figure 30: The game tree for the and even simpler ultimatum game.

We can easily check that there are four NE (and the first two are also SPNE):

- ▷ (S, AR)
- ▷ (K, AA)
- ▷ (K, RR)
- ▷ (K, RA)

P2:

P1:		AA	AR	RA	RR
	S	$5, 5*$	$*5, 5*$	$0, 0$	$*0, 0$
	K	$*10, 0*$	$0, 0*$	$*10, 0*$	$*0, 0*$

Figure 31: A normal form representation of the even simpler ultimatum game.

Now, we can consider the normal form game in figure 31. There are the same game! They have the same NE! The normal form game seems like it has more actions for player 2, but this is just because we need to duplicate actions in order taken into account the contingent nature of P2 strategies. Each action for player 2 is a pair, corresponding to the action she would chose from each of the nodes that belong to her in the extensive form version. We are enumerating all of the possible strategies (complete contingent plans), and allowing each player to pick a pure strategy. But, of course, this is exactly a normal form game.

Notice how each of the Nash Equilibria that are *not* subgame perfect has P2 playing a weakly dominated strategy. In this simple environment, and by definition of not being subgame perfect, there must be some situation (node in the tree, or equivalently, profile of strategies) in which the player is *not* best responding. (This situation, by nature of the profile of strategies being NE, must not occur, and hence, corresponds to a different action by the opponents.)[7]

The next result show that there is nothing special about the ultimatum game, this mapping is always possible. We can make any extensive form game into a normal form game. The entrant game is in normal form is in figure 32.

Theorem. *One can alway represent an extensive form game as a normal form game.*

[7] Technical aside: this is an oversimplification. For example, if a player's *own* strategy precludes her from reaching a node, then her actions may not be subgame perfect (at that node) but are still not weakly dominated. More damagingly, will be the introduction of information sets.

We prefer one construction to the other because it is easier to think of certain problems in certain ways: sequentiality v simultaneity. We want our models to adhere as closely as possible to our intuition, empirical evidence, and narrative motivations, and the representation of timing in the games is important for this reason.

		I: F	I: A
E:	E	$-2,-2$	$*5,5*$
	N	$*0,10*$	$0,10$

Figure 32: A normal form representation of the ENTRANT GAME.

It would be nice if this worked the other way: every normal form game can also be represented by an extensive form counterpart. Then the two definitions would be equivalent, we could use which ever suits the task. But consider the matching pennies MATCHING PENNIES game. We can try to put this in a game tree, as in figure 33, but it doesn't really work. What goes wrong?

- ▷ We are not reflecting that the two player's make a choice at the same time.
- ▷ So g always gets her way, regardless of b's choice because she gets to make a contingent choice.
- ▷ If we switch the order (g moves first), the problem persists in opposite fashion; b always gets his way.
- ▷ This is not an accurate reflection of the original game, where there were no contingent choices.

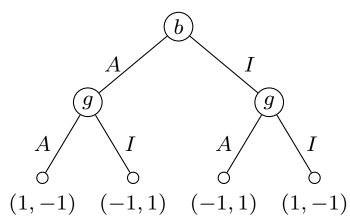

Figure 33: A (not so representative) game tree the MATCHING PENNIES game.

But, we don't really care about timing: we care about information. It is not the literal timing of the choices that matter but that when g makes her choice, she does not know what b choose. Our game tree in figure 33 cannot reflect this. So what if we add an element to the game tree that says: some player cannot tell the difference between nodes (doesn't know which action took place prior).

Definition. *An **information set** is a collection of nodes of the same degree such that the active player cannot distinguish between the nodes therein.*

We now amend the definition of a game, so that each player can only choose one action (or lottery) from each **information set**. If a player cannot distinguish between two nodes, she can only pick a single action for the whole group.

- ▷ In game trees, we will denote an information set as a dotted line between nodes.

- ▷ Of course, this presupposes that node can only be indistinguishable if the available actions are the same and that they occur at the same time –but this is a very natural assumption on knowledge. If the actions available were different, or the nodes took place at different times, the player could use this as the defining characteristic!
- ▷ Note that
  - ▷ There is still a payoff for each *node* –where you are in the information set matters, it just is unknown.
  - ▷ Mixed strategies are calculated the same as ever.
- ▷ A subgame is not defined as any tree that begins from a single node (a singleton information set).

With this additional machinery, we can see the full relation between normal and extensive form games.

Theorem. *When considering games with information sets, normal form and strategic form games are equivalent. That is, for every normal form game there is a corresponding strategic form game, and vice versa.*

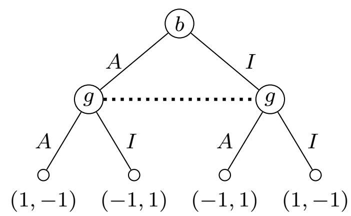

Figure 34: A (representative) game tree the Matching pennies game.

As an example of how to model a normal form game, see the representation of Matching pennies in figure 34. Now g must choose her action without knowing the action b choose. This is a very nice result, since it means we do not have to worry about which game form we are using, and use the most natural method for the job at hand. In addition it allows us to "mix" between the games – we can choose whatever is best interaction we are modeling.

Games with uncertainty

We will now relax another important assumption on the game environment that will allow us to model many more situations. So far we have assumed that the games are unfold according to the players actions and nothing else. This leaves no room for any random component to the game (except of course the realization of mixed strategies, but this is still dictated by the players). This assumption is far too limiting to model many real life games (for example rolling a die, or shuffling cards, etc), not to mention real economic situations (employment, the stock market, the success of a new product, etc). Of course, by this we mean that these phenomena are complicated enough that they appear random, and are not worth modeling in most circumstances.

We model this by letting "nature" be a player. Nature will dictates random events according to a predetermined random strategy. We will assume that, although the choice nature makes in unknown to the players, the strategy (i.e., the distribution over choices) is known. To see how this might work, lets examine the following variate of the COORDINATION GAME.

𝒢ame. (RANDOM COORDINATION GAME). There are two firms, they must work together on project a or b, (but for some clever reason, maybe a legal injunction or something, cannot communicate). It is random which of the two projects is better, with probability p that is it product a. If they work together on the good project they both make a profit, and if they work on the bad project, or mismatch, neither will make a profit.

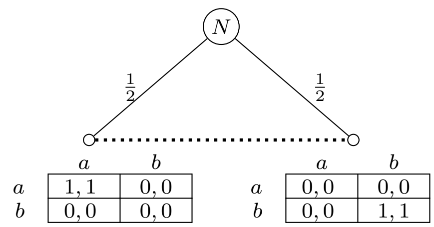

Figure 35: The game form for RANDOM COORDINATION GAME.

Payoffs are as given in figure 35. A few things worth pointing out:

- Since Nature is not a strategic player, it is customary to draw in the probabilities of each branch.
- We are utilizing our ability to mix normal and strategic form games together.
- When they must make their decisions, neither player knows which product will be better, as evidenced by the information set.
- Which strategies are NE might depend on which strategy Nature is playing (it is in this sense that we think of Nature as player –of course we do not need to maximize Nature's payoff, and so that strategy is still just a parameter of the game).
- There are 2 pure strategy NE –they do not depend on p:
  - Both firms play a.
  - Both firms play b.
  - If one firm is always playing a then the other can only get positive profit if playing a (playing b results in 0, regardless of natures choice).
- There is also a mixed strategy equilibrium:
- Now assume that firm 1 randomizes. Then it must be indifferent (by the indifference principle) to playing a or b. To make it indifferent firm 2 must play a with probability q such that:
  - The payoff to a, pq, is equal to the payoff to b, $(1-p)(1-q)$.
    - Why are these the payoffs?
  - So $q = (1-p)$.

- ▷ Symmetry dictates that firm 2 must do the same.
- ▷ So what is the best NE?
  - ▷ The payoff to (a, a) is p.
  - ▷ The payoff to (b, b) is $(1 - p)$.
  - ▷ The payoff to $([(1-p), p], [(1-p), p])$ is $p \times (1-p)^2 + (1-p) \times p^2$
  - ▷ Always strictly worse. Why? Because now there are two ways to mis-coordinate.

***𝒢*ame**. (Random Entrant Game). As in the entrant game, there is an incumbent and an entrant. However, now, there are two states of the world: good economy and bad. The probability the economy is good is p. Neither firm knows. When the economy is good, the incumbent can afford to fight, driving out the entrant and still making a profit. When the economy is bad, fighting is very costly to both firms. Payoffs are as given in figure 36.

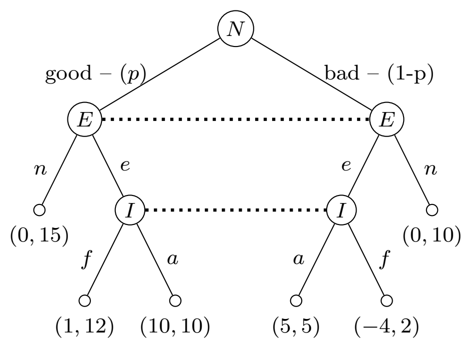

Figure 36: The game tree for the Random Entrant Game.

The economy is good with probability p. We assume that this value is known to the players. We are SPNE, so we start from the bottom most information set, and analyze the incumbents choice. I's payoffs are

- ▷ $\pi_I(F) = 12p + 2(1-p) = 10p + 2$
- ▷ $\pi_I(A) = 10p + 5(1-p) = 5p + 5$
- ▷ It is indifferent if $5p = 3$ or $p = \frac{3}{5}$
- ▷ If $p > \frac{3}{5}$ it will want to play F and if $p < \frac{3}{5}$ it will want to play A

If I plays A, then notice that it a BR for E to play E in either economy. Therefore, if $p \leqslant \frac{3}{5}$ the SPNE is (E, A). What is I plays F?

- ▷ $\pi_E(E) = 1p - 4(1-p) = 5p - 4$
- ▷ $\pi_E(N) = 0$
- ▷ It is indifferent if $5p = 4$ or $p = \frac{4}{5}$

▷ If $p > \frac{4}{5}$ it will want to play E and if $p < \frac{4}{5}$ it will want to play N.

So if $p \geqslant \frac{4}{5} > \frac{3}{5}$ E wants to enter and I wants to fight. The SPNE is (E, F). If $p \in [\frac{3}{5}, \frac{4}{5}]$ then I still wants to fight, but E doesn't want to enter so the SPNE is (N, F). Notice, however, this differs that the normal ENTRANT GAME, in that I's threat to fight is credible at this range of p.

▷ Its almost as if there are two game: the good economy game and the bad economy game. And, two different SPNE, $((E, F)$ and (E, A), respectively).

▷ As the probability shift the players go from one to the other.

▷ They shift a different times relative to their payoffs.

▷ In-between there is a mix where the incumbent would fight but the entrant do not have enough incentive to enter.

Assumptions on beliefs. It is important how we model players' uncertainty regarding the randomness in the game. For example, think for a moment what would happen in the RANDOM ENTRANT GAME if the incumbent knew what the state of the economy was when it made its choice. It is reasonable, especially in this situation, to let players have asymmetric information. An informed incumbent would always fight when the economy was good and acquiesce when it was bad. But then, in the game produced from one level of backwards induction reasoning, it is a dominant strategy for the entrant to *enter*. Regardless of the state of the economy, it will enter. Now consider what happens when $p = \frac{3}{5}$. In the original game, the entrant does not enter, so the incumbent receives a payoff of $\frac{3}{5} \cdot 15 + \frac{2}{5} \cdot 10 = 13$. But now, the entrant alway enters, so it gets a payoff of $\frac{3}{5} \cdot 12 + \frac{2}{5} \cdot 5 = 9\frac{1}{5}$. The incumbent is worse off being informed!

Now the reason is clear: in the original game, the incumbent would have fought in both economies becuase it can "commit" to fighting even when the economy is bad, simply by being uninformed. But the threat is no longer credible when the incumbent is knows the state. More to the point of how beliefs affect payoffs, this could mean that, if the entrant believes the incumbent is informed about the state of the economy, even when this is not the case, our backwards induction reasoning breaks down. The entrant, believing the incumbent to have more information that it actually does, will enter, although this is not admitted by SPNE.

Hence, for us to believe the SPNE predictions in the presence of information sets, we need even more assumptions. The structure of the information, (i.e., who knows what and when do they know it), must be common knowledge. To be clear, this means everyone must know everyone else's information sets, know that everyone knows everyones information sets, and so forth. And as alway when we begin to analyze these restrictions, there are very reasonable objections. In many real life scenarios, people are uncertain of what other people are able to observe –but this means that backward induction reason may not always hold.

REPEATED GAMES

When we first introduced the PRISONERS DILEMMA game, we noted that one key element that is missing is the repeated nature of interactions. Perhaps the reason we see cooperation in real life settings, even in the presence of selfish deviations, is because of the repeated nature; people who cheat get ahead today, but might be punished in future interactions. To flush out these ideas, lets begin by thinking about playing a normal form game over and over again, the total payoffs being the sum of the payoffs in each round. Think about playing the Prisoners Dilemma game twice in a row, moreover, we will allow the players to condition their second round strategies on the outcome in the first round. Will this allow us to sustain new equilibrium?

First off, note that we can represent this game as game tree (although we don't because it would be very big and annoy to make). But we don't really have to, it is easy to analyze this game! Think about SPNE. So we start at the end of the game. There are four different information sets to consider, one for each of the outcomes in the first round (i.e., what do the players do after they both cooperated or one cooperated and and one defected, etc). But they are all exactly the same some payoff plus a Prisoners Dilemma game. But the payoff from the first round makes no difference! It is an affine transformation, so by out expected utility hypothesis, it does not affect predictions. In the second period, no matter what happened in the first period, both players must defect.

But this means that the second periods action is *not* conditional on the first periods outcome (it is always defect), so the game produced by one round of backwards induction reasoning is exactly the Prisoners Dilemma game itself. So the only NE is to defect again. What about adding more periods? Of course this does nothing, backwards induction ensures that the players play the NE in the final round, implying they must also play NE strategies in the second to last round, etc. An unraveling argument; defect in N means defect in $N-1$ etc. So simply adding more periods does not induce (or even allow for) cooperation.

Infinitely Repeated Games

So far we have discussed two types of game (and mixtures between them):

- Normal form games: end immediately.
- Finite Strategic form games: end on terminal nodes.

But what is markedly missing is the ability to model interactions that never end (or at least ones that have no obvious ending: i.e., random ending). For example, our model of two firms choosing prices (as in Prisoners Dilemma and Bertrand Duopoly), assumes implicitly that after they post prices, they can never change them (outcomes are fixed). But this is a very strict assumption. More realistically, firms can post prices and then changes them after some interval of time, and then change them again, and again, until they go out of business or stop making the product. And these types of end points, bankruptcy or a change in market focus, are not known to occur at any specific point in time. The uncertainty regarding the end of the interaction is common in many economic situations where interactions have to fixed end point.

To take into account these situations, we build a flexible model to accommodate such tensions by allowing the game to **never** end. This means we have to make some changes to how payoffs are calculated. Until now, we have assumed payoffs arrive at the end of the game, which is clearly infeasible in games with no end. So instead, we break the game into an infinite number of periods. Each period the players play the same normal form game, referred to as the *stage game*; this is the infinite extension of the finitely repeated model introduced above. We assume that the players receive payoffs according to each periods outcome, at each period of the game. But this leads to another problem; if we simply add together the payoffs at each period, we will end up with payoffs of infinity, and this is no good. For example, consider a the infinite repetition of the COORDINATION GAME. Now fix W's strategy as playing T in every period. Now, does H prefer to always play T or always play S? Intuitively, he prefers to always play T, since this leads to the payoff of 10 every period, whereas playing S leads to the payoff 1 every period. But, in total,

$$\sum_{i=1}^{\infty} 10 = \infty = \sum_{i=1}^{\infty} 1$$

So does the husband not care which strategy he plays? Surely this methodology is untenable.

So as not to run into problems of this nature, we need to **discount** future payments. This means that we will assume that, the further away (in time) a payoff is received the less value it has to the player. In particular we will assume *geometric discounting*. Specifically this means that each player has a specific *discount rate*, denoted by $\delta \in (0,1)$, such in any period $t+1$, the value to payoffs is δ of the value to the same payoff in period t.[8] Specifically, suppose that for some strategies, a game yields payoff $\pi_i(t)$ is period t will have a total payoff of

$$\sum_{t=1}^{\infty} \delta^{t-1} \pi_i(t) \tag{16}$$

There are two interpretations to discounting, although they do not need to be mutually exclusive.

1. Players are impatient. This is the literal interpretation. They care more about their payoffs today then they do about payoffs tomorrow. There is certainly an intuitive justification for this. Imagine you are offered \$100 today, or \$$x$ one year from now. For what values of x would you prefer to wait? Likely, it would have to be larger than 100. What if you had to wait 10 years? Would the amount needed to induce waiting increase?

2. That players do not know when the game will end, and there is a $(1-\delta)$ chance that it ends in the next period. So players care about their future payoffs just the same, but there is only a δ chance that these payoffs will occur.

For the moment we will put the payoff structure on hold (examples and discussion forthcoming), and think about strategies. As an example, recall the above discussion about playing the PRISONERS DILEMMA game twice. Players we able to condition their second period strategies on the outcome in the first period. This meant that there were far more strategies than exist in to *independent* plays of the game (Why? This all relates back to they way we defined strategies in extensive form game as being complete contingent plans). As we add more and more periods, the strategy space (i.e., the set of all strategy profiles, or S), becomes incredibly, and unwieldy complex. Moving to an infinity repeated game just exacerbates things. Just for fun, what *do* strategies look like (say for the infinity repeated PRISONERS DILEMMA).?

- ▷ A strategy the first period is just the same as before: an action or lottery over actions.
- ▷ But after the end of the first period, the players can condition on being in *each* information sets. That is, they can condition on the previous games' outcomes.

[8] It is entirely possible that two players have different discount rates, as it is a reflection on their subjective preferences. However, for most of this class we will talk about a single discount rate for all players. This is just to make things more simple.

- A strategy (for P1) for the second round could be:
  - C if (C, C)
  - D if (C, D)
  - C if (D, C)
  - D if (D, D)
- This strategy says, do whatever P2 did in period 1.
- Or, we have the strategy, play C in round 2 all the time:
  - C if (C, C)
  - C if (C, D)
  - C if (D, C)
  - C if (D, D)
- Or, do whatever P1 did the last round:
  - C if (C, C)
  - C if (C, D)
  - D if (D, C)
  - D if (D, D)
- There are many strategies and we are only on the second round. What would a strategy in the third round look like?
  - C if $(C, C; C, C)$
  - D if $(C, C; D, C)$
  - D if $(D, C; C, C)$
  - C if $(D, C; D, C)$
  - D if $(C, D; C, C)$
  - C if $(C, D; D, C)$
  - C if $(D, D; C, C)$
  - etc.
- Now imagine a strategy in the 130^{th} round, or the $134{,}240{,}344^{th}$ round. Strategies can get *very* confusing.
- We will come up with ways to deal with this problem, but it is still there in the background.

In general, we can think of a function $H(t) : \mathbb{R} \to S$ that returns the outcome of the game in period t. For the Prisoners Dilemma game $H(1)$ might look like $H(1) = (c, c)$ and $H(1) \times H(2) = (c, c; d, c)$. Them formally, a strategy in time t is a function from the history (all outcomes up to t) into the action set. $S_i(t) : \bigtimes_{k=1}^{t-1} H(k) \to \Delta(A_i)$. Then a strategy for the whole game is

$$s_i \in \bigtimes_{t=1}^{\infty} S_i(t)$$

A player chooses a strategy for each period, and can condition on all the outcomes in previous periods. This is ridiculously confusing,[9] so we will limit the options. But before we do, we can jump back to our discussion of payoffs. Give this setup, with strategies over the larger (infinite) space, players want to maximize their *total* expected discounted payoff. We wont worry to much about specific strategies, so it suffices to know that every pure strategy induces a unique sequence of outcomes, and every strategy (pure or mixed) induces a unique lottery over sequences of outcomes.

Imagine that, for some strategy in the infinitely repeated Prisoners Dilemma, the induced sequence of outcomes is (C, C) every period (for example, if both players play C no matter what the history is). Then what is the corresponding (total discounted) payoff to each player?

[9] seriously, ridiculous

- The payoff, as given by (16), to cooperating over and over (playing C forever) is $\sum_{t=1}^{\infty} \delta^{t-1}1$. Where does this come from?
  - Let $G(C, C)$ denote the total payoff to the infinite discounted sequence.
  - From the first period's perspective you get $1 + \delta G(c, c)$.
  - This is because the continuation sequence (i.e., from period 2 onward) is exactly the same. So we have
  - $G(C, C) = 1 + \delta G(c, c) = 1 + \delta[1 + \delta G(c, c)] = 1 + \delta 1 + \delta^2 G(c, c)$.
  - We can keep doing this to get $\sum_{t=1}^{\infty} \delta^{t-1}$.
  - Moreover, we can solve for $G(C, C) = \frac{1}{1-\delta} = \sum_{t=1}^{\infty} \delta^{t-1}$.

Above, we alluded to the fact that discounting fix the problem on infinite payoffs? In order for this to be true, we have to show that the discounted sum is always finite.

- This is true for any game with bounded payoffs in each period (and so any game with finite strategies). For example, in the Prisoners Dilemma game, the maximum payoff a player can get is 5 per period. So the highest (total discounted) payoff is $\sum_{t=1}^{\infty} 5\delta^{t-1}$, as given by (16).
- Once again we have:

$$\begin{aligned}\pi^{max} = \sum_{t=1}^{\infty} 5\delta^{t-1} &= 5\delta^0 + \sum_{t=2}^{\infty} 5\delta^{t-1} \\ &= 5 + \delta \sum_{t=1}^{\infty} 5\delta^{t-1} \\ &= 5 + \delta\pi^{max}\end{aligned}$$

  so solving for π^{max} we get $\pi^{max} = \frac{5}{1-\delta} < \infty$.
- This is always true (replace the 5 with any number). Geometric series are well defined for bounded intervals.

Stationary Strategies

Because the space of strategies is so large (infinite) and so confusing to work with, we will make a major simplification. We will only consider stationary strategies.

Definition. *A strategy is* ***stationary*** *if it depends only of the most recent periods outcome. That is* $S_i(t) : H(t-1) \to \Delta A_i$.

For example, consider the following stationary strategy for the infinity repeated Prisoners Dilemma.

$$s_i = \begin{cases} C \text{ if } H(t-1) = (C, C) \\ D \text{ if } H(t-1) \neq (C, C) \end{cases} \tag{17}$$

This strategy says: cooperate if last period everyone (including myself) cooperated, otherwise defect. Of course, for the strategy to be fully defined, we also need to consider what the players will do in the first period, But with the first period specified, this is a full characterization of the game. It fully specifies what the player should do at every situation and is therefore a complete contingent plan. We can represent these strategies with diagrams: see figure 37.

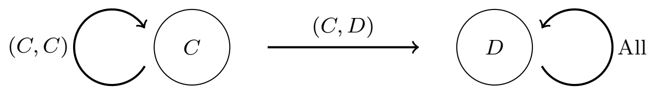

Figure 37: A diagram showing the stationary strategy of (17).

Now that we have a handle both on the strategies and the payoffs of a game, we can ask the central question: what outcomes are possible? Does the move from finite to infinite games change fundamentally change what behavior is feasible, or do we just get Nash behavior at every period like the finite case? In particular, can we now support cooperation. We will show that the answer is yes by means looking at the 3.1.

We need to balance cooperation with punishment for deviating. The worst possible punishment for deviating from (C, C) would force the deviator to get -5 forever (the worst outcome in the game). But we can't do this, since the deviator could herself deviate from the punishment. So, what about the punishment of playing D forever, the best the deviator can do is to also play D forever and get the payoff $u(D, D) = 0$. This is exactly the strategy we described in (17). We will call this strategy the *grim trigger* strategy, since once punishment has been triggered it never ends.

Without worrying about maximality (i.e, best responses) just yet, lets look at the sequence of outcomes that the generated by the grim trigger strategy.

- ▷ If the players start playing C, C then, following the strategy, the outcome is C, C forever.
- ▷ Why? Assume they both play C, C the first round. Then the next round they follow the strategy (look at the diagram) and they will continue to play C, C.
- ▷ This process continues each round. So by induction it goes on forever.
- ▷ If one or both begin with playing D, then every subsequent period the outcome is (D, D).

What is the payoff to following this strategy (and starting with C), assuming the other player is doing the same? Well the sequence of outcomes is (C, C) for ever, so they get $\pi(C, C) = 1$ each round, and as shown above, the payoff is

$$\sum_{t=1}^{\infty} \delta^{t-1} = \frac{1}{1-\delta} \tag{18}$$

Now we have to check if there are deviations to this strategy? This seems like it would be a very confusing thing to do, since there are so many (infinite) strategies with which to deviate. But its actually very simple!

- ▷ Say $P1$ first deviates at period T, at any period $t > T$ P2 will play D.

- ▷ P1's best response is clearly to play D forever as well.
- ▷ The payoff associated with this strategy is therefore

$$\underbrace{\sum_{t=i}^{T-1} \delta^{t-1} 1}_{\text{Before deviation}} \; + \; \underbrace{\delta^{T} 5}_{\text{Deviation}} \; + \; \underbrace{\sum_{t=T+1}^{\infty} \delta^{t-1} 0}_{\text{Punishment}} \tag{19}$$

- ▷ But look!, we only have to check the deviation in the *first* period. Indeed,

$$(19) > (18) \tag{20}$$
$$\Longleftrightarrow$$
$$5 > (18) \tag{21}$$

- ▷ This is because of the nature of geometric discounting (the difference in the way payoffs are treated is a function only of their distance from one another). To convert (20) into (21), we are simply subtracting $\sum_{t=i}^{T-1} \delta^{t-1} 1$ and dividing by δ^{T}.
- ▷ But equation (21) is just $5 > \frac{1}{1-\delta}$.
- ▷ So as long as $\delta > \frac{4}{5}$, this game can sustain cooperation.

This result completely hinges on the game being infinite (there can be no last period). If there was a last period backwards induction logic would kick in and unravel the whole game. But without a last period, there is no time at which each player knows the other player will deviate, and so, deviation, at any time period comes at a cost. Of course the cost of deviation is intimately related to the discount rate.

- ▷ This discount rate can be interpreted as the amount the agents care about their future payoffs.
- ▷ When the discount rate is higher, they care more about the future payoffs.
- ▷ A deviation from C, C comes with an immediate benefit, and a future cost –so a more myopic player will be more inclined to take the immediate benefit, while a more patient player will internalize the cost and stick to cooperation.
- ▷ In the extreme, when $\delta = 1$, players do not care about the future at all, so of course they will play the static NE.
- ▷ So it is only when the agents care about the future enough that the future cost can act as an incentive to continue cooperating.

If we make the immediate benefit to deviation larger, then it will require more patience (higher discount rate) to sustain cooperation. This is because, they will have to care about the punishment enough to curtail to increase temptation of deviating. On the other hand, if we make the benefit to cooperation larger, then it will require less patient agents to sustain cooperation. This is because, the punishment is relatively more sever, and the temptation is diminished. These effects are shown by example in figure 38.

P1 \ P2	C	D
C	$1, 1$	$-5, 10$
D	$10, -5$	$0, 0$

P1 \ P2	C	D
C	$3, 3$	$-5, 5$
D	$5, -5$	$0, 0$

Figure 38: On the left: a PRISONERS DILEMMA payoff matrix, with an increased benefit to deviation. Following the derivation above, we see that cooperation is sustainable only if $10 < \frac{1}{1-\delta}$ or $\delta > \frac{9}{10}$. On the right: a PRISONERS DILEMMA payoff matrix, with an increased benefit to cooperation. Here, cooperation is sustainable only if $5 < \frac{3}{1-\delta}$ or $\delta > \frac{3}{5}$.

What about sustaining things other than (C, C). First off, to make sure the math works out nice, lets assume the game is as in figure 39. Can we sustain alternating between (C, D) and (D, C)? First, can this be accomplished by a stationary strategy? Yes, it can. Check that the diagram in figure 40 will produce such a sequence of outcomes when P1 plays C in round 1 and P2 plays D.

P1 \ P2	C	D
C	$4, 4$	$0, 10$
D	$10, 0$	$1, 1$

Figure 39: An alternate PRISONERS DILEMMA payoff matrix.

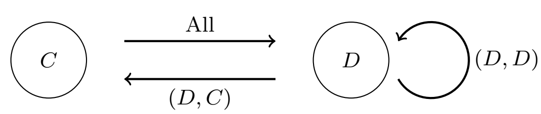

Figure 40: Show that this strategy results in alternation .

Lets say we start with (C, D) then alternate. What is the value to each player?

▷ For player 1:

$$\begin{aligned}
\pi_1 &= \delta^0 0 + \delta 10 + \delta^2 0 + \delta^3 10 + \dots \\
&= \delta^0 (0 + \delta 10) + \delta^2 (0 + \delta 10) + \dots \\
&= \sum_{t=1}^{\infty} \delta^{2(t-1)} (0 + \delta 10) \\
&= \frac{\delta 10}{1 - \delta^2}
\end{aligned}$$

▷ For player 2:

$$\begin{aligned}\pi_2 &= \delta^0 10 + \delta 0 + \delta^2 10 + \delta^3 0 + \dots \\ &= \delta^0(10 + \delta 0) + \delta^2(10 + \delta 0) + \dots \\ &= \sum_{t=1}^{\infty} \delta^{2(t-1)}(10 + \delta 0) \\ &= \frac{10}{1-\delta^2}\end{aligned}$$

▷ They both get 10 alternating for ever. Since they both get the same sequence, but P2 gets the better outcome one period earlier, it makes sense that they receive the same value just discounted back one period.

Now we just need to know if this is a sustainable strategy? Is there a deviation? Let's check for player 1.

▷ When he plays D it is a dominate strategy –so the there is no deviation.

▷ The only deviation would be to play D when he is supposed to play C. By the same argument as above, we can just check the first period.

▷ The payoff to this deviation is you get 1 today, and then, in the punishment state, 1 forever after:

$$\sum_{t=1}^{\infty} 1\delta^{t-1} = \frac{1}{1-\delta}$$

▷ This is not a deviation if

$$\begin{aligned}\frac{1}{1-\delta} &\leqslant \frac{10\delta}{1-\delta^2} \\ 1-\delta^2 &\leqslant 10\delta - 10\delta^2 \\ \frac{1}{9} &\leqslant \delta\end{aligned}$$

▷ For P2, there is no deviation in the first period, but there is in the second.

▷ The deviation is the same as P1's (everything gets discounted) so we already checked it.

So we can sustain this reciprocal cooperation. In fact if $\delta > \frac{2}{3}$ then this is a more profitable strategy (for *both* players) than always cooperating.

Folk Theorems

The multiplicity of SPNE outcomes above raises the question: What outcomes are sustainable in an infinitely repeated game? Can we characterize the set of NE or SPNE? This is indeed possible, but first we need some definitions.

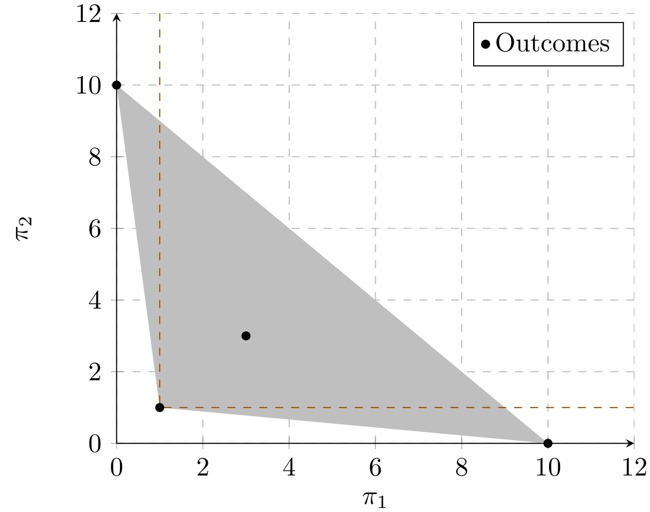

Figure 41: The set of feasible payoffs for the PRISONERS DILEMMA game as shown in figure 39 is in gray. The area northeast of the dashed line are payoffs above $\underline{v} = (1, 1)$.

Definition. *A payoff vector, $(x_1 \ldots x_N)$, is called **feasible** if it lies in the convex combination of game outcomes. That is, a strategy is feasible if it can be created by alternating between outcomes of the game.*

For each player we can define the lowest payoff she could receive in a *Nash Equilibrium*, denoted by $\underline{v}_i$. Then let $\underline{v} = (\underline{v}_1 \ldots \underline{v}_N)$.

Definition. *A payoff vector, $(x_1 \ldots x_N)$, is called **enforceable** if it lies point-wise above $\underline{v}$.*

To elucidate these examples, we show the feasible and enforceable payoffs for the PRISONERS DILEMMA game in figure 41. These two concepts fully characterize the set of SPNE average payoffs in the game. Note that we talk about average payoffs, rather than discounted payoffs, because it is easier to connect payoffs in the stage game with average payoffs.

Theorem (Folk Theorem)**.** *If (x_1, x_2) is a feasible and enforceable payoff vector, then there exists a $\hat{\delta} \in (0, 1)$ such that for all $\delta > \hat{\delta}$, (x_1, x_2) is the average payoff with respect to a SPNE profile.*

The folk theorem has two requirements on the payoffs:

- The payoff has to be **feasible**.
  - We cannot hope to get a payoff that is not possible to generate given the outcomes of the game. This is just a basic possibility requirement.
- The payoff has to be **enforceable**.
  - This is the important qualification. The other players must be able to punish deviations. Further, because we are interested in SPNE, the punishment part of the strategy has to be a Nash Equilibrium in the subgame induced by the deviation. This means that our punishments must be NE themselves, or else the punishment would not be credible.

		P2: C	D	X
P1:	C	$4,4$	$0,10$	$-5,-5$
	D	$10,0$	$1,1$	$-4,-5$

Figure 42: An incredible punishment.

One immediate (and rather obvious) consequence of the FOLK THEOREM is that it is always a NE in the larger (infinite) game to play a NE in each stage game. To see this, note that a NE is always feasible (by being an outcome of the game) and always enforceable (by being a NE). Hence the folk theorem is saying that we can always accommodate more outcomes.

To see why enforceability is necessary, consider the variant of the PRISONERS DILEMMA displayed in figure 42. Here, P2 has the additional option of doing something universally horrible, that hurts both players. We know that for sufficiently low discount rates ($\delta < \frac{2}{3}$, verify this), (C,C) is not a sustainable outcome in the infinity repeated version of the game in figure 39. However, perhaps by threatening to play X, P2 can induce cooperation in $P1$. The grim trigger strategy, using X rather than D, as punishment is shown in figure 43.

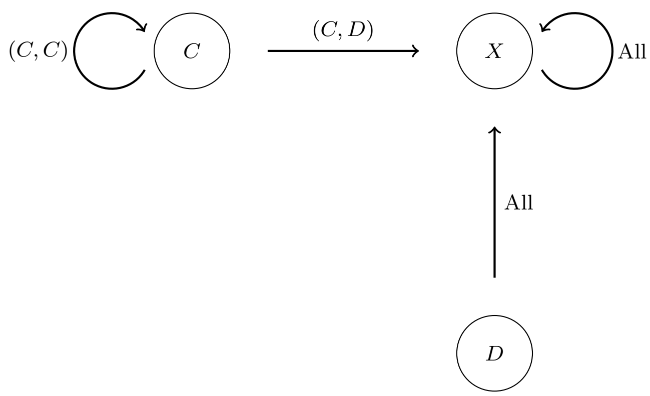

Figure 43: A diagram showing grim trigger with action X.

P1 would not want to deviate from cooperation so long as $\frac{4}{1-\delta} > 10 - \frac{\delta 4}{1-\delta}$ or $\delta > \frac{3}{7}$. By the introduction of outcome X, P2 can induce cooperation in P1 where before she could not, namely when P1's discount is between $[\frac{3}{7}, \frac{2}{3})$. But the strategy outlined in figure 43 is not credible. If P1 deviates, then a forward looking P2 will not punish her, as she would not be best responding to do so (since X is dominated by D). So the strategy outlined above constitutes a NE, but does not constitute a SPNE.

$\mathscr{G}$**ame**. (INFINITE BERTAND). There are two firms producing the same product. Each period (for and infinite number of periods) each firm competes as the BERTRAND DUOPOLY model of competition. Given prices, the profit of each firm is given by

$$\pi_i(p_i, p_j) = \begin{cases} (a-p_i)(p_i-c) & \text{if } p_i < p_j \\ \frac{(a-p_i)(p_i-c)}{2} & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

We know what the (static) NE is: alway play $p_i = c$ and there both firms ear a profit of 0. It should be no surprise that we can induce cooperation. In particular: split monopoly profits as long as the other firm continues to do so and play $p_i = c$ forever if the other firm deviates.

The monopoly price maximizes will maximize

$$(a - p_i)(p_1 - c) = ap - ac + cp - p^2$$

which, as shown in the static analysis implies that, $p^M = \frac{a+c}{2}$. Plugging this into the profit function we see that the profit in each period, as the result of collusion is

$$\pi_1 = \frac{(a - (\frac{a+c}{2}))((\frac{a+c}{2}) - c)}{2} = \frac{(\frac{a-c}{2})(\frac{a-c}{2})}{2} = \frac{(a-c)^2}{8}$$

So the total discounted utility to the cooperative strategy is give by:

$$\sum_{t=1}^{\infty} \frac{(a-c)^2}{8}\delta^t = \frac{(a-c)^2}{8(1-\delta)}$$

And the payoff to deviating is getting the full monopoly posits for one period then 0 forever.[10] So there is no deviation if

$$\begin{aligned} \frac{(a-c)^2}{8(1-\delta)} &> \frac{(a-c)^2}{4} \\ \frac{1}{8(1-\delta)} &> \frac{1}{4} \\ 8(1-\delta) &< 4 \\ \frac{1}{2} &< \delta \end{aligned}$$

Assumptions on Preferences. Each time we have introduced a new concept we have seen that in required that we make more (often hidden) assumptions on the way people behave. Discounting is no different. Because preferences are now being aggregated in a particular way, we need additional axioms, (beyond the expected utility axioms A1-4) to ensure that adhere preferences can be represented by geometric discounting. In this domain preference are not over single items of consumption (i.e., a or b), but over infinite streams of consumption (i.e., $\boldsymbol{a} = a_1a_2a_3\ldots$ or $\boldsymbol{b} = b_1b_2b_3\ldots$, elements in $\times^{\infty} X$. Note the entries in $\boldsymbol{a}$ are not necessarily all the same element. Then $c\boldsymbol{a}$ is the sequence where in the first period c is consumed followed by $\boldsymbol{a}$, i.e., in the second period a_1 is consumed, in the third period a_2, etc.)

To make sure preferences here can be aggregated by geometric discounting there are two additional axioms needed. They are straightforward and intuitive, although arguably not very realistic.

[10]Technical aside: recall that there is in-fact no best response. The deviating firm would like to get as close to the monopoly price as possible, but there is no closest number. However, this doesn't really matter for our analysis, since if we can show that there is no deviation when *all* monopoly profits are captured, then we have also showed that there is no deviation for any other deviation. Even more technically, since the profit is continuous with respect to price everywhere but at p^M, we can find a deviation that gets as close to the full monopoly profits as we like, and so with the set of discount rates such that there is no deviation will be fully characterized.

A5 **Stationarity** $\forall \boldsymbol{a}, \boldsymbol{b} \in \times^{\infty} X$ and $c \in X$:

$$\boldsymbol{a} \succcurlyeq \boldsymbol{b} \iff c\boldsymbol{a} \succcurlyeq c\boldsymbol{b}$$

A6 **Separability** $\forall \boldsymbol{a}, \boldsymbol{b} \in \times^{\infty} X$ and $c \in X$ if $a_n = b_n$ then

$$\boldsymbol{a} \succcurlyeq \boldsymbol{b} \iff \hat{\boldsymbol{a}} \succcurlyeq \hat{\boldsymbol{b}}$$

where $\hat{\boldsymbol{a}}$ is identical to $\boldsymbol{a}$ except a_n is replaced with c, and $\hat{\boldsymbol{b}}$ is identical to $\boldsymbol{b}$ except b_n is replaced with c.

Stationarity says that pushing consumption back by single period should not change preferences; that preferences are stationary in the sense that they do not change over time or because of previous consumption. This assumption gives us the recursive nature of the geometric discounting utility function. Separability dictates that when making a decision between two sequences of consumption, the decision maker ignores periods where the sequences provide the same alternative. This gives rise to the linear (or, additively separable) aspect of the utility function. Notice that the motivation for separability is very similar to the motivation for the independence axiom, and indeed, there is some redundancy in the two restrictions. Nonetheless, A1-6 provide a utility function of the form (16).

Baysian Games

Recall that in the discussion following the introduction of random games, we discussed the possibility that in the Random Entrant Game, one player (the incumbent) knew the state of nature and the other player (the entrant) did now. We will now make formal this general idea, allowing players to hold asymmetric information. This is a very important generalization, since it allows us to talk about the benefits and costs of having information relative to other players.

The most natural circumstance in which information might be asymmetric, reflected largely by the methodology we adopt presently, is where players have different possible *types*, and where each player knows her own type but not the types of others. We will assume that players types are drawn from a commonly known distribution, and the resulting structures will be called *Baysian games* (Baysian because players will be assumed to update their information according to Bayes' Rule rule). Some examples:

- Two competing firms might know their own marginal costs, and not the other firm's marginal cost.
- A firm hiring a worker might not know the skill (intelligence, productivity) of the worker, but the worker does know her own skill level.
- A used car salesman might know the quality of a car, whereas a prospective buyer does not.
- Each bidder in an auction might know her own personal value for a painting, but not the value to other bidders.

How should we model this? We already have the framework in place from random games (in fact, these games are just a particular type of random game). In random games, nature was considered a player, and its choice determined the payoffs for the other players. Now, we will think of nature as choosing each players *type* (this of course, influences the payoffs, it could also influence what actions are available to a player). In a strict mathematical sense the following are equivalent:

1. There are multiple states of the world, players can tell distinguish from different sets of states.

2. There are different types of players (each of which knows her own type but not the types of her opponents).

and so our distinction between random and Baysian games is just a mater of interpretation.

Lets see how such an environment looks with the following simple example.

$\mathscr{G}$**ame**. (Incomplete information Battle of the Sexes). Our couple is going through a rough patch, but decided to go out to the theatre. As usual, there was some coordination issues (perhaps this explains the marital tensions). The husband is just like before as before, he prefers B to S, and prefers prefers over all else that the two players meet. However, there are two "types" of wife. One type wants to meet (and prefers S) and the other wants to avoid her husband (but still prefers S). The man believes the two types are equally probable.

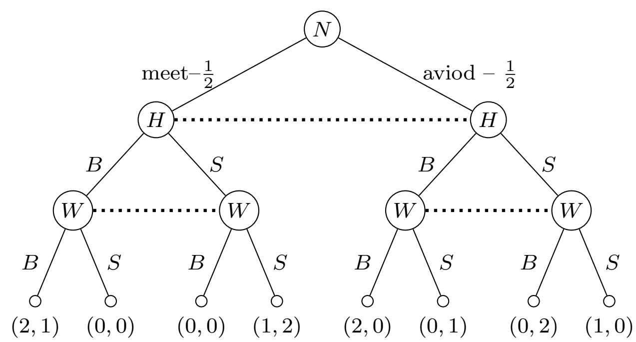

Figure 44: The game form for Incomplete information Battle of the Sexes.

The game tress is represented in figure 44. Notice that it cannot be represented as nature choosing between two normal form game as in 35. Why? Because they wife knows game they are playing but the husband does not.

So, what is a strategy in for this game? As in general extensive form game, a strategy is a complete contingent plan. In Baysian games, that could mean two equivalent things (corresponding to the equivalence above):

1. Information sets. This is the interpretation in that nature choose a state of the world, and some players receive information about it. Each player must choose an action (or lottery) from each information set.

   ▷ For H, this is a single action. He cannot distinguish between the two types of opponent, so he can only make a single action.

   ▷ For the W, this is two actions, one for each information set.

2. Types. This is the interpretation that each player has a type, which is privately known.

    - ▷ For H, this is a single action. He has a single type.
    - ▷ For the W, this is two actions, one for each of her two types.

Notice that, under the Baysian interpretation there is a true type of W, but she must choose an action in the counterfactual world in which she was the other type. This seems to be a philosophical conundrum (and perhaps at a deeper level it is). But for now, this is easily remediable: although only one one type of W will actually be realized, H must be best responding to his belief about what will happen in *all* possible worlds. That is to say, W will only be one type or the other, so she will only take 1 action, but we have to model both types' actions so that H can entertain a well formed belief. The play of the counterfactual type might just be in H's mind, but all the same, if he believes such a type is possible, we must model it as well. We will have more to say about this subtly at the end of the section.

What is a NE in this game? Just what you would think: strategies that are mutual best responses. It is the same as solving a extensive form game with randomness: we need to make sure to calculate payoffs according to the probability of each state of the world.

- ▷ The payoff to W is deterministic (given actions). However, for H we need to calculate according to the probability of each state.
- ▷ The best response for H is $BR_H(BB) = B$, $BR_H(BS) = B$, $BR_H(SB) = B$, $BR_H(SS) = S$
- ▷ The best response for W is $BR_W(B) = BS$, $BR_H(S) = SB$.
- ▷ We can check if any of these is mutual best responses: there is a NE at T, TS

The interpretation is that the husband will always go to the Bach concert, and the wife will meet him if the she prefers to, or go to the Stravinsky concert if she wished to avoid him. There are, also, mixed strategy NE.

Definition. *A game of incomplete information is a set of players, N, a set of actions, A_i, a set of states of the world Ω, a belief over the state space μ, a mapping from actions and states to payoffs $\phi : \Omega \times A \to \mathbb{R}^N$, and an set information sets, one for each player. (Information sets correspond to a partition of states).*

Mapping the previous game into this language:

- ▷ Players: H, W.
- ▷ Actions: $\{B, S\}$.
- ▷ States: meet (m), avoid (a).
- ▷ Payoffs are assigned by given by the matrix or extensive form game in figure 44.
- ▷ Informations sets were: H: $\{m, a\}$, W: $\{m\}, \{a\}$.

A NE in this environment, also called a *Bayes Nash Equilibrium* (BNE), in addition to the usual requirements, also requires that beliefs are consistent.

▷ That was easy in the above game. We didn't even notice we did it.

▷ It means that H places probability $\frac{1}{2}$ on each state (as given by the prescribed beliefs) and W placed probability 1 on each state after seeing which state she is in.

▷ So after entering an information set, the belief of the player changes: W now places probability 1 on the state, instead of the original belief.

▷ This process is called Baysian Updating.

Baysian Updating

Baysian updating is a statistical procedure for changing beliefs after the arrival of information. It states that says that the probably of a state occurring after a signal is in proportion to the probability before and the likelihood of the signal. Specifically, Bayes rule states that, the probability of an event A (a subset of the state space) after receiving a signal that the true state is contained in event B is given by

$$Pr(A|B) = \frac{Pr(A \text{ and } B)}{Pr(B)}$$

For example. There are 4 states: $\omega_1, \omega_2, \omega_3, \omega_4$. Each has probably $\frac{1}{4}$ of occurring.

1. What is the probability of ω_1 occurring if we get a signal telling us the state is either ω_1 or ω_2?

   ▷ The new probability is (the probability of "A and B" in this context is $Pr(\omega_1 \wedge (\omega_1, \omega_2)) = \frac{1}{4}$, since this event requires both ω_1 occurs and $\{\omega_1, \omega 2\}$ occurs, which is true only if ω_1 is the true state).

$$Pr(\omega_1|\omega_1, \omega_2) = \frac{Pr(\omega_1 \wedge (\omega_1, \omega_2))}{Pr(\omega_1, \omega_2)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

2. What is the probability of ω_2, ω_3 occurring if we get a signal telling us the state is either ω_2, ω_3 or ω_4?

   ▷ The new probability is (the probability of A and B in this context is $Pr((\omega_2, \omega_3) \wedge (\omega_2, \omega_3, \omega_4)) = \frac{1}{2}$, since this happens when either ω_2 or ω_3 is the true state).

$$Pr(\omega_2, \omega_3|\omega_2, \omega_3, \omega_4) = \frac{Pr((\omega_2, \omega_3) \wedge (\omega_2, \omega_3, \omega_4))}{Pr(\omega_2, \omega_3, \omega_4)} = \frac{\frac{2}{4}}{\frac{3}{2}} = \frac{2}{3}$$

3. What is the probability of ω_3, ω_4 occurring if we get a signal telling us the state is either ω_2 or ω_3?

   ▷ The new probability is (the probability of A and B in this context is $Pr((\omega_3, \omega_4) \wedge (\omega_2, \omega_3)) = \frac{1}{4}$, since this happens only if ω_3 is the true state).

$$Pr(\omega_3, \omega_4|\omega_2, \omega_3) = \frac{Pr((\omega_3, \omega_4) \wedge (\omega_2, \omega_3))}{Pr(\omega_2, \omega_3)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$$

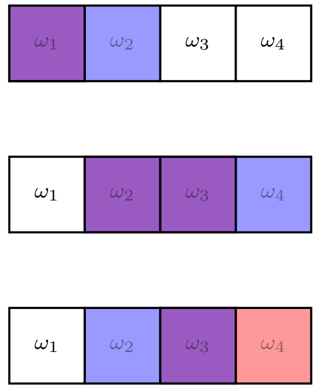

Figure 45: The Baysian updating on the signal structure described in the above example. The event in question (the event A) is in blue: , the signal, (the event B) is in red: , and the over lap (the event $A \wedge B$) is in purple . The resulting posterior is alway the ratio of purple to purple and blue.

A graphical representation of the above process is shown in figure 45.

$\mathscr{G}$**ame**. (Mugger Game). There are two players. A pedestrian and a mugger. The pedestrian can be of three types: indolent (with probability $\frac{1}{3}$), athletic ($Pr = \frac{1}{2}$), or a black belt ($Pr = \frac{1}{6}$). The mugger can choose to mug the person or not, and the person can choose to fight or acquiesce. The pedestrian knows his type for sure, where as the mugger only receives a signal: there are also two signals: if the person is small or big. The signal big occurs only (and always) if the person works out. The signal small happens if the person does not work out or is a black belt.

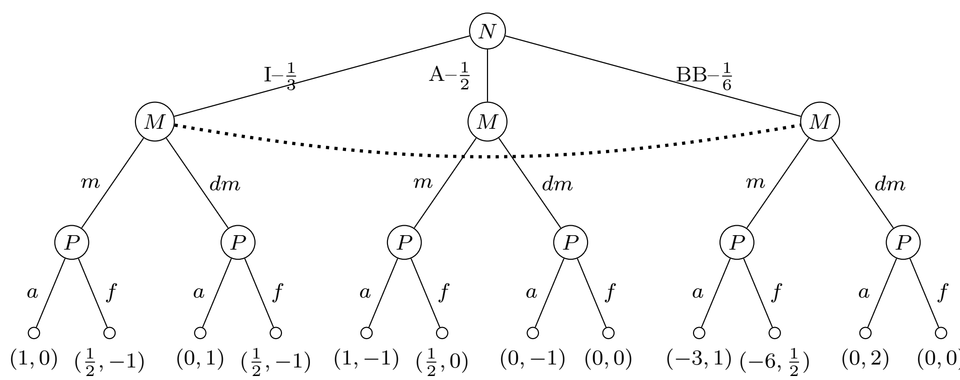

Figure 46: The game tree for the Mugger Game.

They payoffs are given by figure 46. Let's examine what happens if the mugger *knows* who which type of opponent he is playing against. We can solve each game using backwards induction. The indolent type will always play a as it is a dominate strategy. As a result, the mugger will always play m. The athletic type will always fight, as for him, this is his dominant strategy –the mugger will also play m. The black belt prefers to acquiesce (again a dominate strategy) but the mugger wants nothing to do with him! So the mugger wants to rob lazy and athletic people, but not black belts.

But the mugger does not know with whom he is dealing! What are the consistent beliefs each signal.

- For the Pedestrian, he knows with certainty, so after the signal, he has a belief concentrated on the state
- For the mugger it depends on his signal:
  - $$Pr(bb|s) = \frac{Pr(bb \wedge s)}{Pr(s)} = \frac{\frac{1}{6}}{\frac{1}{6} + \frac{1}{3}} = \frac{1}{3}$$
  - $$Pr(n|s) = \frac{Pr(n \wedge s)}{Pr(s)} = \frac{\frac{1}{3}}{\frac{1}{6} + \frac{1}{3}} = \frac{2}{3}$$
  - $$Pr(wo|s) = \frac{Pr(wo \wedge s)}{Pr(s)} = \frac{0}{\frac{1}{6} + \frac{1}{3}} = 0$$
  - $$Pr(bb|b) = \frac{Pr(bb \wedge b)}{Pr(b)} = \frac{0}{\frac{1}{2}} = 0$$
  - $$Pr(n|b) = \frac{Pr(n \wedge b)}{Pr(b)} = \frac{0}{\frac{1}{2}} = 0$$
  - $$Pr(wo|b) = \frac{Pr(wo \wedge b)}{Pr(b)} = \frac{12}{\frac{1}{2}} = 1$$

What do we need for an equilibrium? A strategy for each type.

- Each type of pedestrian
- AND each type of mugger (mugger after signal small and after signal big)
- There are 5 actions we need to specify. $(P, n), (P, wo), (P, bb), (M, s), (M, b)$

If the mugger sees the signal big he knows for sure that the pedestrian works out. (Since his belief is 1). Then his dominate strategy is to mug. Likewise, a pedestrian of type workout has a dominate strategy to fight so we have:

- $(P, wo) = F$

▷ $(M, b) = M$

Further we know that there is the small type has a dominate strategy to acquiesce and the blackbelt type has a dominate strategy to acquiesce so we also have

▷ $(P, n) = A$

▷ $(P, bb) = A$

We have a single thing left to specify: what does the mugger do when he encounters a small person? His payoff is the expected value given where he is. If he sees a small person there is a $\frac{2}{3}$ chance it is a not work out person and a $\frac{1}{3}$ chance it is a black belt. So the expected payoffs are

$$\pi(M, s) = \frac{2}{3}u_m(M, A, wo) + \frac{1}{3}u_m(M, A, bb) = \frac{2}{3}1 + \frac{1}{3} - 3 = -\frac{1}{3}$$

$$\pi(D, s) = \frac{2}{3}u_m(D, A, wo) + \frac{1}{3}u_m(D, A, bb) = \frac{2}{3}0 + \frac{1}{3}0 = 0$$

So it is better to not mug. Thus the NE strategies are

▷ $(P, n) = A$, $(P, wo) = F$, $(P, bb) = A$.

▷ $(M, b) = M$, $(M, s) = D$.

Applications – Adverse Selection

Adverse selection refers to the observation that when the quality (of something) is unobservable, the highest quality types will not want to participate in the market because they will, on average, receive compensation less then their value. This leads to the market being propagated by low quality types –hence the selection into the market is *adverse*. In this section, we will formalize this idea through two examples. Adverse selection models all have the following components:

1. Goods are of heterogeneous quality.
2. One side of the market is informed about the quality. (Baysian Game).
3. Goods have value to both sides of the market.

$\mathscr{G}$ame. ((Discrete) Market for Lemons). This game considers the used car market. There are two players: a buyer, b, and seller, s. Cars come in two qualities, high h, and low, l. Further assume that the probability a car is high quality is the same as the probability it is low quality is $\frac{1}{2}$. The seller knows the quality of the car, while the buyer knows only the distribution. They buyer can make any offer to the seller, who can accept of reject. Utilities are as follows, given an offer of x:

$$\begin{array}{cc|cc} u_b(x, a, h) = 1000 - x & u_b(x, r, h) = 0 & u_s(x, a, h) = x & u_s(x, r, h) = 750 \\ u_b(x, a, l) = 0 - x & u_b(x, r, l) = 0 & u_s(x, a, l) = x & u_s(x, r, l) = 0 \end{array}$$

In other words, the value of the high quality car is 1000 to the buyer but only 800 to the seller, and the value of the low quality car is 0 to the buyer and 0 the the seller. This means that for either quality of the car, trade is (weakly) welfare improving: the best possible outcome in terms of joint utility is to trade.

Lets examine the Bayes Nash Equilibrium. To begin, note that for any offer x, such that the high quality seller accepts a trade, it must also be that the low quality seller accepts (since his reservation wage is strictly lower). So, in any equilibrium, the buyer's beliefs regarding the quality of the car (conditional on accepting an offer) can never be greater than $p(h) = \frac{1}{2}$.

So what is the maximal offer the buyer is going to be willing to make? If he believes all sellers will accept, the the expected value of the car is $\frac{1000}{2} + \frac{0}{2} = 500$. Since this is the most optimistic belief possible, the buyers will never offer more than $x = 500$.

Since beliefs must be correct in equilibrium, it must be that both the high quality seller and the low quality seller accept the offer. But,

$$u_s(500, a, h) = 500 < 750 = u_s(500, r, h). \tag{22}$$

High quality sellers will not accept an offer of 600! But then the buyers expected value to offering 600 is

$$p(h)u_b(500, r, h) + p(l)u_b(500, a, l) = \frac{1}{2}0 + \frac{1}{2} - 500 = -250.$$

which is clearly worse than offering nothing. So offering $x \geqslant 500$ is not an equilibrium.

We know that for any offer $x < 500$, (22) is still satisfied, and so beliefs in equilibrium must be that only the low quality types accept. Clearly, and $x > 0$ is not a best response to this belief, and so, we are left with the degenerate equilibrium where the buyer offers $x = 0$ and the low quality seller accepts while the high quality rejects.

There is a large welfare loss:

- ▷ When trade is efficient (high quality cars) it does not occur.
- ▷ This is because of adverse selection. The high quality types did not want to be lumped in with the low quality types, and so select out of the market.
- ▷ The is because of *asymmetric information*. Classical market models (intro to micro) assume everyone in the market knows the quality of the goods –the result is that prices exactly reflect value, and trade is efficient. Here we get only trivial trade.

What if there we more than two types? Is the problem an artifact that if the high types exit the market, then the only trade left is when the value is 0?

$\mathscr{G}$**ame**. ((Continuous) Market for Lemons). The game mechanics (players, actions, information, etc) are the same as in (Discrete) Market for Lemons, but with a continuum of qualities. In particular, the quality is uniformly distributed from 0 to 1000, $q \sim_u [0, 1000]$. They buyer can make any offer to the seller, who can accept of reject. Utilities are as follows, given an offer of x and quality, q:

$$u_b(x, a, q) = q - x \quad u_b(x, r, q) = 0 \;\Big|\; u_s(x, a, q) = x \quad u_s(x, r, q) = \tfrac{3q}{4}$$

Lets skip to the result. Assume that in an equilibrium the buyer offers any $x > 0$. It might not be the case that all sellers accept the offer, in particular, a seller will only accept if

$$u_s(x, a, q) > u_s(x, r, q)$$
$$x > \frac{3}{4}q$$

or if $q < \frac{4}{3}x$.

We can calculate the expected value of q conditional on accepting x. This is equal to $E(q|q < \frac{4}{3}x)$, or

$$\int_0^{\min\{1000, \frac{4}{3}x\}} \frac{q}{1000} dq. \tag{23}$$

There are two cases.

▷ If $\frac{4}{3}x > 1000$, (i.e., if $x > 750$), then (23) evaluates to 500.

▷ If $1000 \geqslant \frac{4}{3}x$, (i.e., if $x < 750$), then (23) evaluates to $\frac{16x^2}{18000} = \frac{x^2}{1125}$.

Notice that for all $0 < x < 750$ we have that $\frac{x}{1125} < 1$ and so, multiplying both sides by x, we have $\frac{x^2}{1125} < x$. Therefore, for any $x > 0$, the expected quality conditional on trade occurring will be less than x and therefore the expected utility of such a strategy is negative. A classic unraveling argument. Again we are left only with the trivial equilibrium in which the buyer offers 0 and only sellers of the lowest possible quality accept (and strictly efficient trade does not take place, and trade (of any type) takes place only with probability 0.)

So, the mere existence of low types pushed the equilibrium towards no trade. Notice, we are no considering dynamics, reputation, etc., but the idea is intuitive nonetheless. The problem is that sellers can take advantage of buyers buy exploiting the extra information they posses. In the end, the possibility of this exploitation makes buyers reluctant, and hurts *everyone* in the market.

This is clearly a problem; but, it is potentially rectifiable by introducing *signaling*. Signaling is when different types differentiated from one another by taking different actions, this allows the other side of the market circumvent the full blown adverse selection problem. Of course, it these actions must be optimal responses, i.e., part of an equilibrium.

$\mathscr{G}$ame. (SPENCE SIGNALING MODEL). There are two kinds of players, a worker, w and a two (identical) firms, f. Workers come in two types high, θ_h, and low θ_l, productivity. We will assume that these are numbers such that $\theta_h > \theta_l > 0$. Workers perfectly know their type, while firms only know the distribution of types given by $p(h)$. The firms' value of hiring a high type worker is θ_h and low type is l. Workers can obtain education or not, i.e., $e = \{0, 1\}$. Education does not effect productivity at all, but is costly to the workers, assume that $c_l(1) > c_h(1) > c_l(0) = c_h(0) = 0$. The cost depends on the productivity of the worker: c_l or c_h. First the workers choose how much education to obtain, then firms offers wage contracts based on education: $w : \{e, ne\} \to \mathbb{R}$. We assume that the firms are perfectly competitive, and so, we constrain ourselves to consider wages that are the expected value of the worker to the firm. Then given e and w and the productivity, the payoffs are:

$$u_w(e, w, h) = w(e) - c_h(e) \quad u_w(e, w, l) = w(e) - c_l(e) \;\Big|\; u_f(e, w, p(h)) = E_{p(h)}[\theta|e] - w(e)$$

Lets first examine the case where there is no education (or for example, where education is so costly no one will get any). Then the two types of workers are indistinguishable, and so the wage is $w = E_{p(h)}[\theta] = p(h)\theta_h + (1 - p(h))\theta_l$.

This is still an equilibrium in the game with education. For example,

$$w(e) = \begin{cases} E_{p(h)}[\theta] \text{ if } e = 0 \\ \theta_l \text{ if } e = 1 \end{cases}$$

and $e = 0$ for all workers.

We have to check that beliefs are correct in equilibrium. It is obviously a best response for all workers to get no education, therefore, the beliefs of the firm at the beginning of the game are such that all players choose $e = 0$. So $e = 1$ is a zero probability event, Bayes' rule cannot be used, and beliefs can be arbitrary. This is called a pooling equilibrium since it pools together the workers of different productivities.

We know two things immediately about any such separating equilibrium:

1. Let $e(h)$, $e(l)$ be the education choice of h and l, respectively. Then $w^\star(e(h)) = \theta_h$ and $w^\star(e(l)) = \theta_l$. Since there in no uncertainty.

2. This means that it must be that $e(h) = 1$ and $e(l) = 0$. If not, l could deviate by pretending to be a high type.

So then we need to make sure that there are no deviations: In particular, $u_w(1, w^\star, h) = \theta_h - c_h(1) \geqslant \theta_l = u_w(0, w^\star, h)$. That is $c_h(1) \leqslant \theta_h - \theta_l$. Also, $u_w(0, w^\star, l) = \theta_l \geqslant \theta_h - c_l(1) = u_w(1, w^\star, l)$, or that $c_h \geqslant \theta_h - \theta_l$. So if

$$c_h(1) \leqslant \theta_h - \theta_l \leqslant c_l(1) \tag{24}$$

a separating equilibrium exists. This might not happen all the time. But, we could think, instead of choosing a discrete level of education, choosing from a continuous quantity. In this case, we can replace (24) with a more general assumption, called the *single crossing property*.