EC5110: MICROECONOMICS

LECTURE 1: PREFERENCE AND UTILITY

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Autumn 2018

Front Matter:

- Time:
  - Lectures: Tuesdays 11:00–13:00
  - Seminar: Tuesdays 10:00–11:00

- Office hours: Tuesdays 14:00 – 16:00

- Book: W. Nicholson and C. Snyder. Microeconomic theory: Basic principles and extensions. Nelson Education, 2011. ISBN 9781111525514

What is Microeconomics:

- The study of individual agents making decisions.
- Agents can be:
  - Single Humans
  - Firms, Companies, Universities, etc.
  - Govenments
- Each agent has desires (goals) and actions she can take.

We will use models to study individual decisions, markets, and resourcet allocation. A **model** is

- A set of assumptions and predictions.
  - We examine how our assumptions map into predictions.
- A simplification of the world.
  - Try to capture universal properties.
  - Exclude inessential complexity.
- An abstraction.
  - Things we learn in one scenario will carry over to others.

We can test models by

- Verifying the assumptions hold.
- Testing the predictions (i.e., statistical analysis).

We should always be thinking about the value of our assumptions!

What is special about Economic Thinking:

- Assume agents have goals that can be characterized mathematically.
- In general, study choices via optimization problems.
  - This allows us to study behavior via **marginal** considerations.
  - Efficiency can be characterized by marginal conditions.

There are many other paradigms to think about human choices:

- Psychology / neuroscience:
  - How do human brains work? How do they influence our actions.
  - Often concerned with constraints: memory, emotion, cognitive bias, confusion.

- Sociology:
  - How does culture affect choices. How do humans build social structures?
  - Often concerned with values and non-tangible/mathematizable goals.

- Statistics:
  - Atheoretical account of choices. Organizing data without models.
  - Does not require human input. Deals mostly with prediction.

Roadmap

We will build a theory of resource allocation by markets.

1. Demand: How do consumers decide what to buy?
2. Supply: How do producers decide what to supply?
3. Markets: How do markets form to allow exchange?

Consumers:

- Most economic decisions you have made are probably as a consumer.
- Choose what to eat, then eat it.

What is consumed?

- Objects, experiences, services, . . .

We want to talk about the implications of consumption choices:

- What gets chosen and by whom? What is the demand for goods?
- How does demand change with respect to changes in prices?
- Are people better off under regulation that changes supply/prices?

In order to address these questions, we need a **model** of consumption.

This has three parts:

1. A formalization of what choices can be made: a description of the objects.
2. A notion of what choices are feasible: the constraints of the consumer.
3. A notion of desirability or value: the objective of the consumer.

Part 1: Objects of Choice

We will examine the problem where **choice objects** or **consumption bundles** are points in $\mathbb{R}^n$.

Math Refresh: $\mathbb{R}^n$

Most of the course will take place in $\mathbb{R}^n$ the n-dimensional Euclidean space.

- Vectors $\boldsymbol{x} \in \mathbb{R}^n$ are n real numbers:

$$\boldsymbol{x} = (x_1, \ldots x_n)$$

- We have addition, scalar multiplication, and an inner product:
  - $\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \ldots, x_n + y_n)$.
  - for $a \in \mathbb{R}$ let $a\boldsymbol{x} = (ax_1, \ldots, ax_n)$.
  - $\boldsymbol{x} \cdot \boldsymbol{y} = \sum_n x_i y_i$.

Math Refresh: $\mathbb{R}^n$

If B is a collection of vectors, then we say that B is **convex** if for all $\boldsymbol{x}, \boldsymbol{y} \in B$, $s\boldsymbol{x} + (1 - s)\boldsymbol{y} \in B$ for any $s \in [0, 1]$.

Math Refresh: $\mathbb{R}^n$

We have the ordering

- $\boldsymbol{x} \geq \boldsymbol{y}$ if $x_1 \geq y_1 \ldots x_n \geq y_n$.
- $\boldsymbol{x} > \boldsymbol{y}$ if $x_1 \geq y_1 \ldots x_n \geq y_n$ with some strict.
- $\boldsymbol{x} \gg \boldsymbol{y}$ if $x_1 > y_1 \ldots x_n > y_n$.

If $\boldsymbol{x} \in \mathbb{R}^n$ represents a consumption object, then the interpretation is there are n different ‘types’ of consumption and x_i is the magnitude of the i^{th} type.

What is meant by type?

- Different consumption goods—wine and beer being an oft cited example for some odd reason.
    - (x_w, x_b): the agent consumes x_w worth of wine and x_b worth of beer.
- Each dimension could be a point in time:
    - (x_1, x_2) being the amount of wine at time 1 and at time 2.
- Each dimension is a hypothetical state upon which consumption can be conditioned:
    - (x_r, x_s) being the amount of wine when it rains and is sunny.

We will use different interpretations to answer different questions. For now, we can just think of

$$\boldsymbol{x} = (x_1, \ldots, x_n)$$

as the amount of n different goods.

Part 2: Constraints

A **decision problem** or **choice problem** or **budget** is a subset of $\mathbb{R}^n$.

- $B \subset \mathbb{R}^n$ is the set of consumption bundles which are feasible.
  - Affordable given income and prices.
  - Available in the current store.
  - Technologically possible.

Let x_1 represent beer and x_2, wine. Assume beer costs $\$1$ per unit, and wine $\$3$ per unit. Let B denote the set of all affordable bundles given a $\$10$ budget.

Is B convex?

What the consumer is facing B, she must choose a bundle $\boldsymbol{x} \in B$ to consume.

We want to understand:

- What does she choose?
- What is the consumer's objective?
- How well off is she given her choice?

Part 3: Utility

We will assume the consumer has a utility function

$$U : \mathbb{R}^n \to \mathbb{R}.$$

- Describes the value or desirability of $\boldsymbol{x}$.
- $U(\boldsymbol{x}) > U(\boldsymbol{y})$ indicates the consumer likes $\boldsymbol{x}$ more than $\boldsymbol{y}$.
- Thus, her objective is to maximize U subject to $\boldsymbol{x} \in B$.

Math Refresh: Functions on $\mathbb{R}^n$

$f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if

$$f(s\boldsymbol{x} + t\boldsymbol{y}) = sf(\boldsymbol{x}) + tf(\boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

- Linear functions are generalizations of lines.
- It is easy to see that if f is linear then $f(0) = 0$ (why?)
- If $f : \mathbb{R}^n \to \mathbb{R}$ is linear than for any finite collection of m vectors, we have $f(\sum_m t_i \boldsymbol{x}_i) = \sum_m t_i f(\boldsymbol{x}_i)$.

Math Refresh: Functions on $\mathbb{R}^n$

$f : \mathbb{R}^n \to \mathbb{R}^m$ is affine if and only if

$$f(s\boldsymbol{x} + (1-s)\boldsymbol{y}) = sf(\boldsymbol{x}) + (1-s)f(\boldsymbol{y})$$

for all $s \in [0,1]$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

- Clearly all linear functions are affine.
- Affine functions are also generalizations of lines, but they do not necessarily pass through the origin.

Are the following linear, affine (but not linear) or neither:

- $f(x_1, x_2) = x_1$
- $f(x_1, x_2) = x_1 x_2$
- $f(x_1, x_2) = 4x_1 + 3x_2$
- $f(x_1, x_2) = 4x_1 + 3x_2 + 2$
- $f(x_1, x_2) = \ln(x_1 + x_2)$

Are the following linear, affine (but not linear) or neither:

- $f(x_1, x_2) = x_1$
  - Linear
- $f(x_1, x_2) = x_1 x_2$
- $f(x_1, x_2) = 4x_1 + 3x_2$
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  - Neither
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  - Linear
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  - Linear
- $f(x_1, x_2) = x_1 x_2$
  - Neither
- $f(x_1, x_2) = 4x_1 + 3x_2$
  - Linear
- $f(x_1, x_2) = 4x_1 + 3x_2 + 2$
  - Affine
- $f(x_1, x_2) = \ln(x_1 + x_2)$

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  - Linear
- $f(x_1, x_2) = x_1 x_2$
  - Neither
- $f(x_1, x_2) = 4x_1 + 3x_2$
  - Linear
- $f(x_1, x_2) = 4x_1 + 3x_2 + 2$
  - Affine
- $f(x_1, x_2) = \ln(x_1 + x_2)$
  - Neither

Math Refresh: Functions on $\mathbb{R}^n$

If $f : \mathbb{R}^n \to \mathbb{R}$ is a function, and k is an integer, then f is said to be **homogeneous of degree k** if

$$f(a\boldsymbol{x}) = a^k f(\boldsymbol{x})$$

for all $\boldsymbol{x} \in \mathbb{R}^n$ and $\alpha > 0$.

- Homogeneity is a generalization of linearity
  - All linear functions are homogenous of degree 1

Are the homogeneous, and if so, of what degree?:

- $f(x_1, x_2) = \max\{x_1, x_2\}$
- $f(x, y, z) = x^5 y^2 z^3$
- $f(x) = ln(x)$ (defined over R_+)
- $f(x, y) = (x^2 + y^2)^{\frac{1}{2}}$

Are the homogeneous, and if so, of what degree?:

- $f(x_1, x_2) = \max\{x_1, x_2\}$
  - H.d.1
- $f(x, y, z) = x^5 y^2 z^3$
- $f(x) = ln(x)$ (defined over R_+)
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  - H.d.1
- $f(x, y, z) = x^5 y^2 z^3$
  - H.d.10
- $f(x) = ln(x)$ (defined over R_+)
- $f(x, y) = (x^2 + y^2)^{\frac{1}{2}}$

Are the homogeneous, and if so, of what degree?:

- $f(x_1, x_2) = \max\{x_1, x_2\}$
  - H.d.1
- $f(x, y, z) = x^5 y^2 z^3$
  - H.d.10
- $f(x) = ln(x)$ (defined over R_+)
  - Not homogeneous
- $f(x, y) = (x^2 + y^2)^{\frac{1}{2}}$

Are the homogeneous, and if so, of what degree?:

- $f(x_1, x_2) = \max\{x_1, x_2\}$
  - H.d.1
- $f(x, y, z) = x^5 y^2 z^3$
  - H.d.10
- $f(x) = ln(x)$ (defined over R_+)
  - Not homogeneous
- $f(x, y) = (x^2 + y^2)^{\frac{1}{2}}$
  - H.d.1

Remark.

If f is differentiable and homogenous of degree k, then $\frac{\partial f}{\partial x_i}$ is homogenous of degree $k-1$.

- Differentiate both sides of $f(a\boldsymbol{x}) = a^k f(\boldsymbol{x})$ with respect to x_i.

Math Refresh: Functions on $\mathbb{R}^n$

$f : \mathbb{R}^n \to \mathbb{R}^m$ is convex if

$$f(s\boldsymbol{x} + (1-s)\boldsymbol{y}) \leq sf(\boldsymbol{x}) + (1-s)f(\boldsymbol{y})$$

for all $s \in [0,1]$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and strictly so if the inequality is strict.

$f : \mathbb{R}^n \to \mathbb{R}^m$ is concave if

$$f(s\boldsymbol{x} + (1-s)\boldsymbol{y}) \geq sf(\boldsymbol{x}) + (1-s)f(\boldsymbol{y})$$

for all $s \in [0,1]$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and strictly so if the inequality is strict.

Functions on $\mathbb{R}^n$

Relatedly, a function is **quasi-convex** if

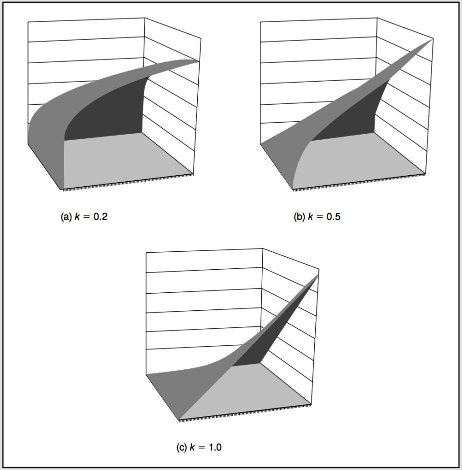
$$f(s\boldsymbol{x} + (1-s)\boldsymbol{y}) \le \max\{f(\boldsymbol{x}), f(\boldsymbol{y})\}$$

for all $s \in [0,1]$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and strictly so if the inequality is strict.

Quasi-concave can be defined using pattern matching skills!

FIGURE 2.4 Concave and Quasi-Concave Functions

In all three cases these functions are quasi-concave. For a fixed y, their level curves are convex. But only for $k = 0.2$ is the function strictly concave. The case $k = 1.0$ clearly shows nonconcavity because the function is not below its tangent plane.



Are the following convex, concave, quasi-convex, quasi-concave:

- $f(x_1, x_2) = x_1^2$

- $f(x_1, x_2) = x_1 + x_2$

- $f(x) = ln(x)$ (defined over R_+)

Are the following convex, concave, quasi-convex, quasi-concave:

- $f(x_1, x_2) = x_1^2$
  - Convex, quasi-convex
- $f(x_1, x_2) = x_1 + x_2$
- $f(x) = ln(x)$ (defined over R_+)

Are the following convex, concave, quasi-convex, quasi-concave:

- $f(x_1, x_2) = x_1^2$
  - Convex, quasi-convex
- $f(x_1, x_2) = x_1 + x_2$
  - All (its linear)
- $f(x) = ln(x)$ (defined over R_+)

Are the following convex, concave, quasi-convex, quasi-concave:

- $f(x_1, x_2) = x_1^2$
  - Convex, quasi-convex
- $f(x_1, x_2) = x_1 + x_2$
  - All (its linear)
- $f(x) = ln(x)$ (defined over R_+)
  - Concave, quasi-concave, and quasi-convex.

Remark.

Every convex function is quasi-convex.

By the definition of max

$$\begin{aligned} f(s\boldsymbol{x} + (1-s)\boldsymbol{y}) &\leq sf(\boldsymbol{x}) + (1-s)f(\boldsymbol{y}) \\ &\leq s\max\{f(\boldsymbol{x}), f(\boldsymbol{y})\} + (1-s)\max\{f(\boldsymbol{x}), f(\boldsymbol{y})\} \\ &= \max\{f(\boldsymbol{x}), f(\boldsymbol{y})\}. \end{aligned}$$

Remark.

Let f be a quasi-convex function, then $\{\boldsymbol{x} \mid f(\boldsymbol{x}) \leq a\}$ is a convex set for all $a \in \mathbb{R}$.

Given our utility function, and constraint sets we have a plan of attack. Given a policy that changes the feasible set, how does

$$\arg\max U(\boldsymbol{x}) \text{ subject to } \boldsymbol{x} \in B$$

change.

- Looks a lot like the optimization problems from math camp.

But, what is utility?

- We (at least I) do not have utility functions that we try to maximize.
- Where do these numbers come from?
- How do they relate to ‘actual’ decision making?
- How can we falsify this model?

A brief foray into Decision Theory

How do people value things? By making comparisons.

- I prefer “1 Magic Hat IPA” to “3 Budweisers”
- I like Tame Impala’s albums “Innerspearker” and “Lonerism” equally.
  - And both better than “Currents”
- I cannot decide weather I want “a Toyota for $18000” or “a Honda for $21000”

A brief foray into Decision Theory

Questions:

1. Can we formalize the above preference statements?
2. Can we link the real, observable “preferences” with the unobservable “utilities”
   - Utilities are just way, way easier to work with. We can use calculus!

We will define a preference $\succcurlyeq$ over elements of $\mathbb{R}^n$:
The statement

$$\boldsymbol{x} \succcurlyeq \boldsymbol{y}$$

is read,

"The consumer likes the bundle $\boldsymbol{x}$ at least as much as she likes the bundle $\boldsymbol{y}$"

"The consumer likes the bundle x at least as much as she likes the bundle y"

- This is **weak** preference.
  - Does not rule out that x and y are valued equally.

- Define $x \succ y$ as $x \succcurlyeq y$ but not $y \succcurlyeq x$.
  - **Strict** preference; x is strictly better than y.

- Define $x \sim y$ as $x \succcurlyeq y$ and $y \succcurlyeq x$.
  - **Indifference**; x is valued the same as y.

- I prefer “1 Magic Hat IPA” to “3 Budweisers”
  - $(1_{mh}, 0_{bw}) \succcurlyeq (0_{mh}, 3_{bw})$
- I like Tame Impalas albums “Innerspearker” and “Lonerism” equally.
  - $(1_I, 0_L) \sim (0_I, 1_L)$
- I cannot decide weather I want “a Toyota for $18000” or “a Honda for $21000”
  - Not $(1_t, 0_h, -18000_\$) \succcurlyeq (0_t, 1_h, -21000_\$)$ nor $(0_t, 1_h, -21000_\$) \succcurlyeq (1_t, 0_h, -18000_\$)$

So $\succcurlyeq$ constitutes the consumers preference. Then given a decision problem B the consumer chooses:

$$x \in B \text{ such that } \boldsymbol{x} \succcurlyeq \boldsymbol{y} \text{ for all } \boldsymbol{y} \in B$$

How can we relate $\succcurlyeq$ with a utility function U?

We say that U **represents** $\succcurlyeq$ if

$$\boldsymbol{x} \succcurlyeq \boldsymbol{y} \iff U(\boldsymbol{x}) \geq U(\boldsymbol{y})$$

Theorem.

When U represents $\succcurlyeq$ then

$$x \in B \text{ such that } \boldsymbol{x} \succcurlyeq \boldsymbol{y} \text{ for all } \boldsymbol{y} \in B$$

$$=$$

$$\arg\max U(\boldsymbol{x}) \text{ subject to } \boldsymbol{x} \in B$$

- When U represents the preferences of a consumer, we can maximize U to predict her behavior.
- If U represents $\succcurlyeq$ then:
  - $\boldsymbol{x} \sim \boldsymbol{y} \iff U(\boldsymbol{x}) = U(\boldsymbol{y})$
  - $\boldsymbol{x} \succ \boldsymbol{y} \iff U(\boldsymbol{x}) > U(\boldsymbol{y})$

Not every $\succcurlyeq$ is representable:

- $x \succ y$, $y \succ z$, $z \succ x$.

So when can we move from $\succcurlyeq$ to U? We need 3 restrictions

Axiom 1: Completeness

Say that $\succcurlyeq$ is complete if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ either $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ or $\boldsymbol{y} \succcurlyeq \boldsymbol{x}$ or both.

- The consumer can make decisions.

Axiom 2: Transitivity

Say that $\succcurlyeq$ is transitive if for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$ if $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ and $\boldsymbol{y} \succcurlyeq \boldsymbol{z}$ then $\boldsymbol{x} \succcurlyeq \boldsymbol{z}$.

- The consumer is consistent.
- This was violated by the example above.

Axiom 2: Continuity

Say that $\succcurlyeq$ is continuous if for all convergent sequences $\{\boldsymbol{x}_n\}_{n\in\mathbb{N}}$ with $\boldsymbol{x}_n \succcurlyeq \boldsymbol{y}$ (respectively, $\boldsymbol{y} \succcurlyeq \boldsymbol{x}_n$) for all n then $\lim_{n\to\infty} \boldsymbol{x}_n \succcurlyeq \boldsymbol{y}$ (rsp, $\boldsymbol{y} \succcurlyeq \lim_{n\to\infty} \boldsymbol{x}_n$)

- As bundles get really really close, preferences do not change much.
- This is mostly technical.

Theorem.

$\succcurlyeq$ is complete, transitive and continuous then there exists a continuous $U : \mathbb{R}^n \to \mathbb{R}$ that represents it.

- This is a pretty amazing result.
- We will not prove it in the this class.
- In seminar, we can show it for finite sets.

Consider 3 bundles $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, with

$$U(\boldsymbol{x}) = 0,\ U(\boldsymbol{y}) = 1,\ U(\boldsymbol{z}) = 10000$$

- It feels like $\boldsymbol{z}$ is **much** better than $\boldsymbol{x}$, and $\boldsymbol{y}$ is just **slightly** better.
- All we know is: $\boldsymbol{z} \succcurlyeq \boldsymbol{y} \succcurlyeq \boldsymbol{x}$. Nothing else!
- Notice $U(\boldsymbol{x}) = 0,\ U(\boldsymbol{y}) = 1,\ U(\boldsymbol{z}) = 1.0001$ also represents the same preferences.

Theorem.

Let U represent $\succcurlyeq$. Then if $h : \mathbb{R} \to \mathbb{R}$ is strictly increasing, $V = h \circ U$ also represents $\succcurlyeq$.

- We can actually strengthen: all representations are monotone transforms of each other.
- This is just like the result that if x maximizes U then x maximizes $h \circ U$.

Proof

Let U represent $\succcurlyeq$. Then

- $\boldsymbol{x} \succcurlyeq \boldsymbol{y} \iff U(\boldsymbol{x}) \geq U(\boldsymbol{y})$
  - By definition of representation.
- $U(\boldsymbol{x}) \succcurlyeq U(\boldsymbol{y}) \iff h(U(\boldsymbol{x})) \geq h(U(\boldsymbol{y}))$
  - Because h is strictly increasing.

This result means

- We cannot read much into the numbers, only their order.
- We can choose which representation to use. Make the math simpler!

In addition to the axioms needed for a representation there are other restrictions that we can employ.

Say that $\succcurlyeq$ is **strictly monotone** if $\boldsymbol{x} > \boldsymbol{y}$ implies $\boldsymbol{x} \succ \boldsymbol{y}$.

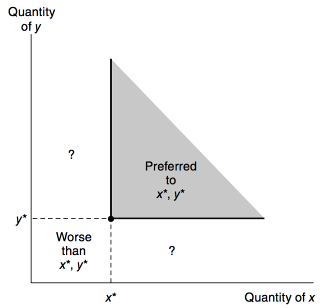
- We will sometimes say U is strictly monotone.
- More is always better.
- Is this reasonable?

Lets consider $\mathbb{R}^2$, where we can graph things.

- $U(x, y) = x + y$ is SM.
- $U(x, y) = xy + 2$ is SM over $\mathbb{R}^2_+$ but not over $\mathbb{R}^2$.
- $U(x, y) = (x - y)^2$ is not SM.

FIGURE 3.1 More of a Good Is Preferred to Less

The shaded area represents those combinations of x and y that are unambiguously preferred to the combination x^*, y^*. Ceteris paribus, individuals prefer more of any good rather than less. Combinations identified by "?" involve ambiguous changes in welfare because they contain more of one good and less of the other.



When U is differentiable, the **marginal utility** of x is

$$\frac{\partial U}{\partial x}$$

- How much additional utility do we get from increasing consumption.
- SM says that it is positive.
- We generally think it is diminishing. That U is concave.
  - The 1^{st} glass of milk is better than the 80^{th}

But, diminishing marginal utility is not a ordinal concept. It does not come from preferences.

Consider $\succcurlyeq$ over $\mathbb{R}_+$ such that $x \succcurlyeq y$ iff $x \geq y$.

- $U(x) = x$
- $U(x) = x^2$
- $U(x) = \ln(x)$

all represent this preference.

We are not generally interested in the utility of a single good, but comparing between them.

- How does a consumer choose?
- Does trade improve welfare?
- How do prices determine allocations?

Our graph helps us consider the tradeoff between good x and good y.

The indifference curve through $\boldsymbol{x}$ is the set of all points indifferent to x

$$IC(\boldsymbol{x}) = \{\boldsymbol{y} : \boldsymbol{y} \sim \boldsymbol{x}\}$$

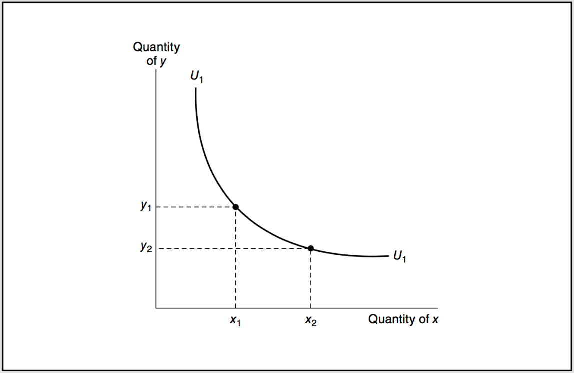
- Given U: for every $z \in \mathbb{R}$

  $$\{\boldsymbol{x} : U(\boldsymbol{x}) = z\}$$

  forms an indifference curve.

FIGURE 3.2 A Single Indifference Curve

The curve U_1 represents those combinations of x and y from which the individual derives the same utility. The slope of this curve represents the rate at which the individual is willing to trade x for y while remaining equally well off. This slope (or, more properly, the negative of the slope) is termed the ***marginal rate of substitution***. In the figure, the indifference curve is drawn on the assumption of a diminishing marginal rate of substitution.



- Indifference curves never cross (if $\succcurlyeq$ is representable).

FIGURE 3.4 Intersecting Indifference Curves Imply Inconsistent Preferences

Combinations *A* and *D* lie on the same indifference curve and therefore are equally desirable. But the axiom of transitivity can be used to show that *A* is preferred to *D*. Hence, intersecting indifference curves are not consistent with rational preferences.

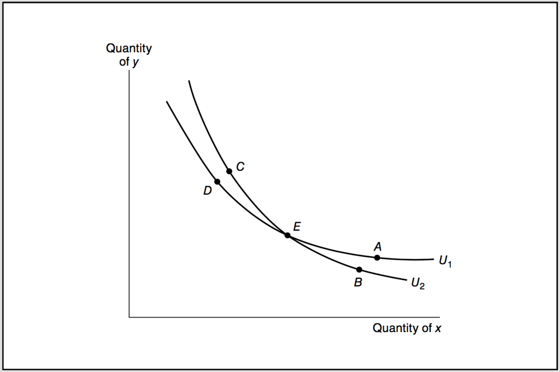
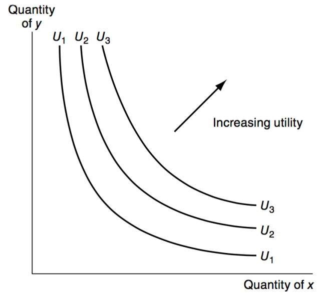


FIGURE 3.3 There Are Infinitely Many Indifference Curves in the x–y Plane

There is an indifference curve passing through each point in the x–y plane. Each of these curves records combinations of x and y from which the individual receives a certain level of satisfaction. Movements in a northeast direction represent movements to higher levels of satisfaction.



How much is y worth in terms of x?

- To get 1 unit of y, how much x are you willing to give up (to remain as well off)?
- We can visualize this on the graph.
  - The tangent to the indifference curve.
  - (This depends on the current consumption)

The negative of the slope of an indifference curve at some point is termed the **marginal rate of substitution** (of y for x) at that point. That is,

$$MRS_x^y = -\frac{dy}{dx}$$

the slope of the tangent line to the indifference curve at the point in question.

- How much y am I willing to give up to get more x.

Example

A consumer preference over of hamburgers, y, and soft drinks x, could be represented by the utility function

$$U(x, y) = (xy)^{\frac{1}{2}}$$

What is the MRS_x^y at $\boldsymbol{x} = (20, 5)$?

Example

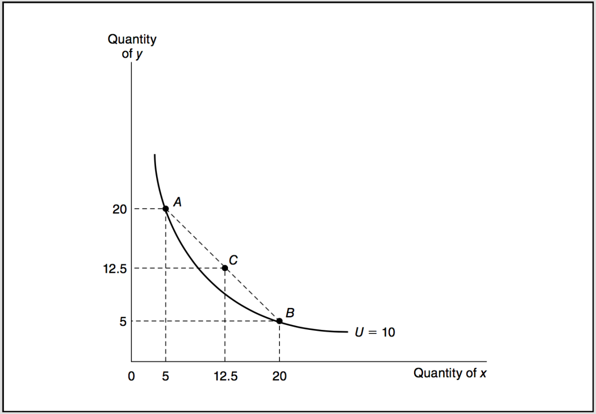
Recall, we can just as well work with any monotone transformation. So lets pick

$$\bar{U}(x, y) = U(x, y)^2 = xy$$

At $\boldsymbol{x} = (20, 5)$ the consumer has utility 100.

FIGURE 3.7 Indifference Curve for Utility $= \sqrt{x \cdot y}$

This indifference curve illustrates the function $10 = U = \sqrt{x \cdot y}$. At point A (5, 20), the MRS is 4, implying that this person is willing to trade $4y$ for an additional x. At point B (20, 5), however, the MRS is 0.25, implying a greatly reduced willingness to trade.



The equation for this particular curve is

$$y = \frac{100}{x}$$

Hence the MRS (of y with respect to x) is

$$MRS_x^y = -\frac{dy}{dx} = \frac{100}{x^2}$$

At our point $\boldsymbol{x} = (20, 5)$ the MRS (of y with respect to x) is

$$MRS_x^y(\boldsymbol{x}) = -\frac{dy}{dx}|_{\boldsymbol{x}} = \frac{100}{x^2} = \frac{1}{4}$$

- The consumer will give up $\frac{1}{4}$ hamburgers (y) to get 1 soft drink (x)
  - Burgers are rare; she is not very willing to trade them.
- At $(5, 20)$ the MRS is 4.
  - She has a surfeit of y, so it is not that valuable.

The total differential of the utility function is

$$\partial U = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy$$

When moving along an indifference curve, this is identically 0, so

$$MRS_x^y(\boldsymbol{x}) = -\frac{dy}{dx}|_{\boldsymbol{x}} = \frac{\frac{\partial U}{\partial x}|_{\boldsymbol{x}}}{\frac{\partial U}{\partial y}|_{\boldsymbol{x}}}$$

Example

With $\bar{U}(x, y) = xy$, we have

$$MRS_x^y = \frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}} = \frac{y}{x}$$

- $MRS_x^y(20, 5) = \frac{5}{20} = \frac{1}{4}$
- $MRS_x^y(5, 20) = \frac{20}{5} = 4$

- Did squaring the utility matter?
- Did we change the MRS?
- Is the MRS ordinal?

If U and V both represent x then there is a strictly increasing h such that $h \circ U = V$.

$$\frac{\frac{\partial V}{\partial x}}{\frac{\partial V}{\partial y}} = \frac{\frac{\partial h}{\partial U}\frac{\partial U}{\partial x}}{\frac{\partial h}{\partial U}\frac{\partial U}{\partial y}} = \frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}}.$$

- The MRS is invariant to the choice of utility.

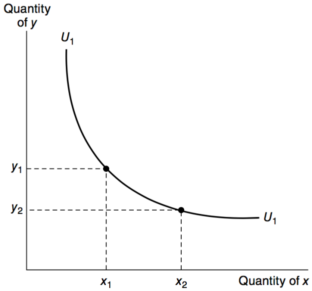
Just like with marginal utility, it feels like the MRS should be diminishing.

- As I trade more and more y for x, I become less willing to give up y.
- I am willing to forego less y for the same amount of x.
- I.e., as x increases the MRS decreases.
  - Staying on the same IC, of course.

Unlike decreasing MU, decreasing MRS is ordinally meaningful.

FIGURE 3.2 A Single Indifference Curve

The curve U_1 represents those combinations of x and y from which the individual derives the same utility. The slope of this curve represents the rate at which the individual is willing to trade x for y while remaining equally well off. This slope (or, more properly, the negative of the slope) is termed the ***marginal rate of substitution***. In the figure, the indifference curve is drawn on the assumption of a diminishing marginal rate of substitution.



- If $-\frac{dy}{dx}$ is decreasing as x increases, then
- $\frac{dy}{dx}$ is increasing, then
- $\frac{d^2y}{dx^2} > 0$, then
- $IC(\boldsymbol{x})$ is convex!

The following are equivalent:

- Decreasing MRS
    - as the amount of a good consumed increases its relative value decreases

- Indifference curves are convex (as functions).

- While the curvature (convexity/concavity) of the utility function is meaningless...
  - It is not invariant to monotone translations.
  - Is not associated with a property of $\succcurlyeq$.

- The curvature of indifference curves is meaningful.
  - Diminishing marginal rate of substitution.
  - Identified by $\succcurlyeq$.

Recall the definition of a convex function:
$\alpha f(\boldsymbol{x}) + (1-\alpha) f(\boldsymbol{y}) \geq f(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y})$.

An equivalent definition:

f is **convex** if the set of points above the graph $f(x)$ is a convex set.

- Recall a set, A, is convex is $\boldsymbol{x}, \boldsymbol{y} \in A$ implies $\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in A$.

What does it mean that $IC(\boldsymbol{z})$ is convex? Apply the definition:

- The set of points above $IC(\boldsymbol{z})$ is a convex set.
- If $\boldsymbol{x}, \boldsymbol{y}$ are both above $IC(\boldsymbol{z})$ so too is $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}$.
    - $\boldsymbol{x}$ above the curve $IC(\boldsymbol{z})$ means $\boldsymbol{x}$ is preferred to $\boldsymbol{z}$.

What does it mean that $IC(\boldsymbol{z})$ is convex?

If $\boldsymbol{x}$ and $\boldsymbol{y}$ are both preferred to $\boldsymbol{z}$ then $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}$ is preferred to $\boldsymbol{z}$.

- Call such preferences **convex** (holds for all $\boldsymbol{z}$).
- The set of points better than any $\boldsymbol{z}$ is a convex set.

Convex preferences prefer balanced bundles to extreme ones. Don’t want all the consumption in

- One product
- One time period
- Contingent on a single event (dislike risk).

Theorem.

Let U represent $\succcurlyeq$. Then $\succcurlyeq$ is convex if and only if U is quasi concave.

- U is quasi-concave if $U(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \geq \min\{U(\boldsymbol{x}), U(\boldsymbol{y})\}$

Proof

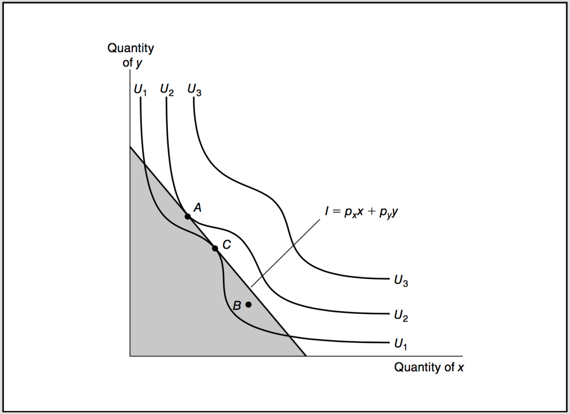
Assume that $\succcurlyeq$ is convex. Consider $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n_+$.

- Either $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ or $\boldsymbol{y} \succcurlyeq \boldsymbol{x}$ (assume the former) so that $U(\boldsymbol{x}) \geq U(\boldsymbol{y})$
  - By completeness.
- Then $\boldsymbol{x} \succcurlyeq \boldsymbol{y}$ and $\boldsymbol{y} \succcurlyeq \boldsymbol{y}$ implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} \succcurlyeq \boldsymbol{y}$
  - By convexity.

Therefore $U(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \geq U(\boldsymbol{y})$. Hence U is quasi-concave.

FIGURE 4.3 Example of an Indifference Curve Map for Which the Tangency Condition Does Not Ensure a Maximum

If indifference curves do not obey the assumption of a diminishing *MRS*, not all points of tangency (points for which $MRS - p_x/p_y$) may truly be points of maximum utility. In this example, tangency point *C* is inferior to many other points that can also be purchased with the available funds. In order that the necessary conditions for a maximum (that is, the tangency conditions) also be sufficient, one usually assumes that the *MRS* is diminishing; that is, the utility function is strictly quasi-concave.



Some prototypical utilities...

Cobb-Douglas

$$U(x, y) = x^{\alpha} y^{\beta}$$

- The example was where $\alpha = \beta = \frac{1}{2}$.
- Since we can take monotone transforms, we usually set $\beta = (1 - \alpha)$.
- The ratio $\frac{\alpha}{\beta}$ is the relative importance of the two factors.
- $MRS_x^y = \frac{\alpha x^{(\alpha-1)} y^{\beta}}{\beta x^{\alpha} y^{(\beta-1)}} = \frac{\alpha}{\beta} \frac{y}{x}$

Perfect Substitutes

$$U(x, y) = \alpha x + \beta y$$

- $MRS_x^y = \frac{\alpha}{\beta}$
- The MRS is constant. y can be exchanged for x at the same cost always.
  - Can always **substitute** y for $\frac{\alpha}{\beta}x$.
- There is not (strictly!) decreasing MRS.

Perfect Compliments

$$U(x, y) = \min\{\alpha x, \beta y\}$$

- MRS_x^y, either 0 or ∞.
- Either one good is in excess or $\alpha x = \beta y$.
- At the point $\alpha x = \beta y$, must get more of both goods.

FIGURE 3.8 Examples of Utility Functions

The four indifference curve maps illustrate alternative degrees of substitutability of x for y. The Cobb-Douglas and CES functions (drawn here for relatively low substitutability) fall between the extremes of perfect substitution (panel b) and no substitution (panel c).