EC5110: MICROECONOMICS

LECTURE 4: RISK / UNCERTAINTY

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So far, all objects of choice have been deterministic.

- A car, or a tangerine.

But many decisions effect uncertain outcomes

- Investments
- Savings decisions
- New experiences
- Research and development.

We will expand our model to account for risk: We need

- a formal way to represent risk (randomness).
- to consider how consumers respond to risk.
- how does this affect markets.

A lottery ticket could be compared to a can of beer!

- This ignores the extra structure of risky prospects.

Formalizing Risk

A simple lottery is a probability distribution over consumption such that only a finite number of elements obtain with positive probability.

Formally a simple lottery p is a function $p : \mathbb{R}^n \to [0, 1]$ such that

$$\text{supp}(p) = \{\boldsymbol{x} \in \mathbb{R}^n \mid p(x) > 0\} \text{ is finite}$$

and

$$\sum_{\text{supp}(p)} p(\boldsymbol{x}) = 1$$

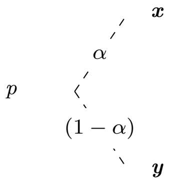
$$[\boldsymbol{x}, \alpha; \boldsymbol{y}, (1-\alpha)]$$

is the lottery that yields $\boldsymbol{x}$ with prob α, $\boldsymbol{y}$ with prob $(1-\alpha)$.

- I.e., this is a way of writing

$$p : \begin{cases} \boldsymbol{x} & \mapsto \alpha \\ \boldsymbol{y} & \mapsto (1-\alpha) \\ \boldsymbol{z} & \mapsto 0, \text{ for } \boldsymbol{z} \neq \boldsymbol{x}, \boldsymbol{y} \end{cases}$$

The lottery $p = [\boldsymbol{x}, \alpha; \boldsymbol{y}, (1-\alpha)]$



A special case is $\mathbb{R}^n = \mathbb{R}$: lotteries over money:

$$[0, \frac{1}{2}; 5, \frac{1}{4}; 10, \frac{1}{4}]$$

Let P denote the set of all simple lotteries over $\mathbb{R}^n$.

A compound lottery is a mixture of simple lotteries. For example

$$\alpha p + (1 - \alpha) q$$

is the lottery that yields the lottery p with prob α; q with $(1 - \alpha)$.

We identify compound lotteries with their reduction; the simple lottery with the same distribution of outcomes. For each $\boldsymbol{x}$:

$$(\alpha p + (1-\alpha)q)(\boldsymbol{x}) = \alpha p(\boldsymbol{x}) + (1-\alpha)q(\boldsymbol{x})$$

What is the reduction of the following lotteries:

- $\frac{1}{2}[0, \frac{1}{2}; 25, \frac{1}{4}; 100, \frac{1}{4}] + \frac{1}{2}[0, \frac{1}{2}; 100, \frac{1}{2}]$

- $\frac{1}{4}[0, \frac{1}{2}; 100, \frac{1}{2}] + \frac{3}{4}[100, \frac{1}{2}; 144, \frac{1}{2}]$

- $\frac{1}{2}[0, \frac{1}{2}; 100, \frac{1}{2}] + \frac{1}{2}[0, \frac{1}{2}; 100, \frac{1}{2}]$

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  - $[0, \frac{4}{8}; 25, \frac{1}{8}; 100, \frac{3}{8}]$
- $\frac{1}{4}[0, \frac{1}{2}; 100, \frac{1}{2}] + \frac{3}{4}[100, \frac{1}{2}; 144, \frac{1}{2}]$
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  - $[0, \frac{1}{2}; 100, \frac{1}{2}]$

We assume consumers have a utility function over consumption:

$$u : \mathbb{R}^n \to \mathbb{R}$$

- Called a vNM **utility index**
- This is the value of “degenerate lotteries”
- $[\boldsymbol{x}, 1] \succcurlyeq [\boldsymbol{y}, 1] \iff u(\boldsymbol{x}) \geq u(\boldsymbol{y})$.

We assume that consumers are expected utility maximizers. They value a lottery by its expected utility.

$$U(p) = \sum_{\text{supp}(p)} p(\boldsymbol{x}) u(\boldsymbol{x})$$

represents the consumers preference.

- We can axiomatize this via restrictions on $\succcurlyeq$.
  - This is pretty elegant.

- Utility is linear in probability.
  - $U(\alpha p + (1-\alpha)q) = \alpha U(p) + (1-\alpha)U(q)$

- Not (necessarily) linear in consumption.
  - $U(\alpha \boldsymbol{x} + (1-\alpha)\boldsymbol{y}; 1) \neq \alpha U(\boldsymbol{x}) + (1-\alpha)U(\boldsymbol{y})$

Lets say the utility index over money is $u(x) = x^{\frac{1}{2}}$. What is the utility of

- $[0, \frac{4}{8}; 25, \frac{1}{8}; 100, \frac{3}{8}]$

- $[0, \frac{1}{8}; 25, \frac{4}{8}; 144, \frac{3}{8}]$

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- $[0, \frac{4}{8}; 25, \frac{1}{8}; 100, \frac{3}{8}]$
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- $[0, \frac{1}{8}; 25, \frac{4}{8}; 144, \frac{3}{8}]$
  - $0 + \frac{4}{8}5 + \frac{3}{8}12 = \frac{56}{8} = 7$
- $[0, \frac{1}{2}; 100, \frac{1}{2}]$

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  - $0 + \frac{4}{8}5 + \frac{3}{8}12 = \frac{56}{8} = 7$

- $[0, \frac{1}{2}; 100, \frac{1}{2}]$
  - $0 + \frac{1}{2}10 = 5$

There is myriad evidence that people do not behave according to expected utility. So why study it?

1. It is normatively appealing. One can argue that it is the philosophically correct way of making decisions under risk.

2. It is simple, and useful in applications. Often the deviations from the theory are not detrimental to the results of an application, so cautiously using EU can simplify analysis.

3. Deviations are often systematic, and therefore EU gives a baseline model to compare other more complicated behavioral models.

Recall: if U represents $\succcurlyeq$ the so does *every* strictly increasing transformation.

With EU we can say more: the utility index is unique up-to affine transformations.

- The above result is still true!
- This stronger identification is because we only consider linear representations.
- Given linearity we can identify cardinal information.

If all we know is $\boldsymbol{z} \succcurlyeq \boldsymbol{y} \succcurlyeq \boldsymbol{x}$ then both

$$U(\boldsymbol{x}) = 0,\, U(\boldsymbol{y}) = 1,\, U(\boldsymbol{z}) = 10000$$

and

$$U(\boldsymbol{x}) = 0,\, U(\boldsymbol{y}) = 1,\, U(\boldsymbol{z}) = 2$$

represent our preferences. But if we have access to risk we can say more:

- In the first: $[\boldsymbol{z}, \frac{1}{10000}; \boldsymbol{x}, \frac{9999}{10000}] \sim [\boldsymbol{y}, 1]$
- In the second: $[\boldsymbol{z}, \frac{1}{2}; \boldsymbol{x}, \frac{1}{2}] \sim [\boldsymbol{y}, 1]$

Theorem.

Let

$$U(p) = \sum_X p(x)u(x)$$

be an EU representation. Then

$$V(p) = \sum_X p(x)v(x)$$

represents the same preferences if and only if $v = au + b$ for some $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$.

Proof

Assume $v = au + b$. We have

$$\begin{aligned}
V(p) \geq V(q) &\iff \sum_X p(x)(au(x) + b) \geq \sum_X q(x)(au(x) + b) \\
&\iff a \sum_X p(x)u(x) + b \geq a \sum_X q(x)u(x) + b \\
&\iff \sum_X p(x)u(x) \geq \sum_X q(x)u(x) \\
&\iff U(p) \geq U(q)
\end{aligned}$$

Proof

Now assume $v \neq au + b$ for $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$.
Choose any x, y, z such that $u(x) > u(y) > u(z)$, but that the affine relation does not hold.

- Let $a = \frac{v(x)-v(z)}{u(x)-u(z)}$ (which is necessarily strictly positive) $b = v(z) - bu(z)$.
- It is easy to check that $v(x) = au(x) + b$ and $v(z) = au(z) + b$.
- Let $\alpha \in (0, 1)$ be the unique number such that $\alpha u(x) + (1 - \alpha)u(z) = u(y)$.

Proof

$$
\begin{aligned}
v(y) &\neq au(y) + b \\
&= aU(\alpha x + (1-\alpha)z) + b \\
&= a(\alpha u(x) + (1-\alpha)u(z)) + b \\
&= \alpha(au(x) + b) + (1-\alpha)(au(z) + b) \\
&= \alpha v(x) + (1-\alpha)v(z) \\
&= V(\alpha x + (1-\alpha)z)
\end{aligned}
$$

So V and U represent different preferences.

We can now turn our attention risk attitudes, or, how to quantify a consumer's tolerance for risk.

- Generally, we think that consumers are risk *averse*, that they prefer less risk, keeping expected consumption levels constant

- The insurance industry exists entirely to reduce exposure to risk

- stocks and other risky securities must pay a premium to entice investors

We will only consider lotteries over money: consumption takes place in $\mathbb{R}^n$.

Let u be continuous and strictly increasing.

For any lottery $p \in P$, there is a unique $x \in \mathbb{R}$ such that

$$u(x) = U(p) = \sum_{\mathbb{R}} p(x)u(x)$$

We will call such an amount of money the certainty equivalent, and denote it by c_p.

- Since u is continuous, we can apply the intermediate value theorem to obtain existence, and strict monotonicity delivers uniqueness.

- c_p, is the amount of money such that the consumer is indifferent between receiving the risky lottery p or c_p with certainty
- $c_p = u^{-1}(U(p))$
- Considering degenerate lotteries $c_x = u^{-1}(U(x)) = u^{-1}(u(x)) = x$: a degenerate lottery is its own certainty equivalent.

For any $p \in P$ the expected payoff of p:

$$e_p = \sum_{\mathbb{R}} p(x)x$$

- The expected payoff of a lottery is exactly what it sounds like: the amount of money the consumer can expect to receive on average

Consider a lottery p and its expected payoff e_p. These two alternatives provide the same expected consumption level, but the latter is risk free.

- A consumer is risk averse if $U(p) \leq U(e_p)$ for all p.
  - She is strictly risk averse of the inequality is strict (for non-degenerate lotteries).
  - risk seeking if the inequality is reversed.
  - risk neutral if both risk seeking and risk averse, so $U(p) = U(e_p)$
- Equivalently, a consumer is risk averse if $c_p \leq e_p$.

Lets say the utility index over money is $u(x) = x^{\frac{1}{2}}$. What is the expected payoff of

- $[0, \frac{4}{8}; 25, \frac{1}{8}; 100, \frac{3}{8}]$
  - Utility: $0 + \frac{1}{8}5 + \frac{3}{8}10 = \frac{35}{8}$
- $[0, \frac{1}{8}; 25, \frac{4}{8}; 144, \frac{3}{8}]$
  - Utility: $0 + \frac{4}{8}5 + \frac{3}{8}12 = \frac{56}{8} = 7$
- $[0, \frac{1}{2}; 100, \frac{1}{2}]$
  - Utility: $0 + \frac{1}{2}10 = 5$

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- $[0, \frac{4}{8}; 25, \frac{1}{8}; 100, \frac{3}{8}]$
    - Utility: $0 + \frac{1}{8}5 + \frac{3}{8}10 = \frac{35}{8}$
    - Expected Payoff $\frac{325}{8}$
    - $U(\frac{325}{8}) \cong 6.4$

- $[0, \frac{1}{8}; 25, \frac{4}{8}; 144, \frac{3}{8}]$
    - Utility: $0 + \frac{4}{8}5 + \frac{3}{8}12 = \frac{56}{8} = 7$

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  - Utility: $0 + \frac{1}{8}5 + \frac{3}{8}10 = \frac{35}{8}$
  - Expected Payoff $\frac{325}{8}$
  - $U(\frac{325}{8}) \cong 6.4$

- $[0, \frac{1}{8}; 25, \frac{4}{8}; 144, \frac{3}{8}]$
  - Utility: $0 + \frac{4}{8}5 + \frac{3}{8}12 = \frac{56}{8} = 7$
  - Expected Payoff $\frac{532}{8}$
  - $U(\frac{532}{8}) \cong 8.5$

- $[0, \frac{1}{2}; 100, \frac{1}{2}]$
  - Utility: $0 + \frac{1}{2}10 = 5$

Lets say the utility index over money is $u(x) = x^{\frac{1}{2}}$. What is the expected payoff of

- $[0, \frac{4}{8}; 25, \frac{1}{8}; 100, \frac{3}{8}]$
  - Utility: $0 + \frac{1}{8}5 + \frac{3}{8}10 = \frac{35}{8}$
  - Expected Payoff $\frac{325}{8}$
  - $U(\frac{325}{8}) \cong 6.4$

- $[0, \frac{1}{8}; 25, \frac{4}{8}; 144, \frac{3}{8}]$
  - Utility: $0 + \frac{4}{8}5 + \frac{3}{8}12 = \frac{56}{8} = 7$
  - Expected Payoff $\frac{532}{8}$
  - $U(\frac{532}{8}) \cong 8.5$

- $[0, \frac{1}{2}; 100, \frac{1}{2}]$
  - Utility: $0 + \frac{1}{2}10 = 5$
  - Expected Payoff : 50
  - $U(50) \cong 7$

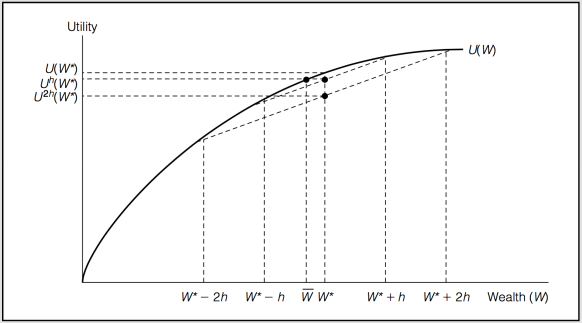
Theorem.

If a consumer has a concave preference over $\mathbb{R}$ then she is risk averse, if she has linear preferences she is risk neutral.

- The definition of concavity/linearity delivers this immediately for lotteries with 2 elements. For a general proof, appeal to Jensen's inequality.

FIGURE 7.1 Utility of Wealth from Two Fair Bets of Differing Variability

If the utility-of-wealth function is concave (i.e., exhibits a diminishing marginal utility of wealth), then this person will refuse fair bets. A 50–50 bet of winning or losing h dollars, for example, yields less utility $[U^h(W^*)]$ than does refusing the bet. The reason for this is that winning h dollars means less to this individual than does losing h dollars.



Example

- A risk averse consumer has a utility index over wealth given by $u(x) = x^{\frac{1}{2}}$.
- She currently has a wealth of 100.
- She might suffer a loss (say her house is on fire) of 64 with probability $\frac{1}{2}$.

How much is she willing to pay to insure herself fully against the loss?

Example

- If she does not insure herself her expected utility is

$$\frac{1}{2}100^{\frac{1}{2}} + \frac{1}{2}(100 - 64)^{\frac{1}{2}} = \frac{10}{2} + \frac{6}{2} = 8$$

- If she does insure herself, at cost c, her expected utility is

$$(100 - c)^{\frac{1}{2}}$$

- Setting these equal and solving for c we see that the consumer is willing to pay $c = 36$.
  - This is more than the expected loss: $\frac{1}{2}64 = 32$.

Example

What if the DM had linear preferences? How does c change?

- If she does not insure herself her expected utility is

$$\frac{1}{2}100 + \frac{1}{2}(100 - 64) = 68$$

- If she does insure herself, at cost c, her expected utility is

$$(100 - c)$$

- Setting these equal and solving for c we see that the consumer is willing to pay $c = 32$.
  - This is exactly expected loss: $\frac{1}{2}64 = 32$.

Example

You are an expected utility maximizer with a utility over money is given by $u(x) = x^{\frac{1}{2}}$, current wealth is 0.

- You are sending a package worth 64.
- There is an $\pi \in [0, 1]$ chance that it gets destroyed in the post, resulting in a valuation of 0.
- The cost for full insurance is 15.

For what values of π will you weakly prefer to purchase the insurance?

Example

You will purchase the insurance if

$$\pi u(0) + (1-\pi)u(64) \leq u(64-15)$$

or if $(1-\pi)8 \leq 7$. Therefore whenever $\pi \geq \frac{1}{8}$.

Insurance Premiums

At wealth w, facing risk embodied by the lottery p, the insurance premium, $\mathsf{ip}(w, p)$, is how much she is willing to pay to insure against risk.

$$U(w + p) = u(w - \mathsf{ip}(w, p))$$

Therefore,

$$\mathsf{ip}(w, p) = w - u^{-1}(U(w + p))$$