MODELING THE MODELER:
A NORMATIVE THEORY OF EXPERIMENTAL DESIGN

Evan Piermont
Royal Holloway, University of London

joint with

Fernando Payró
UAB, BSE & CODE

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- Decision theoretic analysis of experiments: analyst's preference over experiments
- The value of an exp is determined by what it allows to identify
- In principle any kind of experiment, but adapt the analysis to ***revealed preference***
  - Laboratory economics experiments
  - Any agent who can probe at the preference of other agents

- We propose three normative principles for experimental design
  - minimal rationality properties, independent of specific motivations
  - We will specifically think about revealed preference experiments
- We show that they imply a particular representation
  - Relates a experiment to the expected value of identification
  - Unifies many distinct models of experimentation
  - Axiomatic characterization for Bayesian Experimental Design
  - Test for analyst to make sure they do not have an "agenda"

- A space of parameters $\mathcal{U}$
- Experiment A has possible observable outcomes $\{P_1, \ldots, P_n\}$
- Observing P ***identifies*** a set of parameters:
  - $W_{A,P} \subseteq \mathcal{U}$ consistent with observation
- An "ideal" experiment should induce a partition over $\mathcal{U}$
  - Not always possible (some parameters might not uniquely determine an observation)

Ranking over experiments should reflect the value of potential identification

Normative Principles

Structural Invariance: Two experiments that identify the sets of parameters are equally valued

Information Monotonicity: Experiments that induce sharper identification are (weakly) better

Identification Separability: The value of identifying a set of parameters should *not* depend on counterfactuals

Expected Identification Value

These principles characterize ranking according to *expected identification value*

- Exists some τ: for $W \subseteq \mathcal{U}$, $\tau(W)$ is the value of identifying W
- Experiments are valued according to:

$$F(A, \mathscr{P}) = \sum_{P \in \mathscr{P}} \tau(W_{A,P})\mu(W_{A,P})$$

- where μ is the (exogenous) prior probability

Special Case: Entropy

$$\tau(W) = -\log(\mu(W))$$

- Value of experiment is expected reduction in entropy
- Axiomatized in paper

Special Case: Hypothesis Testing

$$\tau(W) = \begin{cases} 1 & \text{if } W \subseteq W^* \text{ or } W^* \subseteq W^c \\ 0 & \text{otherwise} . \end{cases}$$

- Hypothesis: the parameter lies in W^*
- Value of exp is the probability the hypothesis can be accepted or rejected

Special Case: Actions

$$\tau(W) = \max_{a \in \mathbb{A}} \int_W \xi(a, u)\, d\mu.$$

- The analyst will take action $a \in \mathbb{A}$
- Utility of outcome depends on the parameter: $\xi(a, u)$
- Value of exp is expected value of conditionally optimal action

Related literature

Decision Theory

- Dekel, Lipman and Rustichini (2001) “Representing Preferences with a Unique Subjective State Space”
- Gilboa and Lerher (1991) “The value of information-An axiomatic approach”
- Ergin and Sarver (2015) “Hidden actions and preferences for timing of resolution of uncertainty”

Statistics

- Lindley (1972) “Bayesian statistics: A review”
- Chaloner and Verdinelli (1995) “Bayesian experimental design: A review”

An analyst (she) wishes to infer a subject's (he) utility function over X:

- Revealed Preference: she should offer a menu $A \subseteq X$ and observe the subject's choice
- Different menus offer different "inference" opportunities
- Ranking over menus will depend on the goals for the analyst

Experimental Environment

- Z set of alternatives
- $\mathcal{U} \subseteq \{u : Z \to \mathbb{R}\}$ set of utility functions over Z
- Ω algebra of measurable sets of $\mathcal{U}$
- μ prior over $(\mathcal{U}, \Omega)$

The tuple $(Z, \mathcal{U}, \Omega, \mu)$ constitutes a theory for a Bayesian experimenter

An **experiment** $e = (A, \mathscr{P})$ is a pair:

- $A \subseteq Z$ is finite decision problem
- $\mathscr{P}$ is a partition of A
  - Represents observability constraints
  - Allows for dynamic experiments, non-lab settings, etc

Given an experiment, $(A, \mathscr{P})$, define the ***identified set***:

$$W_{A,P} = \{u \in \mathcal{U} \mid P \cap \operatorname{argmax}_{x \in A} u(x) \neq \varnothing\}$$

- Observing $P \in \mathscr{P}$ identifies that the subject's utility is in $W_{A,P}$
- We require for an experiment $(A, \mathscr{P})$ that for any $P, Q \in \mathscr{P}$

(1) $W_{A,P} \in \Omega$ — measurability

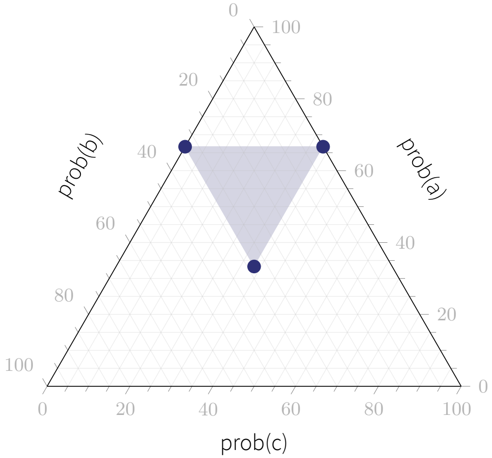
(2) $\mu(W_{A,P} \cap W_{A,Q}) = 0$ — zero μ-prob of times

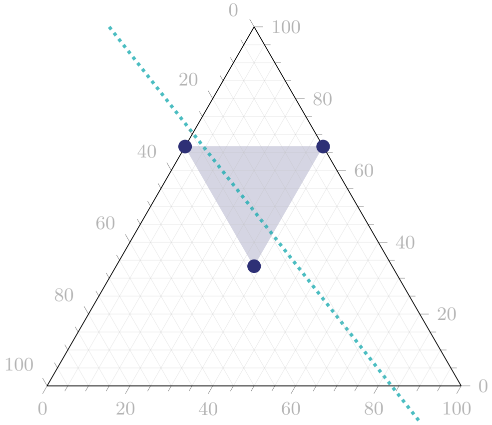
Expected Utility Preferences

We can example identifying EU preferences as an example:

- Z is lotteries over $\{a, b, c\}$
- $\mathcal{U}$ is affine functions

$$\{\tfrac{2}{3}a + \tfrac{1}{3}b, \tfrac{2}{3}a + \tfrac{1}{3}c, \tfrac{1}{3}a + \tfrac{1}{3}b + \tfrac{1}{3}c\}$$

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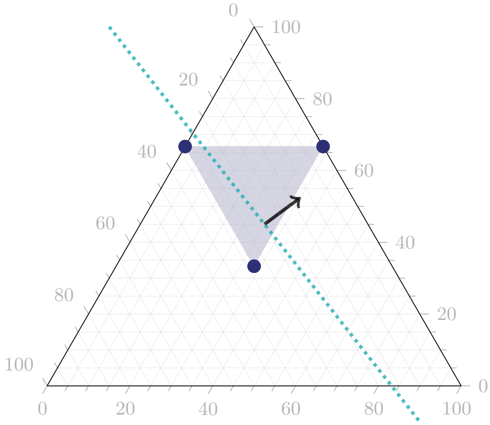

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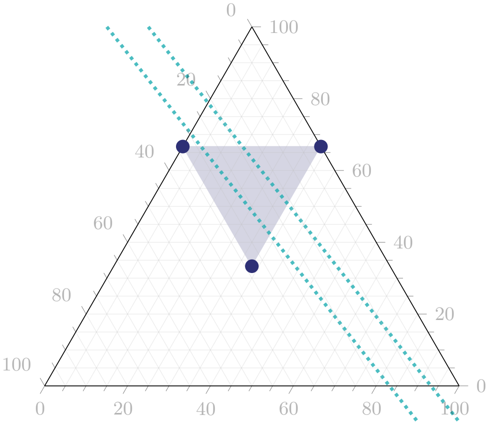
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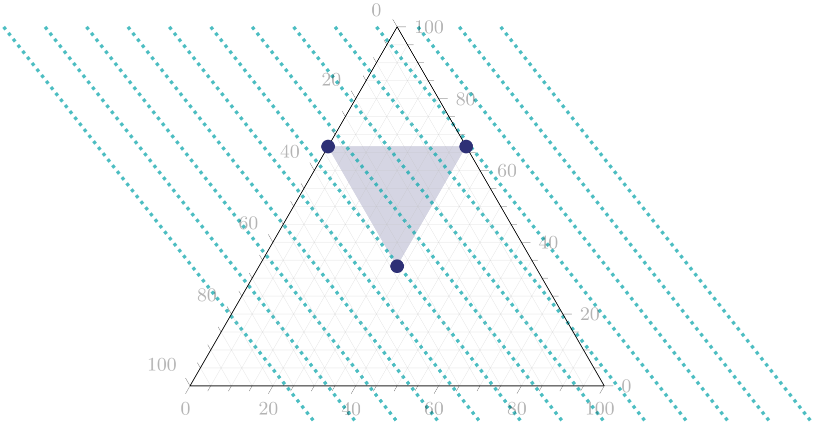
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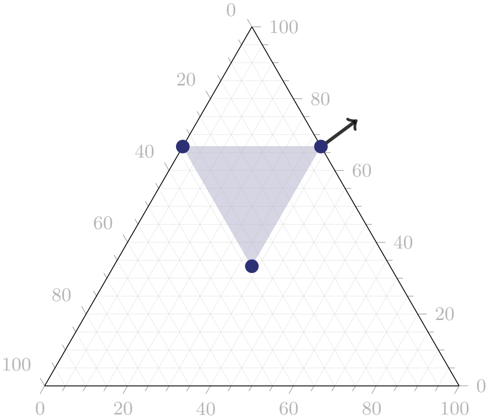

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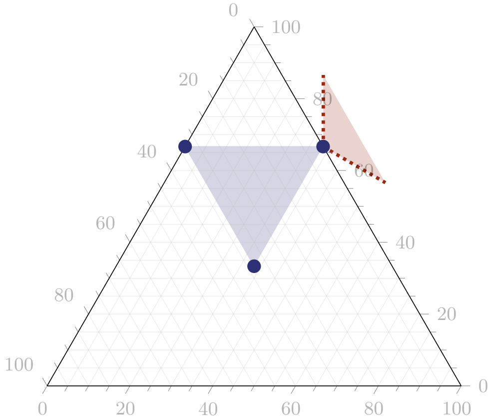

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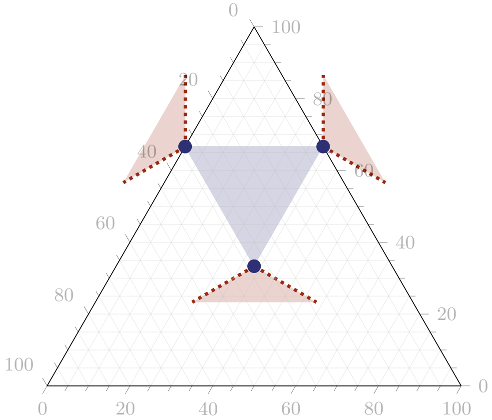

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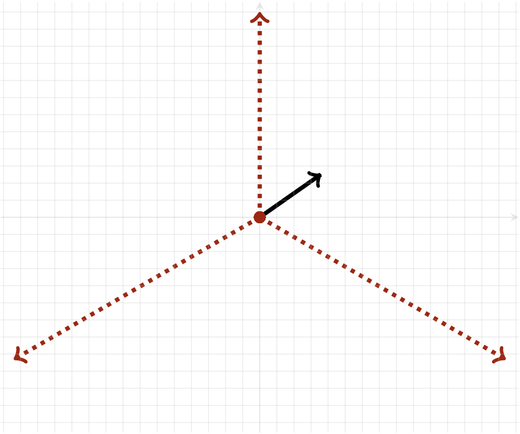
$$\mathcal{U} \cong \mathbb{R}^2$$



μ-equivalence

Call $\{W_1, \ldots, W_n\}$ and $\{V_1, \ldots, V_m\}$, families of subsets of $\mathcal{U}$, *μ-equivalent* if

$$\mu(W_i) > 0 \to \mu(W_i) = \mu(W_i \cap V_j) \text{ for some } j \quad \text{and,}$$
$$\mu(V_j) > 0 \to \mu(V_j) = \mu(W_i \cap V_j) \text{ for some } i$$

- Such collections identify the same sets of utilities up to a measure zero
- Take $[0, 1]$ with λ the Lebesgue measure. The following are λ-equivalent:
  - $\{[0, \frac{1}{2}), (\frac{1}{2}, 1]\}$; $\{[0, \frac{1}{2}), \{\frac{1}{2}\}, (\frac{1}{2}, 1]\}$, $\{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$

Rich Experimental Settings

We say a set of experiments $\mathbb{E}$ is ***rich*** if

(1) $(A, \mathscr{P}) \in \mathbb{E} \to (A, \mathscr{Q}) \in \mathbb{E}$ whenever $\mathscr{Q}$ is a coarsening of $\mathscr{P}$

(2) For any finite Ω-measurable partition of $\mathcal{U}$, there exists an experiment $(A, \mathscr{P})$ such that $\{W_{A,P}\}_{P \in \mathscr{P}}$ is μ equivalent

- Any partition can be approximated up to 0 probability events
- For the EU model, the set of all experiments is rich for any "regular" μ

Primitive

- Our primitive is a ranking $\succcurlyeq$ over the set of all random experiments
- A *random experiment* is a lottery over some (fixed) rich set $\mathbb{E}$

Normative Principles of Experimental Design

"Two experiments that identify the sets of parameters are equally valued"

(P1) - Structural Invariance

If $\{W_{A,P} | P \in \mathscr{P}\}$ is μ-equivalent to $\{W_{B,Q} | Q \in \mathscr{Q}\}$ then $(A, \mathscr{P}) \sim (B, \mathscr{Q})$.

- Structural properties of experiments are irrelevant
- Also, 0-probability events are irrelevant

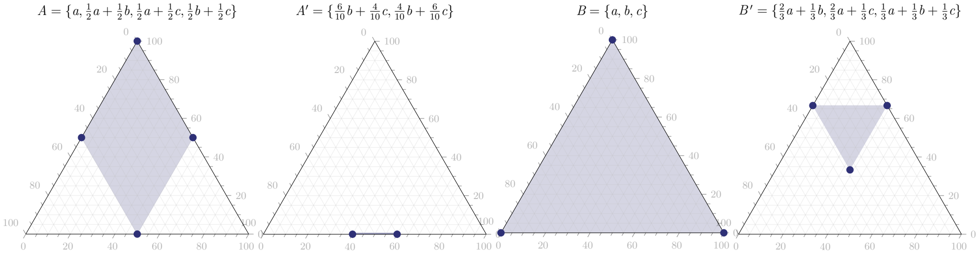
Consider our EU maximizing subject choosing lotteries over $\{a, b, c\}$.

$$\text{EXP A}: A = \{a,\ \tfrac{1}{2}a + \tfrac{1}{2}b,\ \tfrac{1}{2}a + \tfrac{1}{2}c,\ \tfrac{1}{2}b + \tfrac{1}{2}c\}$$

$$A' = \{\tfrac{6}{10}b + \tfrac{4}{10}c,\ \tfrac{4}{10}b + \tfrac{6}{10}c\}.$$

$$\text{EXP B}: B = \{a,\ b,\ c\}$$

$$B' = \{\tfrac{2}{3}a + \tfrac{1}{3}b,\ \tfrac{2}{3}a + \tfrac{1}{3}c,\ \tfrac{1}{3}a + \tfrac{1}{3}b + \tfrac{1}{3}c\}.$$


$A = \{a, \frac{1}{2}a + \frac{1}{2}b, \frac{1}{2}a + \frac{1}{2}c, \frac{1}{2}b + \frac{1}{2}c\}$
$A' = \{\frac{6}{10}b + \frac{4}{10}c, \frac{4}{10}b + \frac{6}{10}c\}$
$B = \{a, b, c\}$
$B' = \{\frac{2}{3}a + \frac{1}{3}b, \frac{2}{3}a + \frac{1}{3}c, \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c\}$
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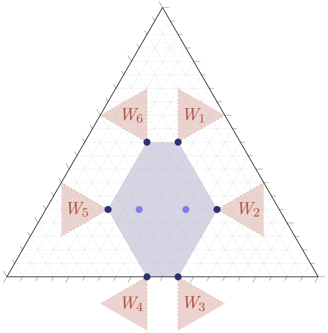
- Linearity states that

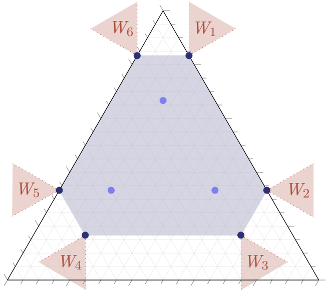
$$\left.\begin{array}{l} x \in \arg\max_A u(\cdot) \\ y \in \arg\max_B u(\cdot) \end{array}\right\} \quad \text{if and only if} \quad \alpha x + (1-\alpha) y \in \arg\max_{\alpha A + (1-\alpha) B} u(\cdot)$$

- Therefore, observing A followed by A' is equivalent to observing $\frac{1}{2}A + \frac{1}{2}A'$

$\frac{1}{2}A + \frac{1}{2}A'$



$\frac{1}{2}B + \frac{1}{2}B'$



- Structural invariance reflects the symmetries of the given domain
- With linear utility, the symmetry is ***translation invariance***:

Structural Invariance for Expected Utility

$$(A, \{P_1, \ldots P_n\}) \sim (A + B, \{P_1 + B, \ldots P_n + B\})$$

This isn't exactly correct, since $\{P_1 + B, \ldots P_n + B\}$ might have overlaps....

"Experiments that induce sharper identification are (weakly) better"

(P2) - Information Monotonicity

If $\mathscr{P}$ refines $\mathscr{Q}$ then $(A, \mathscr{P}) \succcurlyeq (A, \mathscr{Q})$.

- Preference respects Blackwell order

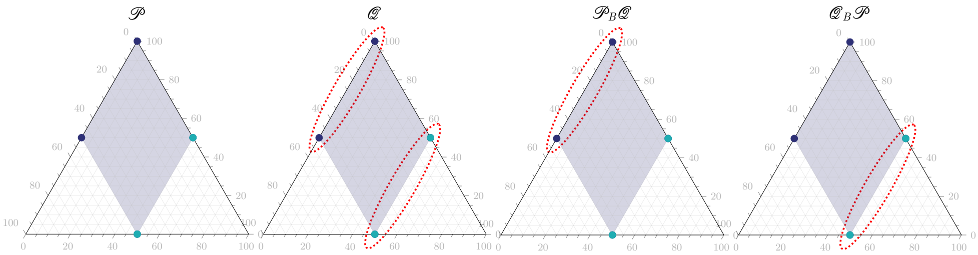
“The value of identification not depend on counterfactuals”

(P3) - Identification Separability

$$\tfrac{1}{2}(A,\mathscr{P}) + \tfrac{1}{2}(A,\mathscr{Q}) \sim \tfrac{1}{2}(A,\mathscr{P}_B\mathscr{Q}) + \tfrac{1}{2}(A,\mathscr{Q}_B\mathscr{P}).$$

- $\mathscr{P}$ and $\mathscr{Q}$ partitions of A and $B \subseteq A$, then $\mathscr{P}_B\mathscr{Q}$ denotes the partition that coincides with $\mathscr{P}$ over B and with $\mathscr{Q}$ over $A \setminus B$

Consider decision problem A (from before) with the following partitions



- The set B is the two south-east lotteries (in teal)

Theorem

Let $\succcurlyeq$ be an expected utility preference, represented by index $F : \mathbb{E} \to \mathbb{R}$.

Then $\succcurlyeq$ satisfies **P1-3** if and only if there exists a $\tau : \Omega \to \mathbb{R}$ such that:

$$F(A, \mathscr{P}) = \sum_{P \in \mathscr{P}} \tau(W_{A,P})\mu(W_{A,P})$$

with $W \subseteq V$ implies

- $\tau(W)\mu(W|V) + \tau(V \setminus W)(1 - \mu(W|V)) \geq \tau(V)$
- $\mu(W) = \mu(V)$ implies $\tau(W) = \tau(V)$

Representation reflects our normative principles:

$$F(A, \mathscr{P}) = \sum_{P \in \mathscr{P}} \tau(W_{A,P})\mu(W_{A,P})$$

- Only dependents on $W_{A,P} \rightarrow$ Structural Invariance
- Additive $\rightarrow$ Identification Separability
- $\tau(W)\mu(W|V) + \tau(V \setminus W)(1 - \mu(W|V)) \geq \tau(V) \rightarrow$ Monotonicity

In the paper we:

- Specify our axioms to the entropy minimization case
- Consider a prior free model
    - τ “includes” the probability
- Show how our model can capture dynamic experiments

Thank You!