Asymmetric Gain-Loss Reference Dependence and Attitudes towards Uncertainty*

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Abstract

This paper characterizes a model of reference-dependence, where a state-contingent contract (act) is evaluated by its expected value and its expected gain-loss utility. The expected utility of an act serves as the reference point, hence gains (resp., losses) occur when the act provides an outcome that is better (worse) than expected. The utility representation is characterized by a belief regarding the state space and a degree of reference-dependence; both are uniquely identified from behavior. We establish a link between this type of reference-dependence and attitudes towards uncertainty. We show that loss aversion and reference dependence are equivalent to max-min and concave expected utility.

Keywords: Reference Dependent Preferences, Endogenous Reference Points, Gain-Loss Attitudes, Subjective Expected Utility, Belief Distortion.

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1 Introduction

In many circumstances, a decision maker (DM) may evaluate an uncertain prospect not only in absolute terms but also in relative relation to some reference point. Kahneman and Tversky [1979] first introduced the notion of reference-dependence, in the seminal *Prospect Theory*, to explain experimental violations of expected utility. Within Prospect Theory, deviations from the reference point are weighted by a gain-loss value function, which has the feature, referred to as *loss aversion*, that losses have more negative value than equal sized gains have positive value.

A different resolution for empirical deviations from expected utility proposes models of multiple priors, in particular *MaxMin Expected Utility (MMEU)*. MMEU, axiomatized by Gilboa and Schmeidler [1989] as an explanation of the Ellsberg Paradox,[1] considers a DM who holds a family of beliefs regarding the likelihood of events. She evaluates uncertain prospects by the minimum expected utility consistent with any of her beliefs. As such, a MMEU DM displays *uncertainty aversion* (or, ambiguity aversion), the feature that she prefers to minimize her exposure to uncertainty.

At a purely intuitive level, there seems to be a connection between loss aversion and uncertainty aversion; both behaviors characterize some form of pessimism in comparison to a subjective expected utility (SEU) maximizer. A loss averse DM places more weight on the utility of "bad" events but leaves the probabilities undistorted, whereas a uncertainty averse DM places more weight on the probability of "bad" events but leaves the utilities undistorted. We show in this paper that this connection is more than superficial; there exists a formal connection between reference dependence and attitude towards uncertainty. In particular, we axiomatize a simple class of reference dependent preferences, called *asymmetric gain-loss (AGL) preferences*, which can be equivalently represented by a MMEU functional. Within our framework, loss aversion and uncertainty aversion produce identical choice data.

1.1 AGL Prefernces

In addition to formalizing the connection between reference dependence and ambiguity aversion, AGL preferences provide a simple model of endogenous reference dependence. Our object of choice is a state contingent contract, or act, which is an assignment of

[1]Because it is well established in the literature, we refer the reader to Gilboa and Schmeidler [1989] for a formal discussion of the Ellsberg Paradox and its resolution by MMEU.

consumption (in utility terms) to each state of the world. AGL preferences evaluate an act according to two components: *first-order expected utility*, and *gain-loss utility*. The first order expected utility of an act is as in the standard SEU model, where the decision maker holds a subjective belief, μ, over the state-space. Her first-order expected utility of an act $f : S \to \mathbb{R}$ is

$$\mathbb{E}_\mu[f] = \sum_{s \in S} \mu(s) f(s), \tag{1.1}$$

where S is the state space and $s \in S$ is a generic state. The AGL DM takes this assessment of acts both as the reference point by which gains and losses are measured, and as the baseline level of utility on which gains and losses act as distortions. The main result of this paper is the behavioral characterization of asymmetric gain-loss preferences, which are preferences that can be represented by the functional

$$V(f) = \underbrace{\mathbb{E}_\mu[f]}_{\text{First Order Expected Utility}} + \underbrace{\lambda \sum_{s: f(s) < \mathbb{E}_\mu[f]} \mu(s)[\mathbb{E}_\mu[f] - f(s)]}_{\text{Expected Utility Penalty}}. \tag{1.2}$$

The first term is the DM's subjective expected utility without any reference considerations, and the second is the gain-loss utility. The gain-loss term captures the reference effects. Depending on the sign of λ, the expected gain-loss utility either adds utility in states when gains occur or subtracts utility in states where losses occur. Notice, in (1.2), when $\lambda < 0$, the decision maker receives a utility penalty when the realized utility falls short of the expectation.

On the other hand, when $\lambda > 0$, the decision maker receives a utility bonus in states when the realized utility exceeds the expected utility. Exploiting the symmetry of gains and losses, we can express equation (1.2) in the following way which more clearly illustrates the role of the gain-loss term when λ is positive

$$V(f) = \underbrace{\mathbb{E}_\mu[f]}_{\text{First Order Expected Utility}} + \underbrace{\lambda \sum_{s: f(s) > \mathbb{E}_\mu[f]} \mu(s)[f(s) - \mathbb{E}_\mu[f]]}_{\text{Expected Utility Bonus}}. \tag{1.3}$$

The utility bonus or penalty is linearly scaled by λ. In our representation results, all the elements are identified from choice behavior: μ and λ are identified uniquely.[2]

The decomposition of outcomes into positive and negative events was first done by Gul [1991], wherein the DM is averse to disappointing outcomes given objectively risky lotteries. Note, in this model any gain-loss considerations take place because of uncertainty. The gain-loss parameter λ expresses how the DM will feel for every realization of uncertainty compared to the reference point. Hence, in the absence of uncertainty, the DM behaves as an expected utility maximizer.

1.2 A Simple Example of Asymmetric Gain-Loss Preferences

We employ the following numerical example to explain the intuition behind the representation, and show how asymmetric gain-loss preferences can explain different types of behavior regarding uncertainty.

Consider the environment of a seller selling a single good to a buyer who makes a take-it-or-leave-it offer.[3] The value to the seller is $\hat{v}$ which the buyer believes takes the values $\{5, 3, 2\}$, with probability .2, .3, and .5, respectively. The buyer has an independent private value for the object given by $v = 10$. The buyer will submit her offer, b, and the seller will accept or reject the offer. The seller will accept any offer which (weakly) exceeds her value. The buyer's utility associated with the bid b is given by

$$U(b) = \begin{cases} v - b & \text{if} \quad b \geq \hat{v}, \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that the optimal bid will always be in $\{5, 3, 2\}$. If the buyer is a risk neutral expected utility maximizer, her optimal bid solves

$$\max_{b \in \{5,3,2\}} \quad (v - b)\text{Prob}(b \geq \hat{v}).$$

The optimal bid is $b^{RN} = 3$, which has an expected utility of 5.6 before the bid is placed.

Now suppose that the buyer has gain-loss preferences: in addition to the expected

[2] It is possible to decompose λ into a gain component λ_g and a loss component λ_l, such that $\lambda_g + \lambda_l = \lambda$. Then λ_g scales the utility difference in gain states and λ_l in loss states. While the equivalent representation based on such a decomposition may feel more intuitive, such a decomposition is not unique, and so, we choose to focus on the mathematically simpler representation.

[3] We further explore the implications of our model in standard auction environments in a supplement to this paper available at `http://www.pitt.edu/~ehp5/AGL_SM.pdf`.

value she wants to avoid losses, so she subtracts any expected losses from the expected first order utility to determine the valuation (this corresponds to the parametrization $\lambda = -1$). A loss for her is any outcome where her ex-post utility is worse than the expected value; therefore, outcomes in which she does not obtain the item are considered losses.

Her expected AGL utility, taking into account her gain-loss preferences, of making the bid $b = 3$ is given by:

$$\underbrace{5.6}_{\text{First Order Expected Utility}} \quad - \quad \underbrace{(.2)\,(5.6 - 0)}_{\text{Expected Loss for } b^{RN}=3} \quad ,$$

which is equal to 4.48. The optimal AGL bid, however, is $b^{AGL} = 5$, which provides a constant utility, and hence an AGL utility of 5.

When the buyer in this simple take-it-or-leave-it example takes into consideration expected gains and losses in addition to the standard expected utility, she is better off increasing her bid. Intuitively, she sacrifices her payoff in good outcomes (where she obtains the item) in order to decrease the chance of bad outcomes (not obtaining the item). While her payoff is smaller contingent on obtaining the good, the outcome is favorable more often and she increases her ex-ante utility.

A similar behavior could be captured by a buyer who was averse to ambiguity. To see this, assume that the buyer, instead of having AGL preferences, has MMEU preferences. Further, the buyer believes that the distribution of the seller's valuation is contained in the following set of distributions over $\{5, 3, 2\}$:

$$C = \left\{ \begin{array}{l} [.75, .15, .10], [.35, .51, .14], [.40, .24, .36], \\ [.60, .36, .04], [.65, .09, .26], [.25, .45, .30] \end{array} \right\}.$$

As such, the buyer chooses a bid according to:

$$\max_b \left(\min_{\nu \in C} \; (v - b)\mathrm{Prob}_\nu(b \geq \hat{v}) \right).$$

It is straightforward to check that the optimal bid is $b^{MMEU} = 5$. Therefore a AGL bidder with $\mu = [.2, .3, .5]$ and $\lambda = -1$ and a MMEU bidder with multiple priors given by C will make the same optimal bid. While at first glance this connection may seem contrived, in fact, the AGL and MMEU DMs choose identically not only in this bidding game but in *all* decision problems –they have identical preferences.[4] We show in Section 3 that AGL behavior can always be equivalently described by a MMEU decision maker.

1.3 Reference Point Determination

There are two important features that set AGL preferences apart from other models of endogenous reference dependence, such as Shalev [2000], Kőzsegi and Rabin [2006] (henceforth KR) or Sarver [2011], and make it more tractable for use in applied work. First, in our model the DM has no control over the reference point: the reference point does not depend on any actions taken by the DM. The reference point is identified from beliefs and preferences over outcomes, neither of which are affected by reference dependence. In the aforementioned models, the DM is aware of her reference dependence and how her choices affect the reference point. These papers require that the DM's choices are optimal given the reference point they induce, and so, require an equilibrium condition to account for the mutual relationship between reference points and choices. The model presented here has the benefit that it captures gain-loss attitudes and elicits the reference point, reference effects, and beliefs simultaneously from choices in a way that is easy to compute or estimate, without the need of equilibrium conditions.

The second, and closely related difference, is that in our model, the domain of uncertainty is fixed across decision problems. KR provide the first decision-theoretic model of reference dependence that tackles the determination of an endogenous reference point. In their model, the uncertainty concerns the distribution over possible choice problems. On the other hand, in our model, uncertainty is over states that determine the outcomes of acts, which is the standard domain to analyze choice under uncertainty.

The identification of beliefs about the possible choice problems that a DM might face poses a challenge from a theoretical and empirical point of view. If the DM is uncertain about the problems she might face, she must form expectations about the environment long before seeing the choice set, let along making any choices. There is no standard choice setting that allows for the identification of these beliefs. Any observed choice for a DM in the KR framework will be the choice conditional on

[4] The set of priors, C, coincides with the distributions obtained from equation (2.1) and are visualized in Figure 1.

a reference point, and the reference point is always the chosen alternative, via the equilibrium requirement. This feedback between choices and the reference point can lead to intransitivity of the revealed preference, as shown in Gul and Pesendorfer [2006]. Therefore, under standard choice settings, beliefs about the possible choice sets (and hence the endogenous reference point) cannot be independently identified from behavior.

1.4 Structure of the Paper

The rest of the paper is structured as follows. Section 2 provides an axiomatic characterization of the preferences, discusses the concept of the alignment of acts, which is instrumental for the endogenous determination of a reference point, and formally defines the utility representation. Section 3 explores the link between gain-loss and ambiguity attitudes. Section 4 contains a literature review. Section 5 concludes. All proofs are contained in the appendix.

2 Axiomatization

In this section, we formally present the choice environment and a set of axioms which prove to be necessary and sufficient for the representation presented later in this section.

Let $S = \{s_1, s_2, ..., s_n\}$ be a finite set of states of the world that represent all possible payoff-relevant contingencies for the DM; any $E \subseteq S$ is called an event. Define $\mathcal{E} = \mathcal{P}(S) \setminus \{\emptyset, S\}$ as the set of all non-trivial events. Denote by $\mathcal{F} = \mathbb{R}_+^S$ the set of all acts, that is, functions $f : S \to \mathbb{R}_+$ (endowed with the standard Euclidean topology). We interpret the act f as providing the payoff $f(s)$ in state $s \in S$ and assume it is the utility received by the DM when f is chosen and s is realized.[5]

Take the mixture operation on $\mathcal{F}$ as the standard pointwise mixture, where for any $\alpha \in [0, 1]$, $\alpha f + (1-\alpha)g \in \mathcal{F}$ gives $\alpha f(s) + (1-\alpha)g(s) \in \mathbb{R}_+$ for any $s \in S$. Abusing notation, any $c \in \mathbb{R}_+$ can be identified with the constant act $c(s) = c$ for all $s \in S$. Let $\mathcal{F}_c \cong \mathbb{R}_+$ be the set of constant acts. Preferences on $\mathcal{F}$ are denoted by the binary relation $\succsim$; $\succ$ and $\sim$ represent respectively the asymmetric and symmetric

[5] Note, we are tacitly assuming the decision makers cardinal utility has already been identified via standard means, i.e., the examination of preferences over *objective* lotteries. We could just as easily add a second stage of objective randomization into acts, à la Anscombe and Aumann [1963], but this would require additional notation, and the elicitation of utility values is not central to our model.

components of $\succsim$. For each $f \in \mathcal{F}$, if there is some $c_f \in \mathcal{F}_c$, such that $f \sim c_f$, then call c_f the *certainty equivalent* of f. Before we can specify the behavioral restrictions on preference that correspond to the AGL utility representation, we need to consider some particular structures in the choice domain.

Balanced Pairs of Acts

A particularly important type of act to study AGL preference is given by those that provide perfect hedges against uncertainty. Hedging gets rid of uncertainty, and therefore it also removes all possible gain-loss considerations from the act. Call a pair of acts $(f, \bar{f})$ *balanced* if they provide a perfect hedge and are indifferent to each other.[6] The importance of balanced acts is that eliminating subjective gain-loss considerations allows an analyst to identify beliefs from preferences.

Definition 1. Two acts f and $\bar{f}$ are *balanced* if $f \sim \bar{f}$, and for any states $s, s' \in S$

$$\frac{1}{2}f(s) + \frac{1}{2}\bar{f}(s) = \frac{1}{2}f(s') + \frac{1}{2}\bar{f}(s').$$

If there exists $e_f \in \mathcal{F}_c$ such that $e_f = \frac{1}{2}f(s) + \frac{1}{2}\bar{f}(s)$ for all $s \in S$, we call e_f the *hedge* of f. $(f, \bar{f})$ is referred to as a *balanced pair*, and $\bar{f}$ is a *balancing act* of f (and vice-versa).

When the notation $\bar{f}$ is used, it is always in reference to the balancing act of $f \in \mathcal{F}$. The conditions imposed on preferences below guarantee that c_f and e_f are unique and well defined for each f.

Act Alignment: Separating Positive and Negative States

Balanced acts will provide a behavioral way of separating gains and losses. We require that when the outcome in state s is considered a gain for f, the outcome on state s is considered a loss for $\bar{f}$. This is a natural requirement given that f and $\bar{f}$ provide a perfect hedge to the DM. Hence, $\bar{f}$ has the exact opposite gain-loss composition of f. For an act, define positive states as those states that deliver gains, and negative states as those states that deliver losses.

[6] Siniscalchi [2009] calls a pair of acts that provide perfect hedging as *complementary acts*. We strengthen the definition of complementary acts to further require the acts to be indifferent.

Definition 2. Let $(f, \bar{f})$ be a balanced pair. Say $s \in S$ is a *positive state* for f if $f(s) \geq \bar{f}(s)$, and a *negative state* for f if $\bar{f}(s) \geq f(s)$. If a state is both positive and negative (i.e. $f(s) = \bar{f}(s)$) say s is a *neutral state* for f.

Any balanced pair of acts induces a set of partitions, each of which splits the state space into two events: one event that contains only positive states for f ($\{s \in S | f(s) \geq \bar{f}(s)\}$) and one event that contains only negative states for f ($\{s \in S | \bar{f}(s) \geq f(s)\}$). We use the convention that neutral states can be labeled as either positive or negative.[7] When there are no neutral states, each act has a unique way of partitioning the states into positive and negative. These partitions associated with each act are called the *alignment* of the act. We use the convention that the alignment of the act is represented by the event that includes the positive states E (the complement is the negative states) rather than saying that the alignment is represented by the partition $\{E, E^c\}$.

Definition 3. For any $f \in \mathcal{F}$, say f is aligned with the event $E \in \mathcal{E}$ if for all $s \in E$, $f(s) \geq \bar{f}(s)$ and for all $s \in E^c$, $\bar{f}(s) \geq f(s)$.

For every $E \in \mathcal{E}$, there is a set of acts that is aligned with E.

Definition 4. Given any event $E \subset S$, define $\mathcal{F}^E$ be the set of acts where the positive states are contained in E, i.e.

$$\mathcal{F}^E = \{f \in \mathcal{F} : \ \forall \ s \in E, f(s) \geq \bar{f}(s) \text{ and } \forall s \notin E, \ \bar{f}(s) \geq f(s)\}.$$

Note that any constant act is its own balancing act, therefore constant acts are aligned with all partitions of the state space. It is useful to consider acts that have only one alignment, which are called *single alignment* acts. These acts are important because they are acts where small perturbations on outcomes do not change the alignment.

Definition 5. Given $f \in \mathcal{F}$, f is *single-alignment* act if for no $s \in S$, $f(s) = \bar{f}(s)$.

If the event E represents an alignment of f, every subset of E or E^c is called a *non-overlapping event.* These are the events where all the states are either all positive or all negative for f, so there is no overlap between positive and negative states for f. Non-overlapping events provide a way of specifying situations where there is no tradeoff between positive and negative states, only across one type of state.

[7] We do not allow the neutral states to be labeled both positive and negative, instead when there are neutral states a balanced pair induces more than one partition.

Definition 6. Given $f \in \mathcal{F}$, event $F \subset S$ is a *non-overlapping event* for f if every state in F is aligned in the same way.

If follows that F is non-overlapping for $f \in \mathcal{F}^E$ if

$$F \subseteq E \text{ or } F \subseteq E^c.$$

Abusing terminology, we say that F is non-overlapping for E whenever F is non-overlapping for all $f \in \mathcal{F}^E$.

With these definitions in mind we can now specify the behavioral restrictions that are necessary and sufficient to be represented by the AGL-functional, as given by equation (1.2).

Standard Axioms

The first 3 conditions, A1-A3, are standard axioms in the literature of choice under uncertainty.[8]

A1. (Weak Order). $\succsim$ is complete and transitive.

A2. (Continuity). For all $f \in \mathcal{F}$, the sets $\{g \in \mathcal{F} : g \succsim f\}$ and $\{g \in \mathcal{F} : f \succsim g\}$ are closed.

A3. (Strict Monotonicity). If for all s, $f(s) \geq g(s)$ then $f \succsim g$. If $f(s) > g(s)$ for some s and $f(s) \geq g(s)$ for all s, then $f \succ g$.

New Axioms: Mixture Conditions

The standard subjective expected utility model from Anscombe and Aumann [1963] is characterized by some version of A1-A3, plus the independence axiom. Independence requires that $f \succsim g$ if and only if $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$ for any $h \in \mathcal{F}$, and any $\alpha \in (0, 1)$.

The independence axiom does not hold for AGL preferences because it does not allow for gains and losses to be evaluated differently; convex combinations of acts can change the gain-loss composition of acts, therefore changing the assessments as well. AGL preferences relax independence, but impose three consistency requirements for mixtures of acts.

[8] Strict monotonicity implies non-triviality, state-independence, and that no state is null, i.e. the DM puts positive probability on every state occurring.

The first new axiom states that as long as the alignment of acts remains the same when mixing, then independence is preserved. If two acts have the same alignment, taking any mixture of them does not change the composition of gains and losses, so the tradeoff between gains and losses should not change.

A4. (Alignment Independence). If $f, h \in \mathcal{F}^E$ and $g, h \in \mathcal{F}^F$ for some $E, F \in \mathcal{E}$, then $f \succsim g$ if and only if $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$ for all $\alpha \in [0,1]$.

The second condition imposes structure on mixtures for pairs of acts that are not mutually aligned. The condition is motivated by two observations. First, when any act h is neutral on an event E, then mixing f with h should not change the gain-loss consideration of f within E directly. Second, for a single alignment act f, perturbing it slightly (i.e. mixing f with any other act, where the weight on f is close to 1) does not change the alignment of f, if preferences are continuous and monotonic.

Under the full independence axiom, the preference between f mixed with h or f mixed with h' would depend only on the preference between h and h'. Here, on the other hand, mixing acts may change the alignment of states, and therefore alter the gain-loss consideration in non-linear ways. However, if we consider only mixtures that do not alter the alignment of positive and negative states (for example when α is very close to 1), then the preference between $\alpha f + (1-\alpha)h$ or $\alpha f + (1-\alpha)h'$ should reflect only the differences between h and h'. This condition is called *Local Mixture Consistency.*

A5. (Local Mixture Consistency). For any single-alignment acts $f \in \mathcal{F}^E$ and $g \in \mathcal{F}^{E'}$, any event F, which is non-overlapping for *both* E, and E', and any $h, h' \in \mathcal{F}$ such that $h(s) = \bar{h}(s) = h'(s) = \bar{h}'(s)$ for all $s \notin F$, there exists $\alpha^* < 1$ such that

$$\alpha f + (1-\alpha)h \succsim \alpha f + (1-\alpha)h' \iff \alpha g + (1-\alpha)h \succsim \alpha g + (1-\alpha)h',$$

for all $\alpha \in (\alpha^*, 1)$.

The last axiom imposes a consistency condition on mixtures of acts when reversing the role of gains and losses. Intuitively, the condition requires that the effect of mixing h and f is the opposite to the effect of mixing h and $\bar{f}$. This condition is called *Antisymmetry.*

A6. (Antisymmetry). For any acts f and g such that $f \sim g$, for every $h \in \mathcal{F}$, and for any $\alpha \in (0,1)$, $\alpha h + (1-\alpha)f \succsim \alpha h + (1-\alpha)g$ implies

(i) $\alpha h + (1-\alpha)\bar{f} \precsim \alpha h + (1-\alpha)\bar{g}$, and,

(ii) $\alpha \bar{h} + (1-\alpha) f \precsim \alpha \bar{h} + (1-\alpha) g$,

where $\bar{f}$, $\bar{g}$, and $\bar{h}$ are balancing acts of f, g, and h respectively.

Since f is indifferent to g, the DM has a strict preference between the mixtures only if h is changing the gain-loss component of utility. Consider the case where the DM is loss biased. Then, $\alpha h + (1-\alpha) f \succsim \alpha h + (1-\alpha) g$ whenever f "smooths" out consumption of h more than g does $-f$ provides a better hedge against the loss states of h. Of course, when f and g are replaced with $\bar{f}$ and $\bar{g}$, the losses and gains are reversed, and so, $\bar{f}$ now exaggerates the loss states of h, breaking the indifference between f and g in the opposite direction. The same intuition applies when h is replaced with $\bar{h}$. Notice, when h is constant, there is no room for hedging, and so, the mixtures with f and g will be indifferent (as dictated by Alignment Independence).

2.1 Representation Results

This section provides the main representation results of the paper. Theorem 2.1 introduces the AGL representation as characterized by the above axioms. This section also outlines important preliminary results that highlight the role of particular axioms and elucidate the relation between the AGL representation and other decision theoretic models.

Theorem 2.1. *The preference $\succsim$ satisfies A1-A6 if and only if there exists a probability distribution $\mu \in \Delta(S)$ and a real number $\lambda \in \mathbb{R}$ such that $f \succsim g \iff V(f) \geq V(g)$ where $V : f \to \mathbb{R}$ is defined by*

$$V(f) = \mathbb{E}_\mu[f] + \lambda \sum_{s: f(s) < \mathbb{E}_\mu[f]} \mu(s)[\mathbb{E}_\mu[f] - f(s)]. \tag{AGL}$$

Moreover, (i) μ is a unique probability distribution, (ii) $\lambda \in \mathbb{R}$ is unique, and $|\lambda| < \min_{s \in S} \left(\frac{1}{\mu(s)}\right)$.

The bound on λ is a consequence of strict monotonicity, since an act is never be deemed worse than the outcome on the worst possible state. The parameter λ captures the difference between the weight placed on gains and the weight placed on losses, which for the representation is unique. An important application of the representation result from Theorem 2.1 is that it provides an index for reference dependence (λ) that is decoupled from risk attitudes, and can be easily estimated.

2.1.1 Sketch of the Proof and Preliminary Results

The result is proven in two steps. First, Lemma 2.2 provides a SEU representation on $\mathcal{F}^E$ established by axioms A1-A4 (that is, excluding Local Mixture Consistency and Antisymmetry), which can be extended to aggregate preferences across families of mutually aligned acts. Then, we utilize the properties of axioms A5 and A6 to generate the final result.

Lemma 2.2. *The preference $\succsim$ satisfies A1-A4 if and only if there exists a set of probability distributions over S, $\{\mu_E\}_{E\in\mathcal{E}}$, such that for $f \in \mathcal{F}^F$ and $g \in \mathcal{F}^{F'}$,*

$$f \succsim g \iff \mathbb{E}_{\mu_F}[f] \geq \mathbb{E}_{\mu_{F'}}[g].$$

Moreover the set $\{\mu_E\}_{E\in\mathcal{E}}$ is unique.

Every prior in $\{\mu_E\}_{E\in\mathcal{E}}$ is different and Alignment Independence does not imply any structure on the priors. To derive the main result from the representation of Lemma 2.2, Local Mixture Consistency and Antisymmetry are used to guarantee that every prior in the set $\{\mu_E\}_{E\in\mathcal{E}}$ can be written as functions of one unique prior μ.

Adding Local Mixture Consistency guarantees that for all $E \in \mathcal{E}$, the conditional distributions are the same for all μ_E, when conditioned on any F which is non-overlapping with E. Hence for any E and $E' \in \mathcal{E}$, $\mu_E(\cdot|F) = \mu_{E'}(\cdot|F)$ whenever F is non-overlapping for E and E'. Moreover, the axioms imply that there exists a unique distribution $\mu \in \Delta(S)$ that generates these conditionals: μ is the unique distribution that represents e_f for all $f \in \mathcal{F}$. In other words, if e_f is the hedge of f then $\mathbb{E}_\mu[f] = e_f$. This implies that for any $E \in \mathcal{E}$, μ_E can be written as a constant distortion γ_E^+ of μ on all $s \in E$ (positive states), and a constant distortion γ_E^- of μ on all $s \in E^c$ (negative states), i.e.

$$\mu_E(s) = \begin{cases} \gamma_E^+\mu(s) & \text{if} \quad s \in E, \\ \gamma_E^-\mu(s) & \text{if} \quad s \in E^c. \end{cases}$$

Antisymmetry implies a particular relationship between distributions indexed by complementary alignments, which is that for any $E, F \in \mathcal{E}$,

$$\mu_E + \mu_{E^c} = \mu_F + \mu_{F^c}.$$

Using this observation we show that for $s \in E$ the distributions on μ_E and $\mu_{E \setminus s}$ depend only on s. In particular, whenever $s \in E \cap F$,

$$\mu_E - \mu_{E \setminus s} = \mu_F - \mu_{F \setminus s}.$$

Then, using these conditions about the family $\{\mu_E\}_{E \in \mathcal{E}}$ we show that the difference between the distortion on negative states and positive states is always constant, thus $\gamma_E^+ - \gamma_E^- = \lambda$ for all $E \in \mathcal{E}$. Therefore it is possible to characterize any μ_E, as functions of μ, and this constant λ that captures the difference between the negative and positive distortions. That is,

$$\begin{aligned} &\mu_E(s) = \mu(s)\left(1 + \lambda\mu(E^c)\right) && \text{if } s \in E, && \text{and,} \\ &\mu_E(s) = \mu(s)\left(1 - \lambda\mu(E)\right) && \text{if } s \in E^c. && \end{aligned} \tag{2.1}$$

Finally this representation of μ_E is used to rewrite the representation from Lemma 2.2 in terms of μ, which yields the desired result.

3 Relation to Ambiguity Attitude

Maxmin Expected Utility

According to Theorem 2.1, the DM who abides by the AGL axioms is probabilistically sophisticated but displays some reference effect. That is, she holds some unique belief, μ, regarding the state space, and evaluates each act according to this belief and her preferences for outcomes. Nonetheless, Lemma 2.2 states that the same preferences can be represented by a family of distributions, $\{\mu_E\}_{E \in \mathcal{E}}$, each of which is a distortion of the original belief, μ. This alludes to a possible relationship between reference effects and attitudes towards uncertainty, which has classically been modeled by a DM who considers a (non-singleton) set of priors.

Definition 7. $\succsim$ has an MMEU representation if there exists a convex set of priors $C \subseteq \Delta(S)$ such that

$$V^{MM}(f) = \min_{\nu \in C} \mathbb{E}_{\nu(s)}[f] \tag{3.1}$$

represents $\succsim$.

MMEU, axiomatized by Gilboa and Schmeidler [1989] is characterized by two key conditions: *certainty independence* and *uncertainty aversion*. Certainty independence requires that $f \succsim g$ if and only if for all $\alpha \in [0, 1]$, $\alpha f + (1 - \alpha)c \succsim \alpha g + (1 - \alpha)c$ where $c \in \mathcal{F}_c$. Mixing two acts with a common constant act does not reverse the preference between them. Since constant acts are aligned with every $E \in \mathcal{E}$, Alignment Independence implies certainty independence.

Uncertainty aversion requires that for all f, g such that $f \sim g$ for any $\alpha \in (0, 1)$, $\alpha f + (1 - \alpha)g \succsim f$. If uncertainty aversion is changed for *uncertainty seeking* preferences,[9] then the representation is a Maxmax representation, where the DM evaluates an act according to the prior that maximizes her expectations. It is clear from the representation that if $\lambda \leq 0$ then the DM is uncertainty averse, and if $\lambda \geq 0$ then she is uncertainty seeking.

Uncertainty aversion can be characterized as a preference for hedging, as hedging reduces the exposure to uncertainty. Moreover, hedging reduces the exposure to negative states. Pushing the utility value in each state closer to the average has more effect on the negative states (because of the loss bias) and hence weakly improves the act.

The formal connection is captured by the following result, which states that asymmetric gain-loss preferences always admit a Maxmin or Maxmax representation and that the set of priors C has a specific structure that is related to the distortion of the (unique) beliefs of the DM.

Theorem 3.1. *Suppose $\succsim$ admits an AGL representation (μ, λ), with $\lambda < 0$, then $\succsim$ admits a MMEU representation. Moreover, $C = \text{conv}(\{\mu_E\}_{E \in \mathcal{E}})$, as defined in Lemma 2.2.*

Theorem 3.1 has several implications. First, it shows that this form of reference-dependence is always tied to a particular attitude towards uncertainty.[10] So, preferences studied in this paper will always be either uncertainty averse or uncertainty seeking. Second, it gives a precise form to the belief distortion that takes place when gain-loss consideration affect a probabilistically sophisticated DM.

While every AGL representation can be faithfully captured within the MMEU framework the converse is not true. In the AGL framework, the distorted beliefs keep the relative likelihood of states among gains and among losses unchanged, but, depending on the sign of λ, increase or decrease the total weight given to gains (and losses)

[9] For all $f, g \in \mathcal{F}$, $f \sim g$ implies for all $\alpha \in (0, 1)$ $f \succsim \alpha f + (1 - \alpha)g$

[10] Of course, if $\lambda > 0$, then Theorem 3.1 holds when Maxmin is replaced with Maxmax.

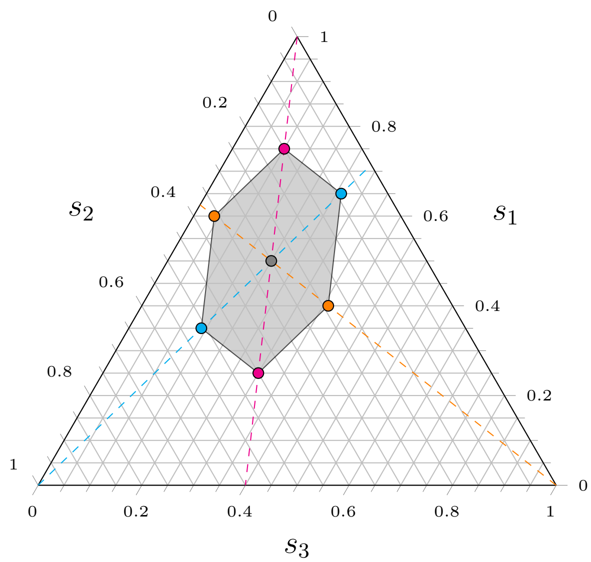

Figure 1: The set of priors for $S = \{s_1, s_2, s_3\}$ and $\mu = (.5, .3, .2)$, and $\lambda = 1$.

proportional to the baseline belief. This distortion is a function only of the degree of reference dependence, λ, and the baseline prior μ. In addition, Antisymmetry implies the set of priors is symmetric with respect to all hyperplanes (in the $|S|-1$ dimensional simplex) which divide the state-space into positive and negative states and which pass through the baseline prior. See Figure 1; the dashed lines show such symmetries.

Intuitively, this additional symmetric structure imposed on MMEU stems from the fact the the reference effects distort utility *relative to a reference point*. Hence, when translating the utility distortions in the AGL model into the equivalent probabilistic distortions, the symmetries around some reference point is preserved. An arbitrary convex set of priors would not necessarily admit such a baseline prior, and so, could not be translated into a model of reference effects.

Concave Expected Utility

Alignment Independence imposes more structure than certainty independence, and therefore AGL also shares a connection to a class of ambiguity models outside of MMEU. In particular, any loss adverse AGL preference is also a *concave expected utility* (cavEU) preference. CavEU is a capacity based model, which considers all possible decompositions of an act into bets over events (where a bet of magnitude $a_E \in \mathbb{R}_{++}$ on E, is an act that is constant on E and 0 off E, i.e., $a_E \mathbb{1}_E$ where $\mathbb{1}_E$ is the characteristic function on E). The preference $\succsim$ is cavEU if it can be represented by a concave integral introduced by Lehrer [2009].

Definition 8. $\succsim$ has a cavEU representation if there exists a capacity $v : 2^S \to [0,1]$ such that

$$V^{cav}(f) = \max \left\{ \sum_{E \in \mathcal{E}} a_E v(E) \middle| \sum a_E \mathbb{1}_E = f, a_E \in \mathbb{R}_{++} \right\} \tag{3.2}$$

represents $\succsim$.

The concave integral returns the maximum value of all possible decompositions, when aggregated according to the capacity v. Lehrer and Teper [2015] show that $\succsim$ is cavEU if and only if it satisfies A1-A3 plus uncertainty aversion, independence with respect to the constant act 0, and co-decomposable independence. This last requirement states, for every non-bet act f, there exist a bet a_E and an act f' such that (i) $f = \alpha a_E + (1-\alpha) f'$ for $\alpha \in (0,1)$, and (ii) $\succsim$ satisfies independence over $\{\alpha a_E + \beta f' | \alpha, \beta \in \mathbb{R}_+\}$.

Theorem 3.2. *Suppose $\succsim$ admits an AGL representation (μ, λ), with $\lambda < 0$, then $\succsim$ admits a cavEU representation with $v : \mathcal{P}(S) \to [0,1]$ defined by $v : E \mapsto \min_{F \in \mathcal{E}} \mu_F(E)$.*

The property that AGL preference admit cavEU representations stems from the fact that each act, $f \in \mathcal{F}^E$ can always be decomposed into a bet on E and another act in $\mathcal{F}^E$. Since all these acts share the same alignment, independence holds within the convex-cone generated thereby. As with the set of priors in the MMEU representation, the capacity v is characterized by the lower envelope of the distorted beliefs arising themselves from Lemma 2.2. Of course, this must be, since these functionals represent the same preferences! Note, cavEU and MMEU are *not* nested models; AGL preferences reside in the non-trivial intersection.

4 Related Literature

This paper links two topics in the literature that have not been formally linked yet: reference-dependent preferences, and attitudes towards variation and ambiguity. We show that the idea of Kőzsegi and Rabin [2006] where the reference point depends on expectations, provides a clean way to link these two concepts in the domain of choice under uncertainty.

In many decision theory models, the status quo has been interpreted as a reference point. Giraud [2004a], Masatlioglu and Ok [2005], Sugden [2003], Sagi [2006], Rubinstein and Salant [2007], Apesteguia and Ballester [2009], Ortoleva [2010], Riella and Teper [2014] and Masatlioglu and Ok [2013] provide models of reference dependence, where the reference point is exogenously given. Along with Kőzsegi and Rabin [2006] and Kőszegi and Rabin [2007], three other papers that tackle the problem of endogenous reference point determination are Giraud [2004b], Ok et al. [2014] and Sarver [2011].

The approach in Ok et al. [2014] investigates reference point determination problem under a very general framework, where they do not need an equilibrium condition to characterize reference dependence. Nonetheless in their framework it is impossible to identify reference points and reference effects uniquely. They identify reference dependence through WARP violations; their representation is a family of functionals. This difference is characterized by the source of reference dependence: in Ok et al. [2014], the reference point is a function of the choice problem (the menu), whereas in the AGL model, the reference point is a function of the object of choice (the act).

For gain-loss preferences, the evaluation of acts depends on the state by state variation of the act. Although some papers have studied attitudes toward variation in the context of risk and uncertainty, none relates such attitudes to reference-dependence. In the risk domain, Quiggin and Chambers [1998, 2004] measure attitudes towards risk, which depend on the expectation of the lottery and a risk index of the lottery that depends on the variation of the distribution. A different approach is taken by Gul [1991], who provides a theory of disappointment aversion. In Gul [1991], variation depends on the composition of a lottery among disappointment and elation outcomes. Dillenberger et al. [2012], as we do, interpret distortions of a prior belief as the behavioral markers of optimism or pessimism. However, in Dillenberger et al. [2012], the probabilistic distortions preserve the expected utility structure, whereas AGL preferences clearly do not.

In the uncertainty domain some models use the idea of a *reference prior*, which is a baseline for adjusting imprecise information, or to measure different specifications of a model like in the multiplier preferences of Hansen and Sargent [2001]. Wang [2003], Gajdos et al. [2004], Gajdos et al. [2008], Siniscalchi [2009], Strzalecki [2011] and Chambers et al. [2014] take this approach. In contrast, in this paper the DM has a unique prior over the states, but there is some contamination due to gain-loss considerations.

Grant and Polak [2013] provide a general model of mean-dispersion preferences, where deviations from the expectation affect utility as well. They show that many well-known families of preferences such as Choquet EU [Schmeidler, 1989], Maxmin EU [Gilboa and Schmeidler, 1989], invariant biseparable preferences [Ghirardato et al., 2004], variational preferences [Maccheroni et al., 2006], and Vector EU [Siniscalchi, 2009], belong to this family of preferences. The asymmetric gain-loss preferences belong to this family of preferences as well, since deviations from the reference point are in fact deviations from the expected utility of each act. However, the general mean-dispersion preferences are so general that it is not possible to provide clear comparative statics results like the ones presented in the supplemental material to this paper. Chambers et al. [2014] model mean-dispersion preferences where absolute uncertainty aversion is allowed to vary across acts, unlike in the mean-dispersion preferences of Grant and Polak [2013] where absolute uncertainty aversion is constant.[11]

5 Conclusion

This paper provides a model of reference dependent preferences in the domain of choice under uncertainty, which is characterized by the family of preferences called asymmetric gain-loss preferences. This model provides a behavioral way to identify a unique reference point, and uniquely capture reference-dependent attitudes from choice behavior.

AGL model establishes a simple and intuitive link between reference dependence and uncertainty attitudes. Whenever a DM exhibits gain-biased preferences she will be exhibit uncertainty averse preferences, and if she exhibits loss-biased preferences then she must exhibit uncertainty averse preferences. In addition even though the DM in this model has a unique prior over the states, which she uses to assess the reference point, gain-loss asymmetries can contaminate the beliefs in a way that she will behave as if she has multiple priors in mind.

[11]Other models where the uncertainty aversion is allowed to vary are Klibanoff et al. [2005] and Cerreia-Vioglio et al. [2011].

References

F. J. Anscombe and R. J. Aumann. A definition of subjective probablity. *Annals of Mathematical Statistics*, 34:199–205, 1963.

J. Apesteguia and M. Ballester. A theory of reference-dependent behavior. *Economic Theory*, 40(3):427–455, September 2009.

S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci, and L. Montrucchio. Uncertainty averse preferences. *Journal of Economic Theory*, 146(4):1275–1330, July 2011.

R. G. Chambers, S. Grant, B. Polak, and J. Quiggin. A two-parameter model of dispersion aversion. *Journal of Economic Theory*, 150:611–641, 2014.

D. Dillenberger, A. Postlewaite, and K. Rozen. Optimism and pessimism with expected utility. 2012.

T. Gajdos, J.-M. Tallon, and J.-C. Vergnaud. Decision making with imprecise probabilistic information. *Journal of Mathematical Economics*, 40(6):647–681, September 2004.

T. Gajdos, T. Hayashi, J.-M. Tallon, and J.-C. Vergnaud. Attitude toward imprecise information. *Journal of Economic Theory*, 140(1):27–65, May 2008.

P. Ghirardato and M. Marinacci. Ambiguity made precise: A comparative foundation. *Journal of Economic Theory*, 102(2):251–289, February 2002.

P. Ghirardato, F. Maccheroni, and M. Marinacci. Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory*, 118(2):133–173, October 2004.

I. Gilboa and D. Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141–153, April 1989.

R. Giraud. Reference-dependent preferences : rationality, mechanism and welfare implications. Sept. 2004a.

R. Giraud. Framing under risk : Endogenizing the reference point and separating cognition and decision. Cahiers de la maison des sciences economiques, Sept. 2004b.

S. Grant and B. Polak. Mean-dispersion preferences and constant absolute uncertainty aversion. *Journal of Economic Theory*, 148(4):1361–1398, 2013.

F. Gul. A theory of disappointment aversion. *Econometrica*, 59(3):667–86, May 1991.

F. Gul and W. Pesendorfer. The revealed preference implications of reference dependent preferences. Mimeo, 2006.

L. P. Hansen and T. J. Sargent. Robust control and model uncertainty. *American Economic Review*, 91(2):60–66, May 2001.

I. N. Herstein and J. Milnor. An Axiomatic Approach to Measurable Utility. 21(2): 291–297, 1953.

D. Kahneman and A. Tversky. Prospect theory: An analysis of decision under risk. *Econometrica*, 47(2):263–91, March 1979.

B. Kőszegi and M. Rabin. Reference-dependent risk attitudes. *American Economic Review*, 97(4):1047–1073, September 2007.

B. Kőzsegi and M. Rabin. A model of reference-dependent preferences. *The Quarterly Journal of Economics*, 121(4):1133–1165, November 2006.

P. Klibanoff, M. Marinacci, and S. Mukerji. A smooth model of decision making under ambiguity. *Econometrica*, 73(6):1849–1892, November 2005.

E. Lehrer. A new integral for capacities. *Economic Theory*, 39(1):157–176, 2009.

E. Lehrer and R. Teper. Subjective independence and concave expected utility. *Journal of Economic Theory*, 158:33–53, 2015.

F. Maccheroni, M. Marinacci, and A. Rustichini. Ambiguity aversion, robustness, and the variational representation of preferences. *Econometrica*, 74(6):1447–1498, November 2006.

Y. Masatlioglu and E. A. Ok. Rational choice with status quo bias. *Journal of Economic Theory*, 121(1):1–29, March 2005.

Y. Masatlioglu and E. A. Ok. A canonical model of choice with initial endowments. *The Review of Economic Studies*, page rdt037, 2013.

E. A. Ok, P. Ortoleva, and G. Riella. Revealed (p) reference theory. *The American Economic Review*, 105(1):299–321, 2014.

P. Ortoleva. Status Quo Bias, Multiple Priors and Uncertainty Aversion. *Games and Economic Behavior*, 69:411–424, 2010.

J. Quiggin and R. G. Chambers. Risk premiums and benefit measures for generalized-expected-utility theories. *Journal of Risk and Uncertainty*, 17(2):121–37, November 1998.

J. Quiggin and R. G. Chambers. Invariant risk attitudes. *Journal of Economic Theory*, 117(1):96–118, July 2004.

G. Riella and R. Teper. Probabilistic dominance and status quo bias. *Games and Economic Behavior*, 87:288–304, 2014.

A. Rubinstein and Y. Salant. (a, f) choice with frames. Technical report, Jul 2007. Mimeo.

J. S. Sagi. Anchored preference relations. *Journal of Economic Theory*, 130(1):283–295, September 2006.

T. Sarver. Optimal anticipation. Mimeo, September 2011.

D. Schmeidler. Subjective probability and expected utility without additivity. *Econometrica*, 57(3):571–87, May 1989.

J. Shalev. Loss aversion equilibrium. *International Journal of Game Theory*, 29(2): 269–287, 2000.

M. Siniscalchi. Vector expected utility and attitudes toward variation. *Econometrica*, 77(3):801–855, 05 2009.

E. Stiemke. *Matemathische Annalen*, 76(2-3):340–342, 1915.

T. Strzalecki. Axiomatic foundations of multiplier preferences. *Econometrica*, 79:47–73, 2011.

R. Sugden. Reference-dependent subjective expected utility. *Journal of Economic Theory*, 111(2):172–191, August 2003.

T. Wang. A class of multi-prior preferences. Mimeo, 2003.

A Appendix: Proofs

A.1 Proofs of the Main Results

This section provides proofs for the main results. The proofs for the auxiliary lemmas and propositions are in Appendix A.2.

Proof of Lemma 2.2.

This is an obvious consequence of the Herstein and Milnor [1953] Mixture Space Theorem. Fix $E \in \mathcal{E}$. $\succsim$ satisfies Alignment Independence, $\mathcal{F}^E$ is convex, and it includes all the constant acts. Therefore $\succsim_E$ and $\mathcal{F}^E$ define a mixture space, so by the Mixture Space Theorem, the conditions for a SEU representation of $\succsim_E$ are satisfied. Therefore there exists a cardinally unique expected utility function $U_E : \mathbb{R} \to \mathbb{R}$, and an unique probability distribution $\mu_E : 2^S \to [0,1]$, such that for any $f, g \in \mathcal{F}^E$,

$$f \succsim g \iff V_E(f) \geq V_E(g).$$

Where

$$V_E(f) = \sum_{s \in S} \mu_E(s) U(f(s)),$$

By strict monotonicity, $\mu_E(s) > 0$ for all s, so every state is non-null. Moreover, by strict monotonicity, $U_E = id_{\mathbb{R}}$ clearly represents $\succsim_E$ over the constant acts, and therefore, such a normalization is without loss.

Since any constant $c \in \mathcal{F}_c$ is in $\mathcal{F}^E$ for all $E \in \mathcal{E}$, and every $f \in \mathcal{F}$ has a certainty equivalence c_f, (and $\succsim$ is complete and transitive), we have, for any $f \in \mathcal{F}^E$, $g \in \mathcal{F}^{E'}$, $f \succsim g$, if and only if $c_f \succsim c_g$, if and only if $V_E(f) \geq V_F(g)$, if and only if

$$\sum_{s \in S} \mu_E(s) f(s) \geq \sum_{s \in S} \mu_F(s) g(s),$$

proving the result. ■

Proof of Theorem 2.1.

Start with the representation of Lemma 2.2, which is guaranteed by A1-A4. Hence, there is a set of probability distributions over S, indexed by $\mathcal{E}$, $\{\mu_E\}_{E \in \mathcal{E}}$.

Step 1: Show that for every E, E' the conditional distributions of μ_E and $\mu_{E'}$- conditional on any event F which is non-overlapping for E and E', are the same. And show that there is a unique distribution μ over S, that generates all the conditionals. In other words, there exists a unique μ, such that $\mu_E(\cdot|F) = \mu(\cdot|F)$ for all $E \in \mathcal{E}$ and F as long as F is non-overlapping for E. This is achieved by the addition of Local Mixture Consistency (A5).

Proposition A.1. *Suppose $\{\mu_E\}_{E \in \mathcal{E}}$ constitute SEU representations for $\succsim_E$ on $\mathcal{F}^E$ for all $E \in \mathcal{E}$. If $\succsim$ satisfies A1-A6, then there exists a unique distribution $\mu : 2^S \to [0,1]$,* *such that for any $F, E \in \mathcal{E}$ such that $F \subseteq E$ or $F \cap E = \emptyset$, $\mu_E(\cdot|F) = \mu(\cdot|F)$, where for all $s \in F$, $\mu(s|F) = \frac{\mu(s)}{\mu(F)}$.*

Proof. In Appendix A.2. ∎

By Proposition A.1, given $E \in \mathcal{E}$, for any $s, s' \in E$, $\frac{\mu(s)}{\mu(s')} = \frac{\mu_E(s)}{\mu_E(s')}$. This holds if for all $s \in E$, $\mu(s) = \gamma\mu_E(s)$, where $\gamma \in \mathbb{R}_{++}$. So on E, μ_E is just a constant perturbation of the original prior μ. Then the distribution μ_E can be written in the following way

$$\mu_E(s) = \begin{cases} \gamma_E^+\mu(s) & \text{if} \quad s \in E, \\ \gamma_E^-\mu(s) & \text{if} \quad s \in E^c, \end{cases}$$

where γ_E^+ represents how the original prior is perturbed on the positive states (i.e. E), and γ_E^- represents how the original prior is modified on the negative states. Both γ_E^+ and γ_E^- are positive numbers from monotonicity of $\succsim$, where $\gamma_E^+ \geq 1 \Leftrightarrow \gamma_E^- \leq 1$ for every E because μ_E is a probability distribution and the sum need to add up to 1. Therefore if μ_E and $\mu_{E'}$ agree on the probability assigned to a state they are exactly the same.

Lemma A.2. *Given $E \neq E' \in \mathcal{E}$. Suppose $\mu_E(s) = \mu_{E'}(s)$ for some s, then $\mu_E = \mu_{E'}$.*

Proof. In Appendix A.2. ∎

Step 2: Adding Antisymmetry, implies some consistency between the distributions induced on $\mathcal{F}^E$ and $\mathcal{F}^{E^c}$. In which the average probability attached to each s is always the same for the pair μ_E, μ_{E^c}, or in other words the distortions on E and E^c exactly balance out.

Proposition A.3. *Let $\succsim$ satisfy A1-A6, then for any $E, F \in \mathcal{E}$,*

$$\mu_E + \mu_{E^c} = \mu_F + \mu_{F^c}.$$

Proof. In Appendix A.2. ∎

Lemma A.4. *Let $\succsim$ satisfy A1-A6, then for all $E \in \mathcal{E}$,*

$$\frac{\mu_E + \mu_{E^c}}{2} = \mu.$$

Where μ_E is the distribution from Lemma 2.2 that represents preferences over $\mathcal{F}^E$.

Proof. This is an immediate consequence of Propositions A.1 and A.3. ∎

From Lemma A.4, further conclude that $\gamma_E^+ + \gamma_{E^c}^- = 2$ for all $E \in \mathcal{E}$. A more relevant implication is that μ, uniquely defines e_f for all $f \in \mathcal{F}$. Recall that e_f is defined as the constant where $e_f = \frac{1}{2}f(s) + \frac{1}{2}\bar{f}(s)$ for all $s \in S$. Let $f \in \mathcal{F}^E$ and hence $\bar{f} \in \mathcal{F}^{E^c}$.

Proposition A.5. *Let $\succsim$ satisfy A1-A6, then for every $f \in \mathcal{F}$, $e_f \in \mathcal{F}_c$ is an act such that $e_f = \mathbb{E}_\mu[f]$.*

Proof. In Appendix A.2. ■

Step 3: Show that the distribution induced on $\mathcal{F}^E$ and $\mathcal{F}^F$ only depend on the states they do not have in common.

Proposition A.6. *Let $\succsim$ satisfy A1-A6, then for any $E, F \in \mathcal{E}$ such that $|E|, |F| \geq 2$ and $s \in E \cap F$,*

$$\mu_E - \mu_{E\setminus s} = \mu_F - \mu_{F\setminus s}. \tag{A.1}$$

Proof. In Appendix A.2. ■

Step 4: Based on the previous results, provide a characterization of the distortions γ_E^+ and γ_E^-, as functions of μ and μ_E. Further, show that for any particular $E \in \mathcal{E}$, the difference between the negative and the positive distortion is always constant.

Proposition A.7. *If $\succsim$ satisfies A1-A6, then for any $E, F \in \mathcal{E}$, $\gamma_E^+ - \gamma_E^- = \gamma_F^+ - \gamma_F^-$. Where γ_E^+, γ_E^- are defined as*

$$\gamma_E^+ = \frac{\mu_E(s)}{\mu(s)} \qquad \text{for } s \in E,$$
$$\gamma_E^- = \frac{\mu_E(s)}{\mu(s)} \qquad \text{for } s \in E^c.$$

Proof. In Appendix A.2. ■

Therefore $\gamma_E^+ - \gamma_E^-$ is a constant (independent of E), which can de defined as

$$\gamma_E^+ - \gamma_E^- \equiv \lambda.$$

The next step is to characterize λ.

Proposition A.8. *If $\gamma_E^+ - \gamma_E^- = \lambda$ for all $E \in \mathcal{E}$, then for any $E \in \mathcal{E}$,*

$$\begin{aligned} \mu_E(s) &= \mu(s)\left(1 + \lambda\mu(E^c)\right) && \text{if } s \in E, && \text{and,} \\ \mu_E(s) &= \mu(s)\left(1 - \lambda\mu(E)\right) && \text{if } s \in E^c. \end{aligned} \tag{A.2}$$

Proof. In Appendix A.2. ■

Step 5: Use the definition of μ_E from Proposition A.8 into the representation from Lemma 2.2.

For for any $f \in \mathcal{F}^E$, then $V(f) = \mathbb{E}_{\mu_E}[f]$ which is equivalent to

$$\begin{aligned} \mathbb{E}_{\mu_E}[f] &= \sum_{s\in E} \mu(s)\left(1 + \lambda\mu(E^c)\right) f(s) + \sum_{s\in E^c} \mu(s)\left(1 - \lambda\mu(E)\right) f(s) \\ &= \mathbb{E}_\mu[f] + \lambda\mu(E^c) \sum_{s\in E} \mu(s) f(s) - \lambda\mu(E) \sum_{s\in E^c} \mu(s) f(s) \\ &= \mathbb{E}_\mu[f] + \lambda\mu(E^c) \left(\sum_{s\in E} \mu(s) f(s) + \sum_{s\in E^c} \mu(s) f(s) \right) - \lambda(\mu(E) + \mu(E^c)) \sum_{s\in E^c} \mu(s) f(s) \\ &= \mathbb{E}_\mu[f] + \lambda \sum_{s\in E^c} \mu(s)\left(\mathbb{E}_\mu[f] - f(s)\right). \end{aligned} \tag{A.3}$$

The representation follows from the observation that $E^c = \{s \in S | f(s) < e_f = \mathbb{E}_\mu[f]\}$. Finally, μ and every μ_E are unique, $\lambda = \gamma_E^+ - \gamma_E^-$ is unique as well from the definition of γ's from (A.15) and (A.16). ■

Proof of Theorem 3.1.

Let $(\{\mu_E\}_{E\in\mathcal{E}}, \lambda)$ be an AGL representation. Let $C = \mathrm{conv}(\{\mu_E\}_{E\in\mathcal{E}})$. Then from equation (A.3), we know, for any $f \in \mathcal{F}$ and $E \in \mathcal{E}$,

$$\mathbb{E}_{\mu_E}[f] = \mathbb{E}_\mu[f] + \lambda \sum_{s\in E^c} \mu(s) \left(\mathbb{E}_\mu[f] - f(s)\right).$$

Hence, if $\lambda < 0$, the functional, $\mathbb{E}_{\mu_E}[f]$, for a fixed f, is minimized at $E = \{s \in S | f(s) < \mathbb{E}_\mu[f]\}$, and if $\lambda > 0$ is maximized at $E = \{s \in S | \mathbb{E}_\mu[f] < f(s)\}$. In either case, this is exactly (AGL). ■

Proof of Theorem 3.2.

That $\succsim$ satisfies A1-A3, uncertainty aversion and independence with respect to the constant act 0 is immediate. So it remains to show $\succsim$ satisfies co-decomposable independence. Fix some non-bet act $f \in \mathcal{F}^E$ and assume without loss of generality that E includes all neutral states for f. It is clear that the bet $a_E = e_f$ (that is equals e_f on E and 0 otherwise) is also in $\mathcal{F}^E$. For each $\alpha \in (0, 1)$ denote f' as the bet $2f - a_E$, so $f' = \frac{1}{2}f + \frac{1}{2}a_E$. Now, let $g = \alpha a_E + \beta f'$ with $\alpha, \beta \in \mathbb{R}_+$. Then, for each $s \in E$, $g(s) = \alpha f'(s) + \beta a_E(s) = \alpha(\frac{1}{2}f(s) + \frac{1}{2}a_E(s)) + \beta a_E(s) \geq \alpha(\frac{1}{2}e_f + \frac{1}{2}e_{a_E}) + \beta e_{a_E} = \alpha e_{f'} + \beta e_{a_E} = e_g$. Likewise, for $s \in E^c$, $g(s) < e_g$. Hence, $g \in \mathcal{F}^E$. So by alignment independence $\succsim$ satisfies independence over $\{\alpha a_E + \beta f' | \alpha, \beta \in \mathbb{R}_+\}$.

Finally, to characterize v, notice that the capacity is fully determined by its valuation over all bets and that it is unique. Further, by Theorem 3.1, $V^{MM}(a_E) = \min_{F\in\mathcal{E}} \mu_F(E) a_E$, for any bet a_E. Hence $v : E \mapsto \min_{F\in\mathcal{E}} \mu_F(E)$ induces that same ranking over bets as $V^{MM}(\cdot)$, and therefore represents $\succsim$. ■

A.2 Proofs of Lemmas and Propositions

Proof of Proposition A.1.

Consider any $E, E' \in \mathcal{E}$, and single alignment acts $f \in \mathcal{F}^E$, $g \in \mathcal{F}^{E'}$. Let F be a non-overlapping event for both E and E'. Consider two distinct $h, h' \in \mathcal{F}$, such that $h(s) = \bar{h}(s) = h'(s) = \bar{h}'(s)$ for all $s \notin F$, which means that their alignment is neutral in F^c. Moreover suppose that for every $s \in F$, $h(s) > \bar{h}(s)$ or $h(s) < \bar{h}(s)$, and $h'(s) > \bar{h}'(s)$ or $h'(s) < \bar{h}'(s)$, so that on every state in F, h and h' are either strictly considered positive or negative. Further assume that for all $s \in F$, $0 < h(s) < h'(s)$ or $0 < h'(s) < h(s)$ (hence on F the acts are always different in terms of preferences). By

Strong Monotonicity and Continuity there is always possible to find such acts h and h'.

Local Mixture Consistency guarantees for f, g, h there exists some $\alpha_{hh'}$ such that for any $\alpha \in (\alpha_{hh'}, 1)$,

$$\alpha f + (1-\alpha)h \succsim \alpha f + (1-\alpha)h' \iff \alpha g + (1-\alpha)h \succsim \alpha g + (1-\alpha)h'. \tag{A.4}$$

Moreover, since f and g are single-aligned continuity of preferences imply that alignment does not change for small perturbations around f and g. Hence, the for α close to one, $\alpha f + (1-\alpha)h \in \mathcal{F}^E$, and $\alpha g + (1-\alpha)h \in \mathcal{F}^{E'}$. From the representation of 2.2, (A.4) implies

$$\mathbb{E}_{\mu_E}[\alpha f + (1-\alpha)h] = \mathbb{E}_{\mu_E}[\alpha f + (1-\alpha)h'] \iff \\ \mathbb{E}_{\mu_{E'}}[\alpha g + (1-\alpha)h] = \mathbb{E}_{\mu_{E'}}[\alpha g + (1-\alpha)h'].$$

Which by linearity and the fact that $h(s) = h'(s)$ for all $s \notin F$, reduces to

$$\sum_F (h-h')\mu_E(s) = 0 \iff \sum_F (h-h')\mu_{E'}(s) = 0. \tag{A.5}$$

Normalize μ_E and $\mu_{E'}$ conditional on F to be probability distributions over F, then equation (A.5) becomes

$$\sum_F (h-h') \underbrace{\frac{\mu_E(ds)}{\mu_E(F)}}_{\mu_E(s|F)} = 0 \iff \sum_F (h-h') \underbrace{\frac{\mu_{E'}(ds)}{\mu_{E'}(F)}}_{\mu_{E'}(s|F)} = 0.$$

Since all states are non-null, $\mu_E(\cdot|F)$ and $\mu_{E'}(\cdot|F)$ are strictly positive $|F|$-dimensional vectors. These vectors are normal to $(h-h') \in \mathbb{R}^{|F|}$, which consists of non-zero elements by the assumption that h and h' are different for all $s \in F$. Therefore $\mu_E(\cdot|F)$ and $\mu_{E'}(\cdot|F)$ are colinear as vectors in $\mathbb{R}^{|F|}$, with norm 1. Then for all $s \in F$,

$$\frac{\mu_E(s)}{\mu_E(F)} = \frac{\mu_{E'}(s)}{\mu_{E'}(F)} \quad \text{or equivalently} \quad \mu_E(s|F) = \mu_{E'}(s|F). \tag{A.6}$$

It remains to show that if (A.6) holds there exists a unique distribution μ that generates the conditional distributions, i.e. such that for all $E \in \mathcal{E}$ and non-overlapping F,

$$\frac{\mu_E(s)}{\mu_E(F)} = \frac{\mu(s)}{\mu(F)}. \tag{A.7}$$

Consider the events $E_i = \{s_i, s_{i+1}\}$ for $i = 1, ..., n-1$. If such a μ exists it follows from (A.7) that for E_i,

$$\frac{\mu(s_i)}{\mu(s_{i+1})} = \frac{\mu_{E_i}(s_i)}{\mu_{E_i}(s_{i+1})}. \tag{A.8}$$

If such a μ exists and it is unique for the set $\{E_i\}_{i=1,\ldots,n-1}$, then by (A.6), it will hold for any $E \in \mathcal{E}$. This is because the set $\{E_i\}_{i=1,..n}$ can be used to find the conditional distributions for any other $E \in \mathcal{E}$ using the results from the first part of the proof. To see this suppose that there is a unique μ such that for all $\{E_i\}_{i=1,\ldots,n-1}$, (A.7) holds. Now consider some $E \notin \{E_i\}_{i=1,\ldots,n-1}$. $E \subset \bigcup_{i=m}^{M} E_i := E_m^M$ for some $m < M$, where $m \geq 1$ and $M \leq n$. Then for any $s_t, s_k \in E$, $t < k$ and $t, k \in \{m, ..., M\}$,

$\frac{\mu_E(s_t)}{\mu_E(s_k)} = \frac{\mu_{E_m^M}(s_t)}{\mu_{E_m^M}(s_k)}$ from (A.6). Moreover,

$$\begin{aligned}
\frac{\mu_{E_m^M}(s_t)}{\mu_{E_m^M}(s_k)} &= \left(\frac{\mu_{E_m^M}(s_t)}{\mu_{E_m^M}(s_{t+1})}\right)\left(\frac{\mu_{E_m^M}(s_{t+1})}{\mu_{E_m^M}(s_{t+2})}\right)...\left(\frac{\mu_{E_m^M}(s_{k-1})}{\mu_{E_m^M}(s_k)}\right) \\
&= \left(\frac{\mu_{E_t}(s_t)}{\mu_{E_t}(s_{t+1})}\right)\left(\frac{\mu_{E_{t+1}}(s_{t+1})}{\mu_{E_{t+1}}(s_{t+2})}\right)...\left(\frac{\mu_{E_{k-1}}(s_{k-1})}{\mu_{E_{k-1}(s_k)}}\right) \\
&= \left(\frac{\mu(s_t)}{\mu(s_{t+1})}\right)\left(\frac{\mu(s_{t+1})}{\mu(s_{t+2})}\right)...\left(\frac{\mu(s_{k-1})}{\mu(s_k)}\right) \\
&= \frac{\mu(s_t)}{\mu(s_k)}.
\end{aligned}$$

Hence it suffices to show that for the set $\{E_i\}_{i=1,...,n}$, a unique distribution exists such that (A.7) holds. Note that (A.8) implies that for any $i = 1, 2, ...n-1$,

$$\mu(s_i) = \left(\frac{\mu_{E_i}(s_i)}{\mu_{E_i}(s_{i+1})}\right)\mu(s_{i+1}).$$

These $n-1$ equations, along with the necessary condition to be a probability distribution:

$$\sum_{i=1}^{n} \mu(s_1) = 1,$$

gives n equations and n unknowns (the $\mu(s_i)$'s), which can be written in the following form:

$$\underbrace{\begin{bmatrix}
1 & -\frac{\mu_{E_1}(s_1)}{\mu_{E_1}(s_2)} & 0 & 0 & \dots & \dots & 0 \\
0 & 1 & -\frac{\mu_{E_2}(s_2)}{\mu_{E_2}(s_3)} & 0 & \dots & \dots & 0 \\
0 & 0 & 1 & -\frac{\mu_{E_3}(s_3)}{\mu_{E_3}(s_4)} & 0 & \dots & 0 \\
\vdots & \vdots & & \ddots & & & \vdots \\
0 & 0 & \dots & & \dots & 1 & -\frac{\mu_{E_{n-1}}(s_{n-1})}{\mu_{E_{n-1}}(s_n)} \\
1 & 1 & \dots & & & \dots & 1
\end{bmatrix}}_{A_n} \mu = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{A.9}$$

Equation (A.9) has a unique solution if and only if the matrix A_n is invertible. We will prove the stronger condition that $\det(A_n) > 0$ instead, by induction on $|S|$. Let $|S| = 3$, then need to show that $\det(A_3) \neq 0$. Let $a_{ij} = \frac{\mu_{E_i}(s_i)}{\mu_{E_i}(s_j)} > 0$. $a_{ij} \in (0, \infty)$ because every state is non-null so $\mu_{E_i}(s_j) > 0$ for all i, j. We have,

$$\det\left(\begin{bmatrix} 1 & -a_{12} & 0 \\ 0 & 1 & -a_{23} \\ 1 & 1 & 1 \end{bmatrix}\right) = 1\left(1 + a_{23}\right) - \left(-a_{12}\right)\left(a_{23}\right) = 1 + a_{23}(1 + a_{12}) > 0.$$

Suppose now that $\det(A_k) > 0$ for all $k < m$. Then

$$A_m = \left[\begin{array}{c|cccccc} 1 & -a_{12} & 0 & 0 & \dots & \dots & 0 \\ \hline 0 & & & & & & \\ 0 & & & & & & \\ \vdots & & & & A_{m-1} & & \\ 0 & & & & & & \\ 1 & & & & & & \end{array}\right].$$

Hence $\det(A_k) = \det(A_{m-1}) + a_{12}\det(B_k)$ where B_k is defined as

$$B_k = \begin{bmatrix} 0 & -a_{23} & 0 & \dots & \dots & 0 \\ 0 & 1 & -a_{34} & \dots & \dots & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & \dots & 1 & -a_{n(n-1)} \\ 1 & 1 & \dots & & \dots & 1 \end{bmatrix} = \left[\begin{array}{c|ccccc} 0 & -a_{23} & 0 & \dots & \dots & 0 \\ \hline 0 & & & & & \\ \vdots & & & A_{m-3} & & \\ 0 & & & & & \\ 1 & & & & & \end{array}\right].$$

Therefore $\det(A_k) = \det(A_{m-1}) + a_{12}(a_{23}\det(A_{m-3})) > 0$, from the induction hypothesis that for all $k < m$, $\det(A_k) > 0$. Hence the system from equation (A.9) has a unique solution, μ. From the previous result, for any $E \in \mathcal{E}$ such that $|E| > 2$, μ_E is also generated by μ. Hence there exists a unique $\mu : 2^S \to [0,1]$ such that every conditional distribution of μ (conditional on event F), is the same as the conditional distribution of μ_E provided that F and E are non-overlapping. ■

Proof of Lemma A.2.

Suppose there exists $s \in S$ such that $\mu_E(s) = \mu_{E'}(s)$ for some $E \neq E'$. Then by Proposition A.1, there are two cases:

(i.) $s \in E \cap E'$ (or $s \in E^c \cap E'^c$).

(ii.) $s \in E \cap (E')^c$ (or $s \in E' \cap E^c$).

For case (i) by Proposition A.1, for every $s' \in E \cup E'$, $\mu_E(s') = \mu_{E'}(s)$ since $\mu_E(s) = \gamma_E^+\mu(s)$ and $\mu_E(s) = \gamma_{E'}^+\mu(s)$. In addition, for any $t \in E \cap (E')^c$, $\mu_E(t) = \mu_{E'}(t)$ by the same argument, which implies that for all $s \in (E')^c$, $\mu_{E'}(s) = \gamma_{E'}^+\mu(s)$, which can only be true if $\gamma_E^+ = \gamma_{E'}^+ = 1$. Hence $\mu_E = \mu_{E'} = \mu$. For the second case the argument is symmetric (replacing $\gamma_{E'}^+$ for $\gamma_{E'}^-$). ■

Proof of Proposition A.3.

Consider some single alignment $f \in \mathcal{F}^E$, and $g \in \mathcal{F}^F$ such that $f \sim g$, where $\bar{f} \in \mathcal{F}^{E^c}$ and $\bar{g} \in \mathcal{F}^{F^c}$ are the respective balancing acts. Given $s \in S$, consider some $h \in \mathcal{F}$ such that $h(t) = 0$ for all $t \neq s$ and $h(s) > 0$.

Suppose $\alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h$, then by Antisymmetry $\alpha\bar{f} + (1-\alpha)h \prec \alpha\bar{g} + (1-\alpha)h$. From the definition of alignment and continuity of $\succsim$, for α close to 1, then $\alpha f + (1-\alpha)h \in \mathcal{F}^E$ and $\alpha g + (1-\alpha)h \in \mathcal{F}^F$; likewise $\alpha \bar{f} + (1-\alpha)h \in \mathcal{F}^{E^c}$ and $\alpha \bar{g} + (1-\alpha)h \in \mathcal{F}^{F^c}$. Therefore by the representation result from Lemma 2.2, from Antisymmetry $\mu_E(s) > \mu_F(s)$ if and only if $\mu_{F^c}(s) > \mu_{E^c}(s)$.

Suppose $\mu_E + \mu_{E^c} \neq \mu_F + \mu_{F^c}$. It must be the case that the following two conditions hold:

$$\begin{aligned} \mu_E(s) - \mu_F(s) &= \theta\left(\mu_{F^c}(s) - \mu_{E^c}(s)\right) \quad \text{for some} \quad \theta < 1 \\ \mu_E(s') - \mu_F(s') &= \theta'\left(\mu_{F^c}(s') - \mu_{E^c}(s')\right) \quad \text{for some} \quad \theta' < 1 \end{aligned} \tag{A.10}$$

Let $h' \in \mathcal{F}$ be such that $h'(t) = 0$ for all $t \neq s, s'$, and $h'(s), h'(s') \neq 0$. According to the above argument, for single alignment $f \in \mathcal{F}^E$ and $g \in \mathcal{F}^F$ where $f \sim g$, for α close to 1, we can appeal to Antisymmetry and to obtain,

$$\begin{aligned} & \mu_E(s)h(s) + \mu_E(s')h(s') > \mu_F(s)h(s) + \mu_F(s')h(s') \\ \Leftrightarrow \quad & \mu_{E^c}(s)h(s) + \mu_{E^c}(s')h(s') > \mu_{F^c}(s)h(s) + \mu_{F^c}(s')h(s'). \end{aligned} \tag{A.11}$$

In other words, there is no solution to the system obtained from equations A.10 and A.11.

$$\begin{bmatrix} \theta\left(\mu_{F^c}(s) - \mu_{E^c}(s)\right) & \theta'\left(\mu_{F^c}(s') - \mu_{E^c}(s')\right) \\ -\left(\mu_{F^c}(s) - \mu_{E^c}(s)\right) & -\left(\mu_{F^c}(s') - \mu_{E^c}(s')\right) \end{bmatrix} \begin{bmatrix} v_s \\ v_{s'} \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{A.12}$$

Since there is no solution to (A.12), there exists some $p > 0$ (Stiemke's Alternative [Stiemke, 1915]) such that

$$\begin{bmatrix} p_1 & p_2 \end{bmatrix} \underbrace{\begin{bmatrix} \theta\left(\mu_{F^c}(s) - \mu_{E^c}(s)\right) & \theta'\left(\mu_{F^c}(s') - \mu_{E^c}(s')\right) \\ -\left(\mu_{F^c}(s) - \mu_{E^c}(s)\right) & -\left(\mu_{F^c}(s') - \mu_{E^c}(s')\right) \end{bmatrix}}_{A} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $\left(\mu_{F^c}(s) - \mu_{E^c}(s)\right)(p_1\theta - p_2) = 0$ and $\left(\mu_{F^c}(s') - \mu_{E^c}(s')\right)(p_1\theta' - p_2) = 0$, where $p > 0$. Since $\left(\mu_{F^c}(s) - \mu_{E^c}(s)\right) \neq 0$ and $\left(\mu_{F^c}(s') - \mu_{E^c}(s')\right) \neq 0$, it must be the case that $(p_1\theta' - p_2) = (p_1\theta - p_2) = 0$, which never holds when at least p_1 or p_2 are non-zero, and $\theta > 1 > \theta'$. A contradiction. Therefore for any $E, F \in \mathcal{E}$,

$$\mu_E + \mu_{E^c} = \mu_F + \mu_{F^c}.$$

■

Proof of Proposition A.5.

Recall e_f is defined as the constant such that for a balanced pair, $(f, \bar{f})$, $\frac{1}{2}f(s) + \frac{1}{2}\bar{f}(s) = e_f$ for all $s \in S$. From lemma A.4 and the representation from Lemma 2.2,

$$\begin{aligned} \mathbb{E}_\mu[f] &= \frac{1}{2}\mathbb{E}_{\mu_E}[f] + \frac{1}{2}\mathbb{E}_{\mu_{E^c}}[f] \\ &= \frac{1}{2}\mathbb{E}_{\mu_{E^c}}[\bar{f}] + \frac{1}{2}\mathbb{E}_{\mu_{E^c}}[f] \\ &= \mathbb{E}_{\mu_{E^c}}[\frac{1}{2}f + \frac{1}{2}\bar{f}] \\ &= e_f. \end{aligned}$$

■

Proof of Proposition A.6.

We will prove the claim in steps. First, for all such E and F such that $E \cup F = S$ and $E \cap F^c \neq \emptyset$ and $F \cap E^c \neq \emptyset$ and $E \cap F = I \neq \emptyset$, we claim $\mu_E - \mu_{E \setminus I} = \mu_F - \mu_{F \setminus I}$. Indeed, by definition $F \setminus I = E^c \neq \emptyset$ and $E \setminus I = F^c \neq \emptyset$. So,

$$\mu_E - \mu_{E\setminus I} = \mu_E - \mu_{F^c}, \qquad \text{and,}$$
$$\mu_F - \mu_{F\setminus I} = \mu_F - \mu_{E^c},$$

Also, from Proposition A.3, $\mu_E + \mu_{E^c} = \mu_F + \mu_{F^c}$ for all $E, F \in \mathcal{E}$. This and the above observation imply $\mu_E - \mu_{E\setminus I} = \mu_F - \mu_{F\setminus I}$.

Next, we claim, for all such E and F such that $E \cap F^c \neq \emptyset$ and $F \cap E^c \neq \emptyset$ and $E \cap F = I \neq \emptyset$, $\mu_E - \mu_{E\setminus I} = \mu_F - \mu_{F\setminus I}$. To see this, notice that $(E, E^c \cup I)$, $(E^c \cup I, F^c \cup I)$, and $(F^c \cup I, F)$ all satisfy (as pairs of subsets), the conditions to apply the first claim. So, $\mu_E - \mu_{E\setminus I} = \mu_{(E^c \cup I)} - \mu_{E^c} = \mu_{(F^c\cup I)} - \mu_{F^c} = \mu_F - \mu_{F\setminus I}$, as desired.

Finally, we use this second claim to prove the proposition. Let E and F be such that $s \in E \cap F$. Notice, if $E \cap F = s$ we can apply the second claim directly. So assume $s \subsetneq E \cap F$. There are two cases. (i) $E^c \cap F^c \neq \emptyset$. Then $(E, (E^c \cap F^c) \cup s)$, and $((E^c \cap F^c) \cup s, F)$, satisfy the conditions of the second claim so, $\mu_E - \mu_{E\setminus s} = \mu_{(E^c\cap F^c)\cup s} - \mu_{E^c \cap F^c} = \mu_F - \mu_{F\setminus s}$. (ii) $E^c \cap F^c = \emptyset$. Then $(E, E^c \cup s)$ satisfy the conditions for the second claim: $\mu_E - \mu_{E\setminus s} = \mu_{(E^c\cup s)} - \mu_{E^c}$. Now notice that it must be that $E^c \cup s \subset F$, hence $(E^c \cup s)^c \cap F^c \neq \emptyset$. Applying case (i), provides $\mu_{(E^c\cup s)} - \mu_{E^c} = \mu_F - \mu_{F\setminus s}$. This completes the proof. ■

Proof of Proposition A.7.

Consider 3 different cases, (i) $E = F^c$, (ii) $F \subsetneq E$ and (iii) $E \cap F \neq \emptyset$ and $E^c \cap F \neq \emptyset$, and $E \cap F^c \neq \emptyset$. It suffices to consider these three conditions since whenever $E \cap F = \emptyset$, and $E^c \cap F^c \neq \emptyset$, Lemma A.4 will get the result for E and F, from E^c and F^c.

First note that the case where $E = F^c$ the result follows straightforwardly from Proposition A.3. For cases (ii) and (iii) notice that there exists some $s \in E \cap F$. It is without loss of generality to assume that $|E|, |F| \geq 2$,[12] for this s and any $t \in S$, we can divide (A.1) (from proposition A.6) by $\mu(t) > 0$ and obtain

$$\frac{\mu_E(t)}{\mu(t)} - \frac{\mu_{E\setminus s}(t)}{\mu(t)} = \frac{\mu_F(t)}{\mu(t)} - \frac{\mu_{F\setminus s}(t)}{\mu(t)}. \tag{A.13}$$

Now, consider the case where $F \subsetneq E$. By the definition of $\gamma_E^+ = \frac{\mu_E(s)}{\mu(s)}$ for $s \in E$, and $\gamma_E^- = \frac{\mu_E(s)}{\mu(s)}$ for $s \in E^c$. Suppose $s \in E \cap F$, then using (A.13) and the definition of states as positive or negative (when viewed from $E, F, E\setminus s$, and $F\setminus s$). Since $F \subsetneq E$,

[12]This cannot be violated in case (iii), and in case (ii), E must have more than two elements. Further, note that if $|F| = 1$, $\gamma_\emptyset$ would not be defined, but in that case if $|S| \geq 3$, $|F^c| = n - 1$ and the result can follow from reversing the roles of F and E, with E^c and F^c and Proposition A.3.

there exists some $s \in E \cap F$ and $t \in E^c \cap F^c$.

$$s \in E \cap F: \qquad \gamma_E^+ - \gamma_{E\setminus s}^- = \gamma_F^+ - \gamma_{F\setminus s}^-, \tag{A.14a}$$

$$t \in E^c \cap F^c: \qquad \gamma_E^- - \gamma_{E\setminus s}^- = \gamma_F^- - \gamma_{F\setminus s}^-. \tag{A.14b}$$

Subtracting (A.14b) from (A.14a), yields $\gamma_E^+ - \gamma_E^- = \gamma_F^+ - \gamma_F^-$.

Now consider the cases where $E \cap F \neq \emptyset$ and $E^c \cap F \neq \emptyset$, and $E \cap F^c \neq \emptyset$. Define $G = E \cap F$, so, $G \subsetneq E$ and $G \subsetneq F$. Applying case (ii) twice (i.e., $\gamma_E^+ - \gamma_E^- = \gamma_G^+ - \gamma_G^- = \gamma_F^+ - \gamma_F^-$), proves the claim. ■

Proof of Proposition A.8.

By definition $\mu_E(s) = \gamma_E^+ \mu(s)$ if $s \in E$ and $\mu_E(s) = \gamma_E^- \mu(s)$ if $s \in E^c$. Since γ_E^+ is the same for all $s \in E$, it follows that for any $E' \subseteq E$, then $\mu_E(E') = \gamma_E^+ \mu(E')$ as well. Then $\gamma_E^+ - \gamma_E^- = \lambda$ implies that

$$\begin{aligned}
& \frac{\mu_E(E)}{\mu(E)} - \frac{\mu_E(E^c)}{\mu(E^c)} = \lambda, \\
\Rightarrow \qquad & \frac{\mu_E(E)}{\mu(E)} - \frac{1 - \mu_E(E)}{1 - \mu(E)} = \lambda, \\
\Rightarrow \qquad & \mu_E(E)(1 - \mu(E)) - (1 - \mu_E(E))\mu(E) = \lambda(1 - \mu(E))\mu(E).
\end{aligned}$$

Solving for $\mu_E(E)$ yields $\mu_E(E) = \mu(E)\left(1 + \lambda(1 - \mu(E))\right)$, hence

$$\gamma_E^+ = \frac{\mu_E(E)}{\mu(E)} = \left(1 + \lambda(1 - \mu(E))\right). \tag{A.15}$$

Likewise we can solve for γ_E^- to get

$$\gamma_E^- = \frac{\mu_E(E^c)}{\mu(E^c)} = (1 - \lambda\mu(E)). \tag{A.16}$$

The fact that $\frac{\mu_E(E)}{\mu(E)} = \frac{\mu_E(s)}{\mu(s)}$ for all $s \in E$, and $\frac{\mu_E(E^c)}{\mu(E^c)} = \frac{\mu_E(s)}{\mu(s)}$ for all $s \in E^c$ follows from Proposition A.1. Therefore

$$\begin{aligned}
\mu_E(s) &= \mu(s)\left(1 + \lambda\mu(E^c)\right) && \text{for } s \in E, \\
\mu_E(s) &= \mu(s)\left(1 - \lambda\mu(E)\right) && \text{for } s \in E^c.
\end{aligned}$$

This proves the result. ■