EXPLORATION AND CORRELATION

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Overview

In dynamic uncertain environments:

- Agents take actions both for immediate payoff and to reduce uncertainty (learn).

- The agents' preferences/beliefs evolve with new info.

- Can we identify these beliefs?

Relevant: innovative industries, voter dynamics, financial regulation, etc

Exploration Problems

Each period a project manager:

- Must choose to invest in project a or b, but not both.
- Observes if the chosen project succeeds or fails.
- Receives a payoff from the outcome.

Learning via Exploration

- The true state of affairs is a joint distribution over

$$S = S_a \times S_b = \{ s_a s_b, s_a f_b, f_a s_b, f_a f_b \}.$$

- The optimal strategy depends on the manager's beliefs regarding the true generating process.
  - Learning: observes outcomes and updates her belief.
  - Tradeoff: immediate "consumption" value / future informational value.

Beliefs

- In applications we specify these beliefs over the outcome space of all projects.
- Bayes' rule determines the dynamic of beliefs.
- We then (try to) solve for the optimal strategy.

This Paper

Conversely we ask:

- What can we learn from the manager's preferences over the different investment strategies?
- Can we identify the beliefs underlying the exploration/exploitation trade off faced by the manager?

Systemic Risk

Systemic risk in an economy/industry depends on the **correlation** between investments.

- Can we understand this correlation by observing investment strategies.

Belief Identification

- We are interested in a manger with exchangeable beliefs.
- Since only one action can be taken in each period, the agent's choices can reveal only the margins of her beliefs.
- We introduce a restriction on marginals, across-arm symmetry, ensuring they arise from an exchangeable process.

Uniqueness?

- We provide a simple example in the finite horizon where the marginals determine the process.
- In the infinite horizon, marginals do not uniquely determine the process.
  - Can always find a (unique) representative for which projects are independent conditional on the true parameter.

Systemic Risk

Strong negative result:

> In the infinite horizon, we cannot learn the correlation between investments from the managers preference over exploration strategies.

Talk Today

1. Statistical Model
    - Construct the proper stochastic model to study the manager's beliefs.
    - How does one determine exchangeability from marginals?
2. (Briefly) Identification
    - We present axioms that allow us to elicit the discounted expected utility representation.
    - Marginals can be uniquely identified.
    - Identification of the joint distribution limited by (1).

Literature

1. **Exploration Problems**: Robbins (1952); Bergemann, Valimaki (2000; price formation in markets), Bergemann, Hege (2005; venture capital), Moroni (2017; Delegated R&D).
2. **Belief Evolution**: de Finetti (1931;1937), Hewitt, Savage (1955), Diaconis (1977).
3. **Belief Identification with Learning**: Dillenberger, Sebastian Lleras, Sadowski, Takeoka (2014), Piermont, Takeoka, Teper (2016), Cooke (2017), Dillenberger, Krishna, Sadowski (2017)

A Single Project: Exchangeability

- State space: S; Time Periods: N (finite or countable).
- A process ζ, over sequences of realizations, S^N, is exchangeable if its distribution is invariant to finite permutations:
  - ζ is exch if, for every $n \in \mathbb{N}$, history $h \in S^N$ and permutation $\pi : n \to n$,
  $$\zeta(h) = \zeta(\pi(h)).$$

Updating and Exchangeability

- To discuss Bayesian updating, one needs to observe the evolution of the joint distribution.
- In exploration models, only a single action can be taken in every period; only the margins of the process can be identified.
  - Beliefs about each individual project conditional on the observed history.

Multi-Dimensional Experiments and Limited Observability

- Two actions "project a" and "project b":

$$S = S_a \times S_b$$

- Let $\mathcal{T}$ be the collection of all sequences of the form $T_1, T_2, T_3, \ldots$, where $T_i \in \{S_a, S_b\}$ for every $i \in N$

Belief Structures

- With every $\mathbf{T} = T_1, T_2, \ldots$ we associate a process $\eta_{\mathbf{T}}$ over $\prod_{i \in N} T_i$.
- $\eta_{\mathbf{T}}$ conveys the distribution of outcomes from taking action T_{n+1} following every history of outcomes $h_i \in T_i$.
    - For a permutation $\pi : n \to n$,

$$\pi\mathbf{T} = (T_{\pi(1)}, T_{\pi(2)}, .., T_{\pi(n)}, T_{n+1}, \ldots)$$

    - Similarly, for a finite history $h = (h_1, \ldots, h_n) \in (T_1, \ldots, T_n)$,

$$\pi h = (h_{\pi(1)}, h_{\pi(2)}, .., h_{\pi(n)})$$

Example 1A

Let $N = 2$. The agent believes that each project will have exactly one success, equally likely to be in either period, and, moreover, believes the two projects will succeed and fail jointly.

$$
\begin{array}{cc|c|c|c|c|}
 & & \multicolumn{4}{c}{n = 1} \\
 & & s_a, s_b & s_a, f_b & f_a, s_b & f_a, f_b \\
\hline
 & s_a, s_b & 0 & 0 & 0 & \frac{1}{2} \\
\hline
n = 0 & s_a, f_b & 0 & 0 & 0 & 0 \\
\hline
 & f_a, s_b & 0 & 0 & 0 & 0 \\
\hline
 & f_a, f_b & \frac{1}{2} & 0 & 0 & 0 \\
\hline
\end{array}
$$

Example 1A

The family of marginal beliefs associated with this joint:

$$\eta_{x,y}(s_x, s_y) = \eta_{x,y}(f_x, f_y) = 0$$
$$\eta_{x,y}(s_x, f_y) = \eta_{x,y}(f_x, s_y) = \frac{1}{2}.$$

where $(x, y) \in \{a, b\} \times \{a, b\}$.

- The joint distribution above was the unique joint consistent with these marginals.

Example 1B

What if the manager believed instead the two projects will succeed and fail independently?

$n = 1$

$n = 0$		s_a, s_b	s_a, f_b	f_a, s_b	f_a, f_b
	s_a, s_b	0	0	0	$\frac{1}{4}$
	s_a, f_b	0	0	$\frac{1}{4}$	0
	f_a, s_b	0	$\frac{1}{4}$	0	0
	f_a, f_b	$\frac{1}{4}$	0	0	0

Example 1B

The family of marginal beliefs associated with this joint:

$$\eta_{x,x}(s_x, s_x) = \eta_{x,x}(f_x, f_x) = 0$$

$$\eta_{x,x}(s_x, f_x) = \eta_{x,x}(f_x, s_x) = \frac{1}{2}$$

$$\eta_{x,y}(s_x, f_y) = \eta_{x,y}(f_x, s_y) = \frac{1}{4} \qquad \text{if } x \neq y.$$

where $(x, y) \in \{a, b\} \times \{a, b\}$.

Example 2

The agent considers two equally probable scenarios: in the first both projects have a $\frac{1}{4}$ likelihood of succeeding in both periods (i.e, i.i.d over time, with probability $\frac{1}{4}$) and in the second the likelihood of success is $\frac{3}{4}$.

Example 2

Consider the following joint distributions:

$n = 1$

$n = 0$		s_a, s_b	s_a, f_b	f_a, s_b	f_a, f_b
	s_a, s_b	$\frac{5}{16}$	0	0	$\frac{3}{16}$
	s_a, f_b	0	0	0	0
	f_a, s_b	0	0	0	0
	f_a, f_b	$\frac{3}{16}$	0	0	$\frac{5}{16}$

	s_a, s_b	s_a, f_b	f_a, s_b	f_a, f_b
s_a, s_b	$\frac{41}{256}$	$\frac{15}{256}$	$\frac{15}{256}$	$\frac{9}{256}$
s_a, f_b	$\frac{15}{256}$	$\frac{9}{256}$	$\frac{9}{256}$	$\frac{15}{256}$
f_a, s_b	$\frac{15}{256}$	$\frac{9}{256}$	$\frac{9}{256}$	$\frac{15}{256}$
f_a, f_b	$\frac{9}{256}$	$\frac{15}{256}$	$\frac{15}{256}$	$\frac{15}{256}$

Example 2

Both joint distributions impart the exact same restrictions on marginal beliefs:

$$\zeta_{x,y}(s_x, s_y) = \zeta_{x,y}(f_x, f_y) = \frac{5}{16}$$

$$\zeta_{x,y}(s_x, f_y) = \zeta_{x,y}(f_x, s_y) = \frac{3}{16}$$

where $(x, y) \in \{a, b\} \times \{a, b\}$.

- In both examples, all joint distributions were exchangeable.

- Only in Example 1 did the marginals expose the manager’s perceived correlation.

- How do we move from marginals to joint?

AA-Symmetry

Definition.

$\{\eta_{\mathbf{T}}\}_{\mathbf{T} \in \mathcal{T}}$ satisfies across arm symmetry if

1. If $h \in \mathbf{T} \cap \mathbf{T}'$, then $\eta_{\mathbf{T}}(h) = \eta_{\mathbf{T}'}(h)$.
2. For every $\mathbf{T} \in \mathcal{T}$, $h \in \mathbf{T}$, and finite permutation π,

$$\eta_{\mathbf{T}}(h) = \eta_{\pi\mathbf{T}}(\pi h).$$

A Non-Symmetric $\{\eta_{\mathbf{T}}\}_{\mathbf{T}\in\mathcal{T}}$

- In Examples 1A/B and 2, the marginals satisfy AASym.
- When does it fail? Consider the following example:
  - As long as project a is chosen, belief regarding both projects is $\frac{1}{2}$
  - Once project b is chosen, the realized outcome occurs with probability 1 for both

$$\eta_{ab}(s_a, f_b) = \frac{1}{4}$$
$$\eta_{ba}(f_b, s_a) = 0$$

Symmetry and Consistency

Theorem.

$\{\eta_{\mathbf{T}}\}_{\mathbf{T}\in\mathcal{T}}$ satisfies across arm symmetry if and only if there exists an exchangeable distribution ζ over S^N such that

$$\text{marg}_{\mathbf{T}}\zeta = \eta_{\mathbf{T}}$$

for every $\mathbf{T} \in \mathcal{T}$

AA-Symmetry and Strongly Exchangeability

AA-symmetry of $\{\eta_{\mathbf{T}}\}_{\mathbf{T}\in\mathcal{T}}$ does not uniquely determine a consistent exch process. From Example 2:

- When the projects are i.i.d. between periods, their contemporary correlation was not pinned down.
- The marginals were consistent with the projects' being contemporaneously independent.

Strong Exchangeability

Definition.

An exch distribution ζ is strongly exchangeable if for every history $h = \prod_{i=1}^{n}(h_{a_i}, h_{b_i})$ and permutations $\pi_a, \pi_b : n \to n$,

$$\zeta(h) = \zeta(\prod_{i=1}^{n}(h_{a_{\pi_a(i)}}, h_{b_{\pi_b(i)}}))$$

- Each dimension can be permuted independently.

de Finetti's Representation

- Let $N = \mathbb{N}$: ζ is exch if and only if there exists a prior distribution $\lambda \in \Delta(\Delta(S))$ such that

$$\zeta = \int_{\Delta(S)} \boldsymbol{\mu} \, \mathrm{d}\lambda(\mu)$$

- As if:
  - A parameter in $\Delta(S)$ is chosen according to λ.
  - The agent does not know the chosen parameter, but knows (or believes) λ.
  - Each period, updates her prior according to the outcome of the experiment.
- Such a representation is unique

A de Finetti like Representation of Strong Exchangeability

Theorem.

ζ over $\prod_{\mathbb{N}} S$ is strongly exchangeable if and only if the support of λ is in $\Delta(S_a) \times \Delta(S_b)$.

- An exch distribution ζ over $S^{\mathbb{N}}$ is a λ-mixture of parameters in $\Delta(S)$.
  - In an exch process, the joint distribution of experiments' outcomes is (inter-temporally) independent conditionally on the true parameter.
- $S = S_a \times S_b$.
  - In a strongly exch process, experiments are also conditionally contemporaneously independent.

Intuition of Proof

- ζ is exch, converges to some $\mu \in \Delta(S_a \times S_b)$ with ζ-probability 1.

- From SE: $\mu(s_a, f_b) \cdot \mu(f_a, s_b) = \mu(s_a, s_b) \cdot \mu(f_a, b_b)$:

$$\mu(s_a|f_b) \cdot \mu(f_b) \cdot \mu(f_a|s_b) \cdot \mu(s_b) = \mu(s_a|s_b) \cdot \mu(s_b) \cdot \mu(f_a|f_b) \cdot \mu(f_b)$$

- $\frac{\mu(s_a|f_b)}{\mu(s_a|s_b)} = \frac{\mu(f_a|f_b)}{\mu(f_a|s_b)}$: true for all events $\implies$ independence.

AA-Symmetry and Strongly Exchangeability

Theorem.

Assume $\{\eta_{\mathbf{T}}\}_{\mathbf{T}\in\mathcal{T}}$ satisfies AA-symmetry. There exists a unique strongly exchangeable distribution ζ over S^N such that

$$\mathrm{marg}_{\mathbf{T}}\zeta = \eta_{\mathbf{T}}$$

for every $\mathbf{T} \in \mathcal{T}$

Intuition of Proof, $N = \mathbb{N}$

$n = 5$	S_a	S_b
$n = 4$	S_a	S_b
$n = 3$	S_a	S_b
$n = 2$	f_a	f_b
$n = 1$	s_a	f_b

- Consider any finite event, E.

Intuition of Proof, $N = \mathbb{N}$

$n = 5$	S_a	S_b
$n = 4$	S_a	S_b
$n = 3$	S_a	f_b
$n = 2$	f_a	f_b
$n = 1$	s_a	S_b

- Permute so that only one restriction per time period.

Intuition of Proof, $N = \mathbb{N}$

$n = 5$	S_a	S_b
$n = 4$	f_a	S_b
$n = 3$	S_a	f_b
$n = 2$	S_a	f_b
$n = 1$	s_a	S_b

- Permute so that only one restriction per time period.

Intuition of Proof, $N = \mathbb{N}$

$n = 5$	S_a	S_b
$n = 4$	f_a	S_b
$n = 3$	S_a	f_b
$n = 2$	S_a	f_b
$n = 1$	s_a	S_b

- Cooresponds to $h \in \mathbf{T} = (S_a, S_b, S_b, S_a, T_5, \ldots)$

Intuition of Proof, $N = \mathbb{N}$

$n = 5$	S_a	S_b
$n = 4$	f_a	S_b
$n = 3$	S_a	f_b
$n = 2$	S_a	f_b
$n = 1$	s_a	S_b

- Set $\zeta(E) = \eta_{\mathbf{T}}(h)$.

Intuition of Proof

- AA-SYM ensures this process is invariant to the permutations chosen.
- There is unique extension of ζ to all events.
- Different (but not that different) proof for finite N.

Eliciting $\eta_{\mathbf{T}}$

- The model above assumes the marginal—but not the joint—distributions are observable.
- We turn to a decision theoretic exercise to understand when and if this is reasonable.

To be shown:

- Assume we have access to the preferences over exploration strategies from a bandit problem.
- Axiomatization of the representation.
- Only $\{\eta_{\mathbf{T}}\}_{\mathbf{T}\in\mathcal{T}}$ can be (uniquely) elicited from the axioms.

Examples, revisited

- Recall: $N = 2$, $\mathcal{A} = \{a, b\}$, $X = \{s_a, f_a, s_b, f_b\}$.
- Let $u(s_a) = 1$, $u(f_a) = -1$, $u(s_b) = 2$, and $u(f_b) = -2$.
- The DM is an EU maximizer
- Total utility is the sum across the two periods.

Examples, revisited

- For $x, y, z \in \{a, b\}$, let $(x, (y, z))$ denote the strategy:
    - x in the first period.
    - y in the second, conditional on x's success, and z on x's failure.

- For example, $(a, (a, b))$ is the strategy dictating taking action a in the first period, and
    - action a in the second period, if it was a success in the first.
    - and action b in the second in case a failed in the first.

Example 1A, revisited

- The agent believes that each project will have exactly one success, equally likely to be in either period, and, moreover, believes the two projects will succeed and fail jointly.
- The agent's valuations for investment plans are given as follows: $V(x,(y,z)) = 0$ if $y = z$, and

$$\begin{aligned} V(a,(a,b)) &= V(b,(a,b)) = \frac{1}{2} \\ V(a,(b,a)) &= V(b,(b,a)) = -\frac{1}{2}. \end{aligned}$$

Example 1B, revisited

If on the other hand, the 2 projects were uncorrelated:
$V(x,(y,z)) = 0$ if $y = z$, and

$$
\begin{aligned}
V(a,(a,b)) &= -\frac{1}{2} & \qquad V(b,(a,b)) &= 1 \\
V(a,(b,a)) &= \frac{1}{2} & \qquad V(b,(b,a)) &= -1.
\end{aligned}
$$

Examples, revisited

Example 2: either projects have a $\frac{1}{4}$ likelihood of succeeding in both periods (i.e, i.i.d over time, with probability $\frac{1}{4}$) and in the second the likelihood of success is $\frac{3}{4}$.

- $V(x,(b,a)) = \frac{1}{8}$ (for $x \in \{a,b\}$) and 0 for all other strategies.
- Does not depend on contemporaneous correlation between projects.

Preference for strategies in bandit problems can identify:

- Marginals, $\{\eta_{\mathbf{T}}\}_{\mathbf{T}\in\mathcal{T}}$—always.
- Joint, ζ—only insofar as given by previous discussion (when $N = \mathbb{N}$, upto strong exch).

Framework

- Let X denote a set of outcomes.
- Let $\mathcal{A}$ denote a set of actions; think, the arms of a bandit problem.
- Each action, a, is associated with a set of possible outcomes, $S_a \subseteq X$.

Histories.

A **history of length n** is a sequence of action/outcome realizations.

- That is, let $h = (a_1, x_1) \dots (a_n, x_n)$.
- Let $\mathcal{H}$ and $\mathcal{H}^\infty$ denote all finite and infinite histories, respectively.

Strategies.

A (mixed) strategy is a mapping from finite histories into randomizations (lotteries) of actions:

$$p : \mathcal{H} \to \Delta(\mathcal{A})$$

- Specifies the action to be taken after each history (including the trivial $\varnothing$).
- Let p_h denote the lottery taken after h with $p_h(a)$ the probably of choosing a.
- Our decision theoretic primitive is a preference relation over all strategies.

Evaluations of Histories

If the manager has a utility index $u : X \to \mathbb{R}$ and discount factor δ, assume she values $h \in \mathcal{H}^\infty$ as

$$U(h) = \sum_{n \in \mathbb{N}} \delta^n u(x_n)$$

Subjective Expected Experimentation

- Let $\mu_{h,a} \in \Delta(S_a)$ denote the manager's belief about action a after having observed history h.
- $\{\mu_{h,a}\}_{h \in \mathcal{H}, a \in \mathcal{A}}$ and p induce a unique measure over $\mathcal{H}$:

$$\text{pr}(h, (a, x)) = \text{pr}(h) \cdot p_h(a) \cdot \mu_{h,a}(x)$$

- Assume $U(p) = \mathbb{E}\, U(h)$.

Subjective Expected Experimentation

Equivalently:

$$U_h(p) = \mathbb{E}_{p_h}\left[\mathbb{E}_{\mu_{h,a}}\left[u(x) + \delta\, U_{h,(a,x)}(p)\right]\right] \qquad \text{(SEE)}$$

- We show $\left\langle u, \{\mu_{h,a}\}_{h \in \mathcal{H}, a \in \mathcal{A}}, \delta \right\rangle$ can be uniquely identified from preferences.

Belief Structures

The family $\{\mu_{h,a}\}_{h\in\mathcal{H},a\in\mathcal{A}}$ is identified with $\{\eta_{\mathbf{T}}\}_{\mathbf{T}\in\mathcal{T}}$

- Consider $\mathbf{T} = S_{a_1}, S_{a_2}, \ldots$ and $h \in \mathbf{T}$.
- Given $\{\mu_{h,a}\}_{h\in\mathcal{H},a\in\mathcal{A}}$

$$\eta_{\mathbf{T}}(x_1 \ldots x_{n+1}) = \prod_{i\leq n} \mu_{h_{i-1},a_i}(x_i)$$

- There exists a unique (σ-addative) extension.
- This mapping is bijective with the set of processes that satisfy (1) of AA-sym.

Standard Axioms

1. VNM: (EU axioms).
2. Stationarity
3. Separability

- A1-3 provide the structure for discounted expected utility.

Following each history, we want to connect the following behaviors:

- The DM treats each action as a (history-dependent) probability distribution over outcomes.
- The probability of x is also the probability of the continuation value when observing x.

H-proportionality.

Idea:

- Treat S_a like a state space.
- The continuation mapping is an “act” in the Anscombe Aumann sense.
- Proportionality ensures beliefs over S_a can be identified, and dictates the likelihood of both current utility and continuation utility are identical.
- For each action a, the outcomes S_a serve both as consumption goods, and both as the state space.
- This is standard in bandit problems. Not an assumption of our model, but implied by axioms.

Proportionality.

Imagine $X = \{x, y\}$. And continuation values are identified so that $f: X \to \mathbb{R}$. Then

$$U(a, f) = \mu_a(x)[u(x) + \delta f(x)] + \mu_a(y)[u(y) + \delta f(y)]$$

So that $U(a, f) \geq U(a, g)$ if and only if $\mathbb{E}_{\mu_a}[f(\cdot)] \geq \mathbb{E}_{\mu_a}[g(\cdot)]$.

Proportionality.

If there is some $\alpha \in [0,1]$ such that for all f, g

$$U(\alpha(a,f) + (1-\alpha)(a',f)) \geq U(\alpha(a,g) + (1-\alpha)(a',g))$$

$$\iff$$

$$U(b,f) \geq U(b,g)$$

it must be that $\mathbb{E}_{\alpha\mu_a + (1-\alpha)\mu_{a'}} = \mathbb{E}_{\mu_b}$.

Proportionality.

Further, if $\mathbb{E}_{\alpha\mu_a+(1-\alpha)\mu_{a'}} = \mathbb{E}_{\mu_b}$, then

$$U(\alpha(a,f) + (1-\alpha)(a',f)) = U(b,f)$$

A4: Proportionality

If two strategies induce the same ranking over continuation values, then when jointly assigned the same continuation value the DM must be indifferent.

- p and q aggregate continuation values the same way.
- The continuation values are a function of the outcome of the actions in p and q.
- Therefore, it must be that p and q aggregate outcomes the same way.
- Probability of outcomes are the same + continuation values are the same = indifference.

Theorem.

$\succcurlyeq$ satisfies A1-4 if and only if there exists and SEE representation: there exits $\left\langle u : X \to \mathbb{R}, \{\mu_{h,a}\}_{h \in \mathcal{H}, a \in \mathcal{A}}, \delta \in (0,1) \right\rangle$ such that

$$U(p) = \mathbb{E}\, U(h). \tag{SEE}$$

represents $\succcurlyeq$. Moreover all parameters are unique in the standard fashion.

Behavioral Markers

- Proportionality holds for any recursive preferences.
- Not a marker of exploration (in general, μ's are unrestricted).
- Exploration models must take a stand of belief evolution.

AA-Sym

- There is an axiomatic version of AA-Sym.
- The value of a bet on an event is invariant to permutations.
- Ensures the family, $\{\eta_{\mathbf{T}}\}_{\mathbf{T}\in\mathcal{T}}$, consistent with elicited $\{\mu_{h,a}\}_{h\in\mathcal{H},a\in\mathcal{A}}$ will arise from a unique strongly exch process.

Conclusion

- Investment strategies reveal inter-temporal correlation.
- In the infinite horizon, this is the limit of identification.
  - Risk/Uncertainty aversion can reveal more, but not everything.
- Bad for regulators; good for investors.