# Rationalization and Robustness in Dynamic Games with Incomplete Information: Extended Abstract

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## 1 Introduction

In the game theoretic environment, there is a clear tension between the strength of a solution concept and its robustness to misspecification. In other words, if the analyst wants his model to be resilient to small errors in the parameters, then he must weaken the predictive power of the model. In fact, this tension can be made formal; under a richness assumption (loosely speaking, for every strategy there is a state such that it is dominant), structure theorems place clear limits to the predictive power of robust solution concepts (Weinstein and Yildiz, 2007; Penta, 2012).

Intuitively, a similar tension arises when considering a solution concept and its epistemic demands.[1] To make sharper predictions, the modeler must place more stringent requirements on the structure of the understanding of agents (at least insofar as to adhere to the requirements of the solution concept). Informally, this observation suggests a possible link between the epistemic demands of a solution concept and it's robustness. The first aim of this paper is to formalize this connection. We show that particular notions of robustness can be thought of as epistemic concerns. In particular, we examine a solution concept's robustness to the misspecification of players' beliefs, the underlying space of payoff uncertainty, and to the joint misspecification of both. In each case, we show that reasonable and common notions of robustness can be described entirely though the epistemic characterization of the solution concept.

Most commonly, robustness has been defined with respect to misspecification of players beliefs; in particular via the upper-hemicontinuity (henceforth, UHC) of the solution concept in question. UHC dictates that if a strategy is ruled out for some type then there is a neighborhood of nearby types for which the strategy is also ruled out. In the absence of UHC, approximations

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[1]Informally, epistemic demand refer to the restrictions placed on players beliefs regarding payoff uncertainty, opponents strategies, and the higher order beliefs over these objects. A more formal, but my no means complete, explanation is found in Section 2.

will not suffice; even if a strategy is selected by every successive approximation, it may not be selected in the limit.

Our first result provides the epistemic characterization of UHC, and hence, a direct method of verification of robustness. The UHC of a a solution concept is related to the closedness of the event that characterizes it. Intuitively, that a solution concept is well behaved with respect to approximation of the players types is implied by the fact that the limit of any sequence in the characterizing event is also in the event.

A second aim of this paper is to develop an appropriate notion of robustness to the misspecification of the space of payoff uncertainty (referred to as the *state space*). Although the richness assumption allows for structure theorems that place clear limits on the robustness of solution concepts and provide for generic dominance solvability, it is often an unreasonable demand on the space of uncertainty. To assume richness is to drop all common knowledge assumptions regarding strategic unceratainty.[2] However, in many, if not most, economic situations, there are clear restrictions on payoffs, and, without assuming major structural ignorance on the part of the players, it seems reasonable to model these payoff restrictions with common knowledge assumptions. Moreover, simply embedding the game in a larger state space (and modeling common knowledge as initial common belief) may distort predictions. This is a problem exclusive to dynamic environments, as in static games, the players initial (and only) beliefs entirely govern actions. In dynamic environments, however, when players are able to update their beliefs, the state space itself becomes relevant to the analysis.

Penta (2012) proposes the robustness notion of *informational invariance*, or, that the predictions of a solution concept do not change when the game is embedded in *any* larger state space. We ague that this is too strong a requirement, as it restricts the solution concept from having any dependence on the state space that is not present in the player's initial beliefs. Instead, we propose the notion of state-space-robustness (s-robustness). To this end, we allow the space of uncertainty that each player considers to become a parameter of the game. We call each players understanding about the state space, and his higher order understanding about his opponents understanding of the state space, his *directory*. A player is described not only by his beliefs (a hierarchy of probability distributions), but also by his directory (a hierarchy of sets of parameters). Then, in analogy to UHC, a solution concept is s-robust if whenever a strategy is ruled out for a directory there is a neighborhood of nearby directories for which the strategy is also ruled out. As in regards to UHC, we provide the epistemic restrictions that coincide with s-robustness.

Lastly, we provide a structure theorem that identifies conditions on directories such that any strict refinement of Extensive-Form Rationalizability (EFR) (introduced in Pearce (1984)) is not UHC. We find conditions that are strictly weaker than the richness assumption in Penta (2012) and its static counterpart in (Weinstein and Yildiz, 2007). In particular, while we require the existence of some objective rich state space, this state space need not be commonly known. Each player could be described by a directory that does *not* contain dominance states. Indeed, it is

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[2]*Relaxing* common knowledge assumptions can be somehow regarded as *strengthening* common awareness ones.

possible to obtain the result even in the case where it is common knowledge that no action is ever dominant.

## 2 UPPER-HEMICONITINUITY: AN EPISTEMIC APPROACH

We consider dynamic environments and our analysis takes place within a formal epistemic framework. Players in a game hold beliefs regarding the state space $\Theta$, the other players beliefs about the payoff uncertainty, and other players strategies $S_{-i}$. Moreover, players hold higher-order beliefs about other players' beliefs about these objects, beliefs about these beliefs and so on.

Following closely the construction due to Battigalli and Siniscalchi (1999), we formally represent each player $i$'s higher-order conditional beliefs on opponents' choices and the payoff state via conditional belief hierarchies, i.e., (epistemic) types drawn from a *universal* type space we denote by $\mathcal{E}$. Note that players' uncertainty in $\mathcal{E}_{-i} \times S_{-i} \times \Theta$ is not modeled only at the beginning of the game, but also at every history along the possible paths of play. We further assume that beliefs are updated in a Bayesian manner whenever possible.

The epistemic analysis is then performed in the set of *states of the world* $\Omega = \mathcal{E} \times S \times \Theta$. Each epistemic type, $e_i \in \mathcal{E}_i$, induces a standard type, $\tau_i \in \mathcal{T}_i$ via the canonical quotient map, $q_i : \mathcal{E}_i \to \mathcal{T}_i$.[3] For each standard type, $\tau_i$, we denote by $[q_i = \tau_i]$ the event that player $i$'s standard hierarchy is exactly $\tau_i$. A solution concept can then be *characterized* by an event contained in this state space. Namely, a solution is characterized by an event $E \subset \Omega$ if the predicted strategies for player $i$ with (initial) type $\tau_i$ are precisely those strategies consistent with $E$ and the event that player $i$'s conditional belief hierarchy on $\Theta$ is $\tau_i$; that is, $\mathrm{Proj}_{S_i}(E \cap [q_i = \tau_i])$.

Throughout the paper we focus on two different dynamic solution concepts: *extensive-form rationalizability* (EFR) (Pearce, 1984), and *interim sequential rationalizable* (ISR) (Penta, 2012). Both are generalizations of the static notion of interim correlated rationalizability; the difference being that EFR requires that players place a higher epistemic priority to rationality than ISR. Our focus on ISR is driven by the structure theorem of Penta (2012), which states that, under the richness assumption, any strict refinement of ISR is not UHC.

Penta (2012) provides the epistemic characterization of ISR as the event composed of Rationality (**R**) and Initial Common Belief in Rationality (**ICBR**). **R** states that each player will always choose a strategy that is a best response to his own belief. **ICBR** for player $i$ states that, at the beginning of the game, player $i$ believes players $j \neq i$ are rational, believes that players $j \neq i$ believe players $k \neq j$ are rational, etc. Since players are Bayesian, this implies that at any history reached with positive probability according to player $i$'s initial belief, player $i$ still holds a common belief in **R**.

In this paper, we recall the characterization of EFR as the event composed of **R** and Common Strong Belief in Rationality (**CSBR**). Strong Belief in Rationality for player $i$ is the event that

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[3]The standard type space consists of all hierarchies of beliefs over payoff uncertainty, modeled at the null history. It is analogous to the canonical space constructed for static environments by Brandenburger and Dekel (1993). The map $q_i$ is simply the hierarchical marginalization on this space.
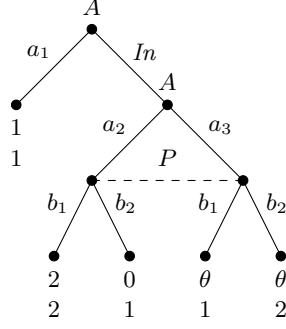
Figure 1: Examples 1 and 2 in Penta (2012).

player $i$ assigns probability 1 to event **R** at *every* history which is not belief-inconsistent with **R**, or, in other words, whenever **R** has not been falsified by observed history. This way, strong belief captures the essence of *forward induction*, in the sense that beliefs about future behavior are, whenever possible, updated according to observed history and the filter rationality imposes. **CSBR** is then the hierarchical iteration of strong belief in rationality; that is, as stated by Battigalli's *best rationalization principle*, the event that players assign to their opponents the highest degree of strategic sophistication consistent with observed behavior (Battigalli, 1996). It is immediate that **ICBR** is implied by **CSBR** and hence EFR is a refinement of ISR. This last observation, together with the structure theorem in Penta (2012), suggests a clear tension between rationalization and robustness: under richness, the robustness of predictions appears to be lost as soon as players are assumed to reason according to the highest rationalization principle. The following example illustrates the conflict.

### 2.1   Alexei and Polina

Example 1 by Penta (2012) studies the sequential game depicted in Figure 1, in which Alexei Ivanovich's (player $A$) utility after history $(In, a_3)$ is represented by unspecified parameter $\theta$. We focus first in the case in which Alexei and Polina Alexandrovna (player $P$) commonly know that $\theta$ equals 0: $\Theta = \{0\}$ and the the corresponding standard type space is $\mathcal{T}^{CK} = \{\tau^{CK}\}$. In this case, strategy $(In, a_3)$ is strictly dominated for Alexei. So, if Polina finds herself at information set $\{(In, a_2), (In, a_3)\}$ (i.e, she is *informed* that Alexei did not choose $a_1$) *and* believes that Alexei is rational, then she must believe he played $(In, a_2)$. Thus, action $b_1$ is optimal for her. Now, if Alexei expects Polina to rationalize his choice, he is able to predict choice $b_1$ in case he plays $(In, a_2)$, and hence, $(In, a_2)$ becomes optimal for him. However, if Polina finds herself at information set $\{(In, a_2), (In, a_3)\}$ *and* does not believe that Alexei is rational, she can rationalize any action. Hence ISR, which does not require Polina to believe Alexei is rational at $\{(In, a_2), (In, a_3)\}$ (if, say, she placed probability 1 on his playing $a_1$), considers strictly more strategies than EFR, which makes such a requirement. Indeed, $\mathrm{ISR}(\tau^{CK}) = \{a_1, (In, a_2)\} \times \{b_1, b_2\}$, and $\mathrm{EFR}(\tau^{CK}) = ((In, a_2), b_2)$, as shown in greater detail in Penta (2012).

Next, consider the model in which $\Theta = \{0, 3\}$ and there is common certainty that $\theta = 0$, that is $\mathcal{T}^{CC} = \{\tau^{CC}\}$, such that type $\tau^{CC}$ assigns probability 1 to payoff state and opponents' type combination $(0, \tau^{CC})$.[4] The analysis in Penta (2012) constructs a set of types $\{\tau^m\}_{m \in \mathbb{N}}$ such that $\tau^m \to \tau^{CC}$ and $\text{ISR}(\tau^m) = \{(a_1, b_2)\}$ for all $m$. Now, if $\text{EFR}(\tau^{CC}) = \text{EFR}(\tau^{CK}) = \{((In, a_2), b_2)\}$ this would indeed be a violation of UHC, as EFR is a refinement of ISR, hence, for all types in the sequence $\{\tau^m\}_{m \in \mathbb{N}}$, $\text{EFR}(\tau^m) \subseteq \text{ISR}(\tau^m) = \{(a_1, b_2)\}$.

Notice however, that $\text{EFR}(\tau^{CC}) \neq \text{EFR}(\tau^{CK})$; the move from common knowledge to common certainty changes the set of EFR strategies. Although Polina must retain his belief that Alexei is rational, even after surprising events, it is *not* the case that she must retain his belief that Alexei places probability 1 on $\theta = 0$. Indeed, consider the following first-order beliefs: Alexei assigns probability 1 to $(b_2, 0)$ when $\theta = 0$, and probability 1 to $(b_2, 3)$ when $\theta = 3$, and Polina assigns probability 1 to $(a_1, 0)$ at the beginning of the game, and probability 1 to $((In, a_3), 3)$ when she observes $In$. Now, assume that Alexei's second-order belief assigns probability 1 to Polina's first-order beliefs above. Polina's second-order beliefs assign probability 1 to Alexei's first-order belief corresponding to $\theta = 0$ at the beginning of the game, and, when she observes $In$, assign probability 1 to Alexei's first-order belief corresponding to $\theta = 3$. Keeping this iteration, it is easy to check that we obtain a profile of conditional belief hierarchies that represent **CSBR** and initial common belief in $\theta = 0$ (when $\theta = 0$, something Alexei is informed about). Moreover, the unique best-response is indeed profile $(a_1, b_2)$. Hence, $(a_1, b_2) \in \text{EFR}(\tau^{CC})$. The mere fact that updated beliefs may take $\theta = 3$ into account, and the fact that it is commonly known that this is possible, mutes the restriction imposed by the high epistemic priority is given to rationality.

## 2.2  CHARACTERIZATION

This example illustrates two key issues. The first is that the aforementioned tension between rationalization and robustness is *not* present: EFR does not fail to be UHC in the example. The tension is mitigated since Polina can revise his beliefs about Alexei's actions without losing his belief in Alexei's rationality because there exists some state ($\theta = 3$) that allowed Alexei to be rational (but perhaps with incorrect beliefs) and take the particular action. This is a special case of a more general phenomena: when all common knowledge assumptions are dropped (i.e., under richness) this is true for *any* action: if an action could be rationalized by assuming an opponent is irrational, then it can be rationalized by changing only that opponents beliefs, but retaining his rationality. These observations can be made formal, and are best examined thorough the following result:

PROPOSITION 1. *Let $E \subseteq \Omega$ be a closed event such that $E \cap [q_i = \tau_i] \neq \emptyset$ for any standard type*

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[4]Penta actually considers a slightly different model in which P1 has private information, but the analysis is undistorted.

$\tau_i$. *Then, for any player $i$, the following correspondence is upper-hemicontinuous:*

$$\mathcal{S}_i^E : \quad \mathcal{T}_i \quad \rightrightarrows \quad S_i$$
$$\tau_i \quad \rightarrow \quad \mathrm{Proj}_{S_i}\left(R_i \cap E \cap [q_i = \tau_i]\right).$$

Proposition 1 provides a general way of verifying if the a solution concept is UHC. Note, by requiring that $E \cap [q_i = \tau_i]$ is non empty, we are implicitly requiring that the solution concept does not place any direct restrictions on the standard type space. In other words, this method of verification only works if the solution concept is well defined for all possible standard types.

Utilizing this result and the characterization of EFR (i.e., by setting $E = \mathbf{CSBR}$), it is straightforward to check that the result holds for EFR. This observation, along with the structure theorem in Penta (2012), easily delivers the following result:

COROLLARY 1. *EFR is UHC. Moreover, under richness,* ISR *and* EFR *coincide.*

Since under richness no strict refinement of ISR is UHC, and since EFR is both UHC and a refinement of ISR, the two solution concepts must coincide under richness. Intuitively this is because **CSBR** only induces a binding restriction on player $i$'s conjectures about his opponents strategies under common knowledge assumptions. This is an important distinction, since it implies that the tension between robustness and rationalization would only be assured to exist (i.e., under richness as to Penta's theorem) when rationalization had no effect at all.

## 3    PERSONAL BASE SPACES OF UNCERTAINTY AND S-ROBUSTNESS

The second issue raised by the example is the importance of considering the state space, and the inability to maintain predictions under embeddings. Indeed, the initial intuition of the example, that EFR is not UHC, is being driven not by a failure of convergence of standard types (since EFR it is in fact UHC), but by the failure to foresee the effects of changing the space of strategic uncertainty. In light of this, we propose a framework where players each have a subjective understanding of the space of payoff uncertainty. This accomplishes two goals simultaneously. First, it allows for the modeler to examine how changes (or mis-specifications) of the true state-space effect predictions. Second, it relaxes the restriction that the state-space is commonly known.

In regards to the second point, it is worth noting that while the richness assumption can be interpreted as relaxing all common knowledge restrictions regarding the payoffs associated with a given strategy, it still imposes, rather restrictively, that all players commonly know that all such common knowledge restrictions are relaxed. In other words, that all players commonly know that it is not common knowledge that any action is not dominant. Hence, by allowing the space of uncertainty to be subjective, we relax the imposition that the state space is commonly known.

To do this we define each players *directory*. Beginning with some fixed, objective state space $\Theta^0$, a directory for player $i$ assigns the subset of $\Theta^0$ that he understands to be the true space of

parameters, the subset he understands each of his opponents to understand to be the true space, and so on. Notationally, let $\mathcal{K}(\Theta^0)$ be the set of all non-empty compact subsets of $\Theta^0$, let $I$ be the set of players, and for each $n \in \mathbb{N}$ let $\Lambda^n = I^n$; finally set $\Lambda = \bigcup_{n \in \mathbb{N}} \Lambda^n$.

DEFINITION 1. *For each player $i \in I$, we say that $\Theta_i : \Lambda \to \mathcal{K}(\Theta^0)$ is a **directory for player i**, if for any $n \in \mathbb{N}$ and any $\lambda \in \Lambda^n$:*

> *D1. If $\lambda_1 = i$, then $\Theta_i(\lambda) = \Theta_i\left((\lambda_k)_{k=2}^n\right)$.*

> *D2. If $\lambda_k = \lambda_{k+1}$ for some $k < n$, then $\Theta_i(\lambda) = \Theta_i\left((\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_n)\right)$.*

*Let $\mathcal{D}(\Theta^0)$ be the set of all directories defined over $\Theta^0$.*

If $\Theta_i(j) = \Theta_1$ then player $i$ only takes into account $\Theta_1$ when conjecturing about $j$'s first order beliefs. Likewise, if $\Theta_i(j, k) = \Theta_2$, then player $i$ only takes into account $\Theta_2$ when conjecturing about $j$'s second order beliefs about $k$'s first order beliefs. Under our interpretation, D1 states that each player correctly understands his own understanding. Then, D1 implies that $\Theta_i(i, j) = \Theta_i(i, i, j) = \Theta_1$, etc. Similarly, D2 dictates that in each player's mind all other players understand the restriction imposed by D1. Hence, player 1's understands that player 2 understands his own understanding. Hence, $\Theta_i(i, j, j, j) = \Theta_i(i, j, j) = \Theta_1$, too. Note that each player $i$'s directory $\Theta_i$ induces a directory for all of his opponents in his mind, namely $\Theta_{j|i}$, where $\Theta_{j|i}(\lambda) = \Theta_i(j, \lambda)$ for any $\lambda \in \Lambda$ and any opponent $j$. As usual, we denote $\Theta_{-i|i} = \prod_{j \neq i} \Theta_{j|i}$.

Each directory induces a type space, $\mathcal{T}_i^{\Theta_i}$, in a natural way. $\mathcal{T}_i^{\Theta_i}$ is the set of types (drawn from the universal space over $\Theta^0$) whose beliefs are concentrated on the appropriate set assigned by the directory. Each player can be described by a pair $(\Theta_i, \tau_i)$, where $\tau_i \in \mathcal{T}_i^{\Theta_i}$.

By directly incorporating each players understanding of the state space, it becomes easy to define s-robustness.

DEFINITION 2. *Let $\mathcal{G}$ be a game with incomplete information. Then, we say that solution concept $\mathcal{S}_i : \mathrm{Graph}(\mathcal{T}_i^{(\cdot)}) \rightrightarrows S_i$ is **s-robust**, if for any player $i$, standard hierarchy $\tau_i$, and sequence $(\Theta_i^n, s_i^n)_{n \in \mathbb{N}} \subseteq \mathcal{D}(\Theta^0) \times S_i$ such that:*[5]

> *(i) $\tau_i \in \mathcal{T}_i^{\Theta_i^n}$ for all $n$,*

> *(ii) $s_i^n \in \mathcal{S}_i(\Theta_i^n, \tau_i)$ for all $n$,*

> *(iii) $\lim_{n \to \infty} \Theta_i^n = \Theta_i$, and,*

> *(iv) $\lim_{n \to \infty} s_i^n = s_i$,*

*then $s_i \in \mathcal{S}_i(\Theta_i, \tau_i)$.*

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[5]In the paper, we formally construct the topology on $\mathcal{D}(\Theta^0)$. Intuitively, it is the product topology generated by sequences of $\mathcal{K}(\Theta^0)$, itself endowed with the Hausdorff metric.

Just as UHC states that the solution concept should be consistent in the limit of successive approximations of player's true type, s-robustenss imparts the same requirement on approximation of the player's true understanding of the true state-space. One can model the situation where some state space, $\hat{\Theta}$, is commonly known by setting $\Theta_i(\lambda) = \hat{\Theta}$ for all $i$ and $\lambda$. It is within this specific case that we can model embeddings of the game and get a more direct comparison to informational invariance (Penta, 2012). S-robustness is a significantly weaker requirement than informational invariance, as it only requires the solution concept to be resilient to small mis-specifications. It should therefore not be too surprising that EFR is also s-robust.

PROPOSITION 2. *EFR is s-robust.*

This result, like its analog for UHC, is proven by examining the epistemic demands of the robustness criterion. However, this involves a large notational burden, and so, is left out of this abstract.

### 3.1    A Structure Theorem

Following Weinstein and Yildiz (2007) and Penta (2012), we provide a structure theorem. While the motivation is similar, the notion of directories raises a new question: is the generic uniqueness of dominance solvability that is being driven by relaxation of common knowledge conditions on strategic uncertainty still present under the relaxation of common knowledge conditions on payoff uncertainty. That is, can we still obtain a structure theorem if we allow players to understand the state space as not being rich.

We provide a (partial) affirmative answer. We find strictly weaker conditions under which a structure theorem still obtains.

ASSUMPTION 1 (Objective Richness). $\Theta^0$ *satisfies the Richness Condition.*[6]

Although we are assuming that the objective space of uncertainty is sufficiently rich, Assumption 1 contains no common knowledge restrictions. This is the direct result of the disentangling of the objective space of uncertainty with players personal base spaces, as given by the directory.

DEFINITION 3. *We say that $\Theta \in \mathcal{K}(\Theta^0)$ has **strongly generic payoffs** if for any strategy $s_i \in S_i$, there is a state $\theta \in \Theta$, and a profile of opponents strategies $s_{-i} \in S_{-i}$, such that $s_i$ is a strict best response to $s_{-i}$ at $\theta$.*

ASSUMPTION 2 (Common Knowledge of Genericity). *For all $i$ and $\lambda \in \Lambda$, $\Theta_i(\lambda)$ has strongly generic payoffs.*

Assumption 2 is, like richness, a common knowledge restriction on the space of uncertainty. However, it is a significantly weaker restriction. For example a simple Battle of the Sexes game would satisfy Assumption 2 but not richness. Indeed, Assumptions 1 and 2 allow for the situation where it is commonly known that *no* action is dominant.[7]

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[6]As defined in Penta (2012).

[7]Note that if $\Theta^0$ is rich and commonly known (i.e., as in previous literature) then Assumptions 1 and 2 are implied.

PROPOSITION 3. *Assumptions 1 and 2, and strict refinement of EFR is not UHC. Moreover, the set $\{(\Theta_i, \tau_i) : |\text{EFR}_i(\Theta_i, \tau_i)| = 1\}$ is open and dense in $\text{Graph}(\mathcal{T}_i^{(\cdot)})$.*

Proposition 3 shows that richness need not be commonly known to arrive at generic dominance solvability. The result relies on the ability to construct a sequence of directories that impose richness at increasingly high order understandings; then using Assumption 2 to cascade the effect down without losing convergence.

## 4 CONCLUSION

In this paper we show a formal connection between the epistemic characterization of a solution concept and its robustness to the misspecification of parameters. This provides both an important conceptual link and a direct method for checking robustness when the epistemic characterization is known. We use this result to show that EFR is UHC. We also present a new framework that relaxes the common knowledge restrictions regarding the *space* of payoff parameters. Then, we propose a new type of robustness, s-robustness, to modeling errors of the player understanding of the space of uncertainty, which is of particular importance in dynamic environments. We then characterize this notion through an our epistemic framework and show that EFR is also s-robust. Finally, we provide a structure theorem for EFR with personal spaces of uncertainty that shows that no common knowledge assumptions regarding the existence of dominance states are required to achieve generic dominance solvability.

## REFERENCES

Battigalli, Pierpaolo (1996). "Strategic rationality order and the best rationalization principle". *Games and Economic Behavior* **13**, 178–200.

Battigalli, Pierpaolo and Marciano Siniscalchi (1999). "Hierarchies of conditional beliefs and interactive epistemology in dynamic games". *Journal of Economic Theory,* **88**, 188–230.

Brandenburger, Adam and Eddie Dekel (1993). "Hierarchies of beliefs and common knowledge". *Journal of Economic Theory* **59**, 189–198.

Pearce, David G. (1984). "Rationalizable strategic behavior and the problem of perfection". *Econometrica* **52**, 1029–1050.

Penta, Antonio (2012). "Higher order uncertainty and information: static and dynamic games". *Econometrica* **80**, 631–660.

Weinstein, Jonathan and Muhamet Yildiz (2007). "A structure theorem for rationalizability with application to robust predictions of refinements". *Econometrica* **75**, 365–400.