Online Appendix: Single Stage Image Conscious Choice

The interpretation of two stage choice is that $\mathscr{C}_1$ represents a choice over 2CPs that is made in the absence of image concerns. Hence, in many scenarios, this choice function will not be observable. This section considers the image conscious model when only second stage choice is accessible to the modeler; it posits axioms only on $\mathcal{C}_2$ equivalent to (C2) of the IC representation. I assume $X = \mathcal{L}$, that consumption takes place in a linear space.

Limited observability bears a cost. First, the uniqueness of Γ is no longer. Second, the axiomatic structure and concomitant proof rely more directly on technical assumptions, and so, are correspondingly more involved. This latter point is self evident, but to understand the failure of uniqueness, consider the following.

Say $I, J \in \mathbb{I}$ are *directly* comparable if there is a D such that $I = I^x_D$, $J = I^y_D$ and either x or y is in $\mathcal{C}_2(D)$. When I and J are directly comparable (and, say x is chosen), we have a bound on the utility difference between I and J in terms of consumption utility:

$$\Gamma(I) - \Gamma(J) \geq u(y) - u(x).$$

Say that $I, J \in \mathbb{I}$ are *indirectly comparable* if they are contained in the transitive closure of the direct comparability relation. If two images are not indirectly comparable, then there is no restriction imposed by the observed choices on the relative values of the images. Indirect comparability is a non-trivial equivalence relation; Γ in the resulting representation can be normalized independently across the classes of this equivalence relation.

B.1 Axioms

Scaling a choice problem may result in non-linear tradeoffs. As λ increases, choice from λD places more importance on consumption utility. The first axiom allows $\mathcal{C}_2(\lambda D)$ to vary non-linearly in λ, but ensures that deviations are consistent with increasing importance on consumption utility.

Axiom 1°—Scale Acyclicity. Let $0 < \lambda < \lambda' < \lambda''$ and $D \in \mathcal{D}$. If $x \in \frac{1}{\lambda}\mathcal{C}_2(\lambda D) \cap \frac{1}{\lambda''}\mathcal{C}_2(\lambda'' D)$ then $x \in \frac{1}{\lambda'}\mathcal{C}_2(\lambda' D)$.

In the limit, as $\lambda \to \infty$, only consumption utility matters. Indeed, this is the manner in which u might be identified. To get at this, we can define the following map, which is well defined given A1° and the finiteness of each D:

$$\mathcal{C}_2^\infty : D \mapsto \lim_{\lambda \to \infty} \frac{1}{\lambda}\mathcal{C}_2(\lambda D).$$

To identify u we need $\mathcal{C}_2^\infty$ to be well behaved; ideally this would just entail the imposition of WARP. Unfortunately, it is possible that $u(x) = u(y)$ but $y \neq \mathcal{C}_2^\infty(\{x, y\})$; this happens whenever $\Gamma(I^x_{\{x,y\}}) > \Gamma(I^y_{\{x,y\}})$. To deal with this, we impose WARP on perturbed choice problems.

Axiom 2°—Sequential Limit Consistency. Let $\{D_k\}_{k\in\mathbb{N}} \subset \mathcal{D}$ converge to $D \supseteq \{y, x\}$ such that $x \in \mathcal{C}_2^\infty(D_k)$ for all $k \in \mathbb{N}$. Then for any D' with $y \in \mathcal{C}_2^\infty(D')$ there exists a sequence $D'_k \to D'$ such that $x \in \mathcal{C}_2^\infty(D'_k \cup \{x\})$ for all k.

Translation invariance (A3) remains, but is transcribed here for completeness.

Axiom 3°—Translation Invariance. For all $x \in X$ and $D \in \mathcal{D}$,

$$\mathcal{C}_2(D + x) = \mathcal{C}_2(D) + x$$

With these three axioms, u can be identified.

Lemma 4. *If $\mathcal{C}_2$ satisfies A1°—3° then there exists a linear $u : X \to \mathbb{R}$ such that*

$$\mathcal{C}_2^\infty(D) \subseteq \arg\max_D u.$$

Moreover, u is unique up to positive linear transformations.

Proof. In section B.2. ∎

From $\mathcal{C}_2$ we can define $\succcurlyeq \subset (X \times \mathbb{I}) \times (X \times \mathbb{I})$ via $(x, I) \succcurlyeq (y, J)$ iff there exists a $D \supseteq \{x, y\}$ with $I^x_D = I$ and $I^y_D = J$, and such that $x \in \mathcal{C}_2(D)$. The next axioms place restrictions on $\succcurlyeq$ but these can be translated back into choice behavior in the obvious, but tedious, manner. Per normal let $\succ$ and $\sim$ denote the asymmetric and symmetric components.

The relation $\succcurlyeq$ will necessarily be highly incomplete; for example, images with overlapping relative interiors will never be comparable. Because of this, $\succcurlyeq$ will not be transitive; it should, however, be extendable to a transitive relation.

Axiom 4°—Acyclicity. $\succ$ is acyclic.

Finally, we impose three restrictions that relate the choice over $\succcurlyeq$ to the consumption utility as identified by Lemma 4: *monotonicity* states that if $(x, I) \succcurlyeq (y, J)$ and $u(x') > u(x)$ then not $(y, J) \succ (x', I)$—ceteris paribus, more consumption is better; *boundedness* states that $(x, I) \succ (y, J)$ cannot hold for all x—I cannot be 'infinitely' better than J; *continuity* states that if $u(x_n) \to u(x)$ and $(x_n, I) \succcurlyeq (y, J)$ for all n, then not $(y, J) \succ (x, I)$—preferences cannot be reversed in the limit.

In the proof of the representation theorem, we will extend $\succcurlyeq$ to a complete binary relation, showing these properties still hold; as such, it is helpful to define things for a general relation R defined over $(X \times \mathbb{I})$.

Definition. Let $v : X \to \mathbb{R}$. Call a relation R (with asymmetric component S) defined over $(X \times \mathbb{I})$ *v-monotone* if whenever

1. $v(z) > 0$ and $(x, I)R(y, J)$, or,
2. $v(z) \geq 0$ and $(x, I)S(y, J)$,

then not $(y, J)R(x + z, I)$.

Definition. Let $v : X \to \mathbb{R}$. Call a relation R defined over $(X \times \mathbb{I})$ *v-bounded* if for all $I, J \in \mathbb{I}$, it is true that $\inf\{v(z) \mid (z, I)R(\mathbf{0}, J)\} > -\infty$.

Definition. Let $v : X \to \mathbb{R}$. Call a relation R defined over $(X \times \mathbb{I})$ *v-continuous* if for all $I, J \in \mathbb{I}$, if whenever there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ such that $v(x_n) \to 0$ and $(x_k, I)R(y, J)$ for all k, then for any x with $v(x) = 0$, if $(y, J)R(x, I)$ then $(x, I)R(y, J)$.

Let $\succcurlyeq^{TC}$ denote the transitive closure of $\succcurlyeq$.

Axiom 5°—Consumption Regularity. The relation $\succcurlyeq^{TC}$ is u-monotone, u-continuous, and u-bounded.

These axioms are equivalent to the existence of an Image Conscious representation $\langle u, \Gamma\rangle$ which represents $\mathcal{C}_2$ as (C2).

Theorem B.1. *The following are equivalent:*

1. *$\mathcal{C}_2$ satisfies A1°—5°*
2. *$\mathcal{C}_2$ has an image conscious representation $\langle u, \Gamma\rangle$.*

Moreover, u is unique up to positive linear translations and Γ is unique up-to an additive constant within each equivalence class generated by the indirect comparability relation.

Proof. In section B.3. ∎

In contrast to the proof of Theorem 2.1, the proof of Theorem B.1 is rather involved. The main difficulty surrounds the intrinsic incompleteness of the induced preference relation on $X \times \mathbb{I}$, owing to the geometric dependence between the set of consumption alternatives and the consequent images. Indeed, imagine that some complete $\succcurlyeq^\star$ over $X \times \mathbb{I}$ was magically identified and preserved the relevant structure and extended $\succcurlyeq$. Then, fixing $I^* \in \mathbb{I}$ and setting $\Gamma(I^*) = 0$, we can recover the entirety of Γ is by simply setting:

$$\Gamma : I \mapsto -u(x^I)$$

where x^I is a consumption alternative such that $(x^I, I) \sim (\mathbf{0}, I^*)$. Such an alternative exists by the u-boundedness and u-continuity assumptions, and its utility is unique by u-monotonicity. Translation invariance and transitivity then ensure the resulting $\langle u, \Gamma\rangle$ actually represents $\succcurlyeq^\star$, and hence $\mathcal{C}_2$.

Guaranteeing that $\succcurlyeq$ can be extended to a complete $\succcurlyeq^\star$ (while preserving the axiomatic structure) turns out to be pain, but mostly for technical reasons. The relatively simple core idea is as follows: we can first extend $\succcurlyeq$ by adding comparisons that were not observed by $\mathcal{C}_2$ but must hold because of transitivity, monotonicity, or continuity. The resulting relation extends $\succcurlyeq$ because of A4° and A5°. Still, there will be images I and J such that no x satisfies $(x, I) \sim (\mathbf{0}, J)$. What can we do? Just pick some x and extend the relation by adding $(x, I) \sim (\mathbf{0}, J)$ (and then again adding all the consequences of transitivity, monotonicity, or continuity). Repeating the process for different I's and J's creates a partial order of extensions of $\succcurlyeq$, which, by Zorn's Lemma, has maximal element that must be complete.[7]

This process also elucidates the exact nature of non-uniqueness. If two images are initially comparable, that is there exists an x and y such that $(x, I) \sim (y, J)$ is implied by the initial choice function, then the difference between $\Gamma(I)$ and $\Gamma(J)$ is identified (up to a common normalization) by the difference between $u(x)$ and $u(y)$. Thus, identification is made over the equivalence classes of initially comparable images (that comparability is an equivalence relation is Lemma 7(i)), but, these equivalence classes can be independently normalized.

B.2 Proof of Lemma 4

Define the preference relation, $\dot{\succcurlyeq}$, on X as follows: $x \dot{\succcurlyeq} y$ if there exists a $\{D_k\}_{k\in\mathbb{N}} \subset \mathcal{D}$ converging to $D \supseteq \{y, x\}$ such that $x \in \mathcal{C}_2^\infty(D_k)$ for all k. We claim that $\dot{\succcurlyeq}$ is an expected utility preference; this would complete the lemma, for if $x \in \mathcal{C}_2^\infty(D)$ then $x \dot{\succcurlyeq} y$ by taking the constant sequence D, and hence $x \in \arg\max_D u$ for any representation of $\dot{\succcurlyeq}$.

Completeness. Fix $x, y \in X$, and take a sequence $\{y_k\}_{k\in\mathbb{N}}$ converging to y. Since $\mathcal{C}_2$ is non-empty there exists a subsequence (w.l.o.g., indexed by the same k) such that for all k either $x \in \mathcal{C}_2^\infty(\{x, y_k\})$ or $x \notin \mathcal{C}_2^\infty(\{x, y_k\})$. If it is the former, we are done and $x \dot{\succcurlyeq} y$. If it is the latter, we can appeal to translation invariance, and for each k, shift by $y - y_k$ to obtain a sequence $\{x + y - y_k, y\}$ such that y is always chosen, so $y \dot{\succcurlyeq} x$.

Transitivity. Let $x \dot{\succcurlyeq} y$ and $y \dot{\succcurlyeq} z$. Consider the choice problem $D = \{x, y, z\}$. If $x \in \mathcal{C}_2^\infty(D)$ then $x \dot{\succcurlyeq} z$ and we are done. If $y \in \mathcal{C}_2^\infty(D)$, then we can appeal to A2° to obtain a sequence $D_k \to D$ such that $x \in \mathcal{C}_2^\infty(D_k \cup \{x\})$ for all k, hence $x \dot{\succcurlyeq} z$ (notice, $x \dot{\succcurlyeq} y$ definitionally implies the antecedent for A2°). Finally, assume $z \in \mathcal{C}_2^\infty(D)$. Then by the above reasoning, we have a sequence $D_k \to D$ such that $y \in \mathcal{C}_2^\infty(D_k \cup \{y\})$. Now since $x \dot{\succcurlyeq} y$, we can, for each D_k find a further sequence $D^k_{k'} \to D_k \cup \{y\}$ such that $x \in \mathcal{C}_2^\infty(D^k_{k'} \cup \{x\})$ for all $k, k' \in \mathbb{N}$. For each $m \in \mathbb{N}$, set $\hat{D}_m$ to be the first element of $\{D^m_{k'}\}_{k'\in\mathbb{N}}$ such that $d_H(D^m_{k'} - D_m) \leq \frac{1}{m}$. This is a sequence converging to D and with x always chosen.

Continuity. Let $\{y_k\}_{k\in\mathbb{N}}$ converge to y and be such that $x \dot{\succcurlyeq} y_k$ for all k. Then by definition, we have a sequence of sequences $\{\{D^k_{k'}\}_{k'\in\mathbb{N}}\}_{k\in\mathbb{N}}$ such that $D^k_{k'} \to D_k$ for all k and $x \in \mathcal{C}_2^\infty(D^k_{k'} \cup \{x\})$. As above, we can find a sequence of sets converging

[7] This is one of the many instantiations of Szpilrajn's extension theorem with additional structure being preserved by the extension.

to $\{x, y\}$ such that x is chosen from each. Closure of the lower contour sets is the analogous.

INDEPENDENCE. Let $x \dot{\succsim} y$; we have $\{D_k\}_{k \in \mathbb{N}} \subset \mathcal{D}$ converging to $D \supseteq \{y, x\}$ such that $x \in \mathcal{C}_2^\infty(D_k)$ for all k. Set $\lambda \in (0,1)$ and $z \in X$. We know $x \in \mathcal{C}_2^\infty(D_k)$ indicates by definition that $x \in \lim_{\gamma \to \infty} \frac{1}{\gamma} \mathcal{C}_2(\gamma D_k) = \frac{1}{\lambda} \lim_{\gamma \to \infty} \frac{1}{\gamma} \mathcal{C}_2(\gamma \lambda D_k)$ or, multiplying by λ, that $\lambda x \in \mathcal{C}_2^\infty(\lambda D_k)$. Then by A3 we have that $\lambda x + \lambda' z \in \mathcal{C}_2^\infty(\lambda D_k + \lambda' z)$. Since $\lambda D_k + \lambda' z$ converges to $\lambda D + \lambda' z$, we have that $\lambda x + \lambda' z \dot{\succsim} \lambda y + \lambda' z$, as desired. ∎

B.3 Proof of Theorem B.1

Definition. Call a relation R defined over $(X \times \mathbb{I})$ *translation invariant* if for all $x, y, z \in X$ and $I, J \in \mathbb{I}$ we have $(x, I)R(y, J)$ if and only if $(x + z, I)R(y + z, J)$.

Since $\mathcal{C}_2$ is translation invariant, $\succsim^{TC}$ is as well.

Definition. Let R_1 and R_2 denote two binary relations on a set X (with asymmetric components S_1 and S_2). We say that R_1 *extends* R_2 if $R_2 \subseteq R_1$ and if xS_2y then also xS_1y.

That is, R_1 includes all comparisons that R_2 includes, but does not break any asymmetric comparison into a symmetric one. Because $\succ$ is acyclic, $\succsim^{TC}$ extends $\succsim$.

Definition. Let $v : X \to R$. Call a relation R (with asymmetric component S) defined over $(X \times \mathbb{I})$ *strongly-v-monotone* if $(x + z, I)R(x, I)$ whenever $v(z) \geq 0$ and $(x + z, I)S(x, I)$ whenever $v(z) > 0$.

Notice that a transitive and strongly-v-monotone relation is also v-monotone. Let $\succsim^{\#}$ denote $\succsim^{TC} \cup \{((x + z, I), (x, I)) \mid x, z \in X, u(z) \geq 0, I \in \mathbb{I}\}$ and $\succsim^*$ its transitive closure.

Lemma 5. *$\succsim^*$ is reflexive, transitive, translation invariant, strongly-u-monotone, u-bounded, u-continuous and extends $\succsim$.*

Proof. That $\succsim^*$ is reflexive follows from the addition of $((x + \mathbf{0}, I), (x, I))$; that it is transitive is immediate in that it is a transitive closure; that it is translation invariant follows from that translation invariance of $\succsim^{TC}$ and the fact that all added relations are added in a translation invariant way. Next, notice that $\succsim^{\#}$ is obviously u-monotone and u-bounded. Further, notice that, because of u-monotonicity, the addition comparisons added to $\succsim^{TC}$ cannot turn a strict preference into an indifference; hence $\succsim^{\#}$ extends $\succsim^{TC}$.

$\succsim^*$ EXTENDS $\succsim^{TC}$. Assume this was not the case so that we have a finite sequence

$$(x_1, I_1) \succsim^{\#} (x_2, I_2) \succsim^{\#} \ldots \succsim^{\#} (x_m, I_m)$$

such that $(x_m, I_m) \succ^{TC} (x_1, I_i)$.

Notice that for at least one $j < m$ we have

$$(x_{j+1}, I_{j+1}) = (x_j - z'_j, I_j) \tag{B.1}$$

for some z'_j with $u(z'_j) \geq 0$. If this was not the case, then each relation holds also for $\succsim^{TC}$, indicating that $(x_1, I_1) \succsim^{TC} (x_m, I_m)$, a clear contradiction.

So, let $B \subseteq \{1 \ldots m\}$ denote the non-empty set of indices where (B.1) holds for some $z'_j \in X$ with $u(z'_j) \geq 0$. We have:

$$(x_1, I_1) \succsim^{\#} \ldots \succsim^{\#} (x_j, I_j) \succsim^{\#} (x_{j+1}, I_{j+1}) = (x_j - z'_j, I_j) \succsim^{\#} \ldots \succsim^{\#} (x_m, I_m)$$

By translation invariance, we can, for the lowest $j \in B$, add z'_j from the all terms after $j+1$ to obtain

$$(x_1, I_1) \succsim^{\#} \ldots \succsim^{\#} (x_{j-1}, I_{j-1}) \succsim^{\#} (x_j, I_j) = (x_{j+1} + z'_j, I_j) \succsim^{\#} \ldots \succsim^{\#} (x_m + z'_j, I_m)$$

Continuing to delete terms in this manner for all $i \in B$, we are left with a sequence, contained within $\succsim^{TC}$, asserting $(x_1, I_1) \succsim^{TC} (x_m + \sum_{i \in B} z'_i, I_m)$, contradicting u-monotonicity.

STRONG-u-MONOTONICITY. By way of contradiction, assume that by taking the transitive closure we generate a violation of strong-u-monotonicity. That $(x+z, I) \succsim^{*} (x, I)$ is immediate, so assume this holds only weakly: for some (x, I), $(x, I) \succsim^{*} (x+z, I)$ for $z \in X$ with $u(z) > 0$.

This requires a sequence of comparisons

$$(x, I) = (x_1, I_1) \succsim^{\#} (x_2, I_2) \succsim^{\#} \ldots \succsim^{\#} (x_m, I_m) = (x + z, I)$$

As above, let $B \subseteq \{1 \ldots m\}$ denote the set of indices where (B.1) holds for some $z'_j \in X$ with $u(z'_j) > 0$. If $B = \varnothing$, we could exhibit this sequence via $\succsim^{TC}$, violating u-monotonicity. Therefore, as above, we can appeal to translation invariance to delete terms for each $i \in B$: the resulting sequence is contained within $\succsim^{TC}$ and asserts $(x, I) \succsim^{TC} (x + z + \sum_{j \in B} z'_j, I)$, contradicting u-monotonicity.

u-BOUNDEDNESS. Fix $z \in X$ and $I, J \in \mathbb{I}$. Let $(z, I) \succsim^{*} (\mathbf{0}, J)$ so that there exists a finite sequence

$$(z, I) = (x_1, I_1) \succsim^{\#} (x_2, I_2) \succsim^{\#} \ldots \succsim^{\#} (x_m, I_m) = (\mathbf{0}, J)$$

Once again, let $B \subseteq \{1 \ldots m\}$ denote the set of indices where (B.1) holds for some $z'_j \in X$ with $u(z'_j) > 0$. If $B = \varnothing$, then this sequence would exist within $\succsim^{TC}$, indicating $\inf\{u(y) \mid (y, I) \succsim^{TC} (\mathbf{0}, J)\} \leq u(z)$. If B is not empty, we can proceed by the usual trick to conclude $(z, I) \succsim^{TC} (\sum_{j \in B} z'_j, J)$, or by translation invariance, $(z - \sum_{j \in B} z'_j, I) \succsim^{TC} (\mathbf{0}, J)$. This indicates that $\inf\{u(y) \mid (y, I) \succsim^{TC} (\mathbf{0}, J)\} \leq u(z) - \sum_{j \in B} u(z'_j) \leq u(z)$. Since $u(z)$ was arbitrary, the infimum with respect to $\succsim^{*}$ can be no lower than with respect to $\succsim^{TC}$, which was bounded below.

u-CONTINUITY. Let x be such that $u(x) = 0$ and a sequence $\{x_k\}_{k \in \mathbb{N}}$ converging with $u(x_n) \to 0$. Let (y, J) be such that $(x_k, I) \succsim^{*} (y, J)$ for all k and $(y, J) \succsim^{*} (x, I)$. We can use the now standard trick to find following relations:

$$(x_k - z_k, I) \succsim^{TC} (y, J)$$

for each k, and

$$(y, J) \succsim^{TC} (x + z, I)$$

with $u(z_k) \geq 0$ for each k and $u(z) \geq 0$. Necessarily, $u(z) = 0$, or else, eventually $u(x_k - z_k) < u(x + z)$ creating a violation of u-monotonicity. For the same reason, it must be that for all $u(z_k) \leq u(x_k)$. Hence $u(x_k - z_k) \to 0$, and by u-continuity $(x + z, I) \succsim^{TC} (y, J)$. Now, since $u(-z) = 0$ we have that $(x, I) \succsim^{\#} (x + z, I) \succsim^{\#} (y, J)$, and hence, $(x, I) \succsim^{*} (y, J)$. ★

Definition. Let $v : X \to R$. Call a relation R defined over $(X \times \mathbb{I})$ *strongly-v-continuous* if for all $I, J \in \mathbb{I}$, if whenever there exists a sequence $\{x_k\}_{k \in \mathbb{N}}$ such that $v(x_n) \to 0$ and $(x_k, I) R (y, J)$ for all k, then for any x with $v(x) = 0$, $(x, I) R (y, J)$.

Lemma 6. *Let $\succsim^{+}$ be the transitive closure of*

$$\succsim^{*} \cup \Big\{ ((x, I), (y, J)) \mid \textit{Exists } \{z_k\}_{k \in \mathbb{N}}, u(z_n) \to 0, (x + z_k, I) \succsim^{*} (y, J) \textit{ for all } k \Big\}$$

Then $\succsim^{+}$ is reflexive, transitive, translation invariant, strongly-u-monotone, u-bounded, strongly-u-continuous and extends $\succsim^{*}$ (hence $\succsim$).

Proof. Reflexivity, transitivity, translation invariance, and strong-u-continuity are all immediate.

$\succsim^{+}$ EXTENDS $\succsim^{*}$. Let $(y, J) \succsim^{+} (x, I)$. Then there must exist a sequence $\{(x_j, I_j)\}_{j=1}^{m}$, with $(x_1, I_1) = (y, J)$ and $(x_m, x_m) = (x, I)$, and such that for each $j < m$ there is a sequence $\{z_k^j\}_{k \in \mathbb{N}}, u(z_k^j) \to 0$ (possibly the constant sequence $\mathbf{0}$, if $(x_i, I_i) \succsim^{*} (x_{i+1}, I_{i+1})$) such that $(x_j + z_k^j, I_j) \succsim^{*} (x_{j+1}, I_j)$ for all k. It is without loss of generality to assume that $u(z_k^j) \geq 0$ for all j, k. But notice we have

$$(x_1 + \sum_{i=1}^{m} z_k^i, I_1) \succsim^{*} (x_2 + \sum_{i=2}^{m} z_k^i, I_2) \succsim^{*} \ldots (x_j + \sum_{i=j}^{m} z_k^i, I_j) \succsim^{*} \ldots \succsim^{*} (x_m, I_m)$$

for each k. This indicates that $(y + \sum_{i=1}^{m} z_k^i, J) \succsim^{*} (x, I)$ where $u(\sum_{i=1}^{m} z_k^i) \to 0$. So by the u-continuity of $\succsim^{*}$, we cannot have $(x, I) \succ^{*} (y, J)$: therefore $\succsim^{+}$ extends $\succsim^{*}$.

STRONG-u-MONOTONICITY. We have that $(x, I) \succsim^{+} (x + z, I)$ immediately; since $\succsim^{+}$ extends $\succsim^{*}$ it cannot be that $(x + z, I) \succsim^{+} (x, I)$.

u-BOUNDEDNESS. Fix $x \in X$ and $I, J \in \mathbb{I}$. Let $(x, I) \succsim^{+} (\mathbf{0}, J)$. Using the same trick as in the proof of extension, we can find a (finite) collection of sequences, $\{\{z_k^j\}_{k \in \mathbb{N}}\}_{j=1}^{m}$ such that $u(z_k^j) \to 0$ for each j and $(x + \sum_{i=1}^{m} z_k^i, I) \succsim^{*} (\mathbf{0}, J)$. Since $u(x + \sum_{i=1}^{m} z_k^i) \to u(x)$ we have that $u(x) \geq \inf\{v(z) \mid (z, I) \succsim^{*} (\mathbf{0}, J)\}$. ★

Lemma 7. *Let $v : X \to \mathbb{R}$ be a linear function and R be a preorder on $(X \times \mathbb{I})$ that is translation invariant, strongly-v-monotone, v-bounded and strongly-v-continuous. Call $I, J \in \mathbb{I}$ R-*comparable *if there exists an $x \in X$ such that $(x, I) R (\mathbf{0}, J)$ and $(\mathbf{0}, J) R (x, I)$. Then*

1. *R-comparability is an equivalence relation.*

2. *If I, J are not comparable, then there exists $\bar{x} \in X$ such that neither $(\bar{x}, I)R(\mathbf{0}, J)$ nor $(\mathbf{0}, J)R(\bar{x}, I)$*

3. *If $\bar{I}, \bar{J}$ are not comparable, and $\bar{x}$ is as in (2), then, R^* defined as the transitive closure of $R^\# = R \cup \{(\bar{x}+z, \bar{I})R(z, \bar{J}), (z, \bar{J})R(\bar{x}+z, \bar{I}) \mid z \in X\}$ is also a translation invariant, strongly-v-monotone, v-bounded, and strongly-v-continuous preorder that extends R.*

Proof. (1) Reflexivity is immediate. Symmetry follows from translation invariance. Transitivity follows from the transitivity and translation invariance of R, in the obvious way.

(2) Consider the sets $\{v(x) \mid (x, I)R(\mathbf{0}, J)\} \subseteq \mathbb{R}$ and $\{v(x) \mid (\mathbf{0}, J)R(x, I)\} \subseteq \mathbb{R}$. By strong-$v$-monotonicity, these are (possibly empty) intervals, the former upward-closed and the later downward-closed. By v-boundedness neither is $\mathbb{R}$ itself. By strong-v-continuity they are closed. If these intervals overlap, then I and J are comparable, so assume they do not overlap. Since $\mathbb{R}$ is connected, there must be a point not in either interval.

(3) Fix $\bar{I}, \bar{J}$ that are not comparable for some R. Let $R^\#$ and R^* be as in the statement of the Lemma, and let S, $S^\#$ and S^* denote respective asymmetric components. Reflexivity, transitivity, and translation invariance are immediate.

R^* extends R. Assume it did not: there exists (x, I) and (y, J) such that $(x, I)S(y, J)$ but $(y, J)R^*(x, I)$. This last relations indicates the existence of a sequence,

$$(y, J)R^\#(x_1, I_1)R^\# \dots R^\#(x_m, I_m)R^\#(x, I).$$

As in the proof of Lemma 5, there must be some relation not contained in R, so that for some $j < m$, we have $(x_j, I_j) = (\bar{x}+z, \bar{I})$ and $(x_{j+1}, I_{j+1}) = (z, \bar{J})$ (or vice versa, with an analogous proof following). It is without loss of generality that there is a single index j such that $(x_j, I_j), (x_{j+1}, I_{j+1}) \notin R$.[8] Capitalizing on the fact that R is transitive, we can further delete all other relations, we have

$$(y, J)R(\bar{x}+z, \bar{I})R^\#(z, \bar{J})R(x, I).$$

We can split the above sequence and swapping the order, recall $(x, I)S(y, J)$, leaving us with:

$$(z, \bar{J})R(x, I)S(y, J)R(\bar{x}+z, \bar{I}).$$

[8]To see why: consider the following sequence

$$(\bar{x}+z, \bar{I})R^\#(z, \bar{J})R(x_i, I_i)R \dots (x_{i+j}, I_{i+j})R(\bar{x}+z', \bar{I})R^\#(z', \bar{J})$$

where the "..." contains only R relations. If $v(z') < v(z)$, then $(z, \bar{J})S(z', \bar{J})$ by strong-v-monotonicity, and we can make the same inference deleting one $R^\#$ relation. If $v(z') \geq v(z)$ we have a contradiction: we have

$$(z', \bar{J})R(z, \bar{J})R(x_i, I_i)R \dots (x_{i+j}, I_{i+j})R(\bar{x}+z', \bar{I})$$

where the first relation is from strong-v-monotonicity. This implies, however, via translation invariance, that $(\mathbf{0}, \bar{J})R(\bar{x}, \bar{I})$, a contradiction to the definition of $\bar{I}, \bar{J}$ and $\bar{x}$.

By the translation invariance of R, this implies $(\mathbf{0}, \bar{J})R(\bar{x}, \bar{I})$, a contradiction to the definition of $\bar{I}, \bar{J}$ and $\bar{x}$.

STRONG-v-MONOTONICITY. We have that $(x, I)R^*(x + z, I)$ immediately; since R^* extends R it cannot be that $(x + z, I)R^*(x, I)$.

v-BOUNDEDNESS. Fix $I, J \in \mathbb{I}$. Define the following constants.

$$a_1 = \inf\{v(z) \mid (z, I)R(\mathbf{0}, J)\}$$
$$a_2 = \inf\{v(z) \mid (z, I)R(\bar{x}, \bar{I})\}$$
$$a_3 = \inf\{v(z) \mid (z, \bar{J})R(\mathbf{0}, J)\}$$

Let $(x, I)R^*(\mathbf{0}, J)$. If this relation can be exhibited by R, then $u(x) \leq a_1$. So, to make things interesting, assume it cannot be; by the above arguments we can find the following sequence of relations:

$$(x, I)R(\bar{x} + z', \bar{I})R^{\#}(z', \bar{J})R(\mathbf{0}, J)$$

By the definition of a_2, and translation invariance, the first relation indicates that $u(x) \geq a_2 + v(z')$. The last relation likewise indicates that $u(z') \geq a_3$; hence $u(x) \geq a_2 + a_3$. In either case, $u(x) \geq \min\{a_1, a_2 + a_3\}$ and is hence bounded from below.

u-CONTINUITY. Let x be such that $u(x) = 0$ and a sequence $\{x_k\}_{k \in \mathbb{N}}$ converging with $u(x_n) \to 0$. Let (y, J) be such that $(x_k, I)R^*(y, J)$ for all k. By taking a subsequence if necessary, it is without loss of generality to restrict attention to the case where either $(x_k, I)R(y, J)$ for all k or not $(x_k, I)R(y, J)$ for all k. The former is a direct application of the strong-u-continuity of R. Assume the latter: we have,

$$(x_k, I)R(\bar{x} + z, \bar{I})R^{\#}(z, \bar{J})R(y, J)$$

By the strong-u-continuity of R, $(x, I)R(\bar{x} + z, \bar{I})$ and hence $(x, I)R^*(y, J)$. ★

Lemma 8. *There exists a translation invariant, strongly-u-monotone, u-bounded, and strongly-u-continuous preorder on $(X \times \mathbb{I})$, $\succcurlyeq^\star$, that extends $\succcurlyeq^+$ such that all $I, J \in \mathbb{I}$ are $\succcurlyeq^\star$ comparable.*

Proof. Consider the set of all translation invariant, strongly-u-monotone, u-bounded, and strongly-u-continuous preorders on $(X \times \mathbb{I})$ that extend $\succcurlyeq^+$. Say that $R \leq R'$ if R' extends R. Clearly, $\leq$ is a partial order, and every chain (totally ordered subset) is bounded by its union. Hence, we can apply Zorn's lemma to conclude the existence of a maximal (with respect to the extension induced order) relation over $(X \times \mathbb{I})$. Call this relation $\succcurlyeq^\star$. By Lemma 7 part (iii), all I, J are $\succcurlyeq^\star$-comparable, or else we could find a further extension, contradicting the maximality of $\succcurlyeq^\star$. ★

For each $I \in \mathbb{I}$, define let x^I denote an element such that $(x^I, I) \sim^\star (\mathbf{0}, \mathbf{0})$. Then define $\Gamma : \mathbb{I} \to \mathbb{R}$ by

$$\Gamma : I \mapsto -u(x^I) \tag{B.2}$$

We now claim that $\langle u, \Gamma \rangle$ forms an IC representation for $\mathcal{C}_2$. Take a menu $D \in \mathcal{D}$. Assume that $x \in \mathscr{C}_1(D)$. Then $(x, I_D^x) \succcurlyeq (y, I_D^y)$ for all $y \in D$. Since $\succcurlyeq^\star$ extends $\succcurlyeq^+$

(Lemma 8), hence $\succcurlyeq$ (Lemma 5), we have $(x, I_D^x) \succcurlyeq^\star (y, I_D^y)$ for all $y \in D$. Therefore, by definition, and translation invariance,

$$(x - x^{I_D^x}, \mathbf{0}) \sim^\star (x, I_D^x) \succcurlyeq^\star (y, I_D^y) \sim^\star (y - x^{I_D^y}, \mathbf{0}).$$

Moreover, by strong-u-monotonicity this indicates that

$$u(x - x^{I_D^x}) \geq u(y - x^{I_D^y}),$$

or, from the definition of Γ and the linearity of u,

$$u(x) + \Gamma(I_D^x) \geq u(y) + \Gamma(I_D^y),$$

for all $y \in D$. So $\mathcal{C}_2(D) \subseteq \arg\max_{x \in D} \big(u(x) + \Gamma(I_D^x)\big)$.

Now assume that $x \notin \mathcal{C}_2(D)$. Then there exists a $y \in D$ such that $(y, I_D^y) \succ (x, I_D^x)$. Since $\succcurlyeq^\star$ extends $\succcurlyeq$, we have $(y, I_D^y) \succ^\star (x, I_D^x)$. From repetition of the above with strict preference/inequality we conclude that

$$u(y) + \Gamma(I_D^y) > u(x) + \Gamma(I_D^x),$$

So $\arg\max_{x \in D} \big(u(x) + \Gamma(I_D^x)\big) \subseteq \mathcal{C}_2(D)$ and we have established the existence of an image conscious representation. ∎