ECON5110: Microeconomics

Lecture 3: Sept, 2017

Contents

1 Overview

Until now, we have assume that prices exogenous, defined outside of the model. We will now analyze the concept of **equlibrium**, in which consumers (and producers) trade with one another. The allocations that result, after trading has been completed, will be the result of demand and supply functions and prices will be set so as to equate these two sides of a the market. In other words, beginning only with initial allocations of good, and preferences over different allocations, we will seek to understand how agents, in the pursuit of more favorable consumption bundles, will trade with one another (and, if there are firms, how firms will produce). A price vector constitutes an equilibrium, if given these prices, no agent wants to change her behavior.

The study of equilibrium is the study of markets. The underlying assumption being that prices reflect market supply and demand and adjust so as to balance them. Throughout this investigation, we will answer the following questions: When does such an equilibrium exist? Is an equilibrium desirable or efficient? In other words, if a benevolent planner was to allocate goods, how would this differ from the resulting equilibrium allocation. How do the initial allocations and production technologies affect the resulting equilibrium.

2 Exchange Economies

To simplify matters we will begin by dispensing with the production side of the economy. We will assume each consumer is endowed with an initial stock of each good and that this initial stock is what can be consumed in total. The question therefore reduces to, given this initial stock of goods, how will consumers trade, keeping in mind that their intention is to end up with the most desirable bundle of goods possible.

There are k goods and n consumers. Each consumer is endowed with an initial stock of each good. We can denote consumers consumers i's **endowment** by $\boldsymbol{\omega}_i = (\omega_i^1 \dots \omega_i^k) \in \mathbb{R}_+^k$. The endowment for the economy is the vector $\boldsymbol{\omega} = (\boldsymbol{\omega}_1 \dots \boldsymbol{\omega}_n) \in \mathbb{R}_+^{nk}$. Likewise an **allocation** (the final consumption) for consumer i is denoted $\boldsymbol{x}_i = (x_i^1 \dots x_i^k) \in \mathbb{R}_+^k$. The allocation for the economy is the vector $\boldsymbol{x} = (\boldsymbol{x}_1 \dots \boldsymbol{x}_n) \in \mathbb{R}_+^{nk}$.

An allocation is **feasible** if $\sum_n \boldsymbol{x}_i \leq \sum_n \boldsymbol{\omega}_i$: if the amount of goods consumed is less than or equal to the amount of goods we started with. Feasibility is a physical constraint.

To understand what how consumer will behave, we need to know what there preferences are. So, each consumer will be defined, in addition to their endowment, by a utility function which we will assume is continuous, concave and strictly monotone.[1] That is, each consumer is defined as the pair $(\boldsymbol{\omega}_i, U_i)$.

Previously, we discussed consumer demand, assuming that the consumers wealth was some exogenously given budget: w. Now, we do not endow the consumer with a budget, but rather with a bundle of goods. However, under the market prices, $\boldsymbol{p} \in \mathbb{R}_+^k$, the consumer can convert her endowment, $\boldsymbol{\omega}_i$ into a budget buy selling it. The resulting wealth is $\boldsymbol{p} \cdot \boldsymbol{\omega}_i$.

Given the economy $\{(\boldsymbol{\omega}_i, U_i)\}_{i \leq n}$, an **equilibrium** is an allocation and a price vector, $(\boldsymbol{x}^\star, \boldsymbol{p}^\star)$, such that

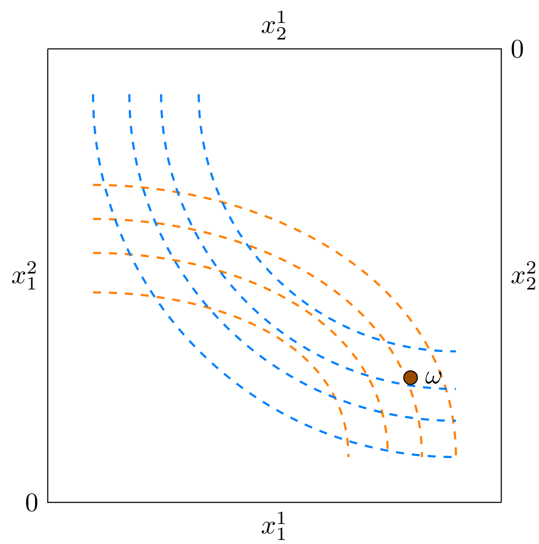
1. $\boldsymbol{x}^\star$ is feasible.
2. $\boldsymbol{x}_i^\star = \boldsymbol{x}_i^\star(\boldsymbol{p}^\star, \boldsymbol{p}^\star \cdot \boldsymbol{\omega}_i)$ for each i.

The first requirement states that the allocation is physically possible. The second is more interesting. Under the equilibrium prices, the consumer is maximizing her utility with respect to the price vector, $\boldsymbol{p}^\star$, and her wealth $\boldsymbol{p}^\star \cdot \boldsymbol{\omega}_i$. The consumption therefore is given by her Walrasian demand function: $\boldsymbol{x}_i^\star(\boldsymbol{p}^\star, \boldsymbol{p}^\star \cdot \boldsymbol{\omega}_i)$.

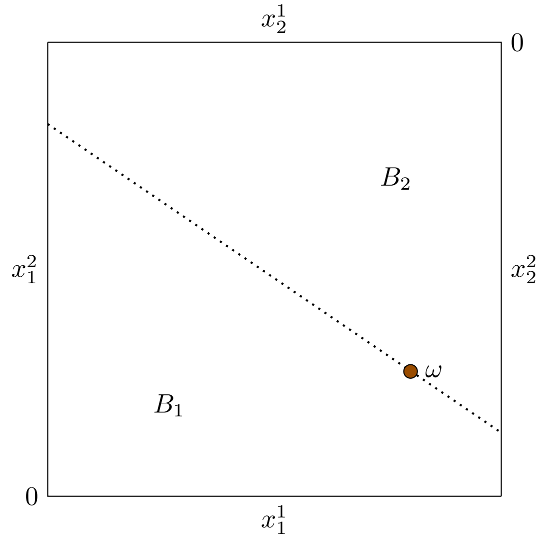
[1] In the resulting analysis, strict monotonicity can be replaced by local non-satiation. Having goods which are undesirable is totally ok with this theory, as we generally allow free disposal. Likewise concavity can be dispensed with, although we may have to deal with set valued demand. However, continuity is *not* a peripheral assumption. If the utility functions are not continuous then demand functions need not be (upper-semi)-continuous. In such a case, equilibria may not exist.

Putting these two restrictions together, an equilibrium is an allocation and a price vector such that under these prices, the consumers are choosing optimally so as to maximize their utility, and the resulting allocation is possible. Starting with a price $\boldsymbol{p}$, we can break this down into two steps: first each consumer sells her endowment to the market and gets a budget of $\boldsymbol{p} \cdot \boldsymbol{\omega}_i$. Then, the consumer optimally spends this wealth to buy back goods, demanding $\boldsymbol{x}_i^\star(\boldsymbol{p}, \boldsymbol{p} \cdot \boldsymbol{\omega}_i)$. This pair $(\boldsymbol{p}, (\boldsymbol{x}_1^\star(\boldsymbol{p}, \boldsymbol{p} \cdot \boldsymbol{\omega}_1) \ldots (\boldsymbol{x}_n^\star(\boldsymbol{p}, \boldsymbol{p} \cdot \boldsymbol{\omega}_n))$ constitute an equilibrium if amount demanded in the second stage is no more than what was sold in the first stage.

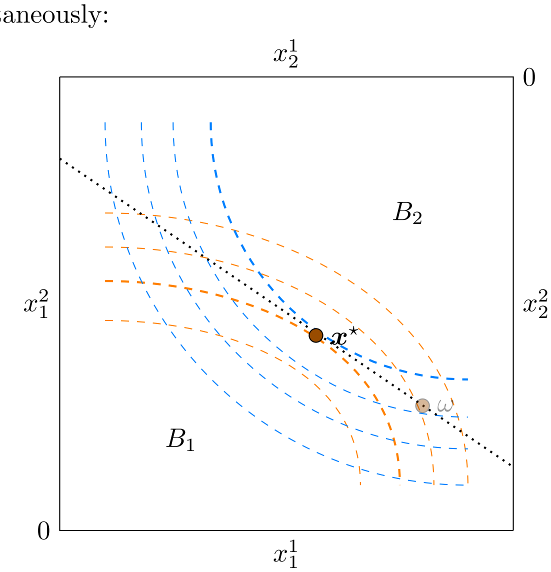
Edgeworth Boxes. A further simplification which allows us to visualize an exchange economy is to stipulate that there are 2 good and 2 consumers. In this case, we can represent the economy as a box, with good 1 on the horizontal axis and good 2 on the vertical axis. The horizontal length of the box is $\omega^1 = \omega_1^1 + \omega_2^1$, the amount of good 1 in the economy; the vertical length of the box is $\omega^2 = \omega_1^2 + \omega_2^2$, the amount of good 1 in the economy. Every point in the box represents a feasible allocation. The point (x^1, x^2) represents the point where consumer 1's allocation is (x^1, x^2) and consumer 2's is $(\omega^1 - x^1, \omega^2 - x^2)$. The endowment is also a point within this box:



Just as within the standard consumer theory problem we can draw the indifference curves within the edgeworth box. Consumer 1 increases her utility by moving to endowments in the northeast, whereas consumer 2 by moving to the southwest. For a given price vector, we can also use the same graph to see which are the affordable bundles (in the sense that, if the consumer liquidates her endowment, she can buy back the demanded bundle):



The slope of the dotted line is the ratio of the prices. Consumer 1's affordable lies below the line, whereas 2's is above. The initial endowment lies on the budget line by construction, since budget is how much the consumers can get for their endowment. By strict monotonicity, we know that the consumers will both spend all of their income, so any equilibrium point must fall on this line. But which point, if any, will be chosen? From our investigation of consumer theory, we know that each consumers Walrasian demand will be the point on the boundary of the budget set where the indifference curve is tangent (the equality of the marginal rate of substitution with marginal cost, via the first order conditions). But now, this must happen for both consumers simultaneously:



Lets see an example:

Example 1. *There are 2 consumers and two goods (an edgeworth box economy). Both consumers have Cobb-Douglas utility functions given by*

$$U_A(x_1, x_2) = x_1^a x_2^{(1-a)}$$
$$U_B(x_1, x_2) = x_1^b x_2^{(1-b)}.$$

The initial endowments are given by $\boldsymbol{\omega}_A = (1, 0)$ *and* $\boldsymbol{\omega}_B = (0, 1)$*. Let* $\boldsymbol{p} = (p_1, p_2)$ *denote a price vector. At these prices, the wealth of each consumer is* $w_A = p_1$ *and* $w_B = p_2$*.*

The demand functions for the two consumers, given their wealth levels are:

$$\boldsymbol{x}_A^\star(\boldsymbol{p}, w_1) = (a\frac{w_A}{p_1}, (1-a)\frac{w_1}{p_2}) = (a, (1-a)\frac{p_1}{p_2})$$
$$\boldsymbol{x}_B^\star(\boldsymbol{p}, w_2) = (b\frac{w_B}{p_1}, (1-b)\frac{w_2}{p_2}) = (b\frac{p_2}{p_1}, (1-b))$$

In order for $\boldsymbol{p}$ *to support and equilibrium, it must be that the demand for each good is equal to the supply (i.e.,* $= 1$*). In the market for good 1, this implies*

$$a + b\frac{p_2}{p_1} = 1$$

or that

$$\frac{p_2^\star}{p_1^\star} = \frac{1-a}{b}$$

Notice, that we cannot push further: only the relative price levels will be determined. This is because of Theorem 2: any price vector can be arbitrarily scaled and still support and equilibrium. As such we have our solution. However, shouldn't we ensure this also make the market for good to clear? Indeed, we have that $p_2^\star = p_1^\star \frac{1-a}{b}$*, so that*

$$(1-a)\frac{p_1}{p_2} + 1 - b = \frac{(1-a)p_1 b}{(1-a)p_1} + 1 - b = 1.$$

Excess Demand. As we saw in the above example, there are essentially 3 different things that had to hold at the equilibrium: (i) the endowment lies of the budget line (because wealth stems from the initial endowment) (ii) consumer 1's indifference curve is tangent to the budget (because she is maximizing her utility given the induced wealth) and (iii) consumer 2's indifference curve is tangent to the budget (because she is also maximizing). It seems reasonable to think that as the number of goods and consumers increased this would become harder and harder to sustain. Heck, even within the edgeworth box, it is not clear that such a point should always exist. Nonetheless, we shall see that an equilibrium will always exists (at least under our assumptions of continuity and monotonicity).

Towards understanding the properties of equilibrium and proving existence, we will define the **excess demand function**, $z : \mathbb{R}^k_+ \to \mathbb{R}^k$ as

$$z(\boldsymbol{p}) = \sum_{i=1}^{n} [\boldsymbol{x}_i^\star(\boldsymbol{p}, \boldsymbol{p} \cdot \boldsymbol{\omega}_i) - \boldsymbol{\omega}_i]$$

The excess demand function calculates how much more of a given good is demanded by the consumers of the economy that exists in that economy. We can recast the definition of equilibrium via the excess demand function: a price vector supports and equilibrium if and only if the corresponding excess demand is weakly negative. Notice that while the excess demand function take as a parameter the initial endowment, we suppress this notation, understanding the endowment to remain fixed throughout the analysis.

Theorem 1 (Walras' Law 2: this time, its equilibrium)**.** *The value of excess demand is* 0*:* $\boldsymbol{p} \cdot z(\boldsymbol{p}) = 0$*.*

Proof. Since each consumer has LNS preferences (via strict monotonicity), we can apply the consumer theory version of Walra's law: the consumer spends all her budget. We have $\boldsymbol{p} \cdot \boldsymbol{x}_i^\star(\boldsymbol{p}, \boldsymbol{p} \cdot \boldsymbol{\omega}_i) = \boldsymbol{p} \cdot \boldsymbol{\omega}_i$. Summing across consumer's provides the result. ■

Theorem 1 states that the value of demanded goods is exactly the value of the endowment. Notice this is not an equilibrium condition, it holds for *any* price vector. We also know:

Theorem 2. *Excess demand is h.d.0:* $z(\alpha \boldsymbol{p}) = z(\boldsymbol{p})$*.*

If we further stipulate the the resulting allocation is an equilibrium, we can say more:

Theorem 3. *If* $\boldsymbol{p}^\star$ *is an equilibrium, and* $z_j(\boldsymbol{p}^\star) < 0$ *then* $\boldsymbol{p}_j^\star = 0$*.*

Proof. By definition of an equilibrium, $z_j(\boldsymbol{p}^\star) \leq 0$. Since prices are non-negative, we have $\boldsymbol{p}_j^\star \cdot z_{j'}(\boldsymbol{p}^\star) \leq 0$ for all j'. Thus, if $z_j(\boldsymbol{p}^\star) < 0$ and $\boldsymbol{p}_j^\star > 0$, we would have $\boldsymbol{p} \cdot z(\boldsymbol{p}) < 0$, contradicting Theorem 1. ■

Theorem 4. *If* $\boldsymbol{p}^\star$ *is an equilibrium (and the consumers preferences are all strictly monotone) then* $z(\boldsymbol{p}^\star) = \boldsymbol{0}$*.*

Proof. Assume this was not the case so that $z_j(\boldsymbol{p}^\star) < 0$ for some good j. By Theorem 3 then we have $\boldsymbol{p}_j^\star = 0$. By the monotonicity of preferences this implies that $x_i^\star, j(\boldsymbol{p}^\star, \boldsymbol{p}^\star \boldsymbol{\omega}_i) = \infty$. A contradiction to our initial assumption that excess demand was negative. ■

Theorem 4 is a just a convoluted way of describing one of the core tenets of economic reasoning: in equilibrium supply equals demand. Notice, however, that this relies on our assumption that goods are desirable: if some goods are bads then in equilibrium the price of that good could be 0 and no consumer wants to consume it—while this is ok, it clutters the analysis with extra cases to check, so we rule it out.

The later observation, that if the first market clears then so does the second *always* holds. This follows directly from Walra's Law (given all prices are strictly positive), since if there was only 1 markets that did not clear (i.e., where excess demand for a single good was non-zero) then clearly the value of the total excess demand is not zero.

Remark 5. *If $z_i(\boldsymbol{p}) = 0$ for all $i < k$ and $p_k > 0$ then $z_k(\boldsymbol{p}) = 0$.*

Example 2. *There are 2 consumers and two goods (an edgeworth box economy). Both consumers have Cobb-Douglas utility functions given by*

$$U_A(x_1, x_2) = \min\{x_1, x_2\}$$
$$U_B(x_1, x_2) = \min\{x_1, x_2\}.$$

The initial endowments are given by $\boldsymbol{\omega}_A = (1, 0)$ and $\boldsymbol{\omega}_B = (0, 1)$. Let $\boldsymbol{p} = (p_1, p_2)$ denote a price vector. At these prices, the wealth of each consumer is $w_A = 2p_1$ and $w_B = p_2$.

What do we know about this equilibrium? In fact, we don't know very much at all. First, note that given that at least 1 price is positive, the both consumers demand an equal amount of each good. So we know that any competitive equilibrium will give (α, α) to consumer 1 and $(1-\alpha), (1-\alpha))$ to consumer 2 for some $\alpha \in [0, 1]$. However, beyond this, we have no restrictions on what the allocation will look like. Indeed, normalize the price of x_1 to 1 and pick any *positive price, p, for good x_2. Then the budgets are 1 and p, respectively.*

Consumer 1 will therefore demand, $\frac{1}{1+p}$ of each good, and consumer 2 will demand $\frac{p}{1+p}$ of each good. Markets clear.

What if we had had $\boldsymbol{\omega}_B = (0, 2)$. Again we have that demand is equal across the two goods. However, notice we cannot have excess demand for the goods. With the same normalizations, the budgets are 1 and $2p$, respectively; the demands will therefore be, $\frac{1}{1+p}$, $\frac{2p}{1+p}$. In order for markets to clear, it must be that $\frac{1}{1+p} + \frac{2p}{1+p} \leq 1$, which can happen only if $p = 0$. Indeed, when $p = 0$ the demand for consumer 1 is 1 unit of each good, and for consumer 2 is 0 of each good. Oddly, by endowing the second consumer with more to begin with, he does worse in equilibrium.

3 Existence

Since excess demand is homogeneous of degree 0, we can restrict our attention to price vectors that sum to 1 (by dividing an arbitrary vector by its magnitude). Thus we are dealing with price vectors that can be associated with points in the $k-1$ dimensional simplex (recall this definition from our study of risk attitudes) (which we call S^{k-1}). Before proceeding the the existence proof, take the following Lemma which, despite being true, will not be proven in this class:

Theorem 6 (Brouwer's Fixed Point Theorem)**.** *Let $f : S^{k-1} \to S^{k-1}$ be continuous. Then there exists an $\boldsymbol{p} \in S^{k-1}$ such that $f(\boldsymbol{p}) = \boldsymbol{p}$.*

We can now prove the existence of a Walrasian equilibrium.

Theorem 7. *If each consumer's demand function satisfies Walra's Law and is continuous then a Walrasian equilibrium exists.*

Proof. Consider the reverse engineered function $g : S^{k-1} \to S^{k-1}$ given by:

$$g_j(\boldsymbol{p}) = \frac{p_j + \max\{0, z_j(\boldsymbol{p})\}}{\sum_{j'=1}^{k} p_{j'} + \max\{0, z_{j'}(\boldsymbol{p})\}}$$

for $j \le k$. Being the composition of a bunch of continuous functions (in particular, z, by proxy of $\boldsymbol{x}^\star$), g is continuous. Applying Brouwer's Fixed Point Theorem we have a $\boldsymbol{p}^\star$ such that $g(\boldsymbol{p}^\star) = \boldsymbol{p}^\star$, or (since $\sum_{j'} p_{j'} = 1$):

$$p_j^\star = \frac{p_j + \max\{0, z_j(\boldsymbol{p}^\star)\}}{1 + \sum_{j'=1}^{k} \max\{0, z_{j'}(\boldsymbol{p}^\star)\}}$$

Cross multiplying and re-arranging gives us:

$$p_j^\star \sum_{j'} \max\{0, z_{j'}(\boldsymbol{p}^\star)\} = \max\{0, z_j(\boldsymbol{p}^\star)\},$$

multiplying by $z(\boldsymbol{p}^\star)$, and summing over the k equations:

$$\sum_{j=1}^{k} z_j(\boldsymbol{p}^\star) p_j^\star \sum_{j'=1}^{k} \max\{0, z_{j'}(\boldsymbol{p}^\star)\} = \sum_{j=1}^{k} z_j(\boldsymbol{p}^\star) \max\{0, z_j(\boldsymbol{p}^\star)\},$$

By Theorem 1, we have that the LHS is 0:

$$\sum_{j=1}^{k} z_j(\boldsymbol{p}^\star) \max\{0, z_j(\boldsymbol{p}^\star)\} = 0.$$

Each term either equals 0 or $(z_j(\boldsymbol{p}^\star))^2$, hence no term is negative. This implies that it must be that every term is identically 0. Now assume that there was some good j such that $z_j(\boldsymbol{p}^\star) > 0$, then $\max\{0, z_j(\boldsymbol{p}^\star)\} = z_j(\boldsymbol{p}^\star)$, so that $z_j(\boldsymbol{p}^\star)\max\{0, z_j(\boldsymbol{p}^\star)\} = (z_j(\boldsymbol{p}^\star))^2 > 0$ a contradiction.

Hence, $z(\boldsymbol{p}) \leq 0$, the characterization of an equilibrium. ■

So given our assumptions, an equilibrium always exists. What is the economic content of the assumptions, however:

- *The demand functions are functions (i.e., single valued).* This can be relaxed by appealing to a stronger fixed point theorem (Kakutani's fixed point theorem). Continuity of demand needs to be replaced with upper-semi-continuity. Recall, convexity of preferences implies that demand is continuous and single valued.
- *The demand functions satisfies Walra's Law.* This is a weak assumption. Local non-satiation implies that it will hold.
- *The demand functions are continuous.* This is a more serious assumption. Plenty of reasonable preferences are not continuous (think about non-dividable goods, for example). Continuity is in fact even more nettlesome than it initially seems: continuity can fail as prices approach zero. When the price of 2 distinct good tends towards 0, at different rates, the consumer might want a bounded amount of one of the two goods. However, given monotonicity, when both goods are at price 0, the demand will be unbounded for both. Dealing with this particular case is possible, but annoying. We are usually safe if we assume that each consumer is endowed with a positive amount of each good.

Lets work with this last point a little more via an example:

Example 3. *There are 2 consumers and two goods (an edgeworth box economy). Utility functions given by*

$$U_A(x_1, x_2) = x_1$$
$$U_B(x_1, x_2) = (x_1)^{\frac{1}{2}} + x_2.$$

Assume that the initial endowments are $\boldsymbol{\omega}_A = (10, 0)$ and $\boldsymbol{\omega}_B = (0, 10)$. We claim that there is no set of prices that makes markets clear. Indeed, neither price can be 0, since then B's demand is infinite. When both prices are strictly positive, A will demand his endowment (since he has no use for x_2 and can therefore afford exactly what he started with). But B will also demand some x_1 (his marginal utility becomes infinite as $x_1 \to 0$). So markets cannot clear.

4 Welfare

If we believe that the consumers in some given economy behave according to the assumptions listed in the last section, then an equilibrium exists. Perhaps, in this economy, prices are the equilibrium prices and markets clear. Economists can make predictions about how prices and allocations will change with the increase or decrease of the endowment (availability) of some goods. But none of this tells us about the welfare of the agents involved. Does trade under equilibrium prices make consumers better off? Is there a system of trade or allocation which in strictly improves on trade under equilibrium prices?

The answer to the first question is immediate.

Theorem 8. *Let $\{U_i, \boldsymbol{\omega}_i\}_{i\leq n}$ be an economy. Let $(\boldsymbol{p}^\star, \{\boldsymbol{x}_i^\star\}_{i\leq n})$ be an equilibrium of this economy. Then for each consumer, i we have $U_i(\boldsymbol{x}_i^\star) \geq U_i(\boldsymbol{\omega}_i)$.*

That is, trade (weakly) improves every consumer's lot in life. Each agent prefers the final allocation to her starting endowment. This indicates that equilibrium trade is better than no reallocation at all. But is there a system of allocation that is better than equilibrium trade?

To answer this question, we must first ask it in a more specific way. What do we mean by better? Is it better to make one consumer much happier at the expense of everyone else? What about only at the expense of on other consumer? In fact, what does it even mean to make a consumer *much* better off, seeing as that our preferences were ordinal to begin with.

Pareto Efficiency. To dispense with the ambiguity and not-so-well-definededness of these subquestions, we will define welfare in a very weak way. One allocation, $\{\boldsymbol{x}_i\}_{i\leq n}$ **Pareto dominates** another $\{\boldsymbol{x}'_i\}_{i\leq n}$ if for all $i \leq n$, $U_i(\boldsymbol{x}_i) \geq U_i(\boldsymbol{x}'_i)$ (strict for some i). That is $\{\boldsymbol{x}_i\}_{i\leq n}$ dominates $\{\boldsymbol{x}'_i\}_{i\leq n}$ if every consumer is at least as happy with the former than with the latter. A **Pareto improvement** is a movement to an allocation which Pareto dominates the status-quo: makes every consumer better off. Given a set of allocations B, we say that $\{\boldsymbol{x}_i\}_{i\leq n} \in B$ is **Pareto efficient** if there is no other allocation, $\{\boldsymbol{x}_i\}_{i\leq n} \in B$, that Pareto dominates it.

This notion is very weak, in the sense that it has nothing to say about many pairs of allocation (in the language of this class: the Pareto domination relation is highly incomplete), so there might be many efficient points. For example, with two (strictly monotone) consumers and one unit of one good, no allocation Pareto dominates any other; every allocation is Pareto efficient. We must take some of the good from one consumer, reducing her utility, to improve her counterpart's.

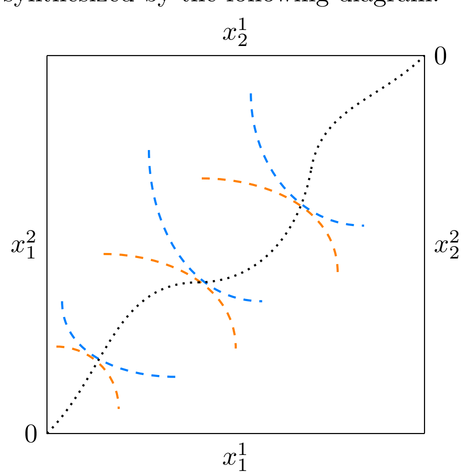
Notice also that Pareto efficiency says nothing about justice or equity. Under strict monotonicity, giving all resources to a single agent is Pareto efficient (the example above is a special case of this). There are in general many Pareto efficient points, some of which may be very undesirable from a social standpoint. However, if an allocation is *not* Pareto efficient than it absolutely should not be socially desirable. If the status quo is not Pareto efficient, then it is possible to make every person better off (at least one person strictly so) with the same resources!

We will first consider what it mean to be Pareto efficient in an Edgeworth box economy. If we fix the utility of consumer 1 at an arbitrary level, then a point along that indifference curve is Pareto efficient if and only if it maximizes the utility of consumer 2 (within the set of point that keep consumer 1 indifferent). Formally, we are solving:

$$\max_{\boldsymbol{x}} \; U_2(\boldsymbol{x}_2) \quad \text{such that } U_1(\boldsymbol{x}_1) \geq \bar{u}$$
$$\text{and } \boldsymbol{x}_1 + \boldsymbol{x}_2 = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$$

Given basic regularity conditions (monotonicity, continuity) the solution to this problem is will be Pareto efficient since by construction we cannot make consumer 2 better off without changing the constraint, hurting consumer 1, and (under the regularity conditions) vice versa. Two things should immediately stand out: (i) the solution to the above maximization problem will be characterized by the tangency of the two consumers indifference curves, and (ii) this process works to find Pareto efficient points giving arbitrary utility to consumer 1.

These two facts can be synthesized by the following diagram:



The dotted line is called the **contract** curve. This line traces out the set of Pareto efficient points for varying levels of utility.

Example 4. *There are 2 goods are 2 agents with identical concave utility functions. Show that $\boldsymbol{\omega}_2$ (and equal split of each good) is Pareto efficient. Assume this was not true, and $\boldsymbol{x}'$ be a Pareto dominating allocation and let (x_1', x_2') denote the allocation to A. Then $U(x_1', x_2') \geq U(\frac{\omega_1}{2}, \frac{\omega_2}{2})$ (because A is better off) and $U(\omega_1 - x_1', \omega_2 - x_2') \geq U(\frac{\omega_1}{2}, \frac{\omega_2}{2})$ (because B is better off), where at least 1 inequality is strict. From the concavity of U:*

$$\begin{aligned} U(\frac{\omega_1}{2}, \frac{\omega_2}{2}) &= U(\frac{1}{2}x_1' + \frac{1}{2}(\omega_1 - x_1'), \frac{1}{2}x_2' + \frac{1}{2}(\omega_2 - x_2')) \\ &\geq \frac{1}{2}U(x_1', x_2') + \frac{1}{2}U(\omega_1 - x_1', \omega_2 - x_2') \\ &> \frac{1}{2}U(\frac{\omega_1}{2}, \frac{\omega_2}{2}) + \frac{1}{2}U(\frac{\omega_1}{2}, \frac{\omega_2}{2}) \\ &= U(\frac{\omega_1}{2}, \frac{\omega_2}{2}) \end{aligned}$$

a contradiction to the fabric of math itself!

The welfare theorems. The next two theorems, the first and second welfare theorems, relate Pareto efficiency and equilibrium outcomes, and are of central importance to economic thinking. First, we will see that every equilibrium outcome is Pareto efficient, and second, that every Pareto efficient allocation is part of *some* equilibrium outcome.

These results are, arguably, the most important results for the foundation of Neoclassical thought. That is not to say that they are realistic or descriptive, but that they allow us to speak in a formal sense about market outcomes in a normative way. Even if you do not buy the premises behind the results, thereby voiding them as valid descriptions of the world, they still serve as a benchmark by which we can compare reality, and a framework to analyze failures of the model.

Theorem 9 (The First Welfare Theorem of Economics)**.** *Let $\{U_i, \boldsymbol{\omega}_i\}_{i\leq n}$ be an economy such that U is locally non-satiated. Let $(\boldsymbol{p}^\star, \{\boldsymbol{x}_i^\star\}_{i\leq n}$ be an equilibrium of this economy. Then $\{\boldsymbol{x}_i^\star\}_{i\leq n}$ is Pareto efficient (over the set of feasible allocations).*

Proof. Assume this was not the case and that some feasible $\{\boldsymbol{x}_i'\}_{i\leq n}$ Pareto dominated $\{\boldsymbol{x}_i^\star\}_{i\leq n}$. This means that for each i, $U_i(\boldsymbol{x}_i') \geq U_i(\boldsymbol{x}_i^\star)$ with the inequality strict for some consumer. This implies that for each i, $\boldsymbol{p}^\star \cdot \boldsymbol{x}_i' \geq \boldsymbol{p}^\star \cdot \boldsymbol{x}_i^\star$, strict for some i (recall, this was a homework question).

Summing across all consumers we have:

$$\sum_i \boldsymbol{p}^\star \cdot \boldsymbol{x}_i' > \sum_i \boldsymbol{p}^\star \cdot \boldsymbol{x}_i^\star \tag{1}$$

Now, x' is also feasible, indicating that $\sum_i \boldsymbol{x}_i' \leq \sum_i \boldsymbol{\omega}_i$. Putting these two observations together we have

$$\begin{aligned} \boldsymbol{p}^\star \sum_i \boldsymbol{\omega}_i \geq \boldsymbol{p}^\star \sum_i \boldsymbol{x}_i' &= \sum_i \boldsymbol{p}^\star \boldsymbol{x}_i' \\ &> \sum_i \boldsymbol{p}^\star \boldsymbol{x}_i^\star \\ &= \sum_i \boldsymbol{p}^\star \boldsymbol{\omega}_i = \boldsymbol{p}^\star \sum_i \boldsymbol{\omega}_i \end{aligned}$$

where the second to last equality is Walra's Law. ■

Intuition behind the proof: each consumer is endowed with a budget that is dictated from the endowments. From this budget each consumer is optimizing her utility *without considering the allocations of others*. That is, we have implicitly fixed the each consumers utility level (the indirect utility $v(\boldsymbol{p}^\star, \boldsymbol{p}^\star \cdot \boldsymbol{\omega})$). Since each consumer is optimizing, making any consumer better off would entail increasing her budget. But to increase one consumers budget is to reduce another's, and under the local non-satiation assumption, this must make her worse off.

While at first glance (i.e., form the perspective of the modal undergraduate student) the first welfare theorem screams: leave markets alone as the *invisible hand* is efficient. Proponents of the neoclassical conservative economic school of thought will use the first welfare theorem to argue for relaxing regulation, while critics will use the persistence of market failures to argue that the general economic style of thinking must be flawed. These views totally miss the point. First, Pareto efficiency $\neq$ socially desirable, as discussed above. There is still work to be done to effect the *best* efficient point. Moreover, as we will see below, market failures can arise for reasons that can be formulated within the present framework. Thus, the FWT tells us which situations we should attend to, which situations will most likely be remedied by simple regulation (externalities) and which are more deeply rooted in the structure of the problem (highly unequal initial endowments).

Theorem 10 (The Second Welfare Theorem of Economics)**.** *Let $\{U_i, \boldsymbol{\omega}_i\}_{i\leq n}$ be an economy such that U is continuous and locally non-satiated. Let $\{\boldsymbol{x}_i^*\}_{i\leq n}$ be a Pareto efficient allocation. Then there exists a equilibrium such that the resulting allocation is $\{\boldsymbol{x}_i^*\}_{i\leq n}$.*

Proof. Let the endowments be given by $\boldsymbol{\omega}_i = \boldsymbol{x}_i^*$ for each i. By Theorem 7 there exists an equilibrium $(\boldsymbol{p}', \boldsymbol{x}')$. Since $\boldsymbol{x}^*$ is affordable under $\boldsymbol{p}'$ (by construction of the budget) it must be that $U_i(\boldsymbol{x}_i') \geq U_i(\boldsymbol{x}^*)$ for every i. But, since $\{\boldsymbol{x}_i^*\}_{i \leq n}$ is a Pareto efficient allocation, it therefore must be that $U_i(\boldsymbol{x}_i') = U_i(\boldsymbol{x}^*)$. Hence $(\boldsymbol{p}', \boldsymbol{x}^*)$ is an equilibrium. ■

Intuition behind the proof: if we begin with a Pareto optimal allocation, then trading cannot make everyone better off without hurting someone. But no consumer *has* to trade, since she could just as well stick to her initial endowment. Hence, we cannot change any consumers utility—it must be that the original circumstances were an equilibrium to begin with.

What could go wrong. The most economically relevant failure of the welfare theorems stems from externalities. An **externality** is when one agent make a decision that *directly* affects the utility of another agent. The classic example of externalities are pollution: polluting the environment directly and negatively affects those who made no decision to pollute.

Example 5. *There are two consumers, A and B. There are two goods. Unlike the standard edgeworth box, the first consumer is hurt by the consumption of the second agent. Utilities are:*

$$U_A(x_1, x_2) = x_{A1}x_{A2} - x_{B1}$$
$$U_B(x_1, x_2) = x_{B1}x_{B2}$$

Let the initial endowment be $\boldsymbol{\omega}_A = (1,1)$ and $\boldsymbol{\omega}_B = (1,1)$. First, we claim that: $\boldsymbol{p} = (1,1)$ and $\boldsymbol{x}_A^\star = \boldsymbol{x}_B^\star = (1,1)$ constitutes an equilibrium. This is obvious for consumer B, since she has C-D preferences with an equal weight to each good and prices are equal. Moreover, consumer A takes x_{B1} as given, and hence maximizes her demand function seeks to maximize $x_{A1}x_{A2}$. For here we see that clearly $(1,1)$ is also the unique demand function.

The utilities under this allocation are 0 and 1, respectively. Now consider the (feasible) allocation $((\frac{5}{4}, \frac{2}{3}), (\frac{3}{4}, \frac{4}{3})$. The utility for consumer A is now $\frac{5}{4}\frac{2}{3} - \frac{3}{4} = \frac{1}{12}$. The utility for consumer B is still 1. Hence this allocation Pareto Dominates the equilibrium allocation.