EC5110: MICROECONOMICS

LECTURE 2: CONSUMER CHOICE

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We now have a model to describe the choices and preferences of consumers.

Now we want to:

- Apply this to economic/market settings.
- Make predictions about patterns of choices.
  - Relate assumptions on utilities (or $\succcurlyeq$) to market outcomes.
- Analyze welfare implications or policy.

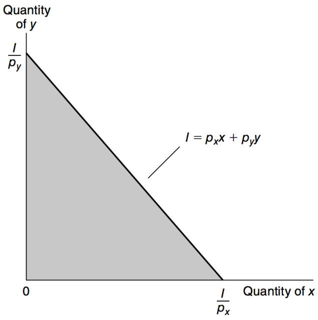
A consumer has an income I and faces prices $\boldsymbol{p} = (p_1, \ldots p_n)$ (all strictly positive).

This defines a decision problem:

$$B(\boldsymbol{p}, I) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{p} \cdot \boldsymbol{x} \leq I\}$$

FIGURE 4.1 The Individual's Budget Constraint for Two Goods

Those combinations of x and y that the individual can afford are shown in the shaded triangle. If, as we usually assume, the individual prefers more rather than less of every good, the outer boundary of this triangle is the relevant constraint where all of the available funds are spent either on x or on y. The slope of this straight-line boundary is given by $-p_x/p_y$.



There is a major tacit assumption: **Prices do not change based on amount purchased.**

- This is an assumption of price taking.
- We will consider where prices come from later.

We know how a consumer with utility U will behave given income I and prices $\boldsymbol{p}$.

$$x \in B(\boldsymbol{p}, I) \text{ such that } \boldsymbol{x} \succcurlyeq \boldsymbol{y} \text{ for all } \boldsymbol{y} \in B$$
$$=$$
$$\arg\max U(\boldsymbol{x}) \;\; \text{subject to } \boldsymbol{x} \in B(\boldsymbol{p}, I)$$

These are the bundles the consumer might demand given $(\boldsymbol{p}, I)$.

The walrasian demand (sometimes called the Marshallian demand) of the consumer is a correspondence $\boldsymbol{x}^\star : \mathbb{R}^n_{++} \times \mathbb{R} \to \mathbb{R}^n_+$ such that

$$\boldsymbol{x}^\star(\boldsymbol{p}, I) = \operatorname*{arg\,max}_{\boldsymbol{y} \in \mathbb{R}^n_+} \; U(\boldsymbol{y}) \quad \text{subject to } \boldsymbol{p} \cdot \boldsymbol{y} \leq I.$$

Example

An agent's utility is $U(x, y) = \ln(x) + \ln(y)$. The price of x is 1 and y is 2. She has I dollars to spend.

She wants to maximize:

$$U(x, y) = \ln(x) + \ln(y) \quad \text{subject to } x + 2y \leq I, x \geq 0, y \geq 0$$

Example

The Lagrangian is:

$$\mathscr{L} = \ln(x) + \ln(y) - \mu_1(x + 2y - I) - \mu_x(-x) - \mu_y(-y)$$

We have the first order conditions:

$$\begin{aligned}
\mathscr{L}_x : &\qquad \frac{1}{x} - \mu_1 - \mu_x = 0 \\
\mathscr{L}_y : &\qquad \frac{1}{y} - 2\mu_1 - \mu_y = 0 \\
\mathscr{L}_{\mu_1} : &\qquad \mu_1(x + 2y - I) = 0 \\
\mathscr{L}_{\mu_x} : &\qquad \mu_x x = 0 \\
\mathscr{L}_{\mu_y} : &\qquad \mu_y y = 0
\end{aligned}$$

We also have our non-negativity constraints of the KKT theorem.

- $\ln(z) \to -\infty$ when $z \to 0$, we know the goods will be consumed in strictly positive amounts.
- Now, since $x > 0$ we have that $\frac{1}{x} > 0$ implying that $\mu_1 > 0$ by $\mathscr{L}_x$.
- Therefore we have by $\mathscr{L}_1$: $x = I - 2y$
- Dividing $\mathscr{L}_x$ by $\mathscr{L}_y$ we have $\frac{y}{x} = \frac{1}{2}$ or, using our previous findings, $\frac{y}{I-2y} = \frac{1}{2}$.

This implies $y^\star = \frac{I}{4}$. Solving for x we get $x^\star = \frac{I}{2}$:
$\boldsymbol{x}(I, (1,2)) = (\frac{I}{2}, \frac{I}{4})$.
The consumer spends an equal amount on each good.

Notice the consumer spent all her money!

$$p_x x^\star + p_y y^\star = \frac{I}{2} + 2\frac{I}{4} = I$$

Theorem. (Walra's Law)

If U is strictly monotone then $\boldsymbol{p} \cdot \boldsymbol{x}^\star(\boldsymbol{p}, I) = I$: the consumer will spend her entire income.

- She has no other value for money.
- More consumption is better.

What would happen if we doubled prices and income?

Theorem.

$\boldsymbol{x}^\star$ is homogenous of degree 0: $\boldsymbol{x}^\star(\boldsymbol{p}, I) = \boldsymbol{x}^\star(\lambda I, \lambda \boldsymbol{p})$ with $(\lambda > 0)$.

- Follows from the fact that $B(\boldsymbol{p}, I) = B(\lambda I, \lambda \boldsymbol{p})$
  - The maximization problem is the same.
- The *money* itself does not matter: just what can be purchased.

Theorem.

If U is quasi-concave (i.e., if $\succcurlyeq$ is convex) then $\boldsymbol{x}^{\star}(\boldsymbol{p}, I)$ is a convex set.

Theorem.

If U is strictly quasi-concave then $\boldsymbol{x}^{\star}(\boldsymbol{p}, I)$ is a singleton.

Proof (of first claim)

- Let $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{x}^\star(\boldsymbol{p}, I)$.
- Since $\boldsymbol{x} \in \boldsymbol{x}^\star(\boldsymbol{p}, I)$ and $\boldsymbol{y} \in B(\boldsymbol{p}, I)$, we have $U(\boldsymbol{x}) \geq U(\boldsymbol{y})$.
- Let $\alpha \in [0,1]$. Then we have $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} \in B(\boldsymbol{p}, I)$ by the convexity of $B(\boldsymbol{p}, I)$
- $U(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \geq U(\boldsymbol{y}) = \min\{U(\boldsymbol{x}), U(\boldsymbol{y})\}$ by quasi-concavity.
- So, $U(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \geq U(\boldsymbol{y}) \geq U(\boldsymbol{z})$ for all $\boldsymbol{z} \in B(\boldsymbol{p}, I)$.

Example

The Lagrangian from the example (taking strictly positive consumption as given) is:

$$\mathscr{L} = \ln(x) + \ln(y) - \mu_1(x + 2y - I)$$

We have the first order conditions:

$$\begin{aligned}
\mathscr{L}_x : &\qquad \frac{1}{x} - \mu_1 = 0 \\
\mathscr{L}_y : &\qquad \frac{1}{y} - 2\mu_1 = 0 \\
\mathscr{L}_{\mu_1} : &\qquad \mu_1(x + 2y - I) = 0
\end{aligned}$$

Dividing $\mathscr{L}_x$ by $\mathscr{L}_y$ yields:

$$\frac{y}{x} = \frac{1}{2}$$

- $\frac{y}{x} = \frac{\frac{\partial U}{\partial x}}{\frac{\partial U}{\partial y}}$ is the marginal rate of substitution.
- $\frac{1}{2}$ is the ratio of prices.

Theorem.

If all goods are chosen in strictly positive quantities then at $\boldsymbol{x}^\star$, the MRS is equal to the ratio of prices.

- This is just the first order condition.
- Has a nice interpretation graphically.

FIGURE 4.2 A Graphical Demonstration of Utility Maximization

Point C represents the highest utility level that can be reached by the individual, given the budget constraint. The combination x^*, y^* is therefore the rational way for the individual to allocate purchasing power. Only for this combination of goods will two conditions hold: All available funds will be spent, and the individual's psychic rate of trade-off (MRS) will be equal to the rate at which the goods can be traded in the market (p_x/p_y).

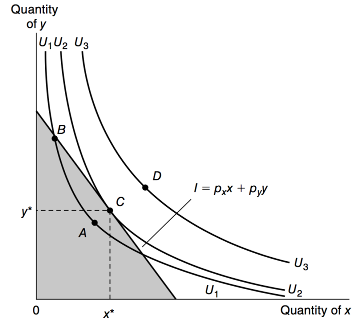
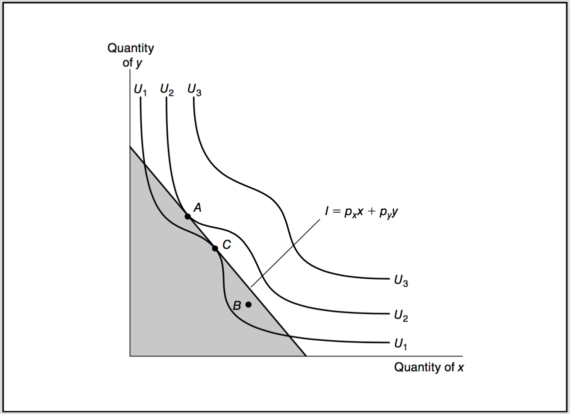


FIGURE 4.3 Example of an Indifference Curve Map for Which the Tangency Condition Does Not Ensure a Maximum

If indifference curves do not obey the assumption of a diminishing *MRS*, not all points of tangency (points for which $MRS - p_x/p_y$) may truly be points of maximum utility. In this example, tangency point C is inferior to many other points that can also be purchased with the available funds. In order that the necessary conditions for a maximum (that is, the tangency conditions) also be sufficient, one usually assumes that the *MRS* is diminishing; that is, the utility function is strictly quasi-concave.



This is all about marginal tradeoffs:

- The consumer could sell a little bit of y (at price p_y) to buy more x (at price p_x) .
    - The amount of y she needs to give up to get 1 unit of x is $\frac{p_x}{p_y}$.
- The consumer faces a “psychic” cost of trade given by the MRS.
    - The amount of y she is *willing* to give up to get 1 unit of x is MRS_x^y.

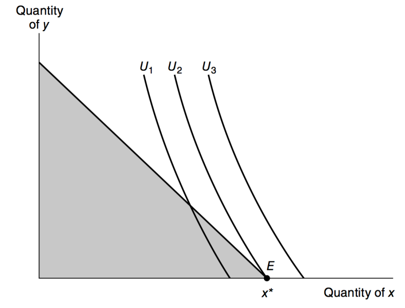
- If the external cost is lower than than her internal cost:
  - $MRS_x^y > \frac{p_x}{p_y}$
  - Better off after trading y for x.

- If the external cost is higher than than her internal cost:
  - $MRS_x^y < \frac{p_x}{p_y}$
  - Better off after trading x for y.

- The optimum will have these balanced!
  - Assumes it is possible to make them equal.

FIGURE 4.4 Corner Solution for Utility Maximization

With the preferences represented by this set of indifference curves, utility maximization occurs at E, where 0 amounts of good y are consumed. The first-order conditions for a maximum must be modified somewhat to accommodate this possibility.



Example

An agent's utility is $U(x, y) = x + y$. The price of x is 1 and y is 2. She has I dollars to spend.

She wants to maximize:

$$U(x, y) = x + y \quad \text{subject to } x + 2y \leq I, x \geq 0, y \geq 0$$

Example

The Lagrangian is:

$$\mathscr{L} = x + y - \mu_1(x + 2y - I) - \mu_x(-x) - \mu_y(-y)$$

We have the first order conditions:

$$\begin{aligned}
\mathscr{L}_x : &\qquad 1 - \mu_1 - \mu_x = 0 \\
\mathscr{L}_y : &\qquad 1 - 2\mu_1 - \mu_y = 0 \\
\mathscr{L}_{\mu_1} : &\qquad x + 2y = I \\
\mathscr{L}_{\mu_x} : &\qquad \mu_x x = 0 \\
\mathscr{L}_{\mu_y} : &\qquad \mu_y y = 0
\end{aligned}$$

If $x^\star, y^\star > 0$, then $\mu_x = \mu_x = 0$. Dividing $\mathscr{L}_x$ by $\mathscr{L}_y$ we get

$$1 = \frac{1}{2}$$

which is probably wrong.

- Thus only one good is consumed.
- We can just check which yields higher utility.

When the consumer allocates consumption optimally, what is her welfare?

The indirect utility function is the function $v : \mathbb{R}^n_{++} \times \mathbb{R} \to \mathbb{R}$ as

$$v(\boldsymbol{p}, I) = U(\boldsymbol{x}^\star(\boldsymbol{p}, I)).$$

- Abusing notation: $\boldsymbol{x}^\star(\boldsymbol{p}, I)$ is a set.
  - OK because v does not depend on which $\boldsymbol{x} \in \boldsymbol{x}^\star(\boldsymbol{p}, I)$ we choose.

Is the consumer better of when prices are $(\boldsymbol{p}, I)$ or $(\boldsymbol{p}', I')$. This is easy now:

Is $v(\boldsymbol{p}, I) > v(\boldsymbol{p}', I')$

Example

The prior example, we had $U(x,y) = \ln(x) + \ln(y)$.

Is the consumer better off with prices $\boldsymbol{p} = (1,2)$ or $\boldsymbol{p'} = (\frac{3}{2}, \frac{3}{2})$.

- Under $\boldsymbol{p}$ we had $\boldsymbol{x}(I,(1,2)) = (\frac{I}{2}, \frac{I}{4})$.
- So $v(I,(1,2)) = \ln(\frac{I}{2}) + \ln(\frac{I}{4}) = \ln(\frac{I^2}{8})$

Under $\boldsymbol{p}' = (\frac{3}{2}, \frac{3}{2})$:

$$\mathscr{L} = \ln(x) + \ln(y) - \mu_1(\frac{3}{2}x + \frac{3}{2}y - I)$$

We have the first order conditions:

$$\begin{aligned} \mathscr{L}_x: &\qquad \frac{1}{x} - \frac{3}{2}\mu_1 = 0 \\ \mathscr{L}_y: &\qquad \frac{1}{y} - \frac{3}{2}\mu_1 = 0 \\ \mathscr{L}_{\mu_1}: &\qquad \frac{3}{2}x + \frac{3}{2}y = I \end{aligned}$$

- $x^\star = y^\star = \frac{I}{3}$
- $v(I, (\frac{3}{2}, \frac{3}{2})) = \ln(\frac{I}{3}) + \ln(\frac{I}{3}) = \ln(\frac{I^2}{9}) < \ln(\frac{I^2}{8}) = v(I, (1, 2))$

Duality

Sometimes it is helpful to think about the 'opposite' problem: what is the most efficient way to keep a consumer at a certain utility level?

- For example: can we trade to make everyone better off?
- In other words, is there a 'cheaper' way to keep consumers just as well off (or better)?

The problem is formally, as a function of the parameters $(\boldsymbol{p}, u)$:

$$\min_{\boldsymbol{x} \in X} \boldsymbol{p} \cdot \boldsymbol{x} \quad \text{subject to } U(\boldsymbol{x}) \geq u.$$

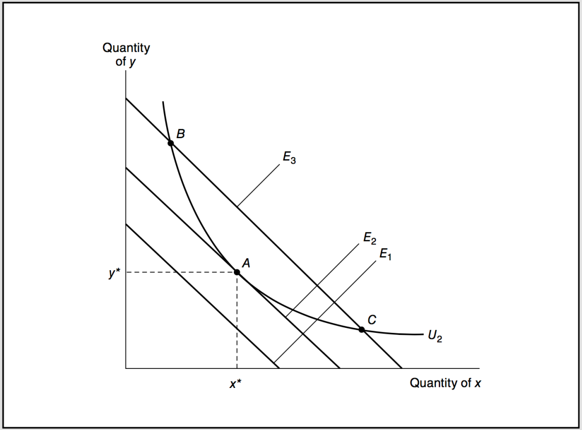
- Cheapest way to ensure utility u or more.

$\boldsymbol{h}^\star : \mathbb{R}^n_{++} \times \mathbb{R} \to \mathbb{R}^n_+$, which specifies the set of minimizers to the above problem, is called the Hicksian demand.

- Hicksian demand is often called compensated demand: the consumer can spend any amount of money so as to attain the same level of utility; she must be compensated in the face of price changes.
- The minimized expenditure function: $e : \mathbb{R}^n_{++} \times \mathbb{R} \to \mathbb{R}$, with $e(\boldsymbol{p}, u) = \boldsymbol{p} \cdot \boldsymbol{h}^\star(\boldsymbol{p}, u)$.

FIGURE 4.6 The Dual Expenditure-Minimization Problem

The dual of the utility-maximization problem is to attain a given utility level (U_2) with minimal expenditures. An expenditure level of E_1 does not permit U_2 to be reached, whereas E_3 provides more spending power than is strictly necessary. With expenditure E_2, this person can just reach U_2 by consuming x^* and y^*.



Notice the same FOC: tangency between MRS and ratio of prices.

- This can be seen from the Lagrangian as well.
- This is why its called the dual problem.

Example

The prior example, we had $U(x, y) = \ln(x) + \ln(y)$.
With prices $\boldsymbol{p} = (1, 2)$ and income I, the indirect utility was $\ln(\frac{I^2}{8})$

What is the cheapest bundle to get the same utility?

Example

So the consumer face the parameters ($\boldsymbol{p} = (1,2), u = \ln(\frac{I^2}{8})$).
The Lagrangian is

$$\mathscr{L} = -(x + 2y) - \mu_1(\ln(\frac{I^2}{8}) - \ln(x) + \ln(y))$$

$$\begin{aligned}
\mathscr{L}_x : \qquad & \frac{\mu_1}{x} - 1 = 0 \\
\mathscr{L}_y : \qquad & \frac{\mu_1}{y} - 2 = 0 \\
\mathscr{L}_{\mu_1} : \qquad & \ln(x) + \ln(y) = \ln(\frac{I^2}{8})
\end{aligned}$$

Example

We have $\frac{y}{x} = \frac{1}{2}$ so $y = \frac{1}{2}x$; implies $\ln(x) + \ln(\frac{1}{2}x) = \ln(\frac{I^2}{8})$

This implies $\frac{x^2}{2} = \frac{I^2}{8}$ or

$$\boldsymbol{h}((1,2), u) = (\frac{I}{2}, \frac{I}{4})$$

Theorem.

If U is continuous and strictly monotone then:

1. $\boldsymbol{h}^{\star}(p, v(\boldsymbol{p}, I)) = \boldsymbol{x}^{\star}(\boldsymbol{p}, I)$
2. $\boldsymbol{x}^{\star}(\boldsymbol{p}, e(\boldsymbol{p}, u)) = \boldsymbol{h}^{\star}(\boldsymbol{p}, u)$
3. $e(\boldsymbol{p}, v(\boldsymbol{p}, I)) = I$
4. $v(\boldsymbol{p}, e(\boldsymbol{p}, u)) = u$

Theorem.

The expenditure function $e(\boldsymbol{p}, u)$ is

1. non-decreasing in $\boldsymbol{p}$
2. homogeneous of degree 1 in $\boldsymbol{p}$
3. concave in $\boldsymbol{p}$
4. continuous in $\boldsymbol{p}$
5. $h_i^\star(\boldsymbol{p}, u) = \frac{\partial e(\boldsymbol{p}, u)}{\partial p_i}$

Proof: Concavity

Consider $(u, \boldsymbol{p})$, $(u, \boldsymbol{p}')$ and $(u, \alpha) = (u, \alpha\boldsymbol{p} + (1-\alpha)\boldsymbol{p}')$. Let $\boldsymbol{x}$, $\boldsymbol{x}'$ and $\boldsymbol{x}^\alpha$ denote the corresponding Hicksian demands. By definition

$$\begin{aligned} \boldsymbol{p} \cdot \boldsymbol{x} &\leq \boldsymbol{p} \cdot \boldsymbol{x}^\alpha \\ \boldsymbol{p}' \cdot \boldsymbol{x}' &\leq \boldsymbol{p}' \cdot \boldsymbol{x}^\alpha \end{aligned}$$

Multiplying the first equation by α and the second by $(1-\alpha)$ and summing we obtain

$$\alpha(\boldsymbol{p} \cdot \boldsymbol{x}) + (1-\alpha)(\boldsymbol{p}' \cdot \boldsymbol{x}') \leq (\alpha\boldsymbol{p} + (1-\alpha)\boldsymbol{p}') \cdot \boldsymbol{x}^\alpha$$

or that $\alpha e(u, \boldsymbol{p}) + (1-\alpha)e(u, \boldsymbol{p}') \leq e(u, \alpha\boldsymbol{p} + (1-\alpha)\boldsymbol{p}')$.

Proof: $\frac{\partial e(\boldsymbol{p},u)}{\partial p_i}$

By the envelope theorem

$$\frac{\partial e}{\partial p_i} = \frac{\partial(p_1 h_1^\star + \ldots + p_n h_n^\star)}{\partial p_i} - \lambda \frac{\partial(U - u)}{\partial p_i} = h_i^\star$$

Theorem. (law of compensated demand)

When price increases, compensated demand decreases:

$$\frac{\partial h_i^\star(\boldsymbol{p}, u)}{\partial p_i} < 0$$

- A similar result does *not* hold for Walrasian demand. We'll see why later.

Proof

- From the prior theorem we have $\boldsymbol{h}_i^\star(\boldsymbol{p}, u) = \frac{\partial e(\boldsymbol{p}, u)}{\partial p_i}$, so

$$\frac{\partial \boldsymbol{h}_i^\star(\boldsymbol{p}, u)}{\partial p_i} = \frac{\partial^2 e(\boldsymbol{p}, u)}{\partial p_i^2}$$

- From the same theorem: e is concave.

Example

An individual with utility function $u(x, y) = \min\{x, y\}$ has income of 150 pounds, and works in a town where the prices of x and y are both 1. The individual's employer proposes sending her to another town where the price of x is 1 pound and the price of y is 2 pounds, with no change in pay. The individual says that having to move is as bad as a cut in pay of A pounds. What is A equal to?

- Her current optimal utility is 75 (where she consumes 75 of each good).
- If she was to move, her optimal utility would be 50 (where she spends 50 on x and 100 of y).
- She could currently get 50 with a wage of 100, so $A = 50$.