Vague Preferences and Contracts

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Abstract

In this paper, I examine decision making in an environment where payoff relevant contingencies are *vague*, that is neither absolutely true nor absolutely false. In this model, an agent values acts that are predicated on *linguistic* statements, rather than an exogenous state-space. I axiomatize a class of preferences under which agent's beliefs about the degree of truth of contingencies is identified from her choices. I then apply this model to a simple contracting environment wherein contracts must be explicitly constructed using said linguistic statements. I show that different restrictions on the contract writing technology can impart different outcomes. However, under mild conditions, as the cost of contractual complexity vanishes, so do the distortionary effects of vagueness.

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1 Introduction

There are many statements that appear neither absolutely true nor absolutely false. That "Banana #4 is ripe" (see Figure 1), for instance, may lie somewhere between true and false. Notice also, such a statement is not a matter of uncertainty, but of vagueness; the actual boundary between ripe and unripe is simply not as crisp as the language describing it. More consequentially, economic decision making—from writing contracts to gathering evidence to enforcing social norms—regards this kind of vagueness as much as it regards probabilistic uncertainty. Judicial decisions, for example, are often predicated on the interpretation of law, deciding the extent to which a particular law applies in a (fully certain) context. A statements like "Gilead Science's Hepatitis-C treatment, *sofosbuvir*, does not infringe on Idenix Pharmaceutical's patent due to the latter's failure to meet the written description and enablement requirements." is not a matter of uncertainty. Rather, this statement—the truth of which was eventually settled by a jury to the tune of 2.5 billion dollars—concerns the interpretation of various legal/contractual clauses, such as what constitutes *common knowledge* or an *industry standard* (Zullow et al., 2021).

This points to a situation in which economic decisions are predicated on contingencies that are not always self-evidently true or false. Just like the ripeness of a banana, there is an inexorable vagueness in the contractual clauses that we write. Such vagueness can arise through some inherent indefinably or from the impracticality of constructing a contractual language that is as complex as the real world. In either case, we find ourselves making decisions on the basis of contingencies that, even in the absence uncertainty, may feel neither absolutely true not absolutely false.

In this paper, I consider an agent who faces payoff relevant uncertainty embodied by a collection of linguistic statements. While the truth of these statements will determine the payoff of a given act, the statements need not be considered exactly true or false, but instead can take intermediate values representing partial truth. Within this environment, I axiomatize a class of preferences under which a unique state-space representing the agent's beliefs about the degree of truth can be identified. This models allows for two orthogonal types of imprecision: uncertainty (not knowing how true a given statement is) and vagueness (a statement being only partially true or false).

I then apply this model to a simple Principal-Agent contracting environment wherein the Principal must explicitly construct contracts using the linguistic statements. In other words, rather than writing contracts that make reference to an objective, and exogenous, state-space,

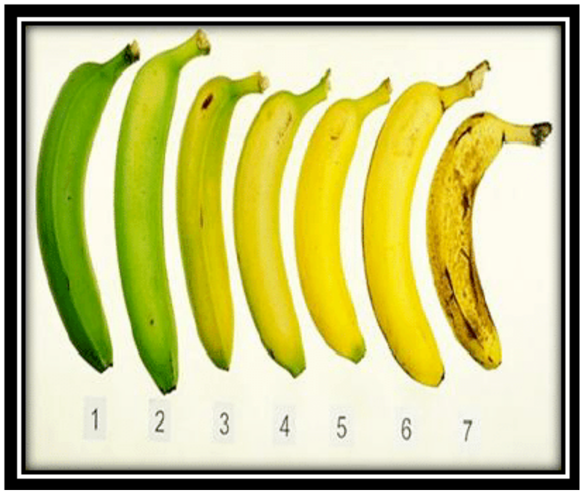


Figure 1: Bananas in various stages of ripeness. Photo credit: Madan et al. (2014).

the Principal must condition payments on the truth of linguistic statements. I show that different restrictions on the contract writing technology can impart different outcomes, but under mild conditions, the distortionary effect of vagueness vanish as the cost of writing complex contracts vanishes: given the technology to write arbitrarily complex contracts (in the given language) the Principal can implement outcomes arbitrarily close to what can be achieved in the baseline model where contracts are simply functions from an exogenous state-space. The take away is striking. Imprecision in language does not inherently imply imprecision in contractual efficacy; despite the fact that the language is innately less precise than the world it is attempting to describe, linguistic contracts can approximate efficient outcomes.

To begin, I assume that the payoff relevant aspects of the decision problem are described by a set of linguistic statements $\mathscr{L}$, which, as is standard, will include statements $\neg\varphi$ (*not* φ), $\varphi \wedge \psi$ (φ *and* ψ), $\varphi \to \psi$ (φ *implies* ψ), etc., for statements $\varphi, \psi \in \mathscr{L}$. Unlike the classical approach, however, these statements can be *partially* true; this is instantiated by allowing truth values to reside anywhere in the unit interval $[0, 1]$. In particular, a function $v : \mathscr{L} \to [0, 1]$ is a *valuation* if it assigns truth values to the statements in $\mathscr{L}$ in a way that respects the logic operations:[1] for example, $v(\neg\varphi) = 1 - v(\varphi)$, or $v(\varphi \wedge \psi) = \min\{v(\varphi), v(\psi)\}$.

The agent's preferences are defined over $\mathscr{L}$-contingent contracts, functions that determine the payoff contingent on the truth of various statements. For example, if $\varphi \in \mathscr{L}$ is "Banana #4 is ripe," then the act f of eating Banana #4 determines a payoff that is modulated

[1] See Section 2.2 for a rigorous definition. Note that if v is $\{0, 1\}$-valued, then this reduces to classical logic.

by the truth of φ—the more true φ the higher f's payoff. If $\psi \in \mathscr{L}$ is "Sofosbuvir infringes on extant intellectual property" then the value of the act g—investing in Sofosbuvir—is likewise modulated by the truth of ψ. Such contracts specify a payoff contingent only on the absolute truth of statements while the actual payoff is tempered by their degree of truth. That is, contracts cannot directly reference the degree of truth: "Receive x in the event φ is $\frac{1}{2}$ true," for example, is prohibited. This is essentially a dictate that truth values themselves are inexpressible (at least in a legally binding way), and only the language itself is available as a contracting tool.

The first main result of the paper is a simple behavioral criteria equivalent to the existence of a unique *vague model of uncertainty*, (Ω, V, μ), that explains the agent's choices. A vague model of uncertainty is:

- Ω a state-space,
- $V = \{v_\omega : \mathscr{L} \to [0,1]\}_{\omega\in\Omega}$ is a *valuation* for each state,
- μ a probability measure over Ω.

The states in Ω differ in the way they value statements. The agent's uncertainty about the degree of truth of statements is then captured by μ.

Such a model explains the agent's preferences when the value of a contract paying x contingent on φ is valued according to[2]

$$\int_\Omega x v_\omega(\varphi)\, \mathrm{d}\mu \tag{$\star$}$$

If the agent knew for sure that the truth value of φ was given by $v(\varphi) \in [0,1]$ then her value for the contract that provides x contingent on the absolute truth of φ would $xv(\varphi)$. She values contracts linearly in both the probability and truth value.

The choice objects are predicated on a discrete language rather than an exogenous state-space, and make no reference whatsoever truth values or anything similar. Nonetheless, not only can a state-space can be inferred from choices, but it is unique! So, the probability of the partial truth of statements, and the correlations between them, can be uniquely recovered from an agent's preferences over simple contractual objects.

I then consider how this model of vagueness, i.e., partial truth, might effect strategic contracting environments. I consider a simple principal-agent environment: The agent chooses

[2]The choice objects considered in the paper are more complicated in that they can depend on multiple statements—for now we examine only the simpler case.

an unobservable, costly action that partially determines the success of a project under the ownership of the principal. Because the agent's choice is unobservable, the principal must incentivize the agent to take costly actions by conditioning the payment to be contractually contingent on the outcome of the project.

The critical distinction from the standard environment is that even if the realization of the success of the project is ex-post verifiable (in the sense that the players, and perhaps some enforcement mechanism such as a legal system, agree on the realization), it is not directly ex-ante contactable. For example, while a court might be able to determine the extent to which a particular project infringed on the intellectual property of an outside party, it is not generally possible that an investment contract could ex-ante depend on this assessment. To make this restriction precise: let $\mathscr{L}$ denote a language describing the outcome of the project, and assume that the value of the project to the principal depends only on the (possibly partial) truth of these statements.

Let (Ω, V, μ) be a vague model of uncertainty for this language. In the usual reduced approach, contracts are specified as functions over Ω; the states of Ω are themselves directly referenced in the contract. Instead, I take a linguistic approach: the principal must actually formulate a literal contract, constructed from the linguistic elements available, $\mathscr{L}$. These contracts are exactly the objects of choice posited in the decision theoretic investigation, and are valued by the players according to ($\star$).

For example, assume the success or failure of the project is accounted for by a single, contactable, statement s:

$$\text{s} = \text{“The project is a success.”}$$

So, a contract that specifics a payment of $x \in \mathbb{R}$ contingent on s and $y \in \mathbb{R}$ on $\neg$s corresponds to the function over Ω, $\omega \mapsto x v_\omega(\text{s}) + y v_\omega(\neg\text{s}) = (x-y)v_\omega(\text{s}) + y$.

Within this framework, I ask two questions: first how does the class of contracts available to the principal affect the set of implementable effort levels, and second, how does it affect the cost of such implementation. In other words, in comparison to the benchmark where contracts can directly specify payments contingent on Ω, how constraining is requirement that contracts be predicated on the underlying language? Unsurprisingly, I show that the principal is often strictly worse off when constrained to write contracts that depend only on the truth or falsity of s: i.e., they take the form $\omega \mapsto x v_\omega(\text{s}) + y v_\omega(\neg\text{s})$, as above.

Remarkably, however, by allowing contracts predicated on arbitrarily complex statements, for example compound statements like $(\text{s} \rightarrow \neg\text{s})$, the distortion introduced relative to the standard model is arbitrarily small. That is, the principal can not only implement the same actions as in the benchmark case, but can do so at the same approximate cost!

This result, beyond illuminating the boundaries of what can be achieved using more realistic contracts, can be seen as a theoretical justification for the use of state-spaces models. Although the modeler might have in mind a particular representation of uncertainty Ω, the contracts themselves make no mention of the states. Nonetheless, outcomes—the action taken by the agent and the ex-post utility to each player—are the same as in the state-space model.

1.1 Discussion and Related Literature

Despite not being well studied within decision theory, or economics more broadly, vagueness is a universal aspect of human language and any decision making predicated thereon. As such, it has long been the study of linguists and philosophers, for a broad introduction see Sorensen (2023). The classical, Boolean approach to logic precludes by fiat the partial truth of propositions, rendering it a poor tool for the study of vagueness. This paper is build on the machinery of multi-valued logics, in particular Łukasiewicz logic, which were motivated at least in part by concerns of vagueness and linguistic ambiguity. For a philosophical overview of these ideas see Gottwald (2022); for a broad, albeit elementary, mathematical treatment see Bergmann (2008); for a rigorous, encyclopedic treatment of Łukasiewicz logic and its related mathematical machinery (namely, MV algebras), see Mundici (2011).

This paper considers inexactness as concerning the truth of contingencies on which decisions are predicated. A distinct way of introducing inexactness is via fuzzy preferences. Here an agent's preferences themselves are inexact: statements like 'x is preferred to y,' are given graded truth values, in the same manner that contingencies entertain graded truth in this paper. Salles (1998) provides an overview of fuzzy preference theory. These idea find their roots in foundational concepts of decision theory: for example Krantz et al., eds (1971), in the seminal work on the foundations of measurement, consider a binary relation over *pairs* of alternatives can be interpreted as the strength of preference; Fishburn (1973) allows for inexactness by entertaining interval valued utility.

Inexactness also enters decision theory through relaxations of probabilistic sophistication. In this literature—under the monikers 'ambiguity' and 'uncertainty'—inexactness regarding probabilities is captured by entertaining multiple probability distributions (Bewley, 1986; Gilboa and Schmeidler, 1989; Maccheroni et al., 2006) or non-additive probability like objects (i.e., capacities) (Schmeidler, 1989). See Machina and Siniscalchi (2014) for an overview.

This paper is a continuance of the recent program of syntactic decision theory, where the primitive relates to statements about the world rather than semantic acts or lotteries Blume et al. (2021); Bjorndahl and Halpern (2021); Piermont and Zuazo-Garin (2023) and Bjorndahl and Halpern (2023) and to a lesser extent Lipman (1999); Piermont (2017); Kochov (2018) and Minardi and Savochkin (2019). Like the current manuscript, within each of these papers, the interpretation of language is itself of sincere interest, for example as mitigated through awareness (Kochov, 2018; Piermont, 2017), causal relationships Bjorndahl and Halpern (2021, 2023), or logical reasoning Lipman (1999); Piermont and Zuazo-Garin (2023).

This paper is also related to the literature on strategic contracting, and, in particular, the role of complexity and language in the design of contracts. It is obvious and unremarkable that real-world contracts are imprecise. It is often argued that is that this imprecision enters optimally, increasing the cost of litigation but reducing the overhead cost of contract writing (Scott and Triantis, 2005; Choi and Triantis, 2009; Suzuki et al., 2020). However, the staggering cost of litigation—estimated as high as 10% of corporate spending (Antill and Grenadier, 2020)—seems to dwarf the plausible cost of writing more precise contracts.

This paper offers a resolution: while (sufficiently complex) vague contracts can approximate any desired outcome, their constituent clauses *are still imprecise*, and thus, would require litigation if challenged. While these ideas are only implicit in the paper, I hope that this framework—in which the language in which a contract is written is made explicit—can be fruitful in exploring strategic contracting in more detail.

2 Model

2.1 Language

Let $\mathscr{P}$ collect a set of propositional variables, intended to represent statements relevant to the decision problem at hand. For example "Anton was negligent" or "The idea was original." Then, $\mathscr{L}$ is the language defined inductively, beginning with $\mathscr{P}$ and such that if φ, ψ are in $\mathscr{L}$ then so too are $\neg\varphi$ and $\varphi \to \psi$. The interpretation is standard: $\neg\varphi$, the negation of φ, is interpreted as the statement that φ is not true; $\varphi \to \psi$, the implication between of φ and ψ, is interpreted as the statement that φ (being true) implies ψ (being true).

It will be notationally convenient to define several other operations from implication and negation:

- $\varphi \vee \psi =_{def} (\varphi \rightarrow \psi) \rightarrow \psi$
- $\varphi \wedge \psi =_{def} \neg(\neg\varphi \vee \neg\psi)$
- $\varphi \oplus \psi =_{def} \neg\varphi \rightarrow \psi$
- $\varphi \odot \psi =_{def} \neg(\varphi \rightarrow \neg\psi)$
- $\mathbf{T} =_{def} \varphi \rightarrow \varphi$
- $\mathbf{F} =_{def} \neg\mathbf{T}$

The operators $\vee$ and $\oplus$ both represent disjunction; the former is called weak disjunction and the latter strong, or Łukasiewicz, disjunction. Analogously, $\wedge$ and $\odot$ represent weak and strong (or Łukasiewicz) conjunction. The interpretation of these operators will be discussed after valuations are introduced in the next subsection.

An *act* is a function $f : \mathscr{L} \rightarrow \mathbb{R}_+$ such with finite support. Endow the set of acts with the subspace topology inherited from the product topology on $\mathbb{R}^{\mathscr{L}}$ (where $\mathscr{L}$ has the discrete topology). The set of all acts, denoted $\mathscr{F}$, is a mixture space under the operation of point-wise mixtures. For $x \in [0,1]$ and $\varphi \in \mathscr{L}$, let x_φ denote the act that maps φ to x and all other statements to 0.

The primitive of the model is a preference relation $\succcurlyeq$ over $\mathscr{F}$. Per usual, let $\sim$ and $\succ$ denote the symmetric and anti-symmetric components of $\succcurlyeq$.

2.2 Valuations

A *valuation* of $\mathscr{L}$ is a function $v : \mathscr{L} \rightarrow [0,1]$ such that

$[\![\neg]\!]$ $v(\neg\varphi) = 1 - v(\varphi)$

$[\![\rightarrow]\!]$ $v(\varphi \rightarrow \psi) = \min\{1, 1 - v(\varphi) + v(\psi)\}$

A valuation represents the degree of truth of each statement in the language; a value of 1 representing absolute truth and 0 absolute falsity. In standard economics models, built from classical logical foundations, only such truth values are ever considered.[3] Thus, it is by the permission of intermediate values that this model can accommodate vagueness or partial truth. Let $\mathcal{V}(\mathscr{L}) \subset [0,1]^{\mathscr{L}}$ collect the set of all valuations over the language $\mathscr{L}$.

[3]Indeed, the set of valuations which take only the values 0 and 1 provide semantics for classical logic.

Endow $\mathcal{V}(\mathscr{L})$ with the product topology. Since each valuation is determined inductively by its value on $\mathscr{P}$, it is clear that $\mathcal{V}(\mathscr{L})$ can be identified with $[0,1]^{\mathscr{P}}$.

When $v(\psi) \geq v(\varphi)$ then ψ is considered more true than φ. Given this interpretation, the restrictions on v states that: (i) negating a statement 'inverts' its truth value. As such, maximal vagueness occurs at a valuation of $\frac{1}{2}$ where a statement and its negation are equally true. (ii) φ implies ψ is absolutely true, that is $v(\varphi \to \psi) = 1$, whenever φ is less true than ψ. When this is not the case, the valuation of the implication is 1 less the gap in truth values: the implication is absolutely false if and only if $v(\varphi) = 1$ and $v(\psi) = 0$.

Say that $\psi \dot{\geq} \varphi$ if $v(\psi) \geq v(\varphi)$ for all $v \in \mathcal{V}(\mathscr{L})$, and let $\doteq$ denote the symmetric component. Notice that $\psi \dot{\geq} \varphi$ exactly when φ implies ψ is absolutely true under *every* possible valuation. The relation $\doteq$ represents logical equivalence in the underlying logic.

Over our derived operators, we have:

$[\![\vee]\!]$ $v(\varphi \vee \psi) = \max\{v(\varphi), v(\psi)\}$

$[\![\wedge]\!]$ $v(\varphi \wedge \psi) = \min\{v(\varphi), v(\psi)\}$

$[\![\oplus]\!]$ $v(\varphi \oplus \psi) = \min\{1, v(\varphi) + v(\psi)\}$

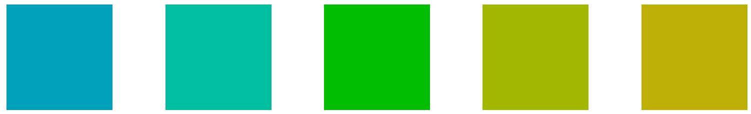
$[\![\odot]\!]$ $v(\varphi \odot \psi) = \max\{0, v(\varphi) + v(\psi) - 1\}$

$[\![\mathbf{T}]\!]$ $v(\mathbf{T}) = 1$

$[\![\mathbf{F}]\!]$ $v(\mathbf{F}) = 0$

Examining the valuation for $\vee$ and $\oplus$, we can see how these two different formalities can capture different colloquial uses of 'or.' Consider, for example, the following statements regarding the squares in Figure 2. It seems plausible that the statements $\beta(b)$ = "Square (b) is blue," $\gamma(b)$ = "Square (b) is green," and $\gamma(d)$ = "Square (d) is green" are all assigned intermediate truth values, perhaps $v(\beta(b)) = v(\gamma(b)) = v(\gamma(d)) = \frac{1}{2}$.

Since square (b) is a clear mixture of green and blue, it further plausible that the statement φ = "Square (b) is blue **or** Square (b) is green" to be absolutely true. This is because the two sub-formula, $\beta(b)$ and $\gamma(b)$, compensate each other: in the space between green and blue, a decrease in blueness corresponds to an increase in greenness. As such, the 'or' in the English language statement φ is playing the role of strong disjunction, so the truth value of φ is the (truncated) sum of the truth values of its constitute parts. φ is interpreted as $\beta(b) \oplus \gamma(b)$. On the other hand, the statement ψ = "Square (b) is green **or** Square (d) is



(a) (b) (c) (d) (e)

Figure 2: Squares of different colors.

green" seems no more true than either of the sub-statements, and so might reasonably be assigned a value of $\frac{1}{2}$. This is because the two sub-formula, $\gamma(b)$ and $\gamma(d)$, do not have any a-priori relation to one another, so the truth value of ψ is maximum of the truth values of its constitute parts. ψ is interpreted as $\gamma(b) \vee \gamma(d)$.

Call φ and ψ *disjoint* , denoted $\varphi \perp \psi$ if $v(\varphi \odot \psi) = 0$ for all $v \in \mathcal{V}(\mathscr{L})$. If $\psi \perp \psi$, then by definition, under all valuations $\varphi \to \neg\psi$ is absolutely true: the truth of φ precludes the truth of φ, substantiating the nomenclature 'disjoint' (of course, this is not a binary restriction, it states that the *more* true φ is the less true ψ is). By examining the $[\![\odot]\!]$, it is clear that $\perp$ is symmetric.

2.3 Representation

A *vague model of uncertainty* is a tuple (Ω, V, μ) such that Ω is a non-empty compact Hausdorff space, referred to as the state space, $V : \Omega \to \mathcal{V}(\mathscr{L})$ is continuous injection that assigns a valuation v_w to each state $\omega \in \Omega$, and μ a regular Borel probability measure over Ω. The dictate that V be injective is simply a prohibition of redundant states, and can be dropped at the cost of increased notational burden.

Given V, each $f \in \mathscr{F}$ defines a function $f^V : \Omega \to [0, 1]$ via the map

$$f^V : \omega \mapsto \sum_{\varphi \in \mathscr{L}} f(\varphi) v_\omega(\varphi). \tag{1}$$

Since V is continuous, so too is f^V, as it is the sum of a finite number of continuous functions.

In each state of the state-space, the map f^V yields the 'weighted' payoff of f. To see this, first consider classical valuations where v_ω is $\{0, 1\}$-valued. Then, in state ω, the act x_φ pays x if and only if φ is true; this is reflected by the map x_φ^V, which sends ω to x whenever $v_\omega(\varphi) = 1$ and to 0 otherwise. For a general V, a statement φ may not be completely true nor completely false, so the payoff is weighted in a continuous manner—x_φ^V sends ω to $x v_\omega(\varphi)$, generalizing the $\{0, 1\}$-valued case. Owing to the obvious linearity of $(\cdot)^V$, we then have that, for more complex acts, $f = \alpha^1 x^1_{\varphi^1} + \ldots + \alpha^n x^n_{\varphi^n}$, the map f^V is linear combination of the indicator functions $x^V_{\varphi^1} \ldots x^V_{\varphi^n}$.

Then, we say such a model represents $\succcurlyeq$ if

$$f \succcurlyeq g \iff \int_\Omega f^V \, \mathrm{d}\mu \geq \int_\Omega g^V \, \mathrm{d}\mu \tag{$\star$}$$

When (Ω, V, μ) represents $\succcurlyeq$, the agent faces two forms of imprecision: vagueness (the fact that the truth of a statement is not absolute) captured by V and uncertainty (the fact that the truth values of statements are unknown) captured by μ. Given these, it is as if the agent maximizing the expectation of f^V, the truth-value-weighted payoff of an act, according to her subjective belief μ.

Of course, the names of the states are irrelevant for the interpretation of behavior. As such, we can identify different models that differ only in the way that states are labeled. Towards this, call a model (Ω', V', μ') a *relabeling* of (Ω, V, μ) if for all measurable $D \subseteq \mathcal{V}(\mathscr{L})$,

$$\mu(\{\omega \in \Omega \mid v_\omega \in D\}) = \mu'(\{\omega' \in \Omega' \mid v_{\omega'} \in D\}).$$

In words, the two models proscribe the same probability to any set of valuations.

3 Axioms

In this section, I present four simple axioms on $\succcurlyeq$ that characterize representation by a vague model of uncertainty. The first three are extremely standard: First, that $\succcurlyeq$ is a continuous preference relation.

A1—ORDER (O). $\succcurlyeq$ is a non-trivial, continuous weak order.

The next axiom states that preferences are reflective of payoffs, in the sense that a higher payoff is preferred.

A2—PAYOFF MONOTONICITY (M). For all $f, g \in \mathscr{F}$ if f point-wise dominates g then $f \succcurlyeq g$.

Next, Independence, provides aggregation from simple to more complex acts in the standard fashion.

A3—INDEPENDENCE (I). For all $f, g, h \in \mathscr{F}$ and $\alpha \in (0, 1)$,

$$f \succcurlyeq g \quad \text{if and only if} \quad \alpha f + (1-\alpha)h \succcurlyeq \alpha g + (1-\alpha)h.$$

The final axiom is the only restriction that merits discussion. Whereas independence, above, relates to aggregation over payoffs, keeping the winning conditions fixed, the next axiom, consistency, dictates how preferences are aggregated across *statements*.

A4—ŁUKASIEWICZ CONSISTENCY (Ł). For all $\varphi, \psi \in \mathscr{L}$ such that $\varphi \perp \psi$, and all $x \in \mathbb{R}_+$,

$$x_\varphi + x_\psi \sim x_\eta,$$

for any $\eta \in \mathscr{L}$ such that $\eta \doteq (\varphi \oplus \psi)$

To understand consistency, it is helpful to first examine the classical case, thinking only of $\{0, 1\}$-valued interpretations. Here the disjointness of φ and ψ is tantamount to $\varphi \to \neg\psi$ (and vice-versa), that the two statements are never true at the same time. Because of this, a bet on φ and a bet on ψ are can never both win simultaneously—receiving both bets should be valued the same as receiving a bet on the joint statement that either φ *or* ψ is true. In classical logic, 'or' would be captured naturally by disjunction, '$\vee$'; so the classical counter part to the axiom would state $\frac{1}{2}_\varphi + \frac{1}{2}_\psi \sim \frac{1}{2}_{\varphi \vee \psi}$.

When considering more general valuations, the notion of disjointness is weaker, since the truth of φ implies the falsity of ψ only relatively—it is permissible that both φ and ψ have non-zero valuations, so long as their sum never exceeds 1. As such, the corresponding notion of disjunction must be commensurately strengthened. The correct notion is *strong disjunction*, the $\oplus$ operator. The truth of $\varphi \oplus \psi$ is given by $\min\{1, v(\varphi) + (\varphi)\}$, which allows for the possibility that neither statement is absolutely true, as long as falsity in one statement is offset but veracity of the other.

These four axioms suffice to ensure that $\succcurlyeq$ is represented by a unique (up to labeling) vague model of uncertainty:

THEOREM 1. *A relation $\succcurlyeq$ satisfies* **A1**-**A4** *if and only if there exists a vague model of uncertainty, (Ω, V, μ), which represents it. Moreover, this representation is unique up to relabeling.*

4 Comparative Truth and Vagueness

In this section, I consider some economically relevant measures of comparative truth and vagueness. That is, I present a methodology for *directly* assessing an agent's subjective belief in the likelihood of one statement φ being more true (or vague) than another ψ.

Given a vague model of uncertainty, (Ω, V, μ), the preferences over bets on a single statement, for example $x_\varphi \succcurlyeq x_\psi$, reflect the agent's beliefs about the expected truth of φ relative to ψ. As a consequence of the rich calculus of formulae in Łukasiewicz logic, similar preferential statements can encode other, more detailed comparisons between statements.

In particular, for a given statement $\varphi \in \mathscr{L}$ and $n \in \mathbb{N}$, let $n \odot \varphi =_{def} \varphi \odot \varphi \ldots \odot \varphi$, where there are n repetitions of φ. From this, we can calculate the likelihood that one statement is more true than another:

Theorem 2. *Let $\succcurlyeq$ be represented by a vague model of uncertainty. (Ω, V, μ). Then:*

$$\inf_{n\in\mathbb{N}} \{x \in [0,1] \mid 1_{(n\odot(\psi\to\varphi))} \sim x_{\mathbf{T}}\} = \mu(\{\omega \in \Omega \mid v_\omega(\varphi) \geq v_\omega(\psi)\}) \tag{2}$$

and in particular, $1_{(\psi\to\varphi)} \sim 1_{\mathbf{T}}$ if and only if $v_\omega(\varphi) \geq v_\omega(\psi)$ μ-almost surely.

Theorem 2 shows that the limit, as $n \to \infty$, of the valuation of the formula $1_{(n\odot(\psi\to\varphi))}$ exactly measures the probability of the event "φ is more true that ψ". This shows that the modeler need not construct the entire state-space in order to make meaningful local comparisons between variables.

Pushing further, we can find simple measures of vagueness. A statement that is valued at 1 or 0—that is either absolutely true or false—is not vague at all, whereas a statement valued at $\frac{1}{2}$ is maximally vague—it is just as true as it is false. Interpolating linearly, we can construct a numerical measure of vagueness given any valuation: Let

$$\textsc{vg} : \mathscr{L} \times \mathcal{V}(\mathscr{L}) \to [0,1] \tag{VG}$$

be defined as $\textsc{vg}(\varphi, v) = 1 - |v(\varphi) - v(\neg\varphi)|$. Given a valuation, vg measures the inverse distance between the truth of a statement and the truth of its negation.

Given a vague model of uncertainty, (Ω, V, μ), we can use this operator to compare the vagueness of distinct statements: say that say that φ *is in expectation more vague* than ψ if $\int_\Omega \textsc{vg}(\varphi, v_\omega)\ \mathrm{d}\mu \geq \int_\Omega \textsc{vg}(\psi, v_\omega)\ \mathrm{d}\mu$. A stronger comparison requires with probably 1, that φ is perceived to be more vague than ψ: say φ is *almost surely more vague* than ψ if

$\mu(\{w \in W \mid \textsc{vg}(\varphi, v_\omega) \geq \textsc{vg}(\psi, v_\omega)\} = 1$. Finally, call φ *almost surely Boolean* if there is μ-probability 0 that φ has a positive vagueness: $\mu(\{w \in W \mid \textsc{vg}(\varphi, v_\omega) = 0\} = 1$.

It turns out that our measure of vagueness, VG, can be succinctly captured as a formula:

- $\Box\varphi =_{def} \neg\big((\varphi \odot \varphi) \oplus (\neg\varphi \odot \neg\varphi)\big)$

The valuation of $\Box$ reflects the bounds of vagueness outlined above via VG, and, leveraging this allows for characterizing the above measures of relative vagueness in terms of preferences:

THEOREM 3. *Let $\succcurlyeq$ be represented by a vague model of uncertainty.* (Ω, V, μ). *Then:*

(i) $v(\Box(\varphi)) = \textsc{vg}(\varphi, v)$

(ii) $1_{\Box(\varphi)} \succcurlyeq 1_{\Box(\psi)}$ *iff φ is in expectation more vague than ψ*

(iii) $1_{(\Box(\psi) \to \Box(\varphi))} \sim 1_{\mathbf{T}}$ *iff φ is almost surely more vague than ψ*

(iv) $1_\varphi \sim 1_{(\varphi \oplus \varphi)}$ *iff φ is almost surely Boolean*

An axiomatization of "classical" expected utility follows from a rather immediate corollary of Theorem 1. Specifically, the additional axiom

A5—BOOLEANISM (B). For all $\varphi \in \mathscr{L}$ and $x \in \mathbb{R}_+$, $x_\varphi \sim x_{(\varphi \oplus \varphi)}$.

COROLLARY 4. *Let $\mathscr{L}$ be countable. Then $\succcurlyeq$ satisfies* **A1**-**A5** *if and only if there exists model of uncertainty,* (Ω, V, μ), *which represents it and such that* $v_\omega(\mathscr{L}) = \{0, 1\}$.

5 VAGUE CONTRACTS

Consider a simple principal-agent environment. The agent chooses a costly action that, in part, determines the success of a project under the ownership of the principal. The agent's possible choices are given by a finite set E; for effort level $e \in E$, she pays a utility cost $c(e) \in \mathbb{R}$. The agent's utility index over money is $u : \mathbb{R} \to \mathbb{R}$ so that, conditional on choosing effort $e \in E$ and receiving payment from the principal $x \in \mathbb{R}$, her ex-post utility is $u(x) - c(e)$. We take u to be continuously differentiable and strictly monotone. Assume her outside option is $\bar{u} \in \mathbb{R}$.

Because the agent's choice is unobservable, the principal must incentive the agent to take costly actions by conditioning the payment to be contractually contingent on the outcome of the project. Towards this, let Ω denote a state-space, the states of which are associated with the various outcomes of the project. The dependence of the project's success on effort can then be captured by a set of conditional probabilities $\{\mu_e\}_{e \in E} \subset \Delta(\Omega)$; after taking action $e \in E$ the distribution of states is given by μ_e.

The critical distinction from the standard environment is that even if the realization of Ω is ex-post verifiable (in the sense that the players, and perhaps some enforcement mechanism such as a legal system, agree on the realization), it is not directly ex-ante contactable. Instead of the usual reduced approach wherein the states are themselves the elements of the contract, here the principal must actually formulate a literal contract, constructed from the linguistic elements available. In what follows, we will examine how the technology to write contracts changes what can be implemented by the principal, and how this technology affects his profit.

In the same way as with the decision theoretic acts defined in Section 2.1, each contract will determine a function $\Omega \to \mathbb{R}$. Thus, for any contract writing technology, we can think of the associated set of contracts as a subset of $C \subseteq \mathbb{R}^\Omega$.

Say that a contract $f \in \mathbb{R}^\Omega$ *implements* $e \in E$ if

$$e = \arg\max_{e' \in E} \int_\Omega u \circ f \, \mathrm{d}\mu_{e'} - c(e') \tag{IC}$$

$$\int_\Omega u \circ f \, \mathrm{d}\mu_e \geq \bar{u} \tag{IR}$$

hold. That is, f implements e when it uniquely entices the agent to exert effort e (IC) and it acceptable to the agent (IR). Moreover, for a fixed set of contracts, $C \subseteq \mathbb{R}^\Omega$, say that C *implements* $e \in E$ if there exists some $f \in C$ that implements e. This defines a mapping κ that determines the cost of implementing a given effort level, provided a specific contract writing technology. Specifically,

$$\kappa : C \mapsto \big(e \mapsto \inf\big\{ \int_\Omega f \, \mathrm{d}\mu_e \mid f \text{ implements } e, f \in C\big\}\big) \tag{3}$$

so that for a set of contracts C, $\kappa(C)$ is a function determining the (infimum) cost of implementing $e \in E$, and where, owing to the fact that $\inf_\varnothing = \infty$, the cost of implementing an non-implementable action is infinite.

5.1 Linguistic Constructability

We now turn our attention to particular classes of contracts, and examine two natural questions: first how does the class of contracts affect the set of implementable effort levels, and second, how does it affect the cost of such implementation.

Let C^{meas} denote the set of (Borel) measurable contracts, i.e., the set of contracts that are measurable functions over Ω (where reliance on Ω is notationally suppressed). Although most of these contracts will not correspond to any linguistically constructable contract, it is instructive to take C^{meas} as the baseline by which we compare a particular contract writing technology. Indeed, the elements of C^{meas} are defined as functions directly on the state space and so delineate the maximal precision of any possible contract technology.[4]

Now, in contrast to arbitrary contracts, we will now examine contracts that are *linguistically constructable*, that is, they are functions from a language into payoffs, in the same sense as our decision theoretic acts defined above. Assume that the outcome of the project (i.e., its value to the principal) is accounted for by a set of propositional statements $\mathscr{P} = \textsc{p}_1 \ldots \textsc{p}_n$, and let $\mathscr{L} = \mathscr{L}(\mathscr{P})$ collect the propositional language there based. Now, consider the set of contracts:

$$C^{\mathscr{P}} = \left\{ xv_\omega(\textsc{p}) + yv_\omega(\neg\textsc{p}) \mid x, y \in \mathbb{R}, \textsc{p} \in \mathscr{P} \right\}.$$

Each contract $f \in C^{\mathscr{P}}$ is predicated directly on the truth of the propositions in $\mathscr{P}$. When $y = 0$, the contract pays linearly in the truth of $\textsc{p}$. Notice that adding an additional payoff criteria based on the negation of $\textsc{p}$ is tantamount to providing some constant payoff in all states. Indeed, a contract of the form $xv_\omega(\textsc{p}) + yv_\omega(\neg\textsc{p})$ induces the affine function $f : \omega \mapsto (x - y)v_\omega(\textsc{p}) + y$. As such, we see that $C^{\mathscr{P}}$ is the set of all affine contracts.

While not particularly surprising, the set of implementable outcomes under $C^{\mathscr{P}}$ can be (far) smaller than under C^{meas}—the limitation that the principal condition the payment only on the truth or falsity of propositions is constraining. This is made clear by the following simple example:

Example 1. Let $\mathscr{P} = \{\text{s} = \text{“The project is a success”}\}$. Assume the agent has linear utility, so that u is the identity. Let $\Omega = [0, 1]$ where $v_w \in \mathcal{V}(\mathscr{L})$ is the unique valuation that sets $v_\omega(\text{s}) = \omega$. Let there be two effort levels $E = \{e, e'\}$ where $\mu_e = Unif([0, 1])$ and $\mu_{e'} = \frac{1}{3}Unif([\frac{1}{4}, \frac{3}{4}]) + \frac{2}{3}Unif([0, 1])$ and where $Unif([a, b])$ is the uniform distribution over the interval $[a, b]$. That is, effort e induces a uniform probability over all truth values,

[4] Up to measurability, of course, which is an indelible constraint.

whereas e' induces a probability wherein outcomes in $[\frac{1}{4}, \frac{3}{4}]$ are twice as likely as outcomes not in this central interval. Further, assume $c(e') > c(e)$. Then there is no contract in $C^{\mathscr{P}}$ that implements e'—indeed no contract will satisfy the (IC) constraint.

Notice that for any contract f corresponding to $xv_\omega(\mathrm{s}) + yv_\omega(\neg\mathrm{s})$, we have

$$\int_\Omega f \, \mathrm{d}\mu_e = \frac{x-y}{2} + y = \int_\Omega f \, \mathrm{d}\mu_{e'}$$

so by the fact that e' is more costly that e, it can never be implemented. Nonetheless, e' can obviously be implemented by a non-linear contract that places incentives in the middle interval of outcomes. For example, for small enough difference in costs, $g = 1-|1-2\omega|$ can incentivize the agent to take e' so as to increase the likelihood of central outcomes—indeed $\int_\Omega g \, \mathrm{d}\mu_e = \frac{1}{2} < \frac{7}{8} = \int_\Omega g \, \mathrm{d}\mu_{e'}$. ■

While the example is contrived so as to make it as simple as possible, the conclusion holds much more generally (i.e., for general u, etc.). This follows from the rather obvious insight that all contracts in $C^{\mathscr{P}}$ provide incentives in a linear manner whereas the effect of effort, or the cost to exert it, can be arbitrary.

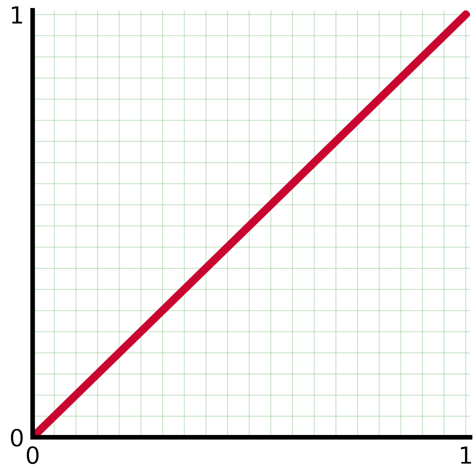
Of course, the language $\mathscr{L}(\mathscr{P})$, despite being predicated only on $\mathscr{P}$, is in fact much richer. Linguistically construable contracts could include clauses beyond simply propositions and their negations. For instance, in the environment of the above example, take $\psi = (\mathrm{s} \to \neg\mathrm{s}) \wedge (\neg\mathrm{s} \to \mathrm{s})$. The value of predicating contracts on this richer set of formulae is immediately apparent by examining ψ. Indeed, the contract that pays 1 contingent on ψ, that is, the function $v_\omega(\psi)$, turns out to be exactly $\omega \mapsto 1 - |1 - 2\omega|$; the derivation of this is shown in Figure 3. So while no contract in $C^{\{\mathrm{s}\}}$ could implement effort e', there are contracts predicated on the language *based on* s that provide the necessary incentives.

This raises the question as to what are the limits of language based contracts. To answer this, consider the fuller set of linguistically constructable contracts:

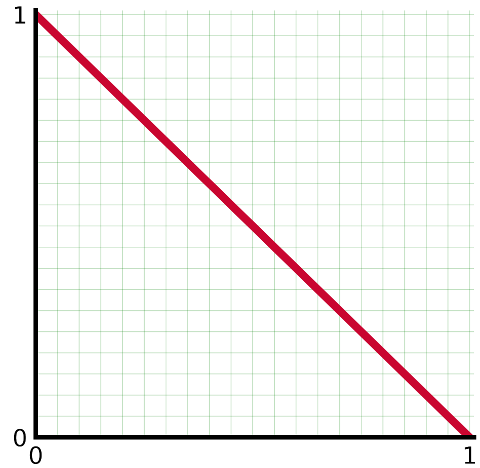
$$C^{\mathscr{L}} = \big\{xv_\omega(\varphi) + yv_\omega(\neg\varphi) \mid \varphi \in \mathscr{L}; x, y \in \mathbb{R}\big\}.$$

Each contract $f \in C^{\mathscr{L}}$ is predicated on a truth of a linguistic statement φ in the language $\mathscr{L}$ (i.e., is based on the truth of elements of $\mathscr{P}$). Just as with the simpler contracts, $C^{\mathscr{P}}$, the contracts $C^{\mathscr{L}}$ are affine in the truth of φ—however, as shown in Figure 3, this does not mean they are affine over Ω.

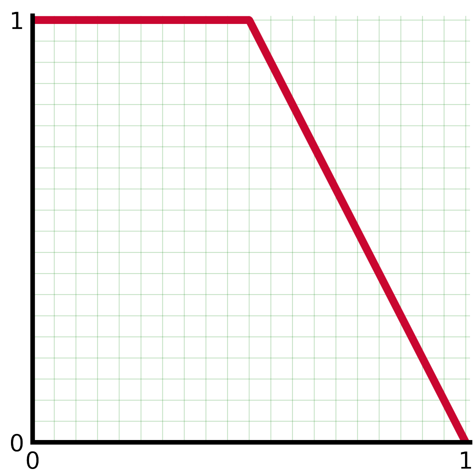
The following theorem shows that under mild conditions on the informativeness of ac-



(a) The valuation of s over $[0,1]$, corresponding to the contract $f : \omega \mapsto \omega$.

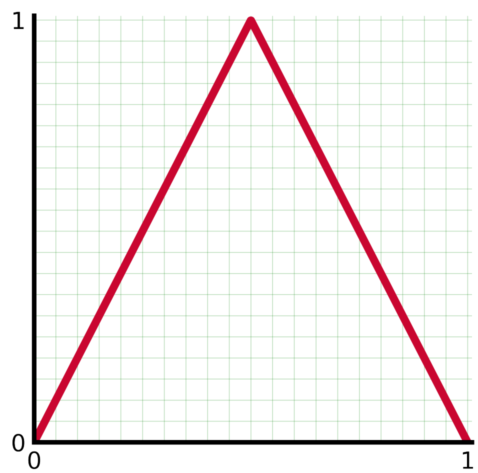


(b) The valuation of $\neg$s over $[0,1]$, corresponding to the functional contract $f : \omega \mapsto 1 - \omega$.



(c) The valuation of s $\to$ $\neg$s over $[0,1]$, corresponding to the functional contract

$$\begin{aligned} f : \omega \mapsto & \min\{1, 1-\omega+(1-\omega)\} \\ &= \min\{1, 2-2\omega\}. \end{aligned}$$



(d) The valuation of (s $\to$ $\neg$s) $\wedge$ ($\neg$s $\to$ s) over $[0,1]$, corresponding to the functional contract

$$\begin{aligned} f : \omega \mapsto \min\{ & \\ & \min\{1, 1-\omega+(1-\omega)\}, \\ & \min\{1, 1-(1-\omega)+\omega\}\} \\ = \min\{ & 1, 2-2\omega, 2\omega\}. \end{aligned}$$

Figure 3: A depiction of the valuation of $\psi = (\mathrm{s} \to \neg\mathrm{s}) \wedge (\neg\mathrm{s} \to \mathrm{s})$ and its sub-formulae. If $v(\mathrm{s}) = x \in [0,1]$ then $v(\psi) = 1 - |1-2x|$. Therefore, the contract that maps ψ to 1, evaluates as the function $g : [0,1] \to [0,1]$ with $g(\omega) = 1 - |1-2\omega|$.

tions, $C^{\mathscr{L}}$ is rich enough to approximate any measurable contract within arbitrary accuracy.

THEOREM 5. *Let λ denote the Lebesgue measure over $[0,1]^n$, then if for all $e \in E$, there exists some $k \in \mathbb{N}$ such that for all Borel-measurable $A \subseteq [0,1]^n$ we have $\mu_e(\{\omega \in \Omega \mid (v_\omega(P_1) \dots v_\omega(P_n)) \in A\}) \leq k\lambda(A)$, then*

$$\kappa(C^{\mathscr{L}}) = \kappa(C^{meas}).$$

Theorem 5 states that not only are linguistic contracts able to implement the same outcomes as arbitrary measurable ones, they can do so at the same approximate cost. This is rather remarkable: the seemingly severe restriction that contracts be explicitly constructed with expressible statements, has, in fact, almost no bite at all. In particular, notice that the linguistic statements on which the contracts in $C^{\mathscr{L}}$ are predicated make no direct mention of the state space. So, Theorem 5 can be seen as a theoretical justification for the use of state-spaces in economic modeling: Although real world contracts are linguistic in nature, the particular *representation of uncertainty*, as a state-space or as the degree of truth of uncertain statements, is irrelevant: the set of implementable actions remains the same.

A few words on the conditions of the Theorem. That $\mu_e(\{\omega \in \Omega \mid v_\omega(\mathrm{P}_1) \dots v_\omega(\mathrm{P}_n)) \in A\})$ is bounded by some constant times the Lebesgue measure of A is essentially a dictate that actions cannot be infinitely informative—no measure zero set of valuations can be given positive probability given a particular provision of effort.

Theorem 5 rests on the fact that the valuations of statements in $\mathscr{L}$ is deeply connected to the calculus of McNaughton functions, piecewise linear functions with integer coefficients. In fact, any such function can be represented as the valuation of some $\varphi \in \mathscr{L}$, letting the value of the propositional variables range over $[0,1]$. From here, we need only show that this class McNaughton functions is rich enough to approximate direct contracts in a way that does not perturb incentives uncontrollably.

A Proofs

A.1 Preliminary facts about MV Algebras

The algebraic counterpart of the Łukasiewicz logic, presented above, is the variety of MV algebras.[5] An MV algebra is an algebraic structure $\langle A, \oplus, \neg, 0\rangle$, where A is a non-empty set, $\oplus$, $\neg$ and 0 are 2-ary, 1-ary and a 0-ary primitive operations, respectively, satisfying the following axioms for all $a, b, c \in A$:

- $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
- $a \oplus 0 = a$
- $x \oplus y = y \oplus x$
- $\neg\neg a = a$
- $a \oplus \neg 0 = \neg 0$
- $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$

It is convenient to introduce the following derived operations (which mirror their syntactic counterparts introduced in Section 2.1).

$$1 =_{def} \neg 0 \qquad a \odot b =_{def} \neg(\neg a \oplus \neg b) \qquad a \to b =_{def} \neg a \oplus b$$

There is a natural partial ordering on A given by $a \leq b$ iff $a \to b = 1$. For any MV algebra, define the distance $d(a, b) =_{def} \big((a \odot \neg b) \oplus (b \odot \neg a)\big)$. It is easy to verify that $d(a, b) = 0$ if and only if $a = b$.

Let Ł denote the canonical MV algebra: $[0, 1]$ endowed with the standard Łukasiewicz operations: $\neg x = 1 - x$ and $x \oplus y = \min\{1, x + y\}$ (for completeness, the other derived operators are: $x \to y = \min\{1, 1 - x + y\}$; $x \vee y = \max\{x, y\}$; $x \wedge y = \min\{x, y\}$; $x \oplus y = \min\{1, x + y\}$; $x \odot y = \max\{0, x + y - 1\}$). The partial ordering on Ł coincides with the usual order of real numbers.

Let $A(\mathscr{L})$ collect the set of $\doteq$-equivalence classes of $\mathscr{L}$, with $[\varphi]$ denoting the class containing $\varphi \in \mathscr{L}$. Let $\mathbf{1} = [\mathbf{T}]$ denote the equivalence class of formulae that evaluate to 1 under any $v \in \mathcal{V}(\mathscr{L})$ and $\mathbf{0} = [\mathbf{F}]$ those that evaluate to 0 (for instance $\neg(\varphi \to \varphi)$). It is will known that $(A, \oplus, \neg, \mathbf{0})$ form an MV algebra, under the operations inherited by the syntax, see for example Chang (1958, 1959). We will used the same syntax for derived

[5] See Mundici (2011) for a rigorous treatment of infinite valued logic and its algebraic counterpart, MV algebras; or, see Bergmann (2008) for a more gentle introduction.

operations as used syntactically: which is without loss of generality since $[\varphi]\oplus[\psi] = [\varphi\oplus\psi]$, $[\varphi] \vee [\psi] = [\varphi \vee \psi]$ etc. The natural ordering on $A(\mathscr{L})$ coincides with $\dot{\geq}$; in an abuse of notation, we write $[\varphi] \mathrel{\dot{\geq}} [\psi]$ whenever $\varphi \mathrel{\dot{\geq}} \psi$. **1** is the top element and **0** the bottom.

A non-empty subset $I \subseteq A$ is called an *ideal* if (i) it is $\dot{\geq}$-downward closed: if $a \in I$ and $a \mathrel{\dot{\geq}} b$, then $b \in I$; and (ii) it is directed: if $a, b \in I$ then $a \oplus b \in I$. An ideal, I, is called *prime* if $I \neq A$ and for all $a, b \in A$, if $a \wedge b \in A$, then either $a \in A$ or $b \in A$. An ideal is said to be *maximal* if it is prime and it is not contained in any other prime ideal.

Let Ω denote the set of all maximal ideals. Endow Ω with the *spectral topology*, the topology generated by the following basis of closed sets: $C_a = \{\omega \in \Omega \mid a \in \omega\}$ for all $a \in A$. Ω is a non-empty compact Hausdorff space (Mundici (2011); Proposition 4.15). For any maximal ideal $\omega \in \Omega$ there is a unique pair (η_ω, A_m) with A_m an MV-subalgebra of $\mathbb{L}$ and η_w an isomorphism of the quotient MV-algebra A/ω onto A_m (Mundici (2011); Proposition 4.16(i)).[6] Now, define the map $(\cdot)^* : A \to \mathbb{L}^\Omega$ such that for each $a \in A$, $a^* : \omega \mapsto \eta_\omega(a/\omega)$.

A.2 Proofs

Proof of Theorem 1. We begin with some preliminary Lemmas

Lemma 1. *If $\varphi \mathrel{\dot{\geq}} \psi$ then $x_\varphi \succcurlyeq x_\psi$ for all $x \in [0, 1]$.*

Proof. Define $\eta = \neg\psi \odot \varphi$. Choose an arbitrary valuation $v \in \mathcal{V}(\mathscr{L})$, we have:

$$\begin{aligned}
v(\psi \odot \eta) &= v(\psi \odot (\neg\psi \odot \varphi)) \\
&= \max\{0, v(\psi) + v(\neg\psi \odot \varphi) - 1\} && \text{by } [\![\odot]\!] \\
&= \max\{0, v(\psi) + \max\{0, 1 - v(\psi) + v(\varphi) - 1\} - 1\} && \text{by } [\![\odot]\!] \text{ and } [\![\neg]\!] \\
&= \max\{0, v(\psi) - v(\psi) + v(\varphi)\} && \text{since } v(\varphi) > v(\psi) \\
&= 0
\end{aligned}$$

[6]The quotient MV-algebra A/ω is defined as the set of equivalence classes generated by $a \sim_\omega b$ if and only if $d(a, b) \in \omega$. It is straightforward to verify that $\sim_\omega$ is a congruence relation, and therefore A/ω is an MV algebra under the inherited operations.

Thus, $\psi \perp \eta$. Moreover, we have:

$$\begin{aligned}
v(\psi \oplus \eta) &= v(\psi \oplus (\neg\psi \odot \varphi)) \\
&= \min\{1, v(\psi) + v(\neg\psi \odot \varphi)\} && \text{by } [\![\oplus]\!] \\
&= \min\{1, v(\psi) + \max\{0, 1 - v(\psi) + v(\varphi) - 1\}\} && \text{by } [\![\odot]\!] \text{ and } [\![\neg]\!] \\
&= \min\{1, v(\psi) - v(\psi) + v(\varphi)\} && \text{since } v(\varphi) > v(\psi) \\
&= v(\varphi)
\end{aligned}$$

Hence $\varphi \doteq \psi \oplus \eta$ and therefore, by (Ł) and (M), respectively, we have

$$x_\varphi \sim x_\psi + x_\eta \succcurlyeq x_\psi$$

The transitivity of $\succcurlyeq$ yields $x_\varphi \succcurlyeq x_\psi$. ★

From this lemma we can conclude $\sim$ respects the $\doteq$-equivalence classes:

LEMMA 2. *If $\varphi \doteq \psi$ then $x_\varphi \sim x_\psi$ for all $x \in [0, 1]$.*

Proof. Follows from two applications of Lemma 1. ★

LEMMA 3. *Let $1 \geq \alpha > \beta \geq 0$ and $\varphi \in \mathbf{1}$. Then $\alpha_\varphi \succ \beta_\varphi$.*

Proof. First, note, it must be that $1_\varphi \succ 0_\varphi$. For suppose this was not the case: For any $\psi \in \mathscr{L}$, Lemma 1 (and (M)) implies $0_\varphi \succcurlyeq 1_\varphi \succcurlyeq 1_\psi \succcurlyeq 0_\psi = 0_\varphi$, or that $\succcurlyeq$ is trivial over $\Delta_{\mathscr{L}} = \{x_\varphi \mid x \in \{0, 1\}, \varphi \in \mathscr{L}\}$, and so by (I) trivial over the convex hull of $\Delta_{\mathscr{L}}$. Now, (O) implies there exists some $f, g \in \mathscr{F}$ such that $f \succ g$, so by (I), $\alpha f + (1-\alpha)0 \succ \alpha g + (1-\alpha)0$; but for sufficiently small α, these acts are in the convex hull of $\Delta_{\mathscr{L}}$, a contradiction.

Now by (I):

$$1_\varphi = (1 - \tfrac{\beta}{\alpha})1_\varphi + \tfrac{\beta}{\alpha}1_\varphi \succ (1 - \tfrac{\beta}{\alpha})0_\varphi + \tfrac{\beta}{\alpha}1_\varphi = \tfrac{\beta}{\alpha}{}_\varphi$$

and then by (I) one more time:

$$\alpha_\varphi = \alpha 1_\varphi + (1 - \alpha)0_\varphi \succ \alpha \tfrac{\beta}{\alpha}{}_\varphi + (1 - \alpha)0_\varphi = \beta_\varphi.$$

★

When there is no danger of confusion, let x_a, for $a \in A(\mathscr{L})$ denote an act from one formula in a. Now define the map $\sigma : A(\mathscr{L}) \to [0,1]$ such that $\sigma(a)$ is unique solution to $1_a \sim \sigma(a)_{\mathbf{1}}$. That this has a solution follows from standard continuity arguments from expected utility representation proofs and the fact that $1_{\mathbf{1}} \succcurlyeq 1_a \succcurlyeq 0_a$; that this solution is unique rests on Lemma 3; that it is well defined (does not depend on the choice of formulae out of a or **1**) follows from Lemma 2. Notice first that since $1_a \sim \sigma(a)_{\mathbf{1}}$, by mixing with the 0-act and appealing to (I), we see that

$$x_a \sim (x\sigma(a))_{\mathbf{1}} \tag{4}$$

for all $x \in [0,1]$.

LEMMA 4. *The map σ satisfies (i) $\sigma(\mathbf{1}) = 1$ and (ii) $\sigma(a) + \sigma(b) = \sigma(a \oplus b)$ whenever $a \odot b = \mathbf{0}$.*

Proof. Point (i) is immediate from the definition. Towards (ii): let $a \odot b = \mathbf{0}$.

Now take some $\varphi \in a$ and $\psi \in b$; $a \odot b = \mathbf{0}$ is equivalent to $\varphi \perp \psi$, so by (Ł):

$$\tfrac{1}{2}_{\varphi} + \tfrac{1}{2}_{\psi} \sim \tfrac{1}{2}_{\varphi \oplus \psi}$$

or by (I) and (4)

$$(\tfrac{1}{2}\sigma(a) + \tfrac{1}{2}\sigma(b))_{\mathbf{1}} = (\tfrac{1}{2}\sigma(a))_{\mathbf{1}} + (\tfrac{1}{2}\sigma(b))_{\mathbf{1}} \sim (\tfrac{1}{2}\sigma(a \oplus b))_{\mathbf{1}}.$$

By Lemma 3, this implies $\frac{1}{2}\sigma(a) + \frac{1}{2}\sigma(b) = \frac{1}{2}\sigma(a \oplus b)$ or that $\sigma(a) + \sigma(b) = \sigma(a \oplus b)$. ★

LEMMA 5. *(Kroupa–Panti theorem) There exists a unique regular Borel probability measure, μ, on Ω, such that for all $a \in A$*

$$\sigma(a) = \int_{\Omega} a^*(\omega) d\mu(\omega). \tag{5}$$

Proof. Mundici (2011) Theorem 10.5. ★

Define the valuation, $V = \{v_\omega : \varphi \mapsto [\varphi]^*(\omega)\}$. It remains to show that (Ω, V, μ) represents $\succcurlyeq$.

We can write each $f \in \mathscr{F}$ as $\alpha^1 x^1_{\varphi^1} + \ldots + \alpha^n x^n_{\varphi^n}$ for $\alpha^1, \ldots \alpha^n \in [0,1]$ summing to 1, $x^1 \ldots x^n \in \mathbb{R}_+$ and $\varphi^1 \ldots \varphi^n \in \mathscr{L}$. Moreover, appealing to the indifference condition (4),

n applications of (I) and transitivity provide that

$$f \sim \alpha^1(x^1\sigma([\varphi^1])_{\mathbf{1}}) + \ldots + \alpha^n(x^n\sigma([\varphi^n])_{\mathbf{1}}) = (\sum_{i\leq n}\alpha^i x^i\sigma(\varphi^i))_{\mathbf{1}}.$$

Thus—writing f as above and g as $\beta^1 y^1_{\psi^1} + \ldots + \beta^m y^m_{\psi^m}$—we have:

$$\begin{aligned}
f \succcurlyeq g \iff & \sum_{i\leq n}\alpha^i x^i\sigma([\varphi^i]) \geq \sum_{j\leq m}\beta^j y^j\sigma([\psi^j]) \\
\iff & \sum_{i\leq n}\alpha^i x^i \int_\Omega [\varphi^i]^*(\omega)d\mu(\omega) \geq \sum_{j\leq m}\beta^j y^j \int_\Omega [\psi^j]^*(\omega)d\mu(\omega) \quad \text{by (5)} \\
\iff & \sum_{i\leq n}\alpha^i x^i \int_\Omega v_\omega(\varphi^i)d\mu(\omega) \geq \sum_{j\leq m}\beta^j y^j \int_\Omega v_\omega(\psi^j)d\mu(\omega) \quad \text{by def of } V \\
\iff & \int_\Omega \sum_{i\leq n}\alpha^i x^i v_\omega(\varphi^i)d\mu(\omega) \geq \int_\Omega \sum_{j\leq m}\beta^j y^j v_\omega(\psi^j)d\mu(\omega) \\
\iff & \int_\Omega \sum_{i\leq n} f(\varphi^i) v_\omega(\varphi^i)d\mu(\omega) \geq \int_\Omega \sum_{j\leq m} g(\psi^j) v_\omega(\psi^j)d\mu(\omega) \quad \text{by def of } f, g \\
\iff & \int_\Omega f^V(\omega)d\mu(\omega) \geq \int_\Omega g^V(\omega)d\mu(\omega) \quad \text{by (1)}
\end{aligned}$$

as desired.

Towards the uniqueness claim, let (Ω', V', μ') denote some other representation of $\succcurlyeq$. We will show that this is a relabeling of (Ω, V, μ), which proves the claim, by observing that *relabeling* is an equivalence relation. First, note that the map $z : w \mapsto (\varphi \mapsto v_\omega(\varphi))$ defines a bijection between Ω and $\mathcal{V}(\mathscr{L})$ (Mundici (2011); Proposition 4.16(ii)).

Consider the map: $\xi : \Omega' \to \Omega$ as $\xi : \omega' \mapsto z^{-1}(v'_{\omega'})$, and the associated pushforward measure $\mu^* : A \mapsto \mu'(\xi^{-1}(A))$. By definition of the representation, and the construction of the measure we have for all $a \in A$

$$\sigma(a) = \int_{\Omega'} v'_\omega(a)d\mu'(\omega') = \int_\Omega v_\omega(a)d\mu^*(\omega).$$

By the uniqueness claim of Kroupa–Panti theorem, we see that $\mu^* = \mu$: Thus, for all measurable $D \subseteq \mathcal{V}(\mathscr{L})$, we have $\mu(\{\omega \mid v_\omega \in D\}) = \mu^*(\{\omega \mid v_\omega \in D\}) = \mu'(\xi^{-1}(\{\omega \mid v_\omega \in D\})) = \mu'(\{\omega' \mid v_{\omega'} \in D\})$. ■

LEMMA 6. *Let $\succcurlyeq$ be represented by a vague model of uncertainty.* (Ω, V, μ) *and let* $\psi \mathrel{\dot{\geq}} \varphi$. *Then if* $1_\varphi \sim 1_\psi$, *it follows that* $\mu(\{\omega \in \Omega \mid v_\omega(\varphi) = v_\omega(\psi)\}) = 1$.

Proof. Let $f = 1_\varphi$ and $g = 1_\psi$. By definition of $\dot{\geq}$, we have that g^V dominates f^V. By the representation, $1_\varphi \sim 1_\psi$ requires

$$\int_\Omega |g^V - f^V| \,\mathrm{d}\mu = \int_\Omega g^V - f^V \,\mathrm{d}\mu = \int_\Omega g^V \,\mathrm{d}\mu - \int_\Omega f^V \,\mathrm{d}\mu = 0$$

and thus, (see Royden and Fitzpatrick (2010) Chapter 4, Proposition 9), requires the set $\big\{\omega \in \Omega \mid f^V(\omega) = g^V(\omega)\big\}$ has μ-measure 1. ∎

Proof of Theorem 2. Fix φ and ψ, and for each $n \in \mathbb{N}$, define $f_n^V : \Omega \to \mathbb{R}$ as $\omega \mapsto v_\omega(n \odot (\psi \to \varphi))$. To establish (2), it thus suffices to show that $f_{n+1}^V \leq f_n^V$ point-wise, and $\{f_n^V\}_{n\in\mathbb{N}}$ converges point-wise to $\mathbb{1}_{\{\omega\in\Omega|v_\omega(\varphi)\geq v_\omega(\psi)\})}$ (where $\mathbb{1}_A$ is the indicator function of a set A). Indeed, the result would then follow from the representation and the monotone convergence theorem.

But these properties follow directly from the semantics from $[\![\odot]\!]$: note that

$$f_n^V(\omega) = \max\{0, nv_\omega(\psi \to \varphi) - (n-1))\} = \max\{0, 1 - n(1 - v_\omega(\psi \to \varphi))\}$$

which is non-increasing in n, converges to 0 if $v_\omega(\psi \to \varphi) < 1$ and is constant and equal to 1 otherwise.

The second claim follows immediately from Lemma 6 and the fact that $v_\omega(\psi \to \varphi) = 1$ if and only if $v_\omega(\varphi) \geq v_\omega(\psi)$. ∎

Proof of Theorem 3. (i) Follows from simple plug and chug but to appease my obsessive side, I'll prove it anyhow. Assume $v(\varphi) \geq \frac{1}{2}$ (where the opposite case follows from interchanging φ and $\neg\varphi$). Notice this implies, via $[\![\odot]\!]$ (and $[\![\neg]\!]$), that

$$\begin{aligned} v(\varphi \odot \varphi) &= \max\{0, 2v(\varphi) - 1\} = 2v(\varphi) - 1 \qquad \text{and} \\ v(\neg\varphi \odot \neg\varphi) &= \max\{0, 1 - 2v(\varphi)\} = 0 \end{aligned} \tag{6}$$

We have:

$$\begin{aligned}
v(\Box\varphi) &= v(\neg\big((\varphi \odot \varphi) \oplus (\neg\varphi \odot \neg\varphi)\big)) \\
&= 1 - \min\{1, v(\varphi \odot \varphi) + v(\neg\varphi \odot \neg\varphi)\} && \text{by } [\![\oplus]\!] \\
&= 1 - \min\{1, 2v(\varphi) - 1\} && \text{by (6)} \\
&= 1 - (v(\varphi) + v(\varphi) - 1) \\
&= 1 - (v(\varphi) - (1 - v(\varphi)) \\
&= 1 - |v(\varphi) - v(\neg\varphi)|.
\end{aligned}$$

(ii) follows directly from (i) and the representation; (iii) follows from Theorem 2; (iv) follows from Lemma 6 and the fact that $v(\varphi) = v(\varphi \oplus \varphi)$ if and only if $v(\varphi) \in \{0, 1\}$. ■

LEMMA 7. *Let $\{f_n\}$ be continuous functions from $[0,1]^n \to [0,1]$. Let λ denote the Lebesgue measure over $[0,1]^n$ and μ be a regular Borel probability measure over $[0,1]^n$ such that there exists some $k \in \mathbb{N}$ such that for all Borel-measurable A we have $\mu(A) \leq k\lambda(A)$. Then*

$$\lim_{n\to\infty} \int_{[0,1]^n} f_n \,\mathrm{d}\lambda = 0 \Longrightarrow \lim_{n\to\infty} \int_{[0,1]^n} f_n \,\mathrm{d}\mu = 0.$$

Proof. Since $\mu(A) \leq k\lambda(A)$, μ is absolutely continuous with respect to λ, and so there exists a Radon–Nikodym derivative $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$. Moreover, $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}$ is λ-essentially bounded by k: if it was not, then it would be possible to find a Borel-set of positive λ-measure, A, such that $\frac{\mathrm{d}\mu}{\mathrm{d}\lambda} > k$ over A. But then $\mu(A) = \int_A \frac{\mathrm{d}\mu}{\mathrm{d}\lambda} \,\mathrm{d}\lambda > \int_A k \,\mathrm{d}\lambda = k\lambda(A)$, a contradiction to our assumption.

Now fix some $\epsilon > 0$. Since $\lim_{n\to\infty} \int_{[0,1]^n} f_n \,\mathrm{d}\lambda = 0$, there exists some $\hat{n}$ such that for all $n > \hat{n}$, $\int_{[0,1]^n} f_n \,\mathrm{d}\lambda < \frac{\epsilon}{k}$. Thus for such $n > \hat{n}$ we have

$$\int_{[0,1]^n} f_n \,\mathrm{d}\mu = \int_{[0,1]^n} f_n \frac{\mathrm{d}\mu}{\mathrm{d}\lambda} \,\mathrm{d}\lambda < k \int_{[0,1]^n} f_n \,\mathrm{d}\lambda < k\frac{\epsilon}{k} = \epsilon,$$

providing the necessary convergence. ■

Proof of Theorem 5. McNaughton (1951) proved that the set of functions $\{\omega \mapsto v_\omega(\varphi) \mid \varphi \in \mathscr{L}\}$ is exactly the set of continuous and piecewise linear functions from $[0,1]^n$ to $[0,1]$ with integer coefficients,[7] now referred to as McNaughton functions. As such, the set of

[7]That is, for each $f \in C^{\mathscr{L}}$, there is a finite set of affine functions with integer coefficients, $a_1\omega +$

contracts $C^{\mathscr{L}}$ correspond to the functions $xm + y$ where $x, y \in \mathbb{R}$ and m is a McNaughton function—in particular, all such contracts are continuous, hence measurable, so $C^{\mathscr{L}} \subseteq C^{meas}$, and we can conclude that $\kappa(C^{meas}) \leq \kappa(C^{\mathscr{L}})$.

We now prove the other direction. It follows from the uniqueness claim of Theorem 1 that it is without loss of generality to assume that $\Omega = [0,1]^n$ and $v_\omega(\mathrm{P}_i) = \omega_i$ (where ω_i is, obviously, the i^{th} component of ω). Take some $f \in C^{meas}$ that implements $e \in E$. Fix $\epsilon > 0$. The theorem follows if we can find some $m \in C^{\mathscr{L}}$ such that m implements e and $\int_\Omega |f - m| \, \mathrm{d}\mu_e < \epsilon$.

It is without loss of generality to assume ϵ is smaller than 1 and smaller slack in the (IC) constraint:

$$\epsilon < \Big(\int_\Omega u \circ f \, \mathrm{d}\mu_e - c(e) \Big) - \Big(\max_{e' \in E \setminus e} \int_\Omega u \circ f \, \mathrm{d}\mu_{e'} - c(e') \Big).$$

Now since f is bounded, the interval $[\inf_\Omega f(\omega), \sup_\Omega f(\omega) + 1] \subset \mathbb{R}$ is compact. Thus, by virtue of continuous differentiability, u is Lipschitz continuous on this set (see Duistermaat and Kolk (2004), Corollary 2.5.5)—let K_u denote a Lipschitz constant. Without loss of generality assume $K_u \geq 1$.

Consider $g = f + \frac{\epsilon}{6K_u}$. By our assumed bounds on ϵ and K, see that u is Lipschitz continuous with constant K_u on the image of g. Since f satisfies the (IR) constraint, and u is strictly monotone, g satisfies it strictly.

By Theorem 2 of Wiśniewski (1994), there exists a sequence of continuous functions, $\{g_k\}_{k \in \mathbb{N}}$ bounded by g and such that $g_n \to g$ point-wise, λ-almost everywhere. By the dominated convergence theorem, $\int_\Omega g_k \, \mathrm{d}\lambda \to \int_\Omega \lim_{k \to \infty} g_k \, \mathrm{d}\lambda = \int_\Omega g \, \mathrm{d}\lambda$. By Lemma 7, $\int_\Omega g_k \, \mathrm{d}\mu_{e'} \to \int_\Omega g \, \mathrm{d}\mu_{e'}$ for all $e' \in E$. We can find an $N \in \mathbb{N}$ large enough so that

$$\int_\Omega |g_N - g| \, \mathrm{d}\mu_{e'} < \frac{\epsilon}{6K_u} \qquad \text{for all } e' \in E \tag{7}$$

$b_1, \ldots a_j\omega + b_j$, with $a_i \in \mathbb{Z}^n$ and $b_i \in \mathbb{Z}$ for all $i \leq j$, such that for all $\omega \in [0,1]$ there exists a $i \leq j$ such that $f(\omega) = a_i\omega + b_i$.

Define $\epsilon' > 0$ as

$$\begin{aligned}\epsilon' = \int_\Omega g_N \; \mathrm{d}\mu_{e'} - \bar{u} &\geq \int_\Omega g_N \; \mathrm{d}\mu_{e'} - \int_\Omega g \; \mathrm{d}\mu_{e'} + \int_\Omega g \; \mathrm{d}\mu_{e'} - \bar{u} \\ &\geq - \int_\Omega |g - g_N| \; \mathrm{d}\mu_{e'} + \int_\Omega g \; \mathrm{d}\mu_{e'} - \bar{u} \\ &> \frac{\epsilon}{6K_u} - \frac{\epsilon}{6K_u} = 0\end{aligned}$$

Since g_N is bounded, there exist $x, y \in \mathbb{R}$ such that $g_N = xh + y$ and h is $[0,1]$-valued. It follows from Theorem 3.4 of Aguzzoli and Mundici (2001) that we can construct a sequence $\{m_n\}_{n\in\mathbb{N}}$ of McNaughton functions such that

$$\int_\Omega |m_n - h| \; \mathrm{d}\lambda < \frac{1}{n},$$

where λ is the Lebesgue measure over $[0,1]$. By Lemma 7, we see that for each $e' \in E$, we also have

$$\lim_{n\to\infty} \int_\Omega |m_n - h| \; \mathrm{d}\mu_{e'} = 0$$

So we can find $N' \in \mathbb{N}$ large enough so that

$$\int_\Omega |m_{N'} - h| \; \mathrm{d}\mu_{e'} < \min\{\frac{\epsilon}{6K_u x}, \frac{\epsilon'}{2K_u x}\} \qquad \text{for all } e' \in E \tag{8}$$

Consider the linguistic contract $xm_{N'} + y \in C^{\mathscr{L}}$. We have

$$\begin{aligned}\int_\Omega u \circ f \; \mathrm{d}\mu_{e'} - &\int_\Omega u \circ (xm_{N'} + y) \; \mathrm{d}\mu_{e'} \\ &\leq \int_\Omega |u \circ f - u \circ g_N| \; \mathrm{d}\mu_{e'} + \int_\Omega |u \circ g_N - u \circ (xm_{N'} + y)| \; \mathrm{d}\mu_{e'} \\ &\leq \int_\Omega K_u |f - g_N| \; \mathrm{d}\mu_{e'} + \int_\Omega K_u |g_N - (xm_{N'} + y)| \; \mathrm{d}\mu_{e'} \\ &< \frac{\epsilon}{3} + \int_\Omega K_u |g_N - (xm_{N'} + y)| \; \mathrm{d}\mu_{e'} \\ &= \frac{\epsilon}{3} + \int_\Omega K_u x |h - m_{N'}| \; \mathrm{d}\mu_{e'} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{6} = \frac{\epsilon}{2}\end{aligned}$$

Thus, under $xm_{N'} + y$ each effort level provides a utility within $\frac{\epsilon}{2}$ of the utility under f, and therefore change in utility between any two effort levels is less than ϵ—the (IC) constraint holds. A similar calculation shows that this contract satisfies the (IR) constraint:

$$\begin{aligned}\int_\Omega u\circ(xm_{N'}+y)\,\mathrm{d}\mu_e-\bar{u} &\geq \int_\Omega u\circ(xm_{N'}+y)\,\mathrm{d}\mu_e-\int_\Omega u\circ g_N\,\mathrm{d}\mu_e+\epsilon' \\ &\geq \epsilon'-\int_\Omega |u\circ(xm_{N'}+y)-u\circ g_N|\,\mathrm{d}\mu_e \\ &\geq \epsilon'-\int_\Omega K_u|(xm_{N'}+y)-g_N|\,\mathrm{d}\mu_e \\ &\geq \epsilon'-\frac{\epsilon'}{2}>0\end{aligned}$$

Taking stock we have that $xm_{N'}+y$ implements e. It remains to show that the change in profit can be suitably bounded: which follows form another simple ϵ-style proof:

$$\begin{aligned}\int_\Omega f\,\mathrm{d}\mu_e-\int_\Omega xm_{N'}+y\,\mathrm{d}\mu_e &\leq \int_\Omega |f-g_N|\,\mathrm{d}\mu_e+\leq \int_\Omega |g_N-(xm_{N'}+y)|\,\mathrm{d}\mu_e \\ &\leq \frac{\epsilon}{3K_u}+\int_\Omega |g_N-(xm_{N'}+y)|\,\mathrm{d}\mu_e \\ &\leq \frac{\epsilon}{3K_u}+\int_\Omega x|h-m_{N'}|\,\mathrm{d}\mu_e \\ &\leq \frac{\epsilon}{3K_u}+x\frac{\epsilon}{6K_u x}\,\mathrm{d}\mu_e \\ &=\frac{\epsilon}{2K_u}\leq\epsilon.\end{aligned}$$

■

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