A REPRESENTATION THEOREM FOR CAUSAL DECISION MAKING

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This talk

We provide a definition of **causal sophistication**:

- When are preferences over *interventions* consistent with a subjective causal model?
- We represent causality via *structural equations*
- In the paper, we provide:
    - An axiomatic characterization
    - An identification Theorem

Causation and Counterfactuals

- Modern theories define causation through counterfactuals.
  - Requires evaluating worlds that do not exist
- We take a structural approach a la Pearl [2000]:
  - Equations directly encode causal mechanisms
  - Provide a succinct way of contemplating counterfactuals

Variables

- $\mathcal{U}$ and $\mathcal{V}$ denote exogenous and endogenous variables, resp.
- $\mathcal{R}(Z) \subset \mathbb{R}$ is the range of $Z \in \mathcal{U} \cup \mathcal{V}$
- A *context* is a vector $\vec{u}$ of values for all the exogenous variables $\mathcal{U}$.
  - Let $\texttt{ctx} = \prod_{U \in \mathcal{U}} \mathcal{R}(U)$ collect all contexts
- A *resolution* is a vector $\vec{r}$ of values for all variables $\mathcal{U} \cup \mathcal{V}$.
  - Let $\texttt{res} = \prod_{Y \in \mathcal{U} \cup \mathcal{V}} \mathcal{R}(Y)$ collect all resolutions

The decision maker cares about the resolution and is uncertain about the context:

- Utility will be defined over `res`
- Beliefs will be defined over `ctx`

Example

The US Federal Reserve is contemplating the economy.

The relevant variables are: the growth rate (gw), the prior interest rate (pr), the current interest rate (rate), inflation (inf), employment rate (emp):

$$\mathcal{U} = \begin{cases} U_{gw} \\ U_{pr} \end{cases} \qquad \mathcal{V} = \begin{cases} Y_{rt} \\ X_{emp} \\ X_{inf} \end{cases}$$

Example

- Utility is determined by the inflation rate and employment level:
  - $\mathbf{u}(\vec{r}) = 2X_{emp} - X_{inf}$.
- Does not know the growth rate:
  - believes $U_{gw} = 1$ with prob α and $U_{gw} = 0$ with prob $(1 - \alpha)$.
- Contemplate interventions that set the interest rate:
  - This will casually effect the resolution
  - But exactly how might depend on the context

Causal Models

Given $\mathcal{U}$ and $\mathcal{V}$ with ranges $\mathcal{R}$, a **causal model M** consists of:

- $\mathcal{F} = \{F_X\}_{X \in \mathcal{V}}$, a set of **structural equations**, where

$$F_X : \prod_{Y \in \mathcal{U} \cup (\mathcal{V} - \{X\})} \mathcal{R}(Y) \to \mathcal{R}(X).$$

- Call **M** *recursive* if there exists a partial order on $\mathcal{V}$:
  - F_X is independent of the variables succeeding X

Example

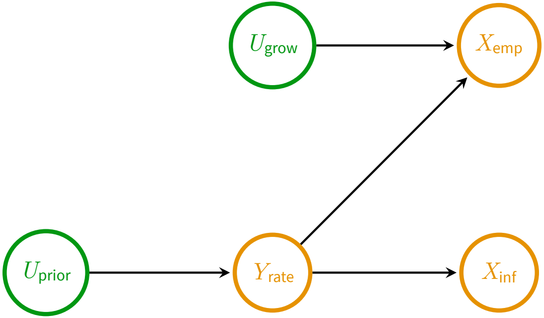
The causal equations are

$$Y_{rt} = U_{pr} \qquad (F_{Y_{rt}})$$

$$X_{inf} = 1 - Y_{rt} \qquad (F_{X_{inf}})$$

$$X_{emp} = 1 - (Y_{rt} \times (1 - U_{gw})) \qquad (F_{X_{inf}})$$

Example



Example

Given a recursive **M**, each context $\vec{u}$ induces a unique resolution $\vec{r}$:

$$\begin{aligned}
U_{gw} &= 0 \\
U_{pr} &= 0 \\
Y_{rt} &= U_{pr} \\
X_{inf} &= 1 - Y_{rt} \\
X_{emp} &= 1 - (Y_{rt} \times (1 - U_{gw}))
\end{aligned}$$

Example

Given a recursive **M**, each context $\vec{u}$ induces a unique resolution $\vec{r}$:

$$
\begin{aligned}
U_{gw} &= 0 \\
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X_{inf} &= 1 - \ Y_{rt} \\
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\end{aligned}
$$

Example

Given a recursive **M**, each context $\vec{u}$ induces a unique resolution $\vec{r}$:

$$\begin{aligned} U_{gw} &= 0 \\ U_{pr} &= 0 \\ Y_{rt} &= 0 \\ X_{inf} &= 1 \\ X_{emp} &= 0 \end{aligned}$$

- When is decision making consistent causal reasoning via some model **M**?
- What kind of data is needed to answer this?
- Preferences over **interventions**

Interventions

An **intervention**

$$\mathbf{do}[\,Y_1 \leftarrow y_1, \ldots, Y_n \leftarrow y_n]$$

is a mediation that sets the values of $Y_1 \ldots Y_n \in \mathcal{V}$:

- $y_i \in \mathcal{R}(\,Y_i)$
- abbreviated as $\mathbf{do}[\,\vec{Y} \leftarrow \vec{y}]$
- interventions only on endogenous variables.

A **conditional intervention** is of the form:

if ϕ **then** A **else** B

- ϕ is a true/false valued question about the variable values
  - such as "the value of X is positive", etc
- A and B are conditional interventions
- These is constructed recursively starting with interventions
- **if** ϕ **then** A shorthand for when $B = \emptyset$

Preference

Observable: preference relation $\succsim$ over conditional interventions:

- Interventions allow the DM to change the resolution
- Conditioning allows contracting away uncertainty about context

A **causally sophisticated** decision maker would understand the effect of conditional interventions via a causal model

Interventions and Causal models

Given the model **M**, the intervention

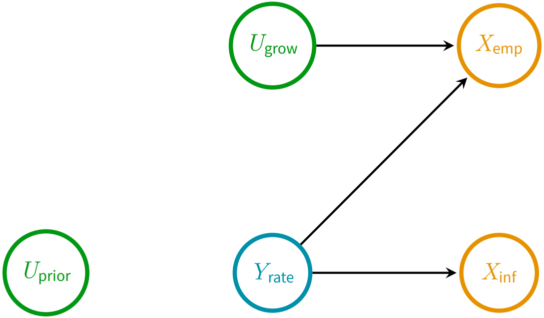
$$\mathbf{do}[\,Y_1 \leftarrow y_1, \ldots, Y_n \leftarrow y_n]$$

induces a *counterfactual model*, $\mathcal{F}_{\mathbf{do}[\,\vec{Y} \leftarrow \vec{y}]}$ where

F_{Y_i} is replaced by the constant function $F'_{Y_i} = y_i$

Example

The intervention $\mathbf{do}[\,Y_{rt} \leftarrow 1]$ sets the current rate to 1:



Example

$$
\begin{aligned}
U_{gw} &= 0 \\
U_{pr} &= 0 \\
Y_{rt} &= 1 \\
X_{inf} &= 1 - Y_{rt} \\
X_{emp} &= 1 - (Y_{rt} \times (1 - U_{gw}))
\end{aligned}
$$

Example

$$
\begin{aligned}
U_{gw} &= 0 \\
U_{pr} &= 0 \\
Y_{rt} &= 1 \\
X_{inf} &= 1 - \ Y_{rt} \\
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\end{aligned}
$$

Example

$$
\begin{aligned}
U_{gw} &= 0 \\
U_{pr} &= 0 \\
Y_{rt} &= 1 \\
X_{inf} &= 1 - 1 \\
X_{emp} &= 1 - (1 \times (1 - 0))
\end{aligned}
$$

Example

$$
\begin{aligned}
U_{gw} &= 0 \\
U_{pr} &= 0 \\
Y_{rt} &= 1 \\
X_{inf} &= 0 \\
X_{emp} &= 0
\end{aligned}
$$

Given a (recursive) model **M** and conditional intervention A, let

$$\beta_A^{\mathbf{M}} : \texttt{ctx} \to \texttt{res}$$

transform contexts into resolutions in the obvious way:

- **M** plus context determines ex-ante resolution
- This resolution determines the ‘clause’ of A in force, hence an intervention
- This intervention determines a (recursive) counterfactual model
- Along with context, this determines the ex-post resolution

Representation

A causally sophisticated agent's preferences are parameterized by

- $\mathbf{M}$ — a recursive model capturing causal relationships
- $\mathbf{u} : \mathtt{res} \to \mathbb{R}$ — value of a resolution of all uncertainty
- $\mathbf{p} \in \Delta(\mathtt{ctx})$ — belief capturing uncertainty about the values of exogenous (hence endogenous) variables

Representation

Subjective Causal Utility

$(\mathbf{M}, \mathbf{p}, \mathbf{u})$ is a **subjective causal utility representation** of $\succsim$:

$$A \succsim B$$

if and only if

$$\sum_{\vec{u} \in \mathtt{ctx}} \mathbf{u}(\beta_A^{\mathbf{M}}(\vec{u}))\mathbf{p}(\vec{u}) \geq \sum_{\vec{u} \in \mathtt{ctx}} \mathbf{u}(\beta_B^{\mathbf{M}}(\vec{u}))\mathbf{p}(\vec{u}).$$

Example

The utility of the Federal Reserve is determined by the inflation rate and employment level, and is given by

$$\mathbf{u}(\vec{r}) = 2X_{emp} - X_{inf}.$$

Example

$$\mathbf{do}[\,Y_{rt} \leftarrow 0]$$

$$\begin{tikzcd}
 & \left\{\begin{array}{ll} Y_{rt} & = 0 \\ X_{inf} & = 1 \\ X_{emp} & = 1 \end{array}\right. \\
\beta \arrow[ur, "U_{gw} = 1"] \arrow[dr, "U_{gw} = 0"'] & \\
 & \left\{\begin{array}{ll} Y_{rt} & = 0 \\ X_{inf} & = 1 \\ X_{emp} & = 1 \end{array}\right.
\end{tikzcd}$$

$$\mathbf{do}[\,Y_{rt} \leftarrow 1]$$

$$\begin{tikzcd}
 & \left\{\begin{array}{ll} Y_{rt} & = 1 \\ X_{inf} & = 0 \\ X_{emp} & = 1 \end{array}\right. \\
\beta \arrow[ur, "U_{gw} = 1"] \arrow[dr, "U_{gw} = 0"'] & \\
 & \left\{\begin{array}{ll} Y_{rt} & = 1 \\ X_{inf} & = 0 \\ X_{emp} & = 0 \end{array}\right.
\end{tikzcd}$$

Example

$\mathbf{do}[Y_{rt} \leftarrow 0]$

$$\begin{tikzcd}
 & 2 \times 1 - 1 = 1 \\
\mathbf{u} \circ \beta \arrow[ur, "U_{gw} = 1"] \arrow[dr, "U_{gw} = 0"'] & \\
 & 2 \times 1 - 1 = 1
\end{tikzcd}$$

Utility is 1

$\mathbf{do}[Y_{rt} \leftarrow 1]$

$$\begin{tikzcd}
 & 2 \times 1 - 0 = 2 \\
\mathbf{u} \circ \beta \arrow[ur, "U_{gw} = 1"] \arrow[dr, "U_{gw} = 0"'] & \\
 & 2 \times 0 - 0 = 0
\end{tikzcd}$$

Utility is 2α

Example

- Preference between setting interest rate at 1 or 0 depends on belief about U_{gw}.
- The conditional intervention

$$\textbf{if}\ (U_{gw} = 1)\ \textbf{then}\ \textbf{do}[Y_{rt} \leftarrow 1]\ \textbf{else}\ \textbf{do}[Y_{rt} \leftarrow 0]$$

  dominates

From preference to causality:

For a resolution $\vec{u} \in \texttt{ctx}$ let

$$\textbf{if}\ \ \vec{u}\ \ \textbf{then}\ \ A\ \ \textbf{else}\ \ B$$

denote the conditional intervention on $\vec{u}$ being true.

- i.e., do A if all variables coincide with $\vec{u}$, else do B

Conditional on $\vec{u}$: if setting $\vec{Y}$ to $\vec{y}$ yields $X = x$, then the agent is indifferent from making such a further intervention on X:

$$\textbf{if } \vec{u} \textbf{ then } \mathbf{do}[\vec{Y} \leftarrow \vec{y}, X \leftarrow x] \sim \textbf{if } \vec{u} \textbf{ then } \mathbf{do}[\vec{Y} \leftarrow \vec{y}].$$

Example

No Intervention

$$\begin{aligned} U_{gw} &= 0 \\ U_{pr} &= 0 \\ Y_{rt} &= U_{pr} \\ X_{inf} &= 1 - Y_{rt} \\ X_{emp} &= 1 - (Y_{rt} \times (1 - U_{gw})) \end{aligned}$$

$\mathbf{do}[Y_{rt} \leftarrow 1]$

$$\begin{aligned} U_{gw} &= 0 \\ U_{pr} &= 0 \\ Y_{rt} &= 1 \\ X_{inf} &= 1 - Y_{rt} \\ X_{emp} &= 1 - (Y_{rt} \times (1 - U_{gw})) \end{aligned}$$

$\mathbf{do}[Y_{rt} \leftarrow 1, X_{emp} \leftarrow 0]$

$$\begin{aligned} U_{gw} &= 0 \\ U_{pr} &= 0 \\ Y_{rt} &= 1 \\ X_{inf} &= 1 - Y_{rt} \\ X_{emp} &= 0 \end{aligned}$$

Example

No Intervention

$$U_{gw} = 0$$
$$U_{pr} = 0$$
$$Y_{rt} = 0$$
$$X_{inf} = 1 - Y_{rt}$$
$$X_{emp} = 1 - (Y_{rt} \times (1 - 0))$$

$\mathbf{do}[Y_{rt} \leftarrow 1]$

$$U_{gw} = 0$$
$$U_{pr} = 0$$
$$Y_{rt} = 1$$
$$X_{inf} = 1 - Y_{rt}$$
$$X_{emp} = 1 - (Y_{rt} \times (1 - 0))$$

$\mathbf{do}[Y_{rt} \leftarrow 1, X_{emp} \leftarrow 0]$

$$U_{gw} = 0$$
$$U_{pr} = 0$$
$$Y_{rt} = 1$$
$$X_{inf} = 1 - Y_{rt}$$
$$X_{emp} = 0$$

Example

No Intervention	$\mathbf{do}[Y_{rt} \leftarrow 1]$	$\mathbf{do}[Y_{rt} \leftarrow 1, X_{emp} \leftarrow 0]$
$U_{gw} = 0$	$U_{gw} = 0$	$U_{gw} = 0$
$U_{pr} = 0$	$U_{pr} = 0$	$U_{pr} = 0$
$Y_{rt} = 0$	$Y_{rt} = 1$	$Y_{rt} = 1$
$X_{inf} = 1 - 0$	$X_{inf} = 1 - 1$	$X_{inf} = 1 - 1$
$X_{emp} = 1 - (0 \times (1 - 0))$	$X_{emp} = 1 - (1 \times (1 - 0))$	$X_{emp} = 0$

Example

No Intervention	$\mathbf{do}[Y_{rt} \leftarrow 1]$	$\mathbf{do}[Y_{rt} \leftarrow 1, X_{emp} \leftarrow 0]$
$U_{gw} = 0$	$U_{gw} = 0$	$U_{gw} = 0$
$U_{pr} = 0$	$U_{pr} = 0$	$U_{pr} = 0$
$Y_{rt} = 0$	$Y_{rt} = 1$	$Y_{rt} = 1$
$X_{inf} = 1$	$X_{inf} = 0$	$X_{inf} = 0$
$X_{emp} = 1$	$X_{emp} = 0$	$X_{emp} = 0$

Thank You!