Distributional Uncertainty and Persuasion*

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Abstract

A Sender designs a signals regarding a state. The distribution of the state is unknown to A Receiver. When information is disclosed many times, accumulated signals change the Receiver?s belief about the distribution. Under mild conditions, the Sender?s private information about the distribution is never fully revealed. I then consider the effect of public commitment to a signal structure. Commitment mechanisms need to be unconditional in order to ensure private information revelation. Hence, my analysis indicates that the metrics by which policy changes are evaluated should be committed to before any preliminary investigation.

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1 Introduction

In many economic environments, one agent, a Sender (she), can strategically disclose information in an attempt to persuade another agent, a Receiver (he), into taking more favorable actions. The Sender chooses a *signal structure* that maps each realization of a state space into a distribution over signals; the Receiver would like to condition his choice of action on the state. This paper extends this model of strategic persuasion to an environment with *distributional uncertainty*, under which only the Sender knows the distribution over the payoff-relevant state space.[1] First, the Sender observes the true distribution over the state space, on which she can condition her choice of signal structure. Then, a profile of states is realized according to the distribution and a profile of signals is generated according to the signal structure chosen by the Sender and the realized states. Finally, after observing the full profile of signals, the Receiver chooses an action regarding each realization of the state.

The Receiver does not observe the profile of states directly, and therefore, signals play a dual role: each signal persuades the Receiver about the realization at hand and also alters his perception of uncertainty regarding the overall distribution of the state space. The Sender can better influence the Receiver's beliefs about any particular realization when the Receiver has less precise information about the distribution. However, the realization-by-realization optimal signal structure will reveal the distribution in equilibrium. The Sender must tradeoff the value of persuading the Receiver about each realization with the value of tailoring his second order beliefs.

I consider agents who care about their average payoffs and answer the central question in such an environment: when will the Receiver (or any third party observer, such as an economist) learn the true distribution over the state space? Under mild conditions, equilibrium behavior is never fully revealing (in the sense that the Receiver will not be able fully infer the private information of the the Sender). In particular, when only two distributions are possible, the Receiver will not learn *anything*; statistical inference regarding the distribution is impossible in equilibrium. I then consider a variant model in which the Sender publicly commits to a signal structure before observing her type. Public commitment alone does not ensure learning.

[1] As such, this paper fits within the growing literature on persuasion, namely Kamenica and Gentzkow (2011), and in particular persuasion with privately informed agents, Kolotilin et al. (2015) and Bergemann and Morris (2016) with a privately informed Receiver; Perez-Richet (2014), like this paper, considers a privately informed Sender, and in dynamic environments Best and Quigley (2017).

When the Sender can publicly commit to actions *contingent* on the arrival of new information, the Sender may still want to keep the Receiver uninformed. I characterize when learning does take place. Loosely speaking, the greater the initial uncertainty the less likely the Sender's optimal signal structure induces learning.

Distributional uncertainty arises naturally when agents do not know the effectiveness of a new policy or program that may change the distribution of an economic variable. Because the rents that can be extracted via signal design increase as uncertainty increases, economic actors have a strong motivation to keep the evaluation of new programs, policies, and institutional changes private. Further, these actors can leverage the induced uncertainty to persuade others all the while keeping them uninformed about the efficacy of the policy change. Thus, the classical intuition that information can be aggregated breaks down; when information disclosure is strategic, the signal structure may be chosen precisely to ensure information is *never* aggregated. Hence, my analysis indicates that the metrics by which policy changes are evaluated should be committed to *before* any preliminary investigation, and, better still, by different agents than those who design the policy change itself.

Example: A New Curriculum. The Superintendent of a high school (the Sender) tries to persuade a University (the Receiver) to accept as many of her students as possible.[2] At the end of the school year, each student is either prepared, x_h, or unprepared, x_l, to attend the University. The Superintendent of the high-school—whose aim is to maximize the proportion of students who get accepted to the University—is responsible for choosing a grading rubric: a (possibly noisy) assessment of each student's ability. The University, after observing applicants' grades, admits students who are sufficiently likely to be prepared: the University will accept a student only if its posterior on x_h is above some threshold, q. If both parties know that the true distribution of prepared and unprepared students in the population is $[\mu, 1-\mu]$, with $\mu < q$, then, without further information, the University will always reject all applicants. Kamenica and Gentzkow (henceforth KG) provide the conditions such that, when the Superintendent can only assign grades according to a publicly-committed rubric but is able to construct any rubric she wants, she is able to *persuade* the

[2] This is a model of bulk simultaneous persuasion; all signals arrive at once, and the Receiver observes the distribution. However, it is mathematically equivalent to assume that signals arrive sequentially and the players maximize long-run average payoffs. This is substantiated by the supplementary material, which can be found at https://evanpiermont.github.io/pdfs/DUaP_supp.pdf.

University to admit a substantial portion of the applications. Intuitively, the Superintendent designs a signal structure with two signals such that after observing the first signal the University's posterior lies above q. Of course, since the University is assumed to be Bayesian, it must reject after seeing the other signal. The optimal signal structure realizes the first signal as often as possible, while keeping the induced posterior above the threshold.

KG assume that the underlying distribution of preparedness and the Superintendent's chosen rubric are both commonly known. When there is a single student, higher order uncertainty regarding the distribution of states is irrelevant, however, it becomes crucial when the interaction is repeated manyfold.[3] The high school is instituting a new curriculum. If the curriculum is good, a high proportion of students will be prepared; if it is bad, a low proportion. Specifically, there are two possible distributions of student ability, $[\mu_g, 1-\mu_g]$ when the curriculum is good, and $[\mu_b, 1-\mu_b]$ when it is bad (where, $\mu_g > \mu_b$, so that the good curriculum induces a more prepared student body). At the time grades are assigned, only the Superintendent knows the true distribution of ability—if the curriculum is good or bad; the University, instead, believes it is good with probability θ and bad with $(1-\theta)$.

I show that more uncertainty is *always* good for the Superintendent in the one-student model. Intuitively, uncertainty is always (weakly) beneficial because it expands the set of inducible beliefs: given an uncertain University, the Superintendent can always provide more information via more accurate grades; conversely, when the University learns something, that information cannot be revoked. When many students apply to the University, however, each student's grade might serve two ends: first, it can persuade the University to admit, and second, it can alter the University's beliefs regarding the efficacy of the curriculum. The KG-optimal strategy (under uncertainty) requires the Superintendent choose a signal structure that will, when aggregated across the student population, reveal her type (rendering the strategy sub-optimal); with many interactions, she necessarily cannot extract the full value of keeping the University uncertain about the distribution.

Turning my attention to the equilibrium strategies, I show that while single student optimal payoff is not achievable, the gains to keeping the University uncertain

[3]Notice also, if there was only a single student, the lack of verifiability regarding the rubric would be detrimental: because the Superintendent's payoff is state independent, no information can be transmitted in equilibrium. However, when there are many signals sent, the University can extract information regarding the rubric from the *profile* of grades it observes.

are still sufficient to preclude full revelation of the Superintendent's private information. Because the rubric is unverifiable, the Superintendent cannot directly benefit from her private information. Why? If the Superintendent was better off when she observes that the curriculum works, then when she observes that it does *not* work, she could enact a completely uninformative grading scheme that mimics, in distribution, the policy that she would have instituted had she learn the opposite. The University will (incorrectly) infer from the distribution of applicants' grades that the curriculum works, and treat applicants as such, increasing the Superintendent's payoff. So, the original strategy was not part of an equilibrium.

Further, given that the Superintendent's payoff cannot depend on the information she observes, she cannot tailor the signal structure to the true distribution of student ability; the best she can do is to keep the University uninformed, extracting as much of the rents from uncertainty as possible. Intuitively, if she revealed her type, the above argument dictates that her payoff would be bounded by her payoff had the worst type of curriculum been commonly known. But, this payoff is always achievable without revealing her type, and, furthermore, the additional uncertainty implies a better payoff is attainable. When the Superintendent is privately informed about the curriculum, but unable to credibly relay what she has observed, the University will never learn fully the effectiveness of the curriculum. Nonetheless, while she cannot reveal what she learned about the curriculum, the Superintendent is still able to persuade the University on a student-by-student basis.

Ex-ante Public Commitment. To what extent does ex-ante public commitment assuage the problem, ensuring the equilibrium reveals the distribution over the state? Within this model, I contrast two possibilities: the case where the Superintendent must choose a *rigid* signal structure and when she can choose a *flexible* one. A rigid signal structure is a single Blackwell experiment, a mapping from each realization of the state space to a distribution over signals. A flexible signal structure can depend on the curriculum—that is, a mapping from distributions over the payoff relevant states into Blackwell experiments. The interpretation here is that the Superintendent can publicly commit to a grading rubric before observing student outcomes, but the rubric is flexible so as to take into account what she might learn.[4]

[4]For example, a syllabus with a known grading curve. If the students are doing poorly part way through the semester their grades will be inflated. The Superintendent can therefore alter the final grades depending on the distribution of student ability, but must commit to the process (i.e., write

Under the restriction to employ rigid signal structures, she cannot extract any of the rents from uncertainty; the Superintendent cannot benefit from persuasion without the University learning if the curriculum was effective. On the other hand, when the Superintendent can employ flexible signal structures, she can capture some of the gains from the University's imprecise knowledge of the distribution. She can choose the rubric in such a way that the distribution of grades does not depend on the curriculum, persuading the University on a student-by-student basis without providing any information about the underlying distribution of student ability. In such circumstances, adding public commitment does not change the resulting equilibrium behavior at all!

So, given flexible rubrics, when does learning take place? The Superintendent keeps the University uninformed when there exist distributions with a very high average student ability (for example, when $\mu_g \gg q$). The intuition is thus. If such a curriculum was revealed by the distribution of grades, the University's perception of the average student would be higher than the threshold for acceptance. But notice, the size of the difference between the University's belief and the threshold (i.e., $\mu_g - q$; the *slack* in the beliefs) does not affect the Superintendent's payoff—every student is getting accepted already. But, when the University is uninformed, the *possibility* of the curriculum being very high quality increases the ex-ante perception of the average student, allowing for a more persuasive rubric. Moreover, this is true even if the curriculum is not actually of high quality, because the University still considers it possible. By keeping the University uninformed, the slack in beliefs is not wasted. Hence, learning is less likely when the possible outcomes of the policy change are more extreme.

In the context of evaluating a policy change, the lack of connection between the distribution of grades and the underlying distribution of student ability is a clear normative failure. The above results suggest a remedy. Public pre-commitment is good, but in many circumstances, flexibility in the commitment mechanism erodes the benefit. Of course, there are other normative reasons to desire flexibility—namely, to allow agents to incorporate new information. Hence, my results delineate the tradeoff between these objectives, and expose when flexibility in commitment mechanisms can be provided without losing the ability to make statistical inference from equilibrium behavior.

the syllabus) before knowing the distribution.

Organization. The next section provides survey of relevant literature. Section 3 introduces the notation and modeling choices of the persuasion game. Equilibrium strategies with private information is found in Section 4. Section 5 discuses the game with public commitment and discusses the optimal strategies for rigid and flexible signal structures. Appendix A contains a numerical representation of the example in the introduction, where the Superintendent's optimal strategy keeps the University uninformed. All proofs are contained in Appendix B.

2 Related Literature

The formal economic study of Persuasion, or optimal signal design, began with Kamenica and Gentzkow (2011). The authors examined a one shot game wherein a Sender designs (with commitment) an signal structure so as to persuade a Receiver. A fundamental insight of their work is, what Ely (2017) terms the *obfuscation principle*: the problem can be greatly simplified, by examining, rather than the space of all signal structures, the space of possible posterior beliefs. In particular, they show the Sender can induce any family of posteriors that integrates back to Receiver's prior belief. Aumann et al. (1995) made a mathematically identical observation in the context of dynamic games.

The persuasion paradigm has been extended in a number of direction relevant to this paper. Ely (2017) and Renault et al. (2014) both examine the case where the Sender observes the evolution of a stochastic process, and wishes to alter the Receiver's dynamic profile of actions. Ely shows, in analogy to the static case, the Sender can induce any (family of) beliefs that (1) integrates to the prior, and (2) evolves according to the known stochastic process at any point addition information is not revealed. Unlike this paper, the both papers assume the underlying stochastic process is commonly known, and that the signal structure is not fixed—that is, can depend on the entire history of realizations. As such, they do not consider what information (about the distribution of the state) is transmitted by optimal disclosure. Bizzotto et al. (2016) also study a persuasion model with dynamic components, where the Receiver can delay taking an action in the hope the exogenous arrival of information. Best and Quigley (2017) considers a dynamic model of Persuasion which shares much with this paper. Chiefly, the authors relax the assumption that the signal structure is publicly committed to, relying on repeated game mechanisms to ensure information transmission. They find that repetitional concerns are not a substitute for ex-ante public commitment, and sentiment that is shared by this paper.

This paper is also related to the literature on persuasion under private information. Kolotilin et al. (2015) consider a privately informed Receiver. Interestingly, they show, allowing the Sender to condition the signal structure on a *report* made by the Receiver (a la mechanism design) does not change the set of feasible outcomes. Here, in contrast, conditional (i.e., flexible) signal structures are, in general, beneficial to the Sender. Alonso and Camara (2016) consider the case where the Sender and Receiver have different priors. They show that, generically, the Sender can benefit from persuasion, even when the Receiver's actions is concave in his beliefs.

Perez-Richet (2014) analyzes the case where a Sender has private information about her type, which is very much related to the analysis of private-persuasion equilibria. In their paper, like this one, the Sender can condition the signal structure on her type. There, however, the Sender's choice of signal structure is public, and so, would be fully informative of the underlying distribution in a repeated interaction. This increases her commitment power, but also, limits the ability for the sender to capitalize on higher order uncertainty. In a result mirroring Theorem 4.2, they show equilibrium conditions constrain the Sender from differentiating her behavior according to the private information she obtains. This is a theme that is evident in Morris (2001), in which agents cannot disclose their private information, in equilibrium, for fear of looking biased.

In the model of private information, I assume the Sender can only credibly reveal the distribution of signals, and not the signal structure itself. As such, the model has clear connection with the literature on unverifiable signals, colloquially referred to as "cheap talk," and pioneered by Crawford and Sobel (1982). I show, with state-independent preferences, a privately informed Sender can persuade Receiver in the realization-by-realization, but not the distributional, dimension. Chakraborty and Harbaugh (2010), somewhat similarly, find that a privately informed Sender (in a pure cheap talk environment) with state independent preferences can be persuasive if the information is multi-dimensional. Margaria and Smolin (2015) show that, when the Sender has state-independent preferences, information can be transmitted in a cheap talk environment by appealing to dynamics. They use repeated game arguments to allow the Receiver to ensure that the Sender's payoff does not depend directly on the messages she sends, thus ensuring some level of cooperation, and yielding Pareto optimal payoffs.

3 Preliminaries

Players, Strategies, and Payoffs. There are two players, a Sender (she) and a Receiver (he). Let X denote the state space, with x a typical element, and $\mathcal{A}$ denote the set of available actions to the Receiver. There exists a true distribution that governs the realization of the state space, $\mu^\star \in \Delta(X)$.[5] Let $\theta \in \Delta(\Delta(X))$ denote the commonly known ex-ante belief regarding the true profile, and assume θ has finite support. Let $D = \text{supp}(\theta)$ denote the set of ex-ante possible distributions over X. It will be helpful to define notation for the average distribution over X, given the ex-ante beliefs: $\mu^{prior} = \sum_D \mu \cdot \theta(\mu)$. Timing is as follows. (1) the Sender privately observes $\mu^\star$. (2) The Sender privately chooses a signal structure. (3) the Receiver observes the profile of signals and chooses an action for each signal $a : \mathcal{S} \mapsto \mathcal{A}$.

A signal structure is a pair, $\langle S, e\colon X \to \Delta(S)\rangle$, with the interpretation that $e(s|x)$ is the probability of seeing signal $s \in S$ when the underlying realization is x. Let $\mathcal{E}$ denote the set of all experiments. It is natural to think of a pair (μ, e) as a distribution over $X \times S$ (where it is understood that S is the signal space associated to e). Indeed, (μ, e) induces $\sigma^{(\mu,e)} \in \Delta(X \times S)$ defined by the following:

$$\sigma^{(\mu,e)}(x, s) = \mu(x)e(s|x). \tag{3.1}$$

When the underlying distribution over X is given by μ, and the Sender chooses $e \in \mathcal{E}$, the Receiver observes $\text{marg}_S \sigma^{(\mu,e)}$, denoted by $\gamma^{(\mu,e)} \in \Delta(S)$.[6]

I assume the per-realization utility index is given by $u_S : \mathcal{A} \to \mathbb{R}$ and $u_R : \mathcal{A} \times X \to \mathbb{R}$, for Sender and Receiver, respectively. Notice that Sender's payoff depends only on the action taken, and not the state. Both players maximize there average payoffs over the profile of realizations, and are assumed not to deviate on measure zero sets. Therefore, if after observing the profile of signals, and conditional on signal $s \in S$, the Receiver holds the posterior belief $\mu \in \Delta(X)$, then $a(s)$ will maximize $\sum_{x \in X} u_R(\cdot, x)\mu(x)$. Let $\mathcal{A}(\mu)$ denote the set of maximizers. Following KG, I assume that when indifferent the Receiver maximizes the Sender's payoff. Therefore, as in KG we can denote by $\hat{v} : \Delta(X) \to \mathbb{R}$ as $\hat{v}(\mu) = \max_{a \in \mathcal{A}(\mu)} u_S(a)$. This is the per-realization value to the Sender of inducing the belief $\mu \in \Delta(X)$.

In what follows, we will examine the Sender's benefit to persuasion and the optimal

[5] Given a measure space $(P, \mathcal{F})$, let $\Delta(P)$ denote the set of distributions thereon. When P is discrete it is assumed $\mathcal{F} = 2^P$.

[6] A remark on notation: μ's will always be used for distributions over X, γ's for distributions over S, and σ's over $S \times X$.

signal structure both in the general case described above and the more simple *threshold* environment.

Definition. *We say that a persuasion game is a* ***threshold environment****, with threshold* $q \in (0,1)$*, if* $X = \{x_h, x_l\}$*,* $\mathcal{A} = \{A, R\}$ *and utilities are such that* $\mathcal{A}(\mu) = \{A\}$ *for all* $\mu > q$ *and* $\mathcal{A}(\mu) = \{R\}$ *for all* $\mu < q$*, where* $\mu \in \Delta(X)$ *is identified with* $\mu(x_h)$*, and* $u_S(A) > u_S(R)$*.*

In threshold environments, there are two states and two actions. The Receiver has preferences such that she wishes to align her action with the state of the world in each realization—if she knew with certainty the realization was x_h her unique strategy would be to choose action A, and if the state was x_l, choose action R. Being an expected utility maximizer, this indicates that the existence of some threshold belief q, such that if her belief the state is x_h is greater than q she will choose action A and if it is less than q, she will choose R. The Sender on the other hand, always prefers the action A to be chosen. Hence, her objective is to send signals that maximize the probability the Receivers belief is above q. Threshold environments capture a multitude of interesting persuasion environments: for example, a college deciding whether to accept a student based on her grades, an investor choosing to buy a security in light of a prospectus, a Judge arbitrating a case based on evidence, etc.

The Baseline Case: A Single Realization. I begin the analysis by examining how the Sender might benefit from the Receiver's uncertainty about the underlying distribution if there was only a single realization. This will introduce notation regarding the KG strategies and payoffs, useful for later analysis. Also, this is the simplest environment in which the tension between persuasion and learning is evident. As such, the observations made in the one-realization model will help illuminate how distributional uncertainty changes the standard persuasion environment and will be useful in understanding strategies over profiles of states.

Because there is only one signal sent, I must also assume here, as in KG, that the Sender publicly commits to a signal structure (in other words, that the Receiver observers the Sender's action). If this was not the case, signals would not be credible and only babbling equilibria would persist. First, to establish notation assume $|D| = 1$, so that $\mu^\star$ is commonly known. This is exactly the KG setup: given a the Sender's choice, $e \in \mathcal{E}$, the Receivers belief must be $\sigma^{(\mu^\star, e)}$. Therefore, the optimal signal structure is defined by the following.

Definition. *Denote by* $e^{KG}(\mu)$*, the equilibrium strategy of the Sender in the one-realization game with a commonly known distribution* μ*. Then* $e^{KG}(\mu)$ *solves*

$$\max_{e \in \mathcal{E}} \sum_{s \in S} \hat{v}\big(\sigma^{(\mu^\star,e)}(\cdot|s)\big)\gamma^{(\mu^\star,e)}(s). \tag{3.2}$$

As in KG, define $V(\mu)$ to be the concave closure of $\hat{v}$. That is $V(\mu) = \sup\{z \in \mathbb{R} | (\mu, z) \in \text{co}(\hat{v})\}$, where $\text{co}(\hat{v})$ is the convex hull of the graph of $\hat{v}(\mu)$.

Remark 3.1 (Kamenica and Gentzkow (2011))**.** *In a one-realization game with commonly known distribution* μ*, the Senders equilibrium payoff is* $V(\mu)$*, so that the Sender benefits from persuasion whenever* $V(\mu) > \hat{v}(\mu)$*. Moreover, if* $\hat{v}$ *is globally strictly concave then* $e^{KG}(\mu)$ *is the uninformative signal for all* μ *and if* $\hat{v}$ *is globally strictly convex then* $e^{KG}(\mu)$ *is the informative signal for all* μ*.*

We can now turn our attention to the case with distributional uncertainty. Abstracting away from private information (and thereby absolving myself of having to deal with off path beliefs until the next section), we can ask, would the Sender prefer the information about the distribution be revealed or not? That is, would she prefer to make a decision according to μ^{prior} or separate decisions according to each $\mu \in D$?

Theorem 3.2. *The Sender's always does better when the true distribution is uncertain. When* $\hat{v}$ *is not strictly convex over* D*, she does strictly better.*

Proof. In appendix B. ∎

The above theorem is a direct application of Jensen's Inequality; see Figures 1 and 2. While in some sense Theorem 3.2 is auxiliary to the main analysis, it allows us to understand when there is a tradeoff between current period persuasion and controlling the information flow to maximize the continuation value. Indeed, when $\hat{v}$ is globally strictly concave, then $V = \hat{v}$. KG point out that in this case, the sender does not benefit from persuasion, and therefore, the optimal signal is complete noise, and so, no learning will take place. Therefore, when $\hat{v}$ is concave, there is no tension between the incentive to persuade and the incentive to control the flow of information. Moreover, if $\hat{v}$ globally strictly convex, then V is linear, and the expected utility maximizing Sender has no incentive to control the flow of information—again the tension between persuasion and higher-order information revelation is muted.

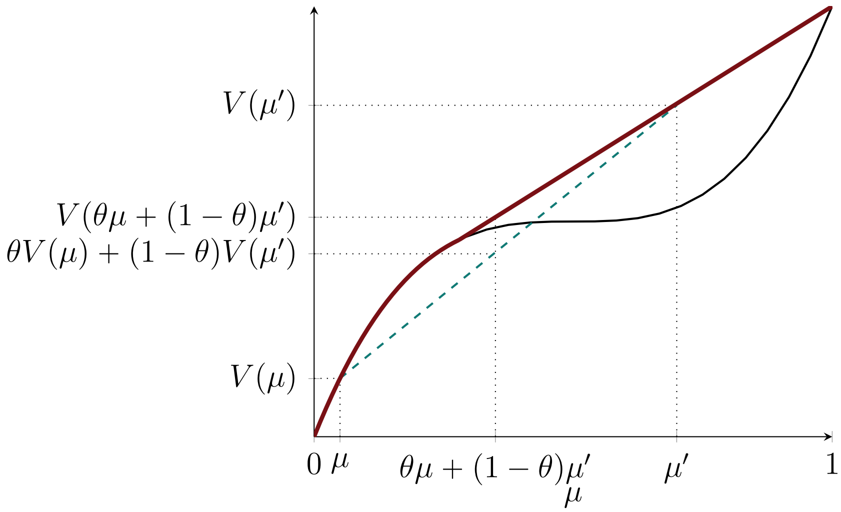

Figure 1: A plot of V where X has two states and $\Delta(X)$ is identified with $[0,1]$. The black curve is $\hat{v}$. The red curve is V. If the receiver believes the true distribution is μ with probability θ, then her prior on X is $\theta\mu + (1-\theta)\mu'$. Ex-ante, the Sender does better when the Receiver's belief is $\theta\mu + (1-\theta)\mu'$ then when there is an θ chance the belief is μ and a $(1-\theta)$ chance it is μ'. The value of the later is represented by the dashed line.

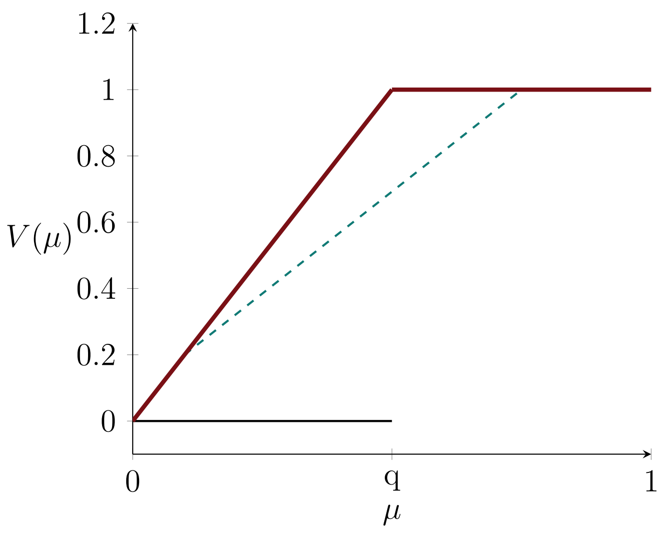

Figure 2: A plot of V in a threshold environment. The dashed line, representing the expected value if the distribution was revealed, lies below V if and only if it intercepts both of the two piecewise linear sections of V.

Remark 3.3. *In a one shot game in a q-threshold environment, the Sender is strictly better off when the true distribution is not revealed if and only if there exists an* $\mu, \mu' \in D$ *such that* $\mu < q < \mu'$.

In a Threshold environment, when all possible distributions are above the threshold, the Sender has no need for persuasion, since the Receiver chooses the action A without additional information. On the other hand, when all possible distributions lie below the threshold, then the payoff to the Sender is linearly related to probability of x_h. In either case, revealing the state has no effect on the Sender's expected payoff. On the other hand, when there are possible distributions such that $\mu < q < \mu'$, then there is *slack* in the problem that benefits the Sender. That is to say, μ' realizes x_h more than is necessary to get the Receiver to choose A. When there is distributional uncertainty, this additional probability on x_h gets mixed in with the other distributions, making it easier to persuade the Receiver to take action A. If the distribution is revealed, the additional realizations of x_h do not make the persuasion problem any easier, and so do not increase the Sender's payoff. This is most easily seen when $\mu = 0$ and $\mu' = 1$ and $\theta(\mu') > q$, so the Receiver takes action A with probability 1 (under a completely uninformative signal). However, under revelation, the Receiver only takes action A if the true distribution was μ'.

4 Equilibrium Strategies Under Private Information

With the above ideas in place, we can now consider the full model. Recall the timing and strategies of the game: (1) the Sender privately observes $\mu^\star \in D$. (2) The Sender chooses (privately, and so, without commitment) a signal structure $e \in \mathcal{E}$. (3) The Receiver observes the profile of signals, $\text{marg}_S \sigma^{(\mu^\star, e)}$, updates his beliefs, and chooses an action for each signal $a : \mathcal{S} \mapsto \mathcal{A}$. Because the Sender can condition her action of her private information, her strategy must dictate her action for each possible piece of information she might observe.

Definition. *A* ***strategy*** *for the Sender is a mapping* $r : D \to \mathcal{E}$.

We write $\sigma^{(\mu,r)}$ rather than $\sigma^{(\mu,r(\mu))}$ when it is not confusing to do so.

Equilibrium and Learning. Notice, because the Sender has private information, the Receiver's beliefs are not fixed as they are in the KG baseline case described above. The Receiver cannot distinguish between two different distributions over X if the resulting profile of equilibrium signals does not differ.

Definition. *Call* (μ, e) *and* (μ', e') ***s-equivalent****, denoted* $(\mu, e) \overset{s}{\sim} (\mu', e')$*, whenever*

$$\text{marg}_S \sigma^{(\mu,e)} = \text{marg}_S \sigma^{(\mu',e')}.$$

Therefore, each equilibrium strategy, r, induces a partition of D, $\mathcal{P}(r)$, where $\{\mu, \mu'\} \subseteq P \in \mathcal{P}(r)$ if $(\mu, r(\mu)) \overset{s}{\sim} (\mu', r(\mu'))$. For example, if the Receiver always learns the true distribution, then $\mathcal{P}(r) = D$, if he never learns $\mathcal{P}(r) = \{D\}$. For each $P \in \mathcal{P}(r)$, identify P with the distribution it induces as an interim belief

$$\sigma^{(P,r)} = \sum_{\mu \in P} \sigma^{(\mu, r(\mu))} \cdot \theta(\mu), \tag{4.1}$$

and define $\gamma^{(P,r)}$ as the marginal on S. Notice that $\gamma^{(P,r)} = \gamma^{(\mu,r)}$ for any $\mu \in P$.

Given the Sender's strategy, $r : D \to \mathcal{E}$, and the observed (marginal) distribution over signals, $\gamma \in \Delta(S)$, the Receiver's equilibrium beliefs must be correct. Let $D(r, \gamma) = \{\mu \in D | \text{marg}_S \sigma^{(\mu,r)} = \gamma\}$. That is, $D(r, \gamma)$ is the set of distributions such that, given the equilibrium strategy r, the induced distribution of signals is γ. So, if $D(r, \gamma)$ is non-empty, then the marginal distribution was possible given the equilibrium strategy and $D(r, \gamma) \in \mathcal{P}(r)$. Therefore, in equilibrium, the Receiver's beliefs must be derived via Bayes' rule:

$$\sigma^{\gamma} = \sum_{\mu \in D(r,\gamma)} \sigma^{(\mu,r)} \cdot \theta(\mu). \tag{4.2}$$

If, on the other hand, $D(r, \gamma)$ is empty, then a deviation has taken place, and the Receivers beliefs are restricted only by consistency with the objective information:

$$\text{marg}_S \sigma^{\gamma} = \gamma \text{ and } \text{marg}_X \sigma^{\gamma} \in \text{co}(D). \tag{4.3}$$

The first condition of 4.3 states that the Reciver's belief over the signals is the observed profile signals; the second condition states that his beliefs about the distribution of the state space is still given by some second order belief over D.

Definition. *A* ***private information distributional persuasion equilibrium*** *(PI-DPE) is a pair* $\langle r^{\star}, \{\sigma^{\gamma}\}_{\gamma \in \Delta(S)} \rangle$*, where* $r^{\star} : D \to \mathcal{E}$ *is the Sender's strategy, and* $\{\sigma^{\gamma}\}_{\gamma \in \Delta(S)}$ *is the Receiver's beliefs indexed by the publicly known distribution over signals, such that:*

(*i*) *Beliefs satisfy* (4.2) *and* (4.3)*, and*

(*ii*) *The Sender's strategy maximizes*

$$U(r|\mu) = \sum_{s \in S} \hat{v}(\sigma^{\gamma}(\cdot|s)) \gamma(s), \tag{4.4}$$

for each $\mu \in D$*, where* $\gamma = \text{marg}_S \sigma^{(\mu,r)}$.

The definition of an equilibrium, as before, takes as given that the Receiver statically optimizes his payoff, given his realization-by-realization beliefs (this is implicit in the use of $\hat{v}$).

Equilibrium Strategies. A first observation: without private information about the distribution, the fact that there are many persuasion interactions does not affect predictions at all.

Proposition 4.1. *When* $|D| = 1$*,* $r^\star : \mu \mapsto e^{KG}(\mu)$ *is the Sender's preferred PI-DPE.*

This follows from the fact that beliefs are fully restricted in such a model, and therefore, (4.4) reduces exactly to (3.2) which dictates the value of a strategy in the KG one-realization game.

With private information, on the other hand, beliefs are induced by equilibrium strategies, so, deviations are not necessarily detectable. Specifically, if the Sender deviates is such a way that the induced distribution over signals was ex-ante possible (i.e., if the Sender had learned a different piece of private information), then the Receiver's beliefs after the deviation will be incorrect. The above observation—that the Sender, after seeing any $\mu \in \Delta$, can deviate so as to effect any belief that was ex-ante possible given the equilibrium strategy—implies that there can be no separating equilibria in which, for different distributions of signals, the Sender receives a different payoff.

Theorem 4.2. *Let* $\langle r^\star, \{\sigma^\gamma\}_{\gamma \in \Delta(S)} \rangle$ *denote a PI-DPE. Then* $U(r^\star|\mu) = U(r^\star|\mu')$ *for all* $\mu, \mu' \in D$*.*

Proof. In appendix B. ■

Theorem 4.2 is somewhat counterintuitive at first glance. The Sender's observation of the underlying distribution, $\mu^\star$, *before* choosing the signal structure—i.e., her private information—presumably affords her the ability to signal to the Receiver that distribution is favorable. However, Theorem 4.2 states that this in never possible; the Sender cannot capitalize on having observed good information.

Upon reflection, the state of affairs is the classical problem of non-credible signaling: if the Sender's payoff *did* depend on her private information, say, $U(r|\mu) > U(r|\mu')$. Then it must be that the Receiver's behavior is different under μ than μ', which in turn implies that, $\gamma^{(\mu,r)} \neq \gamma^{(\mu',r)}$. But the Sender, after observing μ', can choose a completely uninformative signal structure with distribution $\gamma^{(\mu,r)}$. Under this distribution over S, the Receiver behaves *as if* the signal structure was $r(\mu)$, and therefore the Sender's payoff is $U(r|\mu)$. So this is an effective deviation to r, a contradiction to it being an equilibrium.

Given the restrictions Theorem 4.2 placed on the Sender's equilibrium strategies, a full characterization comes easily. Towards this, for any $\gamma \in \Delta(S)$ let $\Sigma(\gamma) \subset \Delta(X \times S)$ denote the set of joint distributions that satisfy (4.3).

Theorem 4.3. *Let $r : D \to \mathcal{E}$. The following are equivalent*

1. *There exists some $U^\star$ such that $U(r|\mu) = U^\star$ for all $\mu \in D$, and for each $\gamma \in \Delta(S)$,*

$$\min_{\sigma \in \Sigma(\gamma)} \sum_{S} \hat{v}(\sigma(\cdot|s))\gamma(s) \leqslant U^\star. \tag{4.5}$$

2. *There exists a set of beliefs $\{\sigma^\gamma\}_{\gamma \in \Delta(S)}$, such that $\langle r, \{\sigma^\gamma\}_{\gamma \in \Delta(S)} \rangle$ is a PI-DPE.*

Proof. In appendix B. ∎

In other words, the fact that the Sender's payoff does not depend on the private information (and of course, that the payoff is high enough that it cannot be beneficial to have the Receiver hold *any other belief*) completely characterizes equilibrium strategies. Notice also that a babbling equilibrium, where signals are completely uninformative, clearly satisfies the requirements, and therefore, existence is no issue.

While Theorem 4.3 is not inherently surprising, given the game theoretic literature on signaling, it provides clear limits on what kind on information can be transmitted by equilibrium strategies. Indeed, the dictate that the Sender's payoff is constant, in many situations, is sufficient to show the equilibrium strategy precludes full learning.

Theorem 4.4. *Assume D can be ordered according to first order stochastic dominance (with respect to $\hat{v}(\cdot)$) and let $v(\cdot)$ be monotone in each dimension of μ. Then in the Sender's preferred PI-DPE, $\langle r^\star, \{\sigma^\gamma\}_{\gamma \in \Delta(S)} \rangle$, $P(r^\star) \neq D$.*

Proof. In appendix B. ∎

Notice that the antecedent of Theorem 4.4 is met whenever $|X| = 2$ and $\hat{v}$ is increasing in the probability of one of the states. In particular, this is true of threshold environments.

The intuition behind Theorem 4.4 is a variation on the now familiar theme: ceteris paribus, uncertainty is good for the Sender. Indeed, consider a threshold environment, where $\mu_1(x_h) > \mu_2(x_h)$. Given that the Sender's payoff is constant, if learning takes place, her payoff is bounded by $V(\mu_2)$. But, this bound is achievable without inducing learning: by appropriately constructing $r(\mu_1)$ to have the same distribution of signals as $r(\mu_2)$ the Receiver takes the same actions as when the Sender learns μ_2. Since the payoff was constant to begin with, this new equilibrium is no worse from the perspective of the Sender. But, it must be that the Receiver's posterior is higher in the new equilibrium than it was after observing μ_2 in the original one (since the prior is now a mixture of μ_1 and μ_2). Hence, by continuity, the Sender can do better by slightly increasing the probability of sending s_A after x_l: this increases false positives, therefore, the Sender's payoff.[7]

So, given that in the Sender's preferred equilibrium does not betray her private information, what does the optimal signal structure look like? In threshold environments, the answer takes the form of a simple optimization problem.

Remark 4.5. *In a threshold environment, with $|D| = 2$ and $\mu_1(x_h) > \mu_2(x_h)$ then the Sender's optimal strategy is characterized by the following. $r^\star(\mu_1)$ solves,*

$$r^\star(\mu_1) \in \operatorname*{argmax}_r \sigma^{(\mu_1,r)}(s_A) \qquad \textit{subject to}$$

$$\frac{\theta\sigma^{(\mu_1,r)}(x_h, s_A) + (1-\theta)\mu_2}{\gamma^{(\mu_1,r)}(s_A)} = q. \tag{OB}$$

And, $r^\star(\mu_2)$ is given by $r^\star(\mu_2)(s_A|x_h) = 1$, $r^\star(\mu_2)(s_r|x_h) = \frac{\gamma^{(\mu_1,r^\star)}(s_A)-\mu_2}{(1-\mu_2)}$.

That such a strategy maximizes the Sender's payoff will turn out to be a consequence of the later analysis (Remark 5.3 in particular) and so is stated without proof. The constraint that the profile of signals is not informative completely determines the information signal structure associated with the second distribution, given the first. Therefore, the optimal signal structure is characterized by signal structure for μ_1 that maximizes the probability of s_A conditional on an obedience constraint, (OB). This constraint ensures the Receiver chooses A after seeing s_A, taking into account that the probability of s_A, under either distribution is $\sigma^{(\mu_1,r)}(x_h, s_A)$ and, the under μ_2, the

[7]Theorem 4.4 can strengthened probabilistically, via a near identical argument, to generic games. That is, whenever the distributions of D are drawn uniform (over the simplex $\Delta(X)$), then with probability 1 *no* information regarding D can be inferred via equilibrium signals. Because the intuition is fully captured by Theorem 4.4, I opted to only formally state and prove the simplest case.

probability of s_a and x_h is μ_2. Notice, this strategy preserves much of the structure of the KG result. In particular, there is no slack in the induce beliefs (when the Receiver accepts, he is indifferent) and the Sender minimizes false negatives (under μ_2 she only send s_R when the state is x_l). Because she must maintain the proportion of signals across distributions, however, false negatives may not be fully eliminated, and, as such, the resulting payoff to the Sender is lower that $V(\mu^{prior})$ (strictly lower whenever $\hat{v}$ is not strictly concave.[8]

Learning with a Non-Monotone $\hat{v}$. This section presents an example in which the Receiver learns the true distribution. When $\hat{v}$ is non-monotone, the Sender and Receiver's preference can be *effectively* aligned, even though the Sender's payoffs are state independent.

Example 1. *Students are either good at reading or math, $X = \{x_m, x_r\}$. There are two curricula, $\mu_m = [\frac{8}{10}, \frac{2}{10}]$ and $\mu_r = [\frac{2}{10}, \frac{8}{10}]$. The University has 3 actions, it can accept a student to an engineering program, to a literature program, or reject him all together, $\mathcal{A} = \{E, L, R\}$. Assume that the Universities preferences are such that it takes action E when it is sure of x_m, and L when it is sure of x_r and take action R whenever there is any residual uncertainty about the student type. Assume further that $u_S(E) > s_S(L) \gg u_S(R) = 0$.*

If either curriculum was commonly known, the KG optimal signal would be the fully informative signal structure, resulting in getting every student accepted. Now consider the case where there is a θ chance of μ_m. If the high school keeps the University uncertain in equilibrium, it can get $\frac{2}{10}$ accepted into the engineering program and $\frac{2}{10}$ into the literature, but no more. Any signal that kept the University uncertain could not fully reveal the ability of the other $\frac{6}{10}$ of the students. The Superintendent's payoff is $\frac{2}{10}u_S(E) + \frac{2}{10}u_S(L)$.

What if the Superintendent allows the University to learn? Then payoffs are bounded by $V(\mu_r) = \frac{2}{10}u_S(E) + \frac{8}{10}u_S(L)$, a clear improvement. Moreover this is a feasible bound. Under μ_r grades are perfectly informative. Under μ_m the student body is split into two groups (uniformly by ability), in the first grades are perfectly informative and all students get accepted. In the second, grades are completely uninformative, and all students are rejected. The proportion of students in each group is such that the resulting payoff is $V(\mu_r)$.

[8]Notice, the optimal signal in the example in Appendix A solves this maximization problem.

5 Distributional Persuasion with Public Commitment

The above characterization relies on the argument that the Sender's payment is flat in her private information, which in turn relies on her lack of ability to publicly commit to a signal structure. This Section contemplates the Sender's problem when she *can* publicly commit before observing the true distribution of the state.[9] In the ex-ante stage, the Sender is responsible for choosing a mapping from the set of distributions to experiments: $r : D \to \mathcal{E}$. (Note, this is functionally equivalent to what she chooses in the previous analysis, with private information). This signal structure is then publicly revealed, as is the profile of signals that it generates (the profile of signals remains unobserved by the Receiver). A *rigid* signal structure is a constant r. A signal structure that is not necessarily rigid is *flexible*.

Equilibrium Notion. Like in PI-DPE, the Receiver's interim beliefs are determined by the partition $\mathcal{P}(r)$. However, because the signal structure (i.e., the Sender's strategy) is publicly announced there are *no* off path beliefs.

Definition. *A* ***commitment distributional persuasion equilibrium*** *(C-DPE) is a strategy* $r^\star : D \to \mathcal{E}$ *that maximizes*

$$U(r) = \sum_{P \in \mathcal{P}(r)} \sum_{S} \hat{v}(\sigma^{(P,r)}(\cdot|s))\gamma^{(P,r)}(s)\theta(P). \tag{5.1}$$

Because the Sender must commit before observing $\mu^\star$, she cares about the expectation of her strategy. Then, conditional on μ, the structure to her payoff resembles to (3.2) (dictating PI-DPE), although here, deviations are not possible. In other words, the Sender only cares about how the signal structure persuades the Receiver given the interim beliefs induced by *that* signal structure. To see how the above characterization embodies this notion, notice the interim belief will be $P \in \mathcal{P}(r)$ with probability $\theta(P)$. Therefore, the Sender's realization specific payoff, when the signal realized was s, will be $\hat{v}(\sigma^{(P,r)}(\cdot|s))$; signal s will be realized with probability $\gamma^P(s)$. Hence $r^\star$ is an equilibrium if it maximizes the average of the persuasion payoff of the possible interim beliefs (i.e., $\mathcal{P}$), weighted by the ex-ante probability of each cell.

5.1 Flexible Signal Structures.

We now turn our attention to equilibrium with flexible signal structures, where the Sender can condition the signal structure on the underlying distribution. One might

[9] Public commitment *after* learning ones type is rather boring. The only possibility is the natural map, $r : \mu \mapsto e^{KG}(\mu)$.

be tempted to blame the lack of information transmission in PI-DPE on the lack of commitment power and, therefore, assume that persuasion in a C-DPE implies revelation of the distribution. This is not the case; the benefit to uncertainty (as captured in the single realization case) can be large enough to so that the Sender prefers to keep the Receiver uncertainty even when she could credibly signal her private information. As such, I pay particular attention to whether learning takes place under the optimal signal structure.

Notice, when the Sender benefits from persuasion in the KG model, the signal cannot be completely uninformative. Now, if these signals are revealing not only on realization-by-realization basis but also about μ, then the Receiver will learn $\mu^{\star}$. So, if the Sender benefits from persuasion, and benefits from the Receiver's uncertainty regarding the distribution, there is a tension: the KG-optimal release of information in pursuit of persuading the Receiver also informs him about the underlying distribution. Of course, if the Sender benefits from only one or the other of these avenues, there is no tension, and so, predictions come easy.

Remark 5.1. *(i) If $\hat{v}$ is strictly globally concave, then the optimal strategy is to send a a completely uninformative signal structure and no learning takes place. (ii) If $\hat{v}$ is strictly globally convex, then the optimal strategy is a fully informative signal structure and full learning takes place.*

Remark 5.1 continues to mirror the observation of KG that when the objective function of the Receiver is concave (resp. convex) the unique optimal strategy in the one-realization game is a completely uninformative signal (resp. completely informative signal). This result is extended cleanly to the distributional-persuasion environment by means of Theorem 3.2, which states that whenever informativeness is desirable about each realization (i.e., under concavity) it is also desirable about the distribution. In other words, if smoothing out the Receiver's beliefs regarding the realization of the state is always beneficial, then smoothing out their second order beliefs is also beneficial.

This is not true however, when the objective function is neither strictly concave nor convex; there is a tension between the benefits of persuasion arising from convexity and the benefits to smoothing the second order belief arising from concavity. The example in Appendix A shows how these tensions can balance out; the optimal signal structure has persuasive content on a realization-by-realization basis but does not change the Receivers belief regarding the underlying distribution. Of course, this echoes the equilibrium analysis of PI-DPE.

We can simplify the problem by looking at each motivation in isolation. To see how this might work, consider some equilibrium strategy r, which induces the partition $\mathcal{P}$ of D. For a given r, the elements of $\mathcal{P}(r)$ can be represented by distributions in $\Delta(X \times S)$. If r is an equilibrium strategy, it must provide a higher payoff than any other strategy, but in particular, higher than any other strategy that *induces the same partition*. Characterizing r is such a manner is simple: it is the maximization of a continuous function over a compact set. Moreover, the set of possible partitions over D is finite by virtue of D being finite. Therefore, by first constructing the set of maximizer for each partition, and then selecting the partition that yields the highest ex-ante average payoff, the equilibrium can be found.

Definition. *Let $\mathcal{P}$ be a partition of D. A strategy r is **$\mathcal{P}$-optimal** if*

$$r \in \underset{r'}{\operatorname{argmax}}\, U(r').$$

subject to $\mathcal{P}(r') \cong \mathcal{P}$. Let $r^{\mathcal{P}}$ denote a $\mathcal{P}$ optimal strategy.

That is, $r^{\mathcal{P}}$ is optimal over all strategies that induce the interim beliefs embodied by the partition $\mathcal{P}$. Existence follows from this concept.

Theorem 5.2. *(i) For each partition $\mathcal{P}$ there exists an $\mathcal{P}$-optimal strategy, $r^{\mathcal{P}}$. (ii) $r^\star$ is a C-DPE if and only if*

$$U(r^\star) = \max_{Partitions} U(r^{\mathcal{P}})$$

Proof. In appendix B. ■

The first claim follows that from the observation that the set of strategies such that $\mathcal{P}(r) \cong \mathcal{P}$ can be compactified under the identification of strategies r with the payoffs they induce. In other words, for any sequence of strategies $\{r_n\}_{n\in\mathbb{N}}$ such that $\mathcal{P}(r_n) = \mathcal{P}$ for all n, there is an corresponding sequence $\{r'_n\}_{n\in\mathbb{N}}$ such that $\mathcal{P}(r'_n) = \mathcal{P}$ and $U(r_n) = U(r'_n)$ for all $n \in \mathbb{N}$. Importantly, this sequence has a convergent subsequence with a limit r' such that that $\mathcal{P}(r') = \mathcal{P}$. This, plus the upper-semicontinuity of $\hat{v}$ is sufficient to guarantee a maximal strategy. The second claim follows directly from the first, and the fact that there are a finite number of partitions.

In addition to ensuring existence, this result also positions us well on our way to characterizing the equilibrium signal structure. While the following results hold in a more general form, intuition is most easily obtained by looking at threshold environments, since by restricting ourselves to threshold environments, the characterization of $\mathcal{P}$-optimal becomes more explicit.

Remark 5.3. *For all $P \in \mathcal{P}$,*

$$r^{\mathcal{P}} \in \underset{r}{\operatorname{argmax}}\ \gamma^{(P,r)}(s_a) \quad \textit{subject to}$$

$$\gamma^{(\mu,r)} = \gamma^{(P,r)} \textit{ for } \mu \in P \tag{SE}$$

$$\frac{\sigma^{(P,r)}(s_a, x_h)}{\gamma^{(P,r)}(s_a)} \geqslant q. \tag{OB}$$

That is, within each cell of $\mathcal{P}$ a $\mathcal{P}$-optimal strategy maximizes the probability of observing s_a subject to two constraints. First, signal equivalence, (SE), ensures that the beliefs are consistent with the partition. That is, for each μ, μ' the marginal on the signal space is the same so that $(\mu, r(\mu)) \overset{s}{\sim} (\mu', r(\mu'))$. Second, the obedience constraint, (OB), ensures that after observing s_a the Receiver chooses action A. Hence, by maximizing the probability of s_a, the Sender is maximizing her payoff.

An important insight is that the maximization takes place within each cell separately, greatly reducing the complexity of the problem. For example, if $\mu \in \mathcal{P}$ is a singleton cell in the partition, then $r(\mu)$ must be $e^{KG}(\mu)$. That is, it is optimal for the Sender to allow the Receiver to learn, the only C-DPE strategy is to use the KG signal structure for each distribution, $r : \mu \mapsto e^{KG}(\mu)$. Since the Receiver's interim beliefs will be $\mu^\star$, any other signal structure will be sub-optimal by definition.

Several other features become apparent, echoing the results of KG, albeit in a slightly weaker form. First, if the Sender can benefit from persuasion, then it must be, whenever the Receiver chooses R, he is indifferent. In other words, (OB) will hold with equality. The reason is intuitive: assume that the Receiver strictly preferred A, then by continuity, the Sender could slightly increase $r(\mu)(s_a|x_l)$ for *every* μ, in such a way as to keep both constraints met. Similarly, for some μ' with $\mu'(x_h) < q$, $r(\mu')(s_a|x_h) = 1$. Again, if this was not true, then we could slightly increase $r(\mu')(s_a|x_h)$, and increase $r(\mu)(s_a|x_h)$ for all other μ, in such a way as to keep (SE) intact.

The connection between C-DPE and PI-DPE. C-DPE is characterized by $\mathcal{P}$-optimality. If in each cell in $\mathcal{P}$ the Sender's payoff is the same, then the Sender's strategy is also part of a PI-DPE.

Theorem 5.4. *Let r denote a C-DPE such that the Sender's payoff is constant across D. Then there exists a set of beliefs $\{\sigma^{\gamma}\}_{\gamma \in \Delta(S)}$, such that $\langle r, \{\sigma^{\gamma}\}_{\gamma \in \Delta(S)} \rangle$ is a PI-DPE that maximizes the Sender's payoff.*

Proof. In appendix B. ∎

In particular, any C-DPE in which $\mathcal{P}(r) = \{D\}$ is a PI-DPE. I.e., if an uninformed Sender with access to flexible signal structures chooses a strategy that precludes learning, then a privately informed Sender (facing the same uncertainty) will choose the exact same strategy. Further, when $|D| = 2$, and so by Theorem 4.4 it must be that $\mathcal{P}(r^{\star}) = \{D\}$, we can solve for solutions only within the space of $\{D\}$-optimal strategies. This is the origin of the maximization problem characterizing PI-DPE, in remark 4.5. The example in Appendix A constitutes an private-persuasion equilibrium by virtue of the fact that the University never learns the efficacy of the curriculum change.

5.2 Rigid Signal Structures.

The previous section indicates that allowing the Sender to publicly commit to a signal structure can help assuage the lack of information transmission (regarding the distribution over the states); but it does not guarantee learning will take place. When there is sufficient benefit to keeping the Receiver uncertain the Sender does so, publicly committing to not reveal and information. Moreover, in many situations, it is a natural to restrict r to be constant. Since the Sender does not know the true distribution when choosing her strategy, perhaps the Sender cannot design a signal structure that depends on the underlying fundamentals that govern uncertainty. This section explores equilibrium in rigid structures. I show that under such a restriction, the Sender cannot persuade the receiver without revealing the true distribution, and so, learning is all but guaranteed.

Definition. *A **rigid distributional persuasion equilibrium** (R-DPE) is a strategy $r \in \mathcal{E}$ such that r maximizes $U(\cdot)$ over all rigid signal structures.*

A theme that is by now quite evident is, when $\hat{v}$ is either concave of convex, the one-realization optimal strategies are very stable. Indeed, under concavity, the Sender prefers the Receiver not to learn the fundamental distribution, and the one-realization strategy—sending complete noise—achieves this. Likewise, under convexity, full separation is both desirable and achievable with the fully informative signal.

Remark 5.5. *(i) If $\hat{v}$ is strictly globally concave, then the optimal rigid strategy is a completely uninformative signal structure. (ii) If $\hat{v}$ is strictly globally convex, then the optimal rigid strategy is a fully informative signal structure.*

This remark is a straightforward corollary of Remark 5.1, which states that the states strategies are optimal when r is not constrained to be constant. Of course, since the completely un/informative signal is constant, it is clearly optimal in the constrained problem.

When the shape of $\hat{v}$ is neither concave nor convex, and when the Sender is constrained to design rigid signal structures, an equilibrium may not exist. While unfortunate, this is enlightening of the mechanics of the model; in particular the following example pin points the problem.

Example 2. *Consider a threshold environment with threshold $\frac{1}{2}$. There are two possible underlying distributions: $\mu = [\frac{1}{2}, \frac{1}{2}]$ and $\mu' = [0, 1]$. Let $\theta(\mu) \in (0, 1)$. Then, under the completely uninformative signal, the Receiver never learns and always chooses B. Let e denote any informative signal, such that signal s_a (and only s_a) induces action A, when the distribution is known to be μ. Since e is informative it will enable learning. Hence, the Receiver will choose action A with ex-ante probability $\theta(\mu)[\frac{1}{2}e(s_a|x_h) + \frac{1}{2}e(s_a|x_l)]$; this is increasing in both $e(s_a|x_h)$ and $e(s_a|x_l)$. However, in the limit, where $e(s_a|x_h) = e(s_a|x_l) = 1$, the signal is completely uninformative.*

Notice that the Receiver was exactly indifferent between his action when he learns the true distribution is μ. This is not a coincidence; an optimal signal structure does not exist because the Sender wants to reveal the distribution without providing any additional (i.e., contemporaneous) information. She wants to reveal the distribution is μ, but any further perturbation, no matter how small, decreases the expected payoff because it could alter the Receivers action. Of course, for this to be the case, it must be that the Receiver was indifferent between two actions at μ.

Assumption 1. *(i) For all nontrivial $E \subset X$ and all $\mu, \mu' \in D$, $\mu(E) \neq \mu'(E)$. (ii) For all $\mu \in D$, $\mathcal{A}(\mu)$ is a singleton.*

Under Assumption 1, when restricted to rigid signal structures, the tradeoff between persuasion and the control of information flow is strict; if the Sender wants to persuade the Receiver, she must allow him to learn the true distribution. This is because, in order to for a signal to be persuasive it must have some informational content. Across underlying distributions, the signals are realized with the same probabilities conditional on the state. Therefore, so long as the relevant the states have different likelihoods of occurring the empirical frequencies of signals will be informative about the underlying distribution. The first part of Assumption 1, therefore, ensures the empirical frequencies of signals are sufficient to identify the states.[10] Without such a restriction, it is possible there are two distributions μ and μ' such that for some nontrivial event E, $\mu(E) = \mu'(E)$ and all persuasion is with respect to E. Then the Receiver will not learn though he may still be persuaded.

Within threshold environments, this cannot happen, so Assumption 1 part (i) is vacuous. Notice, when there are only two states, and $\mu \neq \mu'$ then for *any* $e \in \mathcal{E}$ that is not a completely uninformative signal, (μ, e) and (μ', e) are not $\boldsymbol{s}$-equivalent. This implies that whenever the Sender can benefit from persuasion, it must be that the Receiver learns the underlying distribution! In order to persuade the Receiver, the Sender must provide some information regarding the realizations, and this is enough to ensure revelation of the true distribution.

Remark 5.6. *Under Assumption 1, with rigid signal structures, learning takes place if and only if r is not completely uninformative.*

Proof. In appendix B. ■

The above remark implies that, when Assumption 1 holds, then in any equilibrium in which the Sender benefits from persuasion, the Receiver learns the underlying distribution. Because the conditions are always met it threshold environments, this means that there can be no threshold environment equilibrium in which the Receiver's default action is B, and in which he does not learn.

Of course, this says nothing, so far, about the *existence* of equilibria. It is the second part of Assumption 1 that rules out cases like Example 2. Together these restrictions ensure an equilibrium exists.

Theorem 5.7. *If Assumption 1 holds, then a R-DPE exists.*

Proof. In appendix B. ■

[10] If the distributions are chosen uniformly, this restriction is met with probability 1.

Not only does Remark 5.6 helps provide an understanding of when the Receiver learns, but also of equilibrium signal structure. Because persuasion implies learning will take place with probability 1, the Sender's only motivation in designing the signal is period-by-period persuasion. As such, the optimal signal bears resemblance to the static KG equilibrium signals. In particular, the same tension that underlie the characterization of the one-realization game, also dictate what an optimal signal structure can be.

Theorem 5.8. *Fix some threshold environment such that Assumption 1 holds and the Sender can benefit from persuasion. Then, there is a R-DPE with $S = \{s_a, s_r\}$, and such that, either*

1. *for some $\mu \in D$ with $\mu(x_h) < q$, $r \equiv e^{KG}(\mu')$, or,*
2. *for some $\mu \in D$ with $\mu(x_h) > q$, $q = \mu(x_h|s_r) < \mu(x_h|s_r)$.*

Proof. In appendix B. ■

In threshold environments, where the Receiver learns with probability 1, the optimal signal structure is relative to the expected payoff under each distribution separately. KG show that, in the one-realization game when $\mu(x_h) < q$, whenever the Receiver chooses action B, he knows with certainty the state is x_l; in other words, $r(s_a|x_h) = 1$. Intuitively, increasing $r(s_a|x_h) = 1$ increases both the likelihood *and* the informativeness of s_a,unambiguously increasing the probability of the Receiver taking action A. Since this is true irrespective of the distribution (so long as $\mu(x_h) < q$) it is reasonable to expect the optimal rigid signal structure to also adhere to this rule.

The only potential issue is if, for some μ' with $\mu'(x_h) > q$, increasing $r(s_a|x_h)$ changes the Receivers action, conditional on learning μ' and seeing s_r. That is, by increasing the informativeness of s_a, and thus increasing the posterior on x_h after s_a, the Sender inadvertently lowers the posterior after s_r below the threshold. When $\mu(x_h) < q$, the Receiver chooses B after s_r, and therefore lowering the posterior has no effect on the subsequent action or payoff. When $\mu'(x_h) > q$, however, it might be that a small decrease in $\mu'(x_h|s_r)$ changes the Receiver's action. Of course, this can only happen for arbitrarily small changes, if conditional on learning μ' and seeing s_r, the Receiver was indifferent between the two actions. Hence, if (2) does not hold, $r(s_a|x_h)$ must be 1, as in the optimal one-realization game.

Finally, given that $r(s_a|x_h) = 1$, $r(s_a|x_l)$ must be such that the incentive constraint binds after seeing x_h. That is, an additional increase in $r(s_a|x_l)$ must change the

Receiver's action conditional on learning some μ, or else the Sender could increase the likelihood of s_a. Of course, in the one-realization game, these two constraints fully characterize the optimal signal structure in a threshold environment.

6 Conclusion

In this paper, I introduce a model in which a Sender who tries to persuade a Receiver not to take a single action but rather a profile of actions. The Sender designs a signal structure, which reveals information about a profile of states, drawn according to a distribution which is known privately to the Sender. Because there many signals, each signal, in addition to its persuasive effect regarding its corresponding state realization, also changes the Receiver's belief about the underlying distribution.

I provide conditions, namely monotonicity of $\hat{v}$, under which the Sender's private information is never fully revealed in equilibrium. This stems from the fact that the Sender's chosen signal structure is not public knowledge. Therefore, I also consider a variant of the above model where the Sender has must publicly commit before becoming informed. Public commitment helps mitigate the lack of information transmission, but does not solve the problem entirely when commitment devices are flexible. This is because the rents that can be extracted by keeping the Receiver uninformed can outweigh the loss of loss of precision (in the signal structure) needed to keep him uninformed. Therefore, I argue public commitment mechanisms need to be rigid in order to ensure that the true distribution over the state space will be revealed in equilibrium.

A A Numerical Example

Example: A New Curriculum. Let $X = \{x_h, x_l\}$ denote the two possible states, in which the a student is of high and low ability, respectively. The high school can send two possible grades, $S = \{s_a, s_r\}$. The University can accept of reject each student: $\mathcal{A} = \{accept, reject\}$; the University receives a payoff of 1 if it accepts a high ability student, -1 if it accepts a low ability student and 0 if it rejects the application. The Superintendent receives 1 if the University accepts and 0 if he does not. Notice, the University accepts a student whenever its posterior is above $\frac{1}{2}$.

The high school institutes a new curriculum, perhaps at the behest of the government or some outside actor. If the new curriculum is effective, in which case every student will be prepared for college, $\mu_1 \in \Delta(X) = [1, 0]$, or, the program is very ineffective, in which case only $\frac{2}{10}$ of the students will be prepared, $\mu_2 \in \Delta(X) = [\frac{2}{10}, \frac{8}{10}]$. The likelihood of the program being effective is known to be $\frac{2}{10}$: $\theta = [\frac{2}{10}, \frac{8}{10}]$.

As a point of reference, if the efficacy of the program was known, the optimal signal structure is as derived by KG. In particular:

$$e_1(s_h|x_h) = 1 \qquad e_1(s_a|x_l) = 0 \tag{A.1}$$

$$e_2(s_h|x_h) = 1 \qquad e_2(s_a|x_l) = \frac{2}{8} \tag{A.2}$$

Here, the University accepts a student with probability 1 when the distribution is μ_1 and probability $\frac{4}{10}$ when the distribution is μ_2.

First, consider the case where the Superintendent must design a rigid signal structure. It is clear that under any informative signal structure, the University will learn if the curriculum works. It is easy to see, if the true distribution is μ_1 the University will accept all requests regardless of the signal, and so, the optimal signal must be $r \equiv e_2$.

Now, consider the case where the Superintendent can instruct the the teachers to institute a rubric that depends on the teachers (accurate) assessment of the curriculum. That is, the Superintendents strategy is a function $r : \{\mu_1, \mu_2\} \to \mathcal{E}$, where $\mathcal{E}$ is the set of all grading policies. If the University learns which distribution is true, it is clear that if $r(\mu_i) = e_i$ is optimal (since it is realization by realization optimal). Moreover, this signal structure does induce learning, so that the expected value of such a strategy is $\frac{2}{10}1 + \frac{8}{10}\frac{4}{10} = \frac{52}{100}$.

However, the Superintendent can do better by ensuring the University does not learn! Indeed, consider the following strategy, r, given by,

$$r(\mu_1)(s_a|x_h) = \frac{8}{15} \qquad r(\mu_1)(s_a|x_l) = \frac{8}{15} \tag{A.3}$$

$$r(\mu_2)(s_a|x_h) = 1 \qquad r(\mu_2)(s_a|x_l) = \frac{5}{12} \tag{A.4}$$

This induces the following joint distributions over $X \times S$ It is straightforward to

σ_1

	s_a	s_r
x_h	$1 \cdot \frac{8}{15} = \frac{8}{15}$	$1 \cdot \frac{7}{15} = \frac{7}{15}$
x_l	$0 \cdot \frac{8}{15} = 0$	$0 \cdot \frac{7}{15} = 0$

σ_2

	s_a	s_r
x_h	$\frac{2}{10} \cdot 1 = \frac{2}{10}$	$\frac{2}{10} \cdot 0 = 0$
x_l	$\frac{8}{10} \cdot \frac{5}{12} = \frac{4}{12}$	$\frac{8}{10} \cdot \frac{7}{12} = \frac{7}{15}$

verify that $\text{marg}_S(\sigma_1) = \text{marg}_S(\sigma_2)$, so that the University can make no inference about the true distribution by observing signals Hence, learning does not take place. Further,

$$\mu_{s_a}(x_h) = \frac{\frac{2}{10}\cdot\frac{8}{15}+\frac{8}{10}\cdot\frac{2}{10}}{\frac{2}{10}(\frac{8}{15}+0)+\frac{8}{10}(\frac{2}{10}+\frac{4}{12})} = \frac{1}{2}.$$

So, after seeing the signal s_a the University accepts the application. Whats more, the signal s_a appears in equilibrium with probability $\frac{8}{15}$, so the equilibrium payoff to the Superintendent is $\frac{8}{15} > \frac{52}{100}$.

The Superintendent's optimal strategy is to choose a signal structure in such a way that the University never updates its second order prior. Allowing the Sender to learn erodes the slack in the problem introduced by μ_1. In other words, μ_1 realizes x_h more often than necessary to get the University to accept (notice in Figure 2, once $\mu > q$, $\hat{v}(\mu)$ is constant). When there is uncertainty about the distribution, this additional probability of x_h improves the Superintendent's ability to persuade the University. However, once the distribution has been learned the additional probability on x_h is on no value to the Sender.

Example: A Privately Informed Superintendent. Now, consider what happens the Superintendent knows if the curriculum will be effective or not, and she can base the schools rubric on this decision. Moreover, although the Superintendent must grade the students in a consistent manner, i.e., according she cannot grade two students according to different rubrics, she cannot credibly prove, to the University or some other external agent, what that rubric is. The University sees only the distribution of grades within the applicant pool, but not the generation process that led to such a selection of grades. Because the rubric is not public information, an equilibrium must also specify the beliefs of the University after deviations occur.

Let the parameters of the problem—the distributions μ_1, μ_2 and θ, and the threshold q—be as in the above problem. The first thing to notice is that the Superintendent cannot do better after observing μ_1 than after observing μ_2. If this was the case, the Sender, after observing μ_2, can choose completely uninformative rubric with a distribution $e_2(s|x_h) = e_2(s|x_l) = e_1(s|x_h) + e_1(s|x_l)$ for all $s \in S$, which has the same distribution of grades as the equilibrium strategy after observing μ_1. Therefore, the University believes, incorrectly, that the curriculum is effective, and therefore admits all student. So this is an profitable deviation from the original strategy after observing μ_2, a contradiction to it having been part of an equilibrium.

So, in any equilibrium, the Superintendent's payoff does not depend on her private information. And, what is the best payoff she can sustain while maintaining that the payoff did not depend on her private information? Of course, it must be $\frac{8}{15}$, obtained via the strategies defined by (A.3) and (A.4).[11] By virtue of these strategies constituting an equilibrium strategies in flexible signal structures this is the best the principal could do, with or without commitment power.

If, by contrast, the revelation of μ_1 or μ_2 was *publicly* observed, the Superintendent's optimal strategy would be the statically optimal strategies, defined by (A.1) and (A.2). But, again, without any further calculation we know the Superintendent must do worse than $\frac{8}{15}$, since the equilibrium in flexible signal structures did not induce learning. Hence, by examining such equilibria we know pricelessly when the Superintendent will opt to have distributional uncertainty resolved publicly or privately.

B Proofs

Proof of Theorem 3.2. If the true distribution is revealed to be μ, then by Kamenica and Gentzkow (2011) the equilibrium payoff to the Sender is $V(\mu)$. Hence, when the distribution is revealed the ex-ante payoff is $V^{rev} = \sum_{\mu \in D} V(\mu)\theta(\mu)$. Now assume the distribution is not revealed. Then the Receivers prior is given by $\mu^{prior} = \sum_{\mu \in D} \mu \cdot \theta(\mu)$, and hence, the equilibrium payoff is $V^{norev} = V(\mu^{prior})$. So by Jensen's Inequality, $V^{rev} < V^{norev}$ if and only if V is strictly concave over D. ■

Proof of Theorem 4.2. Suppose to the contrary there was an equilibrium in which $U(r|\mu) > U(r|\mu')$ and $\text{marg}_S \sigma^{(\mu,r)} \neq \text{marg}_S \sigma^{(\mu',r)}$. Let $\hat{\gamma} = \text{marg}_S \sigma^{(\mu,r)}$. Consider the following deviation from r, in which, after observing μ' the Sender chooses a signal structure that is completely uninformative and has same distribution of signals as $\text{marg}_S \sigma^{(\mu,r)}$—that is $e(\cdot|x) = \hat{\gamma}$ for all $x \in X$—and leaves all other signal structures unchanged. After observing μ' and playing according to this deviation, the Receiver has beliefs $\sigma^{\hat{\gamma}}$ as prescribed by the original equilibrium. But then, the Sender's payoff is $\sum_S \hat{v}(\text{marg}_X \sigma^{\hat{\gamma}}(\cdot|s))\hat{\gamma}(s) = U(r|\mu) > U(r|\mu')$. Hence this deviation is profitable, a contradiction to the $\langle r, \{\sigma^{\gamma}\}_{\gamma \in \Delta(S)} \rangle$ constituting an equilibrium. ■

Proof of Theorem 4.3. First, assume that (1) holds. For any γ such that $D(r, \gamma)$

[11] The Universities off path beliefs place probability 1 on μ_1 and a completely uninformative signal structure.

is non-empty, σ^γ is completely determined by (4.2); for any γ such that $D(r,\gamma)$ is empty, define σ^γ to be

$$\operatorname*{argmin}_{\sigma\in\Sigma(\gamma)} \sum_S \hat{v}(\mathrm{marg}_X\sigma)\gamma(s)$$

which satisfies (4.3) by construction. Hence, $\{\sigma^\gamma\}_{\gamma\in\Delta(S)}$ is an admissible belief set for the Receiver. Now let $\mu \in D$ be arbitrary; given that the Sender chooses the signal structure $r(\mu)$ her payoff is $U^\star$. Let $e \in \mathcal{E}$ denote any other signal structure. $\hat{\gamma} = \mathrm{marg}_S\sigma^{(\mu,e)}$. There are two cases: Case (i) $D(r,\hat{\gamma})$ is non-empty. Then there exists some μ' such that $\mathrm{marg}_S\sigma^{(\mu',r)} = \hat{\gamma}$. But then, the Receiver's beliefs are as if the Sender observed μ' and played according to r; the resulting payoff is $U^\star$. Case (ii) $D(r,\hat{\gamma})$ is empty. Then the resulting payoff the the sender is $\sum_S \hat{v}(\mathrm{marg}_X\sigma^\gamma)\hat{\gamma}(s) \leqslant U^\star$ by (1). Therefore, there is no deviation to playing according to r.

Now, assume that (2) holds. Let $\mu \in D$ and set $U^\star = U(r|\mu)$. That $U(r|\mu') = U^\star$ for all $\mu' \in D$ is a direct consequence of Theorem 4.2. Now assume to the contrary that there was γ such that $\min_{\sigma\in\Sigma(\gamma)} \sum_S \hat{v}(\mathrm{marg}_X\sigma)\gamma(s) > U^\star$. Then after observing any piece of private information, the Sender can choose a signal structure that induces γ. Since for every admissible belief of the Receiver, the resulting payoff is above $U^\star$, such a strategy is a deviation from r; a contradiction to r being part of an equilibrium. ■

Proof of Theorem 4.4. Order X according to $\hat{v}$. Order D by stochastic dominance on the index of X. It suffices to show that the lowest type, μ_1, is never revealed. Assume to the contrary that μ_1 was revealed in equilibrium with strategy r so that $\mathcal{P}(r) = \{P_1,\ldots,P_n\}$ with $P_1 = \{\mu_1\}$. By monotonicity of $\hat{v}$ in each dimension we know $V(\mu_i)$ is increasing in i, so by Theorem 4.2, $U(r) = V(\mu_1)$.

Now consider the alternative strategy r'. For $\mu \notin P_n$ the strategy remains unchanged $r'(\mu_i) = r(\mu_i)$. For $\mu \in P_n$, $r'(\mu)(x_1) = \sigma_1(\cdot|x_1)$ and

$$r'(\mu)(x_i) = \sum_{j<i}\pi(\mu,j)\sigma^{P_1}(\cdot|x_j) + (1-\sum_{j<i}\pi(\mu,j))\sigma^{P_1}(\cdot|x_i),$$

where $\pi(\mu,j) = \mu_1(\{x_1,\ldots x_j\}) - \mu(\{x_1,\ldots x_j\})$. Notice $\mathcal{P}(r') = \{P_2,\ldots,P_{n-1},P_n'\}$ where $P_n' = P_n \cup \{\mu_1\}$, and for any is the $\mu \in P_n'$, $\sigma^{(\mu,r')}$ first order stochastically dominates σ_1. By the monotonicity of v this is a weak improvement. Moreover, by slight perturbation, continuity ensures that we can construct a strict improvement (i.e., where $U(r'|\sigma_n') = V(\mu_1) + \epsilon$. Finally, since $r : \mu \mapsto e^{KG}(\mu)$, for each $\mu \neq \mu_1$ provides strictly better than $V(\mu_1)$, it must have been that μ_1 was a binding constraint in light of Theorem 4.2, and hence we can construct the rest of r' so as to provide payoff $V(\mu_1) + \epsilon$ for sufficiently small ϵ. But this is a deviation from r, contradiction it being an equilibrium. ■

Proof of Theorem 5.2. Claim (i). Let $R(\mathcal{P})$ denote the set of all strategies that induce $\mathcal{P}$. Since utilities are bounded, let $\bar{U} = \sup_{r \in R(\mathcal{P})} U(r) < \infty$. Let $\{r_n\}_{n\in\mathbb{N}} \subset R(\mathcal{P})$ be a sequence such that $\bar{U} - U(r_n) \leqslant \frac{1}{n}$, and $U(r_n)$ is monotone increasing. Such a sequence exists by the definition of supremum. For each $P \in \mathcal{P}$, and $a \in \mathcal{A}$ let $S(r, P, a)$ denote the set of signals such that upon observing, the Receiver who has beliefs given by P will chooses action a.

Let $S = \mathcal{P} \times \mathcal{A}$ (with generic signals denoted $s_{P,a}$, rather than (P, a)). For each $r \in \{r_n\}_{n\in\mathbb{N}}$, construct r'_n as follows:

For each $\mu \in P$ let $r'_n(\mu)$ be defined by $r'_n(\mu)(s_{P,a}|x) = r_n(\mu)(S(r, P, a)|x)$. That is, it sends the signals that recommends taking the action a whenever the original strategy would have induce the Receiver to take action a. It is clear that the Receivers actions remain the same (see, Proposition 1 of Kamenica and Gentzkow (2011) for an explicit proof), and so $U(r_n) = U(r'_n)$. Moreover, note that the signal $s_{P,a}$ is completely informative that $\mu^\star \in P$. I.e., the Receiver immediately learns which cell of the partition $\mathcal{P}$ contains the true distribution. Finally, since within any P, actions must be taken with the same probabilities given r, it is clear that $\boldsymbol{s}$-equivalence is inherited within cells of the partition.

The set $Y = \prod_D \prod_X \Delta(S)$ is compact (in the product topology induced by the topology of weak convergence over $\Delta(S)$ itself induced by the discrete topology over the finite $\mathcal{P} \times \mathcal{A}$.) Since $Y(\mathcal{P}) = \{r \in Y | \sigma^{(\mu,r)} \overset{\boldsymbol{s}}{\sim} \sigma^{(\mu',r)}, \forall \mu, \mu' \in P, \forall P \in \mathcal{P}\}$ is a closed subset of Y it is also compact. Hence, $\{r'_n\}_{n\in\mathbb{N}} \subset Y(\mathcal{P})$ has a convergent subsequence $\{r'_{n_k}\}_{k\in\mathbb{N}}$. Denote the limit by r'. By virtue of being in $Y(\mathcal{P})$, r' induces the beliefs $\mathcal{P}$. Finally, since $U : r \mapsto U(r)$ is upper semicontinuous, it follows that $U(r') \geqslant \limsup_{n\to\infty} U(r'_{n_k}) = \limsup_{n\to\infty} U(r_{n_k}) = \bar{U}$. Hence r' is $\mathcal{P}$-optimal.

Claim (ii). Let $\mathcal{P}$ be the highest payoff over all $\mathcal{P}$ optimal strategies but did not constitute an equilibrium: then there exists an r' that has a higher payoff. But r' must induce some beliefs, $\mathcal{Q}$. But then, $U(r^{\mathcal{P}}) \geqslant U(r^{\mathcal{Q}}) \geqslant U(r') > U(r^{\mathcal{P}})$, a contradiction. Likewise, if r is an equilibrium, but not the highest payoff over all $\mathcal{P}$ optimal strategies then there must exist a $\mathcal{P}$ strategies which provides a strictly better payoff, a contradiction to the definition of equilbrium. ■

Proof of Theorem 5.4. Let r denote an C-DPE such that the Sender's payoff is constant across D. Denote this payoff by $U^\star$. It cannot be that for all $\mu \in D$, $U^\star < \hat{v}(\mu)$. To see why, consider the set of signals $\{s_\mu | \mu \in D\}$ and the signal structure $r'(\mu)(s_\mu|x) = 1$ for all x and for all μ. But then, $U(r') = \sum_\mu \hat{v}(\mu)\theta(\mu) > U(r)$. Let $\mu' \in D$ be such that $\hat{v}(\mu') \leqslant U^\star$. Then, for any $\gamma \in \Delta(S)$, the belief that places probably 1 on μ' and the completely uninformative signal with marginal γ, substantiates (4.5). Hence r satisfies condition (1) of Theorem 4.3, and so is part of a PI-DPE. ■

Proof of Theorem 5.6. Let R denote the set of all rigid strategies. Let $R^{un} = \{r \in R | r(s|x) = r(s|y) \forall s \in S, \forall x, y \in X\}$ denote the subset of strategies that are completely uninformative signals. R^{un} is closed in R. Notice, for any $r \in R \backslash R^{un}$, there is some $s \in S$ and $E \subset X$ such that $r(s|E) > r(s|E^c)$. Consider any $\mu, \mu' \in D$, Assumption 1, part (i) we have that $\mu(E) \neq \mu'(E)$, w.o.l.g., $\mu(E) > \mu'(E)$. Hence, the probability of observing s under μ is strictly larger than under μ', so they are no $\boldsymbol{s}$-equivalent under r. Since μ and μ' were arbitrary, it must be that full learning takes place for any strategy in $R \backslash R^{un}$. ■

Proof of Theorem 5.7. Let R and R^{un} be defined as in the proof of Remark 5.6. Now, since utilities are bounded, let $\bar{U} = \sup_{r \in R} U(r) < \infty$. Let $\{r_n\}_{n \in \mathbb{N}} \subset R(\mathcal{P})$ be a sequence such that $\bar{U} - U(r_n) \leqslant \frac{1}{n}$, and $U(r_n)$ is strictly monotone increasing. This is without loss of generality since such a strictly increasing sequence does not exist only if $\bar{U}$ is obtained, in which case the Theorem follows.

By, Remark 5.6, the receivers beliefs are either a point mass on some $\mu \in D$, or given by θ. Enumerate the elements of $D : \mu_1 \ldots \mu_m$. For each r_n, let $S(a_1, \ldots a_m, a_{m+1}, r_n)$ denote the set of signals such that if the Receiver's belief is a point mass on μ_1 the Receiver takes action a_1, when it is μ_2 he takes action a_2, ..., and when no learning takes place (when beliefs are given by θ), he takes action a_{m+1}. Let $S = \prod_{m+1} \mathcal{A}$. Then for $n \in \mathbb{N}$ let $r'_n(\mu)$ be defined by $r'_n(\mu)(s_{a,b,\ldots,c}|x) = r'_n(\mu)(S(a, b \ldots c, r_n)|x)$. That is, it sends the signals that recommends taking the action a if the distribution is μ, whenever the original strategy would have induce the Receiver to take action a when he knows the distribution is μ. It is clear that the Receivers actions remain the same (see, Proposition 1 of Kamenica and Gentzkow (2011) for an explicit proof), and so $U(r_n) = U(r'_n)$.

Since $Y = \prod_D \prod_X \Delta(S)$ is compact,[12] $\{r'_n\}_{n\in\mathbb{N}} \subset Y$ has a convergent subsequence $\{r'_{n_k}\}_{k\in\mathbb{N}}$, with limit r'. If $r' \in R\backslash R^{un}$ then by the closure of R^{un}, the tail of $\{r'_{n_k}\}_{n_k\in\mathbb{N}}$ is also in $R\backslash R^{un}$. Then Remark 5.6 implies that the tail of the sequence, and the limit of the sequence, both induce full learning: therefore $U : R \to \mathbb{R}$ is upper semicontinuous on the domain and the theorem follows. Similar arguments hold if the tail of the sequence in contained in R^{un}.

So, towards a contradiction, let $\{r'_{n_k}\}_{k\in\mathbb{N}} \subset R\backslash R^{un}$, but $r' \in R^{un}$. By Assumption 1 part (ii), and the continuity of u_R, there exists a $\epsilon_1 \dots \epsilon_m$ such that $\mathcal{A}(\mu)$ is a singleton for all $\mu \in B_{\epsilon_i}(\mu_i)$, for $i = 1 \dots m$. Let $\epsilon = \min_{i\leqslant m} \epsilon_i$. Since $\{r'_{n_k}\}_{k\in\mathbb{N}}$ is converging to a completely uninformative signal, there must be some $\bar{k}$ such that for $k \geqslant \bar{k}$, the for all r_{n_k}, $\mu_i(\cdot|s) \in B_{\epsilon_i}(\mu_i)$, for all $i \leqslant m$ and all $s \in S$. But this implies the Receivers actions is constant for the tail of the sequence, a contradiction to $U(r_n)$ being strictly increasing. ■

Proof of Theorem 5.8. Assume that (2) does not hold; we will show that (1) must. Since the Sender can benefit from persuasion, the equilibrium strategy cannot be completely uninformative. Therefore, the Receiver's second order beliefs will be concentrated on the true distribution. Moreover, since the sender benefits from persuasion, it must be that s_a is informative of one state (WLOG, x_h) and s_r of the other. Since r is known to the Receiver in equilibrium, then the characterization of Bayesian updating implies, for every $\mu \in \Delta(X)$, $\mu(x_h|s_a) \geqslant \mu(x_h)$. Moreover, by the martingale property of posteriors, and the fact that $\mu(x_h) < q$ for all ex-ante possible μ, it cannot be that the Receiver takes action A after seeing s_r.

We will first show, in analogy to Proposition 4 of Kamenica and Gentzkow (2011), $r(s_a|x_h)$ must equal 1. By way of contradiction, assume $r(s_a|x_h) = \alpha < 1$. Let $r(s_a|x_l) = \beta \in [0, 1]$. Then, for any $\mu \in D$, we have

$$\mu^\alpha(x_h|s_a) = \frac{\alpha\mu(x_h)}{\alpha\mu(x_h) + \beta\mu(x_l)},$$

which is clearly increasing in α. Hence, for any belief of the Receiver, μ, if $\mu^\alpha(x_h|s_a) \geqslant q$ then $\mu^1(x_h|s_a) \geqslant q$. So increasing α increases the probability of action A conditional seeing s_a, given any μ. Further, increasing α increases the probability of seeing s_a for any μ. If $\mu(x_h) < q$ then by the martingale property of posteriors, it cannot be

[12] In the product topology induced by the topology of weak convergence over $\Delta(S)$ itself induced by the discrete topology over the finite $\{1 \dots m+1\} \times \mathcal{A}$.)

that the Receiver takes action A after seeing s_r. So increasing α strictly increases the Sender's payoff, conditional on the true distribution being μ. If, on the other hand, $\mu(x_h) > q$, then since (2) does not hold, changing α does not change the effect the Receiver's action after seeing s_r, and therefore, also (weakly) increased Sender's payoff, conditional on the true distribution being μ. Thus, an small increase in α increases the payoff to the Sender, conditional on the Receiver learning *any* of the ex-ante possible distributions, and therefore, increases her expected ex-ante payoff.

Next, we show, in analogy to Proposition 4 of Kamenica and Gentzkow (2011), there must be some $\hat{\mu} \in D$, such that $\hat{\mu}(x_h) < q$ and when the Receiver knows the fundamental distribution is $\hat{\mu}$, then conditional on seeing s_a and the is indifferent between actions. Suppose this was not the case: then for all $\mu \in D$, the Receiver strictly prefers one action to the other after seeing x_h (this is immediate from the assumption for all μ with $\mu(x_h) < q$, for all μ' with $\mu'(x_h) > q$ it follows from the fact that x_h is informative of state x_h). We know $r(s_a|x_h) = 1$, and let $r(s_a|x_l) = \beta \in [0, 1)$. Notice, $\beta \neq 1$ by the assumption that the signal is informative. So, by the continuity of preferences, the Sender could change $r(s_a|x_l) = \beta + \epsilon$ for small enough ϵ and not change the Receivers action conditional on s_a, given that the Receiver knows the distribution is μ for any $\mu \in D$. But this increase the probability of s_a, and hence, the probability of action A. The same logic as above implies this also does not change the action after s_r (using the assumption (2) does not hold). Thus a small increase in β increases the payoff to the Sender, conditional on the Receiver learning *any* of the ex-ante possible distributions, and therefore, increases her expected ex-ante payoff.

Since these two properties completely determine the equilibrium strategy in the one-realization game in threshold environments, $r \equiv e^{KG}(\hat{\mu})$, as desired. ■

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