Context Dependent Beliefs*

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June 30, 2016

Abstract

This paper examines a model where the set of available outcomes from which a decision maker must choose alters his perception of uncertainty. Specifically, this paper proposes a set of axioms such that each menu induces a subjective belief over an objective state-space. The decision maker's preferences are dependent on the realization of the state. The resulting representation is analogous to state-dependent expected utility within each menu; the beliefs are menu-dependent and the utility index is not. Under the interpretation that a menu acts as an informative signal regarding the true state, the paper examines the behavioral restrictions that coincide with different signal structures: elemental (where each element of a menu is a conditionally independent signal) and partitional (where the induced beliefs form a partition of the state space).

JEL Classification: D01, D810, D830.
Keywords: Menu-Dependence, Context-Dependence, Framing, Bayesian Signals.

*I thank Luca Rigotti, Juan Sebastián Lleras, Marciano Siniscalchi, Edi Karni, Teddy Seidenfeld, and especially Roee Teper for their helpful and insightful comments and suggestions.

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1 Introduction

Both intuition and psychological evidence insist that a decision maker's (DM's) preference over alternatives is affected by the environment in which the decision is made (Kahneman and Tversky, 1984; Simonson and Tversky, 1992; Sen, 1993). While there are many external factors that potentially exert influence on the decision making process, this paper examines a model in which the set of alternatives that is *currently* available acts as a frame –a process often differentiated from general framing effects under the moniker *context dependence*. I identify the behavioral conditions for context dependent beliefs, when the DM's subjective assessment of the likelihood of events depends on the available alternatives (the menu) from which he must choose, and consider additional restrictions that correspond to particular subjective information structures.

Context dependence is often associated with notions of bounded rationality or psychological heuristics Tversky and Simonson (1993). This paper, however, interprets menu-induced framing as rational, exploring how and when such behavior exists within the subjective expected utility paradigm. If the DM believes the menu itself contains information regarding payoff relevant uncertainty, conditioning his preference on such information is a rational action. Specifically, the model assumes the payoff associated with each alternative is ex-ante uncertain. The DM's utility from consumption depends not only on the chosen outcome, but also on which *state of the world* is realized. The DM, before consumption, is uncertain about the state of the world, but holds a belief (a probability distribution) over the state space; in a given decision problem, the DM maximizes his expected utility according to his belief. When the DM interprets the current selection of alternatives as a signal about the state of the world, his preferences will change across different decision problems in response to his updated beliefs.

Before expounding the finer points of the model, it is worth considering two examples to better illustrate why menu-dependent preferences are indeed necessary to explain many decision making scenarios.

Example 1.A (Luce and Raiffa's diner). *On a first date, Katya finds herself in a restaurant at which she has previously never eaten, and which offers chicken (c) or steak (s). She states her strict preference for chicken ($c \succ s$). However, upon seeing the restaurant also serves frog legs (f), she now states her strict preference for steak ($s \succ c \succ f$).*

While Katya's preference reversal in the face of a (seemingly) irrelevant alternative cannot be accommodated by the standard theory (as it violates the weak axiom of revealed preference (WARP)), it has a simple, intuitive explanation. She prefers steak when the food is well prepared, but considers chicken more resilient to the inept chef. In the typical restaurant, she believes it is unlikely the food will be well cooked, and hence, has a preference for chicken. However, in the presence of an exotic dish, she deems it is more likely the restaurant employs an expert chef and so, reverses her preference.

Example 2.A (Sen's date). *After dinner, Katya's date, Mitya, asks whether she would like to end the date and go home (h) or go next door and get a drink (d). Thinking the date a success, Katya strictly prefers getting a drink ($d \succ h$). However, before she can respond, Mitya offers a third option: the acquisition and consumption of crystal methamphetamine (m). Katya now strictly prefers going home ($h \succ d \succ m$).*

Here, again, Katya's rather intuitive behavior cannot be explained by standard theory. She understands the offer of methamphetamine as a signal regarding Mitya's character. So, while she would prefer to continue the date as long as it is likely Mitya is reputable, his proposition is sufficient to sway her beliefs away from such a outcome.

These vignettes exemplify two main behaviors of the model. First, it is only the DM's perception of uncertainty that is changing; ex-post tastes are fixed. In other words, if the DM knew with certainty which state of the world was to be realized, he would exhibit a constant preference across menus. Second, the uncertainty is *local*. The realization regarding the quality of the food in one restaurant is not informative about the quality in a different restaurant; that a previous date was incorruptible is not evidence that a future date will be.[1]

The first part of this paper axiomatizes a particular type of context-dependence which adheres to these two restrictions. As in Anscombe and Aumann (1963), I examine a DM who ranks *acts* (i.e., functions) from a state space, S, into lotteries over consumption, $\Delta(X)$.[2] Naturally, given the motivation, not all of X will always be available. The DM's entertains a family of preferences over acts, indexed by the subset of X that is currently available. Therefore, for each $A \subseteq X$, we see the decision maker's preference, $\succcurlyeq_A$ over $\{f : S \to \Delta(A)\}$. Then, a *menu-induced belief representation* (MBR) is a single utility index, $u : S \times X \to \mathbb{R}$, and a menu-indexed family of beliefs $\{\mu_A\}_{A\subseteq X} \subseteq \Delta(S)$ such that

$$U_A(f) = \mathbb{E}_{\mu_A}\left(\mathbb{E}_{f(s)}(u(s,x))\right) \tag{MBR}$$

represents $\succcurlyeq_A$, where $\mathbb{E}_\pi(\varphi)$ denotes the expectation of the random variable φ with respect to the distribution π. Fixing the menu, the DM acts as a subjective expected utility maximizer. The utility index, u, is the same across menus. This is the consequence of the main axiom, *menu consistency*. Menu consistency dictates, conditional on the realization of a particular state, the DM's preference for alternatives is fixed across menus. Therefore, the context effect is entirely characterized by the change in the DM's beliefs regarding the state space. This places clear limits on the type of context effects that can be accommodated by a MBR. Since any change in preference is the consequence of shifting beliefs, context dependence cannot reverse preference over outcomes for which the resolution of the state is payoff irrelevant (note, because the tastes are state-dependent, constant acts are not necessarily certain outcomes).

Although the general model imposes *a continuity condition*[3] –if two menus differ only slightly, then so do their associated beliefs– it does not otherwise specify any restriction relating menus with their associated beliefs. The second part of this paper, therefore, explores how menus might correspond to the beliefs they induce. In particular, what restrictions correspond to the DM who, acting in Bayesian manner, interprets each menu as a collection of signals regarding the relative likelihood of each state. And, what further restrictions allow us to identify the structure of these signals. Following the *anything goes* result of Shmaya and Yariv (2015), any MBR can be rationalized by some prior and set of signals. Thus, without imposing any additional structure, it is impossible to rule out the possibility that the DM is acting in a Bayesian fashion.

I consider two more restrictive signal structures and their corresponding behavioral restrictions. In the first, a *partitional signal structure*, the DM entertains a partition of the state space and a prior belief over the likelihood of each state. Each menu induces a belief whose support coincides with some cell of the partition,

[1] Of course, one could tell a different story where there is a dynamic component by which the DM learns about the likelihood of states from experience. This is well outside of the current model.

[2] For a set Y, $\Delta(Y)$ is the set of distributions thereover.

[3] This is a vacuous assumption when X is a discrete space.

and any two menus which induce beliefs with the same support carry the same informational content. In other words, the DM believes each menu can only occur in a particular subset of the state space, but, given a state and the menus possible in that state, the realized menu is chosen uniformly. This signal structure is of particular interest, as it could be seen as arising from endogenously from a separating equilibrium in a game between buyers and sellers (see Section 4).

In the second signal structure I consider, an *elemental signal structure*, the DM takes the elements of the menus as signals rather than the menus themselves. Specifically, he assumes that in each state, s, the inclusion or exclusion of an element x is decided according to the toss of a (potentially biased) (s, x)-coin. Therefore, the collection of included elements (the menu) is the result of a series of conditionally-independent coin tosses.

Revisiting the examples, we can see that MBR preferences and these various information structures can rationalize the choice patterns.

Example 1.B (Luce and Raiffa's diner, revisited). *Katya's has MBR preferences and the following utility index:*

$$u(h,c) = 8 \quad u(h,s) = 16 \quad u(h,f) = 8$$
$$u(m,c) = 8 \quad u(m,s) = 6 \quad u(m,f) = 4$$

where $S = \{h, m\}$ indicates high and medium quality food, respectively. She initially believes it equally likely the food is high quality as it is mediocre: $\mu(h) = \frac{1}{2}$ and $\mu(m) = \frac{1}{2}$. She also believes only (and all) high quality restaurants offer frog legs.

So the menu $\{c, s\}$ indicates with certainty the food is mediocre, so $U_{\{c,s\}}(c) = 8 > 6 = U_{\{c,s\}}(s)$, while the menu $\{c, s, f\}$ indicates with certainty the food is good, so $U_{\{c,s,f\}}(s) = 16 > 8 = U_{\{c,s,f\}}(c)$.

So a MBR and partitional signal structure can rationalize Katya's choices over entrees. Likewise, we can see that an elemental information structure could explain her preferences regarding Mitya.

Example 2.B (Sen's date, revisited). *Katya's has MBR preferences and the following utility index:*

$$u(r,h) = 1 \quad u(r,d) = 5 \quad u(r,m) = -10$$
$$u(d,h) = 1 \quad u(d,d) = -5 \quad u(d,m) = -10$$

where $S = \{r, d\}$ indicates reputable and depraved characters, respectively. She initially believes $\mu(r) = \frac{9}{10}$ and $\mu(b) = \frac{1}{10}$. She also believes that, while all dates will offer going home and getting a drink, depraved characters offer meth with probability $\frac{1}{10}$, with reputable characters with only probability $\frac{1}{100}$.

After updating upon seeing the menu $\{h, d\}$, she holds the beliefs $\mu(r) = \frac{891}{981}$ and $\mu(d) = \frac{90}{981}$; her preference is given by $U_{\{h,d\}}(d) = \frac{5(801)}{981} > 1 = U_{\{h,d\}}(h)$. After the menu $\{h, d, m\}$, she holds the beliefs $\mu(r) = \frac{9}{19}$ and $\mu(d) = \frac{10}{19}$; her preference is given by $U_{\{h,d,m\}}(h) = 1 > \frac{-5}{19} = U_{\{h,d,m\}}(d)$.

1.1 Organization

This paper is organized as follows. The general model is presented in Section 2, with the representation theorem for the main result contained in Section 2.3. Section 2.4 discusses the shortcomings of a variant model with state-independent utilities. Section 3 explores the additional restrictions necessary to capture particular signal structures. Section 4 informally explores how menu-dependent beliefs could arise naturally in a strategic environment. Finally, a survey of relevant literature is found in Section 5. All proofs are contained in the appendix.

2 General Model

2.1 Structure and Primitives

Let X be a compact and metrizable topological space, representing the grand set of consumption alternatives, and with typical elements x, y, z. Define $x^\star$ and $x_\star$ to be two distinguished elements of X, referred to as universal alternatives, and set $\bigstar = \{x^\star, x_\star\}$. Let $\mathcal{P}(X)$ denote the set of compact subsets of X; endow $\mathcal{P}(X)$ with the hausdorff metric (thus, a compact metric space). Let $\mathcal{K}(X)$ denote the subset of $\mathcal{P}(X)$ whose elements contain $\bigstar$.[4] Typical elements are A, B, C. Elements of $\mathcal{K}(X)$ are called menus, with the interpretation that they represent the set of *currently available* consumption alternatives.

For any topological space Y, let $\Delta(Y)$ denote the set of all probability measures on $(Y, \mathcal{B}(Y))$, where $\mathcal{B}(Y)$ denotes the Borel σ-algebra on Y, endowed with the topology of weak convergence. If $\mu \in \Delta(Y)$, and φ is a continuous and bounded function $\varphi : Y \to \mathbb{R}$, then

$$\mathbb{E}_\mu(\varphi(y)) = \begin{cases} \displaystyle\int_Y \varphi(y)\ d\mu(y) & \text{whenever } Y \text{ is infinite, and,} \\ \displaystyle\sum_Y \varphi(y)\mu(y) & \text{whenever } Y \text{ is finite,} \end{cases}$$

denote the expectation of φ with respect to μ.

Notice, for all $A \in \mathcal{K}(X)$, A is compact, (hence separable), so $\Delta(A)$ is metrized by the Lèvy–Prokhorov metric. In the standard abuse of notation, identify $x \in X$ with the degenerate distribution on x. Typical elements of $\Delta(X)$ are denoted π, ρ, τ.

Let S denote a finite state space. Endow $\Delta(X)^S$ with the product topology. The objects of choice will be menu-induced acts: for each $A \in \mathcal{K}(X)$ define $\mathcal{F}_A = \Delta(A)^S \cong \{f : S \to \Delta(X) | f(s)[A] = 1\}$. An act is a commitment to a particular consumption conditional on the realization of the type space, and so, $\mathcal{F}_A$ corresponds to the acts available given the menu A (which put probability 1 on an outcome that is available from A). For each act, $f(s)$ is the distribution over X obtained for realization s. Again, abusing notation, identify each $\pi \in \Delta(X)$ with the degenerate act such that $\pi(s) = \pi$ for all s.

For any $f, g \in \mathcal{F}_X$, and event $E \subseteq S$, let $f_{-E}g$ be the act that coincides with f everywhere except on E, where it coincides with g. Further, for some $\alpha \in (0,1)$ let $\alpha f + (1-\alpha)g$ be the point-wise mixture of f and g (i.e., $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s)$ for each $s \in S$). It is immediate that if $f \in \mathcal{F}_A$ and $g \in \mathcal{F}_B$ then $f_{-E}g$ and $\alpha f + (1-\alpha)g$ both belong to $\mathcal{F}_{A \cup B}$ (in particular, note the case when $A = B$).

The primitive of the model is the family of preference relations $\{\succcurlyeq_A \subset \mathcal{F}_A \times \mathcal{F}_A\}_{A \in \mathcal{K}(X)}$. That is, the DM's preference over the acts which are available given each possible menu. With regards to notation, it is assumed whenever '$f \succcurlyeq_A g$' is written both f and g belong to $\mathcal{F}_A$. For any relation $\succcurlyeq$, let $\succ$ and $\sim$ denote the asymmetric and symmetric components, respectively.

[4] I will interpret $\bigstar$ as a set of outside options, which explains their universal availability.

2.2 Axioms

The goal of the most general representation is to provide the basic framework in which a DM might condition his beliefs regarding the state space on the menu at hand. That is, the DM treats the set of currently available consumption alternatives as a signal regarding the likelihood of different states. Given the menu, the DM acts as a subjective expected utility maximizer, with respect to his menu-induced beliefs. Clearly, each menu-dependent preference should satisfy the expected utility axioms:

[A1: Expected Utility (EU)] *For each $A \in \mathcal{K}(X)$, $\succcurlyeq_A$ satisfies the expected utility axioms, namely,*

1. ***Weak Order***. *$\succcurlyeq_A$ is a non-trivial weak order.*
2. ***Independence***. *For all $f, g, h \in \mathcal{F}_A$ and $\alpha \in (0,1)$, $f \succcurlyeq_A g \iff \alpha f + (1-\alpha)h \succcurlyeq_A \alpha g + (1-\alpha)h$.*
3. ***Continuity***. *For all $f \in F_A$, the sets $\{g \in \mathcal{F}_A | g \succcurlyeq_A f\}$ and $\{g \in \mathcal{F}_A | f \succcurlyeq_A g\}$ are closed in $\mathcal{F}_A$.*

The following well known result (so well known in fact, that it is included only to fix notation for expositional purposes) shows EU provides the expected utility structure for each menu-dependent preference.

Proposition 2.1 (Expected Utility)**.** *$\{\succcurlyeq_A \subset \mathcal{F}_A \times \mathcal{F}_A\}_{A \in \mathcal{K}(X)}$ satisfies EU if and only if for each $A \in \mathcal{K}(A)$ there exists some continuous and bounded $w : S \times X \to \mathbb{R}$ such that*

$$U_A^{VNM}(f) = \sum_s \left(\mathbb{E}_{f(s)}(w_A(s,x)) \right),$$

represents $\succcurlyeq_A$. Moreover, if $w_A(s,x)$ and $\hat{w}_A(s,x)$ both represent $\succcurlyeq_A$, then $w_A(s,x) = a_A \hat{w}_A(s,x) + b_A(s)$ where $a_A \in \mathbb{R}_{++}$ and $b_A(s) \in \mathbb{R}$ for all $s \in S$.

Recall that $f(s)$ and $g(s)$ are given, objective probability measures. The index $w(\cdot)$ can be decomposed into tastes (the utility of consuming an object given the state) and beliefs (the subjective likelihood of each state). Indeed, choose some probability distribution $\mu \in \Delta(S)$ such that $\mu(s) > 0$ and let $u_A(s,x) = \frac{w_A(s,x)}{\mu(s)}$; it is clear that

$$\mathbb{E}_\mu \left(\mathbb{E}_{f(s)}(u_A(s,x)) \right)$$

represents $\succcurlyeq_A$. Of course, this creates the classic problem of multiple rationalizing beliefs: if we consider some other $\nu \in \Delta(S)$ such that $\nu(s) > 0$, then ν and $u'_A(s,x) = \frac{w_A(s,x)}{\nu(s)}$ also represent the same preference. We cannot identify the DM's tastes for ex-post outcomes or his beliefs; the two are jointly determined.

The motivation for expanding our data to include the family of menu-induced preferences is to understand how the menu can alter the beliefs of the DM. In light of this, it becomes obvious further structure is needed to separate the effect on the perception of uncertainty (i.e., menu induced changes in belief) from other internal changes in preference (i.e., a change in tastes).

The first novel axiom, *menu-consistency*, is the first step towards such a disentanglement, and, captures the main behavior behind menu-induced beliefs. It states that the DM's tastes for outcomes do not depend on the menu at hand. This implies, any difference in preferences across menus must be the result of a change in perception of the underlying uncertainty.

Of course, the DM only cares about the assignment to state s if he believes there is a possibility s will be realized. Therefore, menu-consistency should only hold after realizations assigned positive probability according to the DM's subjective assessment. To make such ideas precise, I first need to consider null events.

Definition. *An event, $E \subset S$, is* ***null for menu A*** *(hereafter, null-A) if for all $f, g \in \mathcal{F}_A$,*

$$f_{-E}g \sim_A f.$$

Let N_A denote the set of states that are null-A, and N denote the set of everywhere null states: $N = \bigcap_{A \in \mathcal{K}(A)} N_A$.[5]

Null events, in general, have two indistinguishable interpretations. First, that the DM is indifferent between all available options conditional on the realization of the null event, E; second, that the DM places zero probability on E occurring. However, assuming the DM's tastes are consistent across different menus (the assumption that will be formalized shortly), it is possible to differentiate these two interpretations of null events. If a state, s, is null-A, but there exists a different menu, B, for which the DM displays a strict preference over elements of A (given realization of s), it must be that s was assigned zero probability when facing A. This is because the DM cannot be indifferent to all elements of A (contingent on s) since he displays strict preference in the menu B. This is formalized by evidently-null events, first considered in Karni *et al.* (1983).

Definition. *An event, $E \subset S$, is* ***evidently null for menu A*** *(hereafter, e-null-A) if E is null-A and for all $s \in E$ there exists some menu B such that*

$$(f_{-s}g) \succ_B f$$

for some f, g in $\mathcal{F}_{A \cap B}$. Let E_A denote the union of all e-null-A events.[6]

With this definition in mind we can now define menu consistency.

[A2: Menu Consistency (MC)] *For all $A, B \in \mathcal{K}(X)$ and $s \in S$ with $s \notin E_A \cup E_B$, and all $f \in \mathcal{F}_A$, $g \in \mathcal{F}_B$, $h \in \mathcal{F}_{A \cap B}$, and such that $f(s) = g(s)$,*

$$f_{-s}h \succcurlyeq_A f \iff g_{-s}h \succcurlyeq_B g.$$

If $\{\succcurlyeq_A\}_{A \in \mathcal{K}(X)}$ is menu-consistent, the DM's tastes for outcomes are identical across menus. To see this, let $\pi = f(s) = g(s)$ and $\rho = h(s)$. Then MC states that the DM's preference between ρ and π, in state s, does not depend on the context in which the decision is made (i.e., does not depend on the menu from which the acts were constructed). Behaviorally, this indicates that any context effect does not alter the DM's preferences *conditional* on the realization of a particular state. In other words, if the DM knew the true state, there would be no context effect. It is this restriction that differentiates this model from a more general interpretation of context effects as psychological biases without foundation in rational behavior. The change in behavior across menus is *not* the result of a change in the state-dependent preference for outcomes (objects

[5] Notice, the set of null-A events form a lattice with respect to set inclusion, with N_A the maximal element.

[6] Notice, the set of e-null-A events form a sub-lattice of the lattice of null-A events, with E_A the corresponding maximal element.

about which the DM is ostensibly certain) but of a change in his perception of the between-state-tradeoffs (the domain of uncertainty).

By the very nature of the problem at hand, we are losing structure in comparison to the standard model and so the axioms are weaker in comparison. As such, MC does not characterize a *new* behavior that is the result of context dependent beliefs, but rather places limits on how much structure is lost. What structure is retained by MC guarantees we can find a family of representation for $\{\succcurlyeq_A\}_{A\in\mathcal{K}(X)}$ that shares a common utility index. In other words, the primitive is represented by a single utility index and a family of menu-induced beliefs. It is important to note that this does not rule out preference reversals, even over constant acts. Each menu carries with it a perception of uncertainty, and can therefore change the DM's preferences for acts. However, given menu-consistency, any preference reversal is due entirely to the change in beliefs, and not because of changes in ex-post tastes. Setting $f = g$ in the definition, consistency guarantees that the ordering between $f_{-s}\rho$ and $f_{-s}\pi$ hold regardless of the ambient menu.

Under the definition of a *frame* as (seemingly irrelevant) information which alters the DM's perception of uncertainty, then EU and MC exactly capture the behavior where the DM uses the menu as a frame. Unfortunately, from a practical vantage, this is insufficient, as the problem of non-uniqueness of beliefs persists. When tastes and beliefs cannot be separated, we cannot identify the avenue by which context effects alter the DM's choice process.

To overcome the issue of multiple rationalizing beliefs, Anscombe and Aumann (1963) restricted preferences to be state independent (i.e., in every non-null state, the ranking over distributions is the same). State dependency is a very restrictive assumption; it interprets states as abstract probabilistic events that have no meaning outside of their use as betting devices. Beyond this philosophical issue, state-dependence is a necessary requirement to capture the full gamut of context effects (this necessity is made precise in Remark 3.1). For these reasons, this model weakens state-independence to apply only over $\star$. UV plays the same roles as state independence (equivalently, monotonicity). By ensuring, over the relatively small domain $\star$, that preferences in each state coincide, beliefs can be uniquely recovered from choice data. Under the interpretation of universal elements as outside options, it is natural that the ranking of these elements does not change across different menus.

[A3: Universality (UV)] *For all $A \in \mathcal{K}(X)$ and $s \in S$ with $s \notin N_A$*

$$f_{-s}x^\star \succ_A f_{-s}x_\star.$$

for all $f \in \mathcal{F}_A$.

It is also of interest (when X is infinite) to understand when the context effect acts in a continuous manner.

[A4: Continuity of Context (CC)] *If $\{A_n\}_{n\in\mathbb{N}}$ converges to A in $\mathcal{K}(X)$, then for all $\alpha \in (0,1)$,*

$$\big\{\{g \in \mathcal{F}_\star \mid g \succcurlyeq_{A_n} \alpha x^\star + (1-\alpha)x_\star\}\big\}_{n\in N}, \text{ and}$$
$$\big\{\{g \in \mathcal{F}_\star \mid \alpha x^\star + (1-\alpha)x_\star \succcurlyeq_{A_n} g\}\big\}_{n\in N}$$

converge to $\{g \in \mathcal{F}_\star | g \succcurlyeq_A \alpha x^\star + (1-\alpha)x_\star\}$ and $\{g \in \mathcal{F}_\star | \alpha x^\star + (1-\alpha)x_\star \succcurlyeq_A g\}$, respectively, in $\mathcal{P}(X)$.

In other words, the menu-induced contour sets, when restricted to universal acts, converge whenever the menus converge. Because CC applies only to universal acts, it restricts only that beliefs converge (and has nothing to say about the change in tastes across menus). Convergence, of both the menu and the contour sets, is with respect to Hausdorff metric in the respective ambient spaces. So, CC states that as menus become close, the relative weights placed on each state must also converge. Notice, while the sufficiency of CC is clear, the necessity relies on the fact that the utility derived from the set of universal acts is bounded.

2.3 Menu Induced Belief Representation

Theorem 2.2 (Menu Induced Belief Representation)**.** *(a)* $\{\succcurlyeq_A\}_{A\in\mathcal{K}(X)}$ *satisfies EU, MC, UV, and CC if and only if there exists a state-dependent utility index,* $u : S \times X \to \mathbb{R}$*, such that* $u(\cdot, x^\star) \equiv 1$ *and* $u(\cdot, x_\star) \equiv 0$*, and such that the projections* $u|_A$ *are bounded and continuous for all* $A \in \mathcal{K}(X)$*, and a continuous function,*[7] $\{\mu_A \in \Delta(S)\}_{A\in\mathcal{K}(X)}$*, such that for all* $A \in \mathcal{K}(X)$*,*

$$U_A(f) = \mathbb{E}_{\mu_A}\Big(\mathbb{E}_{f(s)}(u(s,x))\Big) \tag{MBR}$$

represents $\succcurlyeq_A$*, and* $\mu_A(s) = 0$ *if and only if* $s \in E_A \cup N$*.*
(b) Moreover, the family of beliefs $\{\mu_A \in \Delta(S)\}_{A\in\mathcal{K}(X)}$ *is unique and the utility index,* $u(\cdot)$*, is unique up to null states.*

Proof. In appendix B. ∎

The proof is quite straightforward. First, EU provides a linear representation for each $\succcurlyeq_A$. By UV these representations can be decomposed uniquely into a tastes (over A) and beliefs, where the utility index is normalized as in the statement of the theorem. Then, these utility indexes can be stitched together to provide a single u over the whole of X. Finally, MC ensures that this common index will jointly represent each $\succcurlyeq_A$ and CC that beliefs will change continuously.

Because the utility index is fixed across decision problems, the shifting of probabilities is the only avenue for preferences to change. Thus, if an act f is preferred to g on a state-by-state basis, then it is preferred to g in every menu (this is, of course, precisely the content of MC). It is through the menu-dependent beliefs that this structure allows for framing effects, were by the DM's preferences change in the face of new alternatives. It follows that the types of preference reversals that are allowable is limited.

2.4 State-Independence

In light of axiom UV, it may seem parsimonious to quit worrying about the distinguished elements, $x^\star$ and $x_\star$, and require state independence outright. This can, in fact, be accomplished by strengthening MC.

[A2*: Strong Menu Consistency (SMC)] *For all* $A, B \in \mathcal{K}(X)$ *and* $s \in S$ *with* $s \notin E_A$ *and* $s'E_B$*, and all* $f \in \mathcal{F}_A$*,* $g \in \mathcal{F}_B$*,* $h \in \mathcal{F}_{A\cap B}$*, and such that* $f(s) = g(s')$*,*

$$f_{-s}h \succcurlyeq_A f \iff g_{-s'}h \succcurlyeq_B g.$$

[7] i.e., $\mu_{(\cdot)} : \mathcal{K}(X) \to \Delta(S)$ is continuous with respect to $\Delta(S)$ when endowed with the topology of weak convergence.

SMC states that tastes are consistent, not only across menus (if $A \neq B$) but also across states (if $s \neq s'$). As such, it implies the canonical form of state independence for each $\succcurlyeq_A$. When MC is replaced by SMC in Theorem 2.2, the resulting representation coincides except the utility index, $u : X \to \mathbb{R}$ is *state-independent*:

$$U_A^{SI}(f) = \mathbb{E}_{\mu_A}\left(\mathbb{E}_{f(s)}(u(x))\right), \tag{SI-MBR}$$

represents $\succcurlyeq_A$.[8] The existence of the family of beliefs, their uniqueness, and the uniqueness of the utility index are all the same as in Theorem 2.2. While this approach is only a small deviation from the general representation, it implies that there is no uncertainty regarding the preference of constant acts. As discussed before, in order for context effects to have observable content, it must be that the underlying uncertainty is payoff relevant. Together, these facts imply that SMC prohibits the DM from changing his preference over constant acts between different menus.

Remark 2.3. *Let $\{\succcurlyeq_A\}_{A \in \mathcal{K}(X)}$ be represented by (SI-MBR). Then for all $A, B \in \mathcal{K}(X)$, and $\pi, \rho \in \Delta(A \cap B)$, $\pi \succcurlyeq_A \rho \iff \pi \succcurlyeq_B \rho$.*

Remark 2.3 can be seen by noting that $U_A^{SI}(\pi) = \mathbb{E}_\pi(u(x))$, which does not depend on A.

3 Bayesian Frames

The general representation, (MBR), allows for context effects by which the DM's preferences over acts depends on the menu of currently available outcomes. It does not, however, offer any insight into the connection between the menu at hand and the effect it exerts on decision making. This section provides an exploration into the behavioral implications of particular context effects.

It is of interest to identify the restrictions on behavior that ensure the DM is acting rationally with respect to some *information structure* that gives rise to the family of menu-induced beliefs. Consider the interpretation that the DM entertains a prior belief over the state space, $\mu \in \Delta(S)$, and observes, along with the menu, some signals, drawn from a (finite) signal space, Θ. The DM also entertains a likelihood function that specifies the likelihood of a given signal, contingent on the true state, $l : \Theta \times S \to \mathbb{R}_+$. Under this interpretation, we say the information structure (μ, l, Θ) *generates* $\{\mu_A | A \in \mathcal{K}(X)\}$, if the DM's menu-induced beliefs are the posteriors generated by observing the signals. To keep things notationally clean, through this subsection, I assume that X is finite and $N = \varnothing$. I always assume the prior, μ, has full support. These assumptions ensure the updating procedures are binding everywhere, as it alleviates the concern regarding 0 probability events.

Of course, for the posteriors to be indexed by menus there must be a connection between the signals and the menu. At the most general level, the two coincide: $\Theta = \mathcal{K}(X)$.

Definition. *An information structure based on menus, $(\mu, l, \mathcal{K}(X))$, generates $\{\mu_A | A \in \mathcal{K}(X)\}$, if*

$$\mu_A(s) = \frac{\mu(s)l(A, s)}{\mathbb{E}_\mu(l(A, s'))} \tag{3.1}$$

[8]Notice, UV is somewhat redundant in the presence of SMC. In fact, if we are willing to entertain a bit of notational juggling, we can forego it entirely.

and $\sum_{\mathcal{K}(X)} l(A,s) = 1$*,* $\sum_s l(A,s) > 0$ *for all* $A \in \mathcal{K}(X)$ *and* $s \in S$*.*[9]

Unfortunately, the requirement that the DM entertains some generating $(\mu, l, \mathcal{K}(X))$, provides no testable implications. In other words, *every* MBR can be described by some prior and likelihood function over menu realizations. The ability to choose both the signals and the prior provides enough degrees of freedom such that bayesianism can never be ruled out.

Proposition 3.1. *Let* $\{\succcurlyeq_A\}_{A \in \mathcal{K}(X)}$ *be represented by some* (MBR)*, with beliefs* $\{\mu_A | A \in \mathcal{K}(X)\}$*. Then there exists some* $(\mu, l, \mathcal{K}(X))$ *that generates* $\{\mu_A | A \in \mathcal{K}(X)\}$ *as in* (3.1)*.*

Proof. In appendix B. ■

This result is a corollary of Lemma 1 in Shmaya and Yariv (2015). Setting $\Theta = \mathcal{K}(X)$ assumes no relation between the signals associated with different menus, and it is this generality that renders behavior wholly unconstrained. However, under more specific assumptions regarding the structure of the signals, there are falsifiable restrictions on observable preference. Thus, while we can never rule out the possibility that the DM is acting in a Bayesian manner with respect to *some* signal space, we can rule out particular models of information.

First, consider a partitional information structure. Under this model of information, each menu obtains only on a cell of a partition of the state space, and any menu that obtains on the event E contains the same informational content. That is, we set $\Theta = \mathcal{K}(A)/\dot{\sim}$ for some equivalence relation $\dot{\sim}$ (with the equivalence class containing A, denoted by $[A]$).

Definition. *An information structure based on a partition,* $(\mu, l, \mathcal{K}(X)/\dot{\sim})$*, generates* $\{\mu_A | A \in \mathcal{K}(X)\}$ *if*

$$\mu_A(s) = \frac{\mu(s) l([A], s)}{\mathbb{E}_\mu(l([A], s'))} \tag{3.2}$$

and $l([A], s) \in \{0, 1\}$ *for all* $([A], s) \in \mathcal{K}(X)/\dot{\sim} \times S$*, and* $\sum_{s \in S} (l([A], s) \cdot l([B], s)) > 0$ *implies* $[A] = [B]$*.*

The requirement that the likelihoods are binary is indicative of the fact that within equivalence classes all menus have the same informational content. Hence, any two menus that have the same support, must induce the same beliefs (and therefore, we can normalize the likelihood functions to 1). The requirement that $l([A], s) \cdot l([B], s) = 0$ for any $[A] \neq [B]$ ensures that the supports of the distributions are disjoint, and hence, form a partition of S.

[A5: Partitional Signals (PS)] *For all* $A, B \in \mathcal{K}(X)$*, such that* $N_A^c \cap N_B^c \neq \varnothing$*, and* $f, g \in \mathcal{F}_{A \cap B}$*, we have*

$$f \succcurlyeq_A g \iff f \succcurlyeq_B g.$$

PS dictates that any two menus sharing a non-null state must induce the same preference over acts. Because the general representation fixes tastes across different menus, PS implies that if the two menus

[9] The first requirement is equivalent to $\sum_{\mathcal{K}(X)} l(A,s) > 0$, and is included in the current form only for its interpretational content. Under this normalization, we can think of $l(A,s)$ as the probability of seeing menu A in state s. The second requirement states that all menus are ex-ante possible. This ensures that 3.1 is always well defined.

induce beliefs with a overlapping supports, those beliefs must coincide completely. It is clear that this captures the behavior generated by a partitional signal structure.

Theorem 3.2. *Let $\{\succcurlyeq_A\}_{A\in\mathcal{K}(X)}$ be represented by some (MBR), with beliefs $\{\mu_A | A \in \mathcal{K}(X)\}$. Then, there exists some $(\mu, l, \mathcal{K}(X)/\dot{\sim})$ that generates $\{\mu_A | A \in \mathcal{K}(X)\}$ as in (3.2) if and only if $\{\succcurlyeq_A\}_{A\in\mathcal{K}(X)}$ satisfies PS.*

Proof. In appendix B. ∎

An even more restrictive signal structure would be $\Theta = X$ with signals that are conditionally independent. Under this interpretation, the element x is made available (in state s) according to the flip of an (s, x)-coin. Conditional on the state, the coins are independent of one another, although their bias can vary with both the element and the state.

Definition. *An information structure based on elements, (μ, l, X), with conditionally independent signals generates $\{\mu_A | A \in \mathcal{K}(X)\}$ if*

$$\mu_A(s) = \frac{\mu(s)\prod_{x\in A} l(x,s)\prod_{y\notin A}(1 - l(y,s))}{\mathbb{E}_\mu\left(\prod_{x\in A} l(x,s')\prod_{y\notin A}(1 - l(y,s'))\right)}, \tag{3.3}$$

and $l(x, s) \in (0, 1)$ for all $(x, s) \in X\backslash \bigstar \times S$, and $l(x^\star, s) = l(x_\star, s) = 1$ for all $s \in S$.

The requirement that likelihoods lie in the interior of $(0, 1)$ is tantamount to assuming there are no null states, and ensures that (3.3) is well defined for all menus and states. To include null states in such a set up adds little intuition and greatly increases the level of attention to technical detail that needs to be paid. The requirement regarding $x^\star$ and $x_\star$, stems from the fact that they are necessarily realized in every state, and hence, uninformative.

A menu, A, acts as the frame induced by the relative probabilities of inclusion on each x-coin with $x \in A$ and exclusion for each y-coin with $y \notin A$. The fact that signals are independent, indicates that the inclusion or exclusion of a particular element carries the same informational content regardless of the composition of the menu. Of course, even though the informational value is the same, the *effect* of this information on beliefs is relative to the information provided by the other elements included (or excluded) from the menu. This behavior is captured by the following axiom.

[A6: Independent Signals (IS)] *For all $x \in X$, and $A, B \in \mathcal{K}(X)$, such that $x \notin A \cup B$, and states $s, s' \notin N_A \cup N_B$, if for some distributions $\pi^A, \rho^A, \in \Delta(A)$ and $\pi^B, \rho^B, \in \Delta(B)$: $(x_\star)_{-s}\pi^A \sim_A (x_\star)_{-s'}\rho^A$ and $(x_\star)_{-s}\pi^B \sim_B (x_\star)_{-s'}\rho^B$, then for all $\alpha \in (0, 1)$,*

$$(x_\star)_{-s}\pi^A \succcurlyeq_{A\cup x} (x_\star)_{-s'}(\alpha\rho^A + (1-\alpha)x_\star) \iff (x_\star)_{-s}\pi^B \succcurlyeq_{B\cup x} (x_\star)_{-s'}(\alpha\rho^B + (1-\alpha)x_\star). \tag{3.4}$$

IS states that the proportional change in belief, in response to the inclusion of an element x, is the same across all menus. Without x, obtaining π^A in state s and ρ^A in state s' (and $x_\star$ everywhere else) are equally appealing, given menu A. When x is included, the beliefs change (i.e., the observation of the x-coin switches from exclude to include), and therefore, so do preferences. IS states that the same proportional change in preferences must occur, regardless of the initial menu. So if the change in preferences is such that, π^A in state s is now indifferent to $\alpha\rho^A + (1-\alpha)x_\star$ in state s' (and $x_\star$ everywhere else) given A, then the same

α proportional shift preserves indifference when moving from B to $B \cup x$. This behavior, along with the general representation, exactly captures the updating procedure given by (3.3).

Theorem 3.3. *Let $\{\succcurlyeq_A\}_{A \in \mathcal{K}(X)}$ be represented by some (MBR), with beliefs $\{\mu_A | A \in \mathcal{K}(X)\}$, all of which have full support. Then, there exists some (μ, l, X) that generates $\{\mu_A | A \in \mathcal{K}(X)\}$ as in (3.3) if and only if $\{\succcurlyeq_A\}_{A \in \mathcal{K}(X)}$ satisfies IS.*

Proof. In appendix B. ∎

4 From Equilibrium to MBR

This section briefly (and informally) describes how a MBR could arise as the natural consequence of a game between buyers and sellers (i.e., the observed behavior of the buyer in equilibrium satisfies the MBR axioms). Consider the environment where, first, a seller constructs a menu of goods to offer the buyer at posted prices, and then, the buyer decides whether or not to buy any of the offered goods. In other words, the sellers act as *stores*, who can curate their selections. Sellers are privately endowed with a type (read: the seller's quality or ability), and this type governs both the cost of *stocking* a particular good, and also, the utility a buyer derives from its consumption. In this environment, under standard single-crossing conditions, different types of sellers might differentiate themselves in equilibrium by offering different menus of goods. Hence, in such an equilibrium, the sellers beliefs regarding the type of seller, and hence the value of the offered goods, is dependent on the offered menu. Specifically, if the equilibrium is in pure strategies, this induces a MBR with partitional signal structure.

Example 1.C (Luce and Raiffa's diner, one last time). *There are three types of restaurants, high (h), medium (m), and low (l) quality. Each can offer any selection of chicken (c), steak (s), or frog legs (f). The cost for a particular restaurant to keep an item on the menu (train the chef, provide a wine pairing, keep fresh ingredients, etc), is given by the following matrix:*

$$\begin{array}{ccc} c_h(c) = 1 & c_h(s) = 2 & c_h(c) = 6 \\ c_m(c) = 1 & c_m(s) = 4 & c_m(c) = 10 \\ c_l(c) = 2 & c_l(s) = 6 & c_l(c) = 10 \end{array}$$

A patron, given that the quality of the food is known, has preferences (in dollar terms) according to

$$\begin{array}{ccc} u(h,c) = 8 & u(h,s) = 16 & u(h,f) = 8 \\ u(m,c) = 8 & u(m,s) = 6 & u(m,f) = 4 \\ u(l,c) = 4 & u(l,s) = 1 & u(l,f) = 0 \end{array}$$

Each type of restaurant can select any subset of the main courses (along with posted prices) to offer potential diners. Given the observed menu and the subsequently updated beliefs, a diner will select the course that maximizes her utility (her expected utility from consumption less the posted price). All diners can take an outside option with utility 0.

Assume, initially, the diner has a uniform prior over the different types of restaurants. Then the following is a Bayes-Nash equilibrium. The high type offers $\{\langle c{:}8\rangle, \langle s{:}16\rangle, \langle f{:}8\rangle\}$*, the medium type offers* $\{\langle c{:}8\rangle, \langle s{:}6\rangle\}$ *and the low type offers* $\{\langle c{:}4\rangle\}$*. As this is a separating equilibrium, after observing any of these menus, the diner places probability 1 on the corresponding type, and chooses* s*,* c*,* c*, respectively. When seeing any other menu, she places probability 1 on low, and takes the outside option.*

Notice that in this example, both the utilities for outcomes and the beliefs after the observation $\{c, s, f\}$ and $\{c, s\}$ map exactly to Katya's tastes and beliefs given the same observations. As such, the behavior of buyers in such a separating equilibrium would correspond exactly to the MBR with the partitional information structure described in Example 1.B.

5 Literature Review

Tversky and Kahneman (1981) developed the notion of framing –the idea that a decisions are influenced by their surrounding context. Framing has a large literature, both in the theoretical, experimental, and psychological settings (Kahneman and Tversky, 1984; Rubinstein and Salant, 2008; Tversky and Shafir, 1992). A particular type of framing concerns the consideration of menu, or currently available alternatives, referred to in the literature as context dependence. In contrast to this model, context dependence is often associated with particular psychological heuristics such as a basing choice on the difference between the attributes of outcomes or reluctance to choose extreme outcomes (Simonson and Tversky, 1992).

That a menu may contain information relevant to the DM's choice over the objects it contains was first articulated by Luce and Raiffa (1957) and expounded upon by Sen (1993, 1997). Sen describes the notion of the *epistemic value* of a menu with more tact than I could hope to achieve: "What is offered for choice can give us information about the underlying situation, and can thus influence our preference over the alternatives, as we see them. For example, the chooser may learn something about the person offering the choice on the basis of what he or she is offering." It is by paraphrasing/formalizing the vignettes in Luce and Raiffa (1957) and Sen (1997) that I constructed the examples that run throughout this paper.

There have been several decision theory papers which deal with characterizing framing effects that stem from informational sources. Ahn and Ergin (2010) considers a DM whose beliefs, and hence preferences, depend on the description of the state space. There a depiction of the state space is a partition of it, and preferences are defined over all acts measurable with respect to the partition. The interpretation is that different descriptions of the state space might alter the DM of contingencies which he would otherwise be unaware. Bourgeois-Gironde and Giraud (2009) construct a model of "rational" framing in the domain of Bolker–Jeffrey decision theory. They take as motivation, and provide an axiomatic foundation for, the observation of Sher and McKenzie (2006) that (seemingly) logically equivalent statements might in fact contain different information because the choice to use one description over another might itself impart information. As such, Bourgeois-Gironde and Giraud (2009) consider a set of frames and allow two different, but logically equivalent, statements that belong to different frames to induce different beliefs of the DM.

The epistemic aspect of decision problems has been studied by Kochov (2010) in a model that shares many philosophical motivations with this one. Kochov's model defines a decision problem as a collection of menus, and imposes the canonical axioms (i.e., Dekel *et al.* (2001)) on a preference relation over each decision problem to back out a problem-specific subjective state space. The primary mechanism by which epistemic content alters the decision makers preference in Kochov's model is by changing the composition of the subjective state-space (i.e., the difference in preference is mitigated through a change in tastes, rather than beliefs). The interpretation of menus revealing different unforeseen contingencies is problematic from the modelers point of view: it is impossible to observe a decision maker who is both aware and unaware of a particular contingency. This paper, on the other hand, explains the same behavior by confining the context effect to be a local one.

The appeal to a DM who holds multiple beliefs has been explored in the literature on state dependent preferences. Karni *et al.* (1983) propose a DM who ranks alternatives after some hypothetical event with a known probability. Like this paper, there is an imposed consistency in ex-post tastes across different decision problems. Also related, is Karni and Safra (2014), which takes a somewhat converse approach. There, the decision maker has beliefs regarding his state dependent preferences, or *states of mind* which induce a preference over menus, rather than the menu inducing the belief about the state space. As such, it is the DM's beliefs regarding a subjective state space (his *state of mind*) that is invariant across decision problems.

It is also worth noting that models of endogenous reference dependence can be interpreted as context dependence. In these models the decision problem is associated with a reference level of utility by which the DM evaluates each outcome (Koszegi and Rabin, 2006; Ok *et al.*, 2015). As such, adding outcomes that will effect the reference point will thereby change the DM's preferences. These models can be thought of as a specific case of epistemic concerns; the reference point is information about some underlying state variable. A decision problem associated with reference point, r, is an indicator that the state-of-the-world is s_r.

Finally, this paper is related to the decision theoretic literature on identifying the conditions under which a decision maker is Bayesian updating with respect to subjective (and hence unseen) signals, for example, Lehrer and Teper (2015). In particular, the general model can be seen as a special case of the subjective signal structure discussed in Shmaya and Yariv (2015).

A Lemmas

Lemma 1. *If $\{\succcurlyeq_A\}_{A\in\mathcal{K}(X)}$ satisfies UV, then for all $A \in \mathcal{K}(X)$, $N_A = E_A \cup N$.*

Proof. Fix some $A \in \mathcal{K}(X)$. By definition both E_A and N are subsets of N_A, so, $E_A \cup N \subseteq N_A$. Towards the opposite inclusion, let $s \in N_A$. We will show that if $s \notin E_A$ then $s \in N$. So assume further, that $s \notin E_A$. Since s is null-A, $x^\star \sim_A x_\star$. Since s is not e-null-A, for every $B \in \mathcal{K}(X)$, $x^\star \sim_B x_\star$. By the contrapositive of UV we have $s \in N_B$. Since this holds for all B, $s \in N$. ■

Lemma 2. *Let $\{\mu_{A_n} \in \Delta(S)\}_{n\in\mathbb{N}}$ converge weakly to $\mu_A \in \Delta(S)$. Endow $[0,1]^S$ with the $||\cdot||_1$ norm. Fix some $\beta \in (0,1)$. Then, the Kuratowski limit of $\{\hat{g} \in [0,1]^S | \, \mathbb{E}_{\mu_{A_n}}(\hat{g}) \geqslant \beta\}$ is $\{\hat{g} \in [0,1]^S | \, \mathbb{E}_{\mu_A}(\hat{g}) \geqslant \beta\}$.*[10]

[10] Recall, for a metric space $\mathcal{Y}$ the Kuratowski limit exists if and only if the *Kuratowski limit superior* and *Kuratowski limit inferior* coincide. These are defined, respectively, as follows, (for some $\{Y_n \subseteq \mathcal{Y}\}_{n\in\mathbb{N}}$), as

$$\underset{n\to\infty}{LS} Y_n = \{y \in \mathcal{Y} | \forall B_\epsilon(y), B_\epsilon(y) \cap Y_n \neq \varnothing, \text{ for infinitely many } n\},$$
$$\underset{n\to\infty}{LI} Y_n = \{y \in \mathcal{Y} | \forall B_\epsilon(y), B_\epsilon(y) \cap Y_n \neq \varnothing, \text{ for sufficiently large } n\}$$

where $B_\epsilon(y)$ is the ball or radius ϵ around y. Note, these always exists and necessarily $\underset{n\to\infty}{LI} Y_n \subseteq \underset{n\to\infty}{LS} Y_n$.

Proof. Define $\dot{C}(A,\beta) = \{\hat{g} \in [0,1]^S | \mathbb{E}_{\mu_A}(\hat{g}) \geqslant \beta\}$. First, we show, $\hat{g} \in \dot{C}(A,\beta) \implies \hat{g} \in \underset{n\to\infty}{LI}\dot{C}(A_n,\beta)$. Indeed, assume $\hat{g} \in \dot{C}(A,\beta)$, and consider $B_\epsilon(\hat{g})$. Let $in(\hat{g}) = \{s \in S | \hat{g}(s) < 1\}$. Let $M = \sum_{s\in in(\hat{g})} \mu_A(s)$. There are two cases: where $M = 0$ and where $M > 0$. First, let $M = 0$. Since S is a discrete space, and so the indicator functions for each state are continuous, weak convergence implies strong convergence. Thus, $\lim_n \mu_{A_n}(in(\hat{g})) \to \mu_A(in(\hat{g})) = 0$: in particular, for sufficiently large n, $\mu_{A_n}(in(\hat{g})) < 1-\beta$, and so $\mathbb{E}_{\mu_{A_n}}(\hat{g}) = \mathbb{E}_{\mu_{A_n}}(\hat{g}|in(\hat{g})) + \mathbb{E}_{\mu_{A_n}}(\hat{g}|in((\hat{g}))^c) \geqslant \beta$.

So, let $M > 0$ and fix some $\epsilon > 0$. Let $\epsilon' = \frac{\min\{\epsilon,\{1-\hat{g}(s)\}_{s\in in(\hat{g})}\}}{|S|}$. Let, $\hat{f} \in B_\epsilon(\hat{g})$ be given by

$$\hat{f}(s) = \begin{cases} \hat{g}(s) & \text{if } s \notin in(\hat{g}) \\ \hat{g}(s) + \epsilon' & \text{if } s \in in(\hat{g}) \end{cases}$$

Let $m \in \mathbb{N}$ be such that for all $n > m$, $|\mu_A(s) - \mu_{A_n}(s)| < \frac{M\epsilon'}{|S|}$. Then,

$$\begin{aligned} \mathbb{E}_{\mu_{A_n}}(\hat{f}) &> \mathbb{E}_{\mu_A}(\hat{f}) - \sum_s \frac{M\epsilon'}{|S|}\hat{f}(s) \\ &> \sum_{s\notin in(\hat{g})} \mu_A(s)\hat{g}(s) + \sum_{s\in in(\hat{g})} \mu_A(s)\hat{g}(s) + \sum_{s\in in(\hat{g})} \mu_A(s)\epsilon' - M\epsilon' \\ &= \beta. \end{aligned}$$

So, $\hat{f} \in \dot{C}(A_n,\beta)$ for all $n > m$.

Now, we show that $\hat{g} \in \underset{n\to\infty}{LS}\dot{C}(A_n,\beta) \implies \hat{g} \in \dot{C}(A,\beta)$. Indeed, assume $\hat{g} \in \underset{n\to\infty}{LS}\dot{C}(A_n,\beta)$. Then, let n_k denote a subsequence for which $B_{\frac{1}{k}}(\hat{g}) \cap \dot{C}(A_{n_k},\beta) \neq \varnothing$, with $\hat{g}^k$ a point in the intersection. Then $\lim_{n\to\infty}(\mu_{A_{n_k}}(s)\hat{g}^k(s)) = \mu_A(s)\hat{g}(s)$ for all $s \in S$. So, $\mathbb{E}_{\mu_{A_{n_k}}}(\hat{g})$ converges to $\mathbb{E}_{\mu_A}(\hat{g})$, and so, by the preservation of inequalities, $\hat{g} \in \dot{C}(A,\beta)$.

Therefore, $\dot{C}(A,\beta) \subseteq \underset{n\to\infty}{LI}\dot{C}(A_n,\beta) \subseteq \underset{n\to\infty}{LS}\dot{C}(A_n,\beta) \subseteq \dot{C}(A,\beta)$; the Kuratowski limit exists and is equal to $\dot{C}(A,\beta)$. ∎

Definition. *For a menu $A \in \mathcal{K}(X)$, define the* equalizer *of A, $e_A : N_A \times N_A \to R_{++}$ as*

$$e_A(s,s') \mapsto \begin{cases} \frac{1}{\alpha} \text{ such that } (x_\star)_{-s}(\alpha x^\star + (1-\alpha)x_\star) \sim_A (x_\star)_{-s'}x^\star & \text{if } (x_\star)_{-s}x^\star \succcurlyeq_A (x_\star)_{-s'}x^\star \\ \alpha \text{ such that } (x_\star)_{-s'}(\alpha x^\star + (1-\alpha)x_\star) \sim_A (x_\star)_{-s}x^\star & \text{if } (x_\star)_{-s'}x^\star \succ_A (x_\star)_{-s}x^\star \end{cases}$$

That e_A is well defined follows from the following observation.

Lemma 3. *Let $\{\succcurlyeq_A\}_{A\in\mathcal{K}(X)}$ be represented by some (MBR), with beliefs $\{\mu_A | A \in \mathcal{K}(X)\}$, all of which have full support. Then, for all $A \in \mathcal{K}(X)$, $e_A(s,s') = \frac{\mu_A(s)}{\mu_A(s')}$.*

Proof. If $(x_\star)_{-s}x^\star \succcurlyeq_A (x_\star)_{-s'}x^\star$, then for some $\alpha \in (0,1)$, we have $(x_\star)_{-s}\big(\alpha x^\star + (1-\alpha)x_\star\big) \sim_A (x_\star)_{-s'}x^\star$. Using (MBR), we have that $U_A\Big((x_\star)_{-s'}x^\star\Big) = \mu_A(s')$, and $U_A\Big((x_\star)_{-s}\big(\alpha x^\star + (1-\alpha)x_\star\big)\Big) = \alpha\mu_A(s)$. Setting $e_A(s,s') = \frac{1}{\alpha}$, delivers ther result. The other case is similar. ∎

Lemma 4. *Let $\{\succcurlyeq_A\}_{A\in\mathcal{K}(X)}$ be represented by some (MBR) with beliefs $\{\mu_A | A \in \mathcal{K}(X)\}$, all of which have full support. Then $\{\succcurlyeq_A\}_{A\in\mathcal{K}(X)}$ satisfies IS if and only if, for all $x \in X$ and $A, B \in \mathcal{K}(X)$ with $x \notin A \cup B$, and states $s, s' \in S$, we have*

$$\frac{e_A(s,s')}{e_{A\cup x}(s,s')} = \frac{e_B(s,s')}{e_{B\cup x}(s,s')}. \tag{A.1}$$

Proof. **Necessity.** Assume that (A.1) holds, with $x \in X$, $A, B \in \mathcal{K}(X)$, and $s, s' \in S$ satisfying the relevant constraints. Denote by A' and B', $A \cup x$ and $B \cup x$, respectively. Towards a contradiction, assume that there exists some $\pi^A, \rho^A, \in \Delta(A)$, $\pi^B, \rho^B, \in \Delta(A)$, and $\alpha = (0,1)$ be such that,

$$(x_\star)_{-s}\pi^A \sim_A (x_\star)_{-s'}\rho^A, \qquad \text{implying} \quad \frac{\mu_A(s)}{\mu_A(s')} = \frac{(\rho^A \cdot u)}{(\pi^A \cdot u)}, \tag{A.2}$$

$$(x_\star)_{-s}\pi^B \sim_B (x_\star)_{-s'}\rho^B, \qquad \text{implying} \quad \frac{\mu_B(s)}{\mu_B(s')} = \frac{(\rho^B \cdot u)}{(\pi^B \cdot u)}, \tag{A.3}$$

$$(x_\star)_{-s}\pi^A \succcurlyeq_{A'} (x_\star)_{-s'}(\alpha\rho^A + (1-\alpha)x_\star), \qquad \text{implying} \quad \frac{\mu_{A'}(s)}{\mu_{A'}(s')} \geqslant \alpha\frac{(\rho^A \cdot u)}{(\pi^A \cdot u)}, \tag{A.4}$$

$$(x_\star)_{-s}\pi^B \prec_{B'} (x_\star)_{-s'}(\alpha\rho^B + (1-\alpha)x_\star), \qquad \text{implying} \quad \frac{\mu_{B'}(s)}{\mu_{A'}(s')} < \alpha\frac{(\rho^B \cdot u)}{(\pi^B \cdot u)} \tag{A.5}$$

Dividing the implications of (A.2) by (A.4) and (A.3) by (A.5), and applying Lemma 3, we get a direct contradiction to (A.1).

Sufficiency. Assume IS holds. Let $x \in X$, $A, B \in \mathcal{K}(X)$, and $s, s' \in S$ satisfy the relevant constraints for IS. Let $M = \max\{\frac{\mu_A(s)}{\mu_A(s')}, \frac{\mu_B(s)}{\mu_B(s')}, 1\}$. Finally, for any $\beta \in [0, M]$, and $s \in S$, let $f^*(s,\beta) = (x_\star)_{-s}(\frac{\beta}{M}x^\star + (1-\frac{\beta}{M})x_\star)$. Using (MBR), we have

$$U_A\big(f^*(s,1)\big) = U_A\big(f^*(s', \frac{\mu_A(s)}{\mu_A(s')})\big) = \frac{\mu_A(s)}{M}, \text{ and} \tag{A.6}$$

$$U_B\big(f^*(s,1)\big) = U_B\big(f^*(s', \frac{\mu_B(s)}{\mu_B(s')})\big) = \frac{\mu_B(s)}{M}. \tag{A.7}$$

Let $\alpha = \frac{\mu_A(s')\mu_{A'}(s)}{\mu_A(s)\mu_{A'}(s')}$. Case: $\alpha \leqslant 1$. Applying (MBR) again delivers,

$$U_{A'}\big(f^*(s,1)\big) = U_{A'}\big(f^*(s', \alpha\frac{\mu_A(s)}{\mu_A(s')})\big) = \frac{\mu_{A'}(s)}{M}.$$

By (A.6) and (A.7), we can apply IS, so,

$$U_{B'}\big(f^*(s,1)\big) = U_{B'}\big(f^*(s', \alpha\frac{\mu_B(s)}{\mu_B(s')})\big). \tag{A.8}$$

Expanding (A.8) according to (MBR):

$$\mu_{B'}(s) = \mu_{B'}(s')\frac{\mu_A(s')\mu_{A'}(s)}{\mu_A(s)\mu_{A'}(s')}\frac{\mu_B(s)}{\mu_B(s')},$$

which by Lemma 3, is equivalent to (A.1). In the case where $\alpha > 1$, consider $f^*(s, \frac{1}{\alpha})$ and $f^*(s', \frac{\mu_A(s)}{\mu_A(s')})$, and proceed in a similar manner. ∎

B Proofs

Proof of Theorem 2.2. **Part (a), necessity.** The necessity of EU, MC, UV are obvious from the inspection of the representing functionals. CC follows from the continuity of $\mu_{(\cdot)}$. Fix some $\{A_n\}_{n\in\mathbb{N}}$ with limit point A. Notice that $\mathcal{F}_\star$ is homeomorphic to $[0,1]^S$ (endowed with the $||\cdot||_1$ norm), via the identification of $f = \big((\alpha_1 x^\star + (1-\alpha_1)x_\star), \ldots, (\alpha_{|S|}x^\star + (1-\alpha_{|S|})x_\star)\big)$ with $\hat{f} = (\alpha_{s_1}, \ldots, \alpha_{s_{|S|}})$. Fix some $\beta \in (0,1)$. Then, by the representation, we have, for every $B \in \mathcal{K}(X)$,

$$\dot{C}(B,\beta) = \{g \in \mathcal{F}_\star | g \succcurlyeq_B \beta x^\star + (1-\beta)x_\star\} \cong \left\{g \in \mathcal{F}_\star | \mathop{\mathbb{E}}_{\mu_B}(\hat{g}) \geqslant \beta\right\}. \tag{B.1}$$

As $\mu_\cdot$ is continuous, we can apply Lemma 2: the Kuratowski limit of $\dot{C}(A_n,\beta)$ is $\dot{C}(A,\beta)$. Since for all A_n, $\dot{C}(A_n,\beta)$ is convex, they are connected. By the equivalence of Kuratowski convergence and convergence in Hausdorff metric for sequences of connected sets (Salinetti and Wets (1979), Corollary 3A), we are done.

Part (a), sufficiency. It is a direct application of the expected utility theorem that EU delivers for each A the existence of some continuous and bounded $w : S \times X \to \mathbb{R}$ such that

$$U_A^{VNM}(f) = \sum_s \left(\mathop{\mathbb{E}}_{f(s)}(w_A(s,x))\right),$$

represents $\succcurlyeq_A$. Moreover, if $w_A(s,x)$ and $\hat{w}_A(s,x)$ both represent $\succcurlyeq_A$, then $w_A(s,x) = a_A\hat{w}_A(s,x) + b_A(s)$ where $a_A \in \mathbb{R}_{++}$ and $b_A(s) \in \mathbb{R}$ for all $s \in S$.[11]

By exploiting the degrees of freedom from the scalars $b_A(s)$, we can set $w_A(s,x_\star) = 0$, for all A and all $s \in S$. The resulting functionals are unique up to linear transformations. Note, this implies that for all $s \in N_A$, $w_A(s,\cdot)$ is identically 0.

For each $A \in \mathcal{K}(X)$, let $u_A(s,x) : N_A^c \times A \to \mathbb{R}$ be the mapping

$$u_A : (s,x) \mapsto \frac{w_A(s,x)}{w_A(s,x^\star)},$$

and $\mu_A \in \Delta((E_A \cup N)^c)$ as the distribution defined by

$$\mu_A(s) = \frac{w_A(s,x^\star)}{\sum_s w_A(s,x^\star)}.$$

Notice, μ_A is well defined and has full support, since by the non-triviality of $\succcurlyeq_A$, $N_A \neq S$, and for each $s \in (E_A \cup N)^c$, $s \in N_A^c$ (Lemma 1), and so by UV, $w_A(s,x^\star) > w_A(s,x_\star) = 0$. Define,

$$U_A^{MD}(f) = \mathop{\mathbb{E}}_{\mu_A}\left(\mathop{\mathbb{E}}_{f(s)}(u_A(s,x))\right). \tag{B.2}$$

Following standard algebraic manipulations, we can see $\mu_A(s)u_A(s,x) = \frac{1}{\sum_s w_A(s,x^\star)} w_A(s,x)$, and therefore U_A^{MD} represents $\succcurlyeq_A$.

Let $D = \{(s,x) \in S \times X \mid \exists A \in \mathcal{K}(X), x \in A, s \notin N_A\}$. For each $(s,x) \in D$, let $A_{s,x}$ be any menu such that $x \in A_{s,x}$ and $s \notin N_{A_{s,x}}$. Define the mapping $u : D \to \mathbb{R}$ as,

$$u : (s,x) \mapsto u_{A_{s,x}}(s,x).$$

and extend u to $S \times X$, by defining $u(s,x) = 0$ for all $(x,s) \in D^c$.

We now claim, for any $A \in \mathcal{K}(X)$, $s \notin N_A$ and $x \in A$, we have $u(s,x) = u_A(s,x)$. Indeed, for every such $A, B \in \mathcal{K}(X)$ and $s \notin N_A \cup N_B$. Let $\succcurlyeq_{A|B|s} \subseteq (\Delta(A \cap B))^2$ be defined by:

$$\pi \succcurlyeq_{A|B|s} \rho \iff \mathop{\mathbb{E}}_\pi(u_A(s,x)) \geqslant \mathop{\mathbb{E}}_\rho(u_A(s,x)).$$

Since $\succcurlyeq_{A|B|s}$ is represented by a linear utility function, it satisfies EU, and so, by the expected utility theorem, $u_A(s,\cdot)$ is the unique utility index, up to affine transformations.

Fix some A and $s \notin N_A$, and $x \in A$. By (B.2), $\mathbb{E}_\pi(u_A(s,x)) \geqslant \mathbb{E}_\rho(u_A(s,x))$ holds if and only if, for all $f \in F_A$, $f_{-s}\pi \succcurlyeq_A f_{-s}\rho$. Applying MC, we immediately have $g_{-s}\pi \succcurlyeq_{A_{s,x}} g_{-s}\rho$ for any $g \in F_{A_{s,x}}$ (here we use

[11] For a reference using the same framework, see "NM Theorem" of Karni *et al.* (1983).

the fact that $s \notin N_{A_{s,x}}$). From (B.2) again, $\succcurlyeq_{A|A_{s,x}|s} = \succcurlyeq_{A_{s,x}|A|s}$. So $u_A(s,\cdot)$ is an affine transformation of $u_{A_{s,x}}(s,\cdot)$. Moreover, both are twice normalized: $u_A(s,x^\star) = u_{A_{s,x}}(s,x^\star) = 1$ and $u_A(s,x_\star) = u_{A_{s,x}}(s,x_\star) = 0$. Hence they must coincide on $A \cap A_{s,x}$. Finally, since $x \in A \cap A_{s,x}$, we have $u_A(s,x) = u_{A_{s,x}}(s,x) = u(s,x)$. Clearly, since $u_A = u|_A$ and u_A is continuous and bounded, $u|_A$ is continuous and bounded.

Because it eases exposition, we well prove that $\mu_{(\cdot)} : \mathcal{K}(X) \to \Delta(S)$ is continuous after we have shown that it is unique.

Part (b). Uniqueness results are standard. It is clear from the argument above that $u(\cdot,\cdot)$ is unique (given the normalization on $\bigstar$), as it must represent $\succcurlyeq_{A_{s,x}|A_{s,x}|s}$. With regards to beliefs, assume to the contrary, for some $A \in \mathcal{K}(X)$, μ and ν both represent (in conjunction with u, as in (MBR)) $\succcurlyeq_A$. Then there must be some s, s', such that $\mu(s) < \nu(s)$ and $\mu(s') > \nu(s')$. Assume (*with* loss of generality, but the other case follows from the reflected argument) that $\mu(s) \leqslant \mu(s')$. Set π as the probability distribution given by,

$$\pi(x) = \begin{cases} \frac{\mu(s)}{\mu(s')} & \text{if } x = x^\star, \\ 1 - \frac{\mu(s)}{\mu(s')} & \text{if } x = x_\star, \\ 0 & \text{otherwise.} \end{cases}$$

Given that (μ,u) represents $\succcurlyeq_A$, it follows from (MBR) that $(x_\star)_{-s'}\pi \sim_A (x_\star)_{-s}x^\star$. But, since (ν,u) also represents $\succcurlyeq_A$: $(x_\star)_{-s'}\pi \prec_A (x_\star)_{-s}x^\star$, a clear contradiction.

Part (a), sufficiency continued. Assume, by way of contradiction, that $\mu_{(\cdot)} : \mathcal{K}(X) \to \Delta(S)$ was not continuous. Then there exists some $\{A_n\}_{n\in\mathbb{N}}$ converging to A, such that $\{\mu_{A_n}\}_{n\in\mathbb{N}}$ does not converge to A. This implies that there exists some bounded and continuous function, $\hat{f} : S \to \mathbb{R}$ such that $\mathbb{E}_{\mu_{A_n}}(\hat{f})$, does not converge to $\mathbb{E}_{\mu_A}(\hat{f})$, say it is strictly above (if it is below, or does not exist, the arguments are essentially the same). Since f is bounded, it is without loss of generality that we consider $\hat{f} : S \to [0,1]$. Let $\beta = \mathbb{E}_{\mu_A}(\hat{f})$. Then for some $\epsilon > 0$ and all $m \in \mathbb{N}$, there is some $n > m$, such that $\mathbb{E}_{\mu_{A_n}}(\hat{f}) > \beta + \epsilon$. Hence, there is some subsequence, A_{n_k} such that $\hat{f} \in \dot{C}(A_{n_k}, \beta + \frac{\epsilon}{2})$, and $\hat{f} \notin \dot{C}(A, \beta + \frac{\epsilon}{2})$ (notice, $\dot{C}(\cdot,\cdot)$ is defined in (B.1)). But, $\dot{C}(A, \beta + \frac{\epsilon}{2})$ is closed by EU. Hence, $\dot{C}(A_{n_k}, \beta + \frac{\epsilon}{2})$ is bounded away from $\dot{C}(A, \beta + \frac{\epsilon}{2})$, completing the proof as it is tantamount to a contradiction of CC. ■

Proof of Proposition 3.1. This follows directly from Lemma 1 of Shmaya and Yariv (2015) which states (in the language of this paper) that such a given a prior μ and a set of posteriors $\{\mu_A\}_{A\in\mathcal{K}(X)}$ one can find a generating signal structure, that transforms μ into $\{\mu_A\}_{A\in\mathcal{K}(X)}$, so long as the prior beliefs lie in the relative interior of the convex hull of the set of posteriors, i.e., $\mu \in \mathrm{ri}(\mathrm{Conv}(\{\mu_A\}_{A\in\mathcal{K}(X)}))$. Given the additional flexibility in choosing the prior, and the fact the relative interior of a non-empty convex set is non-empty, we can always find such a μ. ■

Proof of Theorem 3.2. **Necessity.** Assume there was some $(\mu, l, \mathcal{K}(X)/\dot{\sim})$, satisfying the relevant assumptions, that generates $\{\mu_A | A \in \mathcal{K}(X)\}$. Let $A, B \in \mathcal{K}(X)$ such that $N_A^c \cap N_B^c \neq \varnothing$. Let $s \in N_A^c \cap N_B^c$. Thus, it must be that $l([A],s) \neq 0$ and $l([B],s) \neq 0$, implying that $[A] = [B]$. But then, $l([A],\cdot) \equiv l([B],\cdot)$. Therefore, $\mu_A = \mu_B$. Clearly, these induce the same preferences over the intersection of A and B, and PS is satisfied.

Sufficiency. Assume PS holds. By assumption $N = \varnothing$, so, for each $s \in S$, choose some menu $A(s)$ such that $s \notin N_A$. Define $\mu \in \Delta(S)$ as

$$\mu(s) = \frac{\mu_{A(s)}(s)}{\sum_{s' \in S} \mu_{A(s')}(s')}.$$

Define $\dot{\sim}$ over $\mathcal{K}(X)^2$, by $A \dot{\sim} B$ if and only if $N_A = N_B$. It is obvious that $\dot{\sim}$ is a equivalence relation, and so, let $\mathcal{K}(X)/\dot{\sim}$ be the resulting quotient set. For each $[A] \in \mathcal{K}(X)/\dot{\sim}$ and $s \in S$, set

$$l([A], s) = \begin{cases} 1 \text{ if } s \notin N_A, \\ 0 \text{ if } s \in N_A. \end{cases}$$

Now, notice that if $l([A], s) = l([B], s) = 1$, then $N_A^c \cap N_B^c \neq \varnothing$. So by PS the projections of $\succcurlyeq_A$ and $\succcurlyeq_B$ onto $\mathcal{F}_\bigstar$ must coincide. Therefore, it must be that $\mu_A = \mu_B$, implying that $[A] = [B]$.

Finally, pick some A in $\mathcal{K}(X)$. Now notice that for all $s \in N_A^c$, $s \in N_{A(s)}^c$ and so, by the above line of reasoning $\mu_A = \mu_{A(s)}$. Clearly, since $l([A], s)|_{N_A^c} \equiv 1$, (3.2) holds. ∎

Proof of Theorem 3.3. **Necessity.** Assume there exists some (μ, l, X), with $l(x, s) \in (0, 1)$ for all $(x, s) \in X \times S$, that generates $\{\mu_A | A \in \mathcal{K}(X)\}$. For some A that does not contain x and $s, s' \in S$, we have $e_A(s, s') = \frac{\mu_A(s)}{\mu_A(s')}$, and $e_{A\cup x}(s, s') = \frac{\mu_{A\cup x}(s)}{\mu_{A\cup x}(s')}$. Using (3.3), we have

$$\mu_{A\cup x}(s) = \frac{\mu_A(s) \frac{l(x,s)}{1-l(x,s)}}{\mathbb{E}_{\mu_A}\left(\frac{l(x,s')}{1-l(x,s')}\right)}$$

for all $s \in S$. So,

$$\begin{aligned} e_{A\cup x}(s, s') &= \frac{\mu_A(s) \frac{l(x,s)}{1-l(x,s)}}{\mu_A(s') \frac{l(x,s')}{1-l(x,s')}}, \\ &= \frac{\frac{l(x,s)}{1-l(x,s)}}{\frac{l(x,s')}{1-l(x,s')}} e_A(s, s'). \end{aligned}$$

Hence the ratio of equalizers does not depend on the menu. By Lemma 4, IS holds.

Sufficiency. Assume IS holds. Let $\alpha(x, s) = \frac{\mu_{\{\bigstar \cup x\}}(s)}{\mu_\bigstar(s)}$, set

$$l(x, s) = \frac{\alpha(x, s)}{1 + \alpha(x, s)}, \tag{B.3}$$

for all $(x, s) \in X \setminus \bigstar \times S$ and $l(x^\star, \cdot) \equiv l(x_\star, \cdot) \equiv 1$. Let $\gamma(s) = \prod_{x \in X \setminus \bigstar} (1 - l(x, s))$. Define $\mu \in \Delta(S)$ by,

$$\mu(s) = \frac{\frac{\mu_\bigstar(s)}{\gamma(s)}}{\mathbb{E}_{\mu_\bigstar}\left(\frac{1}{\gamma(s')}\right)}$$

By construction, $\mu_\bigstar$ is generated according to (3.3).

We will now show that as defined, (μ, l, X) generates the remainder of $\{\mu_A | A \in \mathcal{K}(X)\}$. We proceed by induction on the cardinality of A.

Define $\nu_x \in \Delta(S)$

$$\nu_x(s) = \frac{\mu_\bigstar(s)\frac{l(x,s)}{1-l(x,s)}}{\mathbb{E}_{\mu_\bigstar}\left(\frac{l(x,s')}{1-l(x,s')}\right)}$$

Now, using the algebraic identity $\frac{\frac{\alpha}{1+\alpha}}{1-\frac{\alpha}{1+\alpha}} = \alpha$, we have $\frac{l(x,s')}{1-l(x,s')} = \alpha(x,s) = \frac{\mu_{\{\bigstar\cup x\}}(s)}{\mu_\bigstar(s)}$. Therefore $\nu_x(s) = \mu_{\{\bigstar\cup x\}}(s)$. This completes the base case (for $|A| = 3$).

Now assume that (μ, l, X) generates $\{\mu_A | A \in \mathcal{K}(X), |A| \leqslant n\}$. Fix any A with n elements, and $x \notin A$. Let A' denote $A \cup x$. Set,

$$\nu_{A'} = \frac{\mu_A(s)\frac{l(x,s)}{1-l(x,s)}}{\mathbb{E}_{\mu_A}\left(\frac{l(x,s')}{1-l(x,s')}\right)}$$

Towards a contradiction, assume that $u_{A'} \neq \nu_{A'}$. Therefore, there must exist some s such that $u_{A'}(s) > \nu_{A'}(s)$, and s' such that $u_{A'}(s') < \nu_{A'}(s')$. Therefore we have:

$$\begin{aligned}\frac{e_A(s,s')}{e_{A\cup x}(s,s')} = \frac{\frac{\mu_A(s)}{\mu_A(s')}}{\frac{\mu_{A'}(s)}{\mu_{A'}(s')}} &< \frac{\frac{\mu_A(s)}{\mu_A(s')}}{\frac{\nu_{A'}(s)}{\nu_{A'}(s')}} \\ &= \frac{\frac{l(x,s')}{1-l(x,s')}}{\frac{l(x,s)}{1-l(x,s)}} = \frac{e_\bigstar(s,s')}{e_{\bigstar\cup x}(s,s')}\end{aligned}$$

Which, by Lemma 4 is a contradiction to IS. Therefore, the inductive step holds, and (μ, l, X) generates $\{\mu_A | A \in \mathcal{K}(X)\}$. ■

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