Chapter 7 - Superposition - when waves combine at the same place at the same, the displacements add together

$$\psi = \psi_1 + \psi_2$$

same frequency

$$E_1(x,t) = E_1 \cos(kx_1 - \omega t + \phi_1)$$

$$E_2(x,t) = E_2 \cos(k x_2 - \omega t + \phi_2)$$

different phase

$$\alpha_1 = kx_1 + \phi_1$$

$$\alpha_2 = kx_2 + \phi_2$$

phase difference $\rightarrow$ $\alpha_2 - \alpha_1 = k(x_2 - x_1) + (\phi_2 - \phi_1)$

what if : $\alpha_2 - \alpha_1 = 2\pi \cdot m$ $\} \rightarrow$ even integer of π

any integer

$$E_R = E_1 + E_2 = E_1\cos(\alpha_1 - \omega t) + E_2\cos(\alpha_2 - \omega t)$$

$$= E_1\cos(\alpha_1 - \omega t) + E_2\cos(2\pi\cdot m + \alpha_1 - \omega t)$$

$$\cos(x) = \cos(x + 2\pi\cdot m)$$

$$= \underbrace{(E_1 + E_2)}_{\text{constructive interference}}\cos(\alpha_1 - \omega t)$$

But, what if $\alpha_2 - \alpha_1 = (2m-1)\pi$ → odd integers of π.

any integer

$$E_t = E_1 + E_2 = E_1\cos(\alpha_1 - \omega t) + E_2\cos(\alpha_2 - \omega t)$$

$$= E_1\cos(\alpha_1 - \omega t) + E_2\cos(\alpha_1 + (2m+1)\pi - \omega t)$$

$$-\cos x = \cos(x + (2m-1)\pi)$$

destructive interference

$$= (E_1 - E_2)\cos(\alpha_1 - \omega t)$$

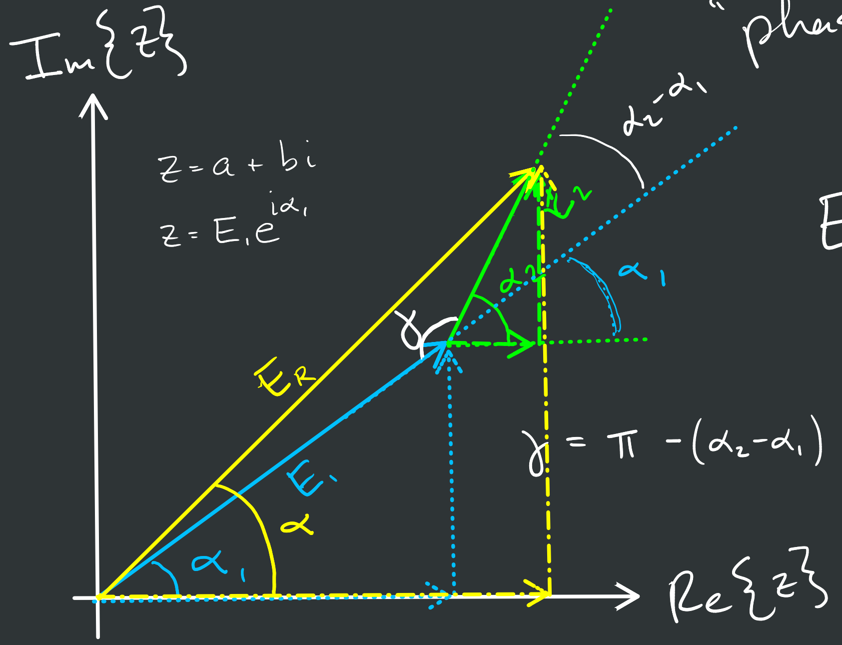
What if any other phase shift?

↳ switch from trig to complex notation

$$E_R = E_1 + E_2 = \mathrm{Re}\left\{E_1 e^{i(\alpha_1 - \omega t)} + E_2 e^{i(\alpha_2 - \omega t)}\right\}$$

$$= \mathrm{Re}\left\{e^{-i\omega t}\underbrace{\left(E_1 e^{i\alpha_1} + E_2 e^{i\alpha_2}\right)}_{}\right\}$$

"phasor diagram" → treat complex number like vectors



$$E_R(x,t) = \mathrm{Re}\left\{E_R e^{i(\alpha - \omega t)}\right\}$$

? ?

$$E_R^2 = E_1^2 + E_2^2 - 2E_1E_2\underbrace{\cos\gamma}_{}$$

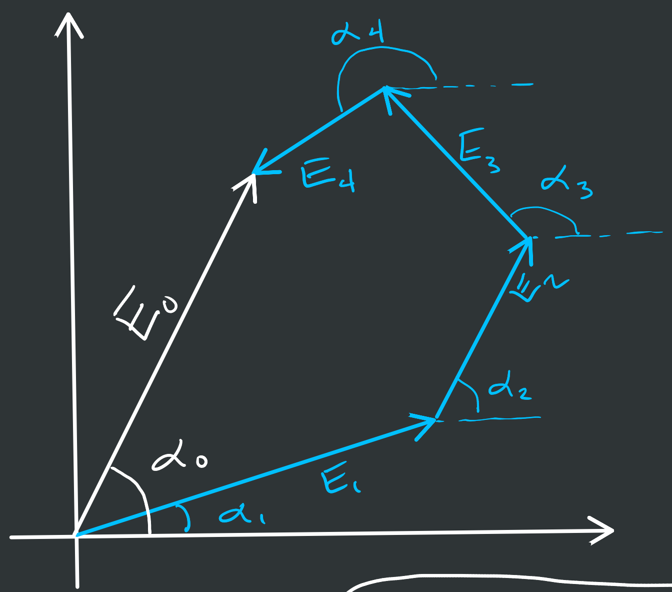
$\cos(\pi - \alpha_2 + \alpha_1)$

$-\cos(-\alpha_2 + \alpha_1)$

$-\cos(\alpha_1 - \alpha_2)$

$$\rightarrow \quad E_R^2 = E_1^2 + E_2^2 + 2E_1E_2\cos(\alpha_1 - \alpha_2)$$

$$\rightarrow \quad \tan\alpha = \frac{E_1\sin\alpha_1 + E_2\sin\alpha_2}{E_1\cos\alpha_1 + E_2\cos\alpha_2}$$



$$\tan\alpha_0 = \frac{\sum_{i=1}^{N} E_i\sin\alpha_i}{\sum_{i=1}^{N} E_i\cos\alpha_i}$$

$$E_0^2 = \underbrace{\left(\sum_{i=1}^{N} E_i\cos\alpha_i\right)^2}_{} + \left(\sum_{i=1}^{N} E_i\sin\alpha_i\right)^2$$

$$\left(\sum_{i=1}^{N} E_i\cos\alpha_i\right)^2 = \sum_{i}^{N} E_i^2\cos^2\alpha_i + \underbrace{\sum_{i}^{N} 2E_i\cos\alpha_i \sum_{j>i}^{N} E_j\cos\alpha_j}_{2\sum_i \sum_{j>1} E_iE_j\cos\alpha_i\cos\alpha_j}$$

$$(a+b+c+d)^2$$

$$E_0^2 = \underbrace{\sum_i^N E_i^2 \cos^2\alpha_i + \sum_i^N E_i^2 \sin^2\alpha_i}_{\sum_i^N E_i^2 \left(\underbrace{\cos^2\alpha_i + \sin^2\alpha_i}_{1}\right)} + \underbrace{2\sum_i \sum_{j>i} E_i E_j \cos\alpha_i \cos\alpha_j + 2\sum_i \sum_{j>i} E_i E_j \sin\alpha_i \sin\alpha_j}_{2\sum_i^N \sum_{j>i}^N E_i E_j \left(\underbrace{\cos\alpha_i \cos\alpha_j + \sin\alpha_i \sin\alpha_j}_{\cos(\alpha_j - \alpha_i)}\right)}$$

$$E_0^2 = \sum_i^N E_i^2 + 2\sum_i \sum_{j>i} E_i E_j \cos(\alpha_j - \alpha_i)$$

if all sources are equal in magnitude $\rightarrow E_i = E_1$

if all sources are random phases and short durations ($< 10\,ns$)

$$\sum_i \sum_{j>i} E_i E_j \cos(\alpha_j - \cos\alpha_i) \rightarrow 0$$

$$\rightarrow E_0^2 = N E_1^2 \Rightarrow E_0 = \sqrt{N} \cdot E_1$$

$$E_0 \propto \sqrt{N}$$

<u>irradiance</u> $\rightarrow I = \frac{1}{2}\epsilon_0 c E_0^2$

$\boxed{I \propto N}$ experimentally verified ✓

if phases are not random, but coherent

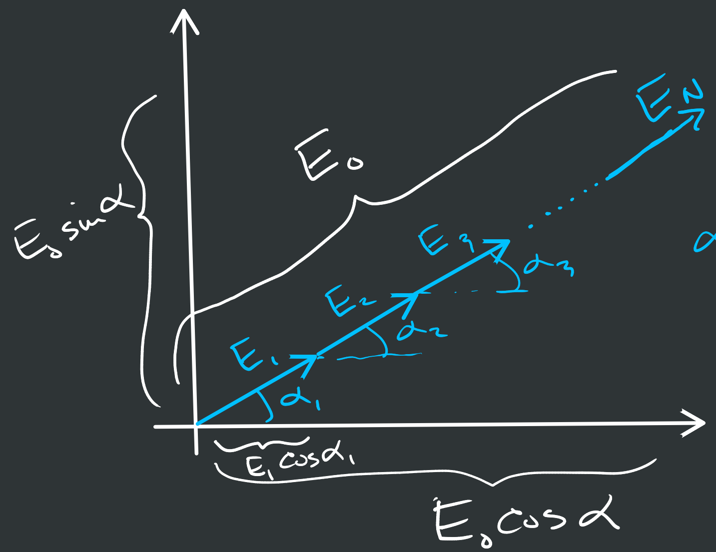
↳ same frequency + waveform

↳ same phase

$$E_0^2 = \sum_i^N E_i^2 + 2\sum_i \sum_{j>i} E_i E_j \underbrace{\cos(\alpha_j - \alpha_i)}_{\cos(0) = 1}$$

$$E_0^2 = \sum_i^N E_i^2 + 2\sum_i \sum_{j>i} E_i E_j$$

→ if sources are equal in magnitude



$\alpha_1 = \alpha_2 = \alpha_3 \dots = \alpha$

$$E_0^2 = (NE_1 \cos\alpha)^2 + (NE_1 \sin\alpha)^2$$

$$E_0^2 = N^2 E_1^2 \underbrace{\left(\cos^2\alpha + \sin^2\alpha\right)}_{1}$$

$$E_0 = N E_1$$

$$I = \frac{1}{2}\epsilon_0 c N^2 E_1^2$$

$$I \propto N^2$$

$$\frac{I_{coherent}}{I_{random}} = \frac{N^2}{N} = N$$

<u>Standing wave</u> $\Rightarrow$ interference of a wave with is own reflection

as two waves: $E_1 = E_0 \sin(-kx + \omega t) \leftarrow$ to the right

$E_2 = E_0 \sin(kx + \omega t) \leftarrow$ to the left

put in a phase shift to account for the reflection at the boundary $\rightarrow -\phi_R$

$$E_R = E_0\left(\sin(\underbrace{-kx + \omega t}_{A}) + \sin(\underbrace{kx + \omega t - \phi_R}_{B})\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{1}{2}(A+B)\right)\cos\left(\frac{1}{2}(A-B)\right)$$

$$E_R = E_0 \cdot 2\sin\left(\frac{1}{2}\left(-\cancel{kx}+wt+\cancel{kx}+wt-\phi_R\right)\right)\cos\left(\frac{1}{2}\left(-kx+\cancel{wt}-kx-\cancel{wt}+\phi_R\right)\right)$$

$$E_R = 2E_0\sin\left(wt-\frac{\phi_R}{2}\right)\cos\left(-kx+\phi_R/2\right)$$

lets take the important case of $\phi_R = \pi$

$$E_R = 2E_0\sin\left(wt-\frac{\pi}{2}\right)\cos\left(-kx+\pi/2\right)$$

$$= 2E_0\left(-\cos(wt)\cdot\sin(kx)\right)$$

$$E_R = \underbrace{2E_0\cdot\sin(kx)}_{\text{spacial amplitude}}\underbrace{\cos(wt)}$$

→ variation of that amplitude as time goes by.

spacial amplitude

→ the places where this are equal to 0 will always be zero

$\underbrace{\text{the places where this are equal to 0 will always be zero}}$

positions are nodes

where do the nodes appear?

$kx = m\pi$ $m = 0, \pm 1, \pm 2, \dots \rightarrow$ is an integer

$k = \frac{2\pi}{\lambda}$

$\omega = \frac{2\pi}{T}$

$\frac{2\cancel{\pi}}{\lambda} \cdot x = m\cancel{\pi}$

$x = m \cdot \frac{\lambda}{2}$ ← positions where amplitude is always zero
positions of the nodes

$x_2 - x_1 = \frac{2\lambda}{2} - \frac{1\lambda}{2}$

$\Delta x = \frac{\lambda}{2}$ ← distance between adjacent nodes

$\lambda = 2\Delta x$

when do the maxima occur?

$\cos(\omega t) = 1$

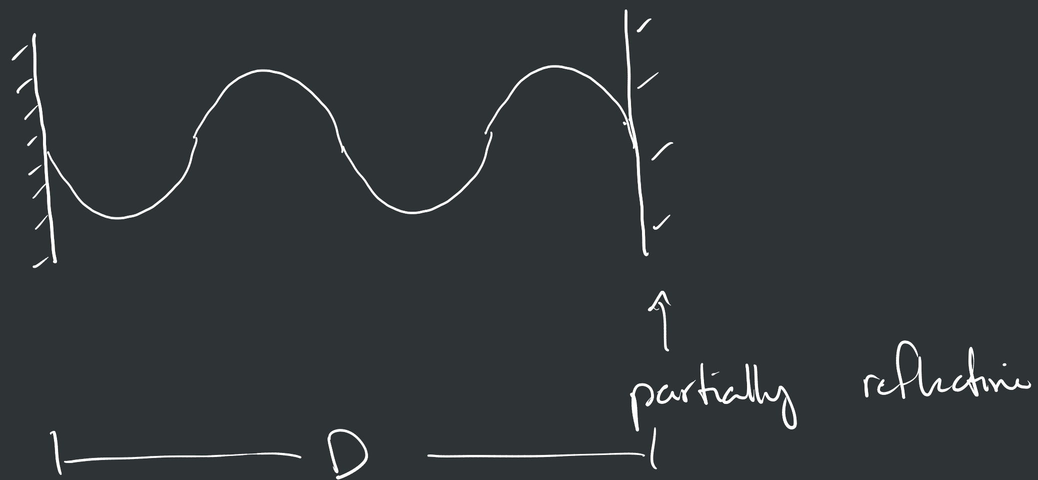
$\omega t = 0, \pi, 2\pi, \dots$ $\omega t = m\pi$ $m = 0, \pm 1, \pm 2 \rightarrow$ integer

$$\omega = \frac{2\pi}{T}$$

$$t_{max} = m\frac{T}{2}$$

$$\Delta t_{max} = \frac{T}{2}$$

How do we do this?



$$D = m\left(\frac{\lambda}{2}\right)$$

number of nodes

$m = 1, 2, 3, 4 \ldots$

What about waves of different frequency / wavelength?

If we have two waves, and the frequencies are close but not exact

↳ Frequency beating

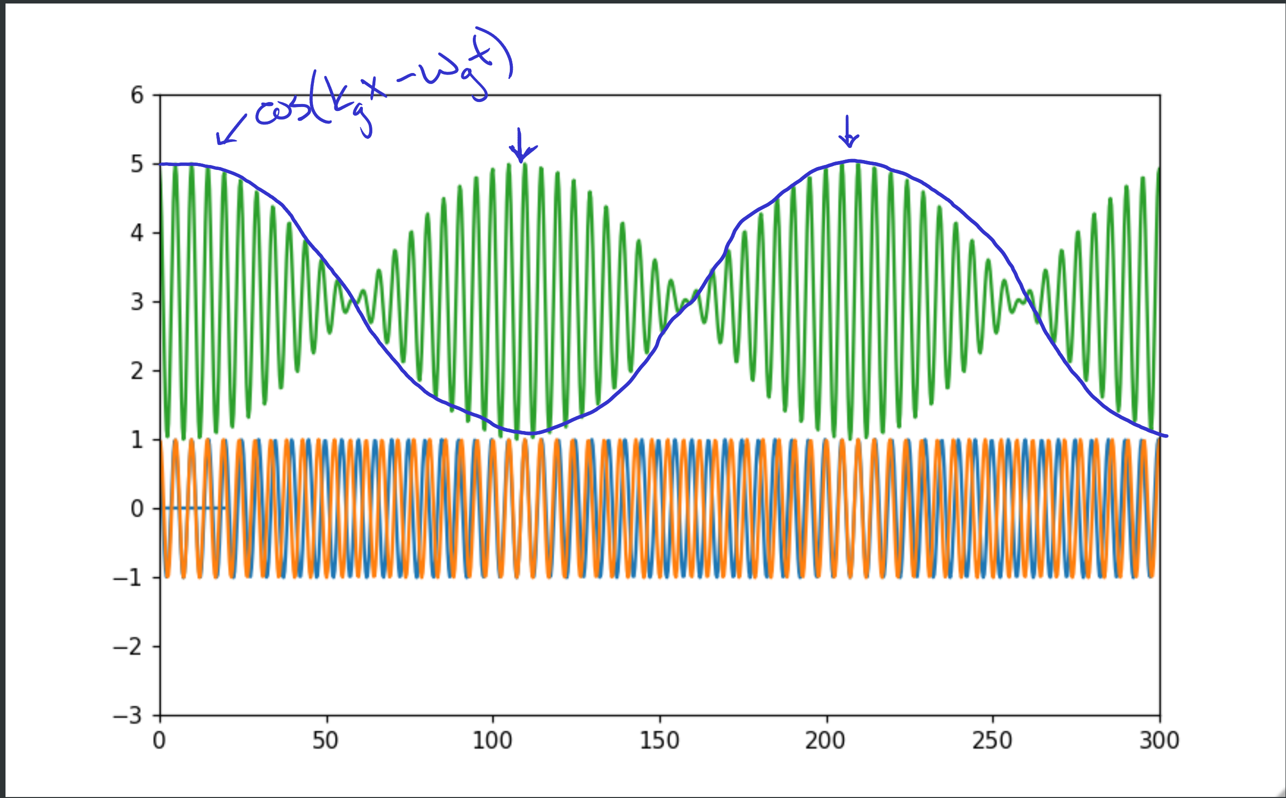
$$
\left.\begin{aligned} E_1 &= E_0 \cos(k_1 x - \omega_1 t) \\ + \quad E_2 &= E_0 \cos(k_2 x - \omega_2 t) \end{aligned}\right\} \cos\beta_1 + \cos\beta_2 = 2\cos\left(\tfrac{1}{2}(\beta_1+\beta_2)\right)\cos\left(\tfrac{1}{2}(\beta_1-\beta_2)\right)
$$

$$E_R = 2E_0 \cos\left(\frac{(k_1+k_2)}{2}x - \frac{(\omega_1+\omega_2)}{2}t\right) \cdot \cos\left(\frac{(k_1-k_2)}{2}x - \frac{(\omega_1-\omega_2)}{2}t\right)$$

$\frac{k_1+k_2}{2} = k_p$ ← phase wave number $\qquad \frac{\omega_1+\omega_2}{2} = \omega_p$ ← phase angular frequency

$k = \frac{2\pi}{\lambda}$ $\qquad \frac{k_1-k_2}{2} = k_g$ ← group wave number $\qquad \frac{\omega_1-\omega_2}{2} = \omega_g$ ← group angular frequency

$$E_R = 2E_0 \underbrace{\cos(k_p x - \omega_p t)}_{\text{"carrier wave" or "signal"}} \cdot \underbrace{\cos(k_g x - \omega_g t)}_{\text{"envelope" or "varying amplitude"}}$$



$$T_b = \frac{T_g}{2}$$

$$\omega = \frac{2\pi}{T}$$

$$\frac{2\pi}{\omega_b} = \frac{2\pi}{2\cdot\omega_g}$$

$$\omega_b = 2\omega_g$$

$$\omega_b = \frac{\cancel{2}\cdot(\omega_1 - \omega_2)}{\cancel{2}}$$

beat frequency is the difference in the two freq

$$\boxed{\omega_b = \omega_1 - \omega_2}$$

$$\boxed{f_b = f_1 - f_2}$$

Dispersion - in materials, EM waves of different frequencies travel w/ different speeds

$$\frac{c}{v} = n(\lambda) \qquad n(\lambda) = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \dots \Bigg\} \begin{array}{l}\text{empirical formula} \\ \text{Cauchy's formula}\end{array}$$

disperson $\quad \dfrac{dn}{d\lambda} = -\dfrac{2B}{\lambda^3} + \underbrace{\dots}_{\text{ignore higher order terms}}$

interpretation for normal materials

larger wavelengths → smaller index

smaller wavelength → larger index
(higher frequency)
↳ higher frequencies → lower speeds

$$v = \lambda \cdot \nu = \frac{\omega}{k}$$

$v_p \leftarrow$ phase velocity

$v_g \leftarrow$ group velocity

higher frequency carrier wave

if these are close in value

$$v_p = \frac{\omega_p}{k_p} = \frac{\omega_1 + \omega_2}{k_1 + k_2} \cong \frac{\omega}{k}$$

lower frequency envelope

↳ group velocity

again, these are close together

$$v_g = \frac{\omega_g}{k_g} = \frac{\omega_1 - \omega_2}{k_1 - k_2}$$

$$v_g \cong \frac{d\omega}{dk}$$

$$V_p = \frac{\omega}{k} \Rightarrow \omega = V_p \cdot k$$

$$V_g = \frac{d(V_p \cdot k)}{dk}$$

$$V_g = V_p + k\frac{dV_p}{dk}$$

does the velocity of waves depend on wavelength?

- in a nondispersive medium, no

$$\frac{dv}{dk} = 0$$

$$\therefore V_g = V_p$$

- but, in a dispersive medium

$$V_p = \frac{c}{n}$$ can be a function of λ so is a function of k

$$v_g = v_p + k \underbrace{\frac{dv_p}{dk}}$$

$$\frac{dv_p}{dk} = \frac{d\left(\frac{c}{n}\right)}{dk} = c\,\frac{d(n^{-1})}{dk} = -c\,n^{-2}\frac{dn}{dk}$$

$$\frac{dv_p}{dk} = -\frac{c}{n^2}\cdot\frac{dn}{dk}$$

$$\underset{\substack{\uparrow \\ \frac{c}{n}}}{v_g = v_p} + k\left(-\frac{c}{n^2}\frac{dn}{dk}\right) = \frac{c}{n} + k\left(-\frac{c}{n^2}\frac{dn}{dk}\right)$$

$$v_g = \frac{c}{n}\left(1 - \frac{k}{n}\underbrace{\frac{dn}{dk}}\right)$$

transform to λ rather than k

$$k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k}$$

$$\frac{dn}{dk} = \frac{dn}{d\lambda}\cdot\frac{d\lambda}{dk} = \frac{dn}{d\lambda}\cdot\underbrace{\frac{d\left(\frac{2\pi}{k}\right)}{dk}}_{-\frac{2\pi}{k^2}}$$

$$\frac{dn}{dk} = -\frac{2\pi}{k^2}\cdot\frac{dn}{d\lambda}$$

$$\frac{dn}{dk} = \frac{-\cancel{2\pi}\cdot\lambda^2}{(2\pi)^{\cancel{2}}}\cdot\frac{dn}{d\lambda}$$

$$\frac{dn}{dk} = -\frac{\lambda^2}{2\pi}\cdot\frac{dn}{d\lambda}$$

$$v_g = \frac{c}{n}\left(1 + \frac{\cancel{2\pi}}{\cancel{\lambda}}\cdot\frac{1}{n}\cdot\left(+\frac{\lambda^{\cancel{2}}}{\cancel{2\pi}}\right)\frac{dn}{d\lambda}\right)$$

$$\boxed{v_g = \frac{c}{n}\left(1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right)}$$

for normal dispersion

$$\frac{dn}{d\lambda} = -\frac{2B}{\lambda^3}$$

$$\frac{dn}{d\lambda} < 0$$

$$v_g < v_p$$

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

HW Ch 7: 1, 5, 8, 12, 15, 16, 18, 35, 38

Simulation:

$$v = \frac{c}{n}$$

Fourier Analysis

Coherence - correlation between phases of light wave

↳ incoherent light → random phase relationship

↳ coherent light → constant phase relationship

↳ longitudinal coherence vs. lateral or spacial coherence

↳ along the length

↳ along the width of the beam

$$\cos\alpha + \cos\beta \neq \cos\gamma \qquad \gamma(\alpha, \beta)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

↑ not harmonic

Take a periodic but anharmonic function. $\rightarrow f(t)$

↳ T

↳ can be expressed as a sum of harmonic waves whose frequencies are multiples of $\frac{2\pi}{T} = \omega$.

$$\underline{f(t)} = \sum_{m=0}^{\infty} a_m \cdot \cos(m \cdot \omega \cdot t) + \sum_{m=0}^{\infty} b_m \sin(m \omega t) \Bigg\}$$

Fourier series → this is possible because sines + cosines form a complete basis set.

$m = 0$, b_0 does not matter

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t)}$$

Many functions do this but some important ones are:

- Legendre polynomials
- Hermite polynomials
- Laguerre polynomials
- Bessel Functions

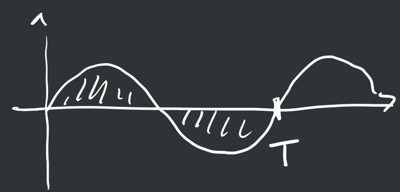
So, the question is what are a_m + b_m?

$$\int_0^T f(t)\,dt = \int_0^T \frac{a_0}{2}\,dt + \underbrace{\int_0^T \sum_{m=1}^{\infty} a_m \cos(m\omega t)\,dt}_{=0} + \int \text{ also } 0$$

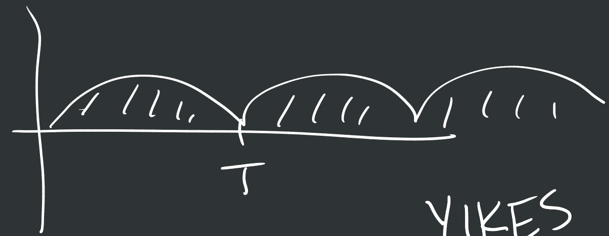
I know this function

$$= \frac{a_0}{2} \cdot T \Rightarrow a_0 = \frac{2}{T} \cdot \int_0^T f(t)\,dt$$

← so that's not too bad!



So now, multiply both sides by $\cos(n\omega t)$ where n is an integer and then integrate over a period.



YIKES

$$\int_0^T f(t)\cdot\cos(n\omega t)\,dt = \underbrace{\int_0^T \frac{a_0}{2}\cos(n\omega t)\,dt}_{=0} + \underbrace{\sum_{m=1}^{\infty}\int_0^T a_m\cos(m\omega t)\cdot\cos(n\omega t)\,dt}_{=0} + \underbrace{\sum_{m=1}^{\infty}\int_0^T b_m\sin(m\omega t)\cdot\cos(n\omega t)\,dt}_{=0}$$

$= 0 \quad m \neq n$

$= \frac{T}{2}a_n \quad m = n$

$$\int_0^T f(t)\cos(n\omega t) = \frac{T}{2}a_n$$

$$a_m = \frac{2}{T}\int_0^T f(t)\cos(m\omega t)\,dt$$

So now we need to find b_m. Same strategy but multiply by $\sin(n\omega t)$.

↳

$$\int_0^T f(t)\cdot\sin(n\omega t)\,dt = \underbrace{\int_0^T \frac{a_0}{2}\sin(n\omega t)\,dt}_{0} + \underbrace{\sum_{m=1}^{\infty}\int_0^T a_m\cos(m\omega t)\cdot\sin(n\omega t)\,dt}_{0} + \underbrace{\sum_{m=1}^{\infty}\int_0^T b_m\sin(m\omega t)\cdot\sin(n\omega t)\,dt}$$

$= 0 \quad n \neq m$

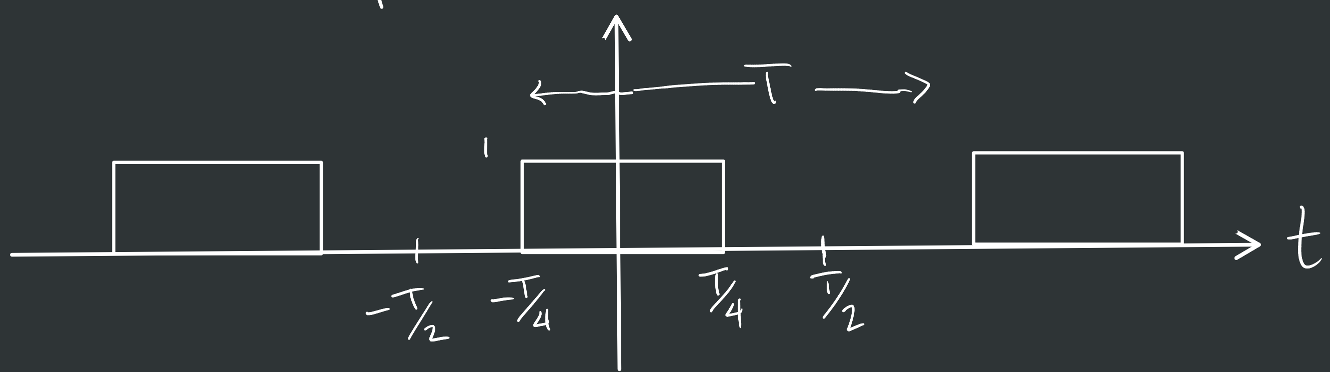
$= \frac{T}{2}a_n \quad n = m$

$$b_m = \frac{2}{T}\int_0^T f(t)\sin(m\omega t)\,dt$$

reminder to talk about Kronicker delta

lets try this!

recticified square wave



$$f(t) = \begin{cases} 0 & -T/2 < t < -T/4 \\ 1 & -T/4 < t < T/4 \\ 0 & T/4 < t < T/2 \end{cases}$$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{2}{T}\int_{-T/4}^{T/4} 1\cdot dt = \frac{2}{T}\cdot \; t\Big|_{-T/4}^{T/4} = \frac{2}{T}\left[\frac{T}{4}-\left(-\frac{T}{4}\right)\right] = \frac{2}{T}\cdot\frac{T}{2} = 1$$

$$a_0 = 1 \qquad f(t) = \frac{1}{2} + \Sigma \qquad \Sigma$$

$$a_m = \frac{2}{T}\int_0^T f(t)\cos(m\omega t)\,dt \Rightarrow a_m = \frac{2}{T}\int_{-T/4}^{T/4} 1\cdot\cos(m\omega t)\,dt = \frac{2}{T}\cdot\frac{\sin(m\omega t)}{m\omega}\Bigg|_{-T/4}^{t=+T/4}$$

$$a_m = \frac{2}{m\omega T}\cdot\left[\sin\left(m\omega\frac{T}{4}\right) - \underbrace{\sin\left(m\omega\left(-\frac{T}{4}\right)\right)}_{\sin\left(-m\omega\frac{T}{4}\right) = -\sin\left(m\omega\frac{T}{4}\right)}\right]$$

$$a_m = \frac{2}{m\omega T}\cdot 2\sin\left(\frac{m\omega T}{4}\right)$$

$$\omega = \frac{2\pi}{T}$$

$$= \frac{4}{m\omega T}\cdot\sin\left(\frac{m\cancel{T}}{4}\cdot\frac{2\pi}{\cancel{T}}\right)$$

$$a_m = \frac{4}{m\omega T} \cdot \sin\left(\frac{m\pi}{2}\right)$$

$m = 1, 2, 3 \ldots$

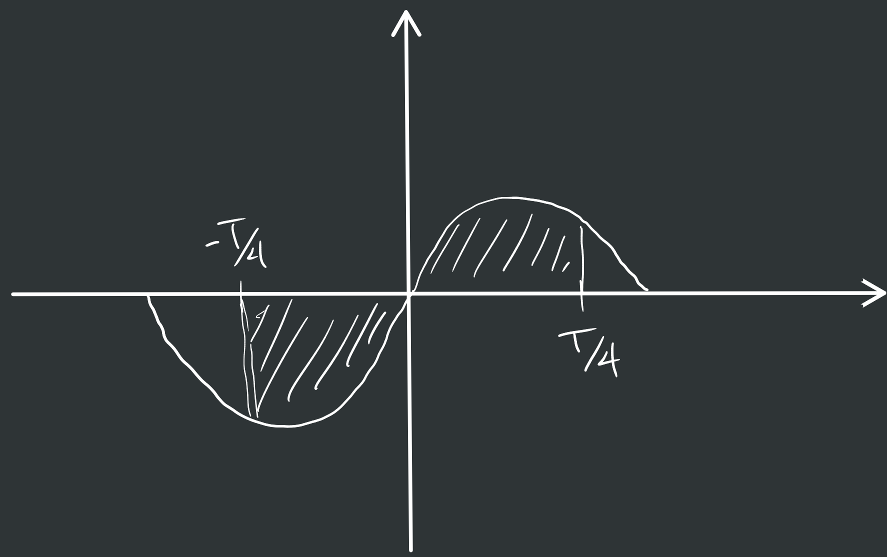
$\sin\left(\frac{\pi}{2}\right), \sin\left(\frac{2\pi}{2}\right), \sin\left(\frac{3\pi}{2}\right),$

$1, \quad 0, \quad -1, \quad 0, \quad 1, \quad 0, \quad -1$

odd m survive

$$b_m = \frac{2}{T}\int_0^T f(t)\sin(m\omega t)\,dt$$

$$b_m = \frac{2}{T}\int_{-T/4}^{T/4} 1 \cdot \sin(m\omega t)\,dt = 0$$



$$f(t) = \frac{1}{2} + \sum_{\substack{m=1 \\ \text{odd}}}^{\infty} \frac{4}{m\omega T} \cdot \sin\left(\frac{m\pi}{2}\right) \cos(m\omega t)$$ ← show me this

OK, now w/ complex notation.....

$$\mathrm{Re}\{z\} = \frac{z + z^*}{2}$$

$$f(t) = \sum_{m=0}^{\infty} a_m \cos(m\omega t) + \sum_{m=0}^{\infty} b_m \sin(m\omega t)$$

$$\frac{a + \cancel{bi} + (a - \cancel{bi})}{2}$$

$$\cos(m\omega t) = \mathrm{Re}\{e^{-im\omega t}\} = \frac{e^{-im\omega t} + e^{im\omega t}}{2}$$

$$\mathrm{Im}\{z\} = \frac{z - z^*}{2i}$$

$$\sin(m\omega t) = \mathrm{Im}\{e^{-im\omega t}\} = \frac{e^{-im\omega t} - e^{im\omega t}}{2i}$$

$$f(t) = \sum_{m=0}^{\infty} a_m e^{-im\omega t} + \sum_{m=0}^{\infty} a_m e^{im\omega t} + \sum_{m=0}^{\infty} b_m e^{-im\omega t} + \sum_{m=0}^{\infty} b_m e^{im\omega t}$$

$$= \sum_{m=0}^{\infty} \underbrace{(a_m + b_m)}_{C_m} e^{-im\omega t} + \sum_{m=0}^{\infty} \underbrace{(a_m + b_m)}_{C_m} e^{im\omega t}$$

$m = -m'$ ← dummy index change

$$= \sum_{m=0}^{\infty} C_m e^{-im\omega t} + \sum_{m'=0}^{-\infty} C_{m'} e^{-im'\omega t}$$

$m = m'$

$$= \sum_{m=0}^{\infty} C_m e^{-im\omega t} + \sum_{m=0}^{-\infty} C_m e^{-im\omega t}$$

$$\boxed{f(t) = \sum_{m=-\infty}^{+\infty} C_m e^{-im\omega t}}$$

Now, we need to find an expression for C_m

$$\int_0^T f(t) e^{+in\omega t} dt = \sum_{m=-\infty}^{+\infty} C_m \underbrace{\int_0^T e^{-im\omega t} e^{+in\omega t} dt}$$

$= T$ when $n = m$

$= 0$ otherwise

$= T \delta_{nm}$

Kronicker delta

$$\delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

$$\boxed{C_m = \frac{1}{T} \int_0^T f(t) e^{+im\omega t} dt}$$

Now, what if I stop adding sines/cosine or $e^{i\omega}$ in frequencis that are multiples of integers, but I add up all frequencies.

$$f(t) = \sum_{m=-\infty}^{\infty} C_m e^{-im\omega t} \cdot \underbrace{\Delta\omega}_{\substack{=1 \\ \text{or} \\ \text{constant}}} \longrightarrow f(t) = \int_{-\infty}^{\infty} g(\omega) \cdot e^{-i\omega t}\, d\omega$$

~~C_n~~ $g(\omega)$ is continuous

$$1 = \Delta\omega = n\omega - (n-1)\omega$$

$$= \omega = \frac{2\pi}{T}$$

$$\Delta\omega = \frac{2\pi}{T}, \quad T \longrightarrow \infty$$

not periodic

$$\downarrow$$

$$d\omega =$$

$f(t)$ is not periodic, so it is any function of t.

So how do we find $g(\omega)$?

↳ employ orthogonality again.

(same thing as before)

$$f(t) = \int_{-\infty}^{\infty} g(\omega) \cdot e^{-i\omega t} \, d\omega$$

$$\int_{-\infty}^{\infty} f(t) e^{+i\omega' t} \, dt = \int_{-\infty \, t}^{\infty} \int_{-\infty \, \omega}^{\infty} g(\omega) \cdot e^{-i\omega t} \cdot e^{+i\omega' t} \, d\omega \, dt$$

$$\int_{-\infty}^{\infty} f(t) e^{i\omega' t} \, dt = \int_{-\infty \, \omega}^{\infty} g(\omega) \cdot \underbrace{\left[\int_{-\infty \, t}^{\infty} e^{-i\omega t} \cdot e^{+i\omega' t} \, dt \right]}_{2\pi \cdot \delta(\omega - \omega')} \cdot d\omega$$

Dirac delta

$$\delta(\omega - \omega') = \begin{cases} \infty & \omega = \omega' \\ 0 & \omega \neq \omega' \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(\omega - \omega') \, d\omega = 1$$

$$\int_{-\infty}^{\infty} f(t) e^{i\omega' t} \, dt = 2\pi \underbrace{\int_{-\infty}^{\infty} g(\omega) \, \delta(\omega - \omega') \, d\omega}_{g(\omega')}$$

$$\int_{-\infty}^{\infty} f(t)e^{i\omega' t}\,dt = 2\pi \cdot g(\omega')$$

$$\int_{-\infty}^{\infty} g(\omega)\cdot\delta(\omega-\omega')\,d\omega = g(\omega')$$

rewrite w/o the prime b/c

$$\int_{-\infty}^{\infty} f(t)e^{i\omega t}\,dt = 2\pi \cdot g(\omega)$$

$$\boxed{g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\cdot e^{i\omega t}\,dt}$$

← Fourier Transform

$$\boxed{f(t) = \int_{-\infty}^{\infty} g(\omega)\cdot e^{-i\omega t}\,dt}$$

← Inverse Fourier Transform

$$\omega = \frac{2\pi}{T} \longrightarrow k$$

$$T \longrightarrow \lambda$$

$$\boxed{\frac{2\pi}{\lambda} = k}$$ ← wavenumber

"spacial frequency"

$$f(x) = \int_{-\infty}^{\infty} g(k) e^{-ikx} \, dk$$ ← Inverse Fourier Transform

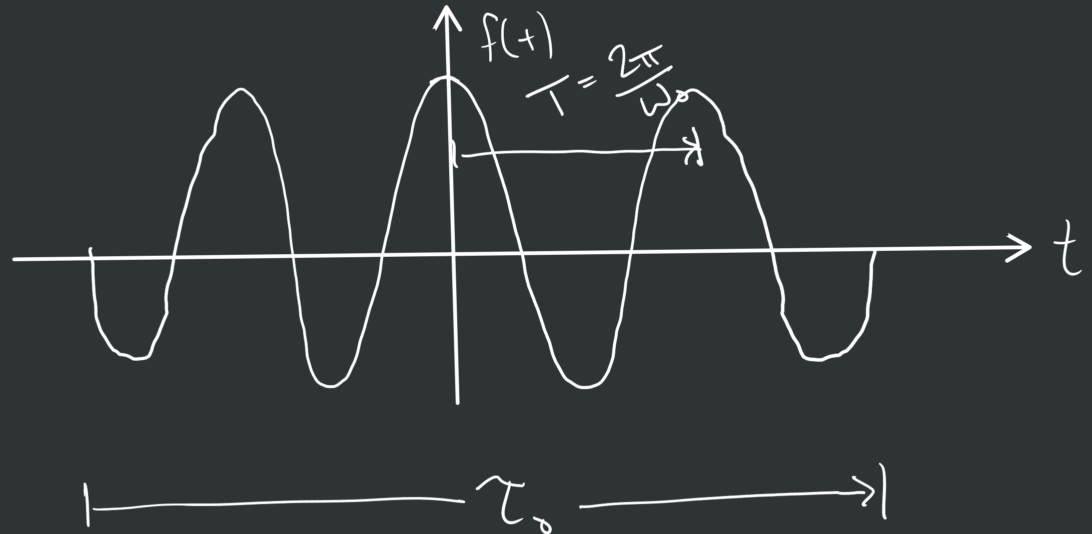
$$g(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx$$ ← Fourier Transform

$$f(t) = \mathrm{IFT}[g(\omega)] = \mathcal{F}^{-1}\{g(\omega)\}$$

$$g(\omega) = \mathrm{FT}[f(t)] = \mathcal{F}\{f(t)\}$$

$$f(t) = \mathrm{IFT}[\mathrm{FT}[f(t)]]$$

$$g(\omega) = \mathrm{FT}[\mathrm{IFT}[g(\omega)]]$$



$$f(t) = \begin{cases} e^{-i\omega_0 t} & -\frac{\tau}{2} < t < \tau/2 \\ 0 & \text{elsewhere} \end{cases}$$

frequency spectrum $\rightarrow$ $g(\omega) = \dfrac{1}{2\pi} \displaystyle\int_{-\tau/2}^{+\tau/2} e^{-i\omega_0 t} \cdot e^{i\omega t} \cdot dt$

$$g(\omega) = \frac{1}{2\pi} \int_{-\tau/2}^{\tau/2} e^{it(\omega - \omega_0)} \, dt$$

$$= \frac{1}{2\pi} \frac{1}{i(\omega-\omega_0)} \cdot e^{it(\omega-\omega_0)} \Bigg|_{-\tau/2}^{+\tau/2}$$

$$\sin(m\omega t) = \frac{e^{-im\omega t} - e^{im\omega t}}{2i}$$

$$= \frac{1}{2\pi} \frac{1}{i(\omega-\omega_0)} \left[e^{i\frac{\tau}{2}(\omega-\omega_0)} - e^{-i\frac{\tau}{2}(\omega-\omega_0)} \right]$$

$$= \frac{1}{\pi(\omega-\omega_0)} \underbrace{\frac{\left[e^{i\frac{\tau}{2}(\omega-\omega_0)} - e^{-i\frac{\tau}{2}(\omega-\omega_0)}\right]}{2i}}_{\sin\left(\frac{\tau}{2}(\omega-\omega_0)\right)}$$

$$g(\omega) = \frac{\sin\left(\frac{\tau}{2}(\omega-\omega_0)\right)}{\pi(\omega-\omega_0)} \left[\frac{\frac{\tau}{2}}{\tau/2}\right]$$

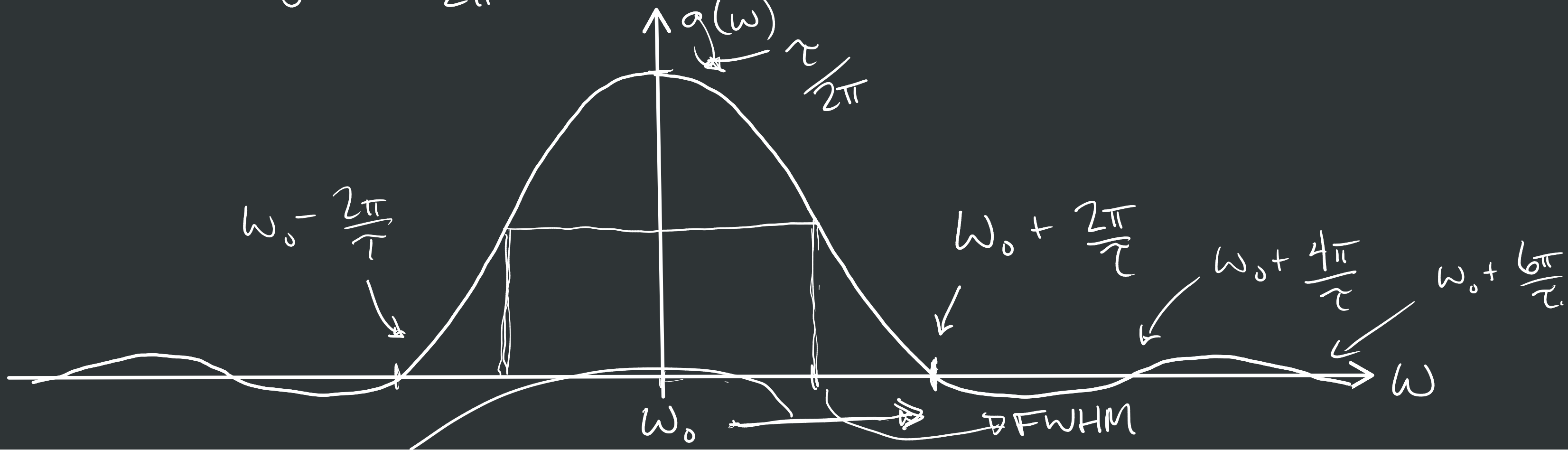
$$g(\omega) = \frac{\frac{\tau}{2}\sin\left(\frac{\tau}{2}(\omega-\omega_0)\right)}{\pi\left(\frac{\tau}{2}(\omega-\omega_0)\right)}$$

$$\frac{\sin(x)}{x} = \operatorname{sinc}(x)$$

$$\operatorname{sinc}(0) = 1$$

$$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right) = 1$$

$$g(\omega) = \frac{\tau}{2\pi}\cdot \operatorname{sinc}\left(\frac{\tau}{2}(\omega-\omega_0)\right)$$



<u>width</u>

$$\Delta\omega = \frac{2\pi}{\tau} \longleftrightarrow \omega = \frac{2\pi}{T}$$

$$\omega = 2\pi\nu$$

$$\Delta\omega = 2\pi \cdot \Delta\nu$$

or

$$\Delta\nu = \frac{1}{\tau}$$

frequency bandwidth

$$0 = \operatorname{sinc}\left(\frac{\tau}{2}(\omega - \omega_0)\right)$$

$$0 = \frac{\sin\left(\frac{\tau}{2}(\omega - \omega_0)\right)}{\frac{\tau}{2}(\omega - \omega_0)}$$

$$0 = \sin\left(\frac{\tau}{2}(\omega - \omega_0)\right)$$

$$\frac{\tau}{2}(\omega - \omega_0) = n \cdot \pi \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\omega = \omega_0 + \frac{2n\pi}{\tau}$$

HW: 43, 44, 45, 46, 47, 51, 52, 53

43)

$$\int_0^{\lambda} \sin(akx)\cos(bkx)\,dx = 0 \rightsquigarrow \int_0^{T} \sin(n\omega t)\cos(m\omega t)\,dt = 0$$ for any n and m

44) $$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t)$$

$$a_m = \frac{2}{T}\int_0^T f(t)\cdot\cos(m\omega t)\,dt$$

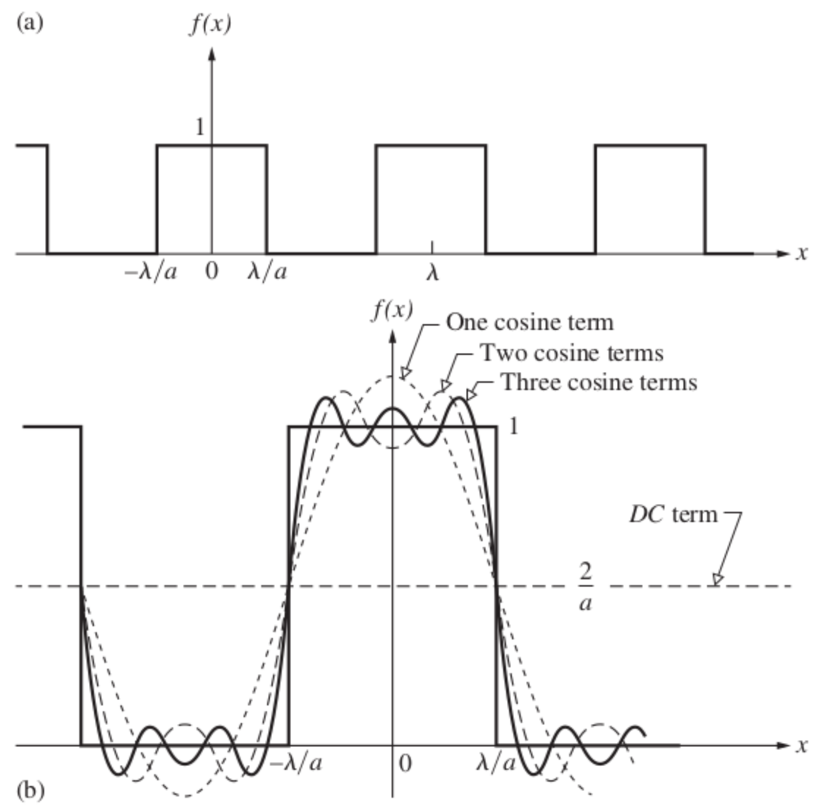


Figure 7.35 An even periodic anharmonic function. In part (*b*) the area under the pulse is $(2\lambda/a) \times 1$ and $A_0 = (2/\lambda)(2\lambda/a) = 4/a$. The *DC* term in the Fourier series is $A_0/2 = 2/a$.