

Chapter 7 - Superposition - when waves combine at the same place at the same, the displacements add together

$$\psi = \psi_1 + \psi_2$$



$$E_1(x,t) = E_1 \cos(kx_1 - \omega t + \phi_1)$$

same frequency

$$E_2(x,t) = E_2 \cos(kx_2 - \omega t + \phi_2)$$

different phase

$$\alpha_1 = kx_1 + \phi_1$$

$$\alpha_2 = kx_2 + \phi_2$$

phase difference $\rightarrow \alpha_2 - \alpha_1 = k(x_2 - x_1) + (\phi_2 - \phi_1)$

What if : $\alpha_2 - \alpha_1 = 2\pi \cdot m$ \rightarrow even integer of π

any integer

$$E_R = E_1 + E_2 = E_1 \cos(\alpha_1 - \omega t) + E_2 \cos(\alpha_2 - \omega t)$$

$$= E_1 \cos(\alpha_1 - \omega t) + E_2 \cos(2\pi \cdot m + \alpha_1 - \omega t)$$

$$\cos(x) = \cos(x + 2\pi \cdot m)$$

$$= (E_1 + E_2) \cos(\alpha_1 - \omega t)$$

constructive interference

But, what if $\alpha_2 - \alpha_1 = (2m-1)\pi$ } \rightarrow odd integers of π .

\uparrow
any integer

$$E_+ = E_1 + E_2 = E_1 \cos(\alpha_1 - \omega t) + E_2 \cos(\alpha_2 - \omega t)$$

$$= E_1 \cos(\alpha_1 - \omega t) + E_2 \cos(\alpha_1 + (2m+1)\pi - \omega t)$$

$$-\cos x = \cos(x + (2m-1)\pi)$$

destructive interference

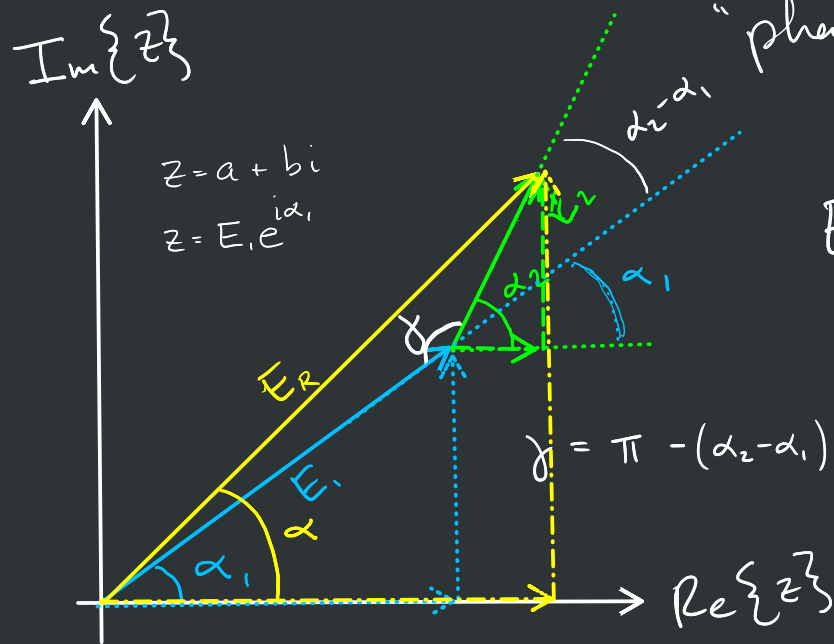
$$= (E_1 - E_2) \cos(\alpha_1 - \omega t)$$

What if any other phase shift?

↳ switch from trig to complex notation

$$E_R = E_1 + E_2 = \operatorname{Re} \left\{ E_1 e^{i(\alpha_1 - \omega t)} + E_2 e^{i(\alpha_2 - \omega t)} \right\}$$

$$= \operatorname{Re} \left\{ e^{-i\omega t} (E_1 e^{i\alpha_1} + E_2 e^{i\alpha_2}) \right\}$$



"phasor diagram" → treat complex number like vectors

$$E_R(x,t) = \operatorname{Re} \left\{ E_R e^{i(\alpha - \omega t)} \right\}$$

\uparrow \uparrow
 ? ?

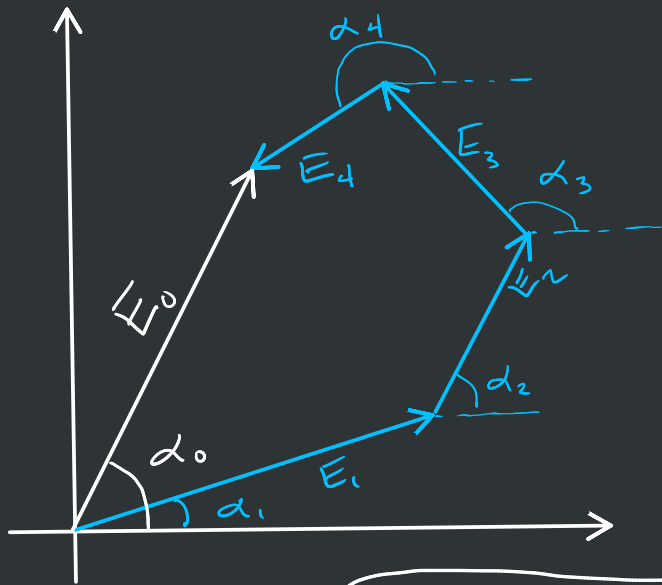
$$E_R^2 = E_1^2 + E_2^2 - 2E_1 E_2 \underbrace{\cos \gamma}_{\cos(\pi - \alpha_2 + \alpha_1)}$$

$$= \cos(-\alpha_2 + \alpha_1)$$

$$= \cos(\alpha_1 - \alpha_2)$$

$$\rightarrow E_R^2 = E_1^2 + E_2^2 + 2E_1E_2\cos(\alpha_1 - \alpha_2)$$

$$\rightarrow \tan \alpha = \frac{E_1 \sin \alpha_1 + E_2 \sin \alpha_2}{E_1 \cos \alpha_1 + E_2 \cos \alpha_2}$$



$$\tan \alpha_0 = \frac{\sum_{i=1}^N E_i \sin \alpha_i}{\sum_{i=1}^N E_i \cos \alpha_i}$$

$$E_0^2 = \left(\sum_{i=1}^N E_i \cos \alpha_i \right)^2 + \left(\sum_{i=1}^N E_i \sin \alpha_i \right)^2$$

$$\left(\sum_{i=1}^N E_i \cos \alpha_i \right)^2 = \sum_i E_i^2 \cos^2 \alpha_i + \underbrace{\sum_i 2E_i \cos \alpha_i \sum_{j>i} E_j \cos \alpha_j}_{2 \sum_i \sum_{j>i} E_i E_j \cos \alpha_i \cos \alpha_j}$$

$$(a + b + c + d)^2$$

$$2 \sum_i \sum_{j>i} E_i E_j \cos \alpha_i \cos \alpha_j$$

$$E_o^2 = \underbrace{\sum_i^N E_i^2 \cos^2 \alpha_i + \sum_i^N E_i^2 \sin^2 \alpha_i}_{\sum_i^N E_i^2 (\underbrace{\cos^2 \alpha_i + \sin^2 \alpha_i}_1)} + \underbrace{2 \sum_i^N \sum_{j>i}^N E_i E_j \cos \alpha_i \cos \alpha_j + 2 \sum_i^N \sum_{j>i}^N E_i E_j \sin \alpha_i \sin \alpha_j}_{2 \sum_i^N \sum_{j>i}^N E_i E_j (\underbrace{\cos \alpha_i \cos \alpha_j + \sin \alpha_i \sin \alpha_j}_{\cos(\alpha_j - \alpha_i)})}$$

$$E_o^2 = \sum_i^N E_i^2 + 2 \sum_i^N \sum_{j>i}^N E_i E_j \cos(\alpha_j - \alpha_i)$$

if all sources are equal in magnitude $\rightarrow E_i = E_1$

if all sources are random phases and short durations ($< 10\text{ns}$)

$$\sum_i^N \sum_{j>i}^N E_i E_j \cos(\alpha_j - \alpha_i) \rightarrow 0$$

$$\rightarrow E_o^2 = N E_1^2 \Rightarrow E_o = \sqrt{N} \cdot E_1$$

$$E_o \propto \sqrt{N}$$

irradiance $\rightarrow I = \frac{1}{2} \epsilon_0 c E_o^2$

$I \propto N$ experimentally verified ✓

if phases are not random, but coherent

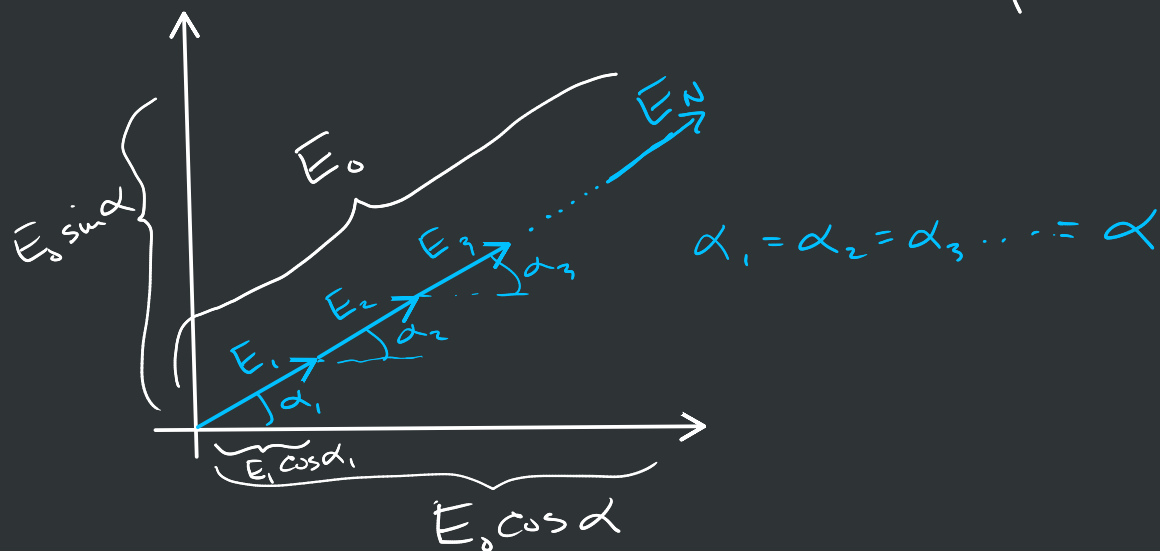
↳ same frequency + waveform

↳ same phase

$$E_o^2 = \sum_i^N E_i^2 + 2 \sum_i \sum_{j>i} E_i E_j \underbrace{\cos(\alpha_j - \alpha_i)}_{\cos(0) = 1}$$

$$E_o^2 = \sum_i^N E_i^2 + 2 \sum_i \sum_{j>i} E_i E_j$$

↳ if sources are equal in magnitude



$$E_o^2 = (NE_1 \cos \alpha)^2 + (NE_1 \sin \alpha)^2$$

$$E_0^2 = N^2 E_1^2 \underbrace{(\cos^2 \alpha + \sin^2 \alpha)}_1$$

$$E_0 = N E_1$$

$$I = \frac{1}{2} \epsilon_0 c N^2 E_1^2$$

$$I \propto N^2$$

$$\frac{I_{\text{coherent}}}{I_{\text{random}}} = \frac{N^2}{N} = N$$

Standing wave \rightarrow interference of a wave with its own reflection

as two waves: $E_1 = E_0 \sin(-kx + \omega t) \leftarrow$ to the right

$E_2 = E_0 \sin(kx + \omega t) \leftarrow$ to the left \leftarrow

$$E_R = E_0 \left(\underbrace{\sin(-kx + \omega t)}_A + \underbrace{\sin(kx + \omega t - \phi_R)}_B \right)$$

put in a phase shift
to account for the
reflection at the
boundary $\rightarrow -\phi_R$

$$\sin A + \sin B = 2 \sin\left(\frac{1}{2}(A+B)\right) \cos\left(\frac{1}{2}(A-B)\right)$$

$$E_R = E_0 \cdot 2 \sin\left(\frac{1}{2}(-\cancel{kx} + \omega t + \cancel{kx} + \omega t - \phi_R)\right) \cos\left(\frac{1}{2}(-\cancel{kx} + \omega t - \cancel{kx} - \omega t + \phi_R)\right)$$

$$E_R = 2E_0 \sin\left(\omega t - \frac{\phi_R}{2}\right) \cos\left(-kx + \frac{\phi_R}{2}\right)$$

lets take the important case of $\phi_R = \pi$

$$E_R = 2E_0 \sin\left(\omega t - \frac{\pi}{2}\right) \cos\left(-kx + \frac{\pi}{2}\right)$$

$$= 2E_0 (-\cos(\omega t) \cdot \sin(kx))$$



$$E_R = 2E_0 \sin(kx) \cos(\omega t)$$

spacial amplitude

→ the places where
this are equal to 0
will always be zero

positions are nodes

variation of that
amplitude as time
goes by.

When do the nodes appear?

$$kx = m\pi \quad m = 0, \pm 1, \pm 2, \dots \rightarrow \text{is an integer}$$

$$k = \frac{2\pi}{\lambda}$$

$$\omega = \frac{2\pi}{T}$$

$$\frac{2\pi}{\lambda} \cdot x = m\pi$$

$$x = m \cdot \frac{\lambda}{2} \quad \leftarrow \text{positions where amplitude is always zero}$$

positions of the nodes

$$x_2 - x_1 = \frac{2\lambda}{2} - \frac{1\lambda}{2}$$

$$\Delta x = \frac{\lambda}{2} \quad \leftarrow \text{distance between adjacent } \underline{\text{nodes}}$$

$$\lambda = 2\Delta x$$

When do the maxima occur?

$$\cos(\omega t) = 1$$

$$\omega t = 0, \pi, 2\pi, \dots$$

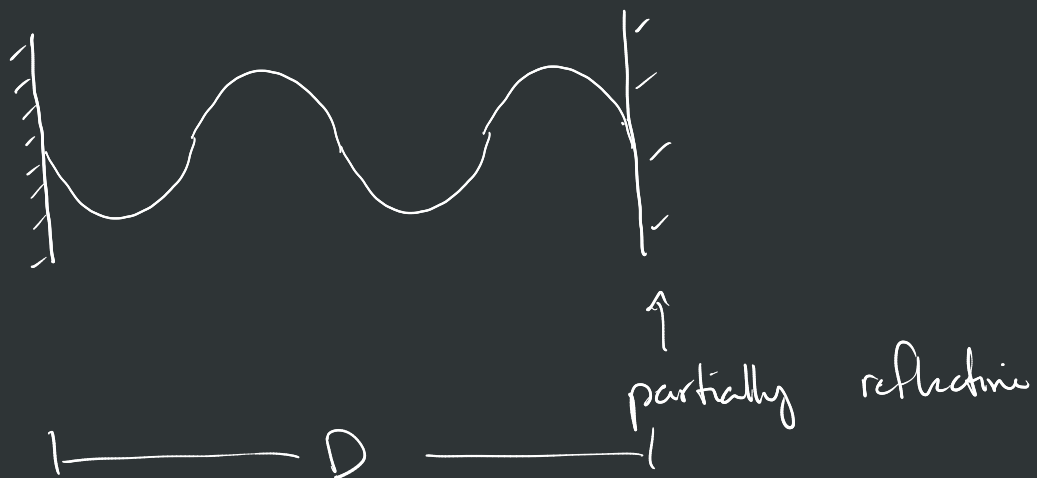
$$\omega t = m\pi \quad m = 0, \pm 1, \pm 2 \rightarrow \text{integer}$$

$$\omega = \frac{2\pi}{T}$$

$$t_{\max} = m \frac{T}{2}$$

$$\Delta t_{\max} = \frac{T}{2}$$

How do we do this?



$$D = m \left(\frac{\lambda}{2} \right)$$

number of nodes
 $m = 1, 2, 3, 4, \dots$

What about waves of different frequency/wavelength?

If we have two waves, and the frequencies are close but not exact

↳ Frequency beating

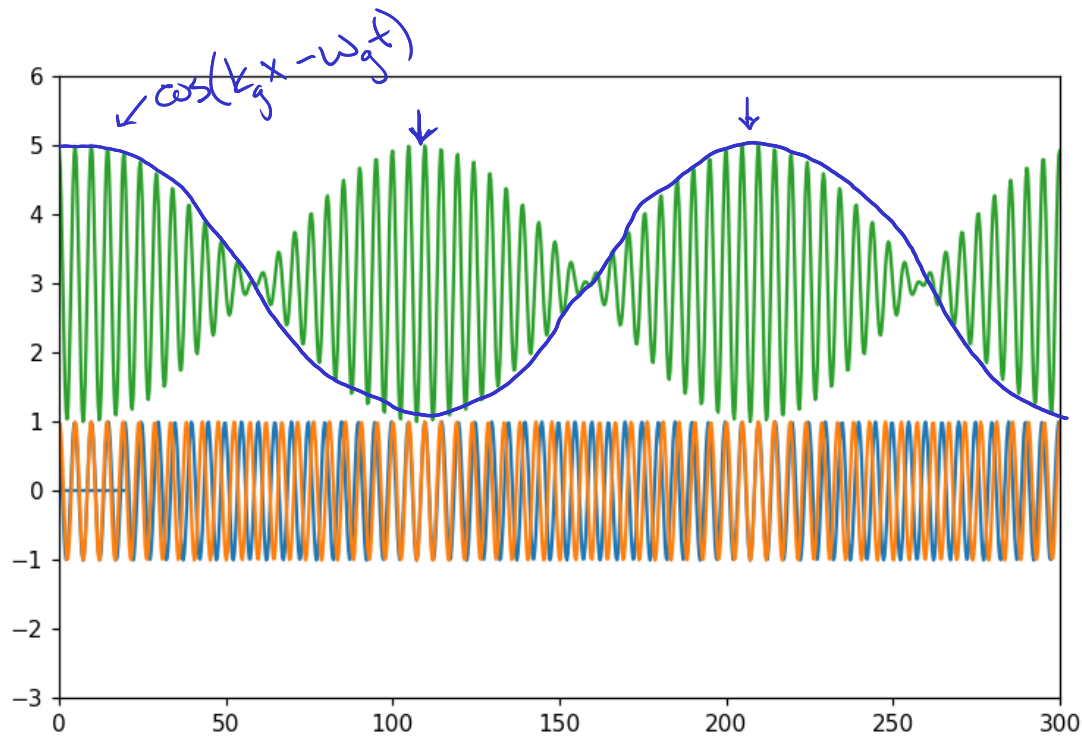
$$\begin{aligned} E_1 &= E_0 \cos(k_1 x - \omega_1 t) \\ + E_2 &= E_0 \cos(k_2 x - \omega_2 t) \end{aligned} \quad \left. \vphantom{\begin{aligned} E_1 &= E_0 \cos(k_1 x - \omega_1 t) \\ + E_2 &= E_0 \cos(k_2 x - \omega_2 t) \end{aligned}} \right\} \cos \beta_1 + \cos \beta_2 = 2 \cos\left(\frac{1}{2}(\beta_1 + \beta_2)\right) \cos\left(\frac{1}{2}(\beta_1 - \beta_2)\right)$$

$$E_R = 2E_0 \cos\left(\frac{(k_1 + k_2)x}{2} - \frac{(\omega_1 + \omega_2)t}{2}\right) \cdot \cos\left(\frac{(k_1 - k_2)x}{2} - \frac{(\omega_1 - \omega_2)t}{2}\right)$$

$$\frac{k_1 + k_2}{2} = k_p \leftarrow \text{phase wave number} \quad \frac{\omega_1 + \omega_2}{2} = \omega_p \leftarrow \text{phase angular frequency}$$

$$k = \frac{2\pi}{\lambda} \quad \frac{k_1 - k_2}{2} = k_g \leftarrow \text{group wave number} \quad \frac{\omega_1 - \omega_2}{2} = \omega_g \leftarrow \text{group angular frequency}$$

$$E_R = 2E_0 \underbrace{\cos(k_p x - \omega_p t)}_{\text{"carrier wave" or "signal"}} \cdot \underbrace{\cos(k_g x - \omega_g t)}_{\text{"envelope" or "varying amplitude"}}$$



$$T_b = \frac{T_g}{2}$$

$$\uparrow \quad \quad \uparrow$$

$$\omega = \frac{2\pi}{T}$$

$$\frac{2\pi}{\omega_b} = \frac{2\pi}{2 \cdot \omega_g}$$

$$\omega_b = 2\omega_g$$

$$\omega_b = \frac{\omega_1 - \omega_2}{2}$$

beat frequency
is the difference
in the two freq

$$\boxed{\omega_b = \omega_1 - \omega_2}$$

$$\boxed{f_b = f_1 - f_2}$$

