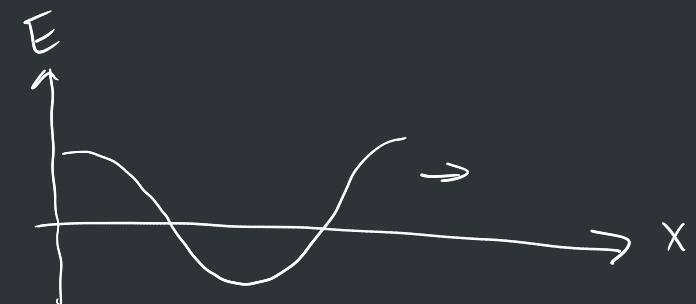


Chapter 7 - Superposition - When waves combine at the same place at the same time, the displacements add together

$$\psi = \psi_1 + \psi_2$$



$$E_1(x,t) = E_1 \cos(kx_1 - \omega t + \phi_1)$$

$$E_2(x,t) = E_2 \cos(kx_2 - \omega t + \phi_2)$$

↳ different phase

$$\alpha_1 = kx_1 + \phi_1$$

$$\alpha_2 = kx_2 + \phi_2$$

phase difference  $\rightarrow \alpha_2 - \alpha_1 = k(x_2 - x_1) + (\phi_2 - \phi_1)$

what if :  $\underbrace{\alpha_2 - \alpha_1}_{\text{any integer}} = 2\pi \cdot m \quad \left. \right\} \rightarrow \text{even integer of } \pi$

$$\begin{aligned}
 E_R &= E_1 + E_2 = E_1 \cos(\alpha_1 - \omega t) + E_2 \cos(\alpha_2 - \omega t) \\
 &= E_1 \cos(\alpha_1 - \omega t) + E_2 \cos(2\pi m + \alpha_1 - \omega t) \\
 \cos(x) &= \cos(x + 2\pi m) \\
 &= (E_1 + E_2) \cos(\alpha_1 - \omega t)
 \end{aligned}$$

constructive interference

But, what if

$$\alpha_2 - \alpha_1 = (2m-1)\pi \quad \left. \begin{array}{l} \rightarrow \text{odd integers} \\ \text{of } \pi. \end{array} \right\}$$

any integer

$$\begin{aligned}
 E_+ &= E_1 + E_2 = E_1 \cos(\alpha_1 - \omega t) + E_2 \cos(\alpha_2 - \omega t) \\
 &= E_1 \cos(\alpha_1 - \omega t) + E_2 \cos(\alpha_1 + (2m+1)\pi - \omega t) \\
 &\quad - \cos x = \cos(x + (2m-1)\pi) \\
 \text{destructive} \quad \curvearrowright &= (E_1 - E_2) \cos(\alpha_1 - \omega t)
 \end{aligned}$$

What if any other phase shift?

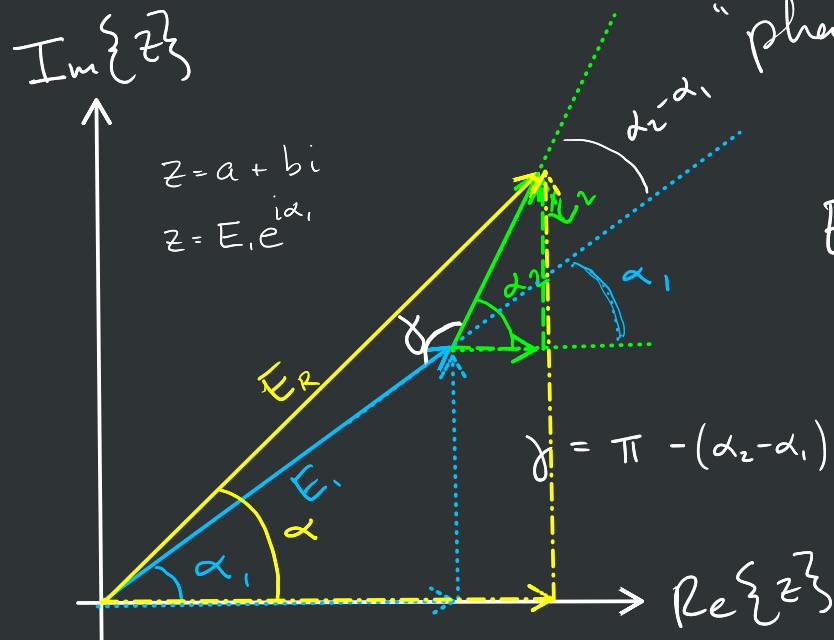
↳ switch from trig to complex notation

$$E_R = E_1 + E_2 = \operatorname{Re} \left\{ E_1 e^{i(\alpha_1 - \omega t)} + E_2 e^{i(\alpha_2 - \omega t)} \right\}$$

$$= \operatorname{Re} \left\{ e^{-i\omega t} \left( E_1 e^{i\alpha_1} + E_2 e^{i\alpha_2} \right) \right\}$$

—————

"phaser diagram" → treat complex numbers like vectors



$$E_R(x,t) = \operatorname{Re} \left\{ E_R e^{i(\alpha - \omega t)} \right\}$$

?      ?

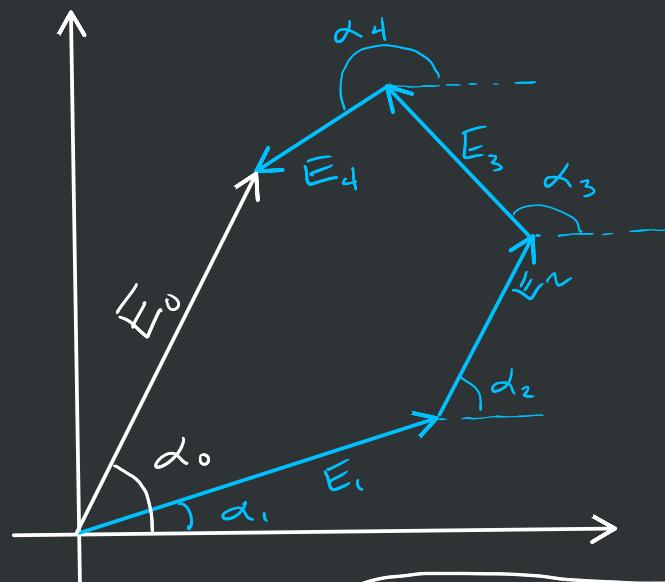
$$E_R^2 = E_1^2 + E_2^2 - 2 E_1 E_2 \underbrace{\cos \gamma}_{\cos(\pi - \alpha_2 + \alpha_1)}$$

$$- \cos(-\alpha_2 + \alpha_1)$$

$$- \cos(\alpha_1 - \alpha_2)$$

$$\rightarrow E_R^2 = E_1^2 + E_2^2 + 2E_1E_2 \cos(\alpha_1 - \alpha_2)$$

$$\rightarrow \tan \alpha = \frac{E_1 \sin \alpha_1 + E_2 \sin \alpha_2}{E_1 \cos \alpha_1 + E_2 \cos \alpha_2}$$



$$\tan \alpha_o = \frac{\sum_{i=1}^N E_i \sin \alpha_i}{\sum_{i=1}^N E_i \cos \alpha_i}$$

$$E_o^2 = \left( \sum_{i=1}^N E_i \cos \alpha_i \right)^2 + \left( \sum_{i=1}^N E_i \sin \alpha_i \right)^2$$

$$\left( \sum_{i=1}^N E_i \cos \alpha_i \right)^2 = \sum_i^N E_i^2 \cos^2 \alpha_i + \underbrace{\sum_i^N 2E_i \cos \alpha_i \sum_{j>i}^N E_j \cos \alpha_j}_{2 \sum_i \sum_{j>i} E_i E_j \cos \alpha_i \cos \alpha_j}$$

$$(a + b + c + d)^2$$

$$E_o^2 = \sum_i^N E_i^2 \cos^2 \alpha_i + \sum_i^N E_i^2 \sin^2 \alpha_i + 2 \sum_i \sum_{j>i} E_i E_j \cos \alpha_i \cos \alpha_j + 2 \sum_i \sum_{j>i} E_i E_j \sin \alpha_i \sin \alpha_j$$

$$\underbrace{\sum_i^N E_i^2 (\cos^2 \alpha_i + \sin^2 \alpha_i)}_1 + 2 \sum_i^N \sum_{j>i} E_i E_j (\cos \alpha_i \cos \alpha_j + \sin \alpha_i \sin \alpha_j)$$

$$\cos(\alpha_j - \alpha_i)$$

$$E_o^2 = \sum_i^N E_i^2 + 2 \sum_i \sum_{j>i} E_i E_j \cos(\alpha_j - \alpha_i)$$

if all sources are equal in magnitude  $\rightarrow E_i = E$ ,  
 if all sources are random phases and short durations ( $< 10\text{ ns}$ )  
 $\sum_i \sum_{j>i} E_i E_j \cos(\alpha_j - \cos \alpha_i) \rightarrow 0$

$$\Rightarrow E_o^2 = N E_i^2 \Rightarrow E_o = \sqrt{N} \cdot E_i$$

$$E_o \propto \sqrt{N}$$

irradiance  $\rightarrow I = \frac{1}{2} E_o C E_o^2$

$|I \propto N|$  experimentally verified ✓

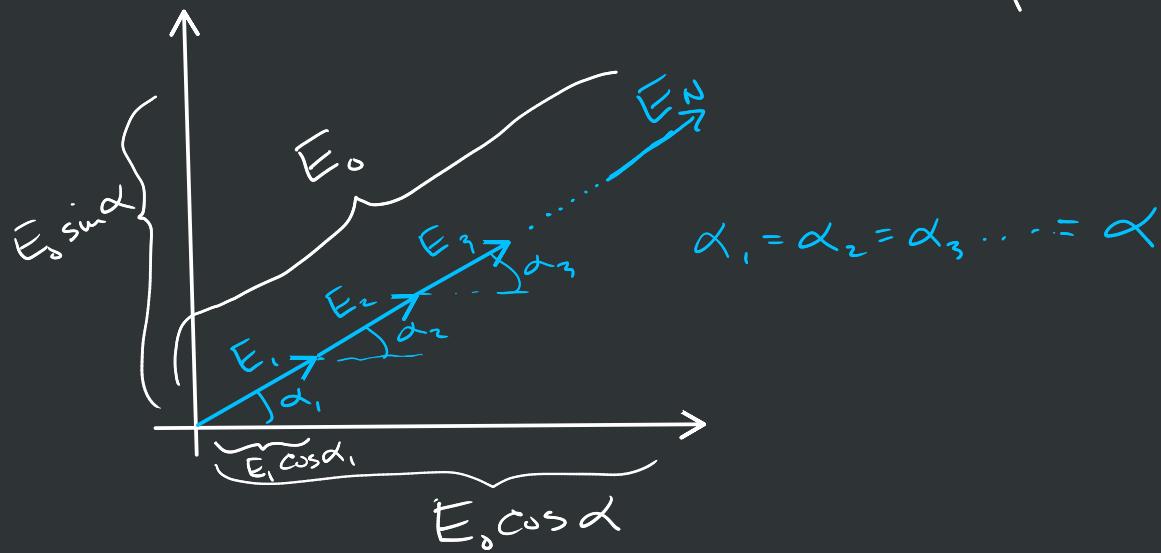
if phases are not random, but coherent  
 ↳ same frequency + waveform  
 ↳ same phase

$$E_o^2 = \sum_i^N E_i^2 + 2 \sum_i \sum_{j>i} E_i E_j \cos(\alpha_j - \alpha_i)$$

$\underbrace{\cos(\delta) = 1}$

$$E_o^2 = \sum_i^N E_i^2 + 2 \sum_i \sum_{j>i} E_i E_j$$

if sources are equal in magnitude



$$E_o^2 = (N E_i \cos \alpha)^2 + (N E_i \sin \alpha)^2$$

$$\bar{E}_o^2 = N^2 \underbrace{E_i^2 (\cos^2 \alpha + \sin^2 \alpha)}_1$$

$$E_o = N E_i$$

$$I = \frac{1}{2} \epsilon_0 c N^2 E_i^2$$

$$I \propto N^2$$

$$\frac{I_{\text{coherent}}}{I_{\text{random}}} = \frac{N^2}{N} = N$$

Standing wave  $\rightarrow$  interference of a wave with its own reflection

as two waves:  $E_1 = E_o \sin(-kx + \omega t)$   $\leftarrow$  to the right

$E_2 = E_o \sin(kx + \omega t)$   $\leftarrow$  to the left

$$E_F = E_o \left( \underbrace{\sin(-kx + \omega t)}_A + \underbrace{\sin(kx + \omega t - \phi_r)}_B \right)$$

put in a phase shift  
to account for the  
reflection at the  
boundary  $\rightarrow -\phi_r$

$$\sin A + \sin B = 2 \sin \left( \frac{1}{2}(A+B) \right) \cos \left( \frac{1}{2}(A-B) \right)$$

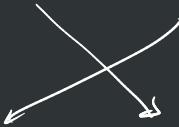
$$E_R = E_0 \cdot 2 \sin\left(\frac{1}{2}(-kx + wt + kx + wt - \phi_r)\right) \cos\left(\frac{1}{2}(-kx + wt - kx - wt + \phi_r)\right)$$

$$E_R = 2E_0 \sin\left(wt - \frac{\phi_r}{2}\right) \cos\left(-kx + \frac{\phi_r}{2}\right)$$

lets take the important case of  $\phi_r = \pi$

$$E_R = 2E_0 \sin\left(wt - \frac{\pi}{2}\right) \cos\left(-kx + \frac{\pi}{2}\right)$$

$$= 2E_0 \left( -\cos(wt) \cdot \sin(kx) \right)$$



$$E_R = 2E_0 \underbrace{\sin(kx)}_{\text{spacial amplitude}} \underbrace{\cos(wt)}_{\text{variation of that amplitude as time goes by.}}$$

$\rightarrow$  the places where  
this are equal to 0  
will always be zero

positions are nodes

When do the nodes appear?

$$kx = m\pi \quad m=0, \pm 1, \pm 2, \dots \rightarrow \text{is an integer}$$

$$k = \frac{2\pi}{\lambda}$$

$$\omega = \frac{2\pi}{T}$$

$$\frac{2\pi}{\lambda} \cdot x = m\pi$$

$$x = m \cdot \frac{\lambda}{2} \quad \leftarrow \begin{array}{l} \text{positions where amplitude is always zero} \\ \text{positions of the nodes} \end{array}$$

$$x_2 - x_1 = \frac{2\lambda}{2} - 1 \frac{\lambda}{2}$$

$$\Delta x = \frac{\lambda}{2} \quad \leftarrow \text{distance between adjacent nodes}$$

$$\lambda = 2\Delta x$$

When do the maxima occur?

$$\cos(\omega t) = 1$$

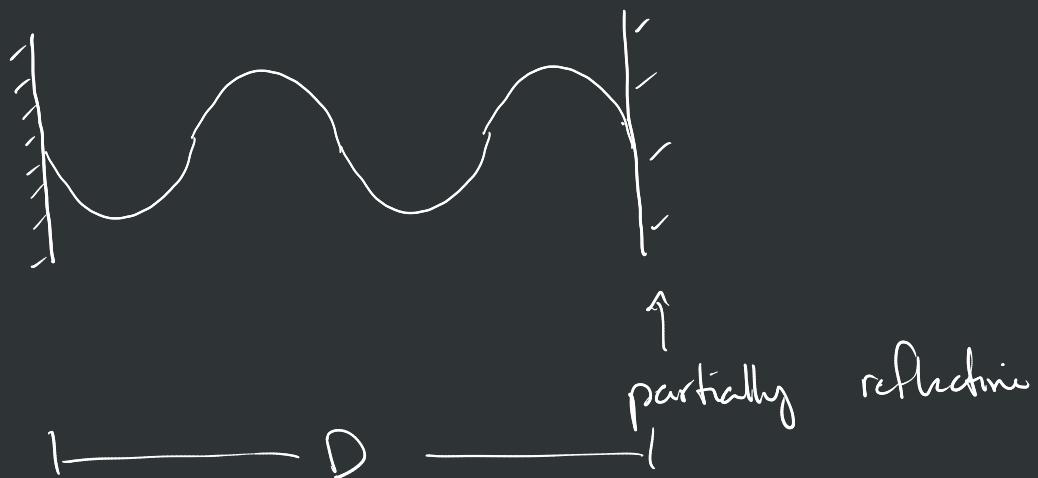
$$\omega t = 0, \pi, 2\pi, \dots \quad \omega t = m\pi \quad m = 0, \pm 1, \pm 2 \rightarrow \text{integer}$$

$$\omega = \frac{2\pi}{T}$$

$$t_{\max} = m \frac{T}{2}$$

$$\Delta t_{\max} = \frac{T}{2}$$

How do we do this?



$$D = m \left( \frac{\lambda}{2} \right)$$

number of nodes

$$m = 1, 2, 3, 4, \dots$$

What about waves of different frequency/wavelength?

If we have two waves, and the frequencies are close but not exact

↳ Frequency beatings

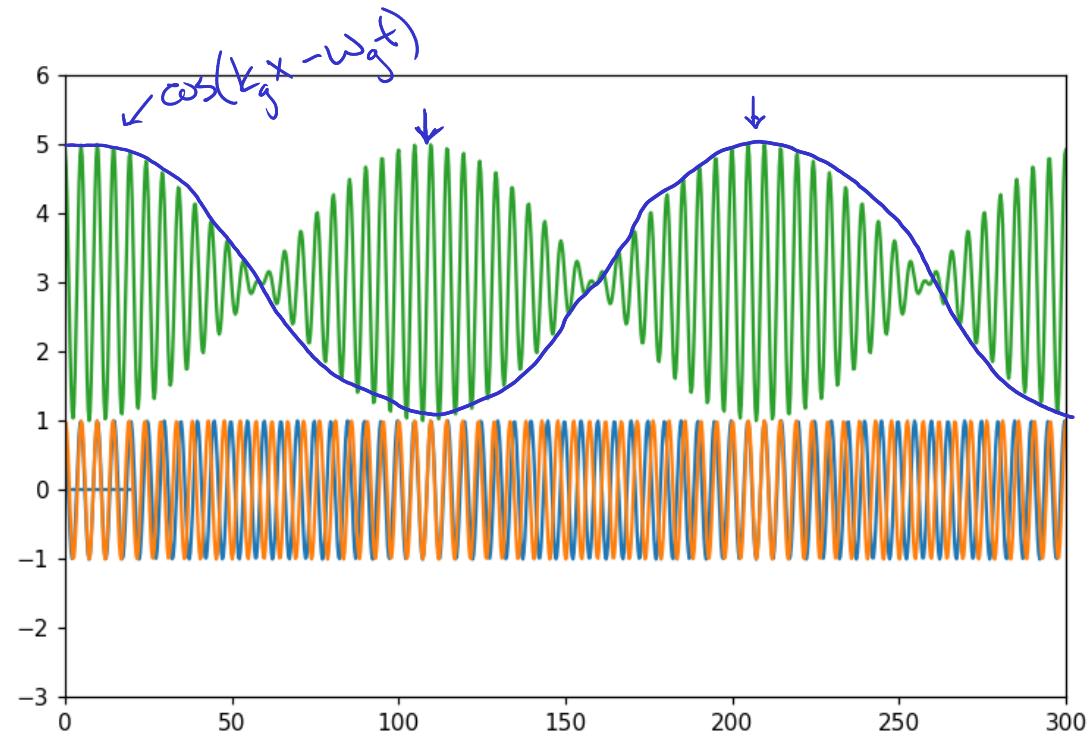
$$\frac{E_1 = E_0 \cos(k_1 x - \omega_1 t) + E_2 = E_0 \cos(k_2 x - \omega_2 t)}{E_R = 2E_0 \cos\left(\frac{(k_1 + k_2)}{2}x - \frac{(\omega_1 + \omega_2)}{2}t\right) \cdot \cos\left(\frac{(k_1 - k_2)}{2}x - \frac{(\omega_1 - \omega_2)}{2}t\right)}$$

$\left. \begin{array}{l} \cos \beta_1 + \cos \beta_2 = 2 \cos\left(\frac{1}{2}(\beta_1 + \beta_2)\right) \cos\left(\frac{1}{2}(\beta_1 - \beta_2)\right) \end{array} \right\}$

$$\frac{k_1 + k_2}{2} = k_p \leftarrow \text{phase wave number} \quad \frac{\omega_1 + \omega_2}{2} = \omega_p \leftarrow \begin{array}{l} \text{phase angular} \\ \text{frequency} \end{array}$$

$$k = \frac{2\pi}{\lambda} \quad \frac{k_1 - k_2}{2} = k_g \leftarrow \text{group wave number} \quad \frac{\omega_1 - \omega_2}{2} = \omega_g \leftarrow \begin{array}{l} \text{group angular} \\ \text{frequency} \end{array}$$

$$E_R = 2E_0 \underbrace{\cos(k_p x - \omega_p t)}_{\text{"carrier wave" or "signal"} \cdot \underbrace{\cos(k_g x - \omega_g t)}_{\text{"envelope" or "varying amplitude"}}$$



$$T_b = \frac{T_g}{2}$$

$\uparrow$                        $\downarrow$

$$\omega = \frac{2\pi}{T}$$

$$\frac{2\pi}{\omega_b} = \frac{2\pi}{2 \cdot \omega_g}$$

$$\omega_b = 2\omega_g$$

$$\omega_b = \frac{\mathcal{K}(\omega_1 - \omega_2)}{\mathcal{K}}$$

beat frequency  
is the difference  
in the two freq

$$\boxed{\omega_b = \omega_1 - \omega_2}$$

$$\boxed{f_b = f_1 - f_2}$$

Dispersion - in materials, EM waves of different frequencies travel w/ different speeds

$$\frac{c}{v} = n(\lambda)$$

$$n(\lambda) = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \dots \quad \begin{cases} \text{empirical formula} \\ \text{Cauchy's formula} \end{cases}$$

dispersion

$$\frac{dn}{d\lambda} = -\frac{2B}{\lambda^3} + \dots$$

ignore higher  
order terms

interpretation for normal materials

larger wavelengths  $\rightarrow$  smaller index

smaller wavelength  $\rightarrow$  larger index  
(higher frequency)

$\hookrightarrow$  higher frequencies  $\rightarrow$  lower speeds

$$V = \lambda \cdot v = \frac{\omega}{k}$$

$v_p \leftarrow$  phase velocity       $v_g \leftarrow$  group velocity

higher frequency carrier wave  
if these are close in value

$$v_p = \frac{\omega_p}{k_p} = \frac{\omega_1 + \omega_2}{k_1 + k_2} \approx \frac{\omega}{k}$$

lower frequency envelope

↳ group velocity

$$v_g = \frac{\omega_g}{k_g} = \frac{\omega_1 - \omega_2}{k_1 - k_2}$$

again, these are close together

$$v_g \approx \frac{d\omega}{dk}$$

$$V_p = \frac{\omega}{k} \Rightarrow \omega = V_p \cdot k$$

$$V_g = \frac{d(V_p \cdot k)}{dk}$$

$$V_g = V_p + k \frac{dV_p}{dk}$$

does the velocity of waves depend on wavelength?

— in a nondispersive medium, no

$$\frac{dV_p}{dk} = 0$$

$$\therefore V_g = V_p$$

— but, in a dispersive medium

$$V_p = \frac{c}{n} \quad \text{can be a function of } \lambda \text{ so is a function of } k$$

$$V_g = V_p + k \underbrace{\frac{dV_p}{dk}}_{\text{?}}$$

$$\frac{dV_p}{dk} = \frac{d(\frac{C}{n})}{dk} = C \frac{d(n^{-1})}{dk} = -C n^{-2} \frac{dn}{dk}$$

$$\frac{dV_p}{dk} = -\frac{C}{n^2} \cdot \frac{dn}{dk}$$

$$V_g = V_p + k \left( -\frac{C}{n^2} \frac{dn}{dk} \right) = \frac{C}{n} + k \left( -\frac{C}{n^2} \frac{dn}{dk} \right)$$

$\frac{C}{n}$

$$V_g = \frac{C}{n} \left( 1 - \frac{k}{n} \underbrace{\frac{dn}{dk}}_{\text{?}} \right)$$

transform to  $\lambda$  rather than  $k$

$$k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k}$$

$$\frac{dn}{dk} = \frac{dn}{d\lambda} \cdot \frac{d\lambda}{dk} = \frac{dn}{d\lambda} \cdot \underbrace{\frac{d(\frac{2\pi}{k})}{dk}}$$

$$\frac{dn}{dk} = -\frac{2\pi}{k^2} \cdot \frac{dn}{d\lambda} \quad -\frac{2\pi}{k^2}$$

$$\frac{dn}{dk} = -\frac{2\pi \cdot \lambda^2}{(2\pi)^2} \cdot \frac{dn}{d\lambda}$$

$$\frac{dn}{dk} = -\frac{\lambda^2}{2\pi} \cdot \frac{dn}{d\lambda}$$

$$V_g = \frac{c}{n} \left( 1 + \frac{2\pi}{\lambda} \cdot \frac{1}{n} \cdot \left( + \frac{\lambda^2}{2\pi} \right) \frac{dn}{d\lambda} \right)$$

$$V_g = \frac{c}{n} \left( 1 + \frac{\lambda}{n} \frac{dn}{d\lambda} \right)$$

for normal dispersion

$$\frac{dn}{d\lambda} = -\frac{2B}{\lambda^3}$$

$$\frac{dn}{d\lambda} < 0$$

$$V_g < V_p$$

$$\text{sinc}(x) = \frac{\sin(x)}{x}$$

HW Ch7: 1, 5, 8, 12, 15, 16, 18, 35, 38

simulation:

$$V = \frac{c}{n} \leftarrow$$

# Fourier Analysis

coherence - correlation between phases of light wave

↳ incoherent light  $\rightarrow$  random phase relationship

↳ coherent light  $\rightarrow$  constant phase relationship

longitudinal coherence  
↳ along the length

vs. lateral or spatial coherence

↳ along the width of the beam

$$\cos\alpha + \cos\beta \neq \cos\gamma \quad \gamma(\alpha, \beta)$$

$$= 2 \cos\left(\frac{\alpha+\beta}{2}\right) \cos\left(\frac{\alpha-\beta}{2}\right)$$

↑ not harmonic

Take a periodic but anharmonic function.  $\rightarrow f(t)$

$$\hookrightarrow T$$

Can be expressed as a sum of harmonic waves whose frequencies are multiples of  $\frac{2\pi}{T} = \omega$ .

$$f(t) = \sum_{m=0}^{\infty} a_m \cos(m\omega t) + \sum_{m=0}^{\infty} b_m \sin(m\omega t) \quad \left. \right\} \text{Fourier series} \rightarrow \text{this is possible because sines + cosines form a complete basis set.}$$

$m=0$ ,  $b_0$  does not matter

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t)$$

So, the question is what are  $a_m$  &  $b_m$ ?

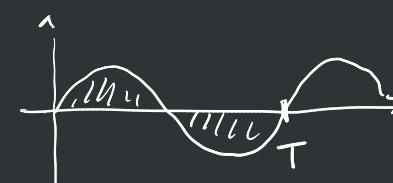
$$\int_0^T f(t) dt = \int_0^T \frac{a_0}{2} dt + \underbrace{\int_0^T \sum_{m=1}^{\infty} a_m \cos(m\omega t) dt}_{=0} + \int_0^T \text{also } 0$$

I know this function

$$= \frac{a_0}{2} \cdot T \Rightarrow a_0 = \frac{2}{T} \cdot \int_0^T f(t) dt \leftarrow \text{so that's not too bad!}$$

Many functions do this but some important ones are:

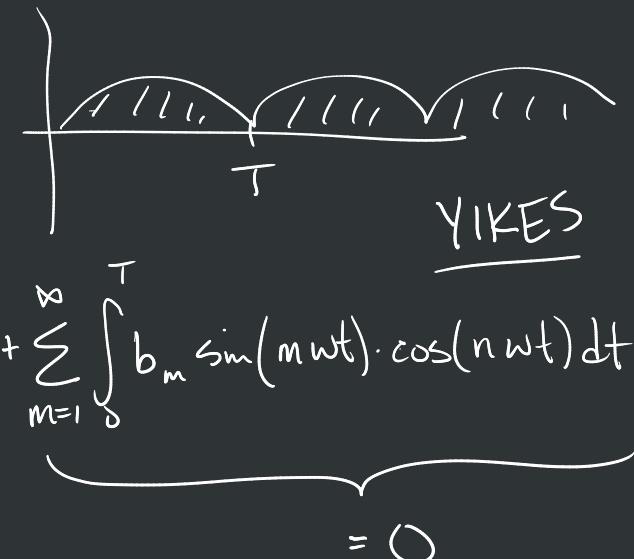
- Legendre polynomials
- Hermite polynomials
- Laguerre polynomials
- Bessel Functions



So now, multiply both sides by  $\cos(nwt)$   
 where  $n$  is an integer and then integrate  
 over a period.

$$\int_0^T f(t) \cdot \cos(nwt) dt = \underbrace{\int_0^T \frac{a_0}{2} \cos(nwt) dt}_= 0 + \sum_{m=1}^{\infty} \underbrace{\int_0^T a_m \cos(mwt) \cdot \cos(nwt) dt}_= 0 \quad m \neq n + \sum_{m=1}^{\infty} \underbrace{\int_0^T b_m \sin(mwt) \cdot \cos(nwt) dt}_= 0$$

$$= \frac{T}{2} a_n \quad m = n$$



$$\int_0^T f(t) \cos(nwt) dt = \frac{T}{2} a_n$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos(nwt) dt$$

So now we need to find  $b_m$ . Same strategy but multiply by  $\sin(nwt)$ .

(a)

$$\int_0^T f(t) \sin(n\omega t) dt = \underbrace{\int_0^T \frac{a_0}{2} \sin(n\omega t) dt}_0 + \underbrace{\sum_{m=1}^{\infty} \int_0^T a_m \cos(m\omega t) - \sin(n\omega t) dt}_0 + \underbrace{\sum_{m=1}^{\infty} \int_0^T b_m \sin(m\omega t) \cdot \sin(n\omega t) dt}_{= 0 \quad n \neq m}$$

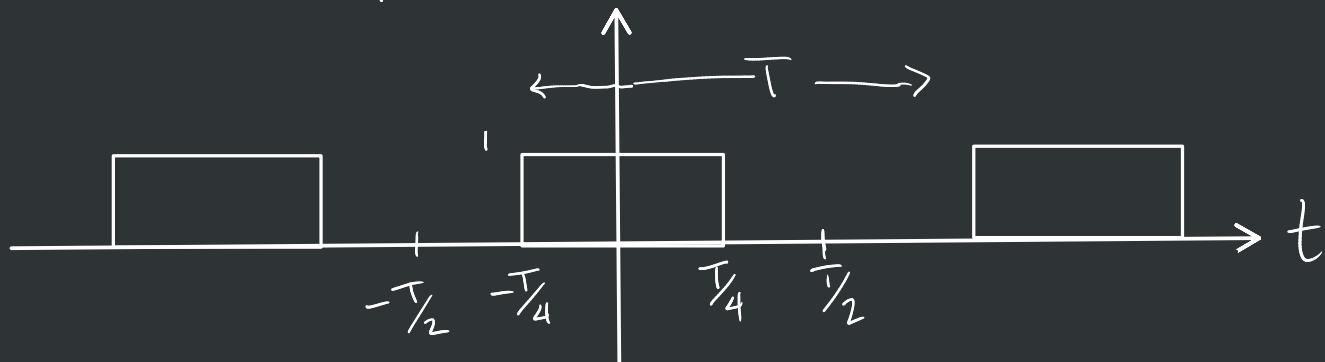
$$b_m = \frac{2}{T} \int_0^T f(t) \sin(m\omega t) dt$$

$$= \frac{T}{2} a_n \quad n = m$$

reminder to talk  
about Kronecker  
delta

lets try this!

rectified square wave



$$f(t) = \begin{cases} 0 & -T/2 < t < -T/4 \\ 1 & -T/4 < t < T/4 \\ 0 & T/4 < t < T/2 \end{cases}$$

$$a_0 = \frac{2}{T} \int_0^T f(t) dt = \frac{2}{T} \int_{-\frac{T}{4}}^{\frac{T}{4}} 1 \cdot dt = \frac{2}{T} \cdot t \Big|_{-\frac{T}{4}}^{\frac{T}{4}} = \frac{2}{T} \left[ \frac{T}{4} - \left( -\frac{T}{4} \right) \right] = \frac{2}{T} \cdot \frac{T}{2} = 1$$

$$a_0 = 1 \quad f(t) = \frac{1}{2} + \varepsilon \quad \{$$

$$a_m = \frac{2}{T} \int_0^T f(t) \cos(m\omega t) dt \Rightarrow a_m = \frac{2}{T} \int_{-\frac{T}{4}}^{\frac{T}{4}} 1 \cdot \cos(m\omega t) dt = \frac{2}{T} \cdot \frac{\sin(m\omega t)}{m\omega} \Big|_{-\frac{T}{4}}^{\frac{T}{4}}$$

$$a_m = \frac{2}{m\omega T} \cdot \left[ \underbrace{\sin\left(m\omega \frac{T}{4}\right)} - \underbrace{\sin\left(m\omega \left(-\frac{T}{4}\right)\right)} \right]$$

$$\sin\left(-m\omega \frac{T}{4}\right) = -\sin\left(m\omega \frac{T}{4}\right)$$

$$a_m = \frac{2}{m\omega T} \cdot 2 \sin\left(\frac{m\omega T}{4}\right)$$

$$\omega = \frac{2\pi}{T}$$

$$= \frac{4}{m\omega T} \cdot \sin\left(\frac{m\omega T}{4} \cdot \frac{2\pi}{T}\right)$$

$$a_m = \frac{4}{mwT} \cdot \sin\left(\frac{m\pi}{2}\right)$$

$m = 1, 2, 3, \dots$

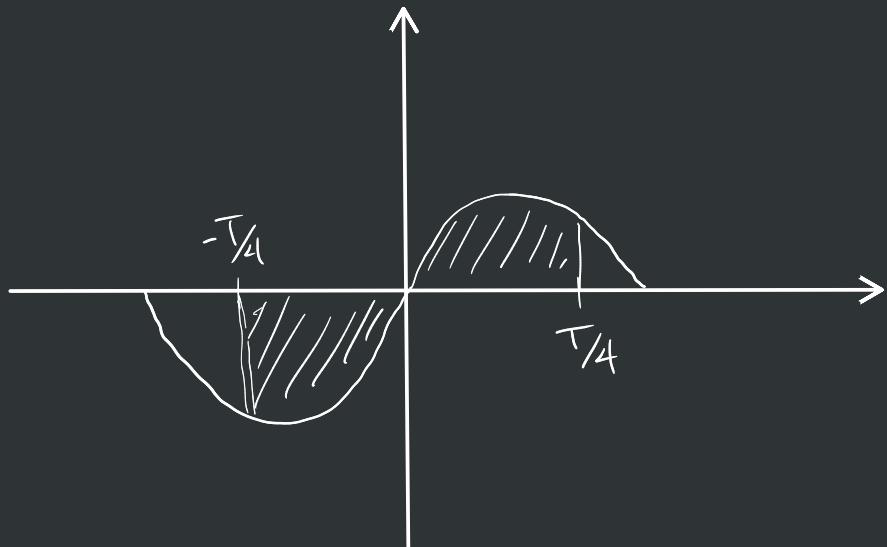
$$\sin\left(\frac{\pi}{2}\right), \sin\left(\frac{2\pi}{2}\right), \sin\left(\frac{3\pi}{2}\right),$$

1, 0, -1, 0, 1, 0, -1

odd m survive

$$b_m = \frac{2}{T} \int_0^T f(t) \sin(mwt) dt$$

$$b_m = \frac{2}{T} \int_{-\frac{T}{4}}^{\frac{T}{4}} 1 \cdot \sin(mwt) dt = 0$$



$$f(t) = \frac{1}{2} + \sum_{\substack{m=1 \\ \text{odd}}}^{\infty} \frac{4}{mwT} \cdot \sin\left(\frac{m\pi}{2}\right) \cos(mwt) \quad \leftarrow \text{show me this}$$

OK, now w/ complex notation.....

$$\operatorname{Re}\{z\} = \frac{z + z^*}{2}$$

$$f(t) = \sum_{m=0}^{\infty} a_m \cos(m\omega t) + \sum_{m=0}^{\infty} b_m \sin(m\omega t)$$

$$\frac{a + b_i + (a - b_i)}{2}$$

$$\cos(m\omega t) = \operatorname{Re}\left\{ e^{-im\omega t} \right\} = \frac{e^{-im\omega t} + e^{im\omega t}}{2}$$

$$\sin(m\omega t) = \operatorname{Im}\left\{ e^{-im\omega t} \right\} = \frac{e^{-im\omega t} - e^{im\omega t}}{2i}$$

$$\operatorname{Im}\{z\} = \frac{z - z^*}{2i}$$

$$f(t) = \sum_{m=0}^{\infty} a_m e^{-im\omega t} + \sum_{m=0}^{\infty} a_m e^{im\omega t} + \sum_{m=0}^{\infty} b_m e^{-im\omega t} + \sum_{m=0}^{\infty} b_m e^{im\omega t}$$

$$= \sum_{m=0}^{\infty} (a_m + b_m) e^{-im\omega t} + \sum_{m=0}^{\infty} (a_m + b_m) e^{im\omega t}$$

$\underbrace{C_m}_{m=-m'} \quad \underbrace{C_m}_{m=m'} \quad \leftarrow \text{dummy index change}$

$$= \sum_{m=0}^{\infty} C_m e^{-im\omega t} + \sum_{m'=0}^{-\infty} C_{m'} e^{-im' \omega t} \quad m = m'$$

$$= \sum_{m=0}^{\infty} C_m e^{-im\omega t} + \sum_{m=0}^{-\infty} C_m e^{-im\omega t}$$

$$f(t) = \sum_{m=-\infty}^{+\infty} C_m e^{-im\omega t}$$

Now, we need to find an expression for  $C_m$

$$\int_0^T f(t) e^{+im\omega t} dt = \sum_{m=-\infty}^{+\infty} C_m \underbrace{\int_0^T e^{-im\omega t} e^{+im\omega t} dt}_{= T} \quad \begin{aligned} &= T && \text{when } n \equiv m \\ &= 0 && \text{otherwise} \end{aligned}$$

$$= T \delta_{nm}$$

$$C_m = \frac{1}{T} \int_0^T f(t) e^{+im\omega t} dt$$

Kronicker delta



$$\delta_{nm} = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$$

Now, what if I stop adding sines/cosines or  $e^{i\omega t}$  in frequencies that are multiples of integers, but I add up all frequencies.

$$f(t) = \sum_{m=-\infty}^{\infty} C_m e^{-im\omega t} \cdot \underbrace{\Delta\omega}_{=1} \quad \rightarrow \quad f(t) = \int_{-\infty}^{\infty} g(\omega) \cdot e^{-i\omega t} d\omega$$

$\nearrow$   ~~$\times$~~   $g(\omega)$  is continuous

$$\Delta\omega = n\omega - (n-1)\omega$$

$$\omega = \frac{2\pi}{T}$$

$$\Delta\omega = \frac{2\pi}{T} \rightarrow \infty \leftarrow \underline{\text{not periodic}}$$



$$d\omega =$$

$f(t)$  is not periodic, so it is any function of  $t$ .

So how do we find  $g(\omega)$ ?

↳ employ orthogonality again.  
(same thing as before)

$$f(t) = \int_{-\infty}^{\infty} g(\omega) \cdot e^{-i\omega t} d\omega$$

$$\int_{-\infty}^{\infty} f(t) e^{+i\omega' t} dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\omega) \cdot e^{-i\omega t} \cdot e^{+i\omega' t} d\omega dt$$

$$\int_{-\infty}^{\infty} f(t) e^{+i\omega' t} dt = \int_{-\infty}^{\infty} g(\omega) \cdot \left[ \int_{-\infty}^{\infty} e^{-i\omega t} \cdot e^{+i\omega' t} dt \right] d\omega$$

Dirac Delta

$$\underbrace{2\pi}_{\sim} \cdot \delta(\omega - \omega')$$

$$\int_{-\infty}^{\infty} f(t) e^{+i\omega' t} dt = \underbrace{2\pi \int_{-\infty}^{\infty} g(\omega) \delta(\omega - \omega') d\omega}_{g(\omega')}$$

$$\delta(\omega - \omega') = \begin{cases} \infty & \omega = \omega' \\ 0 & \omega \neq \omega' \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(\omega - \omega') d\omega = 1$$

$$\int_{-\infty}^{\infty} f(t) e^{i\omega' t} dt = 2\pi \cdot g(\omega')$$

rewrite w/o the prime b/c

$$\int_{-\infty}^{\infty} f(t) e^{i\omega t} dt = 2\pi \cdot g(\omega)$$

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \cdot e^{i\omega t} dt$$

← Fourier Transform

$$f(t) = \int_{-\infty}^{\infty} g(\omega) \cdot e^{-i\omega t} dt$$

← Inverse Fourier Transform

$$\int_{-\infty}^{\infty} g(\omega) \cdot \delta(\omega - \omega') d\omega = g(\omega')$$

$$\omega = \frac{2\pi}{T} \rightarrow k$$

$$T \rightarrow \lambda$$

$$\boxed{\frac{2\pi}{\lambda} = k}$$

Wavenumber  
"spacial frequency"

$$f(x) = \int_{-\infty}^{\infty} g(k) e^{-ikx} dk \leftarrow \begin{matrix} \text{Inverse} \\ \text{Fourier} \\ \text{Transform} \end{matrix}$$

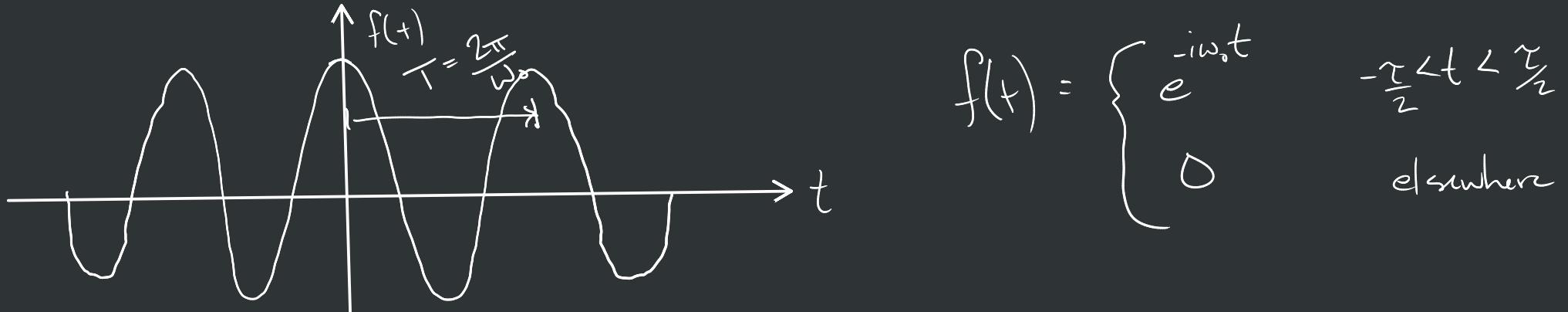
$$g(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{ikx} dx \leftarrow \begin{matrix} \text{Fourier} \\ \text{Transform} \end{matrix}$$

$$f(t) = \text{IFT}[g(\omega)] = \mathcal{F}^{-1}\{g(\omega)\}$$

$$g(\omega) = \text{FT}[f(t)] = \mathcal{F}\{f(t)\}$$

$$f(t) = \text{IFT}[\text{FT}[f(t)]]$$

$$g(\omega) = \text{FT}[\text{IFT}[g(\omega)]]$$



$$\xrightarrow{\tau_s}$$

frequency spectrum  $\rightarrow g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-i\omega_0 t} \cdot e^{i\omega t} \cdot dt$

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{it(\omega - \omega_0)} dt$$

$$= \frac{1}{2\pi} \left. \frac{1}{i(\omega - \omega_0)} \cdot e^{it(\omega - \omega_0)} \right|_{-\infty}^{+\infty}$$

$$\sin(\omega_0 t) = \frac{e^{-im\omega_0 t} - e^{im\omega_0 t}}{2i}$$

$$= \frac{1}{2\pi} \left. \frac{1}{i(\omega - \omega_0)} \left[ e^{i\frac{\pi}{2}(\omega - \omega_0)} - e^{-i\frac{\pi}{2}(\omega - \omega_0)} \right] \right.$$

$$= \frac{1}{\pi(\omega - \omega_0)} \underbrace{\frac{e^{i\frac{\pi}{2}(\omega - \omega_0)} - e^{-i\frac{\pi}{2}(\omega - \omega_0)}}{2i}}_{\sin\left(\frac{\pi}{2}(\omega - \omega_0)\right)}$$

$$g(\omega) = \frac{\sin\left(\frac{\pi}{2}(\omega - \omega_0)\right)}{\pi(\omega - \omega_0)} \begin{bmatrix} \frac{\pi}{2} \\ \frac{\pi}{2} \end{bmatrix}$$

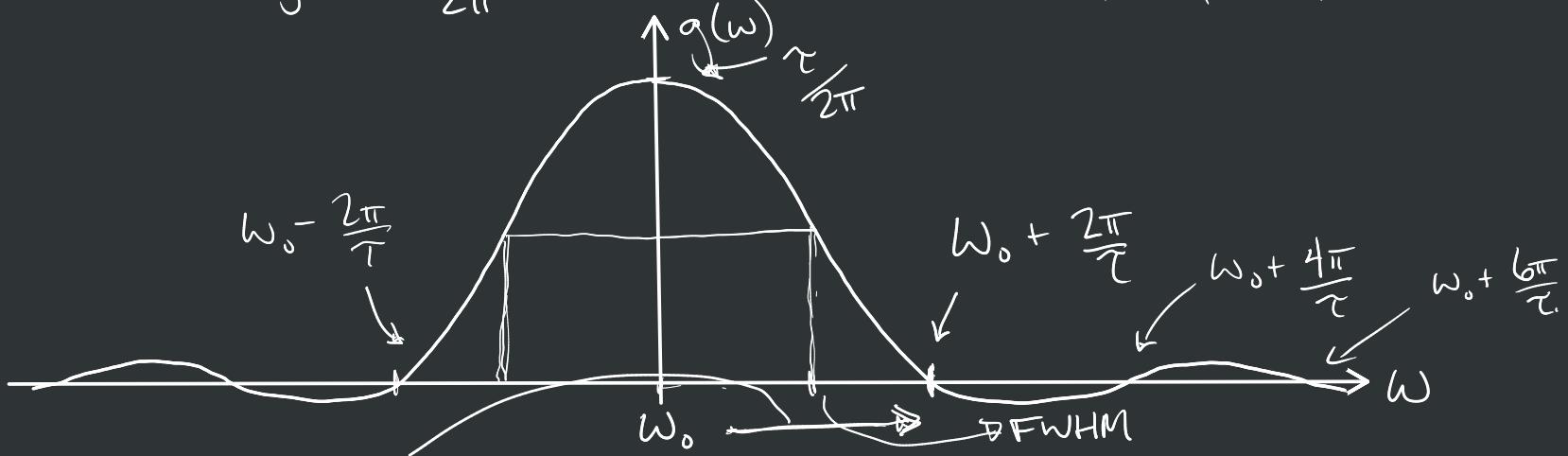
$$g(\omega) = \frac{\frac{\pi}{2} \sin\left(\frac{\pi}{2}(\omega - \omega_0)\right)}{\pi\left(\frac{\pi}{2}(\omega - \omega_0)\right)}$$

$$\frac{\sin(x)}{x} = \text{sinc}(x)$$

$$\text{sinc}(0) = 1$$

$$g(\omega) = \frac{\pi}{2\pi} \cdot \text{sinc}\left(\frac{\pi}{2}(\omega - \omega_0)\right)$$

$$\lim_{x \rightarrow 0} \left( \frac{\sin(x)}{x} \right) = 1$$



Width

$$O = \sin\left(\frac{\pi}{2}(\omega - \omega_0)\right)$$

$$O = \frac{\sin\left(\frac{\pi}{2}(\omega - \omega_0)\right)}{\frac{\pi}{2}(\omega - \omega_0)}$$

$$O = \sin\left(\frac{\pi}{2}(\omega - \omega_0)\right)$$

$$\frac{\pi}{2}(\omega - \omega_0) = n\pi \quad n = 0, 1, 2, \dots$$

$$\Delta\omega = \frac{2\pi}{T} \quad \omega = \frac{2\pi}{T}$$

$$\omega = 2\pi\nu$$

$$\Delta\omega = 2\pi \cdot \Delta\nu$$

or

$$\Delta\nu = \frac{1}{T}$$

$\sim$

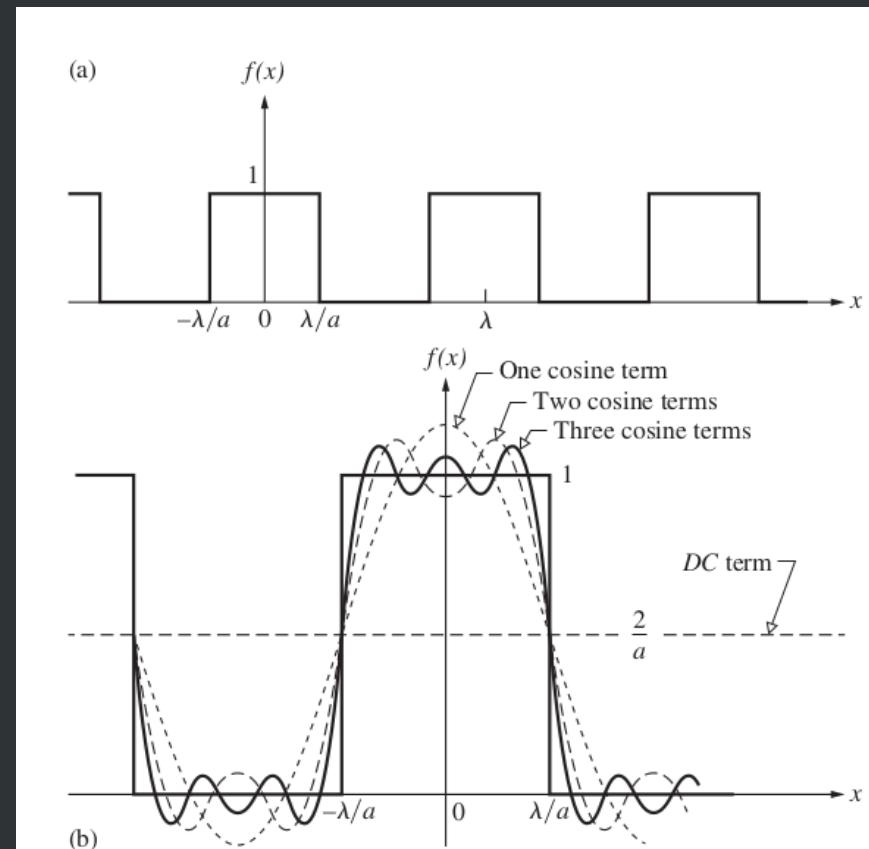
frequency    bandwidth

F/W: 43, 44, 45, 46, 47, 51, 52, 53

$$43) \int_0^{\lambda} \sin(akx) \cos(bkx) dx = 0 \rightarrow \int_0^T \sin(n\omega t) \cos(m\omega t) dt = 0 \quad \text{for any } n \text{ and } m$$

$$44 \quad f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t)$$

$$a_m = \frac{2}{T} \int_0^T f(t) \cdot \cos(m\omega t) dt$$



**Figure 7.35** An even periodic anharmonic function. In part (b) the area under the pulse is  $(2\lambda/a) \times 1$  and  $A_0 = (2/\lambda)(2\lambda/a) = 4/a$ . The DC term in the Fourier series is  $A_0/2 = 2/a$ .

