Chapter 7 - Superposition - when waves combine at the same place at the same, the displacements add together

$$\psi = \psi_1 + \psi_2$$

same frequency

$$E_1(x,t) = E_1 \cos(kx_1 - \omega t + \phi_1)$$

$$E_2(x,t) = E_2 \cos(k x_2 - \omega t + \phi_2)$$

different phase

$$\alpha_1 = kx_1 + \phi_1$$

$$\alpha_2 = kx_2 + \phi_2$$

phase difference $\rightarrow$ $\alpha_2 - \alpha_1 = k(x_2 - x_1) + (\phi_2 - \phi_1)$

what if : $\alpha_2 - \alpha_1 = 2\pi \cdot m$ } $\rightarrow$ even integer of π

any integer

$$E_R = E_1 + E_2 = E_1 \cos(\alpha_1 - wt) + E_2 \cos(\alpha_2 - wt)$$

$$= E_1 \cos(\alpha_1 - wt) + E_2 \cos(2\pi \cdot m + \alpha_1 - wt)$$

$$\cos(x) = \cos(x + 2\pi \cdot m)$$

$$= \underbrace{(E_1 + E_2)}_{\text{constructive interference}} \cos(\alpha_1 - wt)$$

But, what if $\alpha_2 - \alpha_1 = (2m-1)\pi$ → odd integers of π.

any integer

$$E_t = E_1 + E_2 = E_1 \cos(\alpha_1 - wt) + E_2 \cos(\alpha_2 - wt)$$

$$= E_1 \cos(\alpha_1 - wt) + E_2 \cos(\alpha_1 + (2m+1)\pi - wt)$$

$$-\cos x = \cos(x + (2m-1)\pi)$$

destructive interference

$$= (E_1 - E_2)\cos(\alpha_1 - wt)$$

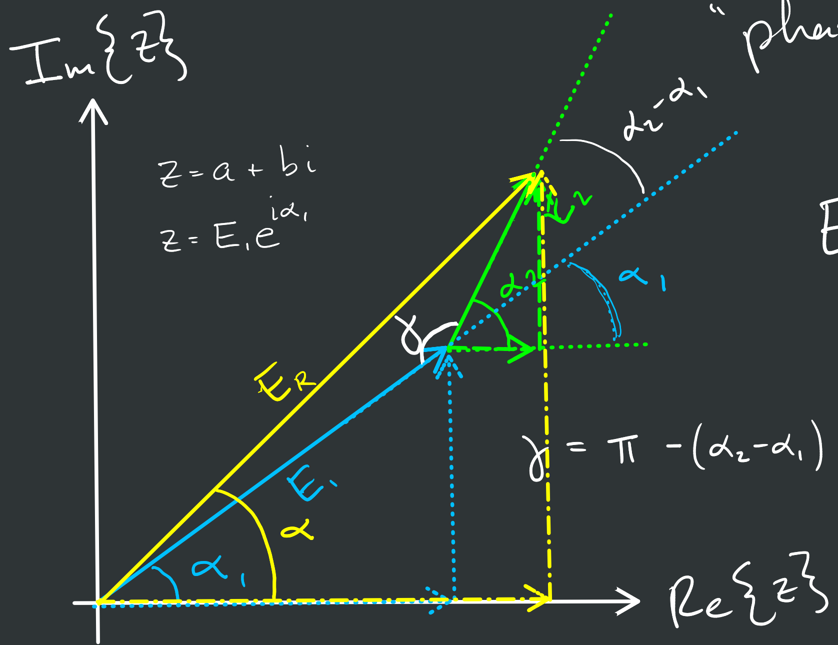
What if any other phase shift?

↳ switch from trig to complex notation

$$E_R = E_1 + E_2 = \mathrm{Re}\left\{E_1 e^{i(\alpha_1 - \omega t)} + E_2 e^{i(\alpha_2 - \omega t)}\right\}$$

$$= \mathrm{Re}\left\{e^{-i\omega t}\underbrace{\left(E_1 e^{i\alpha_1} + E_2 e^{i\alpha_2}\right)}\right\}$$

"phasor diagram" → treat complex number like vectors



$$E_R(x,t) = \mathrm{Re}\left\{E_R e^{i(\alpha - \omega t)}\right\}$$

? ?

$$E_R^2 = E_1^2 + E_2^2 - 2E_1E_2\underbrace{\cos\gamma}$$

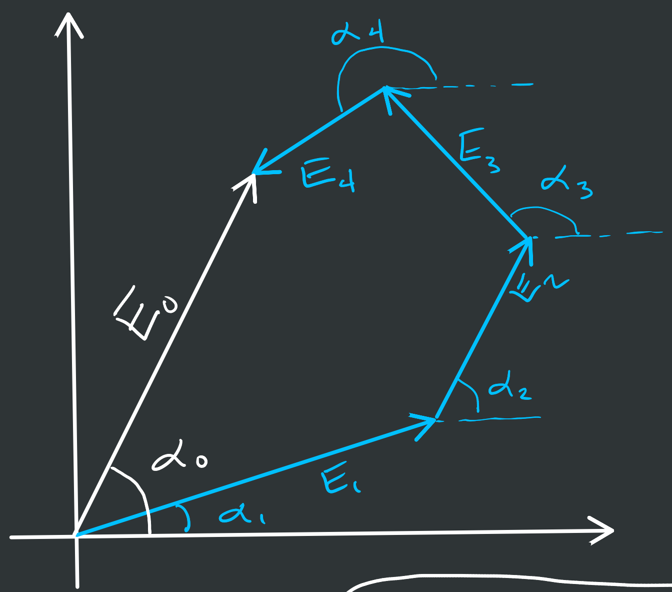
$$\cos(\pi - \alpha_2 + \alpha_1)$$

$$-\cos(-\alpha_2 + \alpha_1)$$

$$-\cos(\alpha_1 - \alpha_2)$$

$$\rightarrow \quad E_R^2 = E_1^2 + E_2^2 + 2E_1E_2\cos(\alpha_1 - \alpha_2)$$

$$\rightarrow \quad \tan\alpha = \frac{E_1\sin\alpha_1 + E_2\sin\alpha_2}{E_1\cos\alpha_1 + E_2\cos\alpha_2}$$



$$\tan\alpha_0 = \frac{\sum_{i=1}^{N} E_i\sin\alpha_i}{\sum_{i=1}^{N} E_i\cos\alpha_i}$$

$$E_0^2 = \underbrace{\left(\sum_{i=1}^{N} E_i\cos\alpha_i\right)^2} + \left(\sum_{i=1}^{N} E_i\sin\alpha_i\right)^2$$

$$\left(\sum_{i=1}^{N} E_i\cos\alpha_i\right)^2 = \sum_{i}^{N} E_i^2\cos^2\alpha_i + \underbrace{\sum_{i}^{N} 2E_i\cos\alpha_i \sum_{j>i}^{N} E_j\cos\alpha_j}_{2\sum_{i}\sum_{j>1} E_iE_j\cos\alpha_i\cos\alpha_j}$$

$$(a + b + c + d)^2$$

$$E_0^2 = \underbrace{\sum_i^N E_i^2 \cos^2\alpha_i + \sum_i^N E_i^2 \sin^2\alpha_i}_{\sum_i^N E_i^2 \underbrace{(\cos^2\alpha_i + \sin^2\alpha_i)}_{1}} + \underbrace{2\sum_i \sum_{j>i} E_i E_j \cos\alpha_i \cos\alpha_j + 2\sum_i \sum_{j>i} E_i E_j \sin\alpha_i \sin\alpha_j}_{2\sum_i^N \sum_{j>i}^N E_i E_j \underbrace{(\cos\alpha_i \cos\alpha_j + \sin\alpha_i \sin\alpha_j)}_{\cos(\alpha_j - \alpha_i)}}$$

$$E_0^2 = \sum_i^N E_i^2 + 2\sum_i \sum_{j>i} E_i E_j \cos(\alpha_j - \alpha_i)$$

if all sources are equal in magnitude $\rightarrow E_i = E_1$

if all sources are random phases and short durations ($< 10\,ns$)

$$\sum_i \sum_{j>i} E_i E_j \cos(\alpha_j - \cos\alpha_i) \rightarrow 0$$

$$\rightarrow E_0^2 = N E_1^2 \Rightarrow E_0 = \sqrt{N} \cdot E_1$$

$$E_0 \propto \sqrt{N}$$

irradiance $\rightarrow I = \frac{1}{2}\epsilon_0 c E_0^2$

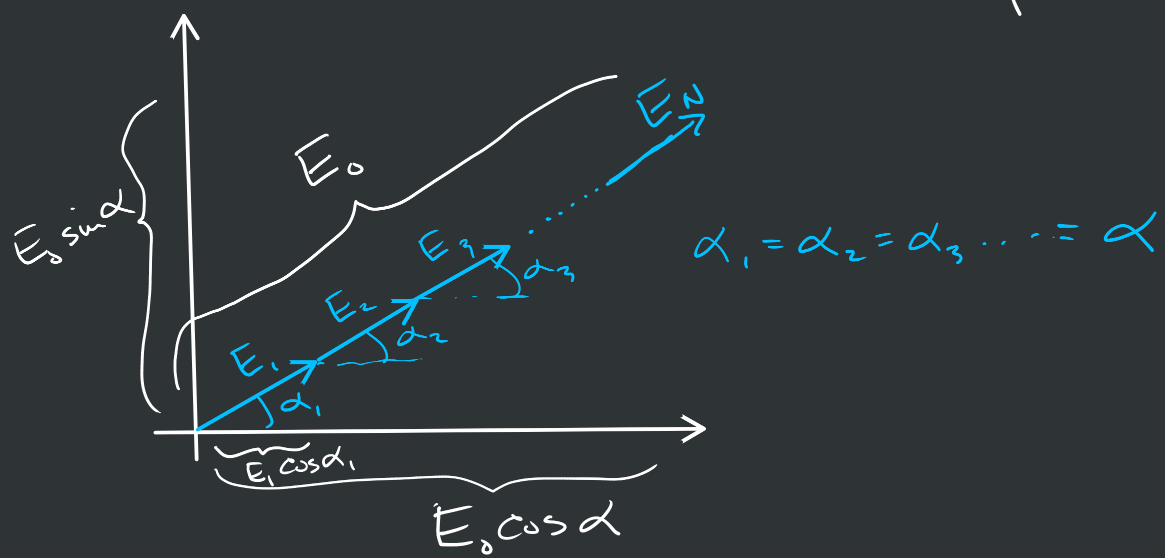
$\boxed{I \propto N}$ experimentally verified ✓

if phases are not random, but coherent
↳ same frequency + waveform
↳ same phase

$$E_0^2 = \sum_i^N E_i^2 + 2\sum_i \sum_{j>i} E_i E_j \underbrace{\cos(\alpha_j - \alpha_i)}_{\cos(0) = 1}$$

$$E_0^2 = \sum_i^N E_i^2 + 2\sum_i \sum_{j>i} E_i E_j$$

→ if sources are equal in magnitude



$$E_0^2 = (NE_1 \cos\alpha)^2 + (NE_1 \sin\alpha)^2$$

$$E_0^2 = N^2 E_1^2 \underbrace{(\cos^2\alpha + \sin^2\alpha)}_{1}$$

$$E_0 = N E_1$$

$$I = \frac{1}{2}\epsilon_0 c N^2 E_1^2$$

$$I \propto N^2$$

$$\frac{I_{coherent}}{I_{random}} = \frac{N^2}{N} = N$$

Standing wave → interference of a wave with is own reflection

as two waves: $E_1 = E_0 \sin(-kx + \omega t)$ ← to the right

$E_2 = E_0 \sin(kx + \omega t)$ ← to the left

put in a phase shift to account for the reflection at the boundary → $-\phi_R$

$$E_R = E_0\left(\sin(\underbrace{-kx + \omega t}_{A}) + \sin(\underbrace{kx + \omega t - \phi_R}_{B})\right)$$

$$\sin A + \sin B = 2\sin\left(\tfrac{1}{2}(A+B)\right)\cos\left(\tfrac{1}{2}(A-B)\right)$$

$$E_R = E_0 \cdot 2 \sin\left(\frac{1}{2}\left(\cancel{-kx}+\omega t + \cancel{kx} + \omega t - \phi_R\right)\right) \cos\left(\frac{1}{2}\left(-kx + \cancel{\omega t} - kx - \cancel{\omega t} + \phi_R\right)\right)$$

$$E_R = 2E_0 \sin\left(\omega t - \frac{\phi_R}{2}\right) \cos\left(-kx + \phi_R/2\right)$$

lets take the important case of $\phi_R = \pi$

$$E_R = 2E_0 \sin\left(\omega t - \frac{\pi}{2}\right) \cos\left(-kx + \pi/2\right)$$

$$= 2E_0\left(-\cos(\omega t) \cdot \sin(kx)\right)$$

$$E_R = \underbrace{2E_0 \cdot \sin(kx)}_{\text{spacial amplitude}} \underbrace{\cos(\omega t)}_{}$$

→ variation of that amplitude as time goes by.

spacial amplitude

↳ the places where this are equal to 0 will $\underline{\text{always}}$ be zero

$\underbrace{\text{will always be zero}}$

positions are nodes

where do the nodes appear?

$$kx = m\pi \qquad m = 0, \pm 1, \pm 2, \dots \rightarrow \text{is an integer}$$

$$k = \frac{2\pi}{\lambda} \qquad\qquad \omega = \frac{2\pi}{T}$$

$$\frac{2\cancel{\pi}}{\lambda} \cdot x = m\cancel{\pi}$$

$$x = m \cdot \frac{\lambda}{2} \leftarrow$$ positions where amplitude is always zero

positions of the nodes

$$x_2 - x_1 = \frac{2\lambda}{2} - \frac{1\lambda}{2}$$

$$\Delta x = \frac{\lambda}{2} \leftarrow$$ distance between adjacent nodes

$$\lambda = 2\Delta x$$

when do the maxima occur?

$$\cos(\omega t) = 1$$

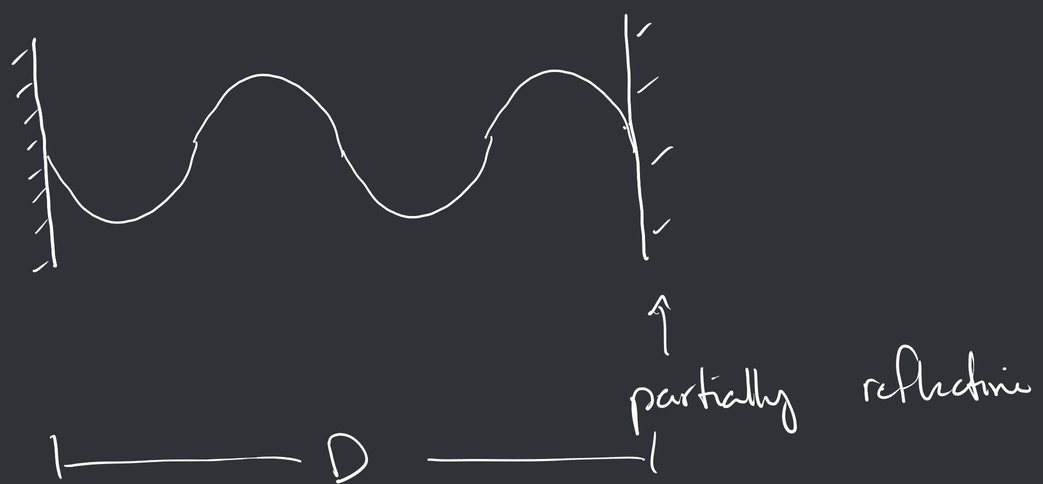
$$\omega t = 0, \pi, 2\pi, \dots \qquad \omega t = m\pi \qquad m = 0, \pm 1, \pm 2 \rightarrow \text{integer}$$

$$\omega = \frac{2\pi}{T}$$

$$t_{max} = m\frac{T}{2}$$

$$\Delta t_{max} = \frac{T}{2}$$

How do we do this?



$$D = m\left(\frac{\lambda}{2}\right)$$

number of nodes

$m = 1, 2, 3, 4 \ldots$

What about waves of different frequency / wavelength?

If we have two waves, and the frequencies are close but not exact

↳ Frequency beating

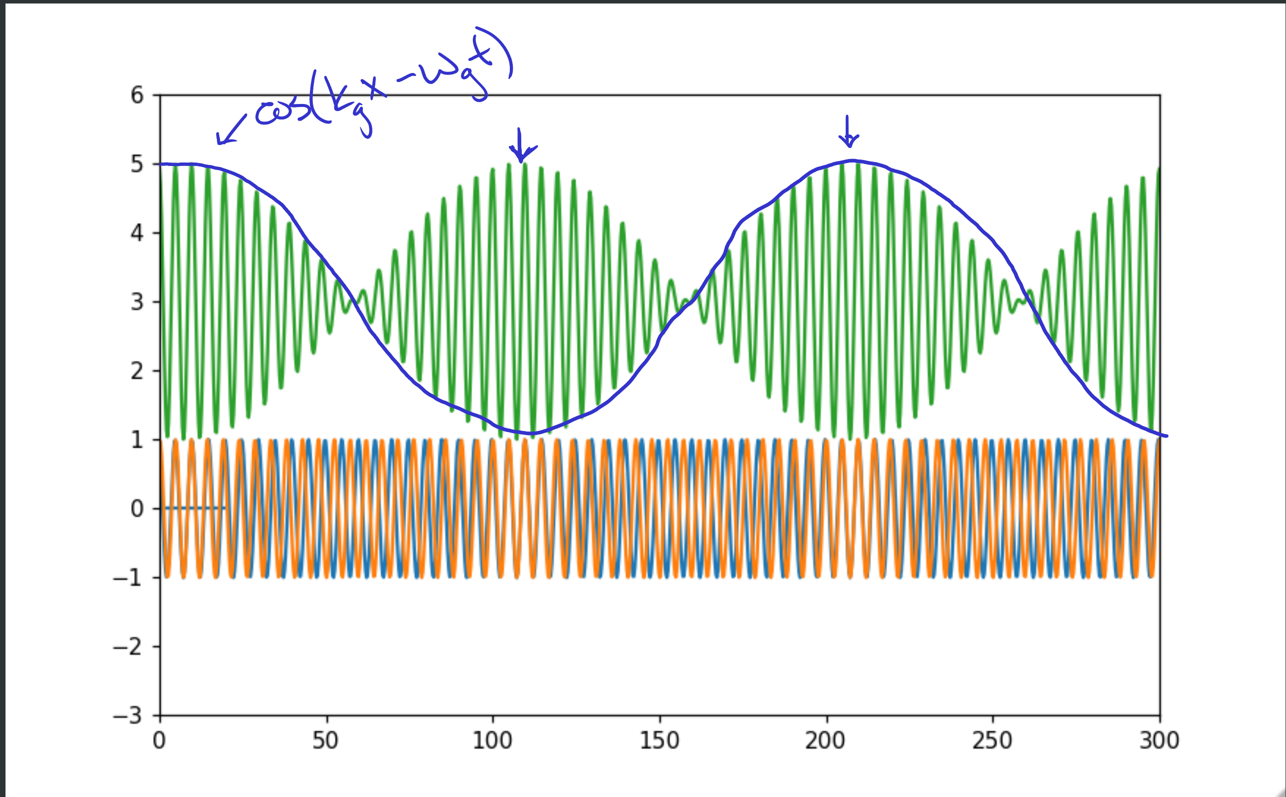
$$\left.\begin{aligned} E_1 &= E_0 \cos(k_1 x - \omega_1 t) \\ + \quad E_2 &= E_0 \cos(k_2 x - \omega_2 t) \end{aligned}\right\} \cos\beta_1 + \cos\beta_2 = 2\cos\left(\tfrac{1}{2}(\beta_1+\beta_2)\right)\cos\left(\tfrac{1}{2}(\beta_1-\beta_2)\right)$$

$$E_R = 2E_0 \cos\left(\frac{(k_1+k_2)}{2}x - \frac{(\omega_1+\omega_2)}{2}t\right) \cdot \cos\left(\frac{(k_1-k_2)}{2}x - \frac{(\omega_1-\omega_2)}{2}t\right)$$

$\frac{k_1+k_2}{2} = k_p$ ← phase wave number $\qquad \frac{\omega_1+\omega_2}{2} = \omega_p$ ← phase angular frequency

$k = \frac{2\pi}{\lambda}$ $\qquad \frac{k_1-k_2}{2} = k_g$ ← group wave number $\qquad \frac{\omega_1-\omega_2}{2} = \omega_g$ ← group angular frequency

$$E_R = 2E_0 \underbrace{\cos(k_p x - \omega_p t)}_{\text{"carrier wave" or "signal"}} \cdot \underbrace{\cos(k_g x - \omega_g t)}_{\text{"envelope" or "varying amplitude"}}$$



$$T_b = \frac{T_g}{2}$$

$$\omega = \frac{2\pi}{T}$$

$$\frac{2\pi}{\omega_b} = \frac{2\pi}{2\cdot\omega_g}$$

$$\omega_b = 2\omega_g$$

$$\omega_b = \frac{\cancel{2}\cdot(\omega_1 - \omega_2)}{\cancel{2}}$$

beat frequency is the difference in the two freq

$$\boxed{\omega_b = \omega_1 - \omega_2}$$

$$\boxed{f_b = f_1 - f_2}$$

Dispersion - in materials, EM waves of different frequencies travel w/ different speeds

$$\frac{c}{v} = n(\lambda) \qquad n(\lambda) = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \dots \Bigg\}$$

empirical formula
Cauchy's formula

dispersion $\displaystyle \frac{dn}{d\lambda} = -\frac{2B}{\lambda^3} + \underbrace{\dots}_{}$

ignore higher order terms

interpretation for normal materials

larger wavelengths → smaller index

smaller wavelength → larger index
(higher frequency)

↳ higher frequencies → lower speeds

$V = \lambda \cdot \nu = \dfrac{\omega}{k}$ $V_p \leftarrow$ phase velocity $V_g \leftarrow$ group velocity

higher frequency carrier wave

if these are close in value

$$V_p = \frac{\omega_p}{k_p} = \frac{\omega_1 + \omega_2}{k_1 + k_2} \cong \frac{\omega}{k}$$

lower frequency envelope

↳ group velocity

again, these are close together

$$V_g = \frac{\omega_g}{k_g} = \frac{\omega_1 - \omega_2}{k_1 - k_2}$$

$$V_g \cong \frac{d\omega}{dk}$$

$$V_p = \frac{\omega}{k} \Rightarrow \omega = V_p \cdot k$$

$$V_g = \frac{d(V_p \cdot k)}{dk}$$

$$V_g = V_p + k\frac{dV_p}{dk}$$

does the velocity of waves depend on wavelength?

— in a nondispersive medium, no

$$\frac{dv}{dk} = 0$$

$$\therefore \; V_g = V_p$$

— but, in a dispersive medium

$$V_p = \frac{c}{n}$$

can be a function of λ so is a function of k

$$v_g = v_p + k \underbrace{\frac{dv_p}{dk}}$$

$$\frac{dv_p}{dk} = \frac{d\left(\frac{c}{n}\right)}{dk} = c\,\frac{d(n^{-1})}{dk} = -c\,n^{-2}\,\frac{dn}{dk}$$

$$\frac{dv_p}{dk} = -\frac{c}{n^2}\cdot\frac{dn}{dk}$$

$$v_g = \underbrace{v_p}_{\frac{c}{n}} + k\left(-\frac{c}{n^2}\frac{dn}{dk}\right) = \frac{c}{n} + k\left(-\frac{c}{n^2}\frac{dn}{dk}\right)$$

$$v_g = \frac{c}{n}\left(1 - \frac{k}{n}\underbrace{\frac{dn}{dk}}\right)$$

transform to λ rather than k

$$k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k}$$

$$\frac{dn}{dk} = \frac{dn}{d\lambda}\cdot\frac{d\lambda}{dk} = \frac{dn}{d\lambda}\cdot\underbrace{\frac{d\left(\frac{2\pi}{k}\right)}{dk}}_{-\frac{2\pi}{k^2}}$$

$$\frac{dn}{dk} = -\frac{2\pi}{k^2}\cdot\frac{dn}{d\lambda}$$

$$\frac{dn}{dk} = \frac{-\cancel{2\pi}\cdot\lambda^2}{(2\pi)^{\cancel{2}}}\cdot\frac{dn}{d\lambda}$$

$$\frac{dn}{dk} = -\frac{\lambda^2}{2\pi}\cdot\frac{dn}{d\lambda}$$

$$V_g = \frac{c}{n}\left(1 + \frac{\cancel{2\pi}}{\cancel{\lambda}}\cdot\frac{1}{n}\cdot\left(+\frac{\lambda^{\cancel{2}}}{\cancel{2\pi}}\right)\frac{dn}{d\lambda}\right)$$

$$\boxed{V_g = \frac{c}{n}\left(1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right)}$$

for normal dispersion

$$\frac{dn}{d\lambda} = -\frac{2B}{\lambda^3}$$

$$\frac{dn}{d\lambda} < 0$$

$$V_g < V_p$$

$$\text{sinc}(x) = \frac{\sin(x)}{x}$$

HW Ch 7: 1, 5, 8, 12, 15, 16, 18, 35, 38

Fourier Analysis

Coherence - correlation between phases of light wave

↳ incoherent light → random phase relationship

↳ coherent light → constant phase relationship

↳ longitudinal coherence vs. lateral or spacial coherence

↳ along the length

↳ along the width of the beam

$$\cos\alpha + \cos\beta \neq \cos\gamma \qquad \gamma(\alpha, \beta)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

↑ not harmonic

Take a periodic $\to T$ but anharmonic function. $\to f(t)$

$\hookrightarrow$ can be expressed as a sum of harmonic waves whose frequencies are multiples of $\frac{2\pi}{T} = \omega$.

$$f(t) = \sum_{m=0}^{\infty} a_m \cdot \cos(m \cdot \omega \cdot t) + \sum_{m=0}^{\infty} b_m \sin(m\omega t)$$

} Fourier series $\to$ this is possible because sines + cosines form a complete basis set.

$m = 0$, b_0 does not matter

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t)$$

Many functions do this but some important ones are:

- Legendre polynomials
- Hermite polynomials
- Laguerre polynomials
- Bessel Functions

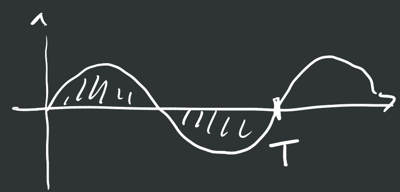
So, the question is what are a_m + b_m?

$$\int_0^T f(t)\,dt = \int_0^T \frac{a_0}{2}\,dt + \underbrace{\int_0^T \sum_{m=1}^{\infty} a_m \cos(m\omega t)\,dt}_{=0} + \int \text{also } 0$$

$\uparrow$ I know this function

$$= \frac{a_0}{2} \cdot T \Rightarrow a_0 = \frac{2}{T} \cdot \int_0^T f(t)\,dt$$

$\leftarrow$ so that's not too bad!



So now, multiply both sides by $\cos(n\omega t)$ where n is an integer and then integrate over a period.

YIKES

$$\int_0^T f(t)\cdot\cos(n\omega t)\,dt = \underbrace{\int_0^T \frac{a_0}{2}\cos(n\omega t)\,dt}_{=0} + \underbrace{\sum_{m=1}^{\infty}\int_0^T a_m \cos(m\omega t)\cdot\cos(n\omega t)\,dt}_{=0} + \underbrace{\sum_{m=1}^{\infty}\int_0^T b_m \sin(m\omega t)\cdot\cos(n\omega t)\,dt}_{=0}$$

$= 0 \quad m \neq n$

$= \frac{T}{2} a_n \quad m = n$

$$\int_0^T f(t)\cos(n\omega t) = \frac{T}{2} a_n$$

$$a_m = \frac{2}{T}\int_0^T f(t)\cos(m\omega t)\,dt$$

So now we need to find b_m. Same strategy but multiply by $\sin(n\omega t)$.

$$\int_0^T f(t)\cdot\sin(n\omega t)\,dt = \underbrace{\int_0^T \frac{a_0}{2}\sin(n\omega t)\,dt}_{0} + \underbrace{\sum_{m=1}^{\infty}\int_0^T a_m\cos(m\omega t)\cdot\sin(n\omega t)\,dt}_{0} + \underbrace{\sum_{m=1}^{\infty}\int_0^T b_m\sin(m\omega t)\cdot\sin(n\omega t)\,dt}$$

$$= 0 \quad n \neq m$$

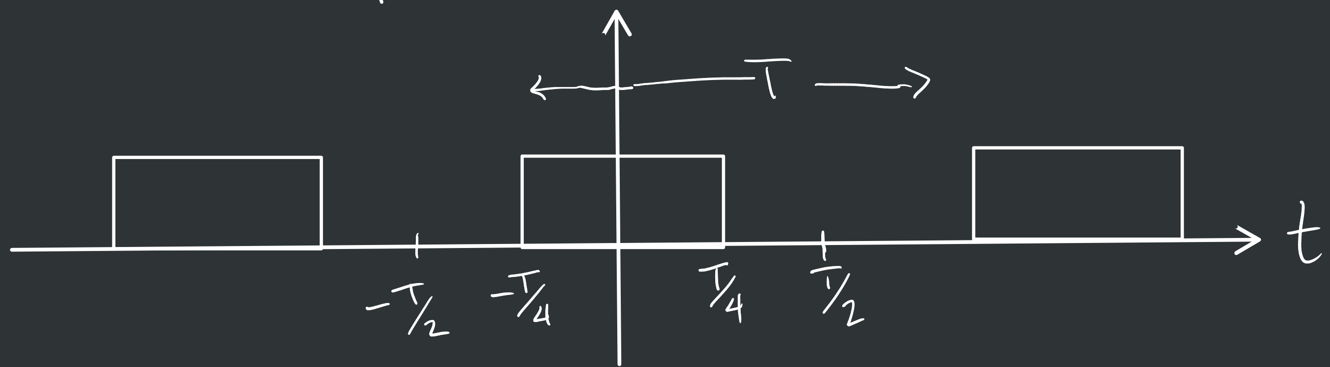
$$= \frac{T}{2}a_n \quad n = m$$

$$b_m = \frac{2}{T}\int_0^T f(t)\sin(m\omega t)\,dt$$

reminder to talk about Kronicker delta

lets try this!

recticified square wave



$$f(t) = \begin{cases} 0 & -T/2 < t < -T/4 \\ 1 & -T/4 < t < T/4 \\ 0 & T/4 < t < T/2 \end{cases}$$

$$a_0 = \frac{2}{T} \int f(t)\, dt = \frac{2}{T} \int_{-T/4}^{T/4} 1 \cdot dt$$

a_m

b_m