

# Chapter 0 - Mathematical Review

## Vectors

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = x\hat{i} + y\hat{j} + z\hat{k}$$

unit vectors

$$\vec{E} = \frac{kq}{r^2} \hat{r} = \frac{kq\vec{r}}{r^3} \left. \vphantom{\frac{kq\vec{r}}{r^3}} \right\} \text{fine, but incomplete} \rightarrow \text{it assumes the charge is located at the origin}$$

$$\left[ \frac{N}{C} \right]$$

more general

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q(\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^3}$$

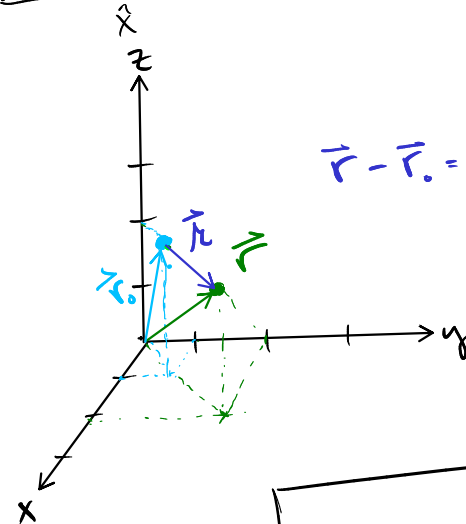
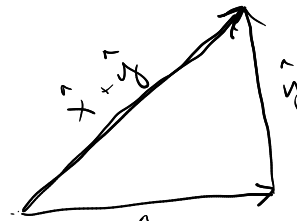
## Example 0.1

$$\vec{r} = (2\hat{x} + 2\hat{y} + 2\hat{z})$$

$$\vec{r}_0 = (1\hat{x} + 1\hat{y} + 2\hat{z})$$

$$\vec{r} - \vec{r}_0 = 1\hat{x} + 1\hat{y} + 0\hat{z} = \hat{x} + \hat{y}$$

$$|\vec{r} - \vec{r}_0|^3 = |\hat{x} + \hat{y}|^3 = (\sqrt{2})^3 = 2^{3/2} = 2\sqrt{2}$$



$$\vec{r} - \vec{r}_0 = \vec{r} \Rightarrow \vec{r}_0 + \vec{r} = \vec{r}$$

$$\vec{E}(\vec{r}) = \frac{q(\hat{x} + \hat{y})}{8\pi\epsilon_0 \sqrt{2}}$$

$$\frac{\vec{r}}{|\vec{r}|} = \hat{r}$$

Day 01  
C = Coulomb 260112 M

$$9 \cdot 10^9 \frac{Nm^2}{C^2}$$

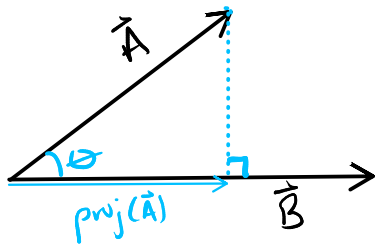
$$k = \frac{1}{4\pi\epsilon_0}$$

$$8.85 \cdot 10^{-12} \frac{C^2}{Nm^2}$$

$\vec{r} \Rightarrow$  location where we want to know the electric field

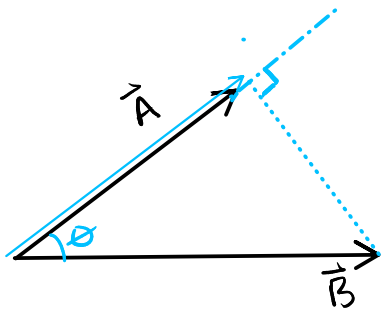
$\vec{r}_0 = \vec{r}_{\text{naught}}$   
location of the charge

## Dot Product (inner product)



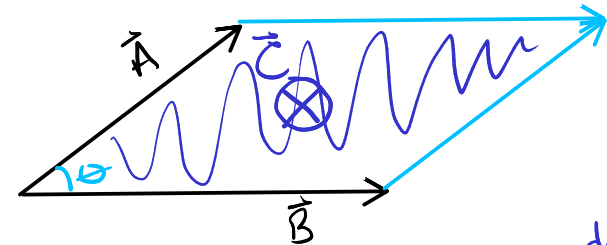
$$\begin{aligned}\vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y + A_z B_z \\ &= |\vec{A}| |\vec{B}| \cos \theta\end{aligned}$$

scalar



## Cross Product (vector product) / Day 02 260124W

(perp. to both  $\vec{A}$  &  $\vec{B}$ )

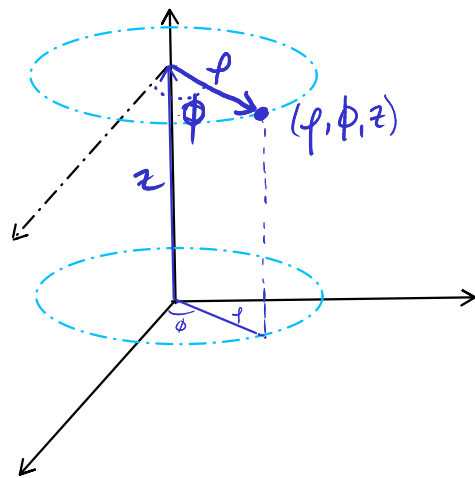


$$\underbrace{\vec{C}}_{\text{vector}} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \leftarrow \text{determinant}$$

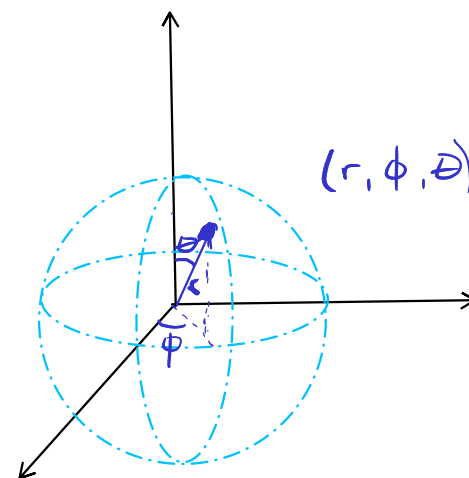
$$\begin{aligned}&= \hat{x}(A_y B_z - A_z B_y) \\ &\quad - \hat{y}(A_x B_z - A_z B_x) \\ &\quad + \hat{z}(A_x B_y - A_y B_x)\end{aligned}$$

$$|\vec{C}| = |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta$$

Cylindrical Coords.



Spherical Coords.



Gradient: of a scalar function  
(directional derivative)

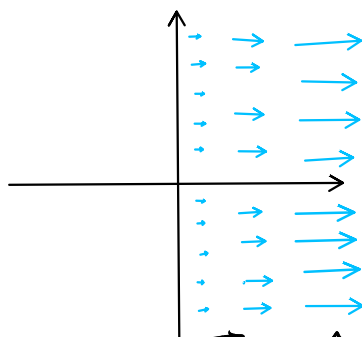
$$\vec{\nabla} f(x, y, z) = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}$$

↑  
scalar  
function

vector result  
points in the  
direction of steepest  
decline

Divergence: of a vector function

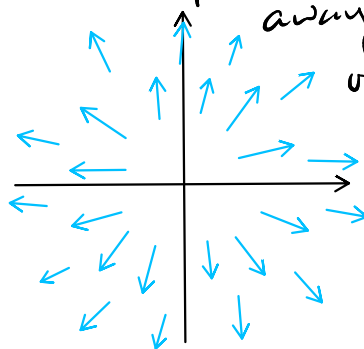
$$\vec{\nabla} \cdot \vec{E} = \underbrace{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}_{\text{scalar}}$$



$$\vec{E} = x \hat{x}$$

$$\vec{\nabla} \cdot \vec{E} = 1$$

uniform field pointed radially  
away from  
origin

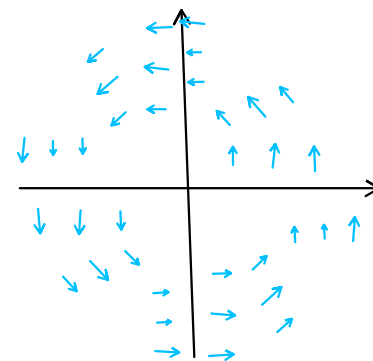


Curl: of a vector function

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \hat{x} \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) - \hat{y} \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) + \hat{z} \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

vector



Day 03  
260116F

Second Derivatives  $\rightarrow$  Laplacian

$$\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

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$$\nabla^2 \vec{E} = \left( \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \right) \hat{x}$$

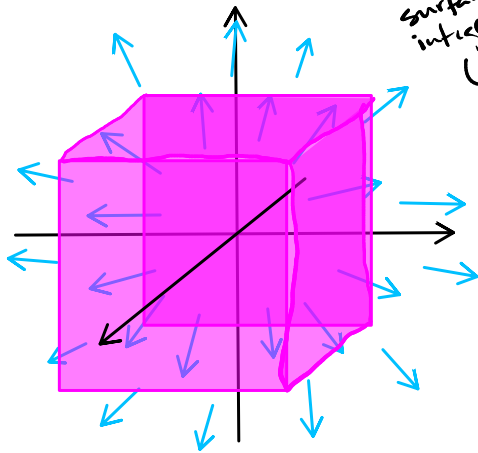
$$\left( \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \right) \hat{y}$$

$$\left( \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) \hat{z}$$

$$\vec{E}(x, y, z) = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$$

$\rightarrow \nabla^2 \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) \leftarrow \text{true in any coordinate system}$

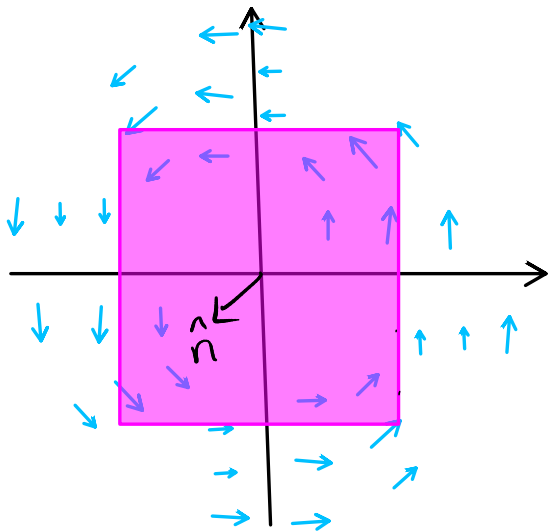
## Divergence Theorem



- closed surface integral  
 $\oint_S \vec{E} \cdot \hat{n} da = \int_V \vec{\nabla} \cdot \vec{E} dv$

direction of the surface  
 break up into surfaces  
 break up into volumes

## Stokes Theorem



$$\int_S (\vec{\nabla} \times \vec{E}) \cdot \hat{n} da = \oint_C \vec{E} \cdot d\vec{\ell}$$

# Complex Numbers

Day 04  
260121 W

imaginary number  $\rightarrow i = \sqrt{-1}$

sum of a real number + an imaginary number

$$\tilde{z} = a + bi$$

$$\operatorname{Re}\{a+bi\} = a$$

$$\operatorname{Im}\{a+bi\} = b$$

$$|\tilde{z}|^2 = |z|^2 = a^2 + b^2$$

$$a = |\tilde{z}| \cos \theta$$

$$b = |\tilde{z}| \sin \theta$$

Now:

$$\tilde{z} = |\tilde{z}| (\cos \theta + i \sin \theta)$$

So:

$$\tilde{z} = |\tilde{z}| \cdot e^{i\theta}$$

complex function  $f + g$

$$\operatorname{Re}\{f\} + \operatorname{Re}\{g\} = \operatorname{Re}\{f+g\}$$

$$\frac{d}{dx} \operatorname{Re}\{f\} = \operatorname{Re}\left\{\frac{df}{dx}\right\}$$

$$\int \operatorname{Re}\{f\} dx = \operatorname{Re}\left\{\int f dx\right\}$$

Taylor Series:

$$f(x) = f(x_0) + \underbrace{\frac{1}{1!} (x-x_0) \cdot \frac{df}{dx} \Big|_{x=x_0}}_{\text{linear term}} + \underbrace{\frac{1}{2!} (x-x_0)^2 \frac{d^2f}{dx^2} \Big|_{x=x_0}}_{\text{quadratic term}} + \dots$$

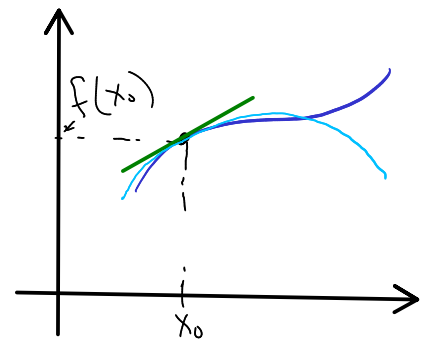
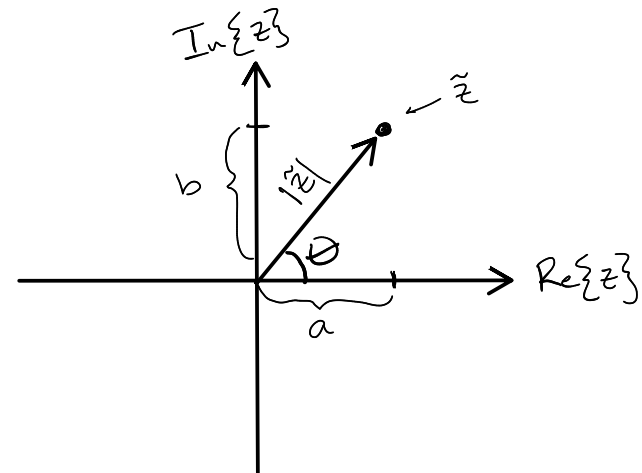
$$\cos \phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \dots$$

$$i \sin \phi = i\phi - \frac{i\phi^3}{3!} + \frac{i\phi^5}{5!} - \dots$$

$$e^{i\phi} = 1 + i\phi - \frac{\phi^2}{2!} - \frac{i\phi^3}{3!} + \frac{\phi^4}{4!} + \dots$$

$$e^{i\phi} = \cos \phi + i \sin \phi \quad \text{Euler's identity}$$

Complex Plane



$$e^{i\pi} = -1$$

$$e^{i\pi} + 1 = 0$$

## Complex Conjugate

$$\tilde{z}^* = a - bi \quad \text{or} \quad \tilde{z}^* = |\tilde{z}| e^{-i\theta}$$

$$\tilde{z} \cdot \tilde{z}^* = |\tilde{z}|^2$$

$$\tilde{z} \cdot \tilde{z}^* = |\tilde{z}| e^{i\theta} |\tilde{z}| e^{-i\theta}$$

$$\tilde{z} \cdot \tilde{z}^* = |\tilde{z}|^2$$

$$\operatorname{Re}\{\tilde{z}\} = \frac{\tilde{z} + \tilde{z}^*}{2} \quad \rightarrow \quad \frac{e^{i\theta} + e^{-i\theta}}{2} = \underline{\underline{\cos\theta}} = \operatorname{Re}\{e^{i\theta}\}$$

$$\operatorname{Im}\{\tilde{z}\} = \frac{\tilde{z} - \tilde{z}^*}{2i}$$

$$\tilde{z} + \tilde{z}^* = A + Bi + A - Bi$$

$$= 2A$$

$$2 \cdot \operatorname{Re}\{\tilde{z}\}$$

HW3: 0.14, 17, 18, 20

0.14 |  $z_1 = 1 - i$     $z_2 = 3 + 4i$

$$\begin{aligned} \text{a) } z_1 - z_2 &= (1 - i) - (3 + 4i) \\ &= 1 - i - 3 - 4i \end{aligned}$$

$$z_1 - z_2 = \underline{\underline{-2 - 5i}} = z_3 = \underline{\underline{|\tilde{z}| e^{i\theta}}} \quad \left. \begin{array}{l} \swarrow \text{magnitude} \\ \nwarrow \text{phase} \end{array} \right\} \text{polar form}$$

$$\text{b) } \frac{z_1}{z_2} = \frac{1-i}{3+4i} \cdot \frac{(3-4i)}{(3-4i)} = \quad \rightarrow \text{polar form}$$

0.17 Show  $\text{Re}\{\tilde{A}\} \times \text{Re}\{\tilde{B}\} = \frac{AB + A^*B}{4} + \text{C.C.} = \frac{AB + A^*B}{4} + \frac{A^*B^* + AB^*}{4}$

?  $\tilde{A} = A_1 + A_2 i$   
 $\tilde{A}^* = A_1 - A_2 i$

$= \frac{AB + A^*B + A^*B^* + AB^*}{4}$

$= \frac{AB + AB^* + A^*B + A^*B^*}{4}$

$= \frac{A(B + B^*) + A^*(B + B^*)}{4}$

$= \frac{(A + A^*)(B + B^*)}{4}$

$= \frac{2\text{Re}\{A\} \cdot 2\text{Re}\{B\}}{4}$

$= \text{Re}\{A\} \cdot \text{Re}\{B\}$

0.18  $E_0 = |E_0| e^{i\delta_E}$   $B_0 = |B_0| e^{i\delta_B}$

$4 \cdot \text{Re}\{E_0 e^{i(kz - \omega t)}\} \text{Re}\{B_0 e^{i(kz - \omega t)}\} =$

$|E_0| e^{i\delta_E} e^{i(kz - \omega t)} \cdot |B_0| e^{i\delta_B} e^{i(kz - \omega t)}$

$+ |E_0| e^{i\delta_E} e^{i(kz - \omega t)} \cdot |B_0| e^{-i\delta_B} e^{-i(kz - \omega t)}$

$+ |E_0| e^{-i\delta_E} e^{-i(kz - \omega t)} \cdot |B_0| e^{i\delta_B} e^{i(kz - \omega t)}$

$+ |E_0| e^{-i\delta_E} e^{-i(kz - \omega t)} \cdot |B_0| e^{-i\delta_B} e^{-i(kz - \omega t)}$

$\left. \begin{array}{l} \\ \\ \\ \end{array} \right\} E_0 B_0^* + E_0^* B_0$

$\text{Re}\{\tilde{A}\} \times \text{Re}\{\tilde{B}\} = \frac{AB + AB^* + A^*B + A^*B^*}{4}$

$$\text{Re}\{E_0 e^{i(kz - \omega t)}\} \text{Re}\{B_0 e^{i(kz - \omega t)}\} = \frac{1}{4}(E_0 B_0^* + E_0^* B_0)$$

$$+\frac{1}{4}|E_0| e^{i\delta_E} e^{i(kz - \omega t)} \cdot |B_0| e^{i\delta_B} e^{i(kz - \omega t)}$$

$$+\frac{1}{4}|E_0| e^{-i\delta_E} e^{-i(kz - \omega t)} \cdot |B_0| e^{-i\delta_B} e^{-i(kz - \omega t)}$$

$$= \frac{1}{4}(E_0 B_0^* + E_0^* B_0)$$

$$+\frac{1}{4}|E_0||B_0| \left( e^{i\delta_E} e^{i\delta_B} e^{2i(kz - \omega t)} + e^{-i\delta_E} e^{-i\delta_B} e^{-2i(kz - \omega t)} \right)$$

$$= \frac{1}{4}(E_0 B_0^* + E_0^* B_0)$$

$$+\frac{1}{4}|E_0||B_0| \left( e^{i(2(kz - \omega t) + \delta_E + \delta_B)} + e^{-i(2(kz - \omega t) + \delta_E + \delta_B)} \right)$$

$$= \frac{1}{4}(E_0 B_0^* + E_0^* B_0)$$

$$+\frac{1}{4}|E_0||B_0| \left( 2 \cdot \cos[2(kz - \omega t) + \delta_E + \delta_B] \right)$$

$$= \frac{1}{4}(E_0 B_0^* + E_0^* B_0) + \frac{1}{2}|E_0||B_0| \cdot \cos[2(kz - \omega t) + \delta_E + \delta_B]$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin\theta$$

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

P0.20  $A \cos(\omega t) + 2A \sin(\omega t + \frac{\pi}{4}) \rightarrow$  write as a single phase shifted cosine

$$\underbrace{A \cos(\omega t) + 2A \sin(\omega t + \frac{\pi}{4})}_{\substack{\text{amplitude} \\ \searrow \\ B}} \underbrace{\phantom{A \cos(\omega t) + 2A \sin(\omega t + \frac{\pi}{4})}}_{\substack{\text{phase} \\ \searrow \\ \phi}} \rightarrow B \cos(\omega t + \phi)$$

if  $\cos \alpha = \operatorname{Re}\{e^{i\alpha}\}$   $\operatorname{Re}\{e^{i\alpha}\} = \{\cos \alpha + i \sin \alpha\} \operatorname{Re} = \cos \alpha$

$$A \cos(\omega t) = A \cdot \operatorname{Re}\{e^{i\omega t}\}$$

if  $\sin(\beta) = \operatorname{Re}\{-ie^{i\beta}\}$

$$2A \sin(\omega t + \frac{\pi}{4}) = 2A \cdot \operatorname{Re}\{-ie^{i(\omega t + \frac{\pi}{4})}\}$$

So now:

$$A \cdot \operatorname{Re}\{e^{i\omega t}\} + 2A \cdot \operatorname{Re}\{-ie^{i(\omega t + \frac{\pi}{4})}\}$$

$$\operatorname{Re}\{A e^{i\omega t} - 2A i e^{i(\omega t + \frac{\pi}{4})}\} \leftarrow$$

$$\operatorname{Re}\{A(e^{i\omega t} - 2i e^{i\omega t} e^{i\frac{\pi}{4}})\}$$

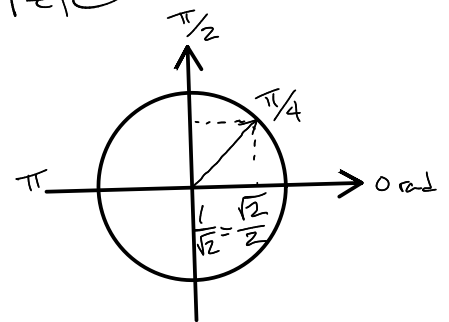
$$\operatorname{Re}\{A e^{i\omega t} (1 - 2i e^{i\frac{\pi}{4}})\}$$

$$\hookrightarrow e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{i\frac{\pi}{4}} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}$$

$$z = |z| e^{i\theta}$$



$$\operatorname{Re}\{(\underline{\hspace{1cm}}) e^{i(\omega t + \underline{\hspace{1cm}})}\}$$

$\uparrow$  amplitude  $B$ 
 $\uparrow$  phase  $\phi$

$$\operatorname{Re}\{Ae^{i\omega t}(1 - \cancel{2}i(\frac{\sqrt{2}}{\cancel{2}}(1+i)))\}$$

$$\operatorname{Re}\{Ae^{i\omega t}(1 - \sqrt{2}(i-1))\}$$

$$\begin{aligned} &1 - i\sqrt{2} + \sqrt{2} \\ &((1+\sqrt{2}) - i\sqrt{2}) \end{aligned}$$

$$\Rightarrow B \cos(\omega t + \phi)$$

$$\phi = 0$$

$$|z| = \sqrt{z \cdot z^*}$$

$$z = A((1+\sqrt{2}) - i\sqrt{2})$$

$$|z| = \sqrt{A((1+\sqrt{2}) - i\sqrt{2}) \cdot A((1+\sqrt{2}) + i\sqrt{2})}$$

$$= \sqrt{A^2[(1+\sqrt{2})^2 + (\sqrt{2})^2]}$$

$$= A\sqrt{1 + 2\sqrt{2} + 2 + 2}$$

$$= A\sqrt{5 + 2\sqrt{2}}$$