

# Chapter 2 - Plane Waves + Index of Refraction

Z60208 M  
Day 12

plane wave:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \phi)$$

"wave vector"  $\vec{k} = k \hat{u} = \frac{2\pi}{\lambda_{\text{vac}}} \hat{u}$   
 "k"  
 direction of propagation

angular frequency  $\rightarrow \omega = \frac{2\pi c}{\lambda_{\text{vac}}} = 2\pi\nu$

$k + \omega$  are related just like  $\lambda + \nu$

$$c = \frac{\lambda_{\text{vac}}}{T} = \lambda_{\text{vac}} \nu = \frac{\omega}{k}$$

$\curvearrowleft$   
 this relationship  
 is known as the  
 "dispersion relation"

"kappa"  $\downarrow$  wavenumber  
 $\frac{1}{\lambda_{\text{vac}}} = K = [\text{cm}^{-1}]$

$c = v_{\text{vac}}$   $\leftarrow$  velocity of light in vacuum

$\nu \nu \leftarrow$  "nu"  
 $\curvearrowleft$  frequency

similarly for magnetic field:

$$\vec{B}(\vec{r}, t) = \vec{B}_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \phi)$$

$$\underbrace{\vec{B}_0 = \frac{\vec{k} \times \vec{E}_0}{\omega}}, \text{ so } \vec{B}_0 \text{ is not independent, but is determined by other parameters.}$$

$$\vec{B}_0 \perp \vec{k} \perp \vec{E}$$

also think about magnitude

$$B_0 = \frac{k E_0}{\omega} = \frac{E_0}{c} \leftarrow \begin{array}{l} \text{since this is so large} \\ \text{we will focus on the } \vec{E} \text{ field.} \end{array}$$

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complex plane waves:

$$\vec{E}(\vec{r}, t) = \operatorname{Re} \left\{ \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right\}$$

$$\vec{E}_0 = \vec{E}_0 e^{i\phi} \leftarrow \text{phase shift}$$

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

Summarize some facts that we know now:

$\lambda \rightarrow$  wavelength

$T \rightarrow$  period

$\gamma \rightarrow$  frequency ( $\frac{1}{T}$ )

$k \rightarrow$  wave vector ( $\frac{2\pi}{\lambda}$ )

$\omega \rightarrow$  angular frequency ( $\frac{2\pi}{T}$ )

$$V = \frac{\omega}{k}$$

$$V = \frac{\lambda}{T} = \lambda \cdot \gamma \leftarrow \text{any wave}$$

$$c = \lambda_{\text{vac}} \cdot \gamma$$

$$\epsilon_0 \mu_0 = \frac{1}{c^2} \Rightarrow c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

permeability  
of  
free space  $\rightarrow \epsilon_0 = 8.85 \cdot 10^{-12} \frac{C^2}{Nm^2}$

permittivity  
of  
free space  $\rightarrow \mu_0 = 4\pi \cdot 10^{-7} \frac{T}{Am^2}$

Speed of light in matter

$\hookrightarrow$  light slows down in materials

$$\frac{c}{v} = n \leftarrow \text{index of refraction}$$

$n=1 \leftarrow$  vacuum

$n=1.0003 \leftarrow$  air

$n=1.33 \leftarrow$  water

$n=1.5 \leftarrow$  glass

$$\nabla^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \frac{k^2}{\omega^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

Q1: what is light? ✓

Q2: where does light come from? ✓

Q3: can the speed of light change? see Q6

Q4: what affects the brightness of light?

Q5: how do human perceive light amount + color

Q6: why does light slow down in materials

Q3

Q7: what is reflection

$\lambda \cdot v = V = \frac{c}{n}$

constant as  $n$  increases  
 $\downarrow$   
 gets smaller as  $n$  increases

$$\lambda_{\text{vac}} \cdot v = c \Leftrightarrow c = n \cdot v$$

$\uparrow$   
 $\lambda \cdot v$

$$\lambda_{\text{vac}} \cdot v = n \cdot \lambda \cdot v$$

$\approx$  wavelength in material

$$\lambda = \frac{\lambda_{\text{vac}}}{n}$$

}  $\lambda$  will be smaller than  $\lambda_{\text{vac}}$  since  $n > 1$

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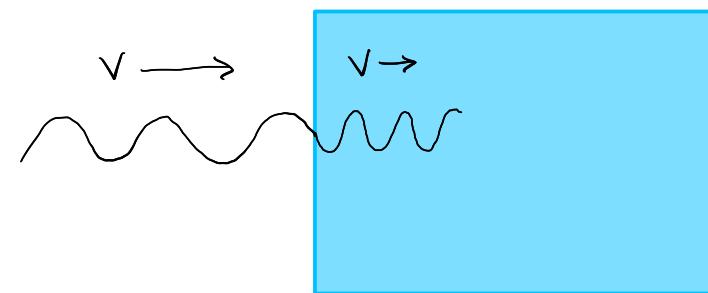
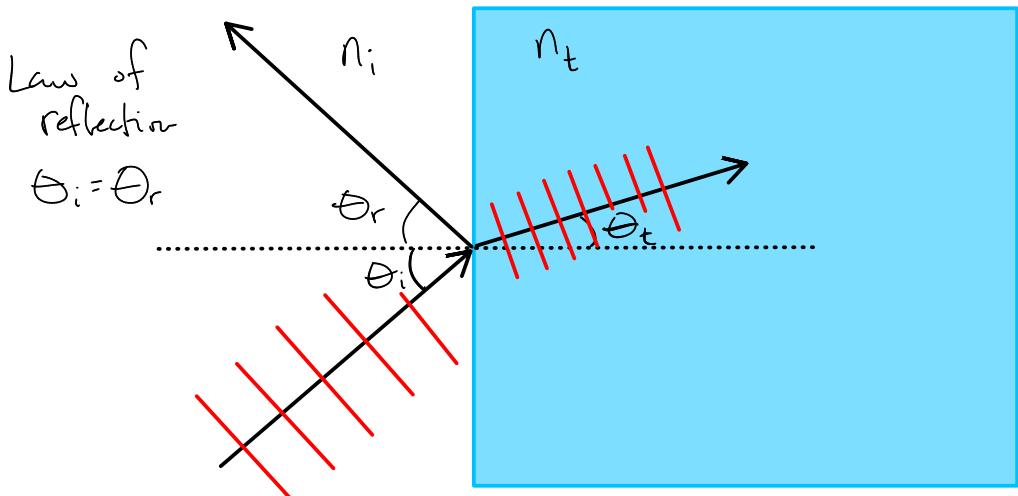


Table 23.1 Indices of Refraction for  $\lambda = 589.3 \text{ nm}$  in Vacuum (at  $20^\circ\text{C}$  Unless Otherwise Noted)

Material	Index
<b>Solids</b>	
Ice (at $0^\circ\text{C}$ )	1.309
Fluorite	1.434
Fused quartz	1.458
Polystyrene	1.49
Lucite	1.5
Plexiglas	1.51
Crown glass	1.517
Plate glass	1.523
Sodium chloride	1.544
Light flint glass	1.58
Dense flint glass	1.655
Sapphire	1.77
Zircon	1.923
Diamond	2.419
Titanium dioxide	2.9
Gallium phosphide	3.5
<b>Liquids</b>	
Water	1.333
Acetone	1.36
Ethyl alcohol	1.361
Carbon tetrachloride	1.461
Glycerin	1.473
Sugar solution (80%)	1.49
Benzene	1.501
Carbon disulfide	1.628
Methylene iodide	1.74



Snell's Law

$$n_i \sin \theta_i = n_t \sin \theta_t$$

260213 F  
Day 14

static case

$$E(\vec{r}) = \frac{kq_0}{r^2}$$



oscillating charge

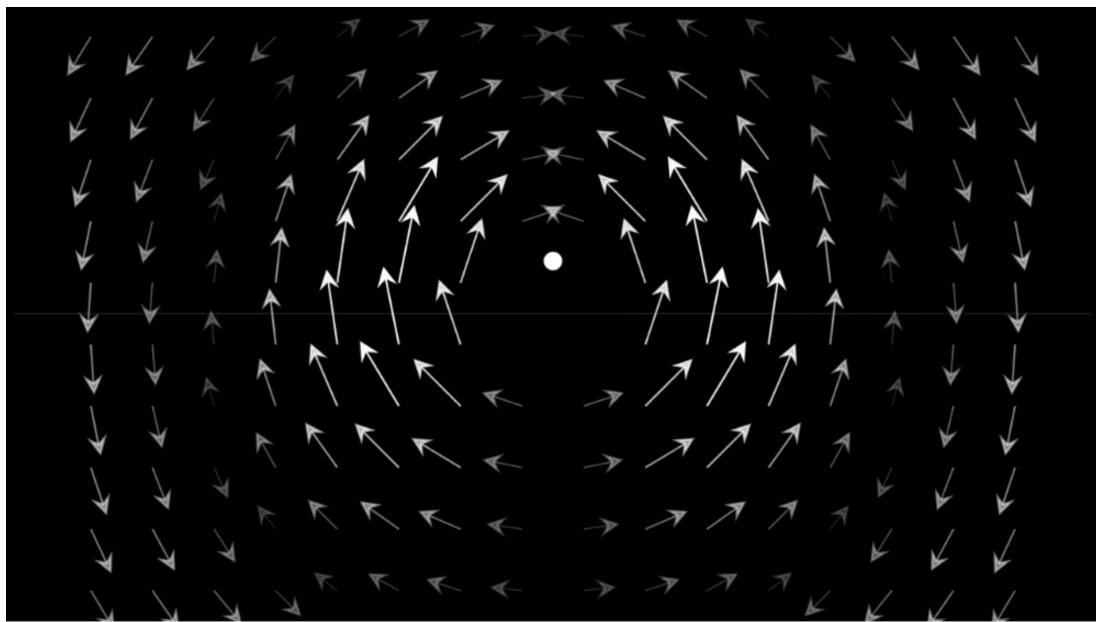
$$\vec{E}(\vec{r}, t) = \vec{E}_{\text{staticish}} + \vec{E}_{\text{rad}}$$

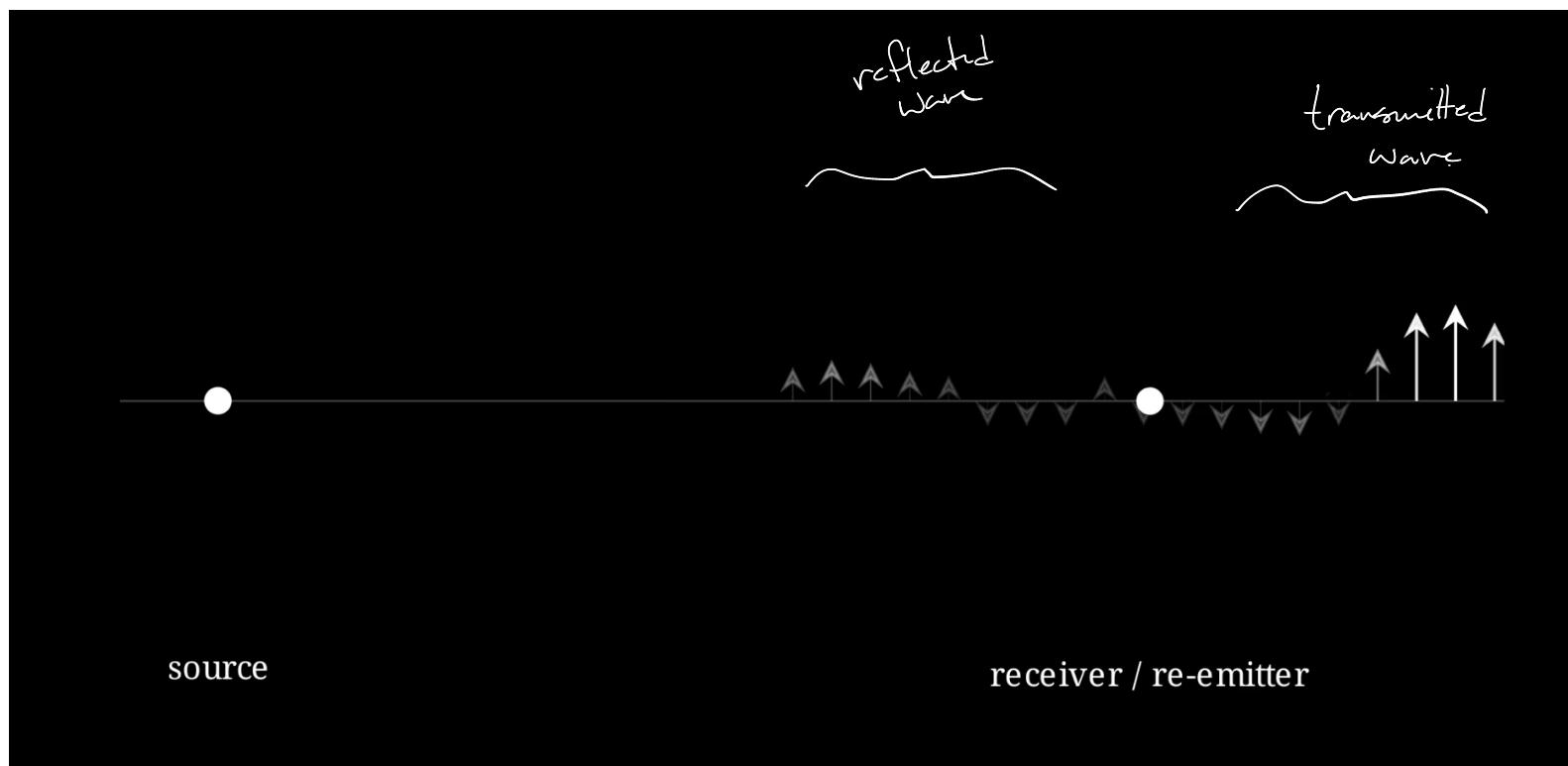
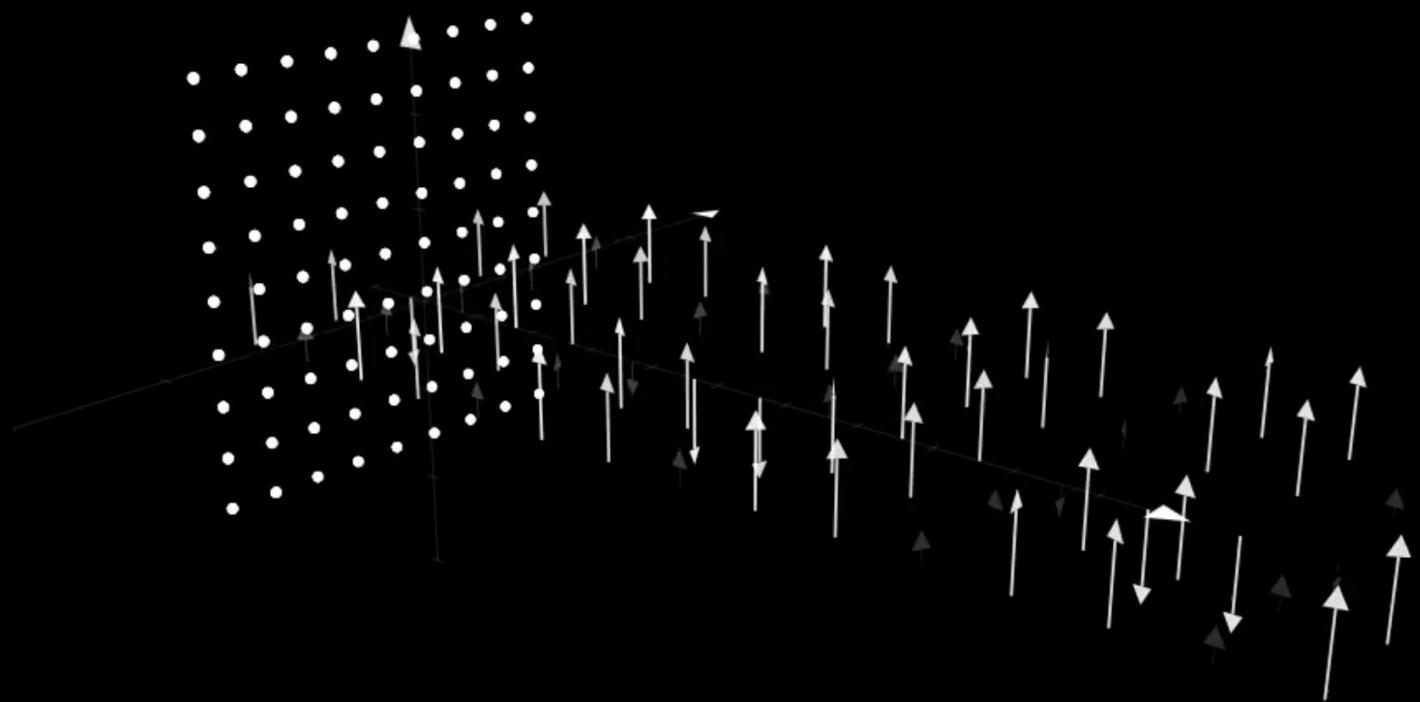
$\sim$

wave behavior

compare  
to  
 $\vec{r} - vt$

$$\vec{E}_{\text{rad}} \propto \frac{\vec{a}_\perp(t - r/c)}{r}$$





Dielectric Media

$\hookrightarrow$  insulator (like glass, water, air)

Assumptions: ① isotropic  $\rightarrow$  same in all directions

② homogeneous  $\rightarrow$  same over distance

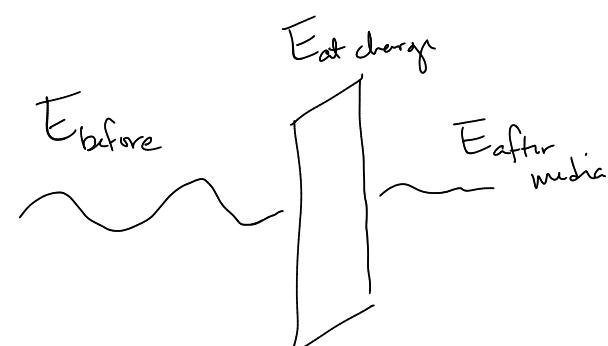
③ non-conducting (insulator/dielectric)  $\vec{J}_{\text{free}} = 0$

④ very thin material  
(air or some other  
thin gas)

} we will  
relax this  
later on

Light in a thin gas  $\rightarrow$  Feynman, vol 1, Ch 31

Light in matter = incident field + field reradiated  
by driven charges



$$\vec{E}_{\text{after}} = \underbrace{\vec{E}_s}_{\text{plate}} + \sum_{\text{all other charges}} \vec{E}_{\text{each charge}}$$

$$\vec{E}_s = \vec{E}_o \cos(kx - \omega t)$$

$$\vec{E}_s = \vec{E}_o \cos\left(k\left(x - \frac{\omega}{c}t\right)\right)$$

$$\vec{E} = \vec{E}_o \cos\left(\omega\left(\frac{k}{\omega}x - t\right)\right)$$

$$\vec{E} = \vec{E}_o \cos\left(-\omega\left(t - \frac{k}{\omega}x\right)\right)$$

$$\vec{E} = \vec{E}_0 \cos\left(\omega(t - \frac{x}{c})\right)$$

$$\vec{E} = \vec{E}_0 e^{i\omega(t - \frac{x}{c})}$$

So now what about the field from other charges?

plate thickness =  $\Delta x$

$$t_{\text{vac}} = \frac{\Delta x}{c}$$

$$t_{\text{plate}} = \frac{n \Delta x}{c}$$

$$\Delta t = t_{\text{plate}} - t_{\text{vac}}$$

$$= (n-1) \frac{\Delta x}{c}$$

$$\begin{aligned} E_{\text{after}} &= E_0 e^{i\omega(t - \Delta t - \frac{x}{c})} \\ \text{plate} &= E_0 e^{i\omega(t - (n-1)\frac{\Delta x}{c} - \frac{x}{c})} \\ &= E_0 e^{-i\omega(n-1)\frac{\Delta x}{c}} E_0 e^{i\omega(t - \frac{x}{c})} \\ &\quad \underbrace{\qquad}_{\text{w/ a phase shift}} \quad \underbrace{\qquad}_{\text{just like the original wave BUT w/ a}} \end{aligned}$$

$$e^{-i\omega(n-1)\frac{\Delta x}{c}}, \Delta x \text{ is very small}$$

$$\left\{ e^S \approx 1 + S, \text{ for small } S \right.$$

$$e^{-i\omega(n-1)\frac{\Delta x}{c}} \approx 1 - i\omega(n-1)\frac{\Delta x}{c}$$

$$E_{\text{after plate}} = E_0 e^{i\omega(t-\frac{x}{c})} - \frac{i\omega(n-1)\Delta x}{c} \cdot E_0 e^{i\omega(t-\frac{x}{c})}$$

$E_s$

$E_a \rightarrow$  new electric field that is the result of all moving charges



$$ma = \sum F = -kz$$

$$m \frac{d^2 z}{dt^2} = -kz$$

$$\boxed{\frac{d^2 z}{dt^2} = -\frac{k}{m} z}$$

~~$$+\cancel{A} \omega^2 \cos(\omega t + \phi) = +\cancel{k} \cdot \cancel{A} \cos(\omega t + \phi)$$~~

$$\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$$

↑ one of our constants

one way

$$\rightarrow z = A \cos(\omega t + \phi)$$

$$\frac{dz}{dt} = -A\omega \sin(\omega t + \phi)$$

$$\rightarrow \frac{dz}{dt} = -A\omega^2 \cos(\omega t + \phi)$$

another way

$$z = A e^{wt + \phi}$$

$$\frac{dz}{dt} = A w e^{wt + \phi}$$

$$\frac{d^2 z}{dt^2} = A w^2 e^{wt + \phi}$$

$$A w^2 e^{wt + \phi} = -\frac{k}{m} \cdot A e^{wt + \phi}$$

$$\omega^2 = -\frac{k}{m}$$

$$z = A \cos(\sqrt{\frac{k}{m}} t + \phi) \leftarrow \text{general solution to that DE}$$

given by initial conditions  
at  $t=0, z=1 \Rightarrow A=1, \phi=0$

$(A + \phi)$  are determined by initial conditions

Now we have a charge connected by a spring being driven by  $\vec{E}(t)$

$$z = 1 \cdot \cos(\sqrt{\frac{k}{m}} \cdot t)$$

$$\frac{d^2 z}{dt^2} = -\frac{k}{m} z + \frac{qeE_0}{m} e^{i\omega t + \phi}$$

the frequency of the light

$$\omega^2$$

$\hookrightarrow$  natural frequency of the electrons

$$\text{Guess: } z = z_0 e^{i\omega t + \phi}$$

$$\dot{z} = z_0 i\omega e^{i\omega t + \phi}$$

$$\ddot{z} = z_0 \underbrace{i^2 \omega^2}_{-1} w^2 e^{i\omega t + \phi}$$

So now we plug in:

$$-\cancel{z_0 w^2} e^{i\omega t + \phi} = -\cancel{w^2 z_0} e^{i\omega t + \phi} + \frac{qeE_0}{m} e^{i\omega t + \phi}$$

$\hookrightarrow$  solve for  $z_0$

$$z_0 (\omega_0^2 - \omega^2) = \frac{qeE_0}{m}$$

$$z_0 = \frac{qeE_0}{m(\omega_0^2 - \omega^2)}$$

↓

so...

$$z(t) = \frac{qeE_0}{m(\omega_0^2 - \omega^2)} \cdot e^{i\omega t + \phi}$$

motion of charges  $\rightarrow$  Electric field that results

$$E_a = -\frac{\eta q_c}{2\epsilon_0 c} i\omega \vec{z}_o e^{i\omega(t-\frac{x}{c})}$$

from Griffiths  
or Feynman

$$E_a = -\frac{\eta q_e}{2\epsilon_0 c} \cdot \frac{i\omega \cdot q_e E_o}{m(\omega_o^2 - \omega^2)} \cdot e^{i\omega(t-\frac{x}{c})}$$

$\Downarrow$  vs.  $E_a = -\frac{i\omega(n-1)\Delta x}{c} \cdot E_o e^{i\omega(t-\frac{x}{c})}$

from before

so putting what survives together

$$\frac{\eta q_e}{2\epsilon_0 m(\omega_o^2 - \omega^2)} = (n-1)\Delta x$$

$\eta$  → number of charges per unit of area

$N$  → number of charges per unit of volume

$$\eta = N \cdot \Delta x$$

$$\frac{N \cdot (\Delta x) q_e}{2\epsilon_0 m(\omega_o^2 - \omega^2)} = (n-1)\Delta x$$

$$\eta = 1 + \frac{N \cdot q_e}{2\epsilon_0 m(\omega_o^2 - \omega^2)}$$

$\omega_o$  is often much larger than  $\omega$  for visible light

$\vec{E}$  from a static sheet of charge

$$\vec{E} = \frac{Q}{2\epsilon_0 A} \vec{z}$$

$$\vec{E} = \frac{\sigma}{2\epsilon_0} \vec{z}$$

$$\vec{E} = \frac{\eta q_c}{2\epsilon_0} \vec{z}$$

} as  $\omega$  rises the denominator gets smaller, so the fraction gets larger!

Let's do this again, but better.  $\rightarrow$  index of refraction for heavier dielectrics

$\rightarrow$  Go back to the wave equation from Maxwell's:

$$\nabla^2 \vec{E} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}$$

Plug in!

$$\left\{ \begin{array}{l} \vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \\ \vec{P} = \vec{P}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \end{array} \right.$$

$$i^2 k^2 \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} - \epsilon_0 \mu_0 (+ i^2 \omega^2) \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \mu_0 (+ i^2 \omega^2) \vec{P}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$\underbrace{-1}_{\text{wave vector, } k}$

$$-k^2 \vec{E}_0 + \epsilon_0 \mu_0 \omega^2 \vec{E}_0 = -\mu_0 \omega^2 \vec{P}_0$$

$$\vec{P}_0 = \epsilon_0 \chi(\omega) \vec{E}_0 \quad \leftarrow \text{constitutive relation}$$

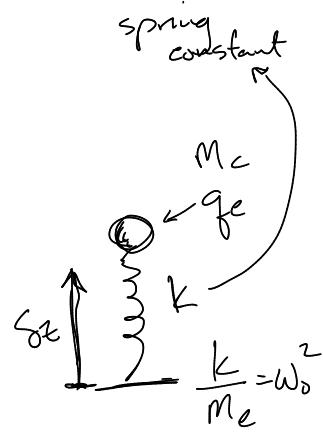
$\hookrightarrow$  susceptibility

$\hookrightarrow$  kicking the can down the road!

$\hookrightarrow$  complex!

$\hookrightarrow$  a function of  $\omega$  (frequency)  
just like  $\vec{P}_0 + \vec{E}_0$ .

$\hookrightarrow$  plug in and let  $\vec{E}_0$  cancel out



$$-\vec{k}^2 + \epsilon_0 \mu_0 \omega^2 = -\epsilon_0 \mu_0 \omega^2 \chi(\omega)$$

↪ solve for  $\vec{k}$ ,  $\epsilon_0 \mu_0 = \frac{1}{c^2}$

$$\epsilon_0 (1 + \chi(\omega)) = \epsilon(\omega) \leftarrow \begin{matrix} \text{permittivity} \\ \text{of a material} \end{matrix}$$

$$\vec{k}^2 = \epsilon_0 \mu_0 \omega^2 (1 + \chi(\omega))$$

still wave vector  $\vec{k}$

$$\boxed{\vec{k} = \frac{\omega}{c} \sqrt{1 + \chi(\omega)}}$$

recall:  $\vec{k} = \frac{\omega}{v}$  and  $v = \frac{c}{n}$

$$\boxed{\vec{k} = \frac{\omega \cdot n}{c}}$$

$$\tilde{\vec{k}} = \frac{\tilde{n} \omega}{c}$$

$$\tilde{\vec{k}} = (n + i\kappa) \frac{\omega}{c}$$

compare!

$$\tilde{n} = \sqrt{1 + \chi(\omega)}$$

$$\tilde{n} = n(\omega) + i\kappa(\omega)$$

↪ light "slows down"

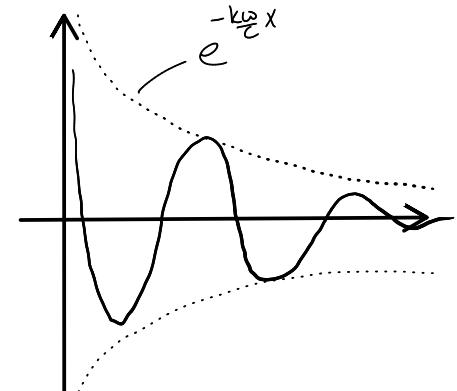
absorption coefficient

$$\tilde{n} = n' + i n''$$

Let's plug this back in to  $\vec{E}$

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \vec{E}_0 e^{i \frac{(n + i\kappa) \omega}{c} \hat{u} \cdot \vec{r} - i\omega t} \\ &= \vec{E}_0 e^{-\kappa \frac{\omega}{c} \hat{u} \cdot \vec{r} + i \left( \frac{n \omega}{c} \right) \hat{u} \cdot \vec{r} - i\omega t} \end{aligned}$$

$$\begin{aligned}
 &= \vec{E}_0 e^{-\frac{k\omega}{c} \hat{n} \cdot \hat{r}} e^{i(\frac{n\omega}{c} \hat{n} \cdot \hat{r} - wt)} \\
 &\quad \leftarrow \text{has phase information} \\
 &= \vec{E}_0 e^{-\frac{k\omega}{c} \hat{n} \cdot \hat{r}} \cdot \cos\left(\frac{n\omega}{c} \hat{n} \cdot \hat{r} - wt + \phi\right) + \text{some imaginary stuff we ignore} \\
 &\quad \leftarrow \text{this wave decays as it goes into this material}
 \end{aligned}$$



$$V_{\text{phase}}(\omega) = \frac{C}{n(\omega)} \quad \leftarrow \text{real part of } \tilde{n}$$

$$P = N \cdot q \cdot \underline{\underline{S}} \cdot \vec{z} \xrightarrow{\text{textbook}} \vec{P} = N q f_e \vec{r}_e$$

$$\text{model} \rightarrow m \ddot{z} = -kz - my \dot{z} + q f_e e^{i\omega t}$$

$$z = \frac{q_f e^{\frac{i\omega}{m_e} t}}{m_e (\omega_0^2 - i\omega y - \omega^2)} \quad \leftarrow \begin{array}{l} \text{inhomogeneous} \\ \text{solution} \end{array}$$

$$P = \frac{N q_f^2 \vec{E}_0 e^{i\omega t}}{m_e (\omega_0^2 - i\omega y - \omega^2)}$$

- or -  
nonhomogeneous

$$\left[ \frac{N q_f^2}{m_e} \right] = \left[ \frac{C^2}{m^3 \text{ kg}} \right]$$

$$F = \frac{q_0 q_1}{4\pi G_0 r^2}$$

$$\frac{k q_0 q_1}{r^2}$$

$$[k] = \left[ \frac{Nm^2}{C^2} \right] \quad C_0 = \left[ \frac{C^2}{Nm^2} \right]$$

$$\frac{N_{qe}^2}{\epsilon_0 M_e} = \omega_p^2 \leftarrow \text{plasma frequency}$$

$$P = \epsilon_0 \left( \frac{\omega_p^2}{\omega_s^2 - i\gamma\omega - \omega^2} \right) \vec{E}_0 e^{i\omega t}$$

Compare  
to this result  
from earlier!

$$\chi(\omega) = \frac{\omega_p^2}{\omega_s^2 - i\gamma\omega - \omega^2}$$

$$\begin{aligned} \left[ \frac{N_{qe}^2}{\epsilon_0 M_e} \right] &= \frac{C}{m^3 kg} \cdot \frac{Nm^2}{C} \\ &= \frac{N}{m kg} = \frac{kam}{s^2} \\ &= \frac{kam}{s^2} \cdot \frac{1}{m kg} \\ &= \frac{1}{s^2} = \frac{rad^2}{s^2} \end{aligned}$$

$$\vec{P}_0 = \epsilon_0 \chi(\omega) \vec{E}_0$$

$$\tilde{n} = n + iK \quad \tilde{n} = \sqrt{1 + \chi(\omega)}$$

$$n + iK = \sqrt{1 + \chi(\omega)}$$

$$(n + iK)^2 = 1 + \chi(\omega)$$

$$(n+iK)^2 = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}$$

$$n+iK = \sqrt{1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}}$$

plot as  
a function  
of frequency

$$n^2(\omega) = 1 + \frac{Nq_e^2}{\epsilon_0 m_e} \left( \frac{1}{\omega_0^2 - \omega^2} \right) \quad (3.70)$$

**3.62\*** Show that Eq. (3.70) can be rewritten as

$$(n^2 - 1)^{-1} = -C\lambda^{-2} + C\lambda_0^{-2}$$

$$\text{where } C = 4\pi^2 c^2 \epsilon_0 m_e / Nq_e^2.$$

**3.63** Augustin Louis Cauchy (1789–1857) determined an empirical equation for  $n(\lambda)$  for substances that are transparent in the visible. His expression corresponded to the power series relation

$$n = C_1 + C_2/\lambda^2 + C_3/\lambda^4 + \dots$$

where the  $C_i$ s are all constants. In light of Fig. 3.41, what is the physical significance of  $C_1$ ?

**3.64** Referring to the previous problem, realize that there is a region between each pair of absorption bands for which the Cauchy Equation (with a new set of constants) works fairly well. Examine Fig. 3.41: what can you say about the various values of  $C_1$  as  $\omega$  decreases across the spectrum? Dropping all but the first two terms, use Fig. 3.40 to determine approximate values for  $C_1$  and  $C_2$  for borosilicate crown glass in the visible.

**3.65\*** Crystal quartz has refractive indexes of 1.557 and 1.547 at wavelengths of 400.0 nm and 500.0 nm, respectively. Using only the first two terms in Cauchy's equation, calculate  $C_1$  and  $C_2$  and determine the index of refraction of quartz at 610.0 nm.

**3.66\*** In 1871 Sellmeier derived the equation

$$n^2 = 1 + \sum_j \frac{A_j \lambda^2}{\lambda^2 - \lambda_{0j}^2}$$

where the  $A_j$  terms are constants and each  $\lambda_{0j}$  is the vacuum wavelength associated with a natural frequency  $\nu_{0j}$ , such that  $\lambda_{0j}\nu_{0j} = c$ . This formulation is a considerable practical improvement over the Cauchy Equation. Show that where  $\lambda >> \lambda_{0j}$ , Cauchy's Equation is an approximation of Sellmeier's. Hint: Write the above expression with only the first term in the sum; expand it by the binomial theorem; take the square root of  $n^2$  and expand again.

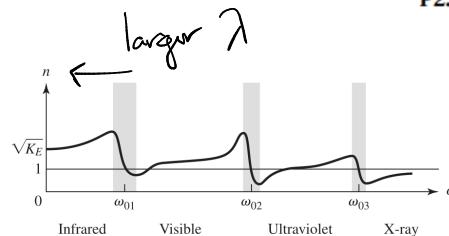


Figure 3.41 Refractive index versus frequency.

### Exercises for 2.4 The Lorentz Model of Dielectrics

**P2.1** Verify that (2.35) is a solution to (2.34).

**P2.2** Derive the Sellmeier equation

$$(n+i\kappa)^2 = 1 + \chi(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2} \quad (2.39)$$

$$n^2 = 1 + \frac{A\lambda_{\text{vac}}^2}{\lambda_{\text{vac}}^2 - \lambda_{0,\text{vac}}^2}$$

from (2.39) for a gas with negligible absorption (i.e.  $\gamma \approx 0$ , valid far from resonance  $\omega_0$ ), where  $\lambda_{0,\text{vac}}$  corresponds to frequency  $\omega_0$  and  $A$  is a constant. Many materials (e.g. glass, air) have strong resonances in the ultraviolet. In such materials, do you expect the index of refraction for blue light to be greater than that for red light? Make a sketch of  $n$  as a function of wavelength for visible light down to the ultraviolet (where  $\lambda_{0,\text{vac}}$  is located).

**P2.3**

In the Lorentz model, take  $N = 10^{28} \text{ m}^{-3}$  for the density of bound electrons in (in the UV)  $(n+i\kappa)^2 = 1 + \chi(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}$  (2.39) I. Assume that the magnitude of  $\mathbf{E}_0$  is  $10^4 \text{ V/m}$ . For three frequencies i)  $\omega = \omega_0 - 2\gamma$ , ii)  $\omega = \omega_0$ , and iii)  $\omega = \omega_0 + 2\gamma$  find:

- (a) the amplitude and phase of the charge displacement  $\mathbf{r}_e$  (2.35) relative to the phase of  $\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ .
- (b) the magnitude and complex phase of the susceptibility  $\chi(\omega)$ . Does  $\chi(\omega)$  depend on the strength of the E-field?
- (c)  $n$  and  $\kappa$  at the three frequencies via (2.29) and (2.27).
- (d) the three speeds of light in terms of  $c$  and how far light penetrates into the material before only  $1/e$  of the amplitude of  $\mathbf{E}$  remains.

**P2.4**

- (a) Use a computer to plot  $n$  and  $\kappa$  as a function of  $\omega$  for a dielectric (i.e. obtain graphs such as the ones in Fig. 2.5). Use the Lorentz model and the following parameters:  $\omega_0 = 10\omega_p$ , and  $\gamma = \omega_p$ ; plot your function from  $\omega = 0$  to  $\omega = 20\omega_p$ . No need to choose a value for  $\omega_p$ ; your horizontal axis will be in units of  $\omega_p$ .
- (b) Plot  $n$  and  $\kappa$  as a function of frequency for a material that has three resonant frequencies:  $\omega_{01} = 10\omega_p$ ,  $\gamma_1 = \omega_p$ ,  $f_1 = 0.5$ ;  $\omega_{02} = 15\omega_p$ ,  $\gamma_2 = \omega_p$ ,  $f_2 = 0.25$ ; and  $\omega_{03} = 25\omega_p$ ,  $\gamma_3 = 3\omega_p$ ,  $f_3 = 0.25$ . Plot the results from  $\omega = 0$  to  $\omega = 30\omega_p$ .

"thin" gas

$$n = 1 + \frac{N q_e^2}{2 \epsilon_0 m_e (\omega_0^2 - \omega^2)}$$

3.62\* Show that Eq. (3.70) can be rewritten as

$$(n^2 - 1)^{-1} = -C\lambda^{-2} + C\lambda_0^{-2}$$

where  $C = 4\pi^2 c^2 \epsilon_0 m_e / N q_e^2$ .

$$n^2(\omega) = 1 + \frac{N q_e^2}{\epsilon_0 m_e} \left( \frac{1}{\omega_0^2 - \omega^2} \right) \quad (3.70)$$

$$\begin{cases} n = \tilde{n} \\ \gamma = 0 \end{cases}$$

$$\tilde{n}(\omega)^2 = 1 + \frac{N q_e^2}{\epsilon_0 M_e (\omega_0^2 - \omega^2)}$$

$$n^2 - 1 = \frac{N q_e^2}{\epsilon_0 M_e (\omega_0^2 - \omega^2)}$$

$$(n^2 - 1)^{-1} = \frac{\epsilon_0 M_e (\omega_0^2 - \omega^2)}{N q_e^2}$$

$$= \frac{\epsilon_0 M_e}{N q_e^2} \cdot \left( \frac{4\pi^2 c^2}{\lambda_0^2} - \frac{4\pi^2 c^2}{\lambda^2} \right)$$

$$= \underbrace{\frac{4\pi^2 c^2 \epsilon_0 M_e}{N q_e^2}}_{C} \left( \frac{1}{\lambda_0^2} - \frac{1}{\lambda^2} \right)$$

$$(n^2 - 1)^{-1} = C \left( \lambda_0^{-2} - \lambda^{-2} \right) \quad \checkmark \checkmark$$

$$\begin{aligned} v &= \lambda \cdot \gamma & v &= \frac{\omega}{k} \\ \gamma &= \frac{v}{\lambda} = \frac{c}{\lambda} & \lambda \gamma &= \frac{\omega}{k} \Rightarrow \omega = \gamma v k \end{aligned}$$

$$\omega = 2\pi v = \frac{2\pi c}{\lambda}$$

$$\omega_0 = \frac{2\pi c}{\lambda_0}$$

$$\begin{aligned} \tilde{n} &= n + ik = \sqrt{1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}} \\ \tilde{n}(\omega)^2 &= 1 + \frac{N q_e^2}{\epsilon_0 M_e (\omega_0^2 - \omega^2)} \\ \frac{N q_e^2}{\epsilon_0 M_e} &= \omega_p^2 \end{aligned}$$

$$\tilde{n}^2 - 1 = 1 + \sum_j \frac{f_j w_p^2}{\omega_{sj}^2 - i\omega \chi_j - \omega^2}$$

**3.63** Augustin Louis Cauchy (1789–1857) determined an empirical equation for  $n(\lambda)$  for substances that are transparent in the visible. His expression corresponded to the power series relation

$$n = C_1 + C_2/\lambda^2 + C_3/\lambda^4 + \dots \quad \text{for longer wavelengths}$$

$$n = \sqrt{K_E} \quad \text{vs}$$

where the  $C_i$ s are all constants. In light of Fig. 3.41, what is the physical significance of  $C_1$ ?

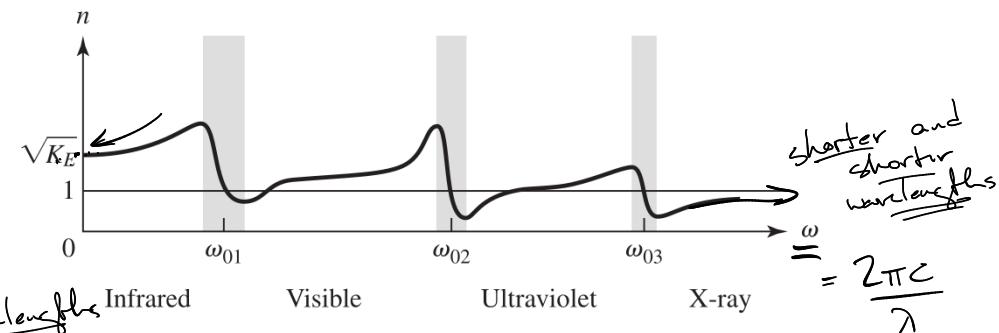


Figure 3.41 Refractive index versus frequency.

material  
permittivity

$$K_E = \frac{\epsilon}{\epsilon_0} \xrightarrow{\text{recall}} \epsilon_0(1+\chi)$$

$$\epsilon_0(1+\chi(\omega)) = \epsilon(\omega) \leftarrow \begin{matrix} \text{permittivity} \\ \text{of a material} \end{matrix}$$

$$n = \sqrt{1 + \chi} \quad \leftarrow \text{that is fine} \rightarrow \chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - i\omega \chi - \omega^2}$$

wl what we  
have said

$$n = \sqrt{1 + \frac{\omega_p^2}{\omega_0^2}}$$

constant stuff

$$\chi(\omega \rightarrow 0) = \frac{\omega_p^2}{\omega_0^2}$$

P2.2

Derive the Sellmeier equation

$$(n + i\kappa)^2 = 1 + \chi(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}$$

$$n^2 = 1 + \frac{A\lambda_{\text{vac}}^2}{\lambda_{\text{vac}}^2 - \lambda_{0,\text{vac}}^2}$$

from (2.39) for a gas with negligible absorption (i.e.  $\gamma \approx 0$ , valid far from resonance  $\omega_0$ ), where  $\lambda_{0,\text{vac}}$  corresponds to frequency  $\omega_0$  and  $A$  is a constant. Many materials (e.g. glass, air) have strong resonances in the ultraviolet. In such materials, do you expect the index of refraction for blue light to be greater than that for red light? Make a sketch of  $n$  as a function of wavelength for visible light down to the ultraviolet (where  $\lambda_{0,\text{vac}}$  is located).

$$\cancel{K} \Rightarrow D \quad \cancel{\chi} \Rightarrow O$$

$$\underbrace{(n + i\kappa)^2}_{\approx 1} = 1 + \chi(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2} \underset{\gamma=0}{=} 0 \quad (2.39)$$

$$\omega_o = \frac{2\pi c}{\lambda_{0,\text{vac}}} \quad \omega = \frac{2\pi c}{\lambda_{\text{vac}}}$$

$$V = C = \lambda_{\text{vac}} \cdot V$$

$$n^2 = 1 + \frac{\omega_p^2}{\frac{4\pi^2 c^2}{\lambda_o^2} - \frac{4\pi^2 c^2}{\lambda^2}}$$

$$= 1 + \frac{\omega_p^2}{\frac{4\pi^2 c^2 \lambda_o^2}{\lambda_o^2 \lambda^2} - \frac{4\pi^2 c^2}{\lambda^2}}$$

$$= 1 + \frac{\lambda_o^2 \omega_p^2 \lambda^2}{4\pi^2 c^2 (\lambda^2 - \lambda_o^2)}$$

↳ A

if  $\omega_o$  is  
a constant,  
then  $\lambda_o$  is  
constant as well

$$n^2 = 1 + \frac{A\lambda^2}{\lambda^2 - \lambda_o^2}$$

$$A = \frac{\lambda_o^2 \omega_p^2}{4\pi^2 c^2}$$

$$A = \frac{\lambda_o^2 N_{qe}^2}{6\pi M_e 4\pi^2 c^2}$$

3.66\* In 1871 Sellmeier derived the equation

$$n^2 = 1 + \sum_j \frac{A_j \lambda^2}{\lambda^2 - \lambda_{0j}^2}$$

where the  $A_j$  terms are constants and each  $\lambda_{0j}$  is the vacuum wavelength associated with a natural frequency  $\nu_{0j}$ , such that  $\lambda_{0j}\nu_{0j} = c$ . This formulation is a considerable practical improvement over the Cauchy Equation. Show that where  $\lambda \gg \lambda_{0j}$ , Cauchy's Equation is an approximation of Sellmeier's. Hint: Write the above expression with only the first term in the sum; expand it by the binomial theorem; take the square root of  $n^2$  and expand again.

$$n^2 = 1 + \frac{A \lambda^2}{\lambda^2 - \lambda_0^2}$$

$$\rightarrow 1 + \frac{A}{1 - \frac{\lambda_0^2}{\lambda^2}}$$

$$\rightarrow 1 + A \left(1 - \frac{\lambda_0^2}{\lambda^2}\right)^{-1}$$

small  $\rightarrow$

$$\lambda_0 \nu_0 = c$$

$$\lambda \gg \lambda_0$$

Show that where  $\lambda \gg \lambda_{0j}$ , C

$$n^2 = 1 + A \left(1 + \frac{\lambda_0^2}{\lambda^2}\right)$$

$$n = \left(1 + A \left(1 + \frac{\lambda_0^2}{\lambda^2}\right)\right)^{1/2} \approx 1 + \frac{A}{2} \left(1 + \frac{\lambda_0^2}{\lambda^2}\right) = \underbrace{\left(1 + \frac{A}{2}\right)}_{C_1} + \underbrace{\frac{A \lambda_0^2}{2 \lambda^2}}_{C_2}$$

again this is small if

A is small

$$(1 \pm x)^p \approx 1 \pm px \quad \leftarrow \text{"expand by binomial theorem"}$$

this is poorly phrased into  
I would say Taylor expansion

Is A small?

$$A = \frac{\gamma^2 N q e^2}{G M_e 4\pi^2 C^2}$$

$\rightarrow 10^{16}$

$$C^2 \sim 10^{16}$$

$$\epsilon_0 \sim 10^{-11}$$

$$M_e = 10^{-31}$$

$$q^2 \sim \text{very small}$$

$$(10^{-19})^2 = 10^{-38}$$

$$A = \frac{10^{-12} \cdot 10^{23} \cdot 10^{-38}}{10^{-11} \cdot 10^{-31} \cdot 10 \cdot 10^{16}} = \frac{10^{-27}}{10^{-25}} = \frac{10^{25}}{10^{21}} = \frac{1}{100}$$

$$\lambda_0 = \frac{c}{\nu} = \frac{3 \cdot 10^8}{10^{14}} = 3 \cdot 10^{-6}$$

$\rightarrow$  small  $\lambda_0$   
 $\rightarrow$  square and its small

$$N = \frac{\text{number}}{\text{volume}} \Rightarrow \frac{N}{V} \rightarrow \begin{array}{l} \text{large number} \\ \sim 10^{23} \end{array}$$

$$q e^2$$

$$\text{but could be bigger}$$

$$n = C_1 + \frac{C_2}{\lambda^2} + \frac{C_3}{\lambda^4} + \dots$$

$$C_2 = \frac{A \lambda_0^2}{2}$$

but I'm not convinced  
that  $A$  is small  
enough... so

$$n^2 = 1 + A \left( 1 + \frac{\lambda_0^2}{\lambda^2} \right)$$

$$= 1 + A + \frac{A \lambda_0^2 (1+A)}{\lambda^2 (1+A)}$$

$$= (1+A) \left( 1 + \frac{A \lambda_0^2}{\lambda^2 (1+A)} \right)$$

this is  
small bc  
 $\lambda^2 \gg \lambda_0^2$

$$n = (1+A)^{1/2} \left( 1 + \frac{A \lambda_0^2}{\lambda^2 (1+A)} \right)^{1/2}$$

so now I'll do my approx:

$$n = (1+A)^{1/2} \left( 1 + \frac{A \lambda_0^2}{2 \lambda^2 (1+A)} \right)$$

then distribute

$$n = (1+A)^{1/2} + \underbrace{\frac{A(1+A)^{1/2} \lambda_0^2}{2(1+A) \lambda^2}}_{C_1} \underbrace{\frac{A \lambda_0^2}{2(1+A)^{1/2}}}_{C_2}$$

that is better!

$$A = \frac{\lambda_0^2 N q e^2}{G M_e 4 \pi^2 C^2}$$

$$C_1 = \left( 1 + \frac{\lambda_0^2 N q e^2}{G M_e 4 \pi^2 C^2} \right)^{1/2}$$







