

Chapter 2 - Plane Waves + Index of Refraction

Z60208 M
Day 12

plane wave:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \phi)$$

"wave vector" $\vec{k} = k \hat{u} = \frac{2\pi}{\lambda_{\text{vac}}} \hat{u}$
"k"
"direction of propagation"

angular frequency $\rightarrow \omega = \frac{2\pi c}{\lambda_{\text{vac}}} = 2\pi\nu$

$k + \omega$ are related just like $\lambda + \nu$

$$c = \frac{\lambda_{\text{vac}}}{T} = \lambda_{\text{vac}}\nu = \frac{\omega}{k}$$

\curvearrowleft
this relationship
is known as the
"dispersion relation"

"kappa" \downarrow wavenumber

$$\frac{1}{\lambda_{\text{vac}}} = K = [\text{cm}^{-1}]$$

$$c = v_{\text{vac}} \leftarrow \text{velocity of light in vacuum}$$

$$\nu \nu \leftarrow "nu"$$

\curvearrowleft frequency

similarly for magnetic field:

$$\vec{B}(\vec{r}, t) = \vec{B}_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \phi)$$

$$\underbrace{\vec{B}_0 = \frac{\vec{k} \times \vec{E}_0}{\omega}}, \text{ so } \vec{B}_0 \text{ is not independent, but is determined by other parameters.}$$

$$\vec{B}_0 \perp \vec{k} \perp \vec{E}$$

also think about magnitude

$$B_0 = \frac{k E_0}{\omega} = \frac{E_0}{c} \leftarrow \begin{array}{l} \text{since this is so large} \\ \text{we will focus on the } \vec{E} \text{ field.} \end{array}$$

complex plane waves:

$$\vec{E}(\vec{r}, t) = \operatorname{Re} \left\{ \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right\}$$

$$\vec{E}_0 = \vec{E}_0 e^{i\phi} \leftarrow \text{phase shift}$$

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

Summarize some facts that we know now:

$\lambda \rightarrow$ wavelength

$T \rightarrow$ period

$\gamma \rightarrow$ frequency ($\frac{1}{T}$)

$k \rightarrow$ wave vector ($\frac{2\pi}{\lambda}$)

$\omega \rightarrow$ angular frequency ($\frac{2\pi}{T}$)

$$V = \frac{\omega}{k}$$

$$V = \frac{\lambda}{T} = \lambda \cdot \gamma \leftarrow \text{any wave}$$

$$c = \lambda_{\text{vac}} \cdot \gamma$$

$$\epsilon_0 \mu_0 = \frac{1}{c^2} \Rightarrow c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

permeability
of
free space $\rightarrow \epsilon_0 = 8.85 \cdot 10^{-12} \frac{C^2}{Nm^2}$

permittivity
of
free space $\rightarrow \mu_0 = 4\pi \cdot 10^{-7} \frac{T}{Am^2}$

Speed of light in matter

\hookrightarrow light slows down in materials

$$\frac{c}{v} = n \leftarrow \text{index of refraction}$$

$n=1 \leftarrow$ vacuum

$n=1.0003 \leftarrow$ air

$n=1.33 \leftarrow$ water

$n=1.5 \leftarrow$ glass

$$\nabla^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \frac{k^2}{\omega^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

Q1: what is light? ✓

Q2: where does light come from? ✓

Q3: can the speed of light change? see Q6

Q4: what affects the brightness of light?

Q5: how do human perceive light amount + color

Q6: why does light slow down in materials

Q3

Q7: what is reflection

$\lambda \cdot v = V = \frac{c}{n}$

constant as n increases
 \downarrow
 gets smaller as n increases

$$\lambda_{\text{vac}} \cdot v = c \Leftrightarrow c = n \cdot v$$

\uparrow
 $\lambda \cdot v$

$$\lambda_{\text{vac}} \cdot v = n \cdot \lambda \cdot v$$

\approx wavelength in material

$$\lambda = \frac{\lambda_{\text{vac}}}{n}$$

} λ will be smaller than λ_{vac} since $n > 1$

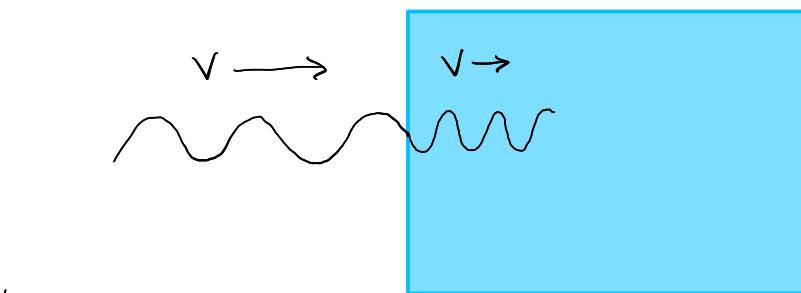
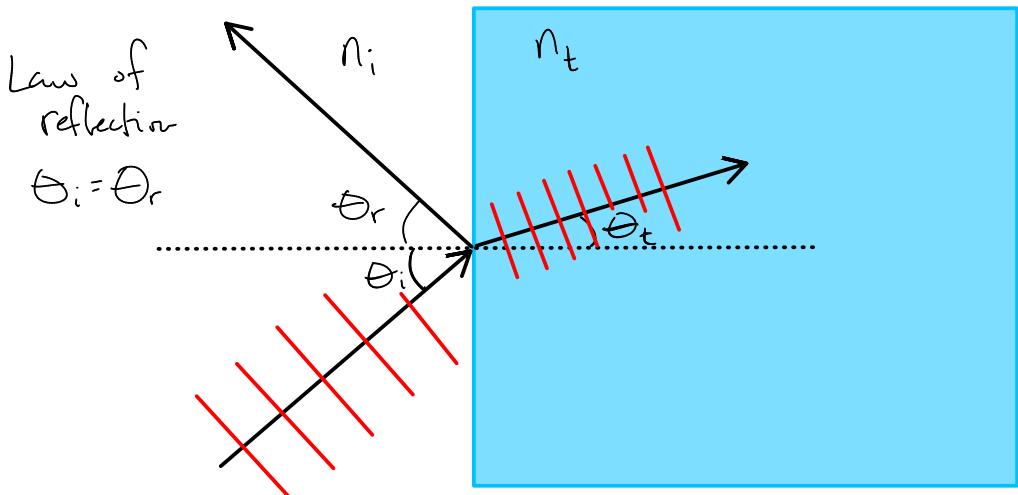


Table 23.1 Indices of Refraction for $\lambda = 589.3 \text{ nm}$ in Vacuum (at 20°C Unless Otherwise Noted)

Material	Index
Solids	
Ice (at 0°C)	1.309
Fluorite	1.434
Fused quartz	1.458
Polystyrene	1.49
Lucite	1.5
Plexiglas	1.51
Crown glass	1.517
Plate glass	1.523
Sodium chloride	1.544
Light flint glass	1.58
Dense flint glass	1.655
Sapphire	1.77
Zircon	1.923
Diamond	2.419
Titanium dioxide	2.9
Gallium phosphide	3.5
Liquids	
Water	1.333
Acetone	1.36
Ethyl alcohol	1.361
Carbon tetrachloride	1.461
Glycerin	1.473
Sugar solution (80%)	1.49
Benzene	1.501
Carbon disulfide	1.628
Methylene iodide	1.74



Snell's Law

$$n_i \sin \theta_i = n_t \sin \theta_t$$

260213 F
Day 14

static case

$$E(\vec{r}) = \frac{kq_0}{r^2}$$



oscillating charge

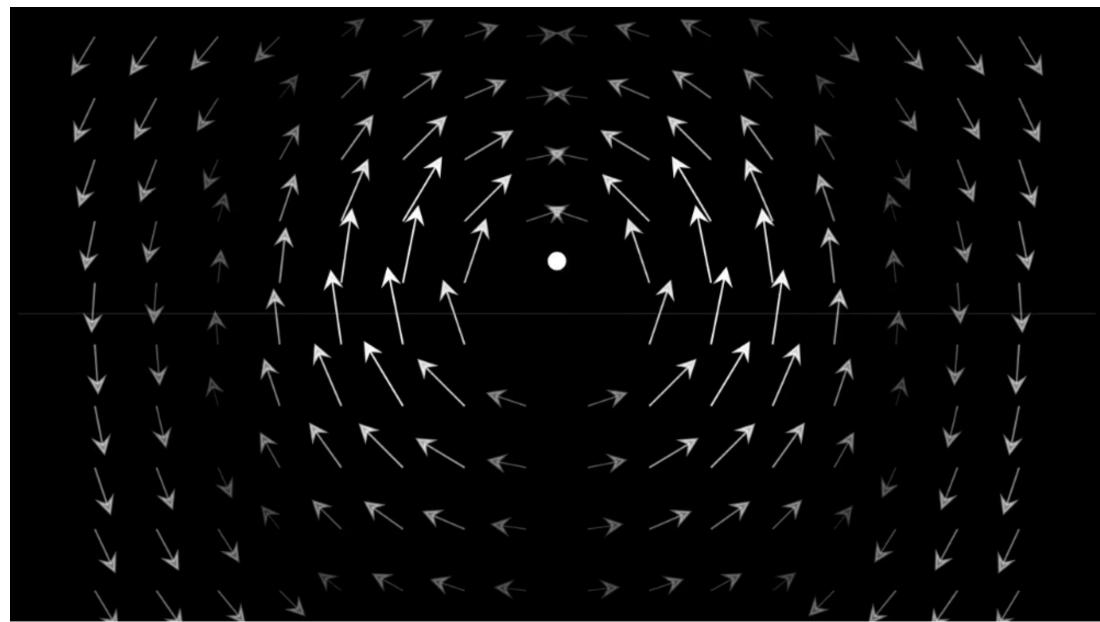
$$\vec{E}(\vec{r}, t) = \vec{E}_{\text{staticish}} + \vec{E}_{\text{rad}}$$

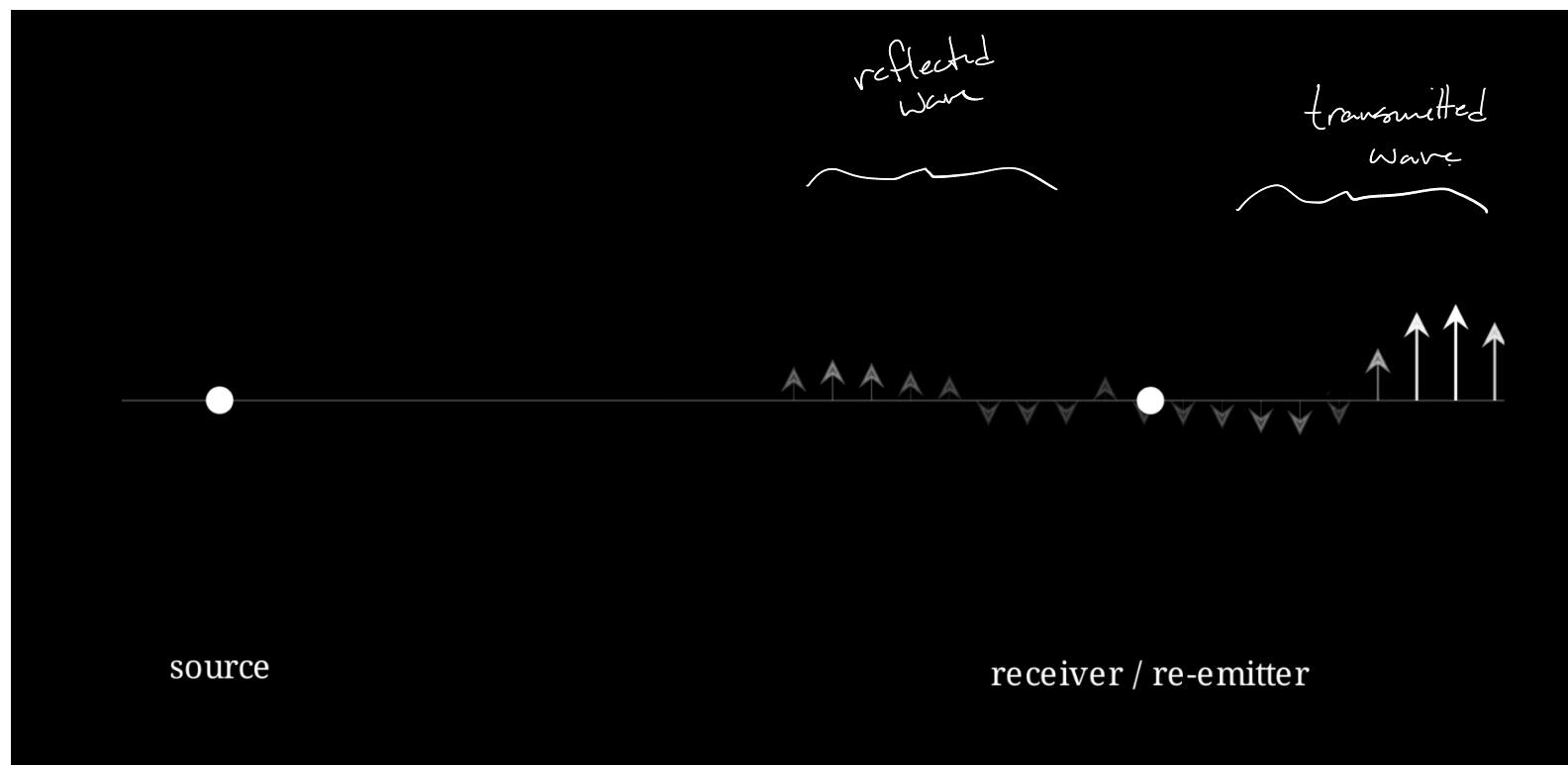
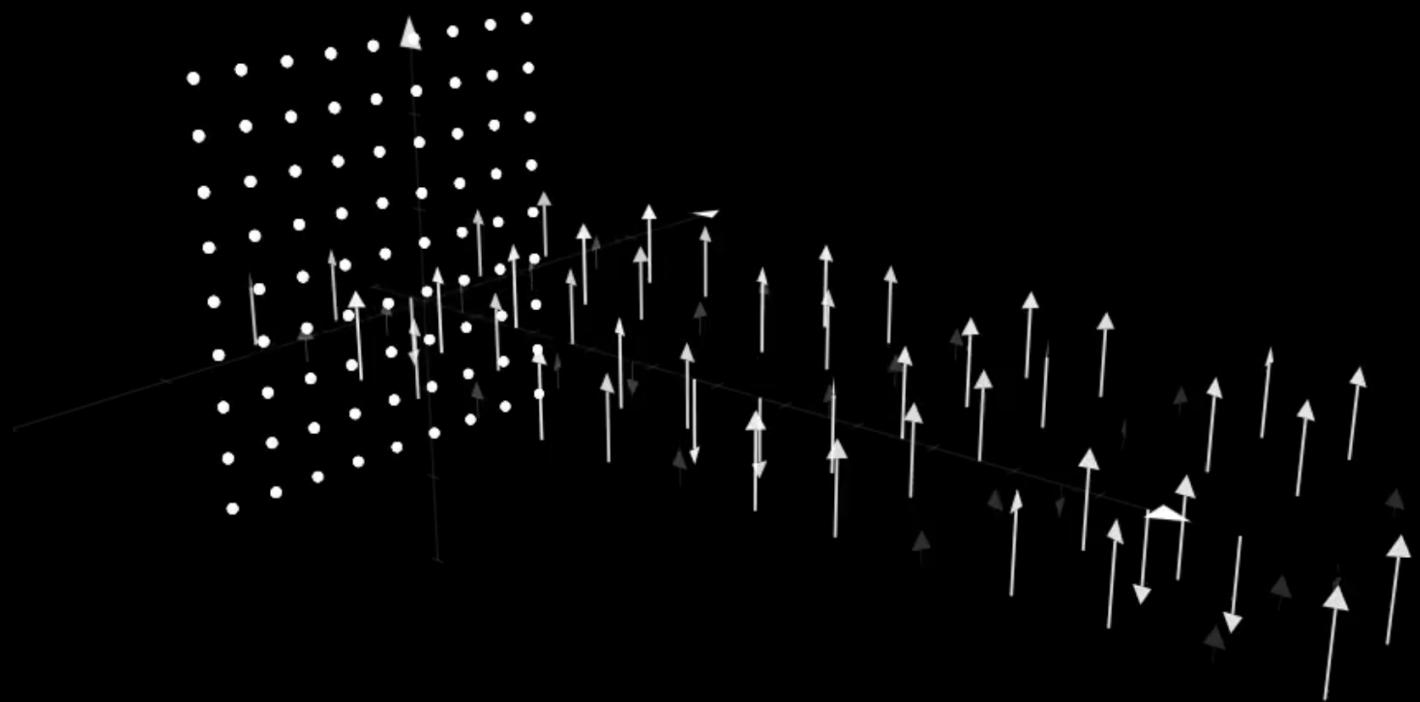
\sim

wave behavior

compare
to
 $\vec{r} - vt$

$$\vec{E}_{\text{rad}} \propto \frac{\vec{a}_\perp(t - r/c)}{r}$$





Dielectric Media

\hookrightarrow insulator (like glass, water, air)

Assumptions: ① isotropic \rightarrow same in all directions

② homogeneous \rightarrow same over distance

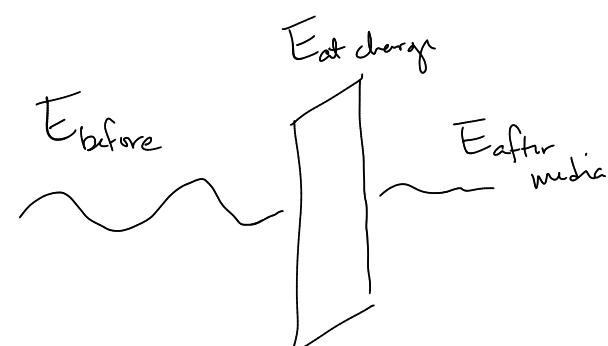
③ non-conducting (insulator/dielectric) $\vec{J}_{\text{free}} = 0$

④ very thin material
(air or some other
thin gas)

} we will
relax this
later on

Light in a thin gas \rightarrow Feynman, vol 1, Ch 31

Light in matter = incident field + field reradiated
by driven charges



$$\vec{E}_{\text{after}} = \underbrace{\vec{E}_s}_{\text{plate}} + \sum_{\text{all other charges}} \vec{E}_{\text{each charge}}$$

$$\vec{E}_s = \vec{E}_o \cos(kx - \omega t)$$

$$\vec{E}_s = \vec{E}_o \cos\left(k\left(x - \frac{\omega}{k}t\right)\right)$$

$$\vec{E} = \vec{E}_o \cos\left(\omega\left(\frac{k}{\omega}x - t\right)\right)$$

$$\vec{E} = \vec{E}_o \cos\left(-\omega\left(t - \frac{k}{\omega}x\right)\right)$$

$$\vec{E} = \vec{E}_0 \cos\left(\omega(t - \frac{x}{c})\right)$$

$$\vec{E} = \vec{E}_0 e^{i\omega(t - \frac{x}{c})}$$

So now what about the field from other charges?

plate thickness = Δx

$$t_{vac} = \frac{\Delta x}{c}$$

$$t_{plate} = \frac{n \Delta x}{c}$$

$$\Delta t = t_{plate} - t_{vac}$$

$$= (n-1) \frac{\Delta x}{c}$$

$$\begin{aligned} E_{\text{after}} &= E_0 e^{i\omega(t - \Delta t - \frac{x}{c})} \\ \text{plate} &= E_0 e^{i\omega(t - (n-1)\frac{\Delta x}{c} - \frac{x}{c})} \\ &= E_0 e^{-i\omega(n-1)\frac{\Delta x}{c}} E_0 e^{i\omega(t - \frac{x}{c})} \\ &\quad \underbrace{\qquad}_{\text{w/ a phase shift}} \quad \underbrace{\qquad}_{\text{just like the original wave BUT w/ a}} \end{aligned}$$

$$e^{-i\omega(n-1)\frac{\Delta x}{c}}, \Delta x \text{ is very small}$$

$$\left\{ e^S \approx 1 + S, \text{ for small } S \right.$$

$$e^{-i\omega(n-1)\frac{\Delta x}{c}} \approx 1 - i\omega(n-1)\frac{\Delta x}{c}$$

$$E_{\text{after plate}} = E_0 e^{i\omega(t-\frac{x}{c})} - \frac{i\omega(n-1)\Delta x}{c} \cdot E_0 e^{i\omega(t-\frac{x}{c})}$$

E_s

$E_a \rightarrow$ new electric field that is the result of all moving charges



$$ma = \sum F = -kz$$

$$m \frac{d^2 z}{dt^2} = -kz$$

$$\boxed{\frac{d^2 z}{dt^2} = -\frac{k}{m} z}$$

~~$$+\cancel{A} \omega^2 \cos(\omega t + \phi) = +\cancel{k} \cdot \cancel{A} \cos(\omega t + \phi)$$~~

$$\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$$

↑ one of our constants

one way

$$\rightarrow z = A \cos(\omega t + \phi)$$

$$\frac{dz}{dt} = -A\omega \sin(\omega t + \phi)$$

$$\rightarrow \frac{dz}{dt} = -A\omega^2 \cos(\omega t + \phi)$$

another way

$$z = A e^{wt + \phi}$$

$$\frac{dz}{dt} = A w e^{wt + \phi}$$

$$\frac{dz}{dt} = A w^2 e^{wt + \phi}$$

$$Aw^2 e^{wt + \phi} = -\frac{k}{m} \cdot A e^{wt + \phi}$$

$$\omega^2 = -\frac{k}{m}$$

$$z = A \cos(\sqrt{\frac{k}{m}} t + \phi) \leftarrow \text{general solution to that DE}$$

given by initial conditions
at $t=0, z=1 \Rightarrow A=1, \phi=0$

$(A + \phi)$ are determined by initial conditions

Now we have a charge connected by a spring being driven by $\vec{E}(t)$

$$z = 1 \cdot \cos(\sqrt{\frac{k}{m}} \cdot t)$$

$$\frac{d^2 z}{dt^2} = -\frac{k}{m} z + \frac{qeE_0}{m} e^{i\omega t + \phi}$$

the frequency of the light

$$\omega^2$$

\hookrightarrow natural frequency of the electrons

$$\text{Guess: } z = z_0 e^{i\omega t + \phi}$$

$$\dot{z} = z_0 i\omega e^{i\omega t + \phi}$$

$$\ddot{z} = z_0 \underbrace{i^2 \omega^2}_{-1} e^{i\omega t + \phi}$$

So now we plug in:

$$-\cancel{z_0 \omega^2} e^{i\omega t + \phi} = -\cancel{\omega^2 z_0} e^{i\omega t + \phi} + \frac{qeE_0}{m} e^{i\omega t + \phi}$$

\hookrightarrow solve for z_0

$$z_0 (\omega_0^2 - \omega^2) = \frac{qeE_0}{m}$$

$$z_0 = \frac{qeE_0}{m(\omega_0^2 - \omega^2)}$$

so...

$$z(t) = \frac{qeE_0}{m(\omega_0^2 - \omega^2)} \cdot e^{i\omega t + \phi}$$

motion of charges \rightarrow Electric field that results

$$E_a = -\frac{\eta q_c}{2\epsilon_0 c} i\omega \vec{z}_o e^{i\omega(t-\frac{x}{c})}$$

from Griffiths
or Feynman

$$E_a = -\frac{\eta q_e}{2\epsilon_0 c} \cdot i\omega \cdot \frac{q_e E_o}{m(\omega_0^2 - \omega^2)} \cdot e^{i\omega(t-\frac{x}{c})}$$

\Downarrow

vs.

$$E_a = -\frac{i\omega(n-1)\Delta x}{c} \cdot E_o e^{i\omega(t-\frac{x}{c})}$$

from before

so putting what survives together →

$$\frac{\eta q_e}{2\epsilon_0 m(\omega_0^2 - \omega^2)} = (n-1)\Delta x$$

η → number of charges per unit of area

N → number of charges per unit of volume

$$\eta = N \cdot \Delta x$$

$$\frac{N \cdot (\Delta x)^2 q_e}{2\epsilon_0 m(\omega_0^2 - \omega^2)} = (n-1)\Delta x$$

$$n = 1 + \frac{N \cdot q_e^2}{2\epsilon_0 m(\omega_0^2 - \omega^2)}$$

ω_0 is often much larger than ω for visible light

\vec{E} from a static sheet of charge

$$\vec{E} = \frac{Q}{2\epsilon_0 A} \vec{z}$$

$$E = \frac{Q}{2\epsilon_0} \vec{z}$$

$$E = \frac{\eta q_e}{2\epsilon_0} \vec{z}$$

} as ω rises the denominator gets smaller, so the fraction gets larger!

Let's do this again, but better. \rightarrow index of refraction for heavier dielectrics

\rightarrow Go back to the wave equation from Maxwell's:

$$\nabla^2 \vec{E} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}$$

Plug in!

$$\vec{E} = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\vec{P} = \vec{P}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$i^2 k^2 \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} - \epsilon_0 \mu_0 (+ i^2 \omega^2) \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \mu_0 (+ i^2 \omega^2) \vec{P}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

wave vector, k

$$-k^2 \vec{E}_0 + \epsilon_0 \mu_0 \omega^2 \vec{E}_0 = -\mu_0 \omega^2 \vec{P}_0$$

$$\vec{P}_0 = \epsilon_0 \chi(\omega) \vec{E}_0 \quad \leftarrow \text{constitutive relation}$$

\hookrightarrow susceptibility

\hookrightarrow kicking the can down the road!

\hookrightarrow complex!

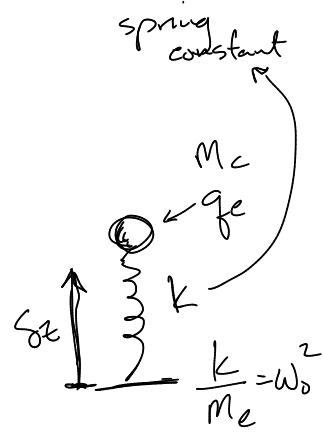
\hookrightarrow a function of ω (frequency)
just like $\vec{P}_0 + \vec{E}_0$.

\hookrightarrow plug in and let \vec{E}_0 cancel out

\rightarrow what is \vec{P} ?
polarization of an atom
(not the polarization
of light)

$$P = N \cdot q \cdot S z$$

↑
number
per
volume



$$-\vec{k}^2 + \epsilon_0 \mu_0 \omega^2 = -\epsilon_0 \mu_0 \omega^2 \chi(\omega)$$

↪ solve for \vec{k} , $\epsilon_0 \mu_0 = \frac{1}{c^2}$

$$\epsilon_0 (1 + \chi(\omega)) = \epsilon(\omega) \leftarrow \begin{array}{l} \text{permittivity} \\ \text{of a material} \end{array}$$

$$\vec{k}^2 = \epsilon_0 \mu_0 \omega^2 (1 + \chi(\omega))$$

still wave vector \vec{k}

$$\boxed{\vec{k} = \frac{\omega}{c} \sqrt{1 + \chi(\omega)}}$$

recall: $\vec{k} = \frac{\omega}{v}$ and $v = \frac{c}{n}$

$$\boxed{\vec{k} = \frac{\omega \cdot n}{c}}$$

$$\tilde{\vec{k}} = \frac{\tilde{n} \omega}{c}$$

$$\tilde{\vec{k}} = (n + i\kappa) \frac{\omega}{c}$$

compare!

$$\tilde{n} = \sqrt{1 + \chi(\omega)}$$

$$\tilde{n} = n(\omega) + i\kappa(\omega)$$

↪ light "slows down"

absorption coefficient

$$\tilde{n} = n' + i n''$$

Let's plug this back in to \vec{E}

$$\begin{aligned} \vec{E}(\vec{r}, t) &= \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \vec{E}_0 e^{i \frac{(n + i\kappa)\omega}{c} \hat{u} \cdot \vec{r} - i\omega t} \\ &= \vec{E}_0 e^{-\kappa \frac{\omega}{c} \hat{u} \cdot \vec{r} + i \left(\frac{n\omega}{c} \right) \hat{u} \cdot \vec{r} - i\omega t} \end{aligned}$$

$$\begin{aligned}
 &= \vec{E}_0 e^{-\frac{\kappa\omega}{c} \hat{n} \cdot \vec{r}} \cdot e^{i(\frac{n\omega}{c} \hat{n} \cdot \vec{r} - \omega t)} \\
 &\quad \leftarrow \text{has phase information} \\
 &= \vec{E}_0 e^{-\frac{\kappa\omega}{c} \hat{n} \cdot \vec{r}} \cdot \cos\left(\frac{n\omega}{c} \hat{n} \cdot \vec{r} - \omega t + \phi\right) + \text{some imaginary stuff we ignore}
 \end{aligned}$$

$\vec{E}_0 = \vec{E}_0 e^{i\phi}$ ← bad notation
 so watch out!

this wave
 decays as it
 goes into this
 material

$$V_{\text{phase}}(\omega) = \frac{c}{n(\omega)} \leftarrow \text{real part of } \tilde{n}$$

