

# Chapter 0 - Mathematical Review

Day 01

## Vectors

unit vectors

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{E} = \frac{kq}{r^2} \hat{r} = \frac{kq\vec{r}}{r^3}$$

fine, but incomplete  $\rightarrow$  it assumes the charge is located at the origin  
[N/C]

more general

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}$$

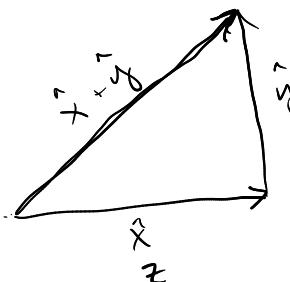
Example 0.1

$$\vec{r} = (2\hat{x} + 2\hat{y} + 2\hat{z})$$

$$\vec{r}_0 = (1\hat{x} + 1\hat{y} + 2\hat{z})$$

$$\vec{r} - \vec{r}_0 = 1\hat{x} + 1\hat{y} + 0\hat{z} = \hat{x} + \hat{y}$$

$$|\vec{r} - \vec{r}_0|^3 = |\hat{x} + \hat{y}|^3 = (\sqrt{2})^3 = 2^{3/2} = 2\sqrt{2}$$



$$\boxed{\frac{\vec{r}}{|\vec{r}|} = \hat{r}}$$

C = Coulomb 260112 M

$$9 \cdot 10^9 \frac{\text{Nm}^2}{\text{C}^2}$$

$$k = \frac{1}{4\pi\epsilon_0}$$

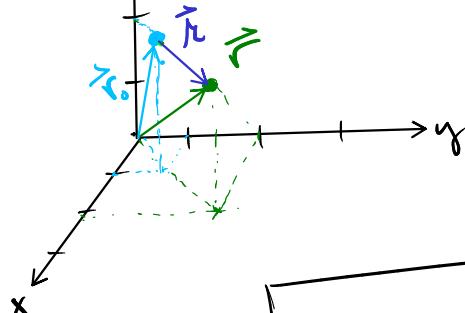
$$8.85 \cdot 10^{-12} \frac{\text{C}^2}{\text{Nm}^2}$$

$\vec{r} \Rightarrow$  location where we want to know the electric field

$\vec{r}_0 = \vec{r}_{\text{naught}}$

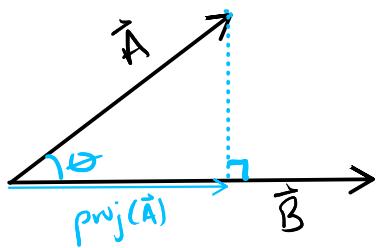
location of the charge

$$\vec{r} - \vec{r}_0 = \vec{r}_0 \Rightarrow \vec{r}_0 + \vec{r} = \vec{r}$$

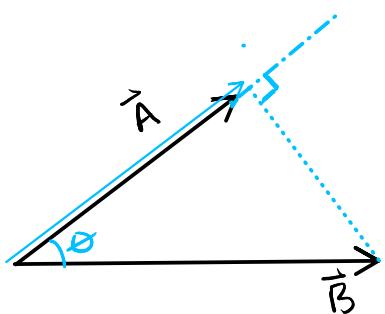


$$\boxed{\vec{E}(\vec{r}) = \frac{q(\hat{x} + \hat{y})}{8\pi\epsilon_0 \sqrt{2}}}$$

## Dot Product (inner product)



$$\vec{A} \cdot \vec{B} = \underbrace{A_x B_x + A_y B_y + A_z B_z}_{\text{scalar}} = |A| |B| \cos \theta$$



## Cross Product (vector product) / Day 02 26.01.24

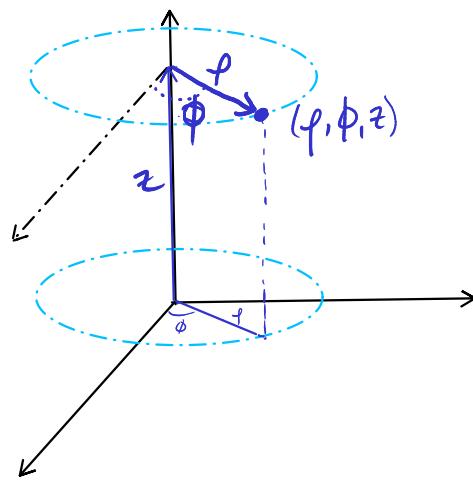
(perp. to both  $\vec{A} + \vec{B}$ )

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \text{determinant}$$

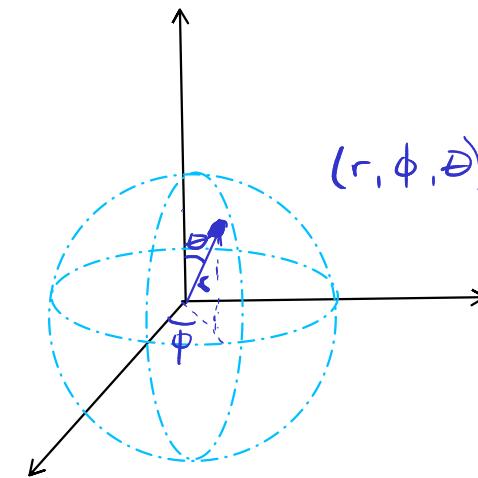
$$\begin{aligned} &= \hat{x}(A_y B_z - A_z B_y) \\ &- \hat{y}(A_x B_z - A_z B_x) \\ &+ \hat{z}(A_x B_y - A_y B_x) \end{aligned}$$

$$|\vec{C}| = |\vec{A} \times \vec{B}| = |A| |B| \sin \theta$$

Cylindrical Coords.



Spherical Coords.



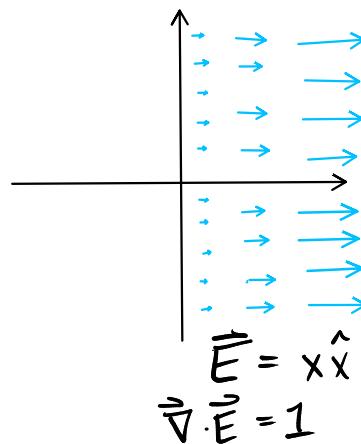
Gradient: of a scalar function  
(directional derivative)

$$\vec{\nabla} f(x, y, z) = \underbrace{\frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}}_{\substack{\text{scalar} \\ \text{function}}} \quad \text{vector result}$$

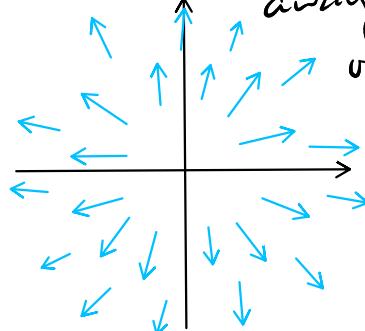
points in the direction of steepest decline

Divergence: of a vector function

$$\vec{\nabla} \cdot \vec{E} = \underbrace{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}_{\text{Scalar}}$$



uniform field pointed radially away from origin



Curl: of a vector function

| Day 03  
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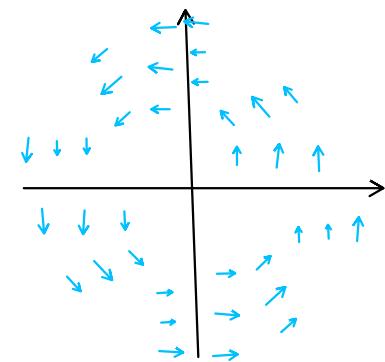
$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \hat{x} \left( \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right)$$

$$- \hat{y} \left( \frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right)$$

$$+ \hat{z} \left( \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

vector



Second Derivatives  $\rightarrow$  Laplacian

$$\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

—

$$\nabla^2 \vec{E} = \left( \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \right) \hat{x}$$

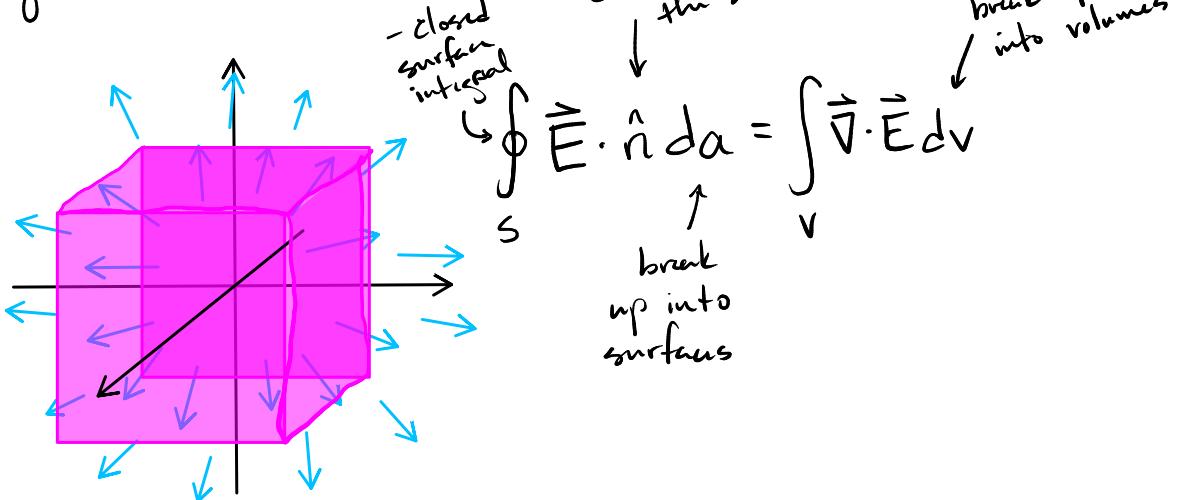
$$\vec{E}(x, y, z) = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$$

$$\left( \frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \right) \hat{y}$$

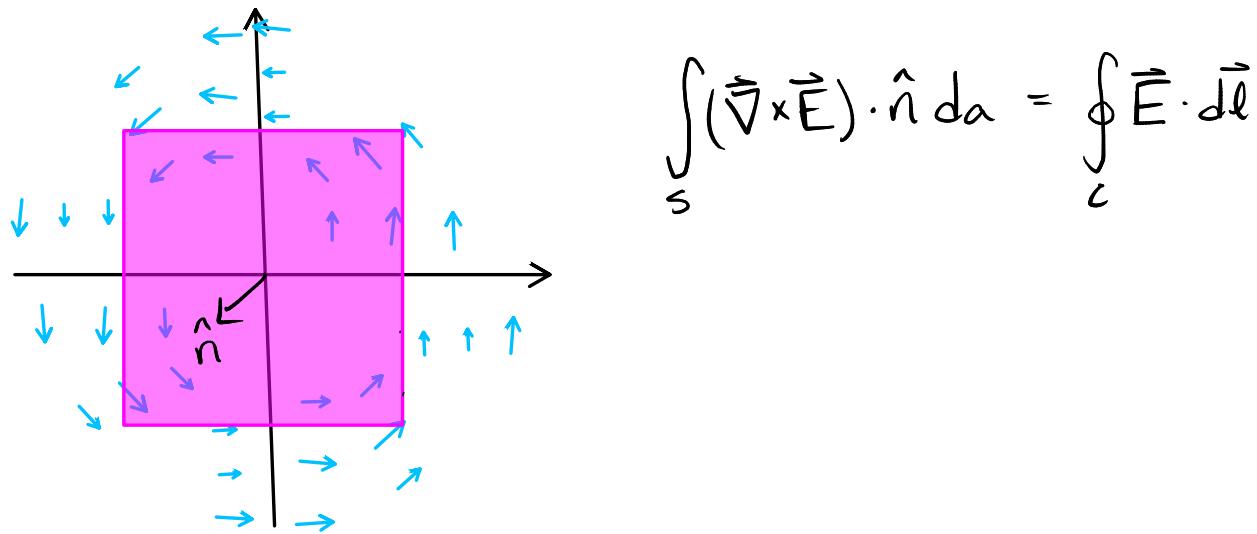
$$\left( \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) \hat{z}$$

→  $\nabla^2 \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{E})$  ← true in any coordinate system

## Divergence Theorem



## Stokes Theorem



## Complex Numbers

imaginary number  $\rightarrow i = \sqrt{-1}$

→ sum of a real number + an imaginary number

complex function  $f + g$

$$\operatorname{Re}\{f\} + \operatorname{Re}\{g\} = \operatorname{Re}\{f+g\}$$

$$\frac{d}{dx} \operatorname{Re}\{f\} = \operatorname{Re}\left\{\frac{df}{dx}\right\}$$

$$\int \operatorname{Re}\{f\} dx = \operatorname{Re}\left\{\int f dx\right\}$$

$$\tilde{z} = a + bi$$

$$\operatorname{Re}\{a+bi\} = a$$

$$\operatorname{Im}\{a+bi\} = b$$

$$|z^2| = |z|^2 = a^2 + b^2$$

$$a = |\tilde{z}| \cos \theta$$

$$b = |\tilde{z}| \sin \theta$$

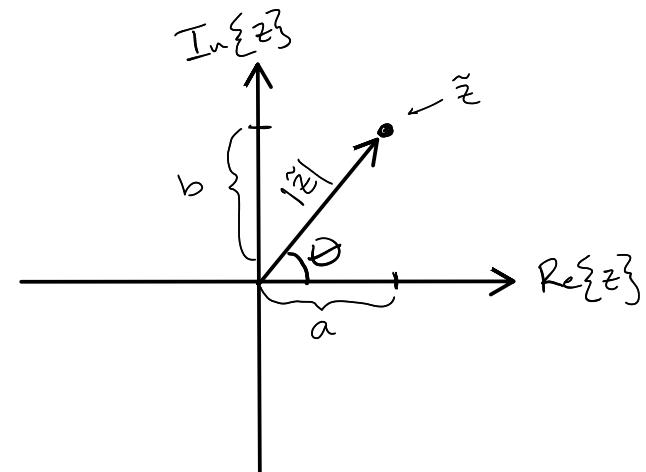
Now:

$$\tilde{z} = |\tilde{z}|(\cos \theta + i \sin \theta)$$

So:

$$\tilde{z} = |\tilde{z}| e^{i\theta}$$

## Complex Plane



## Taylor Series

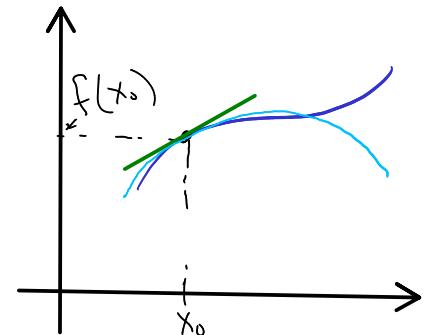
$$f(x) = f(x_0) + \underbrace{\frac{1}{1!} (x-x_0) \cdot \frac{df}{dx}\Big|_{x=x_0}}_{\text{linear term}} + \underbrace{\frac{1}{2!} (x-x_0)^2 \frac{d^2 f}{dx^2}\Big|_{x=x_0}}_{\text{quadratic term}}$$

$$\cos \phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \dots$$

$$i \sin \phi = i\phi - \frac{i\phi^3}{3!} + \frac{i\phi^5}{5!} - \dots$$

$$e^{i\phi} = 1 + i\phi - \frac{\phi^2}{2!} - \frac{i\phi^3}{3!} + \frac{\phi^4}{4!} + \dots$$

$$e^{i\phi} = \cos \phi + i \sin \phi \quad \text{Euler's identity}$$



$$e^{i\pi} = -1$$

$$e^{i\pi} + 1 = 0$$

## Complex Conjugate

$$\underline{\underline{z^*}} = \underline{\underline{a - bi}} \quad \text{or} \quad \underline{\underline{z^*}} = |\tilde{z}| e^{i\theta}$$

$$\tilde{z} \cdot \tilde{z}^* = |\tilde{z}|^2$$

$$\tilde{z} \cdot \tilde{z}^* = |z|e^{i\theta} |\bar{z}|e^{-i\theta}$$

$$\hat{z} \cdot \hat{z}^* = |\hat{z}|^2$$

$$\operatorname{Re}\{\tilde{z}\} = \frac{\tilde{z} + \tilde{z}^*}{2} \quad \xrightarrow{\text{arrow}} \quad \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta = \operatorname{Re}\{e^{i\theta}\}$$

$$\operatorname{Im}\{\tilde{z}\} = \frac{\tilde{z} - \tilde{z}^*}{2i}$$

HW3: 0.14, 17, 18, 20

$$0.14 \left| \begin{array}{l} z_1 = 1 - i \\ z_2 = 3 + 4i \end{array} \right.$$

$$a) z_1 - z_2 = (1-i) - (3+4i)$$

$$= 1 - i - 3 - 4i$$

$$z_1 - z_2 = (1-i) - (3+4i)$$

$$= 1 - i - 3 - 4i$$

$$z_1 - z_2 = \underline{\underline{-2-5i}} = z_3 = \underline{\underline{|z|e^{i\theta}}} \quad \left. \begin{array}{l} \text{magnitude} \\ \downarrow \\ \text{phase} \end{array} \right\} \text{polar form}$$

$$b) \frac{z_1}{z_2} = \frac{1-i}{3+4i} \cdot \frac{(3-4i)}{(3-4i)} = \text{~~~~~} \rightarrow \text{polar form}$$

Q.17 Show  $\operatorname{Re}\{\tilde{A}\} \times \operatorname{Re}\{\tilde{B}\} = \frac{AB + A^*B}{4} + \text{CC.}$

$\underbrace{\quad}_{\text{Complex conjugate}} \rightarrow \frac{A^*B^* + AB^*}{4} = \frac{AB + A^*B}{4} + \frac{A^*B^* + AB^*}{4}$

?  $\tilde{A} = A_1 + A_2i$

?  $\tilde{A}^* = A_1 - A_2i$

$$\begin{aligned} &= \frac{AB + A^*B + A^*B^* + AB^*}{4} \\ &= \frac{AB + AB^* + A^*B + A^*B^*}{4} \\ &= \frac{A(B + B^*) + A^*(B + B^*)}{4} \\ &= \frac{(A + A^*)(B + B^*)}{4} \\ &= \frac{2\operatorname{Re}\{A\} \cdot 2\operatorname{Re}\{B\}}{4} \\ &= \operatorname{Re}\{A\} \cdot \operatorname{Re}\{B\} \end{aligned}$$

Q.18  $E_o = |E_o| e^{i\delta_E} \quad B_o = |B_o| e^{i\delta_B}$

$$4 \cdot \operatorname{Re}\{E_o e^{i(kz-wt)}\} \quad \operatorname{Re}\{B_o e^{i(kz-wt)}\} =$$

$$\begin{aligned} &|E_o| e^{i\delta_E} e^{i(kz-wt)} \cdot |B_o| e^{i\delta_B} e^{i(kz-wt)} \\ &+ |E_o| e^{i\delta_E} e^{i(kz-wt)} \cdot |B_o| e^{-i\delta_B} e^{-i(kz-wt)} \\ &+ |E_o|^{-i\delta_E} e^{-i(kz-wt)} \cdot |B_o| e^{i\delta_B} e^{i(kz-wt)} \\ &+ |E_o|^{-i\delta_E} e^{-i(kz-wt)} \cdot |B_o| e^{-i\delta_B} e^{-i(kz-wt)} \end{aligned} \quad \left. \right\} E_o B_o^* + E_o^* B_o$$

$$\operatorname{Re}\{\tilde{A}\} \times \operatorname{Re}\{\tilde{B}\} = \frac{AB + AB^* + A^*B + A^*B^*}{4}$$

$$R_C \{ E_o e^{i(kz - \omega t)} \} R_C \{ B_o e^{i(kz - \omega t)} \} = \frac{1}{4} (E_o B_o^* + E_o^* B_o)$$

$$+ \frac{1}{4} |E_o| e^{i\delta_E} e^{i(kz - \omega t)} \cdot |B_o| e^{i\delta_B} e^{i(kz - \omega t)}$$

$$+ \frac{1}{4} |E_o|^{-i\delta_E - i(kz - \omega t)} \cdot |B_o|^{-i\delta_B - i(kz - \omega t)}$$

$$= \frac{1}{4} (E_o B_o^* + E_o^* B_o)$$

$$+ \frac{1}{4} |E_o| |B_o| \left( e^{i\delta_E} e^{i\delta_B} \cdot e^{2i(kz - \omega t)} + e^{-i\delta_E} e^{-i\delta_B} \cdot e^{-2i(kz - \omega t)} \right)$$

$$= \frac{1}{4} (E_o B_o^* + E_o^* B_o)$$

$$+ \frac{1}{4} |E_o| |B_o| \left( e^{i(2kz - \omega t) + \delta_E + \delta_B} + e^{-i(2kz - \omega t) + \delta_E + \delta_B} \right)$$

$$= \frac{1}{4} (E_o B_o^* + E_o^* B_o)$$

$$+ \frac{1}{4} |E_o| |B_o| \left( 2 \cdot \cos[2(kz - \omega t) + \delta_E + \delta_B] \right)$$

$$= \frac{1}{4} (E_o B_o^* + E_o^* B_o) + \frac{1}{2} |E_o| |B_o| \cdot \cos[2(kz - \omega t) + \delta_E + \delta_B]$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\bar{e}^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin\theta$$

$$e^{i\theta} + \bar{e}^{-i\theta} = 2\cos\theta$$

P.Q.20]  $A \cos(\omega t) + 2A \sin\left(\omega t + \frac{\pi}{4}\right)$  → write as a single phase shift cosine

↓ amplitude      ↓ phase  
 $\Rightarrow B \cos(\omega t + \phi)$

if  $\cos\alpha = \operatorname{Re}\{e^{i\alpha}\}$        $\operatorname{Re}\{e^{i\alpha}\} = \cos\alpha + i\sin\alpha \quad \operatorname{Re}\{e^{i\alpha}\} = \cos\alpha$

$$A \cos(\omega t) = A \cdot \operatorname{Re}\{e^{i\omega t}\}$$

if  $\sin(\beta) = \operatorname{Re}\{-ie^{i\beta}\}$   
 $2A \sin\left(\omega t + \frac{\pi}{4}\right) = 2A \cdot \operatorname{Re}\{-ie^{i(\omega t + \frac{\pi}{4})}\}$

So now:

$$A \cdot \operatorname{Re}\{e^{i\omega t}\} + 2A \cdot \operatorname{Re}\{-ie^{i(\omega t + \frac{\pi}{4})}\}$$

$$\operatorname{Re}\{Ae^{i\omega t} - 2Aie^{i(\omega t + \frac{\pi}{4})}\} \leftarrow$$

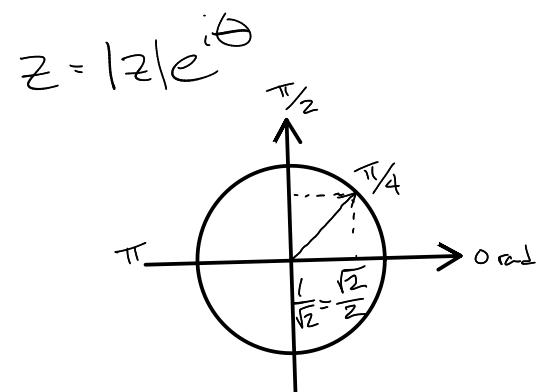
$$\operatorname{Re}\{A(e^{i\omega t} - 2ie^{i\omega t} e^{\frac{i\pi}{4}})\}$$

$$\operatorname{Re}\{Ae^{i\omega t}(1 - 2ie^{\frac{i\pi}{4}})\}$$

$$\Rightarrow e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{\frac{i\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$



$$\operatorname{Re}\{B e^{i(\omega t + \phi)}\}$$

↑ amplitude      ↓ phase  $\phi$   
 $B$

$$Re \left\{ Ae^{i\omega t} \left( 1 - \cancel{2i} \left( \frac{\sqrt{2}}{2} (1+i) \right) \right) \right\}$$

$$Re \left\{ Ae^{i\omega t} \left( 1 - \cancel{\sqrt{2}} (i-1) \right) \right\}$$

$$\left. \begin{array}{c} 1 - i\sqrt{2} + \sqrt{2} \\ ((1+\sqrt{2}) - i\sqrt{2}) \end{array} \right\}$$

$$B \cos(\omega t + \phi)$$

$$\phi = 0$$

$$|z| = \sqrt{z \cdot z^*}$$

$$z = A((1+\sqrt{2}) - i\sqrt{2})$$

$$|z| = \sqrt{A((1+\sqrt{2}) - i\sqrt{2}) \cdot A((1+\sqrt{2}) + i\sqrt{2})}$$

$$= \sqrt{A^2 \left[ (1+\sqrt{2})^2 + (\sqrt{2})^2 \right]}$$

$$= A \sqrt{1 + 2\sqrt{2} + 2 + 2}$$

$$= A \sqrt{5 + 2\sqrt{2}}$$