Chapter 2 - Plane Waves + Index of Refraction

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Day 12

plane wave:

$$\vec{E}(\vec{r},t) = \vec{E}_0 \cdot \cos(\vec{k}\cdot\vec{r} - \omega \cdot t + \phi)$$

"wave vector" "k" → $\vec{k} = k\hat{u} = \frac{2\pi}{\lambda_{vac}}\hat{u}$

direction of propogation

angular frequency → $\omega = \frac{2\pi \cdot c}{\lambda_{vac}} = 2\pi \cdot \nu$

$k + \omega$ are related just like $\lambda + \nu$

$$c = \frac{\lambda_{vac}}{T} = \lambda_{vac} \cdot \nu = \underbrace{\frac{\omega}{k}}$$

this relationship is known as the "dispersion relation"

"kappa" wavenumber

$$\frac{1}{\lambda_{vac}} = \kappa = [cm^{-1}]$$

$C = V_{vac}$ ← velocity of light in vacuum

ν ν ← "nu" ← frequency

similarly for magnetic field:

$$\vec{B}(\vec{r},t) = \underbrace{\vec{B}_0}\cos(\vec{k}\cdot\vec{r} - \omega t + \phi)$$

$$\vec{B}_0 = \underbrace{\frac{\vec{k}\times\vec{E}_0}{\omega}},$$ so $\vec{B}_0$ is not independent, but is determined by other parameters.

$$\vec{B}_0 \perp \vec{k} \perp \vec{E}$$

also think about magnitude

$$B_0 = \frac{kE_0}{\omega} = \frac{E_0}{c}$$ ← since this is so large we will focus on the $\vec{E}$ field.

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complex plane waves:

$$\vec{E}(\vec{r},t) = Re\left\{\tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right\}$$

$$\tilde{\vec{E}}_0 = \vec{E}_0 e^{i\phi}$$ ← phase shift

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

Summarize some facts that we know now:

$\lambda \rightarrow$ wavelength

$T \rightarrow$ period

$\nu \rightarrow$ frequency $\left(\frac{1}{T}\right)$

$k \rightarrow$ wave vector $\left(\frac{2\pi}{\lambda}\right)$

$\omega \rightarrow$ angular frequency $\left(\frac{2\pi}{T}\right)$

$V = \frac{\omega}{k}$

$V = \frac{\lambda}{T} = \lambda \cdot \nu$ ← any wave

$c = \lambda_{vac} \cdot \nu$

$\epsilon_0 \mu_0 = \frac{1}{c^2} \Rightarrow c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$

permittivity of free space $\rightarrow \epsilon_0 = 8.85 \cdot 10^{-12} \frac{C^2}{Nm^2}$

permeability of free space $\rightarrow \mu_0 = 4\pi \cdot 10^{-7} \frac{T}{Am^2}$

- $\nabla^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}$
- $\nabla^2 \vec{E} = \frac{1}{V^2} \frac{\partial^2 E}{\partial t^2}$
- $\nabla^2 \vec{E} = \frac{k^2}{\omega^2} \frac{\partial^2 E}{\partial t^2}$

Q1: what is light? ✓

Q2: where does light come from? ✓

Q3: can the speed of light change? see Q6

Q4: what affects the brightness of light?

Q5: how do human perceive light amount + color

Q6: why does light slow down in materials Q3

Q7: what is reflection

Speed of Light in matter

↳ light slows down in materials

$$\frac{c}{V} = n$$ ← index of refraction

$n = 1$ ← vacuum

$n = 1.0003$ ← air

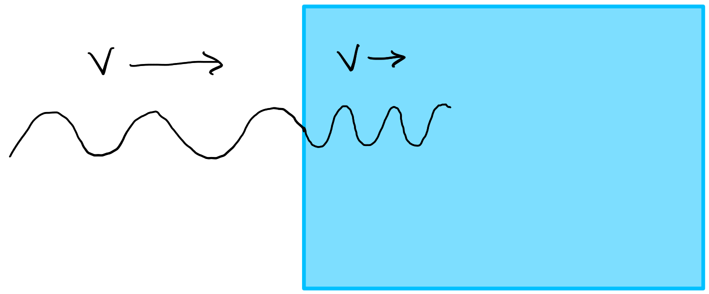
$n = 1.33$ ← water

$n = 1.5$ ← glass

constant as n increases

$$\lambda \cdot \nu = v = \frac{c}{n}$$

λ gets smaller as n increases



$$\lambda_{vac}\,\nu = c \iff c = n \cdot v$$

$$\lambda \cdot \nu$$

$$\lambda_{vac}\,\cancel{\nu} = n \cdot \lambda \cdot \cancel{\nu}$$

← wavelength in material

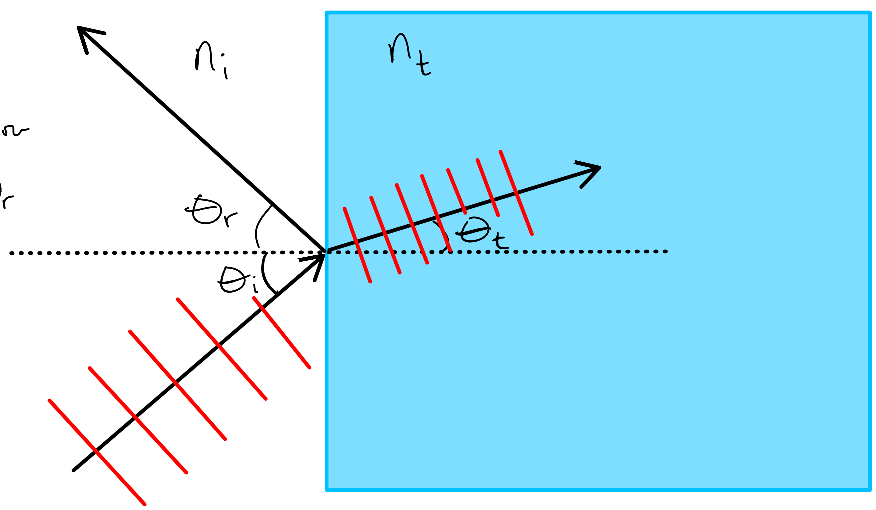
$$\lambda = \frac{\lambda_{vac}}{n}$$ } λ will be smaller than λ_0 since $n > 1$

Table 23.1 Indices of Refraction for $\lambda = 589.3$ nm in Vacuum (at 20°C Unless Otherwise Noted)

Material	Index
Solids	
Ice (at 0°C)	1.309
Fluorite	1.434
Fused quartz	1.458
Polystyrene	1.49
Lucite	1.5
Plexiglas	1.51
Crown glass	1.517
Plate glass	1.523
Sodium chloride	1.544
Light flint glass	1.58
Dense flint glass	1.655
Sapphire	1.77
Zircon	1.923
Diamond	2.419
Titanium dioxide	2.9
Gallium phosphide	3.5
Liquids	
Water	1.333
Acetone	1.36
Ethyl alcohol	1.361
Carbon tetrachloride	1.461
Glycerin	1.473
Sugar solution (80%)	1.49
Benzene	1.501
Carbon disulfide	1.628
Methylene iodide	1.74

Law of reflection

$\theta_i = \theta_r$



Snell's Law

$$n_i \sin\theta_i = n_t \sin\theta_t$$

260213 F

Day 14

static case

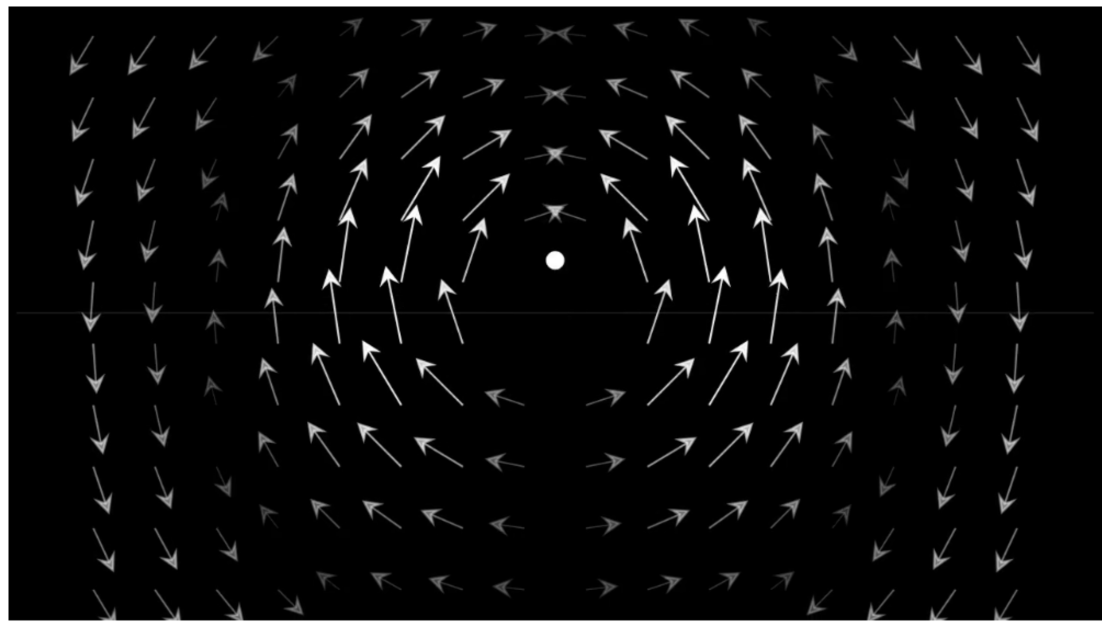
$E(\vec{r}) = \frac{kq_0}{r^2}$ $\longrightarrow$

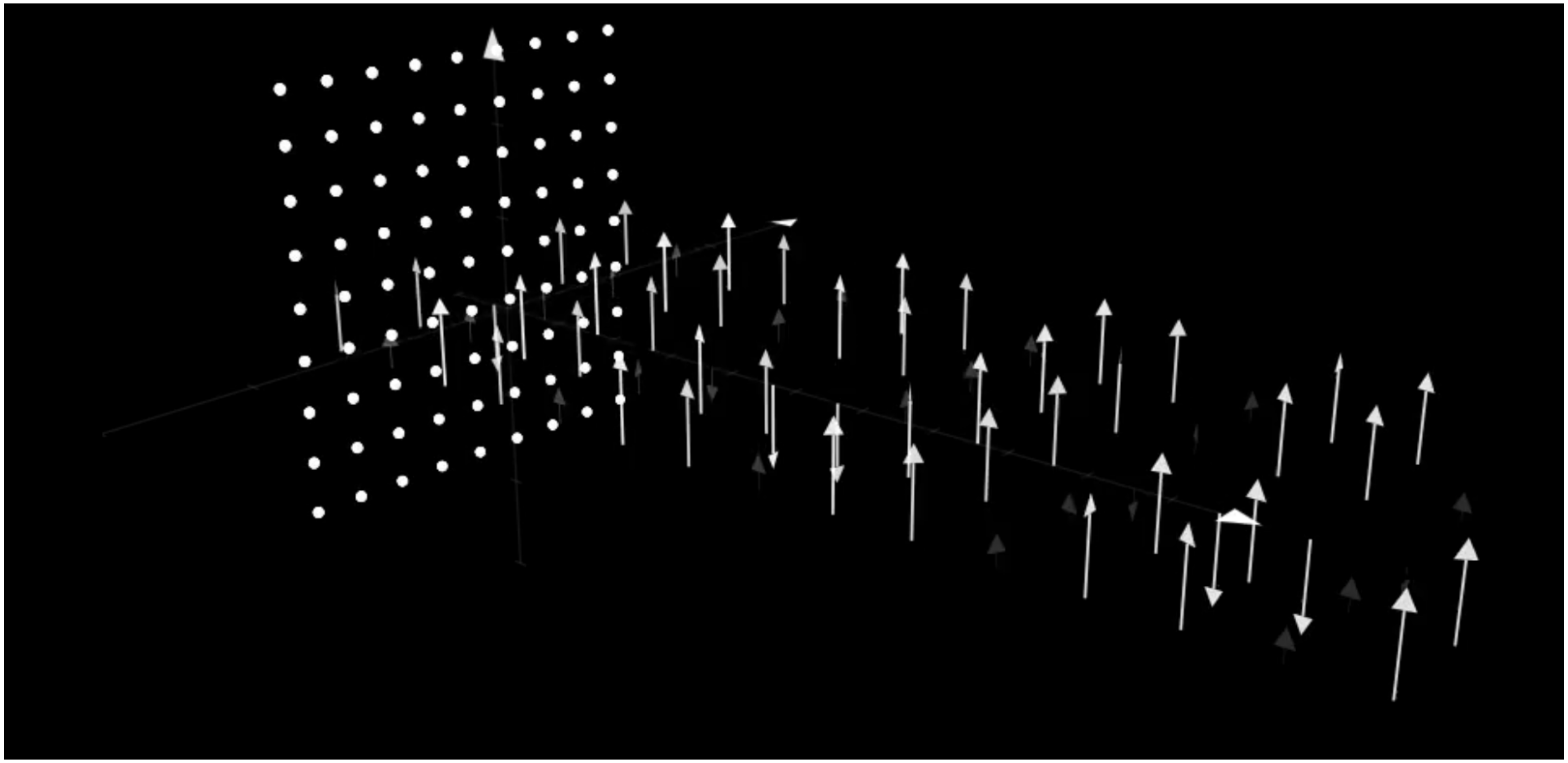
oscillating charge

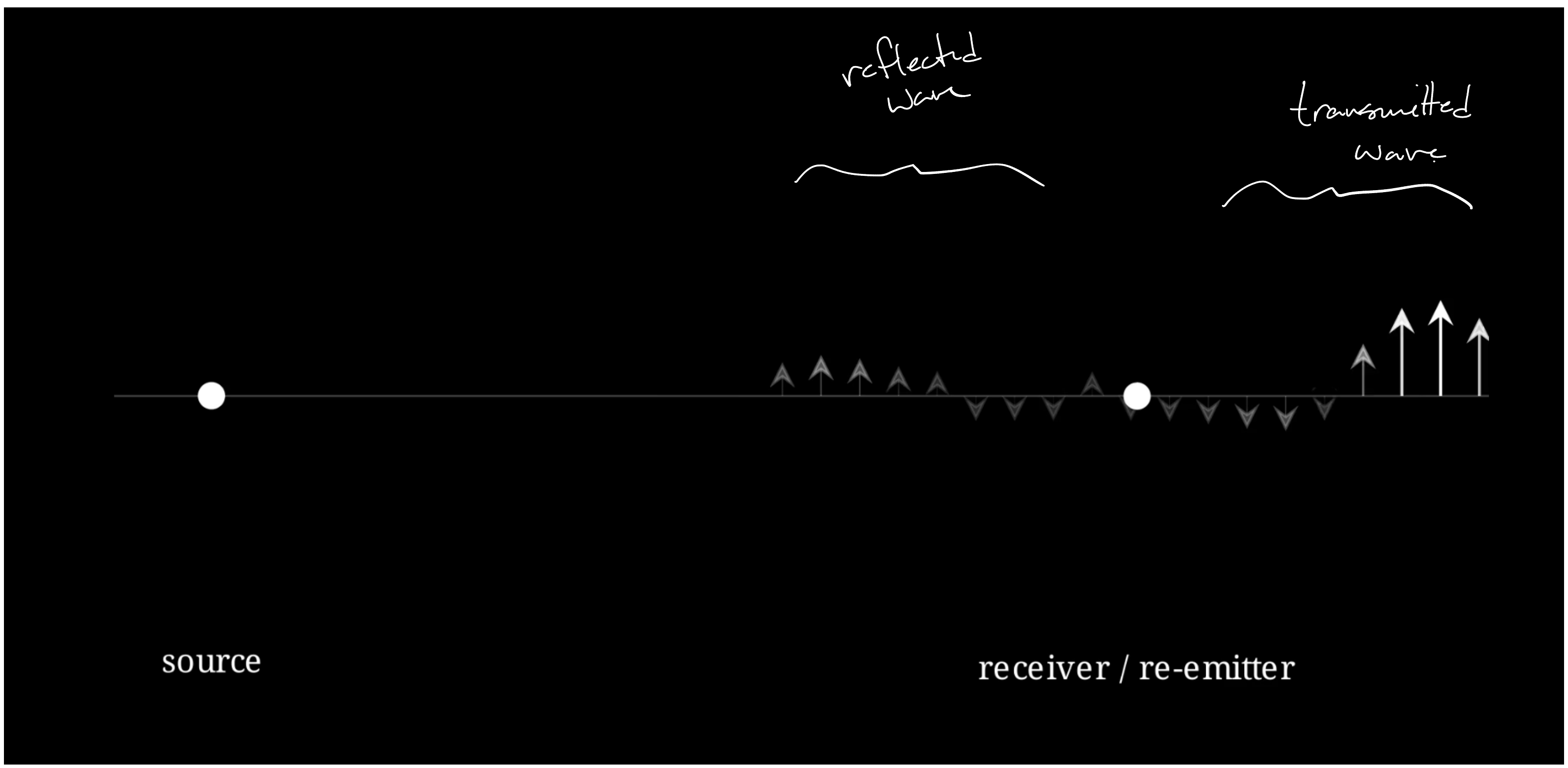
$\vec{E}(\vec{r},t) = \vec{E}_{staticish} + \underbrace{\vec{E}_{rad}}$

$\vec{E}_{rad} \propto \frac{\vec{a}_\perp \overbrace{(t - r/c)}^{\text{wave behavior}}}{r}$

compare to $\vec{r} - vt$






reflected wave
transmitted wave
source
receiver / re-emitter

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Day 15

Dielectric Media

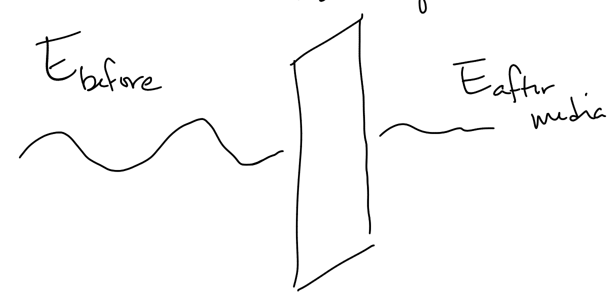
↳ insulator (like glass, water, air)

Assumptions:
1. isotropic → same in all directions
2. homogeneous → same over distance
3. non-conducting (insulator/dielectric) $\vec{J}_{free} = 0$
4. very thin material (air or some other thin gas)

} we will relax this later on

Light in a thin gas → Feynman, vol 1, Ch 31

Light in matter = incident field + field reradiated by driven charges



$$\vec{E}_{\text{after plate}} = \vec{E}_s + \sum_{\text{all other charges}} \vec{E}_{\text{each charge}}$$

$$\vec{E}_s = \vec{E}_0 \cos(kx - \omega t)$$

$$\vec{E}_s = \vec{E}_0 \cos\left(k\left(x - \tfrac{\omega}{k} t\right)\right)$$

$\frac{\omega}{k} = c$

$$\vec{E} = \vec{E}_0 \cos\left(\omega\left(\tfrac{k}{\omega} x - t\right)\right)$$

$$\vec{E} = \vec{E}_0 \cos\left(-\omega\left(t - \tfrac{k}{\omega} x\right)\right)$$

$$\vec{E} = \vec{E}_0 \cos\left(\omega\left(t - \frac{x}{c}\right)\right)$$

$$\vec{E} = \vec{E}_0 e^{i\omega\left(t - \frac{x}{c}\right)}$$

So now what about the field from other charges?

plate thickness = Δx

$$t_{vac} = \frac{\Delta x}{c}$$

$$t_{plate} = \frac{n \Delta x}{c}$$

$$\Delta t = t_{plate} - t_{vac}$$

$$= (n-1)\frac{\Delta x}{c}$$

$$E_{\substack{after \\ plate}} = E_0 e^{i\omega\left(t - \Delta t - \frac{x}{c}\right)}$$

$$= E_0 e^{i\omega\left(t - (n-1)\frac{\Delta x}{c} - \frac{x}{c}\right)}$$

$$= \underbrace{e^{-i\omega(n-1)\frac{\Delta x}{c}}}_{\text{w/ a phase shift}} \underbrace{E_0 e^{i\omega\left(t - \frac{x}{c}\right)}}_{\text{just like the original wave BUT w/ a}}$$

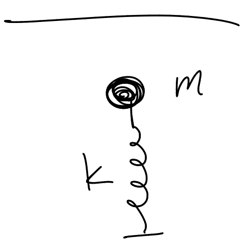
$e^{-i\omega(n-1)\frac{\Delta x}{c}}$, Δx is very small

$\boxed{e^{\delta} \approx 1 + \delta}$, for small δ

$$e^{-i\omega(n-1)\frac{\Delta x}{c}} \approx 1 - i\omega(n-1)\frac{\Delta x}{c}$$

$$E_{\substack{\text{after}\\\text{plate}}} = \underbrace{E_0 e^{i\omega(t-\frac{x}{c})}}_{E_s} - \underbrace{\frac{i\omega(n-1)\Delta x}{c} \cdot E_0 e^{i\omega(t-\frac{x}{c})}}_{E_a}$$

E_a → new electric field that is the result of all moving charges



$$ma = \Sigma F = -kz$$

$$m\frac{d^2z}{dt^2} = -kz$$

$$\boxed{\frac{d^2z}{dt^2} = -\frac{k}{m}z}$$

$$-\cancel{A}\omega^2\cancel{\cos(\omega t+\phi)} = -\frac{k}{m}\cdot\cancel{A\cos(\omega t+\phi)}$$

$\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$ ↖ one of our constants

<u>one way</u>

$\rightarrow z = A\cdot\cos(\omega\cdot t+\phi)$

$$\frac{dz}{dt} = -A\omega\cdot\sin(\omega t+\phi)$$

$\rightarrow \frac{d^2z}{dt^2} = -A\omega^2\cos(\omega t+\phi)$

<u>another way</u>

$$z = Ae^{\omega t+\phi}$$

$$\frac{dz}{dt} = A\omega e^{\omega t+\phi}$$

$$\frac{d^2z}{dt^2} = A\omega^2 e^{\omega t+\phi}$$

$$A\omega^2 e^{\omega t+\phi} = -\frac{k}{m}\cdot Ae^{\omega t+\phi}$$

$$\omega^2 = -\frac{k}{m}$$

$z = A\cdot\cos(\sqrt{\frac{k}{m}}\,t + \phi)$ ← general solution to that DE

↳ given by initial conditions

at $t=0$, $z=1 \Rightarrow A=1$ $\phi=0$

($A + \phi$ are determined by initial conditions)

$z = 1 \cdot \cos\left(\sqrt{\frac{k}{m}} \cdot t\right)$

Now we have a charge connected by a spring being driven by $\vec{E}(t)$

$$\frac{d^2 z}{dt^2} = -\underbrace{\frac{k}{m}}_{\omega_0^2} z + \frac{q_e E_0}{m} e^{i\omega t + \phi}$$

$i\omega t + \phi$ ← the frequency of the light

ω_0^2 ↳ natural frequency of the electrons

Guess: $z = z_0 e^{i\omega t + \phi}$

$$\dot{z} = z_0 i\omega e^{i\omega t + \phi}$$

$$\ddot{z} = z_0 \underbrace{i^2}_{-1} \omega^2 e^{i\omega t + \phi}$$

So now we plug in:

$$-z_0 \omega^2 \cancel{e^{i\omega t+\phi}} = -\omega_0^2 z_0 \cancel{e^{i\omega t+\phi}} + \frac{q_e E_0}{m} \cancel{e^{i\omega t+\phi}}$$

↳ solve for z_0

$$z_0(\omega_0^2 - \omega^2) = \frac{q_e E_0}{m}$$

$$z_0 = \frac{q_e E_0}{\underbrace{m(\omega_0^2 - \omega^2)}}$$

so... ⟹

$$\underbrace{z(t) = \frac{q_e E_0}{m(\omega_0^2 - \omega^2)} \cdot e^{i\omega t + \phi}}_{\text{motion of charges}}$$

motion of charges ⟶ Electric field that results

$$E_a = \frac{-\eta q_e}{2\epsilon_0 c} \, i\omega z_0 \, e^{i\omega(t - \frac{x}{c})}$$

← from Griffiths or Feynman

$\vec{E}$ from a static sheet of charge

$$E = \frac{Q}{2\epsilon_0 A}$$

$$E = \frac{\sigma}{2\epsilon_0}$$

$$E = \frac{\eta q_e}{2\epsilon_0}$$

from before

$$E_a = \frac{-\eta q_e}{2\epsilon_0 c} \cdot i\omega \cdot \frac{q_e E_0}{m(\omega_0^2 - \omega^2)} \cdot e^{i\omega(t-\frac{x}{c})} \quad \text{vs.} \quad E_a = \frac{-i\omega(n-1)\Delta x}{c} \cdot E_0 \, e^{i\omega(t-\frac{x}{c})}$$

So putting what survives together

$$\frac{\eta q_e^2}{2\epsilon_0 m(\omega_0^2 - \omega^2)} = (n-1)\Delta x$$

η → number of charges per unit of area

N → number of charges per unit of volume

$$\eta = N \cdot \Delta x$$

$$\frac{N \cdot \Delta x \, q_e^2}{2\epsilon_0 m(\omega_0^2 - \omega^2)} = (n-1)\Delta x$$

$$n = 1 + \frac{N \cdot q_e^2}{2\epsilon_0 m \underbrace{(\omega_0^2 - \omega^2)}_{\omega_0 \text{ is often much larger than } \omega \text{ for visible light}}}$$

as ω rises the denominator gets smaller, so the fraction gets larger!

Let's do this again, but better. → index of refraction for heavier dielectrics

→ Go back to the wave equation from Maxwell's:

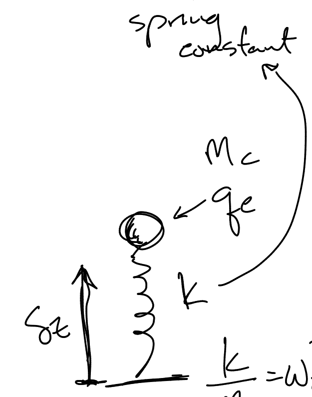
$$\nabla^2 \vec{E} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}$$

→ what is $\vec{P}$?

polarization of an atom (not the polarization of light)

$$P = N \cdot q \cdot \delta z$$

↑ number per volume



$$\frac{k}{m_e} = \omega_0^2$$

plug in!

$$\begin{cases} \vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \\ \vec{P} = \vec{P}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \end{cases}$$

$$\underbrace{i^2}_{-1} k^2 E_0 \cancel{e^{i(\vec{k}\cdot\vec{r} - \omega t)}} - \epsilon_0 \mu_0 (+\underbrace{i^2}_{-1} \omega^2) \vec{E}_0 \cancel{e^{i(\vec{k}\cdot\vec{r} - \omega t)}} = \mu_0 (+\underbrace{i^2}_{-1} \omega^2) \vec{P}_0 \cancel{e^{i(\vec{k}\cdot\vec{r} - \omega t)}}$$

↳ wave vector, k

$$-k^2 \vec{E}_0 + \epsilon_0 \mu_0 \omega^2 \vec{E}_0 = -\mu_0 \omega^2 \vec{P}_0$$

$$\vec{P}_0 = \epsilon_0 \chi(\omega) \vec{E}_0 \quad \leftarrow \text{constitutive relation}$$

↳ susceptibility

↳ kicking the can down the road!

↳ complex!

↳ a function of ω (frequency) just like $\vec{P}_0$ + $\vec{E}_0$

↳ plug in and let $\vec{E}_0$ cancel out

$$-k^2 + \epsilon_0\mu_0\omega^2 = -\epsilon_0\mu_0\omega^2\chi(\omega)$$

$\hookrightarrow$ solve for k, $\epsilon_0\mu_0 = \frac{1}{c^2}$

$$\epsilon_0\left(1 + \chi(\omega)\right) = \epsilon(\omega) \leftarrow \text{permittivity of a material}$$

$$k^2 = \epsilon_0\mu_0\omega^2\left(1 + \chi(\omega)\right)$$

still wave vector k $\rightarrow$ $\boxed{k = \frac{\omega}{c}\sqrt{1 + \chi(\omega)}}$

recall: $k = \frac{\omega}{v}$ and $v = \frac{c}{n}$

$$\boxed{k = \frac{\omega \cdot n}{c}}$$

$\hookrightarrow$ compare!

$$\mathcal{N}(\omega) \equiv n(\omega) + i\kappa(\omega) = \sqrt{1+\chi(\omega)}$$

$$\tilde{n} = \sqrt{1 + \chi(\omega)}$$

$$\tilde{n} = n(\omega) + i\,\kappa(\omega)$$

$\kappa(\omega)$ $\rightarrow$ absorption coefficient

$n(\omega)$ $\hookrightarrow$ light "slows down"

$\hookrightarrow \tilde{n} = n' + in''$

$$\tilde{k} = \frac{\tilde{n}\omega}{c}$$

$$\tilde{k} = (n + i\kappa)\frac{\omega}{c}$$

Let's plug this back in to $\vec{E}$

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} = \vec{E}_0 e^{i(n+i\kappa)\frac{\omega}{c}\hat{u}\cdot\vec{r} - i\omega t}$$

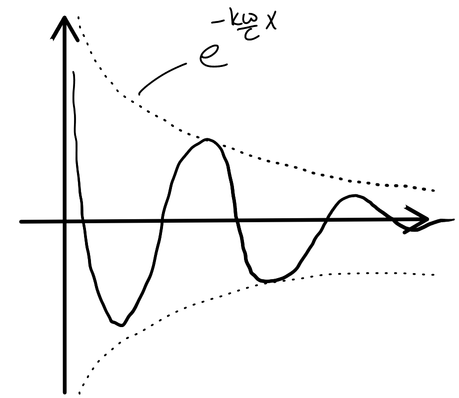
$$= \vec{E}_0 e^{-\kappa\frac{\omega}{c}\hat{u}\cdot\vec{r} + i\left(\frac{n\omega}{c}\right)\hat{u}\cdot\vec{r} - i\omega t}$$

$$= \vec{E}_0 e^{-\frac{\kappa\omega}{c}\hat{u}\cdot\vec{r}} \cdot e^{i\left(\frac{n\omega}{c}\hat{u}\cdot\vec{r}-\omega t\right)}$$

has phase information → $\vec{E}_0 = \vec{E}_0 e^{i\phi}$ ← bad notation so watch out!

$$= \vec{E}_0 \underbrace{e^{-\frac{\kappa\omega}{c}\hat{u}\cdot\vec{r}}} \cdot \cos\left(\frac{n\omega}{c}\hat{u}\cdot\vec{r}-\omega t+\phi\right) + \text{some imaginary stuff the we ignore}$$

this wave decays as it goes into this material



$$V_{phase}(\omega) = \frac{c}{n(\omega)}$$ ← real part of $\tilde{n}$

$$P = N \cdot q \cdot \delta z \xrightarrow{\text{textbook}} \vec{P} = N q_e \vec{r}_e$$

model → $m\ddot{z} = -kz - \underline{\underline{m\gamma}}\dot{z} + q_e E_0 e^{i\omega t}$

$$z = \frac{q_e \vec{E}_0 e^{i\omega t}}{m_e(\omega_0^2 - i\omega\gamma - \omega^2)}$$ ← inhomogeneous solution -or- nonhomogeneous

$$P = \frac{N q_e^2 \vec{E}_0 e^{i\omega t}}{m_e(\omega_0^2 - i\omega\gamma - \omega^2)}$$

$$\left[\frac{N q_e^2}{m_e}\right] = \left[\frac{C^2}{m^3\,kg}\right]$$

ϵ_0 $\quad F = \frac{q_0 q_1}{4\pi\epsilon_0 r^2}$

$$\frac{k q_0 q_1}{r^2}$$

$$[k] = \left[\frac{N m^2}{C^2}\right] \quad \epsilon_0 = \left[\frac{C^2}{N m^2}\right]$$

$$\frac{Nq_e^2}{\epsilon_0 m_e} = \omega_p^2 \leftarrow \text{plasma frequency}$$

$$\left[\frac{Nq_e^2}{\epsilon_0 m_e}\right] = \frac{\cancel{C}^2}{m^3 kg} \cdot \frac{N \cancel{m^2}}{\cancel{C}^2}$$

$$= \frac{N}{m\,kg} = \frac{\frac{kg\,m}{s^2}}{m\,kg}$$

$$= \frac{\cancel{kg\,m}}{s^2} \cdot \frac{1}{\cancel{m\,kg}}$$

$$= \frac{1}{s^2} \doteq \frac{rad^2}{s^2}$$

$$P = \epsilon_0 \underbrace{\left(\frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}\right)}_{\text{compare to this result from earlier!}} \vec{E}_0 e^{i\omega t}$$

$$\vec{P}_0 = \epsilon_0 \chi(\omega) \vec{E}_0$$

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}$$

$$\tilde{n} = n + i\kappa \qquad \tilde{n} = \sqrt{1 + \chi(\omega)}$$

$$n + i\kappa = \sqrt{1 + \chi(\omega)}$$

$$(n + i\kappa)^2 = 1 + \chi(\omega)$$

$$(n + iK)^2 = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}$$

$$n + iK = \sqrt{1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}}$$

plot as a function of frequency