Chapter 1 - Waves + review of E+M

function $\rightarrow \dfrac{3}{10x^2+1}$ $\xrightarrow{\text{wavify}}$ $\dfrac{3}{10(x-vt)^2+1}$

So any function that can be written

$$\psi(x,t) = f(x \mp vt)$$

wave function

↖ specific function that has $x \mp vt$ in it,

$$f(\underbrace{x \mp vt}_{x'}) = f(x')$$

first derivative :
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'} \cdot \underbrace{\frac{\partial x'}{\partial x}}_{1} = \frac{\partial f}{\partial x'}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}$$

second derivative :
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x'}\right)$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial t} = \frac{\partial f}{\partial x'} \cdot (\mp v)$$

$$\frac{\partial f}{\partial t} = \mp v \frac{\partial f}{\partial x'}$$

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial \left(\frac{\partial f}{\partial t}\right)}{\partial x'} \cdot \frac{\partial x'}{\partial t}$$

$$\underset{\frac{\partial f}{\partial t} = \mp v \frac{\partial f}{\partial x'}}{} \qquad \underset{\mp v}{}$$

$$\frac{\partial \left(\frac{\partial f}{\partial x'} \right)}{\partial x'} \cdot \underbrace{\frac{\partial x'}{\partial x}}_{1, \text{ again!}}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x'^2}$$

$$= \mp v \frac{\partial^2 f}{\partial x'^2} \cdot (\mp v)$$

$$\frac{\partial^2 f}{\partial t^2} = \mp v^2 \frac{\partial^2 f}{\partial x'^2}$$

$$\frac{\partial^2 f}{\partial x'^2} = \mp \frac{1}{v^2} \cdot \frac{\partial^2 f}{\partial t^2}$$
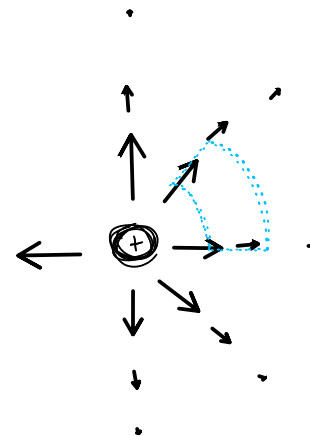
$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x'^2} = \mp \frac{1}{v^2} \cdot \frac{\partial^2 f}{\partial t^2}$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} = \mp \frac{1}{v^2} \cdot \frac{\partial^2 f}{\partial t^2}}$$

Wave equation

compare to → heat equation

→ Schrödinger's

→ Laplace
$$\nabla^2 \phi = 0$$

Let's go back to Maxwell:

$\rho \leftarrow$ charge density

$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ (Gauss's Law) $\quad \leftarrow$ Electric field spreads out if there is charge

$\vec{\nabla} \cdot \vec{B} = 0$ (Gauss's Law for magnetism) $\quad \leftarrow$ Magnetic fields do not spread out (no magnetic monopole — yet!?)

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Maxwell-Faraday's Law) $\quad \leftarrow$ There is electric field if a magnetic field is changing

$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$ (Ampere's Law) $\quad \leftarrow$ There is magnetic field if there is current or if the electric field changes.

For the case of light: no charge in vacuum, no current

$\vec{\nabla} \cdot \vec{E} = 0$

$\vec{\nabla} \cdot \vec{B} = 0$

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla} \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$

$\Rightarrow$ take the curl of Faraday's

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B})$$

$\epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$!

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E}$$

$= 0!$

$$\boxed{\vec{\nabla}^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$$

$$\frac{1}{V^2}$$

$$\varepsilon_0 \mu_0 = \frac{1}{V^2}$$

$$V = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

$$\mu_0 = 4\pi \cdot 10^{-7} \, \frac{m \, kg}{C}$$

$$\varepsilon_0 = 8.85 \cdot 10^{-12} \, \frac{s^2 C^2}{m^2 kg}$$

$$\boxed{V = 2.998 \cdot 10^8 \, \frac{m}{s}}$$

Maxwell's Equations again

$$\vec{\nabla} \cdot \vec{E} = \frac{\vec{\nabla} \cdot \vec{P}}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
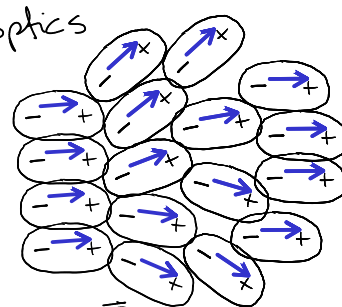
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_{free} + \mu_0 \frac{\partial \vec{P}}{\partial t} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

polarization has units of $\frac{charge \times length}{volume}$

$\vec{J}$ has units of $\frac{charge \times velocity}{volume}$

charge density:

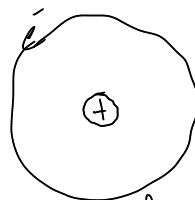$$\rho = \cancel{\rho_{free}}^{\;0,\,for\,us\,in\,optics} + \rho_P$$

$$\vec{\nabla} \cdot \vec{J}_P = \frac{\partial \rho_P}{\partial t} \rightarrow \vec{J}_P = \frac{\partial \vec{P}}{\partial t}$$

current density:

$$\vec{J} = \vec{J}_{free} + \cancel{\vec{J}_m}^{\;0,\,for\,us\,in\,optics} + \vec{J}_P$$

neutral material; no polarization

neutral material polarization

$$\vec{J}_P = \frac{\partial \vec{P}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{J}_P = \frac{\partial \rho_P}{\partial t} \longrightarrow \vec{J}_P = \frac{\partial \vec{P}}{\partial t}$$
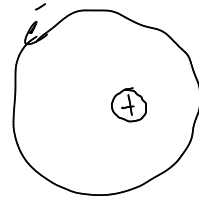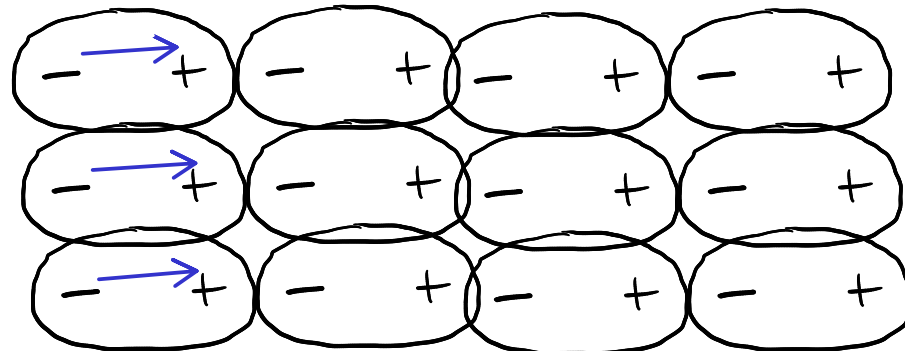
$$\frac{\partial (\vec{\nabla} \cdot \vec{P})}{\partial t} = \frac{\partial \rho_P}{\partial t}$$
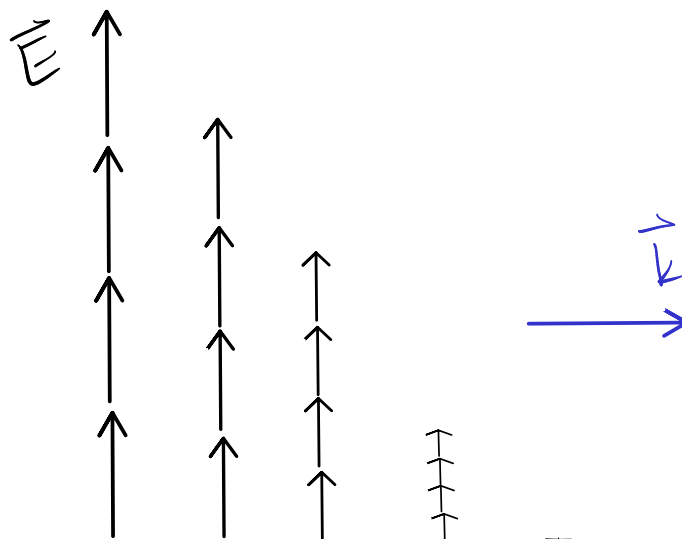
$$\vec{\nabla} \cdot \vec{P} = \rho_P$$

crystals
(anisotropic materials)
↳ not isotropic

$$\vec{\nabla}^2 E - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial J_{free}}{\partial t} + \mu_0 \frac{\partial^2 P}{\partial t^2} - \frac{1}{\epsilon_0} \nabla (\vec{\nabla} \cdot \vec{P})$$

wave equation
that we derived
before.

present when
current of free
charge are flowing.
reflection from mirror.
also light through
plasma.

↳ dipole currents
dipole oscillations

$\vec{E}$

$\vec{k}$

P1.2 Suppose that an electric field is given by $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)$,
where $\mathbf{k}\perp\mathbf{E}_0$ and ϕ is a constant phase. Show that

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mathbf{k}\times\mathbf{E}_0}{\omega}\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi)$$

is consistent with (1.3).

$\hookrightarrow$ assume: $\vec{k} = k_x \hat{x} + 0\hat{y} + 0\hat{z}$

$\hookrightarrow$ assume: $\vec{r} = x\hat{x} + 0\hat{y} + 0\hat{z}$

$\hookrightarrow \vec{E}_0 = 0\hat{x} + E_{oy}\hat{y} + 0\hat{z}$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$ (Faraday's Law) (1.3)

$$\nabla \times \frac{\mathbf{B}}{\mu_0} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}$$ (Ampere's Law revised by Maxwell) (1.4)

$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\left(\vec{E}_0\cos(\vec{k}\cdot\vec{r} - \omega t + \phi)\right)$

can: $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$\vec{\nabla}\times\vec{E} = \hat{x}\left(-\frac{\partial E_y}{\partial z}\,_0\right) - \hat{y}(0) + \hat{z}\left(\frac{\partial E_y}{\partial x}\right)$

$\underbrace{\qquad}_{0}$

$-\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}\left(\frac{\vec{k}\times\vec{E}_0}{\omega}\cos(\vec{k}\cdot\vec{r} - \omega t + \phi)\right)$

$= -\frac{\vec{k}\times\vec{E}_0}{\omega}\left(-\sin(\vec{k}\cdot\vec{r} - \omega t + \phi)\cdot(-\omega)\right)$

$= -\vec{k}\times\vec{E}_0\,\sin(\vec{k}\cdot\vec{r} - \omega t + \phi)$

$= \underbrace{\qquad}\quad \sin(k_x\cdot x - \omega t + \phi)$

$$\vec{\nabla} \times \vec{E} = \hat{z}\left(\frac{\partial}{\partial x}\left(E_{oy}\cos(k_x X - wt + \phi)\right)\right)$$

$$= -\hat{z}\, E_{oy}\sin(k_x X - wt + \phi)\cdot k_x$$

$$= -E_{oy}\cdot k_x \cdot \sin(k_x X - wt + \phi)\,\hat{z}$$

$$\vec{k} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ k_x & 0 & \phi \\ 0 & E_{oy} & \phi \end{vmatrix}$$

$$= k_x E_{oy}\,\hat{z}$$

$$= -E_{oy}\, k_x \sin(k_x X - wt + \phi)\,\hat{z}$$

P1.4 Check that the **E** and **B** fields in P1.2 satisfy the rest of Maxwell's equations:

(a) (1.1). What must ρ be?

(b) (1.2).

(c) (1.4). What must **J** be?

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\rightarrow \quad \frac{\partial E_x^{\,0}}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z^{\,0}}{\partial z}$$

$$\frac{\partial E_y}{\partial y} = \frac{\partial}{\partial y}\underbrace{\left(E_{oy}\cos(k_x x - wt + \phi)\right)}_{\text{does not depend on } y \text{ at all!}} = 0$$

$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$	(Gauss' Law)	(1.1)
$\nabla \cdot \mathbf{B} = 0$	(Gauss' Law for magnetism)	(1.2)
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	(Faraday's Law)	(1.3) ✓
$\nabla \times \dfrac{\mathbf{B}}{\mu_0} = \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} + \mathbf{J}$	(Ampere's Law revised by Maxwell)	(1.4)

$$\underbrace{\mu_0\epsilon_0 = \frac{k^2}{w^2}}\quad \text{HINT!}$$

So $\phi = 0$

Similarly for B since B_z is the only term and B_z does not depend on z.

$$\rightarrow B(\vec{r},t) = \frac{\vec{k} \times \vec{E}}{\omega} \cos(\vec{k} \cdot \vec{r} - \omega t + \phi) = \frac{k_x E_{oy}}{\omega} \hat{z} \cos(k_x x - \omega t + \phi)$$

P1.7 Show that the magnetic field in P1.2 is consistent with the wave equation derived in P1.6. What is the requirement on k and ω?

$\Rightarrow \nabla^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}$ ← this we did in class! see day08!

$\Rightarrow \nabla^2 \vec{B} = \epsilon_0 \mu_0 \frac{\partial^2 B}{\partial t^2}$

→ side note!

Following what we did in class last time:

$\vec{k} = (k_x, 0, 0)$
$\vec{E}_0 = (0, E_{oy}, 0)$ } so $\vec{k} \times \vec{E} = k_x E_{oy} \hat{z}$

which means $\vec{B}$ is entirely in the $\hat{z}$ direction!

also $\vec{k} \cdot \vec{r} = k_x \cdot x$

$\epsilon_0 \mu_0 = \frac{1}{v^2}$ & $\frac{\omega}{k} = v$

$\omega = \left[\frac{rad}{s}\right]$

$k = \left[\frac{rad}{m}\right]$

$\frac{\omega}{k} = \left[\frac{m}{s}\right] = v$

$\epsilon_0 \mu_0 = \frac{k^2}{\omega^2}$

this could be useful!

from chapter 0 notes:

$\nabla^2 \vec{B} = \left(\frac{\partial^2 B_x}{\partial x^2} + \frac{\partial^2 B_x}{\partial y^2} + \frac{\partial^2 B_x}{\partial z^2} \right) \hat{x}$

both zero!

$+ \left(\frac{\partial^2 B_y}{\partial x^2} + \frac{\partial^2 B_y}{\partial y^2} + \frac{\partial^2 B_y}{\partial z^2} \right) \hat{y}$

$+ \left(\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \frac{\partial^2 B_z}{\partial z^2} \right) \hat{z}$

also zero!
B_z only depends on x

$\nabla^2 \vec{B} = \frac{\partial^2 B_z}{\partial x^2} = \frac{k_x E_{oy}}{\omega} \cdot \left(-\cos^2(k_x x - \omega t + \phi) \right) \cdot k_x^2$

$\frac{k_x^2}{\omega^2} \frac{\partial^2 B}{\partial t^2} = \frac{k_x^2}{\omega^2} \cdot \frac{k_x E_{oy}}{\omega} \left(-\cos^2(k x - \omega t + \phi) \cdot \omega^2 \right)$

$$\nabla^2 B = -\frac{k_x^3 E_{oy}}{\omega} \cos^2(k_x \cdot X - \omega t + \phi) \quad \bigg| \quad \frac{k^2}{\omega^2}\frac{\partial^2 B}{\partial t^2} = -\frac{k_x^3 E_{oy}}{\omega} \cos^2(kx - \omega t + \phi)$$

Same!

problem 1.9

P1.9 (a) Show that $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(k(\hat{\mathbf{u}} \cdot \mathbf{r} - ct) + \phi)$ is a solution to the vacuum wave equation (1.41), where $\hat{\mathbf{u}}$ is an arbitrary unit vector, $c = 1/\sqrt{\epsilon_0 \mu_0}$, and k is a constant with units of inverse length.

→ exactly the same as P1.7, just w/ Electric Field instead of Magnetic

(b) Show that each wavefront forms a plane, which is why such solutions are often called 'plane waves'. HINT: A wavefront is a surface in space where the argument of the cosine (i.e. the *phase* of the wave) has a constant value. Set the cosine argument to an arbitrary constant and see what positions are associated with that phase.

$k(\hat{u} \cdot \vec{r} - ct) + \phi$ must be constant at a moment in time. t is constant at a moment of time by definition. c is constant always. ϕ is constant. So what is $\hat{u} \cdot \vec{r}$

(c) Determine the speed $v = \Delta r / \Delta t$ that a wavefront moves in the $\hat{\mathbf{u}}$ direction. HINT: Set the cosine argument to a constant, and consider a change in position along $\hat{\mathbf{u}}$ with its associated change in time.

if $\hat{u}$ is arbitrary, then I can choose:

$$\hat{u} = \hat{x}$$

and then $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

(d) By analysis of this wave, determine the wavelength λ in terms of k. HINT: Holding time constant, find the distance between identical wavefronts by changing the position along $\hat{\mathbf{u}}$ and allowing the cosine argument to evolve through 2π.

so $\hat{u} \cdot \vec{r} = x$

and $k(\hat{u} \cdot \vec{r}) = kx$

(e) Use (1.33) to show that $\mathbf{E}_0$ and $\hat{\mathbf{u}}$ must be perpendicular to each other in vacuum.

$V = \frac{\Delta r}{\Delta t}$

this means that for any x-value (and only x value) the wave has a constant value in the y + z directions.