

Chapter 0 - Mathematical Review

Vectors

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = x\hat{i} + y\hat{j} + z\hat{k}$$

unit vectors

$$\vec{E} = \frac{kq}{r^2} \hat{r} = \frac{kq\vec{r}}{r^3} \left. \vphantom{\frac{kq\vec{r}}{r^3}} \right\} \text{fine, but incomplete} \rightarrow \text{it assumes the charge is located at the origin}$$

$$\left[\frac{N}{C} \right]$$

more general

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q(\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^3}$$

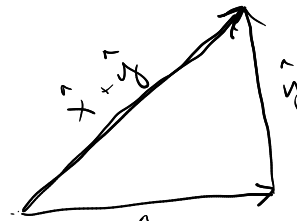
Example 0.1

$$\vec{r} = (2\hat{x} + 2\hat{y} + 2\hat{z})$$

$$\vec{r}_0 = (1\hat{x} + 1\hat{y} + 2\hat{z})$$

$$\vec{r} - \vec{r}_0 = 1\hat{x} + 1\hat{y} + 0\hat{z} = \hat{x} + \hat{y}$$

$$|\vec{r} - \vec{r}_0|^3 = |\hat{x} + \hat{y}|^3 = (\sqrt{2})^3 = 2^{3/2} = 2\sqrt{2}$$

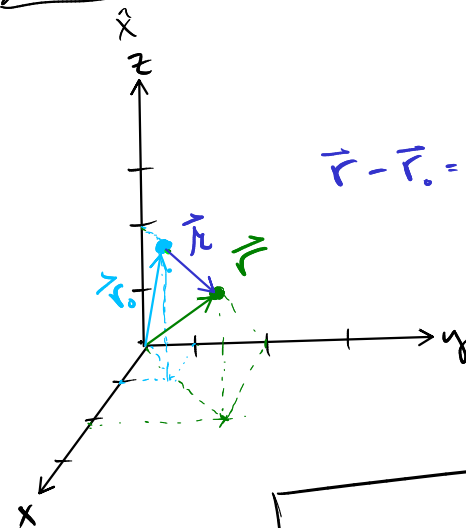


$\vec{r} \Rightarrow$ location where we want to know the electric field

$\vec{r}_0 = \vec{r}_{\text{naught}}$

\rightarrow location of the charge

$$\vec{r} - \vec{r}_0 = \vec{r} \Rightarrow \vec{r}_0 + \vec{r} = \vec{r}$$



$$\vec{E}(\vec{r}) = \frac{q(\hat{x} + \hat{y})}{8\pi\epsilon_0 \sqrt{2}}$$

$$\frac{\vec{r}}{|\vec{r}|} = \hat{r}$$

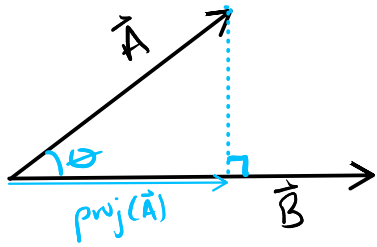
$C \equiv \text{Coulomb}$

$$9 \cdot 10^9 \frac{Nm^2}{C^2}$$

$$k = \frac{1}{4\pi\epsilon_0}$$

$$8.85 \cdot 10^{-12} \frac{C^2}{Nm^2}$$

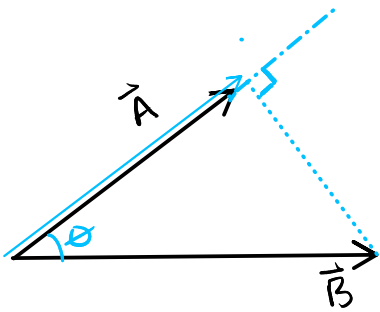
Dot Product (inner product)



$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

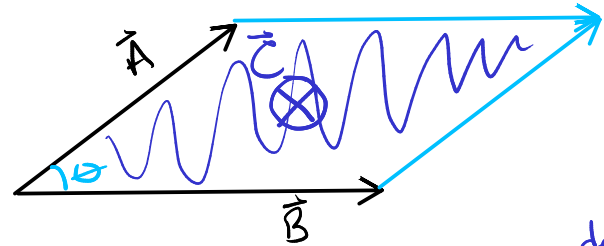
scalar

$$= |\vec{A}| |\vec{B}| \cos \theta$$



Cross Product (vector product)

(perp. to both \vec{A} & \vec{B})

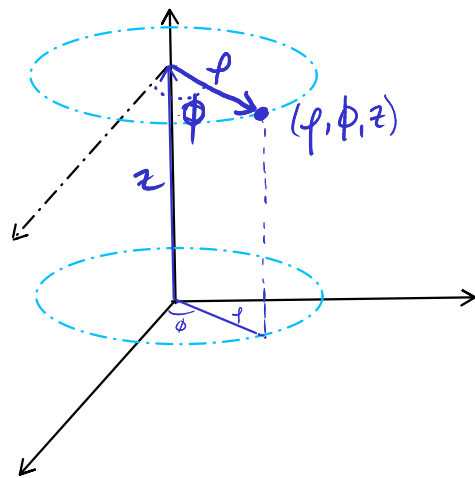


$$\underbrace{\vec{C}}_{\text{vector}} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \leftarrow \text{determinant}$$

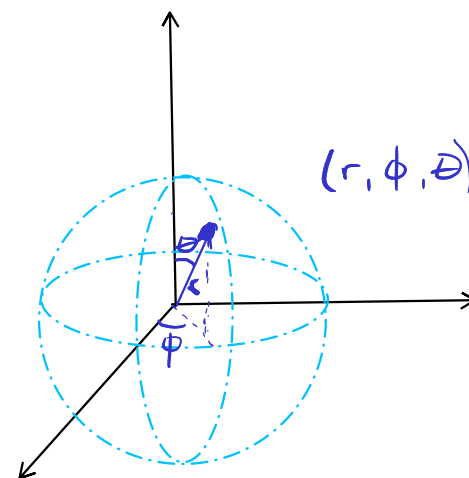
$$= \hat{x}(A_y B_z - A_z B_y) \\ - \hat{y}(A_x B_z - A_z B_x) \\ + \hat{z}(A_x B_y - A_y B_x)$$

$$|\vec{C}| = |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta$$

Cylindrical Coords.



Spherical Coords.



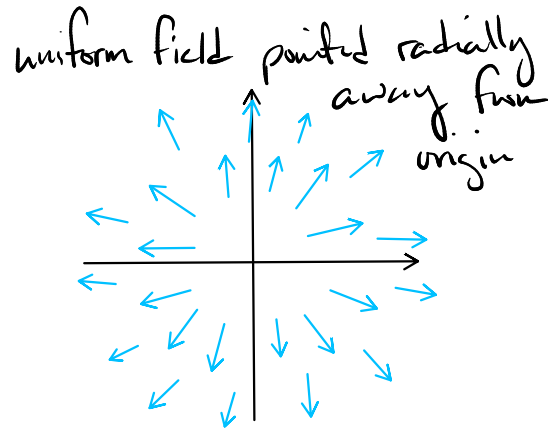
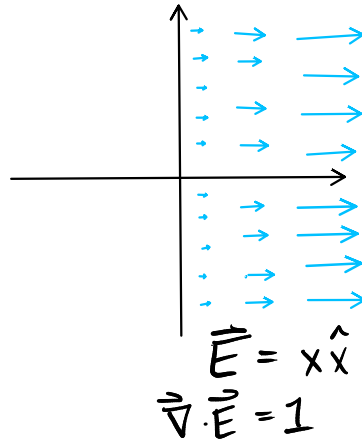
Gradient: of a scalar function
(directional derivative)

$$\vec{\nabla} f(x, y, z) = \underbrace{\frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}}_{\text{vector result points in the direction of steepest decline}}$$

↑
scalar function

Divergence: of a vector function

$$\vec{\nabla} \cdot \vec{E} = \underbrace{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}_{\text{scalar}}$$

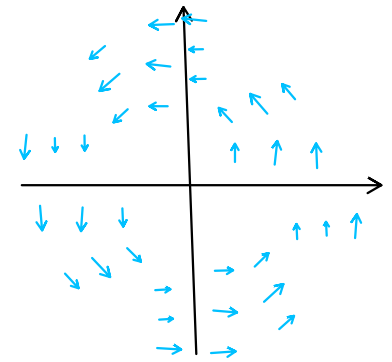


Curl: of a vector function

$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$\begin{aligned} &= \hat{x} \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \\ &\quad - \hat{y} \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) \\ &\quad + \hat{z} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \end{aligned}$$

vector



Second Derivatives \rightarrow Laplacian

$$\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\nabla^2 \vec{E} = \left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \right) \hat{x}$$

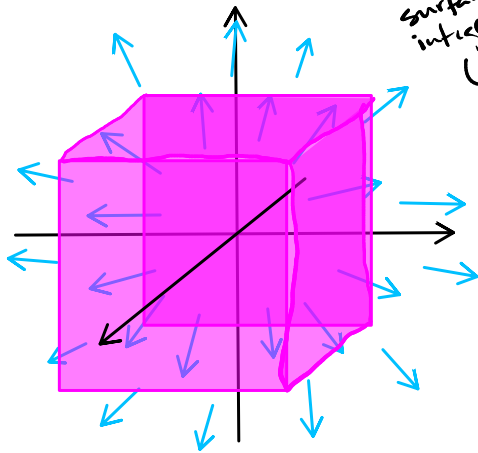
$$\left(\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \right) \hat{y}$$

$$\left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) \hat{z}$$

$$\vec{E}(x, y, z) = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$$

$\rightarrow \nabla^2 \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) \leftarrow \text{true in any coordinate system}$

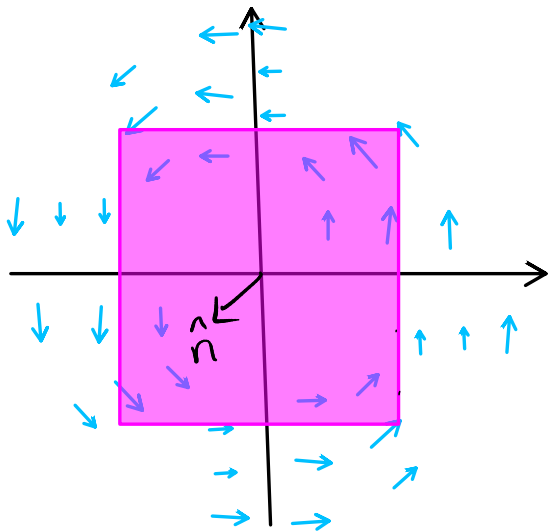
Divergence Theorem



- closed surface integral
 $\oint_S \vec{E} \cdot \hat{n} da = \int_V \vec{\nabla} \cdot \vec{E} dv$

direction of the surface
 break up into surfaces
 break up into volumes

Stokes Theorem



$$\int_S (\vec{\nabla} \times \vec{E}) \cdot \hat{n} da = \oint_C \vec{E} \cdot d\vec{\ell}$$

Complex Numbers

imaginary number $\rightarrow i = \sqrt{-1}$

\rightarrow sum of a real number + an imaginary number

$$\tilde{z} = a + bi$$

$$\text{Re}\{a+bi\} = a$$

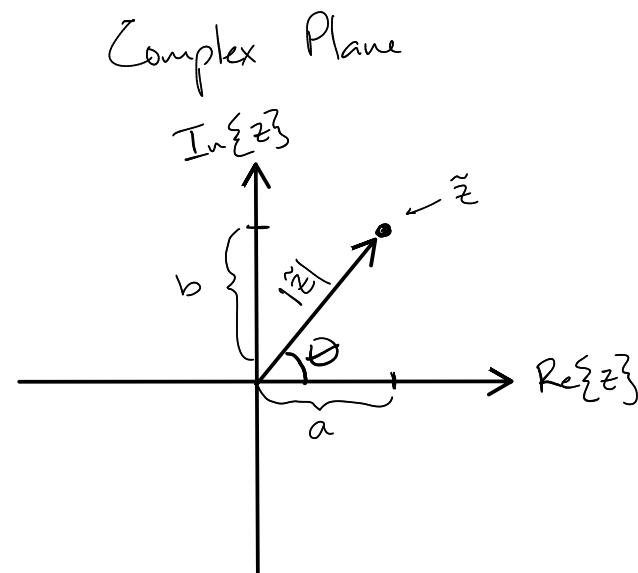
$$\text{Im}\{a+bi\} = b$$

complex function $f + g$

$$\text{Re}\{f\} + \text{Re}\{g\} = \text{Re}\{f+g\}$$

$$\frac{d}{dx} \text{Re}\{f\} = \text{Re}\left\{\frac{df}{dx}\right\}$$

$$\int \text{Re}\{f\} dx = \text{Re}\left\{\int f dx\right\}$$



$$\rightarrow |\tilde{z}|^2 = |\tilde{z}|^2 = a^2 + b^2$$

$$a = |\tilde{z}| \cos \theta$$

$$b = |\tilde{z}| \sin \theta$$

Now:

$$\tilde{z} = |\tilde{z}| (\cos \theta + i \sin \theta)$$

So:

$$\tilde{z} = |\tilde{z}| \cdot e^{i\theta}$$

Taylor Series:

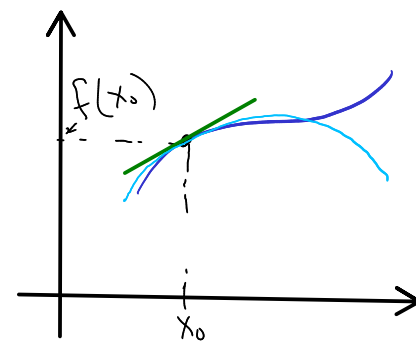
$$f(x) = f(x_0) + \underbrace{\frac{1}{1!} (x-x_0) \cdot \frac{df}{dx} \Big|_{x=x_0}}_{\text{linear term}} + \underbrace{\frac{1}{2!} (x-x_0)^2 \frac{d^2f}{dx^2} \Big|_{x=x_0}}_{\text{quadratic term}} + \dots$$

$$\cos \phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \dots$$

$$i \sin \phi = i\phi - \frac{i\phi^3}{3!} + \frac{i\phi^5}{5!} - \dots$$

$$e^{i\phi} = 1 + i\phi - \frac{\phi^2}{2!} - \frac{i\phi^3}{3!} + \frac{\phi^4}{4!} + \dots$$

$$e^{i\phi} = \cos \phi + i \sin \phi \quad \text{Euler's identity}$$



$$e^{i\pi} = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$

Complex Conjugate

$$\underbrace{\text{complex conjugate}}_{\text{conjugate}} \quad \underbrace{\quad}_{\text{conjugate}}$$

$$\hookrightarrow \tilde{z}^* = a - bi \quad \text{or} \quad \tilde{z}^* = |\tilde{z}| e^{-i\theta}$$

$$\tilde{z} \cdot \tilde{z}^* = |\tilde{z}|^2$$

$$\tilde{z} \cdot \tilde{z}^* = |\tilde{z}| e^{i\theta} |\tilde{z}| e^{-i\theta}$$

$$\hat{z} \cdot \hat{z}^* = |\hat{z}|^2$$

$$\operatorname{Re}\{\tilde{z}\} = \frac{\tilde{z} + \tilde{z}^*}{2} \longrightarrow \frac{e^{i\theta} + e^{-i\theta}}{2} = \underline{\underline{\cos\theta}} = \operatorname{Re}\{e^{i\theta}\}$$

$$\text{Im}\{\tilde{z}\} = \frac{\tilde{z} - \tilde{z}^*}{2i}$$

HW3: 0.14, 17, 18, 20

0.14 $z_1 = 1 - i$ $z_2 = 3 + 4i$

$$\begin{aligned} \text{a) } z_1 - z_2 &= (1-i) - (3+4i) \\ &= 1-i-3-4i \end{aligned}$$

$$\begin{aligned} z_1 - z_2 &= (1-i) - (3+4i) \\ &= 1-i-3-4i \\ z_1 - z_2 &= \underline{\underline{-2-5i}} = z_3 = \underline{\underline{|z|}} e^{i\theta} \end{aligned}$$

$$b) \frac{z_1}{z_2} = \frac{1-i}{3+4i} \cdot \frac{(3-4i)}{(3-4i)} = \text{~~~~~} \rightarrow \text{polar form}$$

0.17 Show $\operatorname{Re}\{\tilde{A}\} \times \operatorname{Re}\{\tilde{B}\} = \frac{AB + A^*B}{4} + \text{C.C.}$

$\underbrace{\hspace{10em}}_{\text{complex conj}} \rightarrow \frac{A^*B^* + AB^*}{4}$

? $\tilde{A} = A_1 + A_2 i$

• $\tilde{A}^* = A_1 - A_2 i$

$$= \frac{AB + A^*B + A^*B^* + AB^*}{4}$$

$$= \frac{B(A + A^*) + B^*(A + A^*)}{4}$$

$$= \frac{(A + A^*)}{2} \frac{(B + B^*)}{2}$$

$$= \operatorname{Re}\{A\} \operatorname{Re}\{B\} \quad \checkmark$$