Chapter 0 – Mathematical Review

$$\boxed{\frac{\vec{r}}{|r|} = \hat{r}}$$

$C \equiv$ Coulomb

Vectors

unit vectors

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{E} = \frac{kq}{r^2}\hat{r} = \frac{kq\vec{r}}{r^3}$$

fine, but incomplete → it assumes the charge is located at the origin

$\boxed{\frac{N}{C}}$

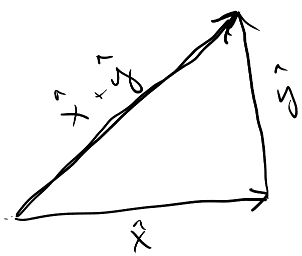
$9 \cdot 10^9 \frac{Nm^2}{C^2}$

$$k = \frac{1}{4\pi\epsilon_0}$$

$8.85 \cdot 10^{-12} \frac{C^2}{Nm^2}$

more general

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}}$$



$\vec{r} \Rightarrow$ location where we want to know the electric field

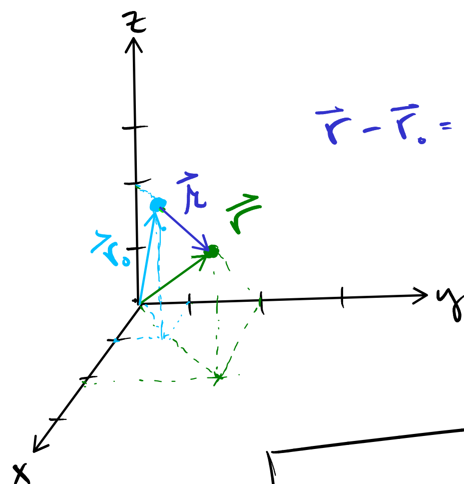
$\vec{r}_0 = \vec{r}$ naught → location of the charge

Example 0.1

$$\vec{r} = (2\hat{x} + 2\hat{y} + 2\hat{z})$$

$$\vec{r}_0 = (1\hat{x} + 1\hat{y} + 2\hat{z})$$

$\vec{r} - \vec{r}_0 = \vec{\mathscr{r}} \Rightarrow \vec{r}_0 + \vec{\mathscr{r}} = \vec{r}$

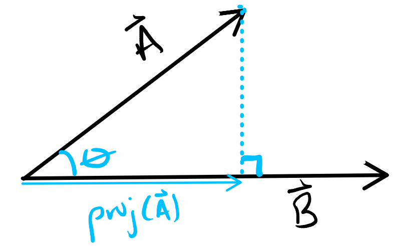


$$\vec{r} - \vec{r}_0 = 1\hat{x} + 1\hat{y} + 0\hat{z} = \hat{x} + \hat{y}$$

$$|\vec{r}-\vec{r}_0|^3 = |\hat{x}+\hat{y}|^3 = (\sqrt{2})^3 = 2^{3/2} = 2\cdot\sqrt{2} \longrightarrow \boxed{\vec{E}(\vec{r}) = \frac{q(\hat{x}+\hat{y})}{8\pi\epsilon_0\sqrt{2}}}$$

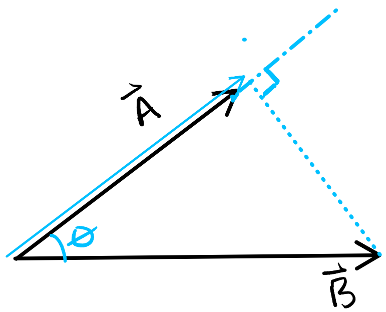
/Day02

Dot Product (inner product)



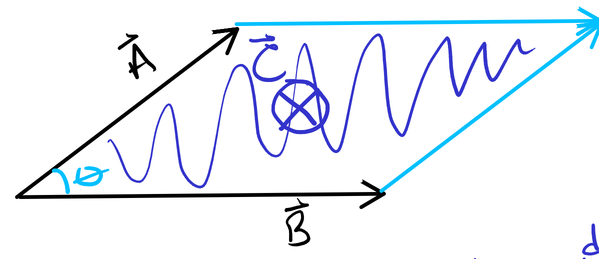
$$\vec{A}\cdot\vec{B} = \overbrace{A_x B_x + A_y B_y + A_z B_z}^{\text{scalar}}$$

$$= |A|\cdot|B|\cdot\cos\theta$$



Cross Product (vector product)

(perp. to both $\vec{A}$ + $\vec{B}$)



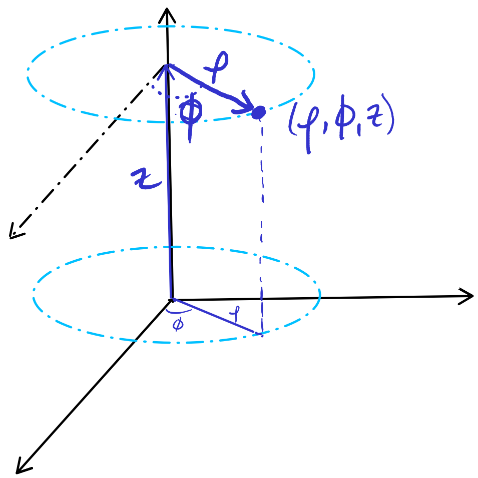
$$\underbrace{\vec{C}}_{\text{vector}} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

determinant

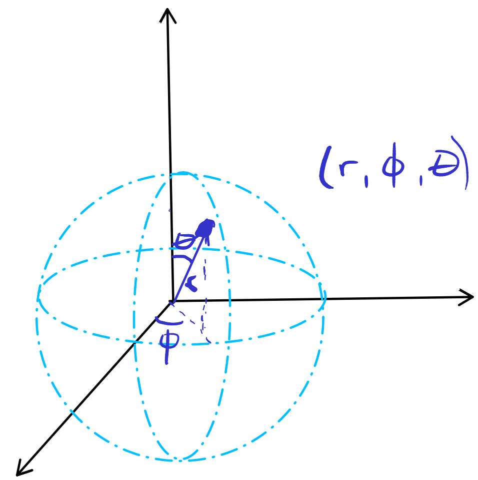
$$= \hat{x}(A_y B_z - A_z B_y)$$
$$- \hat{y}(A_x B_z - A_z B_x)$$
$$+ \hat{z}(A_x B_y - A_y B_x)$$

$$|\vec{C}| = |\vec{A}\times\vec{B}| = |A|\cdot|B|\cdot\sin\theta$$

Cylindrical Coords.



Spherical Coords.



Day03

Gradient: of a scalar function (directional derivative)

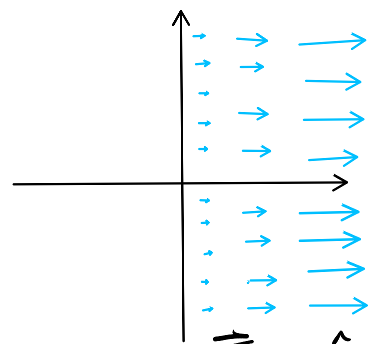
$$\vec{\nabla} f(x,y,z) = \underbrace{\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}}_{\text{vector result}}$$

scalar function

vector result points in the direction of steepest decline

Divergence: of a vector function

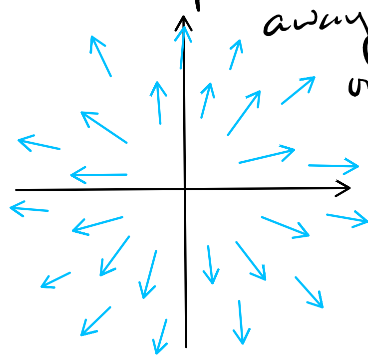
$$\vec{\nabla}\cdot\vec{E} = \underbrace{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}_{\text{scalar}}$$



$\vec{E} = x\hat{x}$

$\vec{\nabla}\cdot\vec{E} = 1$

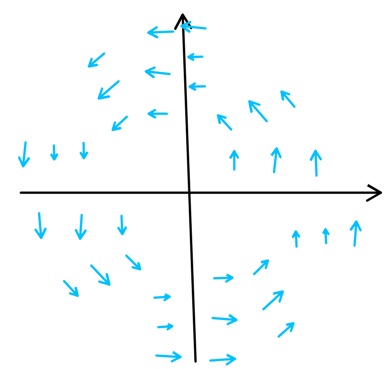
uniform field pointed radially away from origin



Curl: of a vector function

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \underbrace{\hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \hat{y}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)}_{\text{vector}}$$



Second Derivatives → Laplacian

$$\nabla^2 f = \vec{\nabla}\cdot\left(\vec{\nabla} f\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\vec{E}(x,y,z) = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$$

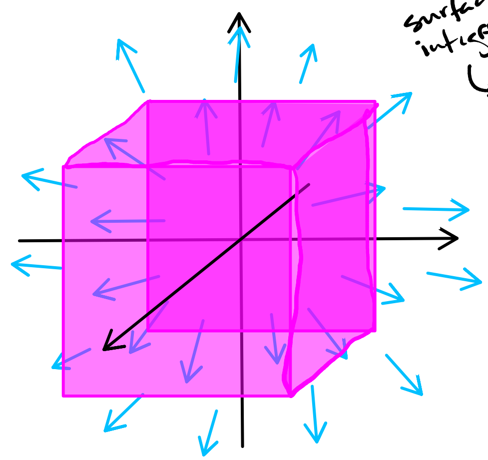
$$\nabla^2 \vec{E} = \left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2}\right)\hat{x}$$

$$\left(\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2}\right)\hat{y}$$

$$\left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2}\right)\hat{z}$$

$$\nabla^2 \vec{E} = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right)$$ ← true in any coordinate system

Divergence Theorem



$$\underset{S}{\oint} \vec{E}\cdot\hat{n}\,da = \underset{V}{\int} \vec{\nabla}\cdot\vec{E}\,dv$$

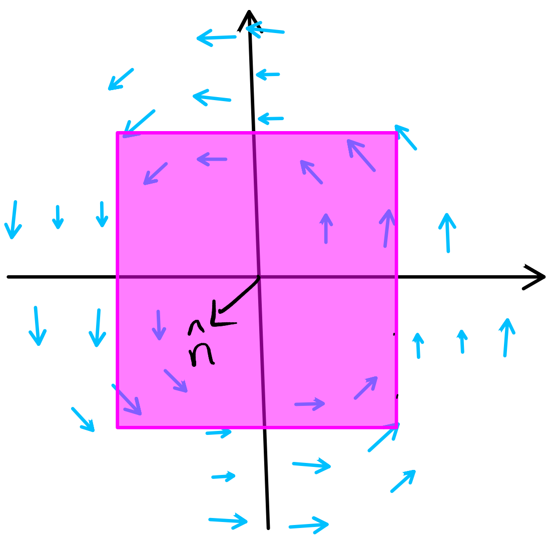
- closed surface integral

direction of the surface

break up into surfaces

break up into volumes

Stokes Theorem



$$\underset{S}{\int} (\vec{\nabla}\times\vec{E})\cdot\hat{n}\,da = \underset{C}{\oint} \vec{E}\cdot d\vec{l}$$

Complex Numbers

imaginary number $\rightarrow i = \sqrt{-1}$

$\hookrightarrow$ sum of a real number + an imaginary number

$$\tilde{z} = a + bi$$

$$Re\{a+bi\} = a$$

$$Im\{a+bi\} = b$$

$$\hookrightarrow |z^2| = |z|^2 = a^2 + b^2$$

$$a = |\tilde{z}|\cos\theta$$

$$b = |\tilde{z}|\sin\theta$$

Now:

$$\tilde{z} = |\tilde{z}|(\cos\theta + i\sin\theta)$$

So:

$$\tilde{z} = |\tilde{z}| \cdot e^{i\theta}$$

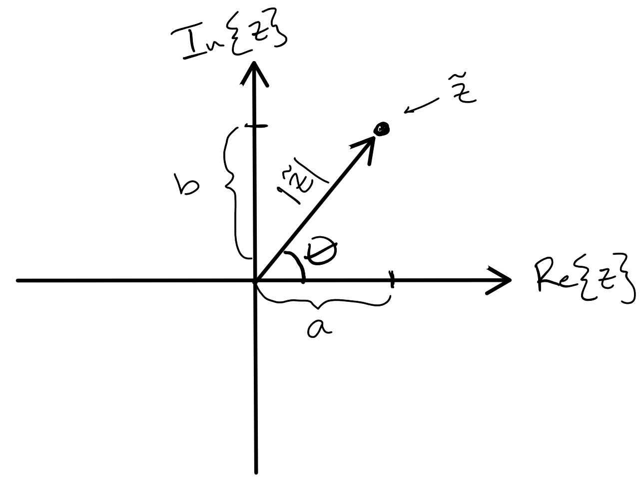
complex function f + g

$$Re\{f\} + Re\{g\} = Re\{f+g\}$$

$$\frac{d}{dx} Re\{f\} = Re\left\{\frac{df}{dx}\right\}$$

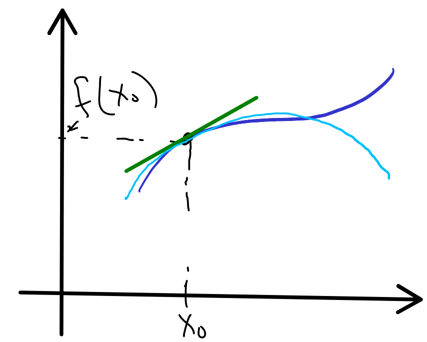
$$\int Re\{f\}\,dx = Re\left\{\int f\,dx\right\}$$

Complex Plane



Taylor Series:

$$f(x) = f(x_0) + \underbrace{\frac{1}{1!}(x-x_0)\cdot\frac{df}{dx}\Big|_{x=x_0}}_{\text{linear term}} + \underbrace{\frac{1}{2!}(x-x_0)^2\frac{d^2f}{dx^2}\Big|_{x=x_0}}_{\text{quadratic term}}$$



$$\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \ldots$$

$$i\sin\phi = i\phi - \frac{i\phi^3}{3!} + \frac{i\phi^5}{5!} - \ldots$$

$$e^{i\phi} = 1 + i\phi - \frac{\phi^2}{2!} - \frac{i\phi^3}{3!} + \frac{\phi^4}{4!} + \ldots$$

$$e^{i\phi} = \cos\phi + i\sin\phi \quad \text{Euler's identity}$$

$$e^{i\pi} = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$

Complex Conjugate

$\tilde{z}^* = a - bi$ or $\tilde{z}^* = |\tilde{z}| e^{-i\theta}$

$\tilde{z} \cdot \tilde{z}^* = |\tilde{z}|^2$

$\tilde{z} \cdot \tilde{z}^* = |\tilde{z}| e^{i\theta} |\tilde{z}| e^{-i\theta}$

$\tilde{z} \cdot \tilde{z}^* = |\tilde{z}|^2$

$Re\{\tilde{z}\} = \dfrac{\tilde{z} + \tilde{z}^*}{2} \rightarrow \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \underline{\underline{\cos\theta}} = Re\{e^{i\theta}\}$

$Im\{\tilde{z}\} = \dfrac{\tilde{z} - \tilde{z}^*}{2i}$

$\tilde{z} + \tilde{z}^* = A + Bi + A - Bi$

$= 2A$

$2 \cdot Re\{\tilde{z}\}$

HW3: 0.14, 17, 18, 20

0.14 $z_1 = 1 - i \quad z_2 = 3 + 4i$

a) $z_1 - z_2 = (1 - i) - (3 + 4i)$

$= 1 - i - 3 - 4i$

magnitude

phase

$z_1 - z_2 = \underline{\underline{-2 - 5i}} = z_3 = \underline{\underline{|z| e^{i\theta}}}$ } polar form

b) $\dfrac{z_1}{z_2} = \dfrac{1 - i}{3 + 4i} \cdot \dfrac{(3 - 4i)}{(3 - 4i)} =$ ~~~ $\rightarrow$ polar form

0.17 Show $Re\{\tilde{A}\} \times Re\{\tilde{B}\} = \underbrace{\frac{AB + A^*B}{4}}_{\text{complex conjugate}} + c.c. \rightarrow \frac{A^*B^* + AB^*}{4}$ $= \frac{AB + A^*B}{4} + \frac{A^*B^* + AB^*}{4}$

? $\tilde{A} = A_1 + A_2 i$

$\tilde{A}^* = A_1 - A_2 i$

$$= \frac{AB + A^*B + A^*B^* + AB^*}{4}$$

$$= \frac{AB + AB^* + A^*B + A^*B^*}{4}$$

$$= \frac{A(B + B^*) + A^*(B + B^*)}{4}$$

$$= \frac{(A + A^*)\cdot(B + B^*)}{4}$$

$$= \frac{\cancel{2} Re\{A\} \cdot \cancel{2} Re\{B\}}{\cancel{4}}$$

$$= Re\{A\} \cdot Re\{B\}$$

0.18 $E_0 = |E_0| e^{i\delta_E}$ $B_0 = |B_0| e^{i\delta_B}$

$Re\{\tilde{A}\} \times Re\{\tilde{B}\} = \frac{AB + AB^* + A^*B + A^*B^*}{4}$

$$4 \cdot Re\{E_0 \cdot e^{i(kz - \omega t)}\} Re\{B_0 \cdot e^{i(kz - \omega t)}\} =$$

$$|E_0| e^{i\delta_E} e^{i(kz-\omega t)} \cdot |B_0| e^{i\delta_B} \cdot e^{i(kz-\omega t)}$$

$$\left.\begin{aligned} &+ |E_0| e^{i\delta_E} e^{i(kz-\omega t)} \cdot |B_0| e^{-i\delta_B} \cdot e^{-i(kz-\omega t)} \\ &+ |E_0| e^{-i\delta_E} e^{-i(kz-\omega t)} \cdot |B_0| e^{i\delta_B} \cdot e^{i(kz-\omega t)} \end{aligned}\right\} E_0 B_0^* + E_0^* B_0$$

$$+ |E_0| e^{-i\delta_E} e^{-i(kz-\omega t)} \cdot |B_0| e^{-i\delta_B} \cdot e^{-i(kz-\omega t)}$$

$$\mathrm{Re}\{E_0 e^{i(kz-\omega t)}\}\,\mathrm{Re}\{B_0 e^{i(kz-\omega t)}\} = \frac{1}{4}\left(E_0 B_0^* + E_0^* B_0\right)$$

$$+\frac{1}{4}|E_0| e^{i\delta_E} e^{i(kz-\omega t)} \cdot |B_0| e^{i\delta_B} e^{i(kz-\omega t)}$$

$$+\frac{1}{4}|E_0| e^{-i\delta_E} e^{-i(kz-\omega t)} \cdot |B_0| e^{-i\delta_B} e^{-i(kz-\omega t)}$$

$$= \frac{1}{4}\left(E_0 B_0^* + E_0^* B_0\right)$$

$$+\frac{1}{4}|E_0||B_0|\left(e^{i\delta_E} e^{i\delta_B} \cdot e^{2i(kz-\omega t)} + e^{-i\delta_E} e^{-i\delta_B} \cdot e^{-2i(kz-\omega t)}\right)$$

$$= \frac{1}{4}\left(E_0 B_0^* + E_0^* B_0\right)$$

$$+\frac{1}{4}|E_0||B_0|\left(e^{i(2(kz-\omega t)+\delta_E+\delta_B)} + e^{-i(2(kz-\omega t)+\delta_E+\delta_B)}\right)$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin\theta$$

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

$$= \frac{1}{4}\left(E_0 B_0^* + E_0^* B_0\right)$$

$$+\frac{1}{4}|E_0||B_0|\left(2\cdot\cos\left[2(kz-\omega t)+\delta_E+\delta_B\right]\right)$$

$$= \frac{1}{4}\left(E_0 B_0^* + E_0^* B_0\right) + \frac{1}{2}|E_0||B_0|\cdot\cos\left[2(kz-\omega t)+\delta_E+\delta_B\right]$$

P0.20 $A\cos(\omega t) + 2A\sin\left(\omega t + \frac{\pi}{4}\right)$ → write as a single phase shifted cosine

$$\underbrace{A\cos(\omega t) + 2A\sin\left(\omega t + \frac{\pi}{4}\right)}_{\rightarrow \overset{\text{amplitude}}{B}\cos(\omega t + \overset{\text{phase}}{\phi})}$$

if $\cos\alpha = Re\{e^{i\alpha}\}$ $\quad Re\{e^{i\alpha}\} = \{\cos\alpha + i\sin\alpha\}_{Re} = \cos\alpha$

$$A\cos(\omega t) = A \cdot Re\{e^{i\omega t}\}$$

if $\sin(\beta) = Re\{-ie^{i\beta}\}$

$$2A\sin\left(\omega t + \frac{\pi}{4}\right) = 2A \cdot Re\{-ie^{i(\omega t + \frac{\pi}{4})}\}$$

So now:

$$A \cdot Re\{e^{i\omega t}\} + 2A \cdot Re\{-ie^{i(\omega t + \frac{\pi}{4})}\}$$

$$Re\{Ae^{i\omega t} - 2Aie^{i(\omega t + \frac{\pi}{4})}\}$$

$$Re\{(\underline{\quad\quad})e^{i(\omega t + (\underline{\quad}))}\}$$

phase ϕ

amplitude B