Chapter 0 – Mathematical Review

Vectors

unit vectors

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\boxed{\frac{\vec{r}}{|\vec{r}|} = \hat{r}}$$

C ≡ Coulomb

$$k = \frac{1}{4\pi\epsilon_0}$$

$9 \cdot 10^9 \ \frac{Nm^2}{C^2}$

$8.85 \cdot 10^{-12} \ \frac{C^2}{Nm^2}$

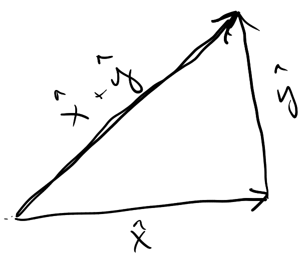
$$\vec{E} = \frac{kq}{r^2}\hat{r} = \frac{kq\vec{r}}{r^3}$$

fine, but incomplete → it assumes the charge is located at the origin

$\left[\frac{N}{C}\right]$

more general

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}}$$



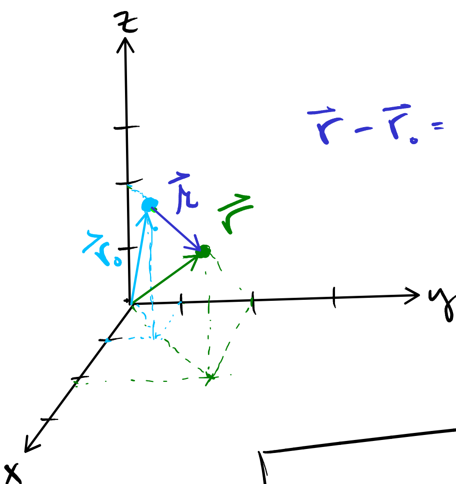
$\vec{r}$ ⇒ location where we want to know the electric field

$\vec{r}_0 = \vec{r}$ naught ⇒ location of the charge

Example 0.1

$$\vec{r} = (2\hat{x} + 2\hat{y} + 2\hat{z})$$

$$\vec{r}_0 = (1\hat{x} + 1\hat{y} + 2\hat{z})$$

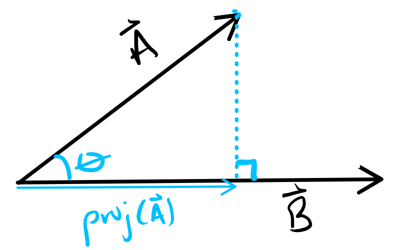


$$\vec{r} - \vec{r}_0 = \vec{\mathscr{r}} \Rightarrow \vec{r}_0 + \vec{\mathscr{r}} = \vec{r}$$

$$\vec{r} - \vec{r}_0 = 1\hat{x} + 1\hat{y} + 0\hat{z} = \hat{x} + \hat{y}$$

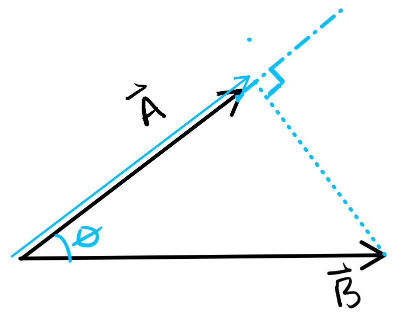
$$|\vec{r}-\vec{r}_0|^3 = |\hat{x}+\hat{y}|^3 = (\sqrt{2})^3 = 2^{3/2} = 2\cdot\sqrt{2} \longrightarrow \boxed{\vec{E}(\vec{r}) = \frac{q(\hat{x}+\hat{y})}{8\pi\epsilon_0\sqrt{2}}}$$

Dot Product (inner product)



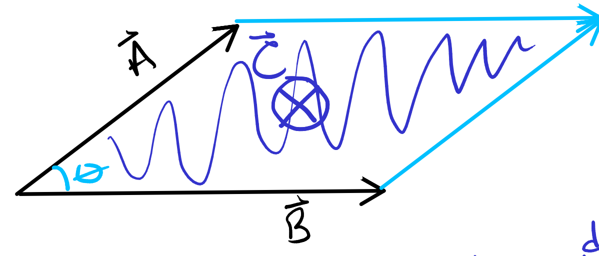
$$\vec{A}\cdot\vec{B} = \underbrace{A_xB_x + A_yB_y + A_zB_z}^{\text{scalar}}$$

$$= |A|\cdot|B|\cdot\cos\theta$$



Cross Product (vector product)

(perp. to both $\vec{A}$ + $\vec{B}$)



determinant

$$\underbrace{\vec{C}}_{\text{vector}} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

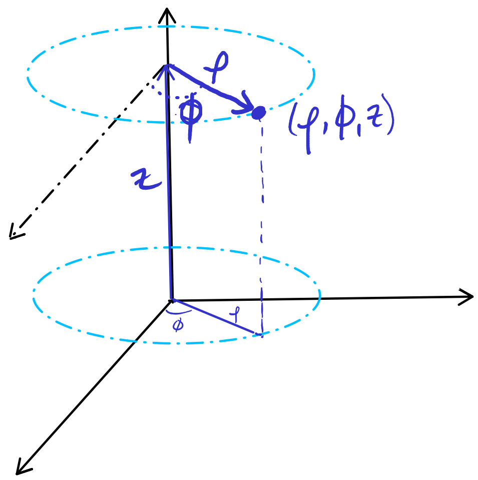
$$= \hat{x}(A_yB_z - A_zB_y)$$

$$-\hat{y}(A_xB_z - A_zB_x)$$

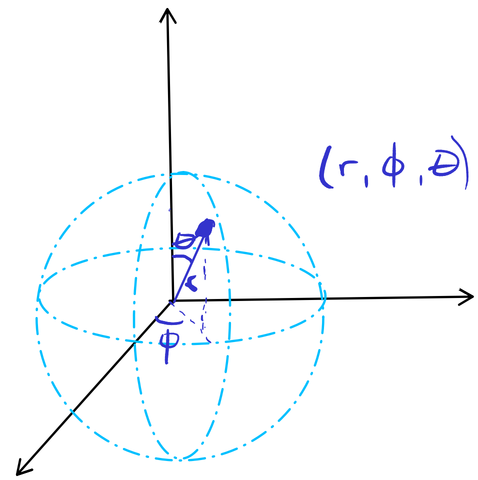
$$+\hat{z}(A_xB_y - A_yB_x)$$

$$|\vec{C}| = |\vec{A}\times\vec{B}| = |A|\cdot|B|\cdot\sin\theta$$

Cylindrical Coords.



Spherical Coords.



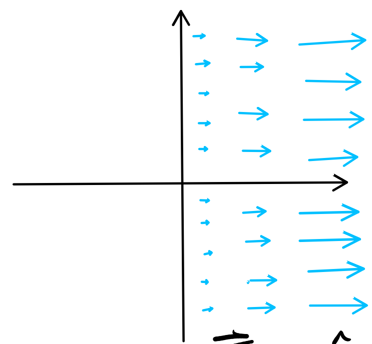
Gradient: of a scalar function (directional derivative)

$$\vec{\nabla} f(x,y,z) = \underbrace{\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}}_{\text{vector result points in the direction of steepest decline}}$$

scalar function

Divergence: of a vector function

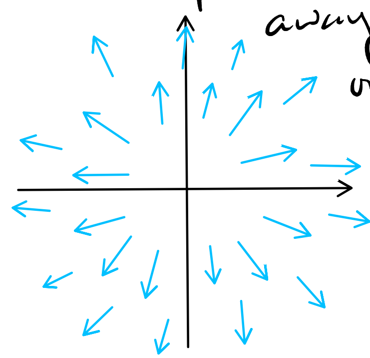
$$\vec{\nabla}\cdot\vec{E} = \underbrace{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}_{\text{scalar}}$$



$$\vec{E} = x\hat{x}$$

$$\vec{\nabla}\cdot\vec{E} = 1$$

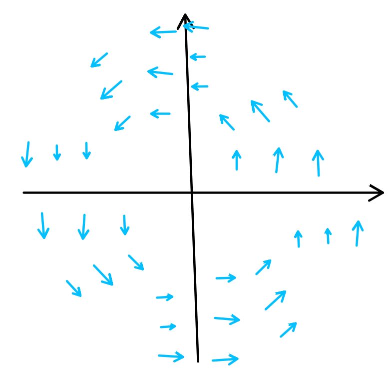
uniform field pointed radially away from origin



Curl: of a vector function

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \underbrace{\hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) - \hat{y}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)}_{\text{vector}}$$



Second Derivatives → Laplacian

$$\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\vec{E}(x,y,z) = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$$

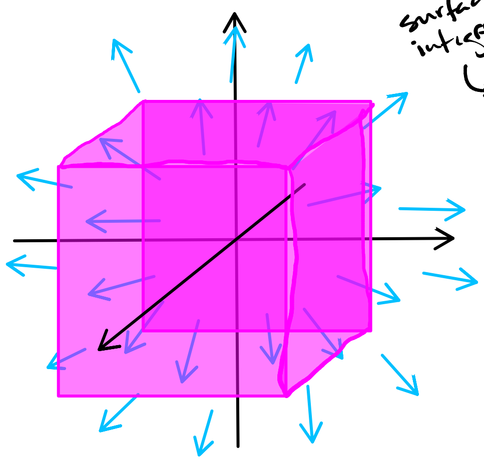
$$\nabla^2 \vec{E} = \left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2}\right)\hat{x}$$

$$\left(\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2}\right)\hat{y}$$

$$\left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2}\right)\hat{z}$$

$$\nabla^2 \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{E})$$ ← true in any coordinate system

Divergence Theorem



- closed surface integral

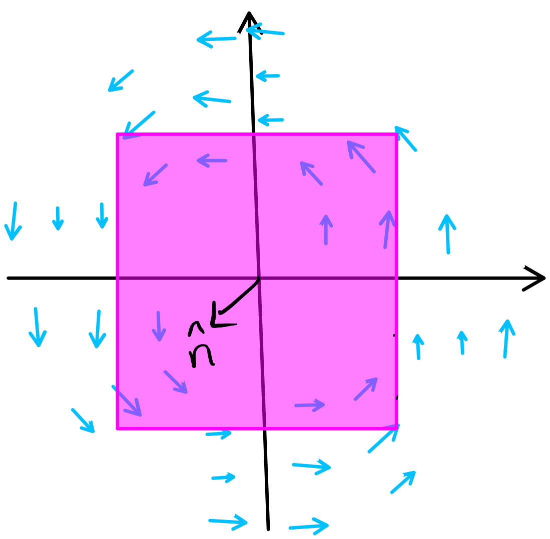
direction of the surface

break up into volumes

$$\oint_S \vec{E} \cdot \hat{n}\, da = \int_V \vec{\nabla} \cdot \vec{E}\, dv$$

break up into surfaces

Stokes Theorem



$$\int_S (\vec{\nabla} \times \vec{E}) \cdot \hat{n}\, da = \oint_C \vec{E} \cdot d\vec{l}$$

Complex Numbers

imaginary number → i

→ sum of a real number + an imaginary number

$$a + bi$$

$$\mathrm{Re}\{a+bi\} = a$$

$$\mathrm{Im}\{a+bi\} = b$$

complex function f + g

$$\mathrm{Re}\{f\} + \mathrm{Re}\{g\} = \mathrm{Re}\{f+g\}$$

$$\frac{d}{dx}\mathrm{Re}\{f\} = \mathrm{Re}\left\{\frac{df}{dx}\right\}$$

$$\int \mathrm{Re}\{f\}\,dx = \mathrm{Re}\left\{\int f\,dx\right\}$$

Euler's formula