Chapter 2 - Plane Waves + Index of Refraction

260208 M

Day 12

plane wave:

$$\vec{E}(\vec{r},t) = \vec{E}_0 \cdot \cos(\vec{k}\cdot\vec{r} - \omega \cdot t + \phi)$$

"wave vector" → $\vec{k} = k\hat{u} = \frac{2\pi}{\lambda_{vac}}\hat{u}$

"k"

$\hat{u}$: direction of propogation

angular frequency → $\omega = \frac{2\pi \cdot c}{\lambda_{vac}} = 2\pi \cdot \nu$

k + ω are related just like λ + ν

$$c = \frac{\lambda_{vac}}{T} = \lambda_{vac} \cdot \nu = \underbrace{\frac{\omega}{k}}$$

this relationship is known as the "dispersion relation"

"kappa" ↓ ↙ wavenumber

$$\frac{1}{\lambda_{vac}} = \kappa = [cm^{-1}]$$

$c = v_{vac}$ ← velocity of light in vacuum

ν ν ← "nu"

← frequency

similarly for magnetic field:

$$\vec{B}(\vec{r},t) = \vec{B}_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \phi)$$

$$\vec{B}_0 = \frac{\vec{k}\times\vec{E}_0}{\omega}$$, so $\vec{B}_0$ is not independent, but is determined by other parameters.

$$\vec{B}_0 \perp \vec{k} \perp \vec{E}$$

also think about magnitude

$$B_0 = \frac{kE_0}{\omega} = \frac{E_0}{c}$$ ← since this is so large we will focus on the $\vec{E}$ field.

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complex plane waves:

$$\vec{E}(r,t) = \mathrm{Re}\left\{\tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}\right\}$$

$$\tilde{\vec{E}}_0 = \vec{E}_0 e^{i\phi}$$ ← phase shift

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

260213 W

Day 13

Summarize some facts that we know now:

$\lambda \rightarrow$ wavelength

$T \rightarrow$ period

$\nu \rightarrow$ frequency $\left(\frac{1}{T}\right)$

$k \rightarrow$ wave vector $\left(\frac{2\pi}{\lambda}\right)$

$\omega \rightarrow$ angular frequency $\left(\frac{2\pi}{T}\right)$

$$V = \frac{\omega}{k}$$

$$V = \frac{\lambda}{T} = \lambda \cdot \nu \leftarrow \text{any wave}$$

$$c = \lambda_{vac} \cdot \nu$$

$$\epsilon_0 \mu_0 = \frac{1}{c^2} \Rightarrow c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

permittivity of free space $\rightarrow \epsilon_0 = 8.85 \cdot 10^{-12} \frac{C^2}{Nm^2}$

permeability of free space $\rightarrow \mu_0 = 4\pi \cdot 10^{-7} \frac{T}{Am^2}$

- $\nabla^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}$
- $\nabla^2 \vec{E} = \frac{1}{V^2} \frac{\partial^2 E}{\partial t^2}$
- $\nabla^2 \vec{E} = \frac{k^2}{\omega^2} \frac{\partial^2 E}{\partial t^2}$

Q1: what is light? ✓

Q2: where does light come from? ✓

Q3: can the speed of light change? see Q6

Q4: what affects the brightness of light?

Q5: how do human perceive light amount + color

Q6: why does light slow down in materials Q3

Q7: what is reflection

Speed of Light in matter

↳ light slows down in materials

$$\frac{c}{V} = n \leftarrow \text{index of refraction}$$

$n = 1 \leftarrow$ vacuum

$n = 1.0003 \leftarrow$ air

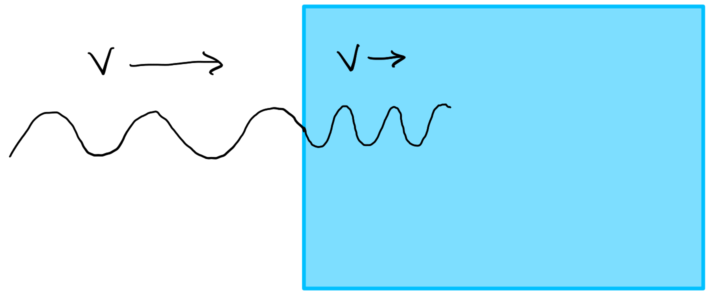
$n = 1.33 \leftarrow$ water

$n = 1.5 \leftarrow$ glass

constant as n increases

$$\lambda \cdot \nu = V = \frac{c}{n}$$

gets smaller as n increases



$$\lambda_{vac} \cdot \nu = c \iff c = n \cdot V$$

$$\lambda \cdot \nu$$

$$\lambda_{vac} \cancel{\nu} = n \cdot \lambda \cdot \cancel{\nu}$$

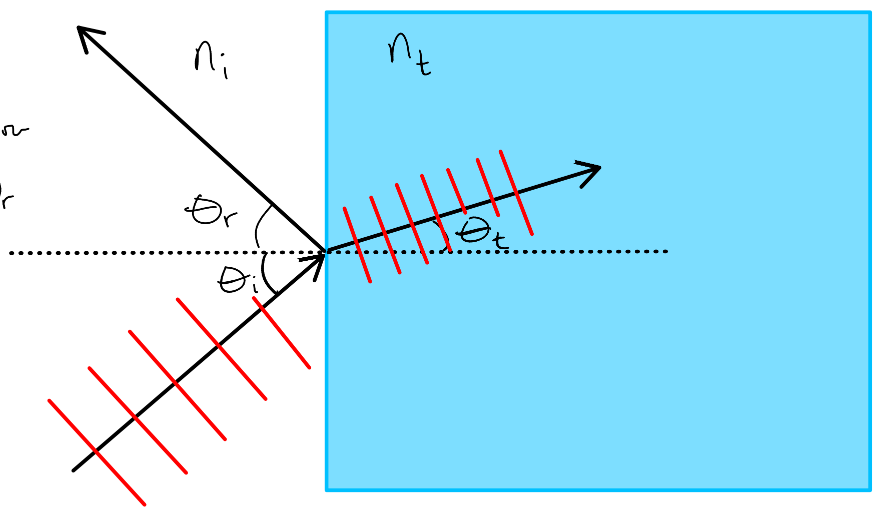
wavelength in material

$$\lambda = \frac{\lambda_{vac}}{n}$$ λ will be smaller than λ_0 since $n > 1$

Table 23.1 Indices of Refraction for $\lambda = 589.3$ nm in Vacuum (at 20°C Unless Otherwise Noted)	
Material	**Index**
Solids	
Ice (at 0°C)	1.309
Fluorite	1.434
Fused quartz	1.458
Polystyrene	1.49
Lucite	1.5
Plexiglas	1.51
Crown glass	1.517
Plate glass	1.523
Sodium chloride	1.544
Light flint glass	1.58
Dense flint glass	1.655
Sapphire	1.77
Zircon	1.923
Diamond	2.419
Titanium dioxide	2.9
Gallium phosphide	3.5
Liquids	
Water	1.333
Acetone	1.36
Ethyl alcohol	1.361
Carbon tetrachloride	1.461
Glycerin	1.473
Sugar solution (80%)	1.49
Benzene	1.501
Carbon disulfide	1.628
Methylene iodide	1.74

Law of reflection

$\theta_i = \theta_r$



Snell's Law

$$n_i \sin\theta_i = n_t \sin\theta_t$$

260213 F
Day 14

static case

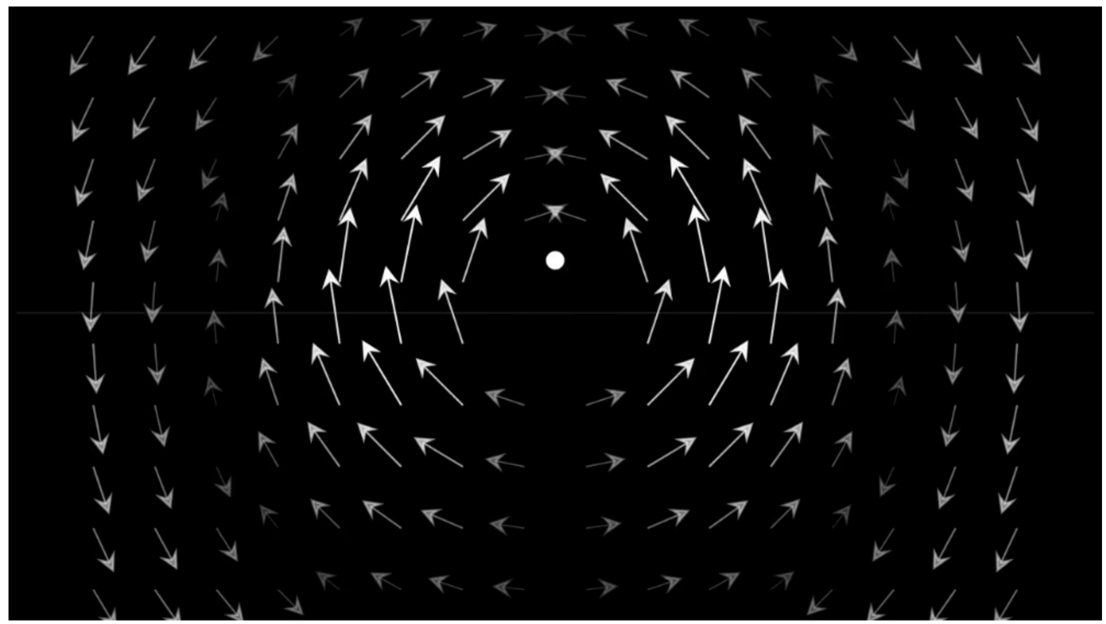
$E(\vec{r}) = \frac{kq_0}{r^2}$ $\longrightarrow$

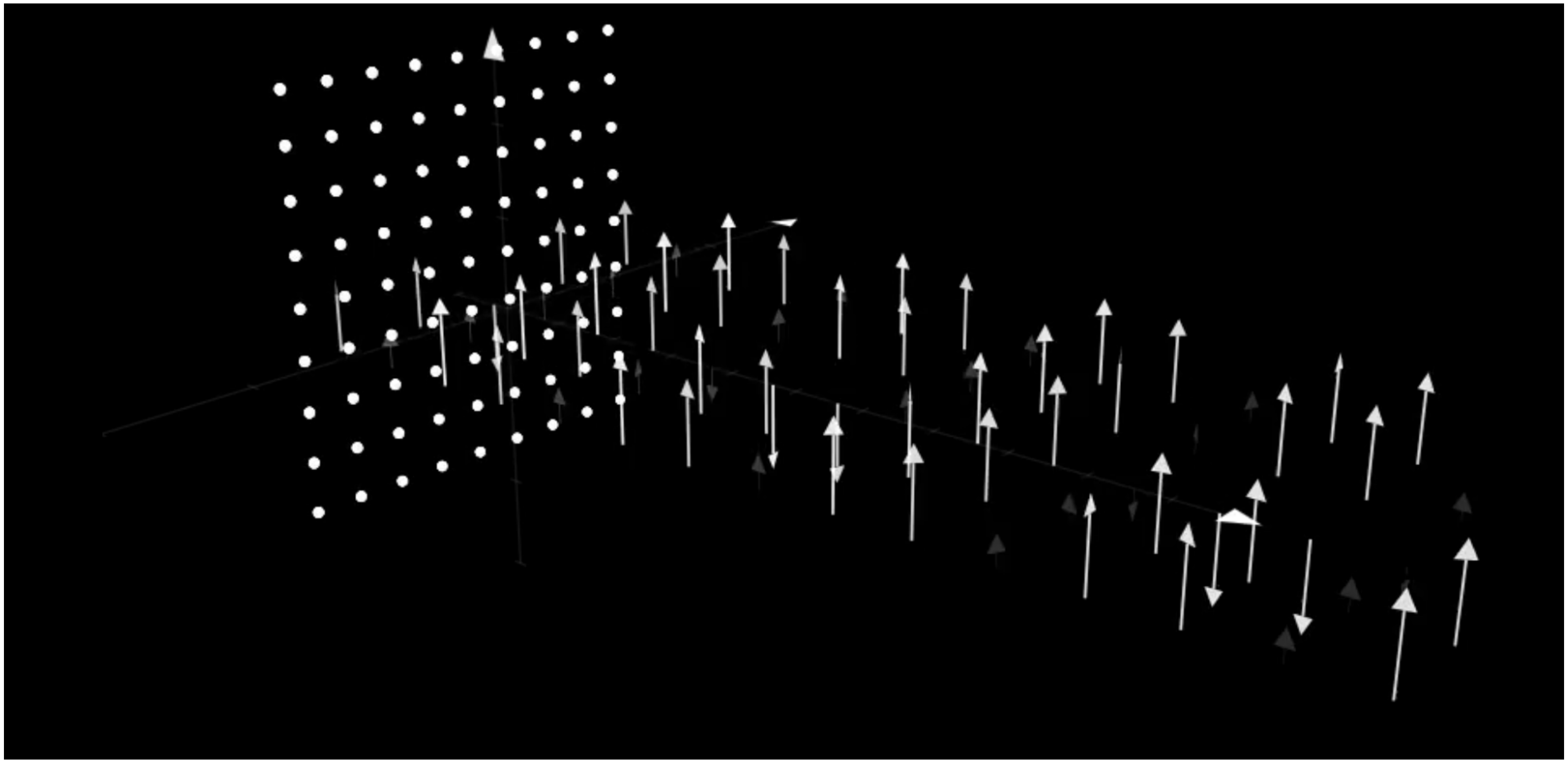
oscillating charge

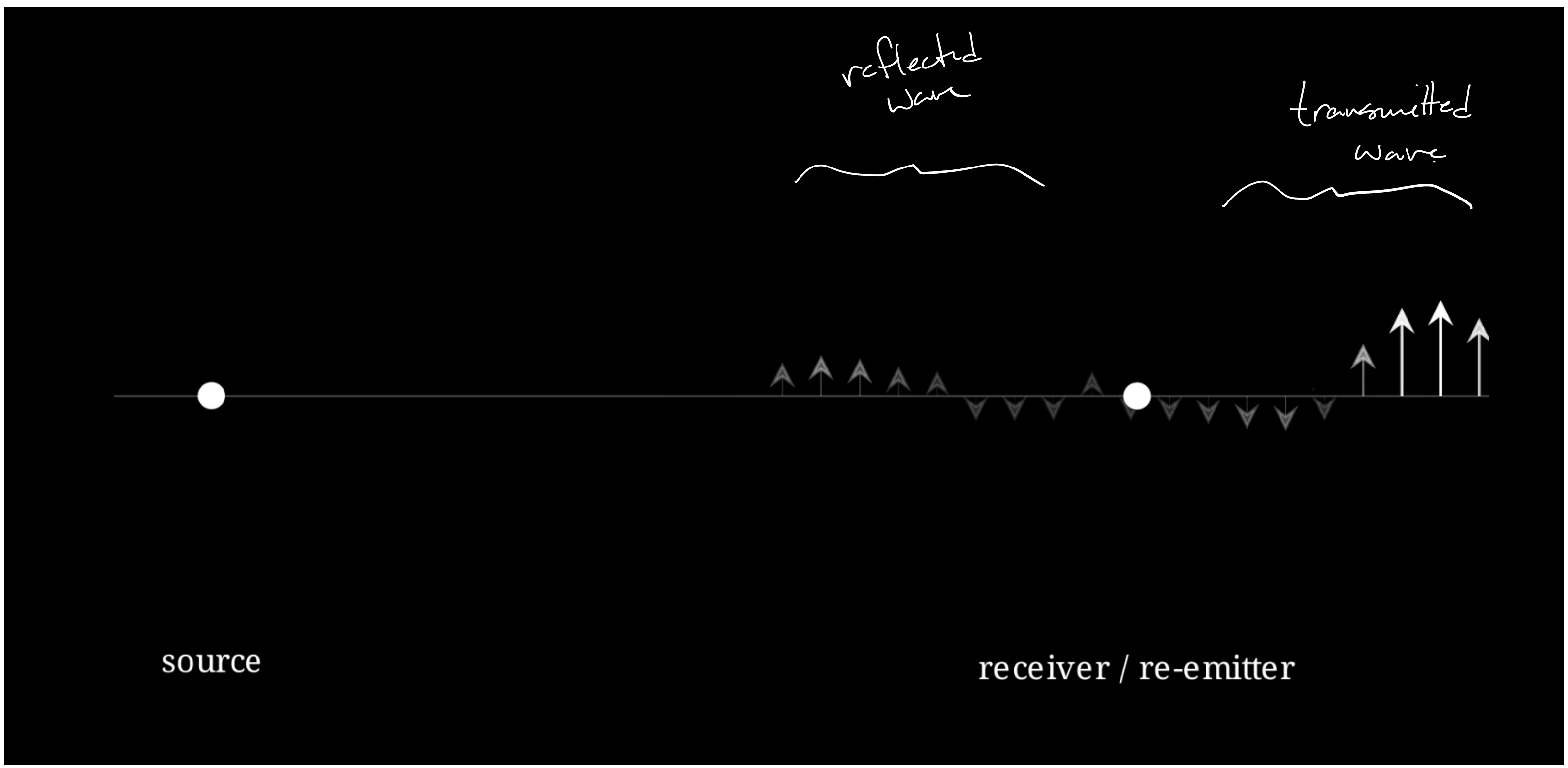
$\vec{E}(\vec{r},t) = \vec{E}_{staticish} + \underbrace{\vec{E}_{rad}}$

$\vec{E}_{rad} \propto \frac{\vec{a}_\perp \overbrace{(t - r/c)}^{\text{wave behavior}}}{r}$

compare to $\vec{r} - vt$






reflected wave
transmitted wave
source
receiver / re-emitter

260216 M
Day 15

Dielectric Media

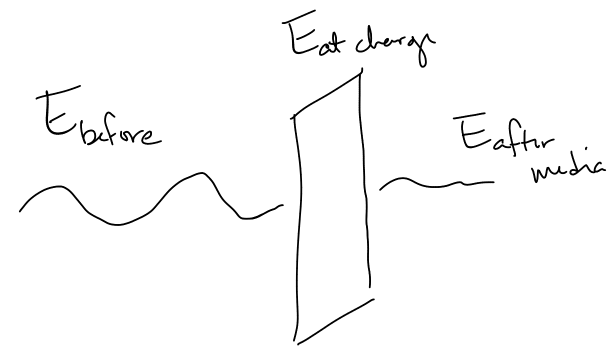
↳ insulator (like glass, water, air)

Assumptions:
1. isotropic → same in all directions
2. homogeneous → same over distance
3. non-conducting (insulator/dielectric) $\vec{J}_{free} = 0$
4. very thin material (air or some other thin gas)

we will relax this later on

Light in a thin gas → Feynman, vol 1, Ch 31

Light in matter = incident field + field reradiated by driven charges



$$\vec{E}_{\text{after plate}} = \vec{E}_s + \sum_{\text{all other charges}} \vec{E}_{\text{each charge}}$$

$$\vec{E}_s = \vec{E}_0 \cos(kx - \omega t)$$

$$\vec{E}_s = \vec{E}_0 \cos\left(k\left(x - \tfrac{\omega}{k} t\right)\right)$$

$\frac{\omega}{k} = c$

$$\vec{E} = \vec{E}_0 \cos\left(\omega\left(\tfrac{k}{\omega} x - t\right)\right)$$

$$\vec{E} = \vec{E}_0 \cos\left(-\omega\left(t - \tfrac{k}{\omega} x\right)\right)$$

$$\vec{E} = \vec{E}_0 \cos\left(\omega\left(t - \frac{x}{c}\right)\right)$$

$$\vec{E} = \vec{E}_0 e^{i\omega\left(t - \frac{x}{c}\right)}$$

So now what about the field from other charges?

plate thickness $= \Delta x$

$$t_{vac} = \frac{\Delta x}{c}$$

$$t_{plate} = \frac{n \Delta x}{c}$$

$$\Delta t = t_{plate} - t_{vac}$$

$$= (n-1)\frac{\Delta x}{c}$$

$$E_{\substack{\text{after}\\ \text{plate}}} = E_0 e^{i\omega\left(t - \Delta t - \frac{x}{c}\right)}$$

$$= E_0 e^{i\omega\left(t - (n-1)\frac{\Delta x}{c} - \frac{x}{c}\right)}$$

$$= \underbrace{e^{-i\omega(n-1)\frac{\Delta x}{c}}}_{\text{w/ a phase shift}} \underbrace{E_0 e^{i\omega\left(t - \frac{x}{c}\right)}}_{\text{just like the original wave BUT w/ a}}$$

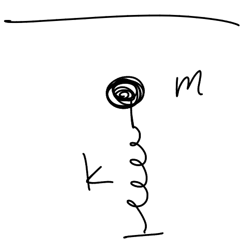
$e^{-i\omega(n-1)\frac{\Delta x}{c}}$, Δx is very small

$e^{\delta} \approx 1 + \delta$, for small δ

$$e^{-i\omega(n-1)\frac{\Delta x}{c}} \approx 1 - i\omega(n-1)\frac{\Delta x}{c}$$

$$E_{\substack{\text{after}\\\text{plate}}} = \underbrace{E_0 e^{i\omega(t-\frac{x}{c})}}_{E_s} - \underbrace{\frac{i\omega(n-1)\Delta x}{c} \cdot E_0 e^{i\omega(t-\frac{x}{c})}}_{E_a}$$

$E_a \to$ new electric field that is the result of all moving charges



$$ma = \Sigma F = -kz$$

$$m\frac{d^2z}{dt^2} = -kz$$

$$\boxed{\frac{d^2z}{dt^2} = -\frac{k}{m}z}$$

one way

$$\to z = A \cdot \cos(\omega \cdot t + \phi)$$

$$\frac{dz}{dt} = -A\omega \cdot \sin(\omega t + \phi)$$

$$\to \frac{d^2z}{dt^2} = -A\omega^2 \cos(\omega t + \phi)$$

another way

$$z = Ae^{\omega t + \phi}$$

$$\frac{dz}{dt} = A\omega e^{\omega t + \phi}$$

$$\frac{d^2z}{dt^2} = A\omega^2 e^{\omega t + \phi}$$

$$A\omega^2 e^{\omega t + \phi} = -\frac{k}{m} \cdot Ae^{\omega t + \phi}$$

$$\omega^2 = -\frac{k}{m}$$

$$\cancel{-}\cancel{A}\omega^2 \cancel{\cos(\omega t + \phi)} = \cancel{-}\frac{k}{m} \cdot \cancel{A} \cancel{\cos(\omega t + \phi)}$$

$$\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$$

one of our constants

$$z = A \cdot \cos\left(\sqrt{\tfrac{k}{m}}\, t + \phi\right) \leftarrow \text{general solution to that DE}$$

A given by initial conditions

at $t=0$, $z=1 \Rightarrow A=1$ $\quad \phi = 0$

($A + \phi$ are determined by initial conditions)

$z = 1 \cdot \cos\left(\sqrt{\frac{k}{m}} \cdot t\right)$

260218 W

Day 16

Now we have a charge connected by a spring being driven by $\vec{E}(t)$

$$\frac{d^2 z}{dt^2} = \underbrace{-\frac{k}{m}}_{\omega_0^2} z + \frac{q_e E_0}{m} e^{i\omega t + \phi}$$

the frequency of the light

ω_0^2 → natural frequency of the electrons

Guess: $z = z_0 e^{i\omega t + \phi}$

$$\dot{z} = z_0 i\omega e^{i\omega t + \phi}$$

$$\ddot{z} = z_0 \underbrace{i^2}_{-1} \omega^2 e^{i\omega t + \phi}$$

So now we plug in:

$$-z_0 \omega^2 \cancel{e^{i\omega t + \phi}} = \underbrace{-\omega_0^2 z_0}\, \cancel{e^{i\omega t + \phi}} + \frac{q_e E_0}{m} \cancel{e^{i\omega t + \phi}}$$

→ solve for z_0

$$z_0(\omega_0^2 - \omega^2) = \frac{q_e E_0}{m}$$

$$z_0 = \frac{q_e E_0}{\underbrace{m(\omega_0^2 - \omega^2)}_{\mid}}$$

$\xrightarrow{\text{so...}}$

$$\underbrace{z(t) = \frac{q_e E_0}{m(\omega_0^2 - \omega^2)} \cdot e^{i\omega t + \phi}}_{\text{motion of charges}}$$

motion of charges → Electric field that results

$$E_a = \frac{-\eta q_e}{2\epsilon_0 c} i\omega z_0 e^{i\omega(t - \frac{x}{c})}$$

← from Griffiths or Feynman

$\vec{E}$ from a static sheet of charge

$$E = \frac{Q}{2\epsilon_0 A}$$

$$E = \frac{\sigma}{2\epsilon_0}$$

$$E = \frac{\eta q_e}{2\epsilon_0}$$

from before

$$E_a = \frac{-\eta q_e}{2\epsilon_0 c} \cdot i\omega \cdot \frac{q_e E_0}{m(\omega_0^2 - \omega^2)} \cdot e^{i\omega(t-\frac{x}{c})} \quad \text{vs.} \quad E_a = \frac{-i\omega(n-1)\Delta x}{c} \cdot E_0 e^{i\omega(t-\frac{x}{c})}$$

So putting what survives together

$$\frac{\eta q_e^2}{2\epsilon_0 m(\omega_0^2 - \omega^2)} = (n-1)\Delta x$$

η → number of charges per unit of area

N → number of charges per unit of volume

$$\eta = N \cdot \Delta x$$

$$\frac{N \cdot \Delta x\, q_e^2}{2\epsilon_0 m(\omega_0^2 - \omega^2)} = (n-1)\Delta x$$

$$n = 1 + \underbrace{\frac{N \cdot q_e^2}{2\epsilon_0 m(\omega_0^2 - \omega^2)}}$$

ω_0 is often much larger than ω for visible light

as ω rises the denominator gets smaller, so the fraction gets larger!

Let's do this again, but better. → index of refraction for heavier dielectrics

→ Go back to the wave equation from Maxwell's:

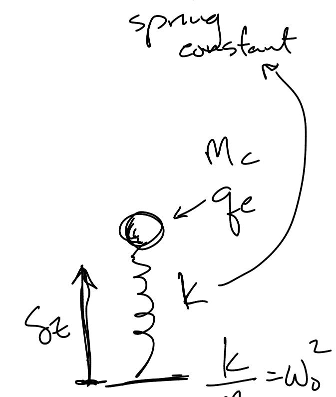
$$\nabla^2 \vec{E} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}$$

→ what is $\vec{P}$?

polarization of an atom (not the polarization of light)

$$P = N \cdot q \cdot \delta z$$

N: number per volume



$$\frac{k}{m_e} = \omega_0^2$$

plug in!

$$\begin{cases} \vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \\ \vec{P} = \vec{P}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \end{cases}$$

$$\underbrace{i^2}_{-1} k^2 E_0 \cancel{e^{i(\vec{k}\cdot\vec{r} - \omega t)}} - \epsilon_0 \mu_0 (+\underbrace{i^2}_{-1} \omega^2) \vec{E}_0 \cancel{e^{i(\vec{k}\cdot\vec{r} - \omega t)}} = \mu_0 (+\underbrace{i^2}_{-1} \omega^2) \vec{P}_0 \cancel{e^{i(\vec{k}\cdot\vec{r} - \omega t)}}$$

k ↳ wave vector, k

$$-k^2 \vec{E}_0 + \epsilon_0 \mu_0 \omega^2 \vec{E}_0 = -\mu_0 \omega^2 \vec{P}_0$$

$$\vec{P}_0 = \epsilon_0 \chi(\omega) \vec{E}_0 \quad \leftarrow \text{constitutive relation}$$

↳ susceptibility
↳ kicking the can down the road!
↳ complex!
↳ a function of ω (frequency) just like $\vec{P}_0$ + $\vec{E}_0$
↳ plug in and let $\vec{E}_0$ cancel out

$$-k^2 + \epsilon_0\mu_0\omega^2 = -\epsilon_0\mu_0\omega^2\chi(\omega)$$

$\hookrightarrow$ solve for k, $\epsilon_0\mu_0 = \frac{1}{c^2}$

$$\epsilon_0(1+\chi(\omega)) = \epsilon(\omega)$$ ← permittivity of a material

$$k^2 = \epsilon_0\mu_0\omega^2(1+\chi(\omega))$$

still wave vector k → $$\boxed{k = \frac{\omega}{c}\sqrt{1+\chi(\omega)}}$$

recall: $k = \frac{\omega}{v}$ and $v = \frac{c}{n}$

$$\boxed{k = \frac{\omega \cdot n}{c}}$$

compare!

$$\tilde{n} = \sqrt{1+\chi(\omega)}$$

$$\tilde{n} = n(\omega) + i\kappa(\omega)$$

$\hookrightarrow$ light "slows down"

κ → absorption coefficient

$\hookrightarrow \tilde{n} = n' + in''$

$$\mathcal{N}(\omega) \equiv n(\omega) + i\kappa(\omega) = \sqrt{1+\chi(\omega)}$$

$$\tilde{k} = \frac{\tilde{n}\omega}{c}$$

$$\tilde{k} = (n+i\kappa)\frac{\omega}{c}$$

Let's plug this back in to $\vec{E}$

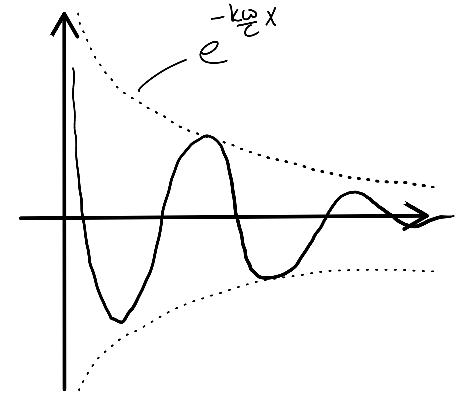
$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} = \vec{E}_0 e^{i(n+i\kappa)\frac{\omega}{c}\hat{u}\cdot\vec{r} - i\omega t}$$

$$= \vec{E}_0 e^{-\kappa\frac{\omega}{c}\hat{u}\cdot\vec{r} + i\left(\frac{n\omega}{c}\right)\hat{u}\cdot\vec{r} - i\omega t}$$

$$= \vec{E}_0 e^{-\frac{\kappa\omega}{c}\hat{u}\cdot\vec{r}} \cdot e^{i(\frac{n\omega}{c}\hat{u}\cdot\vec{r}-\omega t)}$$

has phase information → $\vec{E}_0 = \vec{E}_0 e^{i\phi}$ ← bad notation so watch out!

$$= \underbrace{\vec{E}_0 e^{-\frac{\kappa\omega}{c}\hat{u}\cdot\vec{r}}}_{\text{this wave decays as it goes into this material}} \cdot \cos\left(\frac{n\omega}{c}\hat{u}\cdot\vec{r}-\omega t+\phi\right) + \text{some imaginary stuff the we ignore}$$



$$V_{phase}(\omega) = \frac{c}{n(\omega)}$$ ← real part of $\tilde{n}$

$$P = N \cdot q \cdot \delta z \xrightarrow{\text{textbook}} \vec{P} = N q_e \vec{r}_e$$

model → $m\ddot{z} = -kz - m\gamma\dot{z} + q_e E_0 e^{i\omega t}$

$$z = \frac{q_e \vec{E}_0 e^{i\omega t}}{m_e(\omega_0^2 - i\omega\gamma - \omega^2)}$$ ← inhomogeneous solution -or- nonhomogeneous

$$P = \frac{N q_e^2 \vec{E}_0 e^{i\omega t}}{m_e(\omega_0^2 - i\omega\gamma - \omega^2)}$$

$$\left[\frac{N q_e^2}{m_e}\right] = \left[\frac{C^2}{m^3\,kg}\right]$$

ϵ_0 $\quad F = \frac{q_0 q_1}{4\pi\epsilon_0 r^2}$

$$\frac{k q_0 q_1}{r^2}$$

$$[k] = \left[\frac{N m^2}{C^2}\right] \quad \epsilon_0 = \left[\frac{C^2}{N m^2}\right]$$

$$\frac{N q_e^2}{\epsilon_0 m_e} = \omega_p^2$$ ← plasma frequency

$$\left[\frac{N q_e^2}{\epsilon_0 m_e}\right] = \frac{\cancel{C}^2}{m^3 kg} \cdot \frac{N \cancel{m^2}}{\cancel{C}^2}$$

$$= \frac{N}{m\,kg} = \frac{\frac{kg\,m}{s^2}}{m\,kg}$$

$$= \frac{\cancel{kg\,m}}{s^2} \cdot \frac{1}{\cancel{m\,kg}}$$

$$= \frac{1}{s^2} = \frac{rad^2}{s^2}$$

$$P = \underbrace{\epsilon_0 \left(\frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2} \right)}_{\text{compare to this result from earlier!}} \vec{E}_0 e^{i\omega t}$$

compare to this result from earlier!

$$\vec{P}_0 = \epsilon_0 \chi(\omega) \vec{E}_0$$

$$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}$$

$$\tilde{n} = n + i\kappa \qquad \tilde{n} = \sqrt{1 + \chi(\omega)}$$

$$n + i\kappa = \sqrt{1 + \chi(\omega)}$$

$$(n + i\kappa)^2 = 1 + \chi(\omega)$$

$$(n + iK)^2 = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}$$

$$n + iK = \sqrt{1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}}$$

plot as a function of frequency

$$n^2(\omega) = 1 + \frac{Nq_e^2}{\epsilon_0 m_e}\left(\frac{1}{\omega_0^2 - \omega^2}\right) \qquad (3.70)$$

3.62* Show that Eq. (3.70) can be rewritten as

$$\boxed{(n^2 - 1)^{-1} = -C\lambda^{-2} + C\lambda_0^{-2}}$$

where $C = 4\pi^2 c^2 \epsilon_0 m_e / Nq_e^2$.

3.63 Augustin Louis Cauchy (1789–1857) determined an empirical equation for $n(\lambda)$ for substances that are transparent in the visible. His expression corresponded to the power series relation

$$n = C_1 + C_2/\lambda^2 + C_3/\lambda^4 + \cdots$$

where the Cs are all constants. In light of Fig. 3.41, what is the physical significance of C_1?

3.64 Referring to the previous problem, realize that there is a region between each pair of absorption bands for which the Cauchy Equation (with a new set of constants) works fairly well. Examine Fig. 3.41: what can you say about the various values of C_1 as ω decreases across the spectrum? Dropping all but the first two terms, use Fig. 3.40 to determine approximate values for C_1 and C_2 for borosilicate crown glass in the visible.

3.65* Crystal quartz has refractive indexes of 1.557 and 1.547 at wavelengths of 400.0 nm and 500.0 nm, respectively. Using only the first two terms in Cauchy's equation, calculate C_1 and C_2 and determine the index of refraction of quartz at 610.0 nm.

3.66* In 1871 Sellmeier derived the equation

$$n^2 = 1 + \sum_j \frac{A_j\lambda^2}{\lambda^2 - \lambda_{0j}^2}$$

where the A_j terms are constants and each λ_{0j} is the vacuum wavelength associated with a natural frequency ν_{0j}, such that $\lambda_{0j}\nu_{0j} = c$. This formulation is a considerable practical improvement over the Cauchy Equation. Show that where $\lambda >> \lambda_{0j}$, Cauchy's Equation is an approximation of Sellmeier's. *Hint:* Write the above expression with only the first term in the sum; expand it by the binomial theorem; take the square root of n^2 and expand again.

larger λ

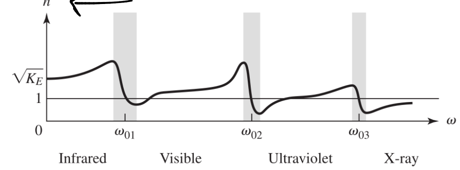


Figure 3.41 Refractive index versus frequency.

n gets closer to C_1 as λ gets larger.

Exercises for 2.4 The Lorentz Model of Dielectrics

P2.1 Verify that (2.35) is a solution to (2.34).

P2.2 Derive the Sellmeier equation

$$(n + i\kappa)^2 = 1 + \chi(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2} \qquad (2.39)$$

$$n^2 = 1 + \frac{A\lambda_{vac}^2}{\lambda_{vac}^2 - \lambda_{0,vac}^2}$$

from (2.39) for a gas with negligible absorption (i.e. $\gamma \cong 0$, valid far from resonance ω_0), where $\lambda_{0,vac}$ corresponds to frequency ω_0 and A is a constant. Many materials (e.g. glass, air) have strong resonances in the ultraviolet. In such materials, do you expect the index of refraction for blue light to be greater than that for red light? Make a sketch of n as a function of wavelength for visible light down to the ultraviolet (where $\lambda_{0,vac}$ is located).

P2.3 In the Lorentz model, take $N = 10^{28}$ m^{-3} for the density of bound electrons in $\omega_0 = 6 \times 10^{15}$ rad/sec (in the UV)

$$(n + i\kappa)^2 = 1 + \chi(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2} \qquad (2.39)$$

l). Assume that the magnitude of $\mathbf{E}_0$ is 10^4 V/m. For three frequencies i) $\omega = \omega_0 - 2\gamma$, ii) $\omega = \omega_0$, and iii) $\omega = \omega_0 + 2\gamma$ find:

(a) the amplitude and phase of the charge displacement $\mathbf{r}_e$ (2.35) relative to the phase of $\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$.

(b) the magnitude and complex phase of the susceptibility $\chi(\omega)$. Does $\chi(\omega)$ depend on the strength of the E-field?

(c) n and κ at the three frequencies via (2.29) and (2.27).

Answer: i) $n = 1.53$, $\kappa = 0.0817$, ii) $n = 1.66$, $\kappa = 1.33$, iii) $n = 0.470$, $\kappa = 0.263$.

(d) the three speeds of light in terms of c and how far light penetrates into the material before only $1/e$ of the amplitude of $\mathbf{E}$ remains.

P2.4 (a) Use a computer to plot n and κ as a function of ω for a dielectric (i.e. obtain graphs such as the ones in Fig. 2.5). Use the Lorentz model and the following parameters: $\omega_0 = 10\omega_p$, and $\gamma = \omega_p$; plot your function from $\omega = 0$ to $\omega = 20\omega_p$. No need to choose a value for ω_p; your horizontal axis will be in units of ω_p.

(b) Plot n and κ as a function of frequency for a material that has three resonant frequencies: $\omega_{01} = 10\omega_p$, $\gamma_1 = \omega_p$, $f_1 = 0.5$; $\omega_{02} = 15\omega_p$, $\gamma_2 = \omega_p$, $f_2 = 0.25$; and $\omega_{03} = 25\omega_p$, $\gamma_3 = 3\omega_p$, $f_3 = 0.25$. Plot the results from $\omega = 0$ to $\omega = 30\omega_p$.

"thin" gas

$\hookrightarrow \quad n = 1 + \frac{N \cdot q_e^2}{2\epsilon_0 m(\omega_0^2 - \omega^2)}$

$$n^2(\omega) = 1 + \frac{Nq_e^2}{\epsilon_0 m_e}\left(\frac{1}{\omega_0^2 - \omega^2}\right) \qquad (3.70)$$

$n = \tilde{n}$

$\gamma = 0$

$\tilde{n} = n + iK = \sqrt{1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}}$

$\frac{Nq_e^2}{\epsilon_0 m_e} = \omega_p^2$

$\tilde{n}(\omega)^2 = 1 + \frac{Nq_e^2}{\epsilon_0 m_e(\omega_0^2 - \omega^2)}$

3.62* Show that Eq. (3.70) can be rewritten as

$$(n^2 - 1)^{-1} = -C\lambda^{-2} + C\lambda_0^{-2}$$

where $C = 4\pi^2 c^2 \epsilon_0 m_e / Nq_e^2$.

$$\tilde{n}(\omega)^2 = 1 + \frac{Nq_e^2}{\epsilon_0 m_e(\omega_0^2 - \omega^2)}$$

$$n^2 - 1 = \frac{Nq_e^2}{\epsilon_0 m_e(\omega_0^2 - \omega^2)}$$

$v = \lambda \cdot \nu \qquad v = \frac{\omega}{k}$

$\nu = \frac{v}{\lambda} = \frac{c}{\lambda} \qquad \lambda\nu = \frac{\omega}{k} \Rightarrow \omega = \lambda \nu k$

$\omega = 2\pi\nu = \frac{2\pi c}{\lambda}$

$\omega_0 = \frac{2\pi c}{\lambda_0}$

$$(n^2 - 1)^{-1} = \frac{\epsilon_0 m_e(\omega_0^2 - \omega^2)}{Nq_e^2}$$

$$= \frac{\epsilon_0 m_e}{Nq_e^2} \cdot \left(\frac{4\pi^2 c^2}{\lambda_0^2} - \frac{4\pi^2 c^2}{\lambda^2}\right)$$

$$= \underbrace{\frac{4\pi^2 c^2 \epsilon_0 m_e}{Nq_e^2}}_{C}\left(\frac{1}{\lambda_0^2} - \frac{1}{\lambda^2}\right)$$

$$(n^2 - 1)^{-1} = C\left(\lambda_0^{-2} - \lambda^{-2}\right) \quad ✓✓$$

$$\tilde{n}^2 - 1 = 1 + \sum_j \frac{f_j \omega_{pj}^2}{\omega_{0j}^2 - i\omega\gamma_j - \omega^2}$$

3.63 Augustin Louis Cauchy (1789–1857) determined an empirical equation for $n(\lambda)$ for substances that are transparent in the visible. His expression corresponded to the power series relation

$$n = C_1 + \underbrace{\cancel{C_2/\lambda^2}^{\,0} + \cancel{C_3/\lambda^4}^{\,0}} + \cdots$$

for longer wavelengths $n = \sqrt{K_E}$ is index at very long wavelengths

where the Cs are all constants. In light of Fig. 3.41, what is the physical significance of C_1?

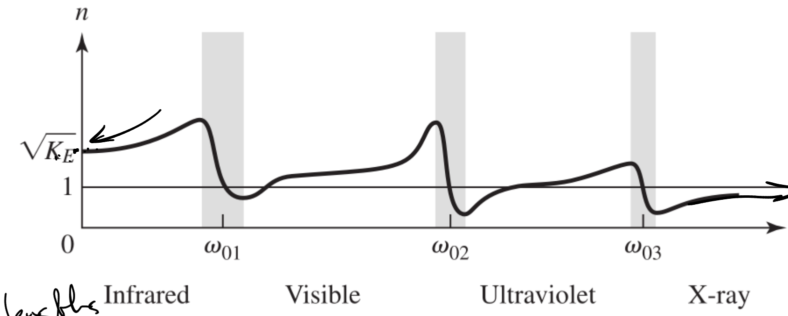


Figure 3.41 Refractive index versus frequency.

shorter and shorter wavelengths

$$\omega = \frac{2\pi c}{\lambda}$$

material permittivity

$$K_E = \frac{\epsilon}{\epsilon_0} \xrightarrow{\text{recall}} \epsilon_0(1+\chi)$$

$$\epsilon_0(1 + \chi(\omega)) = \epsilon(\omega)$$ ← permittivity of a material

$$n = \sqrt{1+\chi}$$ ← that is fine w/ what we have said $\longrightarrow$ $$\chi(\omega) = \frac{\omega_p^2}{\omega_0^2 - \underbrace{i\omega\gamma}_{\to 0} - \omega^2}$$

long λ

$$n = \underbrace{\sqrt{1 + \frac{\omega_p^2}{\omega_0^2}}}_{\text{constant stuff}}$$

$$\chi(\omega \to 0) = \frac{\omega_p^2}{\omega_0^2}$$

P2.2 Derive the Sellmeier equation $(n + i\kappa)^2 = 1 + \chi(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}$

$$n^2 = 1 + \frac{A\lambda_{vac}^2}{\lambda_{vac}^2 - \lambda_{0,vac}^2}$$

from (2.39) for a gas with negligible absorption (i.e. $\gamma \cong 0$, valid far from resonance ω_0), where $\lambda_{0,vac}$ corresponds to frequency ω_0 and A is a constant. Many materials (e.g. glass, air) have strong resonances in the ultraviolet. In such materials, do you expect the index of refraction for blue light to be greater than that for red light? Make a sketch of n as a function of wavelength for visible light down to the ultraviolet (where $\lambda_{0,vac}$ is located).

$\kappa \to 0 \quad \gamma \to 0$

$$\underbrace{(n + i\kappa)^2}_{\tilde{n}^2} = 1 + \chi(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \underbrace{i\omega\gamma}_{=0} - \omega^2} \qquad (2.39)$$

$$\omega_0 = \frac{2\pi c}{\lambda_{0,vac}} \qquad \omega = \frac{2\pi c}{\lambda_{vac}}$$

$v = c = \lambda_{vac}\nu$

$$n^2 = 1 + \frac{\omega_p^2}{\frac{4\pi^2c^2}{\lambda_0^2} - \frac{4\pi^2c^2}{\lambda^2}}$$

$$\frac{Nq_e^2}{\epsilon_0 m_e} = \omega_p^2$$

$$= 1 + \frac{\omega_p^2}{\frac{4\pi^2c^2\lambda^2 - 4\pi^2c^2\lambda_0^2}{\lambda_0^2\lambda^2}}$$

$$= 1 + \frac{\lambda_0^2\omega_p^2\lambda^2}{4\pi^2c^2(\lambda^2 - \lambda_0^2)}$$

$\hookrightarrow A$

if ω_0 is a constant, then λ_0 is constant as well

$$n^2 = 1 + \frac{A\lambda^2}{\lambda^2 - \lambda_0^2}$$

$$A = \frac{\lambda_0^2\omega_p}{4\pi^2c^2}$$

$$A = \frac{\lambda_0^2 Nq_e^2}{\epsilon_0 m_e 4\pi^2c^2}$$

3.66* In 1871 Sellmeier derived the equation

$$n^2 = 1 + \sum_j \frac{A_j \lambda^2}{\lambda^2 - \lambda_{0j}^2}$$

where the A_j terms are constants and each λ_{0j} is the vacuum wavelength associated with a natural frequency ν_{0j}, such that $\lambda_{0j}\nu_{0j} = c$. This formulation is a considerable practical improvement over the Cauchy Equation. Show that where $\lambda >> \lambda_{0j}$, Cauchy's Equation is an approximation of Sellmeier's. *Hint:* Write the above expression with only the first term in the sum; expand it by the binomial theorem; take the square root of n^2 and expand again.

$$(1 \pm x)^p \approx 1 \pm px$$

$\leftarrow$ $\underbrace{\text{"expand by binomial theorem"}}$

this is poorly phrased imo

I would say "binomial approx" or a "Taylor approx"

$\lambda_0 \nu_0 = c$

$\lambda >> \lambda_0$

$$n^2 = 1 + \frac{A\lambda^2}{\lambda^2 - \lambda_0^2}$$

$$\hookrightarrow 1 + \frac{A}{1 - \frac{\lambda_0^2}{\lambda^2}}$$

$$\hookrightarrow 1 + A\Big(1 - \underbrace{\frac{\lambda_0^2}{\lambda^2}}_{\text{small}}\Big)^{-1}$$

small $\rightarrow$ Show that where $\lambda >> \lambda_{0j}$, C

Is A small?

$$A = \frac{\lambda_0^2 N q_e^2}{\epsilon_0 m_e \underbrace{4\pi^2}_{\hookrightarrow 10\text{ ish}} c^2}$$

$$\lambda_0 = \frac{c}{\nu} = \frac{3\cdot 10^8}{10^{14}} = 3\cdot 10^{-6}$$ $\leftarrow$ small ish so square and its small

$c^2 \sim 10^{16}$

$\epsilon_0 \sim 10^{-11}$

$m_e = 10^{-31}$

q_e^2 $\leftarrow$ very small $(10^{-19})^2 = 10^{-38}$

$N = \frac{\text{number}}{\text{volume}} \Rightarrow \frac{N}{V}$ $\leftarrow$ large number $\sim 10^{23}$? but could be bigger

$$A = \frac{10^{-12}\cdot 10^{23}\cdot 10^{-38}}{10^{-11}\cdot 10^{-31}\cdot 10\cdot 10^{16}} = \frac{10^{-27}}{10^{-25}} = \frac{10^{25}}{10^{27}} = \frac{1}{100}$$ $\leftarrow$ not as small as I hoped but still....

$$n^2 = 1 + A\left(1 + \frac{\lambda_0^2}{\lambda^2}\right)$$

$$n = \Big(1 + A\underbrace{\Big(1 + \frac{\lambda_0^2}{\lambda^2}\Big)}_{\text{again this is small if A is small}}\Big)^{1/2} \simeq 1 + \frac{A}{2}\left(1 + \frac{\lambda_0^2}{\lambda^2}\right) = \underbrace{\left(1 + \frac{A}{2}\right)}_{C_1} + \underbrace{\frac{A\lambda_0^2}{2\lambda^2}}_{C_2 = \frac{A\lambda_0^2}{2}} \longrightarrow n = C_1 + \frac{C_2}{\lambda^2} + \frac{C_3}{\lambda^4} + \ldots$$

but I'm not convinced that A is small enough.... so

$$n^2 = 1 + A\left(1 + \frac{\lambda_0^2}{\lambda^2}\right)$$

$$= 1 + A + \frac{A\lambda_0^2}{\lambda^2}\frac{(1+A)}{(1+A)}$$

$$= (1+A)\left(\underbrace{1 + \frac{A\lambda_0^2}{\lambda^2(1+A)}}\right)$$

this is small bc $\lambda^2 >> \lambda_0^2$

$$n = (1+A)^{1/2}\left(1 + \frac{A\lambda_0^2}{\lambda^2(1+A)}\right)^{1/2}$$

so now I'll do my approx:

$$n = (1+A)^{1/2}\left(1 + \frac{A\lambda_0^2}{2\lambda^2(1+A)}\right)$$

then distribute

$$n = \underbrace{(1+A)^{1/2}}_{C_1} + \underbrace{\frac{A(1+A)^{1/2}\lambda_0^2}{2(1+A)\ \lambda^2}}_{C_2 = \frac{A\lambda_0^2}{2(1+A)^{1/2}}}$$

that is better!

$$A = \frac{\lambda_0^2 N q_e^2}{\epsilon_0 m_e 4\pi^2 c^2}$$

$$C_1 = \left(1 + \frac{\lambda_0^2 N q_e^2}{\epsilon_0 m_e 4\pi^2 c^2}\right)^{1/2}$$