

Chapter 0 - Mathematical Review

Vectors

unit vectors

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{E} = \frac{kq}{r^2} \hat{r} = \frac{kq\vec{r}}{r^3}$$

fine, but incomplete \rightarrow it assumes the charge is located at the origin
[N/C]

more general

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \cdot \frac{q(\vec{r}-\vec{r}_0)}{|\vec{r}-\vec{r}_0|^3}}$$

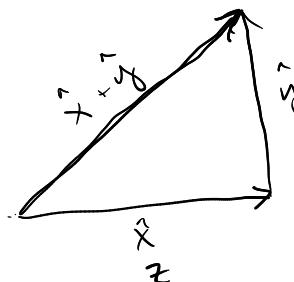
Example 0.1

$$\vec{r} = (2\hat{x} + 2\hat{y} + 2\hat{z})$$

$$\vec{r}_0 = (1\hat{x} + 1\hat{y} + 2\hat{z})$$

$$\vec{r} - \vec{r}_0 = 1\hat{x} + 1\hat{y} + 0\hat{z} = \hat{x} + \hat{y}$$

$$|\vec{r} - \vec{r}_0|^3 = |\hat{x} + \hat{y}|^3 = (\sqrt{2})^3 = 2^{3/2} = 2\sqrt{2}$$



$$\boxed{\frac{\vec{r}}{|\vec{r}|} = \hat{r}}$$

$C = \underline{\text{Coulomb}}$

$$9 \cdot 10^9 \frac{\text{Nm}^2}{\text{C}^2}$$

$$k = \frac{1}{4\pi\epsilon_0}$$

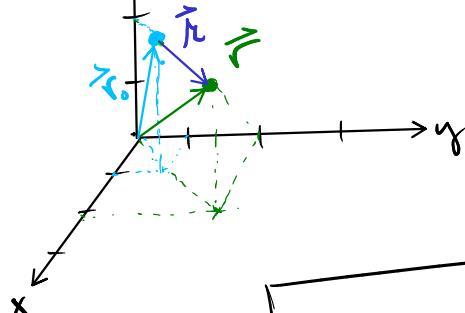
$$8.85 \cdot 10^{-12} \frac{\text{C}^2}{\text{Nm}^2}$$

$\vec{r} \Rightarrow$ location where we want to know the electric field

$\vec{r}_0 = \vec{r}_{\text{naught}}$

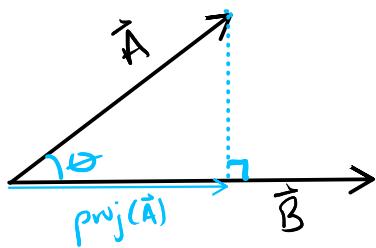
location of the charge

$$\vec{r} - \vec{r}_0 = \vec{r}_l \Rightarrow \vec{r}_0 + \vec{r}_l = \vec{r}$$

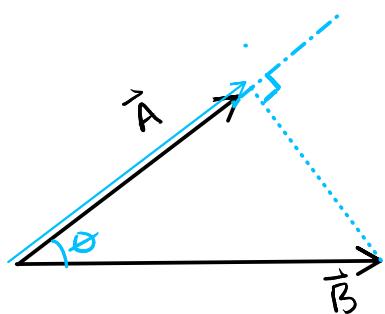


$$\boxed{\vec{E}(\vec{r}) = \frac{q(\hat{x} + \hat{y})}{8\pi\epsilon_0 \sqrt{2}}}$$

Dot Product (inner product)



$$\vec{A} \cdot \vec{B} = \underbrace{A_x B_x + A_y B_y + A_z B_z}_{\text{scalar}} = |A| |B| \cos \theta$$



Cross Product (vector product)

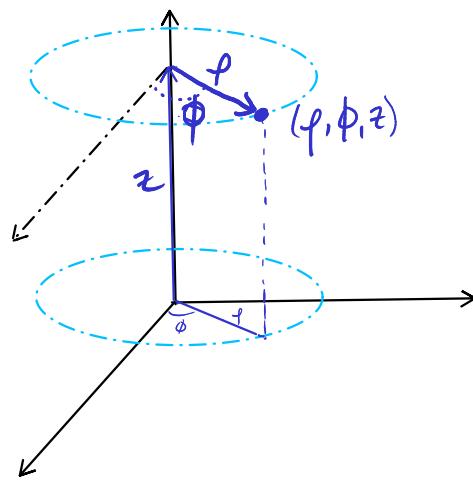
(perp. to both $\vec{A} + \vec{B}$)

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad \text{determinant}$$

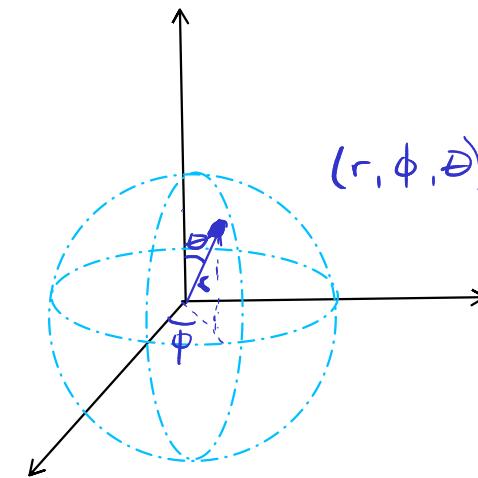
$$\begin{aligned} &= \hat{x}(A_y B_z - A_z B_y) \\ &- \hat{y}(A_x B_z - A_z B_x) \\ &+ \hat{z}(A_x B_y - A_y B_x) \end{aligned}$$

$$|\vec{C}| = |\vec{A} \times \vec{B}| = |A| |B| \sin \theta$$

Cylindrical Coords.



Spherical Coords.



Gradient: of a scalar function
(directional derivative)

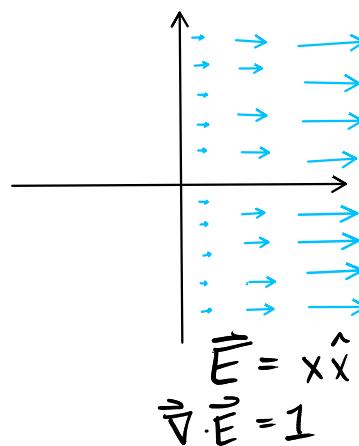
$$\vec{\nabla}f(x, y, z) = \underbrace{\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}}_{\text{vector result}}$$

↑
scalar
function

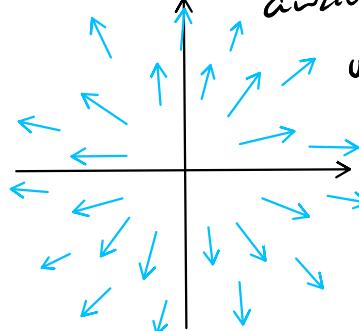
vector result
points in the
direction of steepest
descent

Divergence: of a vector function

$$\vec{\nabla} \cdot \vec{E} = \underbrace{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}_{\text{Scalar}}$$



uniform field pointed radially
away from
origin



Curl: of a vector function

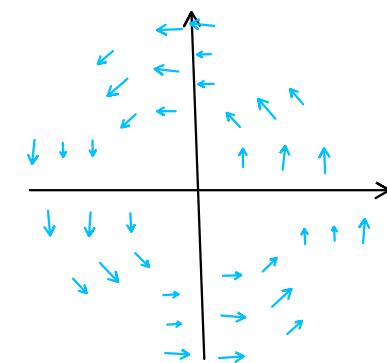
$$\vec{\nabla} \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

$$= \hat{x} \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right)$$

$$- \hat{y} \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right)$$

$$+ \hat{z} \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right)$$

vector



Second Derivatives \rightarrow Laplacian

$$\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

—

$$\nabla^2 \vec{E} = \left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} \right) \hat{x}$$

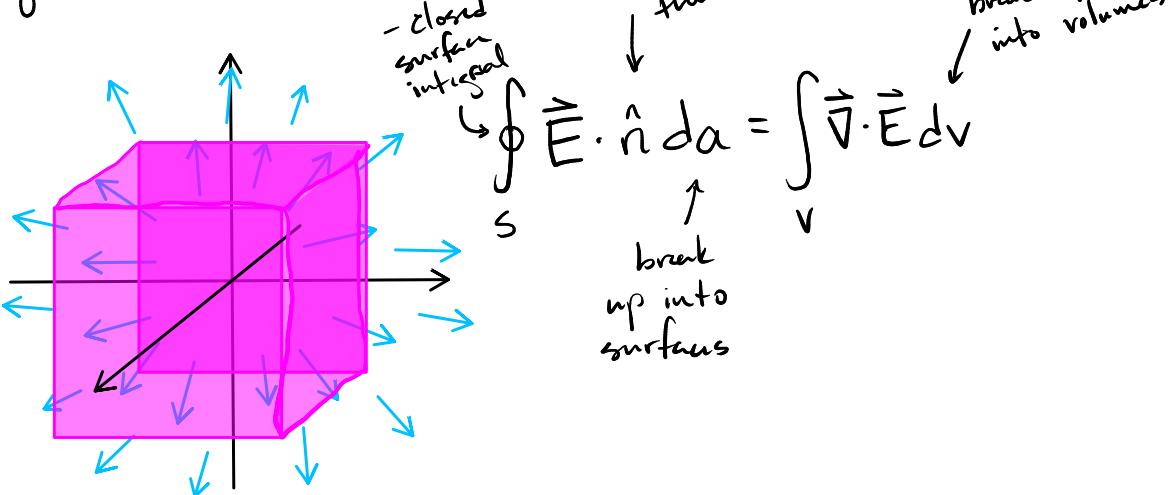
$$\vec{E}(x, y, z) = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$$

$$\left(\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} \right) \hat{y}$$

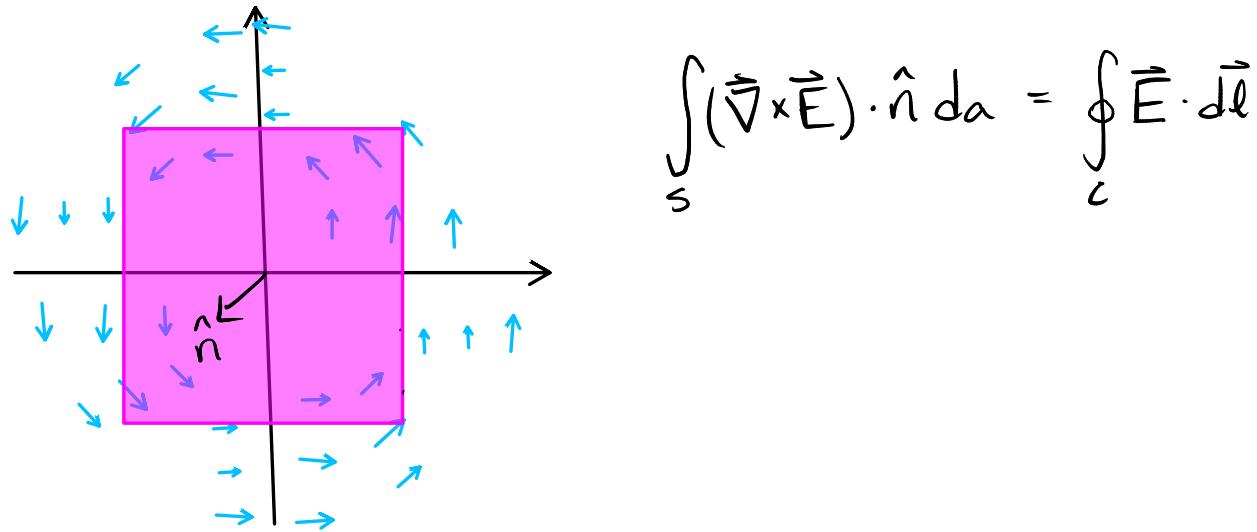
$$\left(\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) \hat{z}$$

→ $\nabla^2 \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{E})$ ← true in any coordinate system

Divergence Theorem



Stokes Theorem



Complex Numbers

imaginary number $\rightarrow i = \sqrt{-1}$

→ sum of a real number + an imaginary number

$$\tilde{z} = a + bi$$

$$\operatorname{Re}\{a+bi\} = a$$

$$\operatorname{Im}\{a+bi\} = b$$

$$\rightarrow |z^2| = |z|^2 = a^2 + b^2$$

$$a = |\tilde{z}| \cos \theta$$

$$b = |\tilde{z}| \sin \theta$$

Now:

$$\tilde{z} = |\tilde{z}|(\cos \theta + i \sin \theta)$$

So:

$$\tilde{z} = |\tilde{z}| e^{i\theta}$$

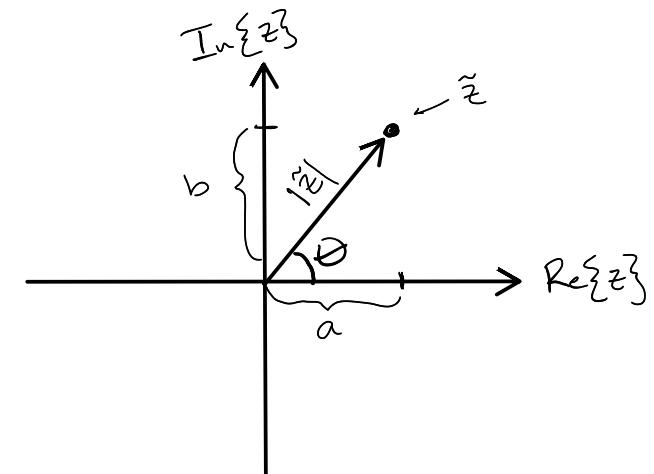
complex function $f + g$

$$\operatorname{Re}\{f\} + \operatorname{Re}\{g\} = \operatorname{Re}\{f+g\}$$

$$\frac{d}{dx} \operatorname{Re}\{f\} = \operatorname{Re}\left\{\frac{df}{dx}\right\}$$

$$\int \operatorname{Re}\{f\} dx = \operatorname{Re}\left\{\int f dx\right\}$$

Complex Plane



Taylor Series:

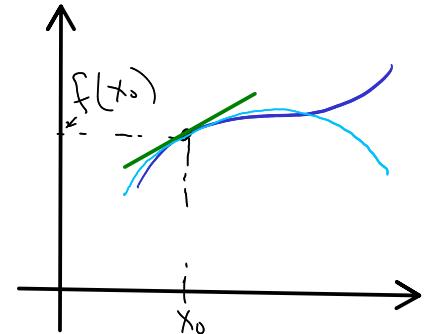
$$f(x) = f(x_0) + \underbrace{\frac{1}{1!} (x-x_0) \cdot \frac{df}{dx} \Big|_{x=x_0}}_{\text{linear term}} + \underbrace{\frac{1}{2!} (x-x_0)^2 \frac{d^2 f}{dx^2} \Big|_{x=x_0}}_{\text{quadratic term}}$$

$$\cos \phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \dots$$

$$i \sin \phi = i\phi - \frac{i\phi^3}{3!} + \frac{i\phi^5}{5!} - \dots$$

$$e^{i\phi} = 1 + i\phi - \frac{\phi^2}{2!} - \frac{i\phi^3}{3!} + \frac{\phi^4}{4!} + \dots$$

$$e^{i\phi} = \cos \phi + i \sin \phi \quad \text{Euler's identity}$$



$$\boxed{e^{i\pi} = -1}$$

$$\boxed{e^{i\pi} + 1 = 0}$$

Complex Conjugate

$$\zeta \tilde{\zeta}^* = a - bi \quad \text{or} \quad \tilde{\zeta}^* = |\tilde{\zeta}| e^{-i\theta}$$

$$\tilde{\zeta} \cdot \tilde{\zeta}^* = |\tilde{\zeta}|^2$$

$$\tilde{\zeta} \cdot \tilde{\zeta}^* = |\tilde{\zeta}| e^{i\theta} |\tilde{\zeta}| e^{-i\theta}$$

$$\tilde{\zeta} \cdot \tilde{\zeta}^* = |\tilde{\zeta}|^2$$

$$\operatorname{Re}\{\tilde{\zeta}\} = \frac{\tilde{\zeta} + \tilde{\zeta}^*}{2} \quad \rightarrow \quad \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta = \operatorname{Re}\{e^{i\theta}\}$$

$$\operatorname{Im}\{\tilde{\zeta}\} = \frac{\tilde{\zeta} - \tilde{\zeta}^*}{2i}$$

HW3: 0.14, 17, 18, 20

0.14 | $\zeta_1 = 1 - i \quad \zeta_2 = 3 + 4i$

a) $\zeta_1 - \zeta_2 = (1 - i) - (3 + 4i)$
 $= 1 - i - 3 - 4i$

\downarrow magnitude \swarrow phase
 $\zeta_1 - \zeta_2 = \underline{\underline{-2 - 5i}} = \zeta_3 = \underline{\underline{|z| e^{i\theta}}}$ } polar form

b) $\frac{\zeta_1}{\zeta_2} = \frac{1-i}{3+4i} \cdot \frac{(3-4i)}{(3-4i)} = \underline{\underline{\quad \quad \quad}}$ → polar form

Q.17] Show $\operatorname{Re}\{\tilde{A}\} \times \operatorname{Re}\{\tilde{B}\} = \frac{AB + A^*B}{4} + \text{CC.}$

$\underbrace{\hspace{1cm}}_{\text{Complex conjugate}} \rightarrow \frac{A^*B^* + AB^*}{4}$

? $\tilde{A} = A_1 + A_2i$

? $\tilde{A}^* = A_1 - A_2i$

$$= \frac{AB + A^*B + A^*B^* + AB^*}{4}$$

$$= \frac{B(A+A^*) + B^*(A+A^*)}{4}$$

$$= \frac{(A+A^*)}{2} \frac{(B+B^*)}{2}$$

$$= \operatorname{Re}\{A\} \operatorname{Re}\{B\} \checkmark$$