Chapter 2

Some combinatorics

coin flips → 5 coins

H H T T H ← microstate

T H H H H ← microstate

3 heads ← macrostates

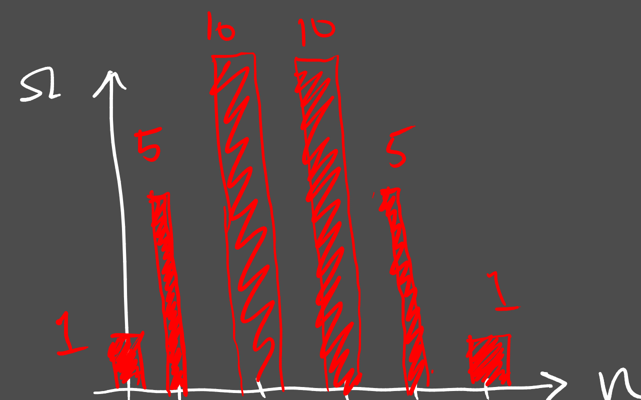
4 heads ← macrostates

how many microstates are in a macrostate?

↳ Depends on the macrostate

multiplicity $\Omega(n) = \dfrac{5!}{n!(5-n)!}$ ← combinations for 5 coins

macrostate
# of heads



$0! = 1$ $\quad n! = n(n-1)\cdots 1$

$1! = 1$

$2! = 2$

↳ generalize to N coins

$$\Omega(N, n) = \frac{N!}{n!(N-n)!} \qquad \binom{N}{n}$$

N: # of coins

n: # of heads

10 atoms → each atom can have 0 or 1 $\underbrace{\text{energy unit}}_{\text{quanta}}$

So how many possible arrangements of 4 quanta (10 quanta)

← microstate

4 energies vs. 10 energies ← macrostate

What if atoms can have more than 1 energy unit at a time

← microstate

$$\Omega(N,q) = \frac{(q+N-1)!}{q!\,(N-1)!} \qquad \binom{q+N-1}{q}$$

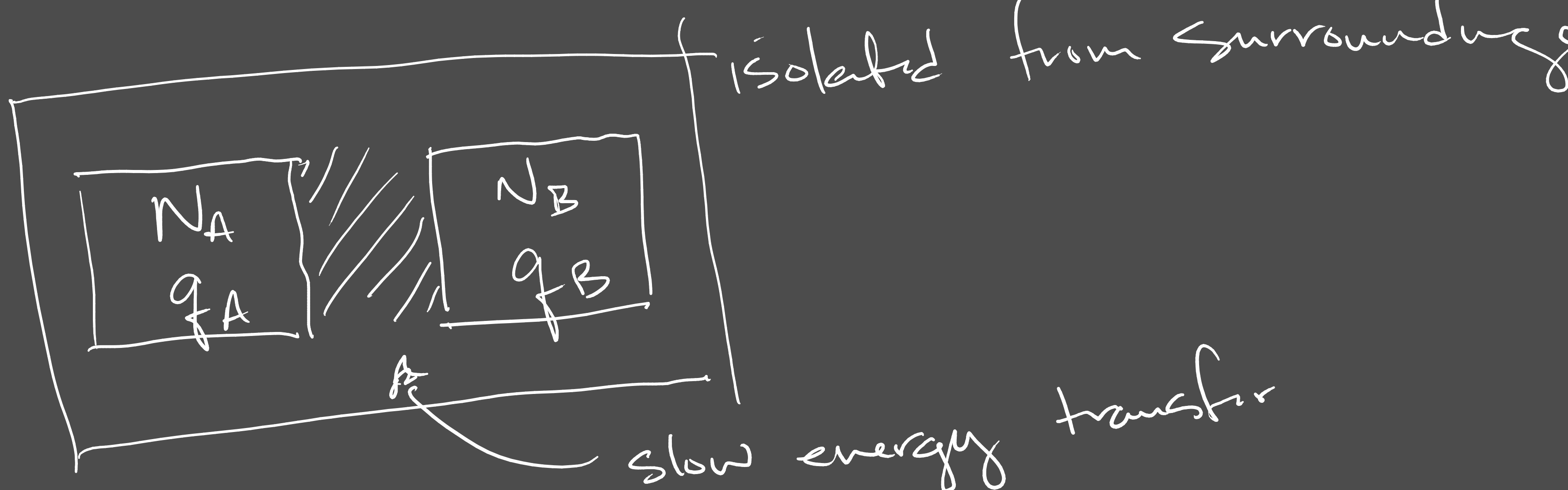
N: # of atoms

q: packets of energy

This model → Einstein Solid

\# 1,5

Two Solids that can exchange energy



A B

← microstate

1 2

macrostate
$q = 3$

$q = 3$

2 1

3

3 2

$q = 1$

$\Omega_A(1) = 3$

$q = 5$

$\Omega_B(5) = 21$

$\Omega_{total} = \Omega_A \cdot \Omega_B$

$\Omega_{total} = 63$

Total number of microstates possible → 462

All microstates are equally likely

→ Fundamental assumption of statistical mechanics. (axiom)

→ But all macrostates are not equally likely.

$$\text{Probability of solid A having } q \text{ energy} = \frac{\Omega_{total}(q_A)}{\sum_{q_A} \Omega_{total}(q_A)}$$

Large Numbers, 10^{23}

Addition of small numbers is not important

$$10^{23} + 50 \approx 10^{23}$$

Very Large Numbers → Multiplication of Large numbers is not important

$$10^{10^{23}}, \quad 10^{10^{100}}$$

$$10^{10^{23}} \cdot 10^{23} = 10^{10^{23}+23} \approx 10^{10^{23}}$$

Stirling's Approximation

$$N! \approx N^N e^{-N} \sqrt{2\pi N}$$

$$\approx \frac{N^N}{e^N} \underbrace{\sqrt{2\pi N}}_{\text{not all that important}}$$

$$\ln N! \approx N \ln N - N$$

$$\Omega(N,q) = \frac{(q+N-1)!}{q!\,(N-1)!} = \frac{(q+N)!}{q!\,N!}$$

$$\ln\Omega = \underbrace{\ln(q+N)!}_{(q+N)\ln(q+N)-(q+N)} - \ln q! - \ln N!$$

$$\ln\Omega = (q+N)\ln(q+N) - \cancel{(q+N)} - q\ln q \cancel{+q} - N\ln N \cancel{+N}$$

$$\ln\Omega = (q+N)\underbrace{\ln(q+N)} - q\ln q - N\ln N$$

$\boxed{\ln(1+x) \approx x \text{ for small } x}$

$\hookrightarrow$ high temperature $\rightarrow q >> N$

$$\ln(q+N)$$

$$\ln\left[q\left(1+\frac{N}{q}\right)\right] = \ln q + \underbrace{\ln\left(1+\frac{N}{q}\right)}_{N/q}$$

$$\ln \Omega = N \ln q + \frac{N^2}{q} + \cancel{q \ln q} + N - \cancel{q \ln q} - N \ln N$$

$$\ln \Omega = N \ln \frac{q}{N} + N + \underbrace{\frac{N^2}{q}}_{\text{small}}$$

$$\Omega(q \gg N) \approx e^{N \ln \frac{q}{N} + N} = \left(\frac{q}{N}\right)^N e^N$$

$$\Omega(q \gg N) = \left(\frac{eq}{N}\right)^N$$

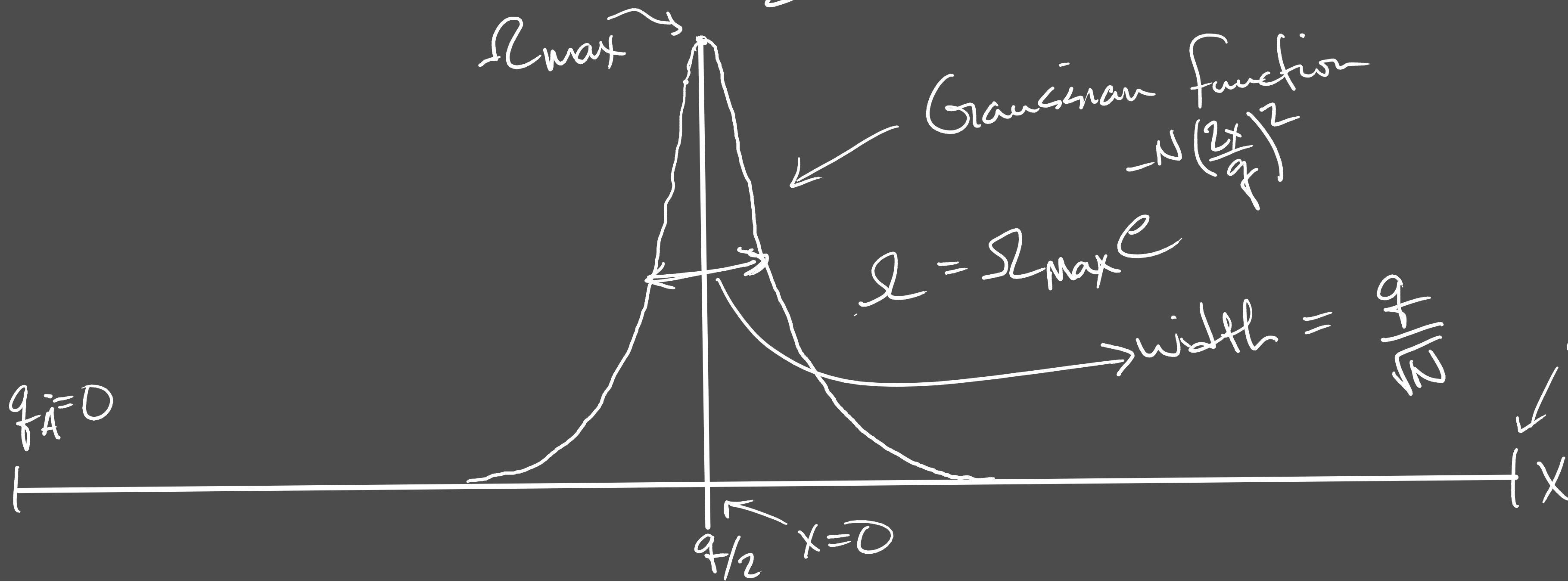
Now go back and use this w/ Einstein solid.

same # particles

$$\Omega_{total} = \left(\frac{eq_A}{N_A}\right)^{N_A}\left(\frac{eq_B}{N_B}\right)^{N_B} = \left(\frac{e}{N}\right)^{2N}(q_Aq_B)^N$$

$$N_A = N_B$$

$$\Omega_{max} \rightarrow q_A = q_B = \frac{q}{2} \qquad \Omega_{max} = \left(\frac{e}{N}\right)^{2N}\left(\frac{q}{2}\right)^{2N}$$



Gaussian function

$$\Omega = \Omega_{max}\, e^{-N\left(\frac{2x}{q}\right)^2}$$

width $= \dfrac{q}{\sqrt{N}}$

$$\frac{\text{width}}{\text{full range}} = \frac{q/\sqrt{N}}{q} = \frac{1}{\sqrt{N}} \longrightarrow N = 10^{20} \qquad \frac{1}{\sqrt{10^{20}}} = \frac{1}{10^{10}}$$

HW: 9, 10, 13, 18, 22

The multiplicity of an ideal gas.

→ macrostate → volume, total energy, number

$$\Omega = \text{distribution of particles in space} \cdot \text{distribution of energy among the particles}$$

$$\Omega_1 \propto V \cdot V_p$$

V: physical volume

V_p: volume of momentum space (surface area of momentum space)

Heisenburg Uncertainty Principle

$$\Delta x \cdot \Delta p_x \approx h$$

$h \rightarrow$ Planck Constant

$\dfrac{L_x}{\Delta x} \rightsquigarrow$ # of distinct places to put particle in x-direction

$\dfrac{L_{p_x}}{\Delta p_x} \rightsquigarrow$ # of distinct momentum states

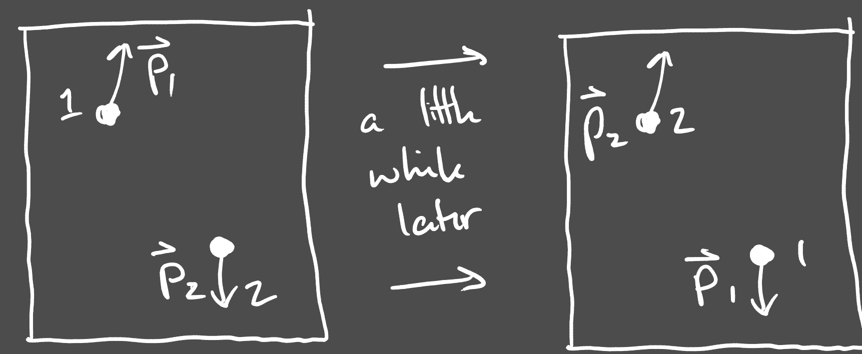
$$\Omega_1 = \overbrace{\frac{L_x}{\Delta x} \cdot \frac{L_y}{\Delta y} \cdot \frac{L_z}{\Delta z}}^{V} \cdot \overbrace{\frac{L_{p_x}}{\Delta p_x} \cdot \frac{L_{p_y}}{\Delta p_y} \cdot \frac{L_{p_z}}{\Delta p_z}}^{V_p}$$

$h \quad h \quad h$

$$\Omega_1 = \frac{V \cdot V_p}{h^3}$$

What about two particles?

$$\Omega_2 = \Omega_A \Omega_B = \underbrace{\frac{V^2}{(h^3)^2} \cdot V_p^{*}}_{\text{double counts}} \cdot \frac{1}{2}$$



what about N particles?

$$\Omega_N = \frac{1}{N!} \cdot \frac{V^N}{(h^3)^N} \cdot V_p^{*N}$$

go back to one particle
has energy U

$$U = \frac{1}{2} m v^2 = \frac{1}{2} m \left(v_x^2 + v_y^2 + v_z^2 \right)$$

$$U = \frac{1}{2m} \left(p_x^2 + p_y^2 + p_z^2 \right)$$

$$p_x^2 + p_y^2 + p_z^2 = \left(\sqrt{2mU} \right)^2$$

$$x^2 + y^2 + z^2 = r^2 \rightsquigarrow 4\pi r^2 \rightsquigarrow 4\pi \left(\sqrt{2mU}\right)^2$$

$$8\pi m U$$

<u>2 particles</u> ?

$$p_{1x}^2 + p_{1y}^2 + p_{1z}^2 + p_{2x}^2 + p_{2y}^2 + p_{2z}^2 = \left(\sqrt{2mU}\right)^2$$

6 dimensional sphere → hypersphere

surface area
for an <u>any</u>
dimensional sphere

3N dimensional hypersphere

1D area → 0

2D area → $2\pi r$

3D area → $4\pi r^2$

D area → $\dfrac{(2\pi)^{d/2}}{\left(\frac{d}{2}-1\right)!} r^{d-1}$

$$\Gamma(n-1) = n!$$

gamma function

$$\Omega_N = \frac{1}{N!}\frac{V^N}{h^{3N}} \cdot \frac{2\pi^{3N/2}}{\left(\frac{3N}{2}-1\right)!} (2mU)^{\frac{3N-1}{2}}$$

↓

$$\Omega_N = \frac{1}{N!}\frac{V^N}{h^{3N}} \cdot \frac{\pi^{3N/2}}{\left(\frac{3N}{2}\right)!} (2mU)^{\frac{3N}{2}}$$

$$\Omega_N = \boxed{f(N)} \cdot V^N \cdot U^{3N/2}$$

2 ideal gasses interacting

$$\Omega_{total} = \Omega_A \cdot \Omega_B = (f(N))^2 (V_A V_B)^N \cdot (U_A U_B)^{3N/2}$$

$$S = k_B \ln \Omega$$

entropy

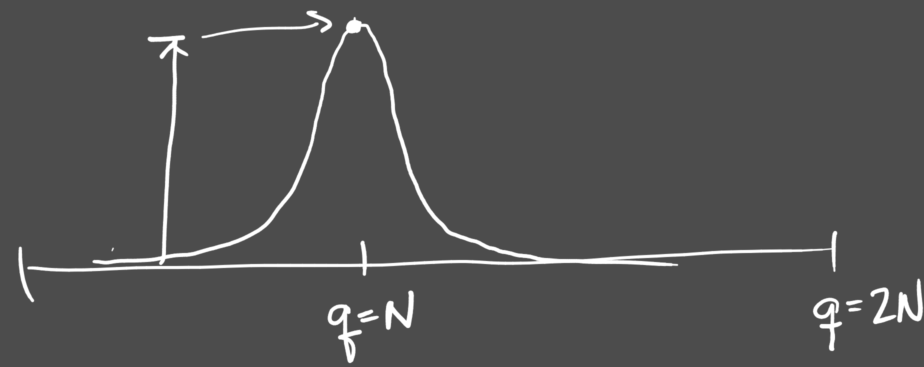
$$q = 2N$$

$$\Omega(N,q) \simeq \frac{\left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{N}\right)^N}{\sqrt{2\pi q(q+N)/N}}$$

$$\Omega_{total} = \Omega_A \Omega_B = (\Omega)^2 =$$

$$\Omega_{total,max} =$$

26, 28, 29, 30


q=N
q=2N