Chapter 2

Some combinatorics

coin flips → 5 coins

H H T T H ← microstate

T H H H H ← microstate

3 heads ← macrostates

4 heads ← macrostates

how many microstates are in a macrostate?

↳ Depends on the macrostate

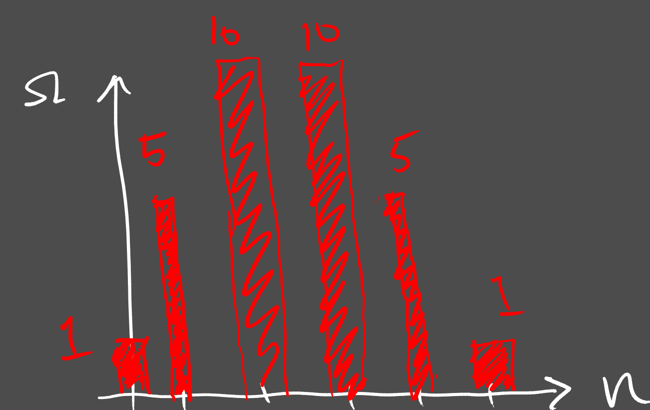
multiplicity $\Omega(n) = \dfrac{5!}{n!(5-n)!}$ ← combinations for 5 coins

macrostate # of heads

$0! = 1$ $n! = n(n-1) \cdots 1$

$1! = 1$

$2! = 2$



↳ generalize to N coins

$$\Omega(\underset{\#\text{ of coins}}{N}, \underset{\#\text{ of heads}}{n}) = \frac{N!}{n!(N-n)!} \qquad \binom{N}{n}$$

<u>10 atoms</u> → each atom can have 0 or 1 $\underbrace{\text{energy unit}}_{\text{quanta}}$

So how many possible arrangements of 4 quanta (10 quanta)

○ ● ● ○ ○ ● ○ ● ○ ○ ← microstate

4 energies vs. 10 energies ← macrostate

What if atoms can have more than 1 energy unit at a time

○	●	●	○	○	○	○	●	○	○	← microstate
	1	2					1			

$$\Omega(N, q) = \frac{(q+N-1)!}{q!\,(N-1)!} \qquad \binom{q+N-1}{q}$$

N: # of atoms

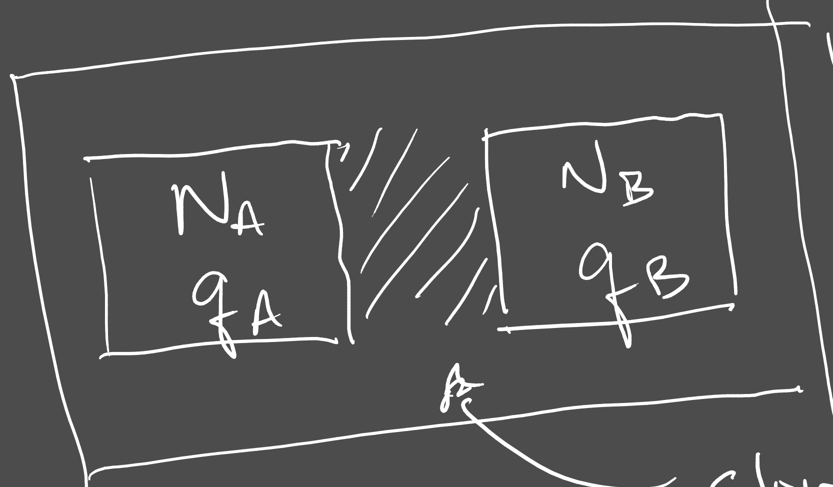
q: packets of energy

→ This model → <u>Einstein Solid</u>

1,5

<u>Two Solids</u> that can exchange energy

isolated from surroundings



slow energy transfer

A	B
1 2	← microstate
macrostate $q=3$	$q=3$
2 1	3
	3 2
$q=1$	$q=5$
$\Omega_A(1)=3$	$\Omega_B(5)=21$

$$\Omega_{total} = \Omega_A \cdot \Omega_B$$

$$\Omega_{total} = 63$$

Total number of microstates possible → 462

All microstates are equally likely

→ Fundamental assumption of statistical mechanics.
(axiom)

→ But all macrostates are not equally likely.

$$\text{Probability of solid A having } q \text{ energy} = \frac{\Omega_{total}(q_A)}{\sum_{q_A} \Omega_{total}(q_A)}$$

Large Numbers, 10^{23}

Addition of small numbers is not important

$$10^{23} + 50 \approx 10^{23}$$

Very Large Numbers $\rightarrow$ Multiplication of Large numbers is not important

$$10^{10^{23}}, \quad 10^{10^{100}}$$

$$10^{10^{23}} \cdot 10^{23} = 10^{10^{23}+23} \approx 10^{10^{23}}$$

Stirling's Approximation

$$N! \approx N^N e^{-N} \sqrt{2\pi N}$$

$$\approx \frac{N^N}{e^N} \underbrace{\sqrt{2\pi N}}_{\text{not all that important}}$$

$$\ln N! \approx N \ln N - N$$

$$\Omega(N,q) = \frac{(q+N-1)!}{q!\,(N-1)!} = \frac{(q+N)!}{q!\,N!}$$

$$\ln \Omega = \underbrace{\ln(q+N)!}_{(q+N)\ln(q+N) - (q+N)} - \ln q! - \ln N!$$

$$\ln \Omega = (q+N)\ln(q+N) - \cancel{(q+N)} - q\ln q + \cancel{q} - N\ln N + \cancel{N}$$

$$\ln \Omega = (q+N)\ln(q+N) - q\ln q - N\ln N$$

$$\boxed{\begin{array}{c}\ln(1+x) \approx x \\ \text{for small } x\end{array}}$$

high temperature $\rightarrow q >> N$

$$\ln(q+N)$$

$$\ln\left[q\left(1+\frac{N}{q}\right)\right] = \ln q + \underbrace{\ln\left(1+\frac{N}{q}\right)}_{N/q}$$

$$\ln \Omega = N \ln q + \frac{N^2}{q} + \cancel{q \ln q} + N - \cancel{q \ln q} - N \ln N$$

$$\ln \Omega = N \ln \frac{q}{N} + N + \underbrace{\frac{N^2}{q}}_{\text{small}}$$

$$\Omega(q \gg N) \approx e^{N \ln \frac{q}{N} + N} = \left(\frac{q}{N}\right)^N e^N$$

$$\Omega(q \gg N) = \left(\frac{eq}{N}\right)^N$$

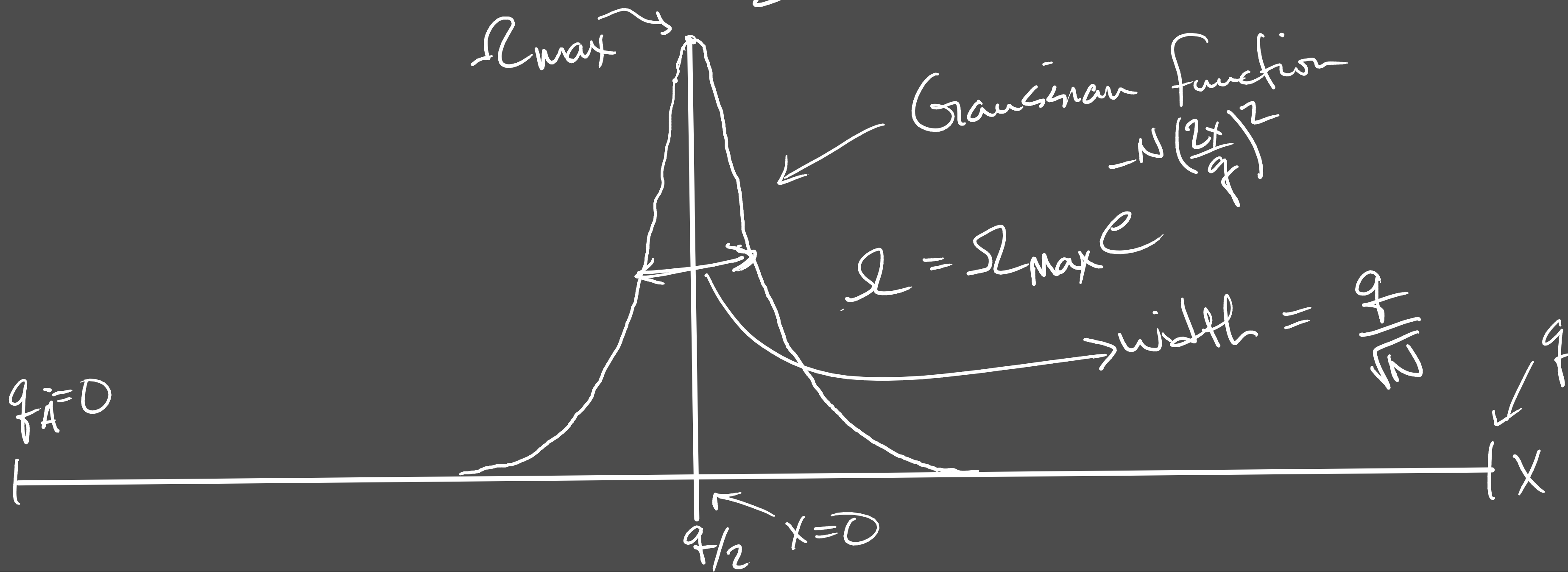
Now go back and use this w/ Einstein solid.

same # particles

$$\Omega_{total} = \left(\frac{e q_A}{N_A}\right)^{N_A} \left(\frac{e q_B}{N_B}\right)^{N_B} = \left(\frac{e}{N}\right)^{2N} (q_A q_B)^N$$

$$N_A = N_B$$

$$\Omega_{max} \rightarrow q_A = q_B = \frac{q}{2} \qquad \Omega_{max} = \left(\frac{e}{N}\right)^{2N} \left(\frac{q}{2}\right)^{2N}$$



Gaussian function

$$\Omega = \Omega_{max}\, e^{-N\left(\frac{2x}{q}\right)^2}$$

$$\text{width} = \frac{q}{\sqrt{N}}$$

$$\frac{\text{width}}{\text{full range}} = \frac{q/\sqrt{N}}{q} = \frac{1}{\sqrt{N}} \longrightarrow N = 10^{20} \qquad \frac{1}{\sqrt{10^{20}}} = \frac{1}{10^{10}}$$

HW: 9, 10, 13, 18, 22

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