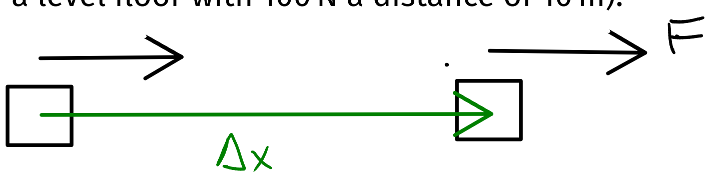
At the end of this worksheet you should be able to

- to discuss the relationships between the quantities of work, energy, displacement, velocity.
- differentiate between a conservative force and a non-conservative force.
- apply the work energy theorem to solve interesting problems that would be hard to use Newton's Laws.
- discuss the principle of conservation of energy and explain when it is useful.

1. *Work* is defined as a transfer of *energy*. This transfer occurs by one object exerting a force on another object *over some displacement*. But the relative directions of these two vector quantities (force and displacement) matters. Summarize the work done in 5 different cases that are represented below by drawing the object and the vectors representing force $\vec{F}$ and displacement $\Delta\vec{x}$. In each case I have provided a simple example to illustrate what I mean. You provide another one.

   - The force and displacement point in exactly the same direction. (I push a box across a level floor with 100 N a distance of 10 m).



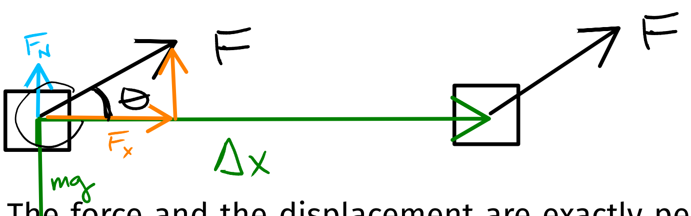
$W = F\cos\theta\,\Delta x$

$\theta = 0°$

$\cos 0° = 1$

$W = F\Delta x = 100\text{N}\cdot 10\text{m} = \underline{1000\text{ J}}$

   - The force and displacement point in different directions, but the angle between them is less than 90°. (I pull a box across a level floor with a string, directing 100 N at an angle of 20° with respect to the floor.)

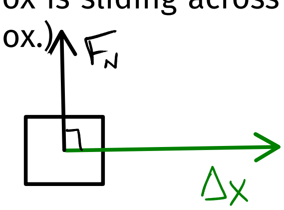


$W = F\cos\theta\,\Delta x$

$= 100\text{N}\cdot\cos 20°\cdot 10\text{m}$

$= 939\text{ J}$

   - The force and the displacement are exactly perpendicular to each other. (A 10 kg box is sliding across a friction-less surface, and the normal force is acting on the box.)

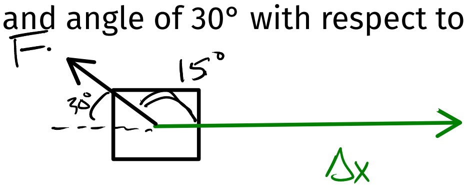


$W = F\cos\theta\,\Delta x$

$\theta = 90°$

$\cos 90 = 0$

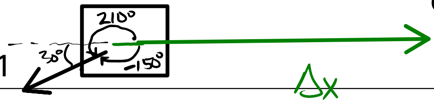
   - The force and displacement point in different directions and the angle between them is greater than 90°. (I bring a sliding box to a stop by exerting a 100 N force on it at and angle of 30° with respect to the horizontal.)



$W = F\cos\theta\,\Delta x$

$= 100\text{ N}\cos 150°\ 10\text{m}$

$= -866\text{ J}$

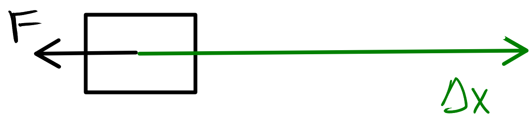


$W = 100\,N \cos 210 \cdot 10\,m = -866\,J$

$W = 100\,N \cos 150 \cdot 10\,m = -866\,J$

$W = 100\,N \cos(-150) \cdot 10\,m = -866\,J$

- The force and displacement point in exactly opposite directions. (I bring a sliding box to a stop by exerting a 100 N force over a distance of 10 m.)

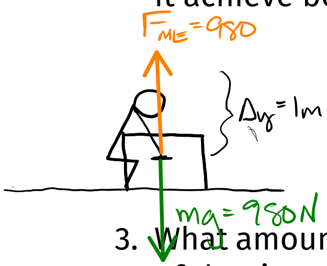


$W = |F| \cos\theta\, |\Delta x|$

$\theta = 180°$

$= 100\,N \cos(180°)\ 10\,m = -1000\,J$

2. What amount of work is done by a person to lift a 100 kg object a distance of 1 meter high? What amount of work is done by the force of gravity? If the person dropped the box what amount of work would the force of gravity do on the box as it fell? What velocity would it achieve before it hit the ground?



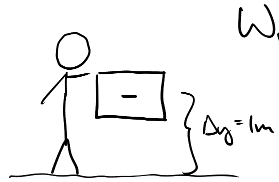
$W_{ME} = |F_{ME}| \cos 0 \cdot |\Delta y|$

$= 980 \cdot 1 \cdot 1m$

$= 980\,J$

$W_{F_g} = |980\,N| \cos(180)\, |1\,m|$

$= -980\,J$



$W_{F_g} = (980\,N) \cos 0\ (1\,m)$

$= +980\,J$

3. What amount of work is required to push a 100 kg object up a friction-less inclined plane of that is 10 m long that's end it 1 meter high? How does this compare to the work done to lift it? Show that this can be used to derive the formula $\frac{F_{push}}{weight} = \frac{height}{l_{plane}}$.

$\sin\theta = \frac{1m}{10m}$

$= 0.1$

$\theta = \sin^{-1}(0.1)$

$= 5.74°$

$mg \sin\theta = F_{ME} = mg \underbrace{\sin(5.74)}_{0.1} = 98\,N$

$W_{ME} = (98\,N) \cos 0\ (10\,m)$

$= 980\,J$

$weight \cdot height = 980\,J$

$F_{push} \cdot l_{plane} = 980\,J$

$weight \cdot height = F_{push} \cdot l_{plane}$

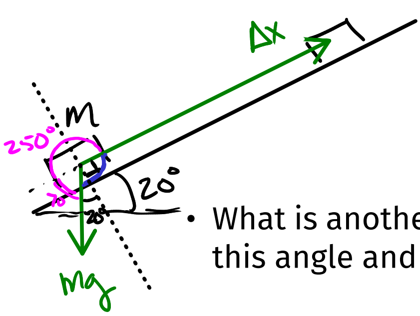
$\frac{F_{push}}{weight} = \frac{height}{l_{plane}}$

4. An object has some initial velocity at the bottom of a friction-less ramp and it begins to slide up the ramp. The force of gravity does negative work here and the object slows down to stop. The ramp has an incline angle of 20° with respect to the horizontal. Calculate the work done by the force of gravity and see that it is a negative value in three ways:

- What is the angle between the displacement and the force of gravity? Use this angle and the definition of work to calculate the work.

$W = F \cos(110) \Delta x$

$= -0.34\, mg\, \Delta x$

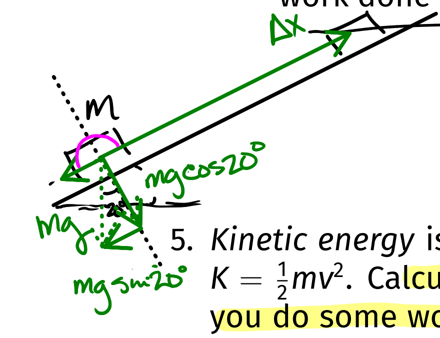


- What is another angle between the displacement and the force of gravity? Now use this angle and the definition of work to calculate the work.

$W = mg \cos(250°) \Delta x$

$= -0.34\, mg\, \Delta x$

- What is the *component* of the force of gravity that is in the direction of the displacement? Are these vectors in the same direction or opposite directions? What is the work done using component and the displacement?



$$W = F\cos\phi\,\Delta x$$

$mg\sin 20°$ $\qquad \phi = 180°$

$$W = mg\,\underbrace{\sin 20°}\,\underbrace{\cos 180°}\,\Delta x$$

$$W = (+0.34)(-1)\,mg\Delta x$$

$$W = -0.34\,mg\Delta x$$

5. *Kinetic energy* is the energy of an object that has velocity. In order to calculate it, use $K = \frac{1}{2}mv^2$. Calculate the kinetic energy of a 10 kg object that has a velocity of 10 m/s. If you do some work to *double* the velocity of the object, what is the new kinetic energy? What is the ratio of the kinetic energy final to the initial kinetic energy? What is the *change* in kinetic energy? How much *work* would be required to cause this change in kinetic energy? If this was done by a force pointed in the direction of the objects motion acting over a distance of 10 meters, what is the magnitude of the force?

$Nm = J$

$\frac{kg\,m}{s^2}\cdot m = J$

$$K_i = \tfrac{1}{2}mv^2$$
$$= \tfrac{1}{2}(10\,kg)(10\,m/s)^2$$
$$= 500\,\frac{kg\,m^2}{s^2} = 500\,J$$
$$K_f = \tfrac{1}{2}(10\,kg)(20\,m/s)^2$$
$$K_f = 2000\,J$$

$K \propto v^2$

$$\frac{K_f}{K_i} = \frac{2000}{500} = 4 = \left(\frac{v_f}{v_i}\right)^2 = (2)^2$$

$$\Delta K = K_f - K_i$$
$$\Delta K = +1500\,J$$
$$\Delta K = W_t$$
$$W = +1500\,J$$

limited to constant force

$$W = F\,\underbrace{\cos\theta}_{1,\ b/c\ \theta=0}\,\Delta x$$

$$W = F\cdot\Delta x$$

$$1500\,J = F\cdot 10\,m \Rightarrow \underline{\underline{F = 150\,N}}$$

6. I push an object at constant velocity of 1 m/s over a friction-full surface. I exert a force of 100 N and do this over a distance of 10 m. What work have I done? What work has the force of friction done? What is the net work done? What is the kinetic energy initially? Does the kinetic energy change? What power am I providing?

not enough info $K_i = ?$

$$W_{ME} = 100\,N\cdot 10\,m\cdot\underbrace{\cos 0°}_{1}$$
$$= 1000\,J$$

NO!

$$\boxed{P = \frac{W}{\Delta t} = \frac{\Delta E}{\Delta t}}$$

$$P_{ME} = \frac{W_{ME}}{\Delta t} = \frac{F_{ME}\cdot\Delta x}{\Delta t} = F_{ME}\cdot\frac{\Delta x}{\Delta t}$$

$$\boxed{P_{ME} = F_{ME}\cdot v}$$

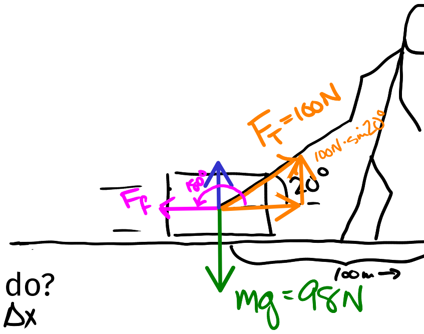
constant velocity

$$W_{F_f} = (100\,N)\cdot\underbrace{\cos(180)}_{-1}(10\,m)$$
$$= -1000\,J$$

$$W_t = W_{ME} + W_{F_f} = 1000\,J + (-1000\,J) = 0 = \Delta K$$

$1\,hp = 746$ Watts $\qquad P = 100\,\frac{J}{s} = \underline{100 \text{ Watt}}$

7. I pull a 10 kg object with a rope at an angle of 20° to a horizontal friction-full surface. I use a force of 100 N on the rope, and the coefficient of kinetic friction is $\mu = 0.1$. I start at rest and exert this force over a distance of 100 m.



	x	y
F_g	0	$-98N$
F_N	0	F_N
F_f	-6.38	0
F_T	$+93.96$	$+34.2N$
$\Sigma F:$	$+87.6$	0

$-98 + F_N + 34.2 = 0$

$\underline{F_N = 63.8\,N}$

$F_f = \mu F_N$

$F_f = 0.1 \cdot 63.8$

$= 6.38\,N$

- What work do I do?

$$W_{ME} = F_T \cos\theta\, \Delta x$$
$$= 100N \cos(20°) \cdot (100m)$$
$$= \underline{9396J}$$

- What work does the force of friction do?

$$W_{F_f} = \overset{F_f}{6.38N} \underbrace{\cos(180)}_{-1} \cdot \overset{\Delta x}{100m}$$
$$= -638\ J$$

- What work does the normal force do?

$$W_{F_N} = F_N \cos(90°) \Delta x$$
$$= 0$$

- What work does the force of gravity do?

$$W_{F_g} = F_g \cos(90°) \Delta x$$
$$= 0$$

- Using these works, what is the net work?

$$W_t = \Sigma W = 9396J + (-638) + 0 + 0 = \underline{\underline{+8758\ J}}$$

- What is the change in kinetic energy?

$$W = \Delta K = +8758$$

- What is the final velocity?

$$8758 = \Delta K = K_f - \cancelto{0}{K_i} \leftarrow \text{starts at rest } v_i = 0$$
$$8758 = \frac{1}{2} m v_f^2$$
$$v_f = \sqrt{\frac{2(8758)}{10}} = 41.8\ m/s$$

- What is the net force? $\Sigma F = 87.6\,N$, +x-direction, right, east

- Using the net force, what is the net work?

$$W_t = (87.6N)\cos(0)(100m)$$
$$= 8760\,N$$

U

U_g

U_s

U_e

8. What forces are conservative forces and what are not conservative forces?

sources of potential energy
- gravity
- springs
- electric

- friction
- applied force

9. When a 10 kg object is 10 m high, what is its potential energy? If it begins it to fall, what is its potential energy after it falls 1 m? How much has its kinetic energy changed?

$$U_g = mgh = mg\Delta y$$
$$U_{gi} = 10kg(9.8\,N/kg)(10m) = 980\,J$$

$$U_{gf} = 10kg(9.8\,N/kg)(9m)$$
$$= 882\,J$$
$$\Delta U_g = U_{gf} - U_{gi} = -98J$$

$$\Delta U + \Delta K = 0$$
$$\Delta U = -\Delta K$$
$$\Delta K = +98J$$

10. An object falls from a height of 100 m then how fast is it going when it hits the ground? Solve this using kinematics an then again using conservation of energy?

$m = 10kg$

Kinematics

$v_{yi} = 0$

$a_y = -9.8$

$\Delta y = -100m$

$v_f = ?$

$$v_{yf}^2 = \cancel{v_{yi}^2}^{\,0} + 2a_y\Delta y$$
$$v_{yf} = \pm\sqrt{2a\Delta y}$$
$$= \pm\sqrt{2(-9.8)(-100)}$$
$$= -44.3\,m/s$$

$$\Delta U + \Delta K = 0$$
$$\Delta U = \cancel{U_f} - U_i$$
$$= -9800J$$
$$\Delta K = +9800J$$
$$\Delta K = K_f - \cancel{K_i}^{\,0}$$
$$+9800J = \frac{1}{2}mv_f^2$$
$$v_f = \sqrt{\frac{2(9800J)}{10kg}} = 44.3\,m/s$$

$$\Delta U + \Delta K = W_{NC}$$
$$U_f - U_i + K_f - K_i = W_{NC}$$
$$K_i + U_i + W_{NC} = K_f + U_f$$

$K_i + U_i + W_{NC} = K_f + U_f$

$W_{NC} = 0$

11. A roller coaster starts from rest at the top of a hill and rolls down its course. Find its kinetic and potential energy at each position marked.

unknown

$m = 100\,kg$

$K_i + U_i$

$0J + \underbrace{(100kg)(9.8\frac{N}{kg})(100)}_{98{,}000J}$

$K_i + U_i = K_f + U_f$

$0 + 98{,}000J = 98000J + 0J = K_f + U_f$

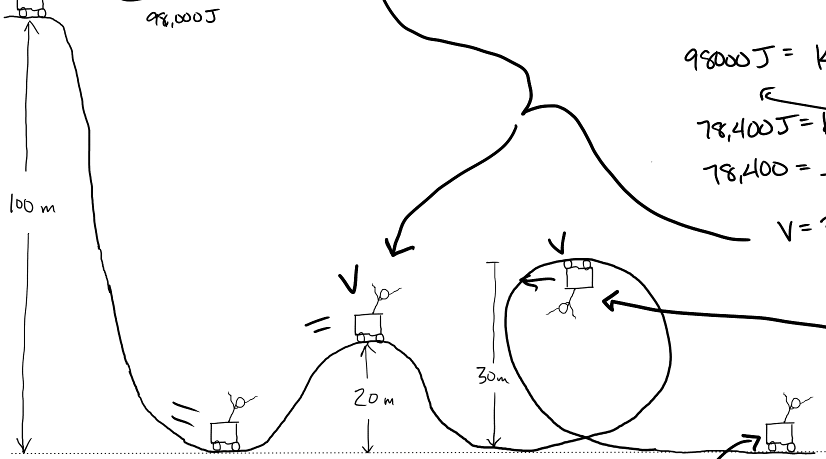
mgh_f

$98000J = K_f + \underbrace{100kg(9.8)(20m)}_{19{,}600J}$

$78{,}400J = K_f$

$78{,}400 = \frac{1}{2}(100kg)v^2$

$v = 39.6\,m/s$



$K_i + U_i = K_f + U_f$

$98{,}000J = \frac{1}{2}mv^2 + mgh$

$98{,}000J = \frac{1}{2}(100)v^2 + (100)(9.8)(30m)$

$v = 37.0\,m/s$

$V = 44.3$

$K_f + U_f$

$\frac{1}{2}mv_f^2 + 0$

$\rightarrow K_i + U_i + W_{NC} = K_f + U_f$

$0J + 98{,}000J + 0 = \frac{1}{2}mv_f^2 + 0$

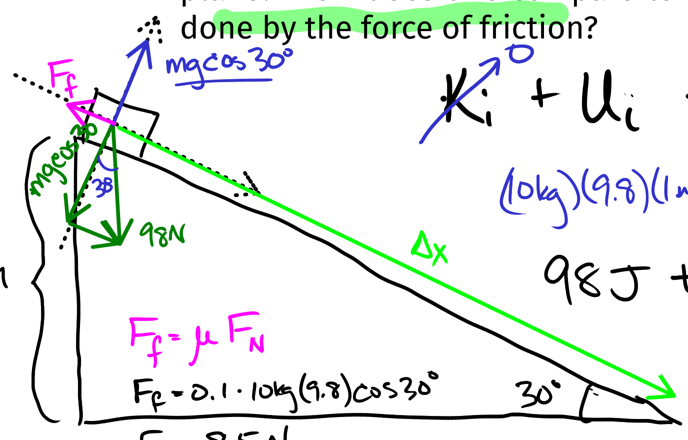
$98000J = \frac{1}{2}(100kg)v_f^2$

$v_f = 44.3\,m/s$

m 2

12. A 10 kg box slides down a friction-full inclined plane ($\mu = 0.1, \theta = 30°$). The height of the plane is 1 m above the horizontal. What is the speed of the box at the bottom of the plane? How does this compare to if there were no friction? How much work has been done by the force of friction?

$V_i = 0\,m/s$



$\cancel{K_i}^{\,0} + U_i + W_{NC} = K_f + \cancel{U_f}^{\,0}$

$(10kg)(9.8)(1m)$ $\quad \frac{1}{2}mv^2$

$98J + (-17J) = \frac{1}{2}10kg \cdot v^2$

$\sqrt{\frac{81(2)}{10}} = v = 4.02\,m/s$

$F_f = \mu F_N$

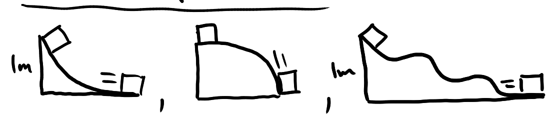
$F_f = 0.1 \cdot 10kg(9.8)\cos 30°$

$F_f = 8.5N$

No friction, down the ramp: $V = 4.4\,m/s$

$U_i = 98J = K_f = \frac{1}{2}mv^2$

No ramp, dropped: $V = 4.4\,m/s$



$\sin 30° = \frac{1m}{\Delta x}$

$\Delta x = \frac{1m}{\sin 30°}$

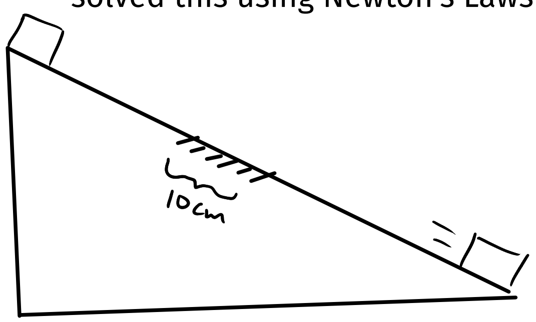
$\Delta x = 2m$

$W_{F_f} = F_f \cos(180)\Delta x$

$= (8.5N)(-1)(2m)$

$= -17J$

13. Here is an example of a problem that would be much more difficult to do with Newton's Laws. Take the same inclined plane as the problem above, but make *most* of the plane friction-less and only a 10 cm portion in the middle of the plane have friction $\mu = 0.1$. Now what is the speed at the bottom of the plane? Think about how you would have solved this using Newton's Laws and kinematics.



$\cancel{K_i}^{0} + U_i + W_{NC} = K_f + \cancel{U_f}^{0}$

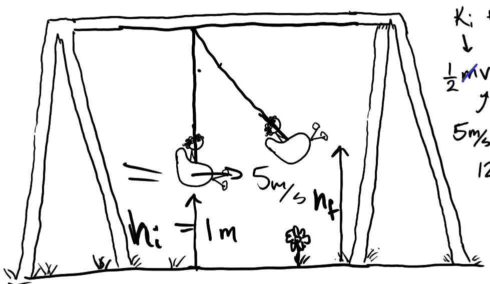
$(10kg)(9.8)(1m)$ $\frac{1}{2}mv^2$

$W_f = -0.84J = 8.5N$

$98J + (-0.84J) = \frac{1}{2} 10kg \cdot v^2$

$v = 4.41\, m/s$

14. The maximum speed of a child on a swing is 5 m/s. At this point the child is 1 m above the ground. What is the maximum height of the child above the ground? Do this two ways, once with $U_g = 0$ at the ground, and once again with $U_g = 0$ at the bottom of the swing's path.



$K_i + U_i + \cancel{W_{NC}}^{0} = \cancel{K_f}^{0} + U_f$

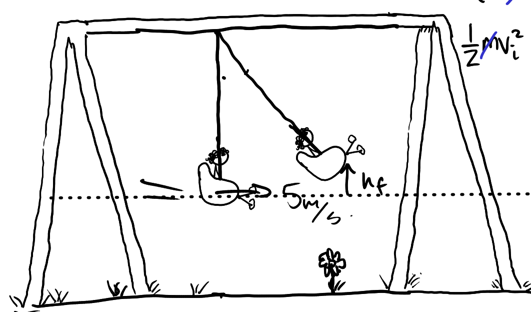
$\frac{1}{2}\cancel{m}v_i^2 + \cancel{m}gh_i = \cancel{m}gh_f$

5m/s 1m

$12.5\, J/kg + 9.8\, J/kg = 9.8 h_f$

$h_f = 2.2m$

$U_g = 0J \leftarrow h = 0m$



$K_i + \cancel{U_i}^{0} = \cancel{K_f}^{0} + U_f$

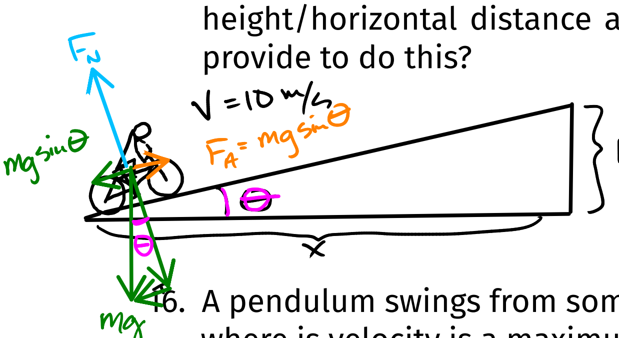
$\frac{1}{2}\cancel{m}v_i^2 + 0 = 0 + \cancel{m}gh_f$

$h_f = 1.2\, m$

$U_g = 0 \leftarrow h = 0$

total height = 2.2m

15. A 100 kg cyclist (and bike) has at constant speed of 10 m/s up an incline of 5% (vertical height/horizontal distance as a percent). What power output does the rider need to provide to do this?



$5\% \rightarrow 0.05 = \frac{h}{x}$

$\tan\theta = \frac{h}{x} = 0.05$

$\theta = \tan^{-1}(0.05) = 2.89°$

$F_{ME} = mg\sin\theta$

$= 100kg(9.8)\sin(2.89°)$

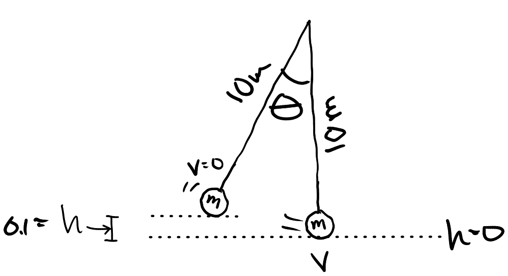
$= 48.9N$

$P = \frac{W}{\Delta t} = F_{ME} \cdot v \cdot \underbrace{\cos 0}_{1}$ (10 m/s)

$P = 48.9 \cdot 10\, m/s$

$P = 489W$

16. A pendulum swings from some maximum height where its velocity is zero to a minimum where is velocity is a maximum and then back up to a maximum height. If the maximum height is 0.1 meters above the minimum height, then what is the speed of the pendulum at the bottom of its swing?



$\cancel{K_i}^{0} + U_i + \cancel{W_{NC}}^{0} = K_f + \cancel{U_f}^{0}$

mgh_i $\frac{1}{2}mv_f^2$

$\cancel{m}gh_i = \frac{1}{2}\cancel{m}v_f^2$

$v_f = \sqrt{2gh_i} = \sqrt{2 \cdot 9.8(0.1)} = 1.4\, m/s$

If L = 10m, what is max θ

$h = L(1-\cos\theta)$

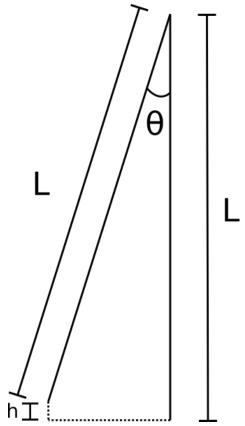
$0.1m = 10m(1-\cos\theta)$

$0.01 = 1-\cos\theta$

$\cos\theta = 0.99$

$\theta = \cos^{-1}(0.99) = 8.2°$

17. For a pendulum, it is hard to measure its maximum height, but it is easy to measure its length and to measure its angle from the vertical. If the maximum angle from the vertical of a 1 m long pendulum is 20° then how high is this above the horizontal? (*Hint: draw a line from the end of the pendulum when it is at its maximum height perpendicularly to the line when it is at is minimum height.*)



$h = L - L\cos\theta$

$h = L\cdot(1-\cos\theta)$

$h = 1m(1-\cos 20°)$

$h = 0.06m$ or $6cm$

18. A horse pulls a 250 kg cart a distance of 1.5 km. The frictional force on the cart is a constant 250 N. The horse eats oats in the morning to prepare for this trip. Each gram of oats provides 10 kJ of energy, but only 10% of this energy can go into work pulling the cart. How many grams of oats must the horse eat?

translate to math

translate to math

$W_{horse} = F_{horse}\cdot\cos(0°)\cdot\Delta x$

$W_{horse} = 250N\cdot(1)\cdot 1500m$

$W_{horse} = 375{,}000J$

10% of $E_{total} = W_{horse}$

$0.10\cdot E_{total} = 375{,}000J$

$E_{total} = 3{,}750{,}000J$

$\frac{10kJ}{gram}\cdot M_{oats} = E_{total}$

$10000\frac{J}{g}\cdot m_{oats} = 3{,}750{,}000J$

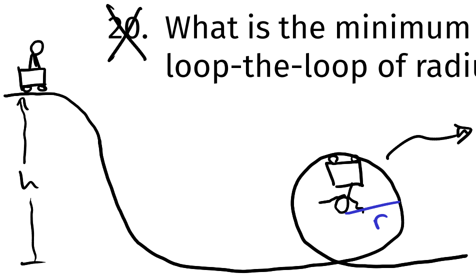
$\boxed{M_{oats} = 375g}$

19. I want to try a weight loss program that involves repeatedly lifting a 50 kg barbell from the ground to over my head at a height of 2 m. I can do this about 5 times per minute. How long will it take me to burn 0.5 kg of fat? "Burning" fat means I have used it to supply energy to do work. Each gram of fat has roughly 39 kJ of energy to the body, but the muscles can only use 10% of this to do work.

$t = 3979$ minutes

$= 2.76$ days → not a good plan

20. What is the minimum height *h* that a roller coaster needs to start at rest in order to do a loop-the-loop of radius *r* and not lose contact with the track?



min speed to stay on the track?

$mg = \frac{mv^2}{r} \Rightarrow v = \sqrt{gr}$

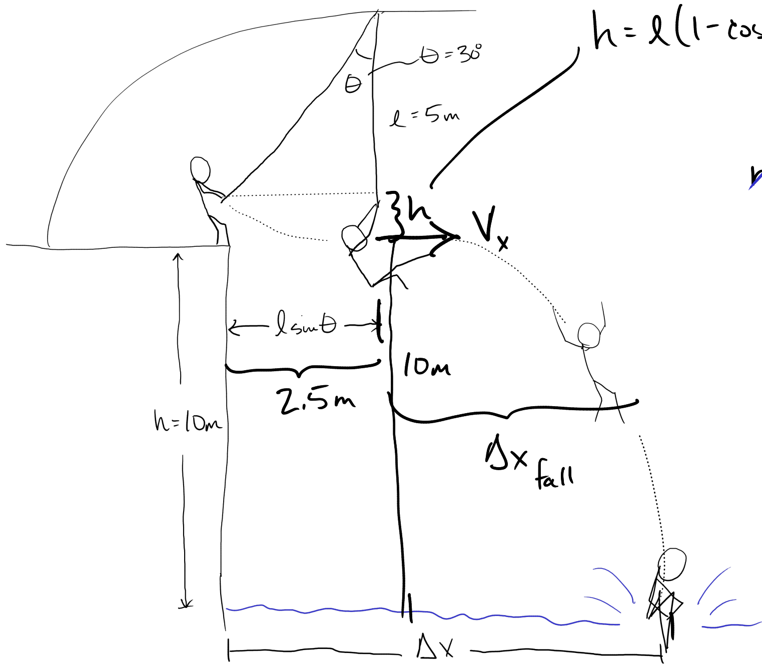
$K_i + U_i = K_f + U_f$

$0 + \cancel{m}gh = \frac{1}{2}\cancel{m}v^2 + \cancel{m}g(2r)$

$\cancel{g}h = \frac{1}{2}(\cancel{g}r) + 2\cancel{g}r$

$h = 2.5r$

21. If you are doing a rope swing and then at the lowest point in the swing you let go and drop into a lake below, how far from the edge of the cliff do you land in the water. See the diagram below for the relevant parameters.



$h = l(1-\cos\theta) = 5m(1-\cos 30^\circ)$

$h = 0.67m$

$\cancel{m}gh = \frac{1}{2}\cancel{m}v^2$

$v = 3.62 \, m/s$ ← all in the horizontal direction

$$\Delta y = -10m = \underbrace{0 \, m/s}_{v_{yi}} \cdot t + \frac{1}{2}\underbrace{(-9.8 \, m/s^2)}_{a_y} \cdot t^2$$

$t = 1.42s$ ← in the air after you let go

$\Delta x_{fall} = V_x \cdot t$

$= 3.62 \, m/s \, (1.42)$

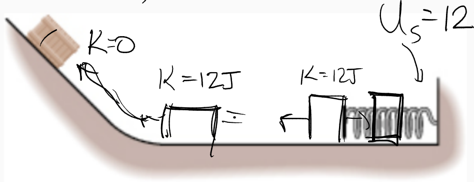
$= 5.14m$

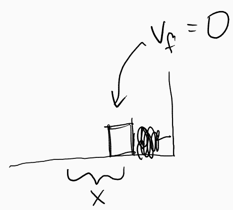
$$\Delta x = \underbrace{2.5m}_{\text{while swinging}} + \underbrace{5.14m}_{\text{while falling}}$$

$\underline{\Delta x = 7.64m}$

A 2.50-kg block is released from rest and allowed to slide down a frictionless surface and into a spring. The far end of the spring is attached to a wall, as shown. The initial height of the block is 0.500 m above the lowest part of the slide and the spring constant is 458 N/m.

$2.5(9.8)(0.5) = 12J$





$$\cancel{K_f}^0 + U_f = \cancel{K_i}^0 + U_i + \cancel{W_{nc}}^0$$

$$\cancel{K_f}^0 + \cancel{U_{fg}}^0 + U_{fs} = \cancel{K_i}^0 + U_{ig} + \cancel{U_{is}}^0 + \cancel{W_{nc}}^0$$

$$\frac{1}{2}k\Delta x^2 = mgh$$

$$\Delta x = \sqrt{\frac{2mgh}{k}}$$

$$\cancel{K_f}^0 + U_{fg} + \cancel{U_{fs}}^0 = \cancel{K_i}^0 + \cancel{U_{ig}}^0 + U_{is} + \cancel{W_{nc}}^0$$

$$mgh = \frac{1}{2}k\Delta x^2$$

$$h = 0.5$$

A 750-kg automobile is moving at 17.6 m/s at a height of 5.00 m above the bottom of a hill when it runs out of gasoline. The car coasts down the hill and then continues coasting up the other side until it comes to rest. Ignoring frictional forces and air resistance, what is the value of *h*, the highest position the car reaches above the bottom of the hill?

V=0

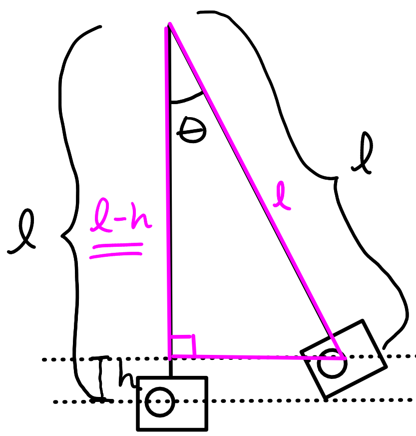
$W_{nc} = 0$

$$\cancel{K_f}^0 + U_f = K_i + U_i + \cancel{W_{nc}}^0$$

where **X** = 17.6.

20.8 ± 2%

m

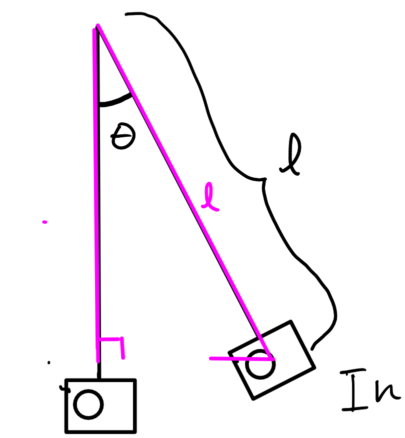


$$\cos\theta = \frac{l-h}{l}$$

$$l\cos\theta = l - h$$

$$h = l - l\cos\theta$$

$$\boxed{h = l(1-\cos\theta)}$$



$$E_i + \cancel{W_{NC}}^{\,0} = E_f$$

$$K_i = U_g$$

$$\frac{1}{2}\cancel{m}v^2 = \cancel{m}gh$$

$$V = \sqrt{2gh}$$

immediately after collision

Conservation of momentum

$$m_{cb} \cdot V_{cb} = (m_{cb} + m_{pend}) \cdot V$$

$$\boxed{V_{cb} = \left(\frac{m_{cb} + m_{pend}}{m_{cb}}\right) \cdot \sqrt{2gh}}$$