Hooke's Law

Force → $F_s = k\Delta x$

At the end of this worksheet you should be able to

- calculate stress and strain and use Young's Modulus to solve for an unknown.
- plot all relevant quantities of simple harmonic motion over time.
- use the quantities of simple harmonic motion and the mathematical description to solve for an unknown.

k: spring constant "stiffness" $\left[\frac{N}{m}\right]$; Δx: distortion

$10^3 \to$ kilo
$10^6 \to$ Mega
$10^9 \to$ Giga
$10^{12} \to$ Tera

$d = 0.0001 \to r = 0.00005\,m$

$A = \pi(0.00005\,m)^2 = 7.85\cdot10^{-9}\,m^2$

1. Consider a wire that is 0.1 mm in diameter and 2 m long and a Young's Modulus of 120 GPa (1 GPa $=$ 10^9 Pa). If you applied 100 N to this wire, then what is the stress on the wire? What is the strain? By how much does its length change? What is its new length? What percent change is this?

Hooke's Law for materials

stress ∝ strain

$$\boxed{\frac{F}{A} = Y\cdot\frac{\Delta L}{L}}$$

Y: young's modulus

$$\text{stress} = \frac{F}{A} = \frac{100N}{7.85\cdot10^{-9}m^2} = 1.27\cdot10^{10}\,Pa$$
$12.7\cdot10^9\,Pa$
$12.7\,GPa$

$$\text{strain} = \frac{\Delta L}{L}$$
fractional length change

$$\frac{\Delta L}{L} = \frac{F/A}{Y} = \frac{12.7\cdot10^9 Pa}{120\cdot10^9 Pa} = 0.1$$

$\Delta L = 0.1\cdot 2m = 0.2m$

$L_2 = L_1 + \Delta L = 2.2m$

$\frac{(2.2-2.0)}{2.0}\times100$

$\left(\frac{2.2}{2}-1\right)\times100$

0.1×100

$\frac{\Delta L}{L}\times100$

10% increase

2. If instead a ~~200~~ 1000N force ~~or~~ a 1000 N force is applied then what is stress, strain, length change and percent change in the length from its original length?

$$\frac{F}{A} = \frac{1000N}{7.85\cdot10^{-9}m^2} = 127\,GPa$$

$$\frac{\Delta L}{L} = \frac{F/A}{Y} = \frac{127\,GPa}{120\,GPa}$$

$\frac{\Delta L}{L} = 1.06$

$\Delta L = 1.06\cdot(2m)$

$\to \Delta L = 2.12$

$\%\Delta L = 106\%$

3. If instead the wire was 1 m long with 100 N applied, then how much does it stretch? What percent change is this?

ΔL

$F = 100N$

$$\frac{F}{A} = 12.7\cdot10^9\,Pa$$

$$\frac{\Delta L}{L} = \frac{F/A}{Y} = 0.1$$

$\Delta L = 0.1(1m) = 0.1m$

4. What if the wire had half of its cross sectional diameter with 100 N applied?

$$\text{stress} = \frac{100\text{N}}{1.96\times10^{-9}\text{m}^2} = 50.1\text{GPa}$$

$$A \propto d^2$$

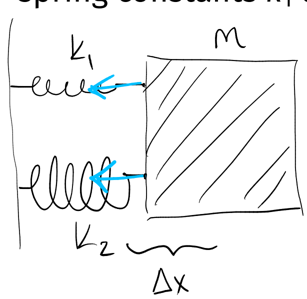
$$\frac{A_2}{A} = \left(\frac{d_2}{d}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} \qquad A_2 = \frac{1}{4}A_1 = \frac{1}{4}\, 7.85\cdot10^{-9}\text{m}^2$$

$$\frac{\Delta L}{L} = \frac{50.1\text{GPa}}{120\text{GPa}} = 0.425$$

$$\Delta L = 0.425\cdot 2\text{m} = 0.85\text{ m}$$

5. Now comparing the form of Hooke's Law for springs $F = kx$ to Hooke's Law for stress and strain $\frac{F}{A} = Y\frac{\Delta l}{L}$, how could you write an expression for spring constant in terms of Young's modulus, length, and cross sectional area? What does this tell you about what would happen to the spring constant of a spring if you cut the spring in half?

6. Speaking of springs, if you attach two springs to an object side by side, then we say the springs are attached *in parallel*. This will result in two spring forces on the object that has been displaced some distance x_1. If you were to model this arrangement of springs in parallel as a single spring with a single spring constant that would have the same effect, then what would this single effective spring constant k_e be in terms of of the original two spring constants k_1 and k_2?



7. If instead of a *parallel* arrangement, we attach one spring to another spring and then to the object, we say that these springs are connected *in series*. Let's find an expressions for an effective spring constant for this arrangement. Each spring will stretch a different

~~amount based on its spring constant, but the object will experience one force and *both of the springs is exerting the same force.*~~

8. ~~A wire of length l_1 and volume V and cross sectional area A_1 is stretched out to length l_2, what is its new cross sectional area?~~ Think about this in terms of proportionality.

9. A 60 kg person upright. By how much does the femur shorten if each femur carries half the weight of the person? The cross sectional area of a femur is about 4 cm^2 and the length is about 30 cm Also find the percent change in length.

$\hookrightarrow 4 \cdot 10^{-4}\, m^2 = A$

$$F_{femur} = \frac{60kg \cdot 9.8 N/kg}{2} = 294 N$$

$$\frac{\Delta L}{L} = \frac{F/A}{Y} = \frac{F}{A \cdot Y} = \frac{294N}{4 \cdot 10^{-4} m^2 \cdot 9.4 \cdot 10^9 Pa} = 7.8 \cdot 10^{-5}$$

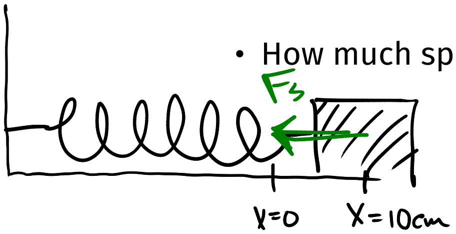
$\underline{\times 100\%}$

$7.8 \cdot 10^{-3}\% = 0.0078\%$

$10^{-3} \rightarrow$ milli m
$10^{-6} \rightarrow$ micro μ
$10^{-9} \rightarrow$ nano n
$10^{-12} \rightarrow$ pico p
$10^{-15} \rightarrow$

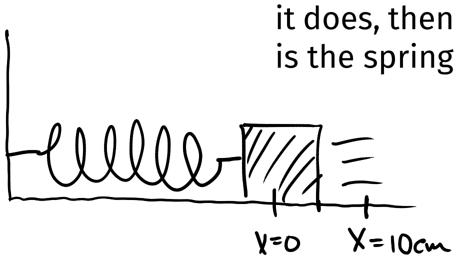
$\Delta L = 7.8 \cdot 10^{5} \cdot (0.3\,m) = 2.3 \cdot 10^{-5}\,m$
$23 \cdot 10^{-6}\,m$
$23\,\mu m$ "micron"

10. A 1 kg mass attached to a spring of spring constant 1000 N/m is positioned so that the spring is stretched 10 cm from its relaxed length. After you finish the following questions, make sure you can write down expressions for all of them in general as well as working them inside out.



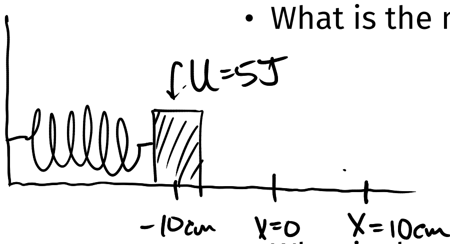
- How much spring potential energy does it have at this position?

$$U_s = \frac{1}{2} k x^2 = \frac{1}{2}\left(1000 \frac{N}{m}\right)(0.10\,m)^2 = 5\,J$$



- When it is released, and it heads back toward equilibrium gaining kinetic energy as it does, then what is the maximum kinetic energy it can achieve? How stretched out is the spring when it has this much kinetic energy?

$$K_{max} := 5\,J$$



- What is the mass's velocity when it has maximum kinetic energy?

$$K = \frac{1}{2} m v^2$$
$$5\,J = \frac{1}{2}(1\,kg) v^2 \rightarrow v = 3.16\,m/s$$

- What is the period of this mass's motion? What is its frequency? What is its angular frequency?

period [s] $\rightarrow$ $T = \frac{1}{f} \Leftrightarrow f = \frac{1}{T}$ frequency $[s^{-1}] \rightarrow [\text{Hertz}] = [\text{Hz}]$

angular frequency $\rightarrow \omega = \frac{2\pi}{T}\,\text{rad} = 2\pi f$

$$\omega = \sqrt{\frac{k}{m}}$$
$$\omega = \sqrt{\frac{1000\,N/m}{1\,kg}} = 31.6\,\frac{\text{rad}}{s}$$
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{31.6} = 0.199\,s$$
$$f = \frac{1}{T} = 5\,Hz$$

- What is the force on the mass when it is at its maximum displacement?

$$F = kx = 1000 \frac{N}{m} \cdot (0.1\,m) = 100\,N$$

- What is the force on the mass when it is at the equilibrium position?

$$F_{eq} = 0$$

$\omega = \sqrt{\frac{k}{m}} \rightarrow \omega^2 = \frac{k}{m}$

- What is the maximum acceleration of the mass? Where does this occur?

$k\,x_{max} = m\,a_{max}$ — $k\,x_{max} = F_{max} = m\,a_{max}$

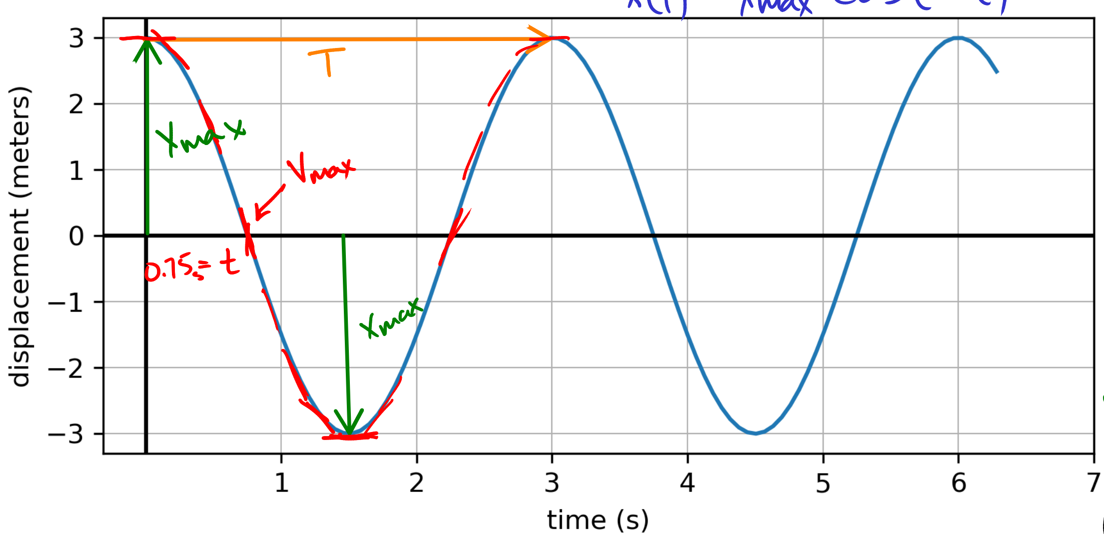
$\frac{k}{m} x_{max} = a_{max}$ $\qquad \frac{100\,N}{1\,kg} = a = 100\,m/s^2$

$\omega^2 \cdot x_{max} = a_{max}$

- Where is the mass when its potential energy and kinetic energy at that position are equal?

11. A 1 kg mass on a spring has the displacement graph that follows. What is the angular frequency, natural frequency, period, spring constant, amplitude, maximum velocity, maximum acceleration, maximum kinetic energy, maximum potential energy, and total energy? Also fill out the rest of the graphs.

$x(t) = x_{max} \cdot \cos(\omega t)$



$T = 3\,s$

$f = \frac{1}{3\,s} = 0.33\,Hz$

$\omega = 2\pi f = 2.08\,\frac{rad}{s} = \frac{2\pi}{T}$

$\omega^2 = \frac{k}{m}$

$k = \omega^2 \cdot m = 4.39\,\frac{N}{m}$

$x_{max} = A = 3\,m$

$U_{max} = \frac{1}{2} k (x_{max})^2 = 19.7\,J$

$K_{max} = \frac{1}{2} m V_{max}^2 \leftarrow K_{max} = 19.7\,J$

$a_{max} = \omega^2 x_{max} = 13.1\,m/s^2$

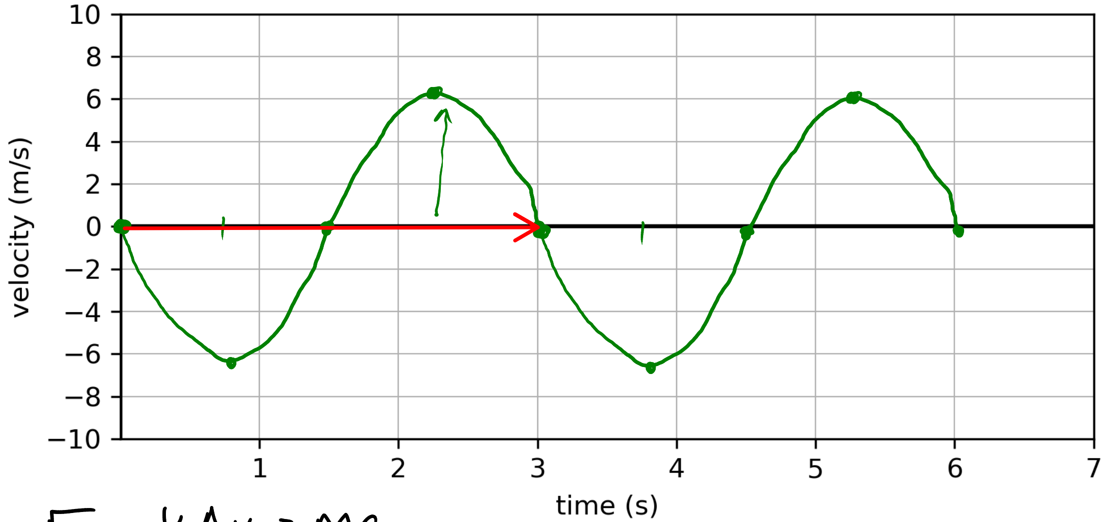
$V_{max} = \sqrt{\frac{2K_{max}}{m}}$

—OR—

$V_{max} = \omega \cdot x_{max} = \underline{6.24\,m/s}$

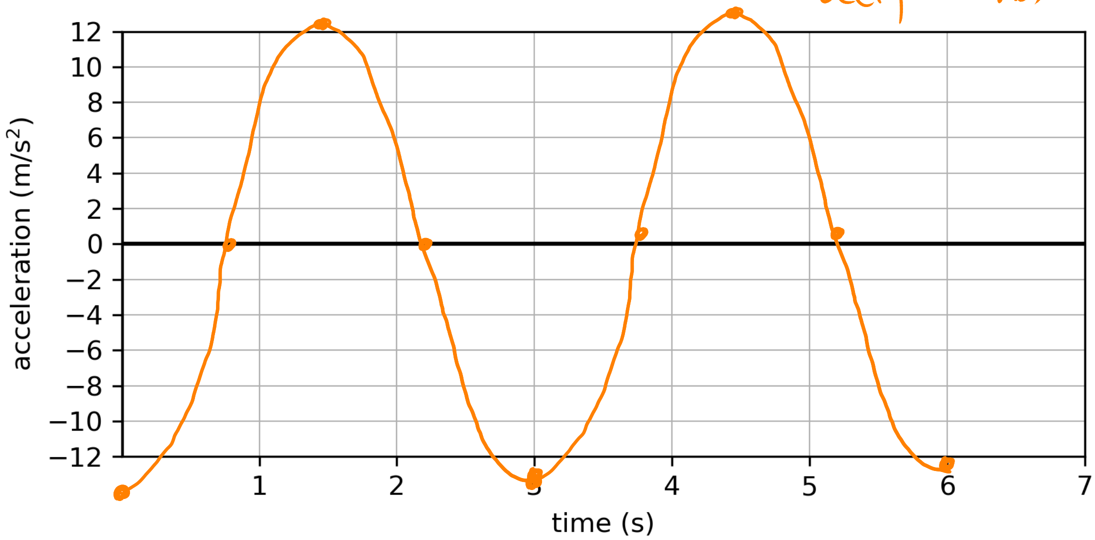
$E_{total} = 19.7\,J$

$v(t) = -v_{max}\sin(\omega t)$



$F = -k\Delta x = ma$

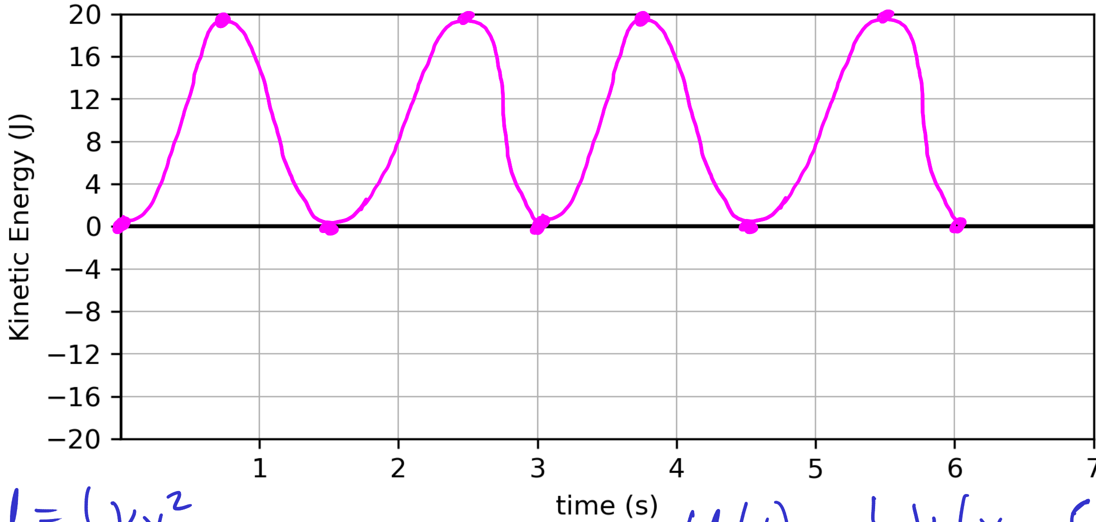
$a(t) = -a_{max}\cos(\omega t)$



$K = \frac{1}{2} m v^2$

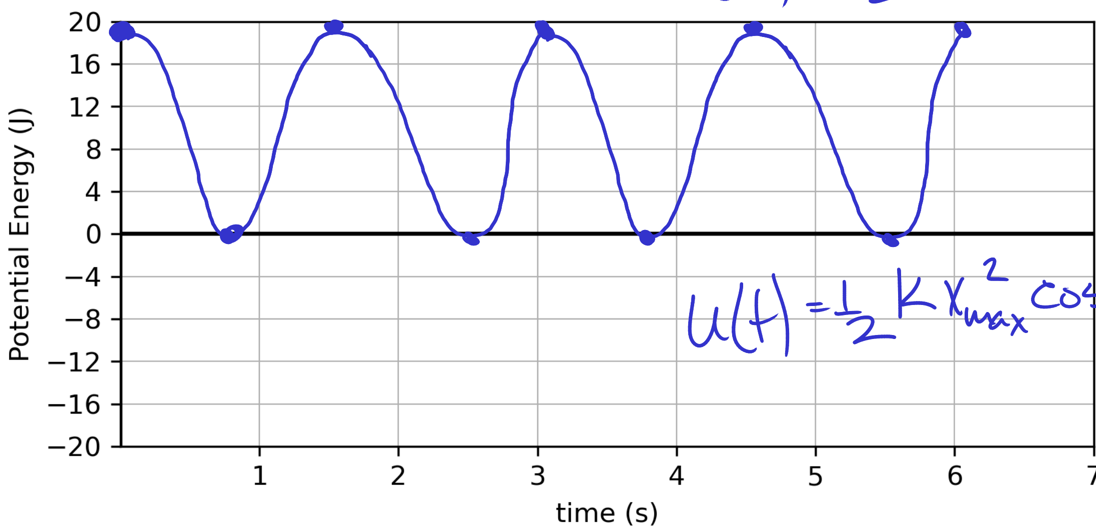
$K(t) = \frac{1}{2} m \left(-v_{max} \sin(\omega t)\right)^2$

$K(t) = \frac{1}{2} m v_{max}^2 \sin^2(\omega t)$



$U = \frac{1}{2} k x^2$

$U(t) = \frac{1}{2} k \left(x_{max} \cos(\omega t)\right)^2$

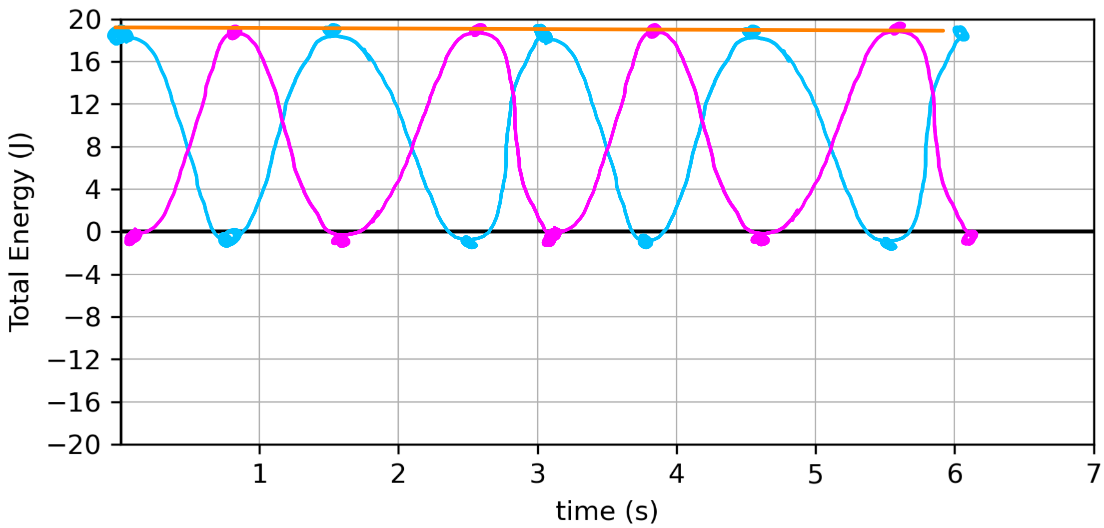


$U(t) = \frac{1}{2} k x_{max}^2 \cos^2(\omega t)$

$$E = K + U = \underbrace{\frac{1}{2} m v_{max}^2}_{19.7\,J} \sin^2(\omega t) + \underbrace{\frac{1}{2} k x_{max}^2}_{19.7\,J} \cos^2(\omega t)$$

$$= 19.7\,J \underbrace{\left(\sin^2(\omega t) + \cos^2(\omega t)\right)}_{1}$$

$$E = 19.7\,J$$



12. According to the table below, which material stretches more, 2 m of steel or 1 m of copper of the same width?

$Y_{Fe-C} = 200\,GPa$

$Y_{Cu} = 120\,GPa$

$$\underbrace{\frac{F}{A}}_{\text{stress}} = Y\frac{\Delta L}{L} \qquad Y_{Fe-C}\cdot\frac{\Delta L_{FeC}}{L_{FeC}} = \frac{F}{A} = Y_{Cu}\cdot\frac{\Delta L_{Cu}}{L_{Cu}}$$

$$Y_{Fe-C}\cdot\frac{\Delta L_{FeC}}{L_{FeC}} = Y_{Cu}\cdot\frac{\Delta L_{Cu}}{L_{Cu}}$$

13. Four brass wires are subjected to the same tensile force ~~stress~~. The wires have the following unstretched lengths and widths. Rank them in order from least to most change in length.

(a) length L, diameter d

(b) length $2L$, diameter d

(c) length $4L$, diameter $d/2$

$$\Delta L_{FeC} = \frac{L_{FeC}\cdot Y_{Cu}\,\Delta L_{Cu}}{Y_{FeC}\,L_{Cu}}$$

$$= 2\cdot\frac{12\cancel{0}\,\cancel{GPa}}{20\cancel{0}\,\cancel{GPa}}\cdot\Delta L_{Cu} \quad .8$$

(d) length $L/4$, diameter $d/2$

$a = d, b, c \longrightarrow$

$\text{stress} = Y \dfrac{\Delta L}{L}$

$\Delta L = L Y^{-1} \text{stress}$

$\Delta L \propto L Y^{-1}$

$\dfrac{\Delta L_{Cu}}{\Delta L_{st}} = \left(\dfrac{L_{Cu}}{L_{st}}\right) \cdot \left(\dfrac{Y_{Cu}}{Y_{st}}\right)^{-1} = \left(\dfrac{1}{2}\right)\left(\dfrac{3}{5}\right)^{-1} = \dfrac{5}{6} = 0.83$

14. A 0.5 m long guitar string of cross sectional area of 1.0×10^{-6} m^2 and Young's modulus $Y = 2.0$ GPa. By how much must you stretch the string to obtain a tension of 20 N.

$\Delta L = ?$

$\dfrac{F}{A} = Y \dfrac{\Delta L}{L}$

$\dfrac{20N}{1 \cdot 10^{-6} m^2} = \dfrac{2 \cdot 10^9 Pa}{0.5} \Delta L$

$\Delta L = 0.005$ m

5 mm

$\dfrac{\Delta L_{Cu}}{\Delta L_{st}} = 0.83$

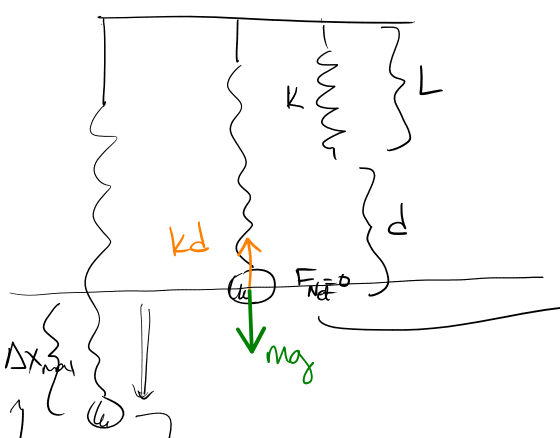
$\underline{\Delta L_{Cu} = 0.83 \cdot \Delta L_{st}}$

15. Young's modulus is like the spring constant for materials, but if you stretch a material beyond a limit then the material will not exactly return to its original length. This point of stress is called the *elastic limit* of a material. After this point, *plastic deformation* begins to occur which means the material will be permanently deformed. More stress can be applied beyond this but the material will fracture and break when the stress reaches the *breaking point*. A hair breaks under a tension of 1.2 N and the tensile stress of the breaking point is 200 MPa. What is the diameter of the hair?

16. A copper wire of length 3.0 m is observed to stretch by 2.1 mm when a weight of 120 N is hung from the end. What is the diameter of the wire and what is the stress in the wire? If the breaking point of copper is 400 MPa, what is the maximum weight that may be hung from this wire?

Table 10.1 Approximate Values of Young's Modulus for Various Substances

Substance	Young's Modulus (GPa)
Rubber	0.002–0.008
Human cartilage	0.024
Human vertebra	0.088 (compression); 0.17 (tension)
Collagen, in bone	0.6
Human tendon	0.6
Wood, across the grain	1
Nylon	2–6
Spider silk	4
Human femur	9.4 (compression); 16 (tension)
Wood, along the grain	10–15
Brick	14–20
Concrete	20–30 (compression)
Marble	50–60
Aluminum	70
Cast iron	100–120
Copper	120
Wrought iron	190
Steel	200
Diamond	1200



amplitude

$$kd - mg = 0$$
$$kd = mg$$

$$k = \frac{mg}{x}$$
$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$F_{NET} = k(d+\Delta x) - mg = ma$$
$$kd + k\Delta x - mg = ma$$
$$\cancel{mg} + k\Delta x - \cancel{mg} = ma$$
$$k\Delta x = ma$$
$$\underline{a = \frac{k}{m}\Delta x}$$

$$2\pi f = \omega = \sqrt{\frac{k}{m}}$$
$$f = \frac{1}{T}$$

still apply to vertical springs

$$k = \frac{mg}{x}$$

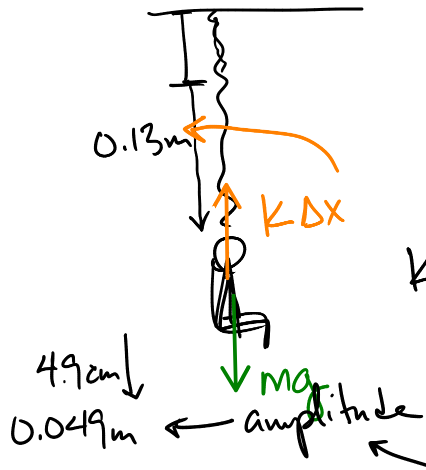
$$V_{max} = \omega\, x_{max}$$

HW#10

$T, \ v_{max}$

$v_{max} = \omega X_{max}$

$\omega = \sqrt{\frac{k}{m}} \rightsquigarrow T$



$k\Delta x = mg$

$k = \frac{mg}{\Delta x}$