

## Chapter 2 - air resistance

$$\Sigma \vec{F} = \vec{F}_{\text{NET}} = m \vec{\ddot{r}}$$

↑

$$\vec{F}_D = -f(v) \hat{v}$$

$$f(v) = a + bv + cv^2$$

↑  
0

↑  
linear  
drag

↑  
quadratic  
drag

$$f_{\text{lin}} = bv$$

$$b \propto D$$

↙ related to viscosity

$$b = \beta D$$

$$f_{\text{quad}} = cv^2$$

$$c \propto D^2$$

↙ density

$$c = \gamma D^2$$

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \propto \frac{D^2 v^2}{Dv} = Dv$$

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \approx \text{Reynolds number} = \frac{\rho D v}{\eta}$$

$\rho$  density  
 $\eta$  viscosity

$$\vec{F}_{\text{net}} = m \ddot{\vec{r}}$$

$$m\vec{g} - b\vec{v} = m\ddot{\vec{r}}$$

$$m\vec{g} - b\vec{v} = m\dot{\vec{v}}$$

really two equations

$$-bv_x = m\dot{v}_x$$

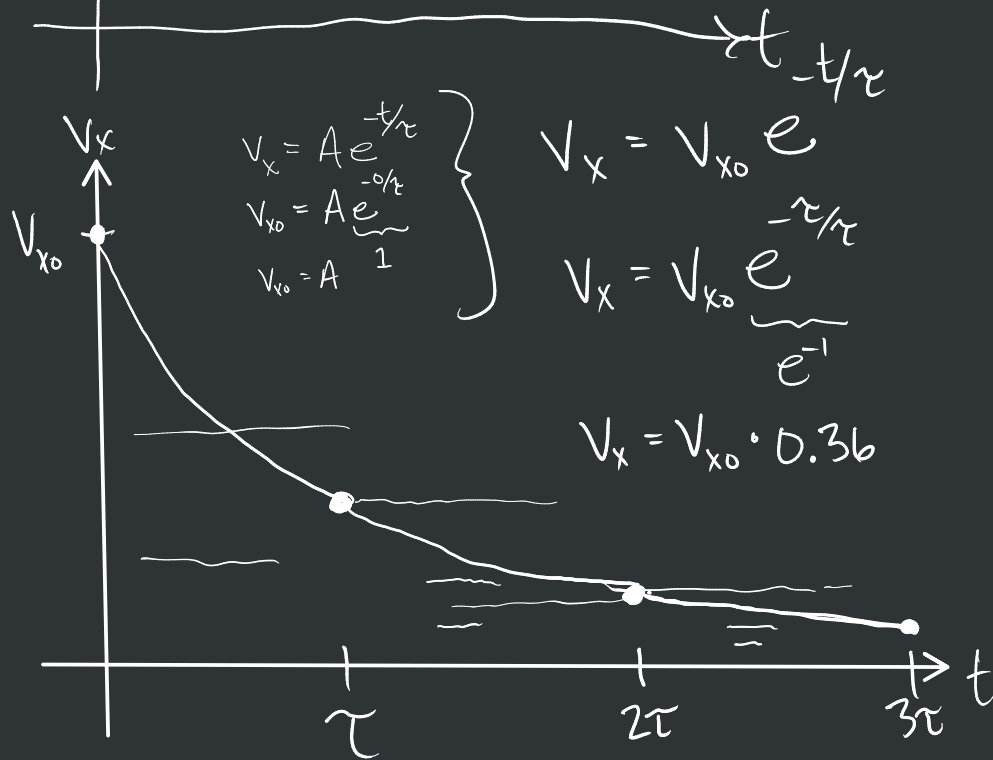
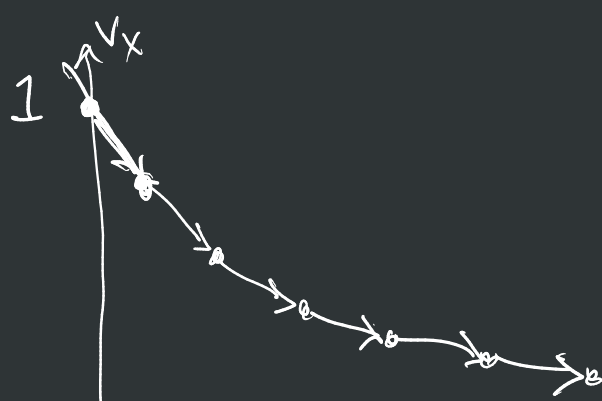
$$\dot{v}_x = -\frac{b}{m}v_x$$

$$\boxed{\frac{dv_x}{dt} = -\frac{b}{m}v_x}$$

$$\frac{dv_x}{v_x} = -\frac{b}{m}dt$$

$$mg - bv_y = m\dot{v}_y$$

$\uparrow$  down  $\oplus$  y-dir



$$\int \frac{dv_x}{v_x} = - \underbrace{\frac{b}{m}} dt$$

$$e^{\ln v_x} = e^{\left(-\frac{b}{m}t + C\right)} = e^{-\frac{b}{m}t} \cdot e^C$$

$$v_x = A \cdot e^{-\frac{b}{m}t}$$

$$\frac{b}{m} = \left[\frac{1}{s}\right] = [s^{-1}]$$

$$\tau = \frac{m}{b} = [s]$$

$$v_x = A e^{-t/\tau} \rightarrow \text{time constant}$$

exponential decay

$-t/\tau$

$\dot{x} = v_{x0} e^{-t/\tau}$

$$\frac{dx}{dt} = v_{x0} e^{-t/\tau}$$

$$\int dx = v_{x0} \int e^{-t/\tau} dt$$

$$\int_{x_0}^x dx' = v_{x0} \underbrace{\int_0^t e^{-t'/\tau} dt'}_{}$$

$$x' \Big|_{x_0}^x \overset{\text{function}}{=} v_{x0} \left( -\tau \right) e^{-t'/\tau} \Big|_0^t$$

$$(x - x_0) = -v_{x0} \tau \left( e^{-t/\tau} - \cancel{e^{-0/\tau}} \right)$$

$$x - x_0 = v_{x0} \tau \left( 1 - e^{-t/\tau} \right)$$

$$x(t) = v_{x0} \tau \left( 1 - e^{-t/\tau} \right) + x_0$$

# Linear Drag Vertically

$$mg - bv_y = m\dot{v}_y$$

if  $\dot{v}_y = 0$

then  $\underline{mg - bv_y = 0}$

$$v_y = \frac{mg}{b} \equiv v_t$$

↑ velocity when  $\dot{v} = 0$

$$m\dot{v}_y = mg - bv_y$$

$$= -b\left(-\frac{mg}{b} + v_y\right)$$

$$m\dot{v}_y = -b(-v_t + v_y)$$

$$u = (-v_t + v_y)$$

$$\dot{u} = \frac{du}{dt} = 0 + \dot{v}_y$$

$$\dot{u} = \dot{v}_y$$

$$m\dot{u} = -b \cdot u$$

$$\dot{u} = -\frac{b}{m} \cdot u \quad \leftarrow$$

$$u = A e^{-t/\tau}$$

$$\tau = \frac{m}{b}$$

$$-v_t + v_y = A e^{-t/\tau}$$

$$V_y = A e^{-t/\tau} + V_t$$

$$V_y(t=0) = V_{y0}$$

$$V_{y0} = A \underbrace{e^{-0/\tau}}_1 + V_t$$

$$V_{y0} = A + V_t$$

$$A = V_{y0} - V_t$$

$$V_y = (V_{y0} - V_t) e^{-t/\tau} + V_t$$

$$V_y = \underbrace{V_{y0} e^{-t/\tau}}_{\substack{t \rightarrow \infty \\ \rightarrow 0}} + \underbrace{V_t (1 - e^{-t/\tau})}_{t \rightarrow \infty}$$

$$V_{y0} = (V_{y0} - V_t) e^{-t/\tau} + V_t$$

$$\cancel{dt} \frac{dy}{\cancel{dt}} = \left( (v_{y_0} - v_t) e^{-t/\tau} + v_t \right) dt$$

$$\int_{y_0}^y dy = \int_0^t \left( (v_{y_0} - v_t) e^{-t'/\tau} + v_t \right) dt'$$

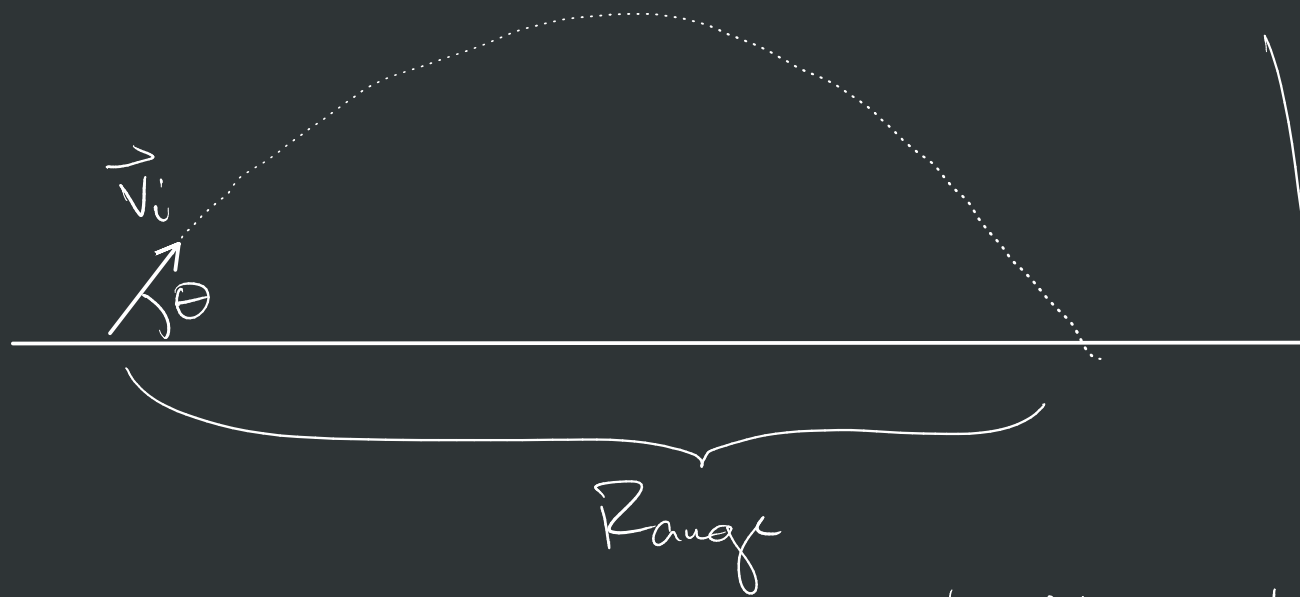
$$y \Big|_{y_0}^y = \left[ (v_{y_0} - v_t) (-\tau) e^{-t'/\tau} + v_t t' \right]_0^t$$

$$y - y_0 = -(v_{y_0} - v_t) \tau e^{-t/\tau} + v_t t - \left[ -(v_{y_0} - v_t) \tau \cdot 1 + 0 \right]$$

algebra

$$y = (v_{y_0} - v_t) \tau (1 - e^{-t/\tau}) + v_t t + y_0$$





## Range of Projectile Motion in vacuum

$$v_x = v_i \cos \theta$$

$$v_{y0} = v_i \sin \theta$$

$$x = v_x \cdot t = v_i \cos \theta \cdot t$$

$$y = y_c + v_{y0} \cdot t + \frac{1}{2} a_y t^2$$

Range  $\rightarrow$  solve  $y=0$  for  $t$   
plug  $t$  in for  $x$

$$0 = v_{y0} t + \frac{1}{2} a_y t^2$$

$$0 = t \left( v_{y0} + \frac{1}{2} a_y t \right)$$

$$t=0 \quad \text{or} \quad t = -\frac{2v_{y0}}{a_y}$$

$$\text{Range} = v_x \left( \frac{-2v_{y0}}{a_y} \right)$$

$$a_y = -g$$

$$R = \frac{2v_x v_{y0}}{g}$$

# Range for Projectile Motion in Linear Drag

$$x = v_{x0} \tau \left( 1 - e^{-t/\tau} \right) + \cancel{x_0} \quad \Bigg| \quad y = (v_{y0} - v_t) \tau \left( 1 - e^{-t/\tau} \right) + v_t t + \cancel{y_0}$$

$$x = v_{x0} \tau \left( 1 - e^{-t/\tau} \right)$$

Solve for  $t$

$$y = (v_{y0} - v_t) \tau \left( 1 - e^{-t/\tau} \right) + v_t t$$

$$= (v_{y0} - v_t) \tau \left( 1 - e^{-t/\tau} \right) + v_t t$$

lots of algebra (to eliminate  $t$ )

$$y = \left( \frac{v_{y0} + v_t}{v_{x0}} \right) x + v_t \tau \ln \left( 1 - \frac{x}{v_{x0} \tau} \right)$$

$$0 = \left( \frac{V_{yo} + V_t}{V_{xo}} \right) R + V_t \tau \ln \left( 1 - \frac{R}{V_{xo} \tau} \right)$$

$$- \left( \frac{V_{yo} + V_t}{V_{xo}} \right) \frac{R}{V_t \tau} = \ln \left( 1 - \frac{R}{V_{xo} \tau} \right)$$

$$e^{- \left( \frac{V_{yo} + V_t}{V_{xo}} \right) \frac{R}{V_t \tau}} = 1 - \frac{R}{V_{xo} \tau}$$

Taylor series approx

$$\leftarrow \ln(1-\epsilon) \hat{=} - \left( \epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 + \dots \right)$$

quadratic equation

$$\boxed{\frac{R}{V_{xo} \tau} = 1 - e^{- \left( \frac{V_{yo} + V_t}{V_{xo}} \right) \frac{R}{V_t \tau}}}$$

$$\begin{cases} \cancel{x = e^x} \\ \rightarrow x = e^{-x} \\ \rightarrow x = (1 - e^{-x}) \end{cases}$$

# Quadratic Drag

## Horizontal

$$F_{\text{net}} = m\dot{v}$$

$$-c v^2 = m\dot{v}$$

$$\vec{v} = v_x \hat{x}$$

$$m \frac{dv}{dt} = -c v^2$$

$$\frac{dv}{v^2} = -\frac{c}{m} dt$$

$$\int \frac{dv}{v^2} = -\frac{c}{m} \int dt$$

$$\int v^{-2} dv = -v^{-1} = -\frac{c}{m} t + D$$

$$x = e^{-x}$$

$$x = 0.567$$

$$x = (1 - e^{-x})$$

$$x = 0$$

$$\left. \begin{aligned} m \frac{dv_x}{dt} &= -c(v_x^2 + v_y^2) \\ m \frac{dv_y}{dt} &= -c(v_x^2 + v_y^2) \end{aligned} \right\} \begin{array}{l} \text{coupled} \\ \text{system} \\ \text{of} \\ \text{DE} \end{array}$$

$$\frac{1}{V} = \frac{C}{m} t + D$$

$$V(t) = \frac{1}{\frac{C}{m} t + D}$$

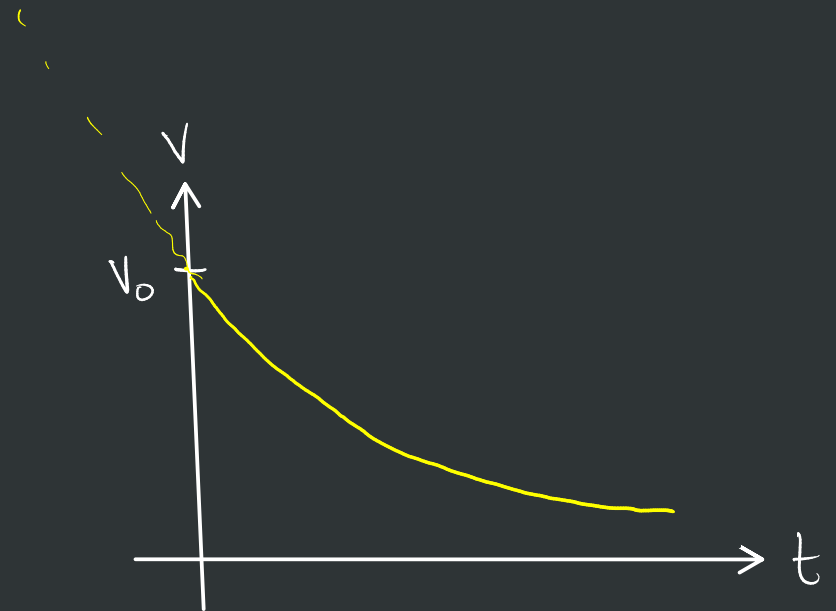
$$V_0 = V(0) = \frac{1}{D} \Rightarrow D = \frac{1}{V_0}$$

$$V(t) = \frac{1}{\frac{C}{m} t + \frac{1}{V_0}}$$

$$V(t) = \frac{V_0}{\frac{C V_0}{m} t + 1}$$

$$\tau = \frac{m}{C V_0}$$

$$V(t) = \frac{V_0}{t/\tau + 1}$$



$$\frac{dx}{dt} = \frac{V_0}{t/\tau + 1}$$

$$\int dx = \int \frac{V_0}{t/\tau + 1} dt$$

$$x = V_0 \int \frac{1}{t/\tau + 1} dt$$

$$x = V_0 \tau \underbrace{\int \frac{1}{u} \cdot du}_{\ln(u)}$$

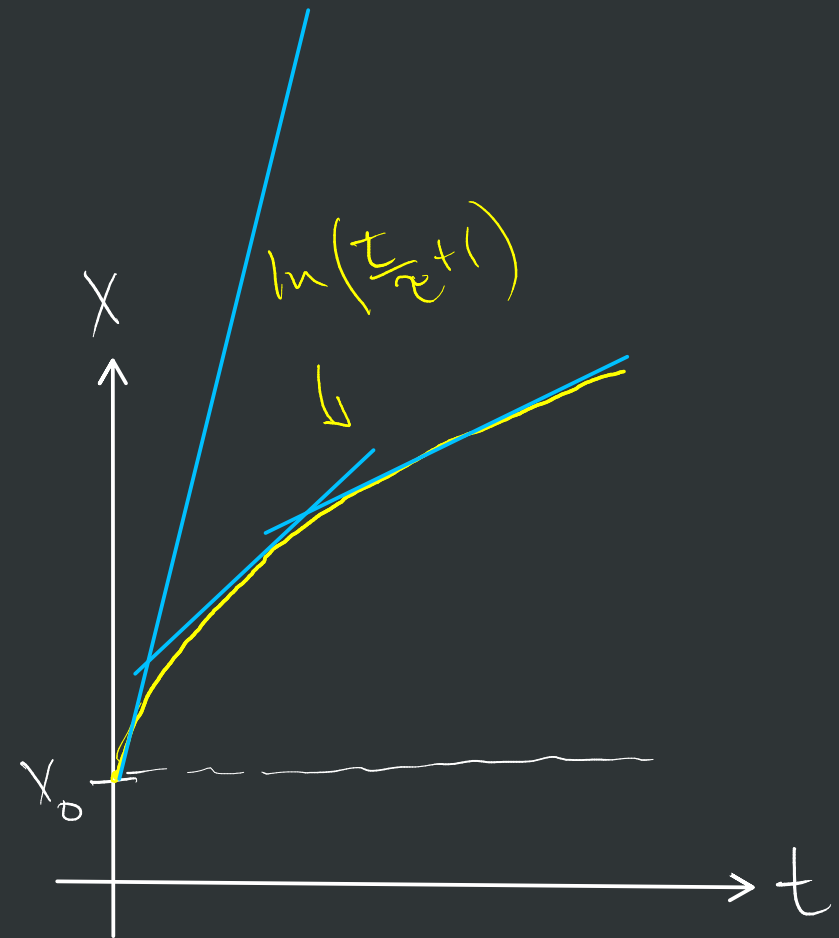
$$\left\{ \begin{array}{l} u = \frac{t}{\tau} + 1 \\ du = \frac{du}{dt} dt \\ du = \frac{1}{\tau} dt \\ \tau du = dt \end{array} \right.$$

$$x(t) = V_0 \tau \ln\left(\frac{t}{\tau} + 1\right) + E$$

$$x_0 = x(0) = V_0 \tau \underbrace{\ln(1)}_0 + E$$

$$x_0 = E$$

$$x(t) = V_0 \tau \ln\left(\frac{t}{\tau} + 1\right) + x_0$$



## Vertical Direction

$$m\dot{v} = mg - cv^2$$

$$\sqrt{\frac{1.98}{.0025}} = \underline{\underline{62.6}}$$

$$\dot{v} = 0 = mg - cv_t^2$$

$$v_t = \sqrt{\frac{mg}{c}}$$

← terminal velocity

$$\dot{v} = g - \frac{c}{m} \frac{g}{g} v^2$$

$$\dot{v} = g \left( 1 - \frac{c}{mg} v^2 \right)$$

$$\dot{v} = g \left( 1 - \frac{v^2}{v_t^2} \right)$$

$$\boxed{\frac{dv}{dt} = g \left( 1 - \frac{v^2}{v_t^2} \right)}$$

← solve for v

$$\int \frac{dv}{\left( 1 - \frac{v^2}{v_t^2} \right)} = \int g dt$$

$$\int \frac{dv}{(1 - v^2/v_t^2)} = \int g dt$$

$$= g \cdot t + C$$

$$\tanh(u) = \frac{v}{v_t}$$

$$\sech^2(u) du = \frac{1}{v_t} dv$$

$$\sech^2(u) + \tanh^2(u) = 1$$

$$\sech^2(u) = 1 - \tanh^2 u$$

$$\int \frac{v_t \sech^2(u) du}{1 - \tanh^2(u)}$$

$$\int \frac{v_t \cancel{\sech^2(u)} du}{\cancel{\sech^2(u)}}$$

$$v_t \int du = v_t \cdot u = g t + C$$

$$\sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

$$\cosh \theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

$$\tanh \theta = \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\tan \theta = \underline{\hspace{2cm}}$$

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{i\theta^3}{3!} + \dots$$

$\downarrow$   
 $i = \sqrt{-1}$   
 $i^2 = \sqrt{-1}^2 = -1$   
 $i^3 = -i$   
 $i^4 = 1$

$$e^{i\theta} = 1 + \underline{i\theta} - \frac{\theta^2}{2!} - \underline{\frac{i\theta^3}{3!}} + \frac{\theta^4}{4!} + \underline{i\frac{\theta^5}{5!}} - \underline{\frac{\theta^6}{6!}} + \dots$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$



$$\tanh(u) = \frac{v}{v_t}$$

$$u = \tanh^{-1}\left(\frac{v}{v_t}\right)$$

$$v_t \cdot \tanh^{-1}\left(\frac{v}{v_t}\right) = gt + C$$

$$\tanh^{-1}\left(\frac{v}{v_t}\right) = \frac{gt}{v_t} + C$$

$$\frac{v}{v_t} = \tanh\left(\frac{gt}{v_t} + C\right)$$

$$v = v_t \cdot \tanh\left(\frac{gt}{v_t} + C\right)$$

$$v(t=0) = v_0$$

$$v_0 = v_t \tanh(C)$$

$$\tanh^{-1}\left(\frac{v_0}{v_t}\right) = C$$

$$e^{i\theta} = \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots}_{\cos\theta} + i \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}_{\sin\theta}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\cosh^2\theta - \sinh^2\theta = 1$$

$$\operatorname{sech}^2\theta + \tanh^2\theta = 1$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\frac{d}{d\theta}(\sinh\theta) = \cosh\theta$$

$$\frac{d}{d\theta}\tanh\theta = \operatorname{sech}^2\theta$$

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta$$

$$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

$$\frac{d}{d\theta}(\tan\theta) = \sec^2\theta = \frac{1}{\cos^2\theta}$$

$$V(t) = V_t \cdot \tanh \left( \frac{gt}{V_t} + \tanh^{-1} \left( \frac{V_0}{V_t} \right) \right)$$

$$\frac{dy}{dt} = V_t \cdot \tanh \left( \frac{gt}{V_t} + \tanh^{-1} \left( \frac{V_0}{V_t} \right) \right)$$

$$dy = V_t \cdot \tanh \left( \frac{gt}{V_t} + \tanh^{-1} \left( \frac{V_0}{V_t} \right) \right) \cdot dt$$

$$\int dy = V_t \int \tanh \left( \frac{gt}{V_t} + \tanh^{-1} \left( \frac{V_0}{V_t} \right) \right) \cdot dt$$

$$u \Rightarrow du = \frac{g}{V_t} dt$$

$$dt = \frac{V_t}{g} du$$

$$y = V_t \int \frac{V_t}{g} \tanh(u) du$$

$$h_f = \frac{V_t^2}{g} \ln(\cosh(u)) + C$$

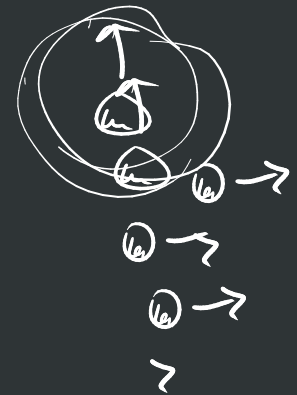
$$\rightarrow h_f = \frac{V_t^2}{g} \ln\left(\cosh\left(\frac{gt}{V_t} + \tanh^{-1}\left(\frac{V_0}{V_t}\right)\right)\right) + C$$

## Project Topic & Timeline:

- Abstract ✓
- Progress Report ✓
- Rough Draft ✓
- Final Project ✓

$v=0$

(u)



$$m \dot{\vec{v}} = -c v^2 \hat{v} + m g \hat{y}$$

$$= -c v \cdot \underbrace{v \hat{v}}_{\vec{v}}$$

$$\vec{v} = v_x \hat{x} + v_y \hat{y}$$

$$v = \sqrt{v_x^2 + v_y^2}$$

$$m \dot{\vec{v}} = -c v \vec{v} + m g \hat{y}$$

$$m \dot{\vec{v}} = -c \sqrt{v_x^2 + v_y^2} \cdot \vec{v} + m g \hat{y}$$

$$\dot{\vec{v}} = \dot{v}_x \hat{x} + \dot{v}_y \hat{y}$$

$$\begin{cases} \dot{v}_x = -\frac{c}{m} \sqrt{v_x^2 + v_y^2} \cdot v_x \\ \dot{v}_y = -\frac{c}{m} \sqrt{v_x^2 + v_y^2} \cdot v_y + g \end{cases} \quad \left. \begin{array}{l} \text{coupled} \\ \text{system} \\ \text{of DE} \end{array} \right\}$$

$$\left\{ x_{n+1} = x_n + \frac{\Delta x_n}{\Delta t} \cdot \Delta t \right\}$$

$$\left\{ \dot{x}_{n+1} = \dot{x}_n + \frac{\Delta \dot{x}_n}{\Delta t} \cdot \Delta t \right\}$$

$$m \frac{dv}{dt} = -cv^2$$

$$\frac{dv}{dt} = -\frac{c}{m}v^2$$

$$F_{\text{quad}} = \underbrace{K \rho_A A \cdot v^2}_{c v^2}$$

$\frac{1}{4}$

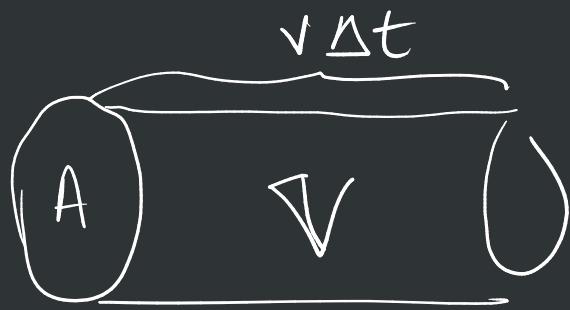












$$m = \rho V = \rho A \cdot v \cdot \Delta t$$

$$\rightarrow \underline{\underline{\frac{m}{\Delta t} = \rho A \cdot v}}}$$

$$\frac{\Delta V}{\Delta t} = A \cdot v$$







#10

$$F_B = \underbrace{\rho_{fl} V_{obj}}_{\text{mass of displaced fluid}} \cdot g$$

density      volume

$$m\dot{v} = mg - bv - \rho_{fl} V g$$

↑

≡  
?

$$0 = mg - bv_{ter} - \rho_{fl} V g$$



