

Chapter 2 - air resistance

$$\Sigma \vec{F} = \vec{F}_{\text{NET}} = m \ddot{\vec{r}}$$

↑

$$\vec{F}_D = -f(v) \hat{v}$$

$$f(v) = a + bv + cv^2$$

↑
0

↑
linear
drag

↑
quadratic
drag

$$f_{\text{lin}} = bv$$

$$b \propto D$$

↙ related to viscosity

$$b = \beta D$$

$$f_{\text{quad}} = cv^2$$

$$c \propto D^2$$

$$c = \gamma D^2$$

↑ density

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \propto \frac{D^2 v^2}{Dv} = Dv$$

$$\frac{f_{\text{quad}}}{f_{\text{lin}}} \approx \text{Reynolds number} = \frac{\rho D v}{\eta}$$

density
 ↓
 viscosity

$$\vec{F}_{\text{net}} = m \ddot{\vec{r}}$$

$$m\vec{g} - b\vec{v} = m\ddot{\vec{r}}$$

$$m\vec{g} - b\vec{v} = m\dot{\vec{v}}$$

really two equations

$$-bv_x = m\dot{v}_x$$

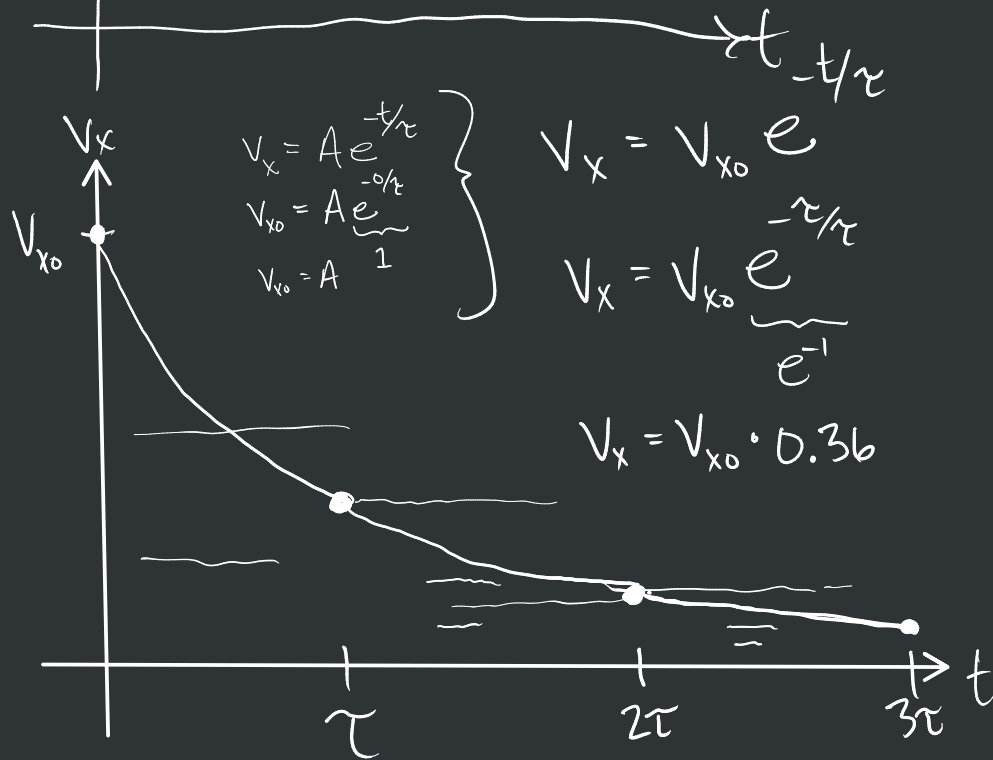
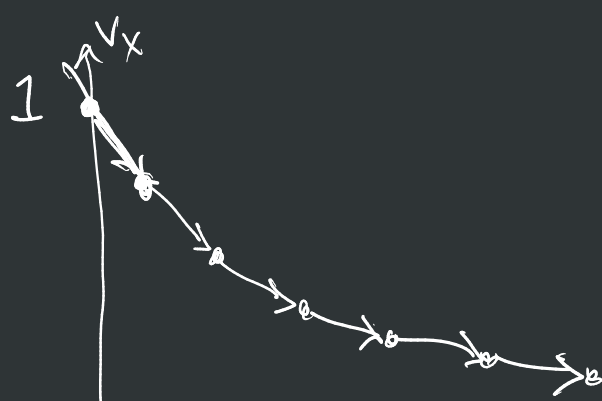
$$\dot{v}_x = -\frac{b}{m} v_x$$

$$\boxed{\frac{dv_x}{dt} = -\frac{b}{m} v_x}$$

$$\frac{dv_x}{v_x} = -\frac{b}{m} dt$$

$$mg - bv_y = m\dot{v}_y$$

↓
 down ⊕ y-dir



$$\int \frac{dv_x}{v_x} = - \underbrace{\frac{b}{m}} dt$$

$$e^{\ln v_x} = e^{\left(-\frac{b}{m}t + C\right)} = e^{-\frac{b}{m}t} \cdot \underbrace{e^C}$$

$$v_x = A \cdot e^{-\frac{b}{m}t}$$

$$\frac{b}{m} = \left[\frac{1}{s}\right] = [s^{-1}]$$

$$\tau = \frac{m}{b} = [s]$$

$$v_x = A e^{-t/\tau} \rightarrow \text{time constant}$$

exponential decay

$-t/\tau$

$\dot{x} = v_{x0} e^{-t/\tau}$

$$\frac{dx}{dt} = v_{x0} e^{-t/\tau}$$

$$\int dx = v_{x0} \int e^{-t/\tau} dt$$

$$\int_{x_0}^x dx' = v_{x0} \underbrace{\int_0^t e^{-t'/\tau} dt'}_{}$$

$$x' \Big|_{x_0}^x \overset{\text{function}}{=} v_{x0} \left(-\tau \right) e^{-t'/\tau} \Big|_0^t$$

$$(x - x_0) = -v_{x0} \tau \cdot \left(e^{-t/\tau} - \cancel{e^{-0/\tau}} \right)$$

$$x - x_0 = v_{x0} \tau \left(1 - e^{-t/\tau} \right)$$

$$x(t) = v_{x0} \tau \left(1 - e^{-t/\tau} \right) + x_0$$

Linear Drag Vertically

$$mg - bv_y = m\dot{v}_y$$

if $\dot{v}_y = 0$

then

$$\underline{mg - bv_y = 0}$$

$$v_y = \frac{mg}{b} \equiv v_t$$

↑ velocity when $\dot{v} = 0$

$$m\dot{v}_y = mg - bv_y$$

$$= -b\left(-\frac{mg}{b} + v_y\right)$$

$$m\dot{v}_y = -b(-v_t + v_y)$$

$$u = (-v_t + v_y)$$

$$\dot{u} = \frac{du}{dt} = 0 + \dot{v}_y$$

$$\dot{u} = \dot{v}_y$$

$$m\dot{u} = -b \cdot u$$

$$\dot{u} = -\frac{b}{m} \cdot u \quad \leftarrow$$

$$u = A e^{-t/\tau}$$

$$\tau = \frac{m}{b}$$

$$-v_t + v_y = A e^{-t/\tau}$$

$$V_y = A e^{-t/\tau} + V_t$$

$$V_y(t=0) = V_{y0}$$

$$V_{y0} = A \underbrace{e^{-0/\tau}}_1 + V_t$$

$$V_{y0} = A + V_t$$

$$A = V_{y0} - V_t$$

$$V_y = (V_{y0} - V_t) e^{-t/\tau} + V_t$$

$$V_y = \underbrace{V_{y0} e^{-t/\tau}}_{\substack{t \rightarrow \infty \\ \rightarrow 0}} + \underbrace{V_t (1 - e^{-t/\tau})}_{t \rightarrow \infty}$$

$$V_{y0} = (V_{y0} - V_t) e^{-t/\tau} + V_t$$

$$\cancel{dt} \frac{dy}{\cancel{dt}} = \left((v_{y_0} - v_t) e^{-t/\tau} + v_t \right) dt$$

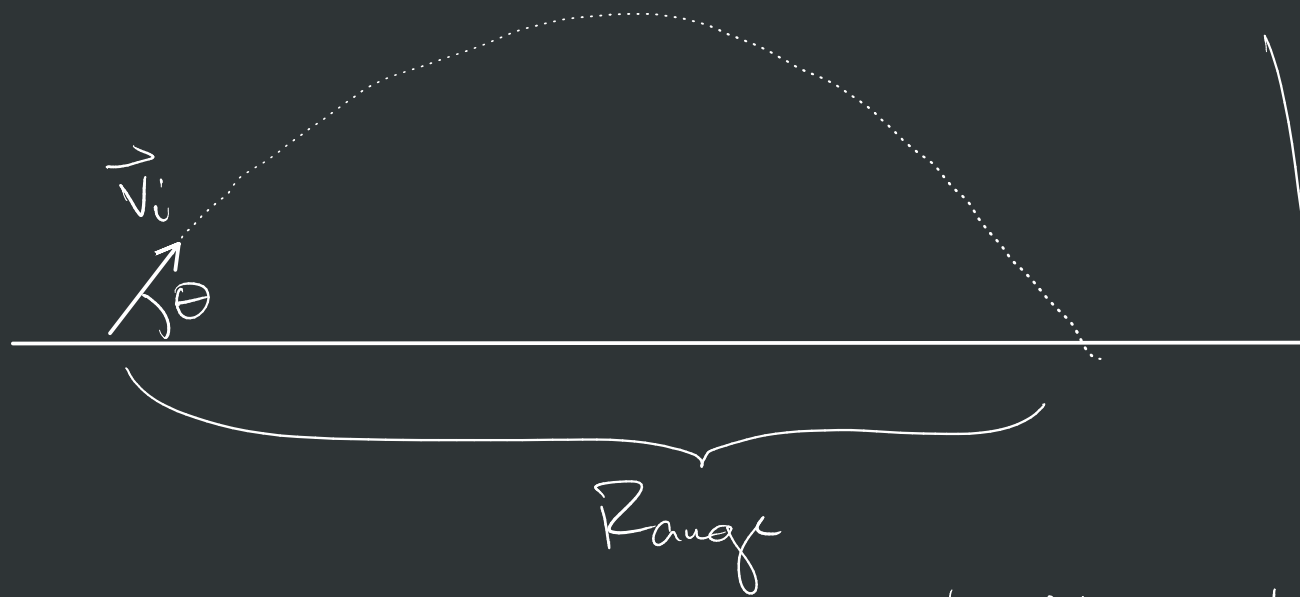
$$\int_{y_0}^y dy' = \int_0^t \left((v_{y_0} - v_t) e^{-t'/\tau} + v_t \right) dt'$$

$$y' \Big|_{y_0}^y = \left[(v_{y_0} - v_t) (-\tau) e^{-t'/\tau} + v_t t' \right]_0^t$$

$$y - y_0 = -(v_{y_0} - v_t) \tau e^{-t/\tau} + v_t t - \left[-(v_{y_0} - v_t) \tau \cdot 1 + 0 \right]$$

algebra

$$y = (v_{y_0} - v_t) \tau \left(1 - e^{-t/\tau} \right) + v_t t + y_0$$



Range of Projectile Motion in vacuum

$$v_x = v_i \cos \theta$$

$$x = v_x \cdot t = v_i \cos \theta \cdot t$$

$$v_{yo} = v_i \sin \theta$$

$$y = y_i + v_{yo} \cdot t + \frac{1}{2} a_y t^2$$

Range \rightarrow solve $y=0$ for t
plug t in for x

$$0 = v_{yo} t + \frac{1}{2} a_y t^2$$

$$0 = t \left(v_{yo} + \frac{1}{2} a_y t \right)$$

$$\begin{matrix} \nwarrow & \searrow \\ t=0 & \text{or} & t = -\frac{2v_{yo}}{a_y} \end{matrix}$$

$$\text{Range} = v_x \left(\frac{-2v_{yo}}{a_y} \right)$$

$$a_y = -g$$

$$R = \frac{2v_x v_{yo}}{g}$$

Range for Projectile Motion in Linear Drag

$$x = v_{x0} \tau \left(1 - e^{-t/\tau} \right) + \cancel{x_0} \quad \Bigg| \quad y = (v_{y0} - v_t) \tau \left(1 - e^{-t/\tau} \right) + v_t t + \cancel{y_0}$$

$$x = v_{x0} \tau \left(1 - e^{-t/\tau} \right)$$

Solve for t

$$y = (v_{y0} - v_t) \tau \left(1 - e^{-t/\tau} \right) + v_t t$$

$$= (v_{y0} - v_t) \tau \left(1 - e^{-t/\tau} \right) + v_t t$$

lots of algebra (to eliminate t)

$$y = \left(\frac{v_{y0} + v_t}{v_{x0}} \right) x + v_t \tau \ln \left(1 - \frac{x}{v_{x0} \tau} \right)$$

$$0 = \left(\frac{V_{yo} + V_t}{V_{xo}} \right) R + V_t \tau \ln \left(1 - \frac{R}{V_{xo} \tau} \right)$$

$$- \left(\frac{V_{yo} + V_t}{V_{xo}} \right) \frac{R}{V_t \tau} = \ln \left(1 - \frac{R}{V_{xo} \tau} \right)$$

$$e^{- \left(\frac{V_{yo} + V_t}{V_{xo}} \right) \frac{R}{V_t \tau}} = 1 - \frac{R}{V_{xo} \tau}$$

Taylor series approx

$$\leftarrow \ln(1-\epsilon) \hat{=} - \left(\epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 + \dots \right)$$

quadratic equation

$$\boxed{\frac{R}{V_{xo} \tau} = 1 - e^{- \left(\frac{V_{yo} + V_t}{V_{xo}} \right) \frac{R}{V_t \tau}}}$$

$$\begin{cases} \cancel{x = e^x} \\ \rightarrow x = e^{-x} \\ \rightarrow x = (1 - e^{-x}) \end{cases}$$

Quadratic Drag

Horizontal

$$F_{\text{net}} = m\dot{v}$$

$$-c v^2 = m\dot{v}$$

$$\vec{v} = v_x \hat{x}$$

$$m \frac{dv}{dt} = -c v^2$$

$$\frac{dv}{v^2} = -\frac{c}{m} dt$$

$$\int \frac{dv}{v^2} = -\frac{c}{m} \int dt$$

$$\int v^{-2} dv = -v^{-1} = -\frac{c}{m} t + D$$

$$x = e^{-x}$$

$$x = 0.567$$

$$x = (1 - e^{-x})$$

$$x = 0$$

$$\left. \begin{aligned} m \frac{dv_x}{dt} &= -c(v_x^2 + v_y^2) \\ m \frac{dv_y}{dt} &= -c(v_x^2 + v_y^2) \end{aligned} \right\} \begin{array}{l} \text{coupled} \\ \text{system} \\ \text{of} \\ \text{DE} \end{array}$$

$$\frac{1}{V} = \frac{C}{m} t + D$$

$$V(t) = \frac{1}{\frac{C}{m} t + D}$$

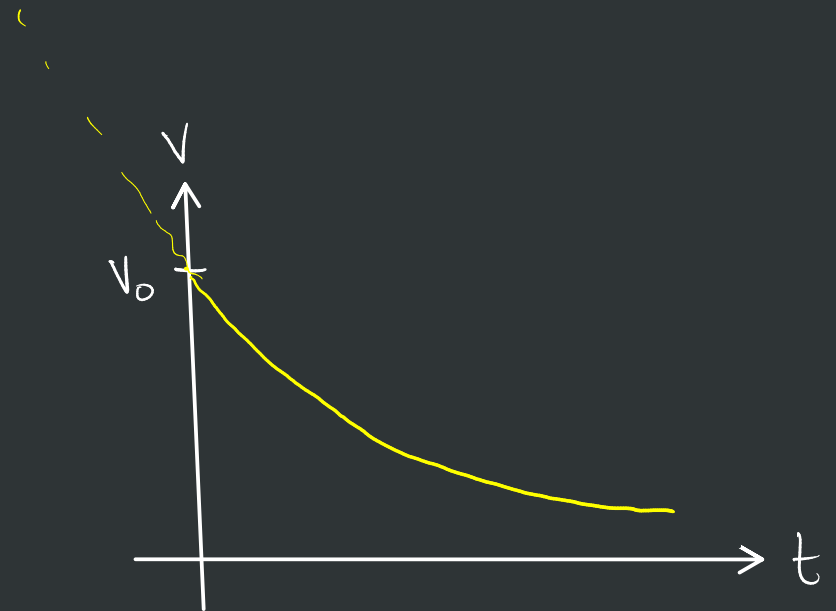
$$V_0 = V(0) = \frac{1}{D} \Rightarrow D = \frac{1}{V_0}$$

$$V(t) = \frac{1}{\frac{C}{m} t + \frac{1}{V_0}}$$

$$V(t) = \frac{V_0}{\frac{C V_0}{m} t + 1}$$

$$\tau = \frac{m}{C V_0}$$

$$V(t) = \frac{V_0}{t/\tau + 1}$$



$$\frac{dx}{dt} = \frac{V_0}{t/\tau + 1}$$

$$\int dx = \int \frac{V_0}{t/\tau + 1} dt$$

$$x = V_0 \int \frac{1}{t/\tau + 1} dt$$

$$x = V_0 \tau \underbrace{\int \frac{1}{u} \cdot du}_{\ln(u)}$$

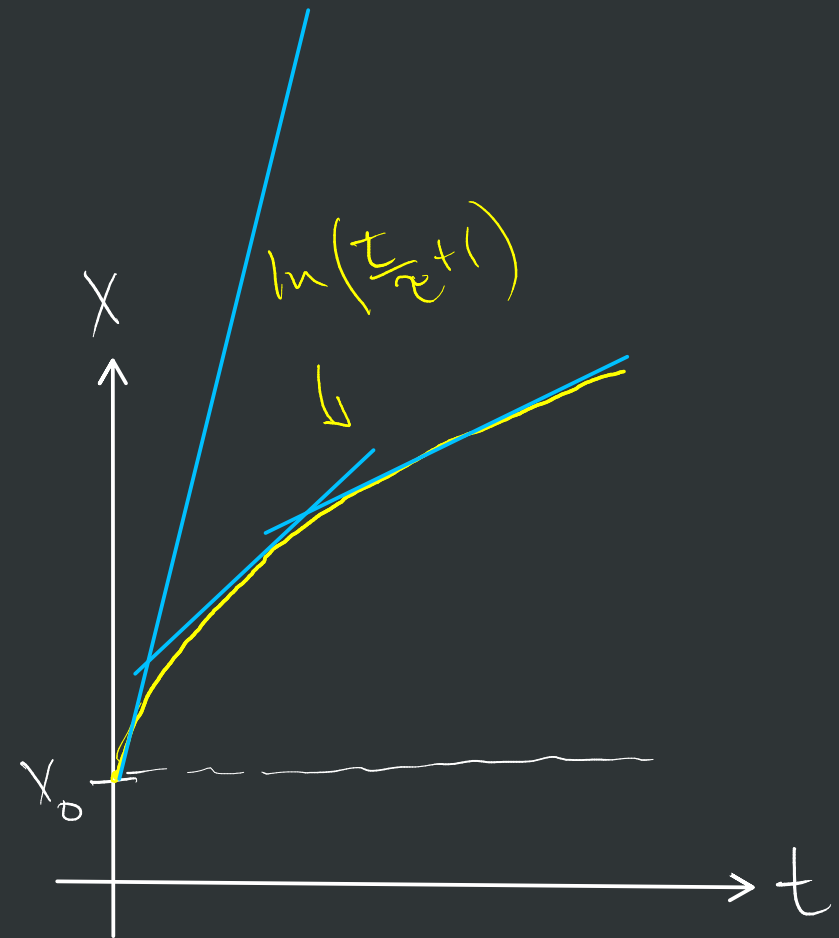
$$\left\{ \begin{array}{l} u = \frac{t}{\tau} + 1 \\ du = \frac{du}{dt} dt \\ du = \frac{1}{\tau} dt \\ \tau du = dt \end{array} \right.$$

$$x(t) = V_0 \tau \ln\left(\frac{t}{\tau} + 1\right) + E$$

$$x_0 = x(0) = V_0 \tau \underbrace{\ln(1)}_0 + E$$

$$x_0 = E$$

$$x(t) = V_0 \tau \ln\left(\frac{t}{\tau} + 1\right) + x_0$$



Vertical Direction

$$m\dot{v} = mg - cv^2$$

$$\dot{v} = 0 = mg - cv_t^2$$

$$v_t = \sqrt{\frac{mg}{c}}$$

← terminal velocity

$$\dot{v} = g - \frac{c}{m} \frac{g}{g} v^2$$

$$\dot{v} = g \left(1 - \frac{c}{mg} v^2 \right)$$

$$\dot{v} = g \left(1 - \frac{v^2}{v_t^2} \right)$$

$$\frac{dv}{dt} = g \left(1 - \frac{v^2}{v_t^2} \right)$$

$$\int \frac{dv}{\left(1 - \frac{v^2}{v_t^2} \right)} = \int g dt$$

$$\int \frac{dv}{(1 - v^2/v_t^2)} = \int g dt$$

$$\tanh(u) = \frac{v}{v_t}$$

$$\text{sech}^2(u) du = \frac{1}{v_t} dv$$

$$\begin{aligned} \sinh \theta &= \frac{e^{\theta} - e^{-\theta}}{2} \\ \cosh \theta &= \frac{e^{\theta} + e^{-\theta}}{2} \\ \tanh \theta &= \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}} \end{aligned}$$

$$\begin{aligned} \sin \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \\ \cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\ \tan \theta &= \end{aligned}$$

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + i\theta^5 - \frac{\theta^6}{6!} + \dots$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$$

$$e^{i\theta} = \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots}_{\cos \theta} + i \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \right)}_{\sin \theta}$$

$$\boxed{e^{i\theta} = \cos \theta + i \sin \theta}$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\cosh^2 \theta - \sinh^2 \theta = 1$$

$$\sinh^2 \theta + \tanh^2 \theta = 1$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sec^2 \theta - \tan^2 \theta = 1$$

$$\sec \theta = \frac{1}{\cos \theta}$$

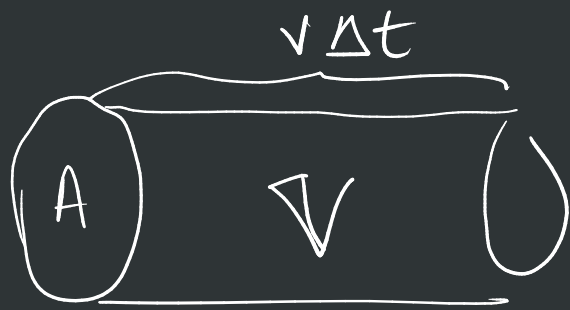
$$\frac{d}{d\theta}(\sinh\theta) = \cosh\theta$$

$$\frac{d}{d\theta} \tanh\theta = \operatorname{sech}^2\theta$$

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta$$

$$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

$$\begin{aligned} \frac{d}{d\theta}(\tan\theta) &= \sec^2\theta \\ &= -\frac{1}{\cos^2\theta} \end{aligned}$$



$$m = \rho V = \rho A \cdot v \cdot \Delta t$$

$$\rightarrow \underline{\underline{\frac{m}{\Delta t} = \rho A \cdot v}}$$

$$\frac{\Delta V}{\Delta t} = A \cdot v$$

#10

$$F_B = \underbrace{\rho_{fl} V_{obj}}_{\text{mass of displaced fluid}} \cdot g$$

density volume

$$m\dot{v} = mg - bv - \rho_{fl} V g$$

↑

≡
?

$$0 = mg - bv_{ter} - \rho_{fl} V g$$

