Chapter 4 → Energy

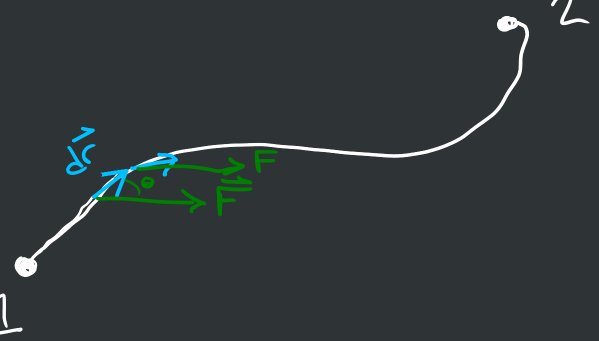
- Kinetic Energy

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{p^2}{2m}$$

- Work is a transfer of energy to the system

$$dW = \vec{F}\cdot d\vec{r} \Rightarrow W = \int_1^2 \vec{F}\cdot d\vec{r}$$

$|\vec{F}|\cdot|d\vec{r}|\cdot\cos\theta$



path integral means work depends on the path between points, unless it doesn't

- Amount of work done (total) is the same as the change in kinetic energy

$$W = \Delta T$$ ← Work-Kinetic energy theorem

- If the work done does not depend on the path taken, (it depends on the end points (positions)), that force is a conservative force

→ work done by a conservative force ⇒ potential energy

$$\Delta U = -W_{\substack{\text{conservative}\\ \text{force}}}$$

Conservative forces
- gravititational force
- electric force
- spring force

non conservative force
- friction
- drag
- "applied force"

* potential energy is defined by a reference point ($\vec{r}$ when $U=0$)

- Conservation of energy

$$W = \Delta K$$

$$W_{NC} = \underbrace{\Delta K + \Delta U}_{\text{mechanical energy}}$$

$$U_g = mgh$$

$$W_{NC} = T_f - T_i + U_f - U_i$$

$$T_i + U_i + W_{NC} = T_f + U_f$$

- if $\Delta U = -\int_1^2 \vec{F} \cdot d\vec{r} \Rightarrow$ $dU = -W = -\vec{F} \cdot d\vec{r}$

$$\cancel{\frac{dU}{d\vec{r}} = -\vec{F}}$$

$$\vec{F} = -\vec{\nabla} U = \frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}$$

gradient → cartesian

$$F = -\frac{dU}{dx}$$ ← one dimension

- What make a F conservative
  - F depends only on position (not v, t, a)
  - Work is path-independent

    ↳ $\vec{\nabla} \times \vec{F} = 0$

    curl

$F_s = -kx$

→ $F_g = mg$

$F_G = \frac{G m_1 m_2}{r^2}$

$F_D = bv$

$= cv^2$

- everything above can be done in spherical coords. or cylindrical coords → often this will be easier central forces

$$\vec{F}(\vec{r}) = f(\vec{r})\,\hat{r}$$

= radial force

$$\vec{F}_G(\vec{r}) = -\frac{G m_1 m_2}{r^2}\,\hat{r}$$ ← in spherical coords.

$$\vec{r} = x\hat{x} + y\hat{y} \qquad \hat{r} = \frac{\vec{r}}{|r|}$$

$$r = \sqrt{x^2 + y^2}$$

$$= -\left(\frac{G m_1 m_2}{x^2 + y^2}\right) \cdot \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2 + y^2}}$$

$$= -\frac{G m_1 m_2}{(x^2 + y^2)^{3/2}} \cdot (x\hat{x} + y\hat{y})$$

$$\vec{\nabla} \times \vec{F}_G =$$

what is curl in spherical?

- All of this applies multiple particle

$$T = T_1 + T_2 + T_3 + \dots$$

$$T = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I \omega_{cm}^2$$

rot. inertia

Recall:

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

$$\frac{dT}{dt} = \frac{1}{2} m \cdot \frac{d}{dt}(\vec{v} \cdot \vec{v})$$

$$= \frac{1}{2} m \left(\frac{d}{dt}(\vec{v}) \cdot \vec{v} + \vec{v} \cdot \frac{d}{dt}(\vec{v}) \right)$$

$$= \frac{1}{2} m (2 \dot{\vec{v}} \cdot \vec{v})$$

$$= m \dot{\vec{v}} \cdot \vec{v}$$

$$\dot{\vec{p}} = \vec{F}$$

$$\frac{dT}{dt} = \vec{F} \cdot \vec{v}$$

$$\frac{dT}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

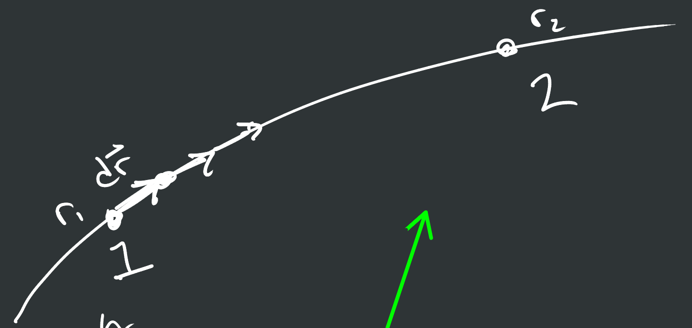
$$\Delta T = W$$

$$dT = \underbrace{\vec{F} \cdot d\vec{r}}$$

define this as a small amount of <u>work</u>

- dW does not make sense.
- $đW$ (in thermo)

$$\Delta T = T_2 - T_1 = \underbrace{\int_1^2 \vec{F} \cdot d\vec{r}}_{\text{line integral}}$$

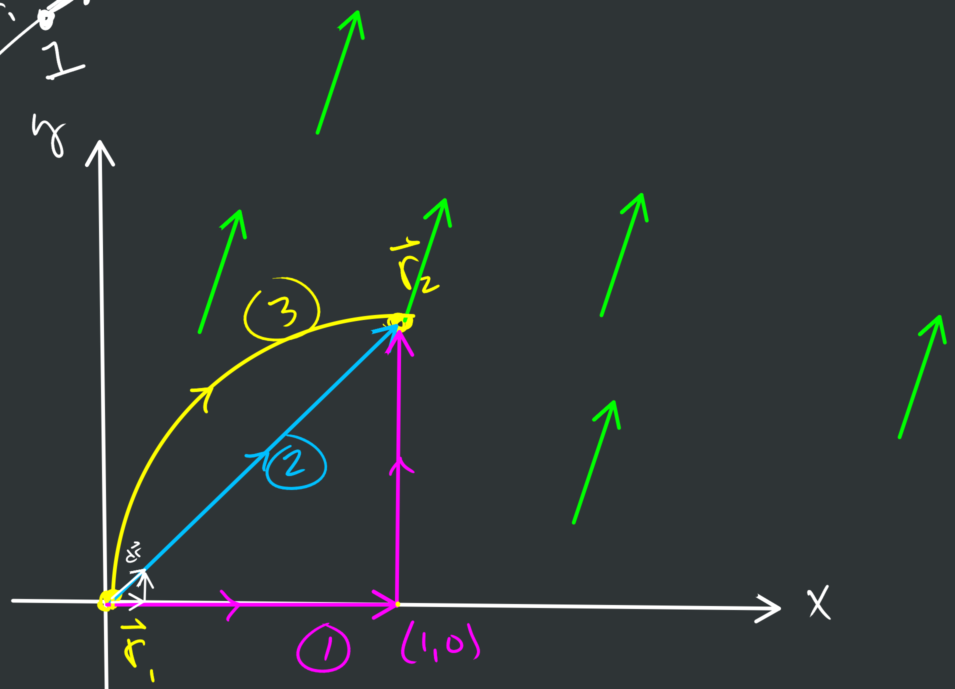


$\int f(x)\,dx$

Ex: $\vec{F} = 1\hat{x} + 2\hat{y}$

$\vec{r}_1 = 0$

$\vec{r}_2 = 1\hat{x} + 1\hat{y}$



Path ①

$$\int_1^2 \vec{F}\cdot d\vec{r} = \int_{(0,0)}^{(1,0)} \underbrace{\vec{F}\cdot d\vec{r}_1} + \int_{(1,0)}^{(1,1)} \underbrace{\vec{F}\cdot d\vec{r}_2}$$

$d\vec{r}_1 = d\vec{x} = dx\,\hat{x}$

$\vec{F}\cdot d\vec{x} = (1\hat{x} + 2\hat{y})\cdot(dx\,\hat{x} + 0\hat{y})$

$= 1\,dx$

$d\vec{r}_2 = dy\,\hat{y}$

$\vec{F}\cdot dy\hat{y} = (1\hat{x} + 2\hat{y})\cdot(0\hat{x} + dy\,\hat{y})$

$= 2dy$

$$\int_1^2 \vec{F}\cdot d\vec{r} = \int_{x=0}^{x=1} dx + \int_{y=0}^{y=1} 2\,dy$$

$$= x\Big|_0^1 + 2y\Big|_0^1$$

Work along path ①

$$= 1 + 2 = 3$$

Path ②

$$\int_1^2 \vec{F}\cdot d\vec{r} =$$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y}$$

$$\vec{F}\cdot d\vec{r} = (1\hat{x} + 2\hat{y})\cdot(dx\,\hat{x} + dy\,\hat{y})$$

$$= dx + 2dy$$

$$\int_1^2 (dx + 2dy) = \int_0^1 dx + \int_0^1 2\,dy = 3$$

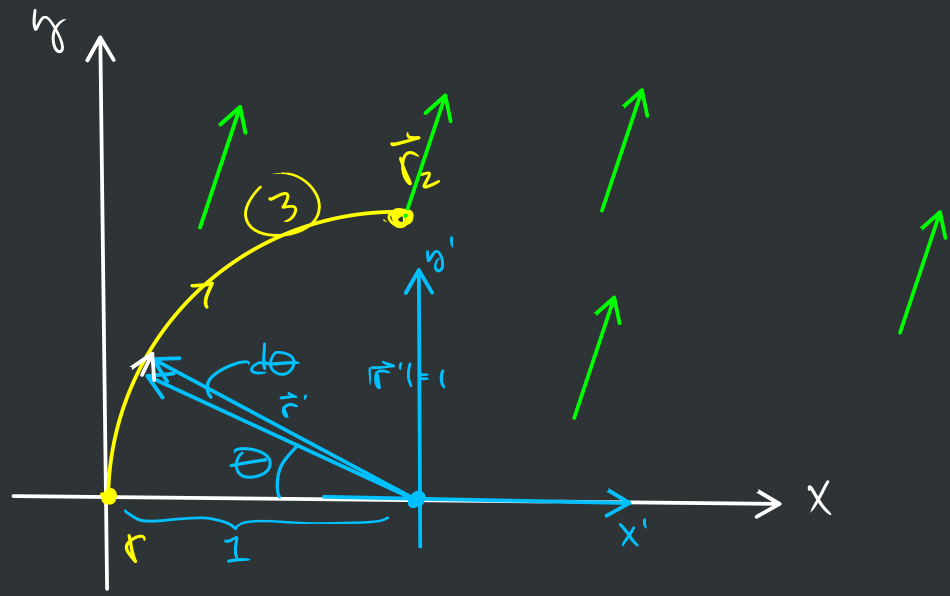
Path ③

$$\int_1^2 \vec{F}\cdot d\vec{r}$$

$$\int_1^2 \left(1\hat{x} + 2\hat{y}\right)\cdot\left(\sin\theta\, d\theta\, \hat{x} + \cos\theta\, d\theta\, \hat{y}\right)$$

$$\int_1^2 \left(\sin\theta\, d\theta + 2\cos\theta\, d\theta\right)$$

$$\int_0^{\pi/2} \sin\theta\, d\theta + \int_0^{\pi/2} 2\cos\theta\, d\theta = 3$$



$$\vec{r}\,' = -\cos\theta\, \hat{x}' + \sin\theta\, \hat{y}'$$

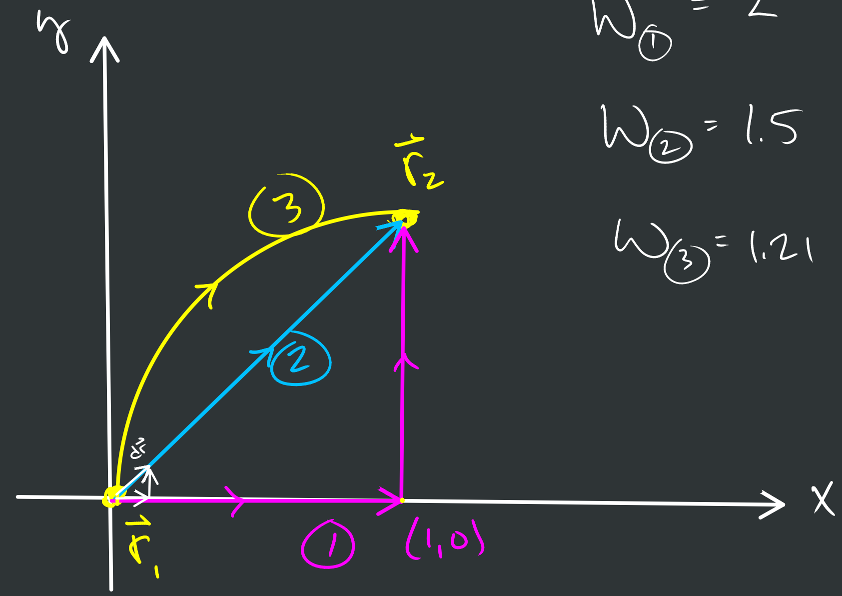
↓ change coordinate systems

$$\vec{r} = (1-\cos\theta)\hat{x} + \sin\theta\, \hat{y}$$

$$d\vec{r} = \sin\theta\, d\theta\, \hat{x} + \cos\theta\, d\theta\, \hat{y}$$

Homework: Same 3 paths

$$\vec{F} = y\hat{x} + 2x\hat{y}$$



$W_{①} = 2$

$W_{②} = 1.5$

$W_{③} = 1.21$

$$\vec{r} = (1-\cos\theta)\hat{x} + \sin\theta\,\hat{y}$$

$$d\vec{r} = \sin\theta\, d\theta\,\hat{x} + \cos\theta\, d\theta\,\hat{y}$$

$$W_{①} = \int_{(0,0)}^{(1,0)} \vec{F}\cdot d\vec{r}_1 + \int_{(1,0)}^{(1,1)} \vec{F}\cdot d\vec{r}_2$$

$d\vec{r}_1 \to dx\,\hat{x}$, $\quad d\vec{r}_2 \to dy\,\hat{y}$

$$(y\hat{x} + 2x\hat{y})\cdot(dx\,\hat{x} + 0\,\hat{y})$$

$$y\,dx + 0$$

$$= \int_{(0,0)}^{(1,0)} y\,dx + \int_{(1,0)}^{(1,1)} 2x\,dy$$

$$= \underbrace{\int_{x=0}^{x=1} 0\cdot dx}_{0} + \int_{y=0}^{y=1} 2\cdot(1)\,dy = 2y\Big|_0^1 = 2(1-0) = 2$$

$$W_{③} = \int_0^{\pi/2} y\sin\theta\, d\theta + \int_0^{\pi/2} 2x\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\, d\theta + 2\int_0^{\pi/2} (1-\cos\theta)\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\, d\theta + 2\int_0^{\pi/2} \cos\theta\, d\theta - 2\int_0^{\pi/2} \cos^2\theta\, d\theta$$

For <u>conservative</u> <u>force</u>

$$U(\vec{r}) = -W(\vec{r}_0 \to \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'$$

Potential energy at $\vec{r}$ in reference to $\vec{r}_0$

conditions for conservative force

time

① $\vec{F}$ only depends on $\vec{r}$ (not $\vec{v}$, not t)

② Work done b/t any two points is independent of path taken

$(\vec{\nabla} \times \vec{F} = 0)$

What about $W(\vec{r}_1 \to \vec{r}_2)$ ← work to go between points not the reference point

$$W(\vec{r}_0 \to \vec{r}_2) = W(\vec{r}_0 \to \vec{r}_1) + W(\vec{r}_1 \to \vec{r}_2)$$

$$W(\vec{r}_1 \to \vec{r}_2) = \underbrace{W(\vec{r}_0 \to \vec{r}_2)}_{-U(\vec{r}_2)} - \underbrace{W(\vec{r}_0 \to \vec{r}_1)}_{-U(\vec{r}_1)}$$

$$= -\underbrace{\left(U(\vec{r}_2) - U(\vec{r}_1)\right)}_{\Delta U}$$

$$\underline{\underline{\Delta U}} = -W(\vec{r}_1 \to \vec{r}_2)$$

Now go back to work-kinetic energy theorem

$$W(\vec{r}_1 \to \vec{r}_2) = \Delta K$$

if only conservative forces are acting

Conservation of Energy

$$-\Delta U = \Delta K$$

$$\Delta K + \Delta U = 0$$

$$E = K + U \quad \} \text{ Mechanical Energy}$$

$$\Delta E = 0$$

Also true for multiple conservative forces.

$\hookrightarrow$ multiple potential energy

$$U = U_g + U_s \; (+ U_e)$$

But, what if also non-conservative forces act?

Conservation of Energy

$$\Delta K + \Delta U \neq 0$$

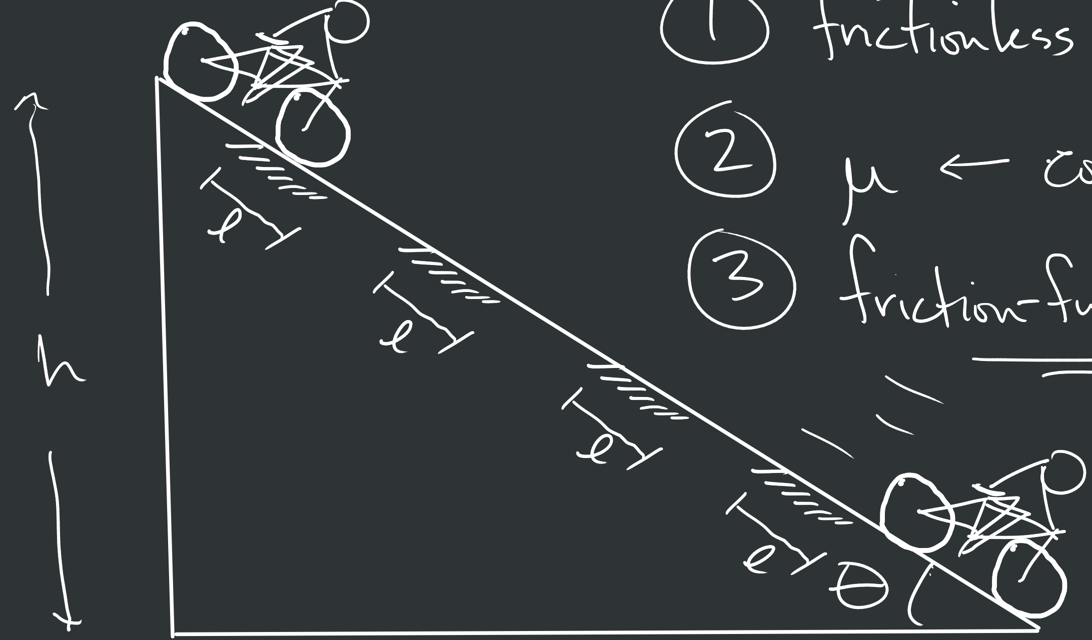
$$\Delta K + \Delta U = W_{NC}$$

$$K_i + U_i + W_{NC} = K_f + U_f$$

Friction

$$W_f = \int \vec{F}_f \cdot d\vec{x}$$

Ex:

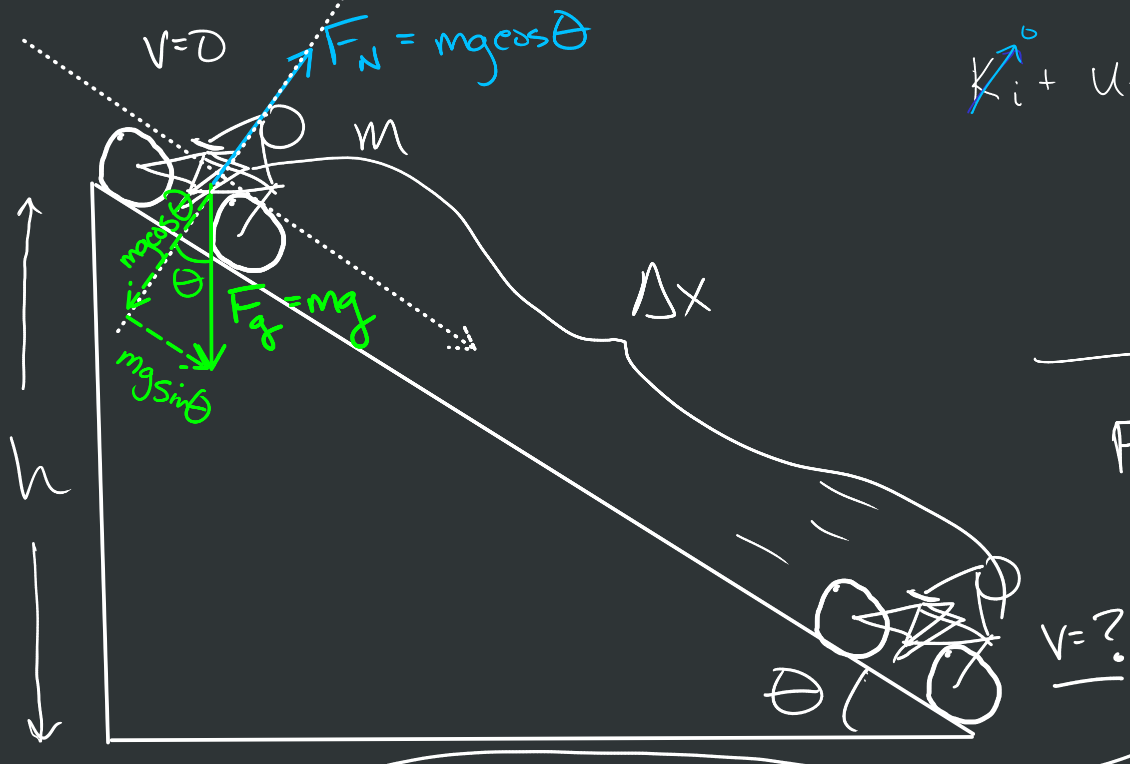


1. frictionless plane
2. $\mu \leftarrow$ constant
3. friction-full patches

← no Newton's

Newton's Laws + Conservation of Energy

1



$$\cancel{K_i}^{0} + U_i + \cancel{W_{NC}}^{0} = K_f + \cancel{U_f}^{0}$$

$$\cancel{m}gh = \frac{1}{2}\cancel{m}v^2$$

$$v = \sqrt{2gh}$$

$$F_{NET,x} = \cancel{m}g\sin\theta = ma = \dot{p}$$

$$g\sin\theta = a$$

$$g\sin\theta = \frac{dv}{dt}$$

$$\int_0^t g\sin\theta \, dt = \int_{v_0=0}^{v} dv'$$

$$g\sin\theta \cdot t = v \qquad v_f = \cancel{v_i}^{0} + at$$

$$v = g\sin\theta \, t$$

$$\int_0^x dx = \int_0^t g\sin\theta \, t' \, dt'$$

$$x = \frac{g\sin\theta \, t^2}{2}$$

$$g\sin\theta = \frac{dv}{dt} = \frac{dv}{dx} \cdot \underbrace{\frac{dx}{dt}}_{v}$$

$$g\sin\theta = v\frac{dv}{dx}$$

$$\int_{x=0}^{x} g\sin\theta \, dx' = \int_0^v v' \, dv'$$

$$g\sin\theta \, x'\Big|_0^x = \frac{v'^2}{2}\Big|_{0=v_i}^{v}$$

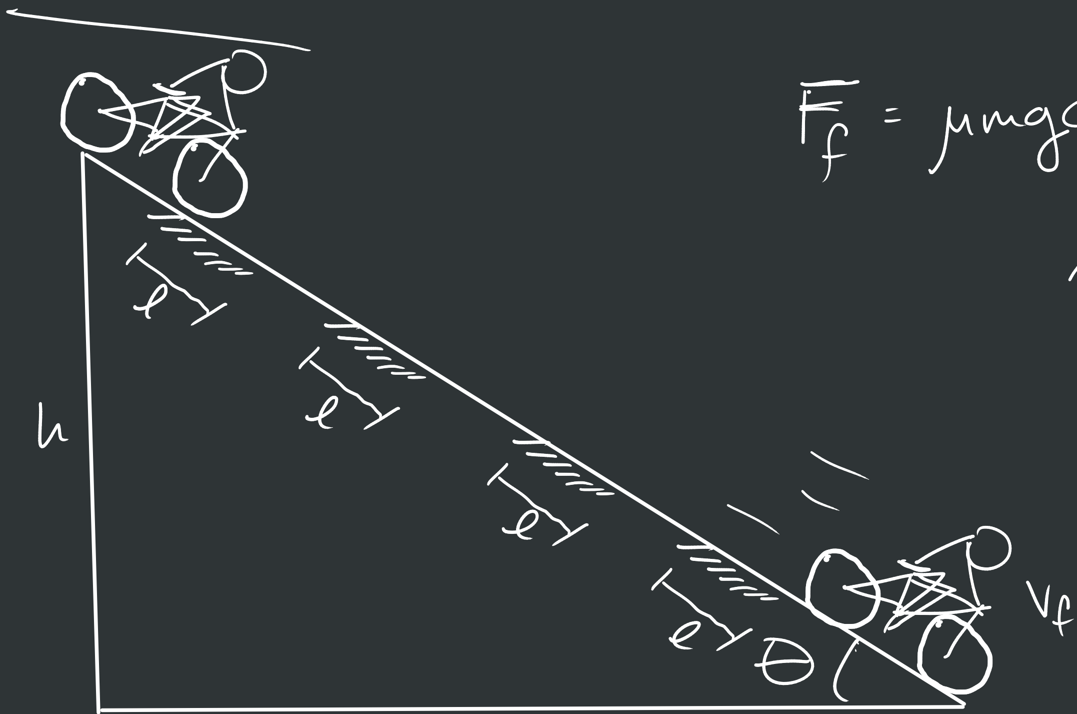
$$g\sin\theta \cdot x = \frac{v^2}{2}$$

$$\sqrt{2g\underbrace{\sin\theta\, x}_{h}} = v$$

$$v = \sqrt{2gh}$$

$$\sin\theta = \frac{h}{x} = \frac{opp}{hyp}$$

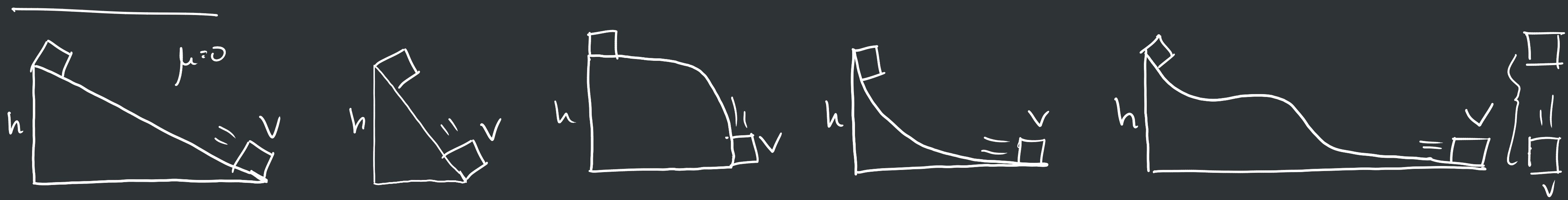
$$\sin\theta \cdot x = h$$



$$F_f = \mu mg\cos\theta$$

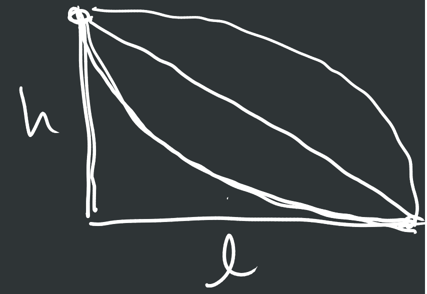
$$\cancelto{0}{K_i} + U_i + W_{NC} = K_f + \cancelto{0}{U_f}$$

$$\cancel{m}gh - 4\cdot\mu\cancel{m}g\cos\theta \cdot l = \frac{1}{2}\cancel{m}\underline{\underline{v_f^2}}$$



Potential Energy

$$U(\vec{r}) = -\int_{r_0}^{r} \vec{F}(\vec{r}\,') \cdot d\vec{r}\,'$$



Gravitational (constant)

$$\vec{F}(\vec{r}\,') = -mg\,\hat{y}$$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$U = -W(0 \to h\hat{y}) = +\int_0^h (+mg\hat{y}) \cdot (dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z})$$

$$= \int_0^h mg\,dy$$

$$= mgy\Big|_0^h$$

$$U(h) = mgh$$

in relation to wherever we call $h=0 \to U=0$

Gravitation (universal)

$$\vec{F} = -\frac{G m_1 m_2}{r^2}\hat{r}$$

$$U(\vec{r}) = -W(0 \to \vec{r}) = -\int_{\infty}^{r} -\frac{G m_1 m_2}{r^2}\hat{r} \cdot dr\,\hat{r}$$

$$= \int_{\infty}^{r} \frac{G m_1 m_2}{r'^2}\,dr' = G m_1 m_2 \int_{\infty}^{r} r'^{-2}\,dr'$$

$$= G m_1 m_2 (-1) r'^{-1}\Big|_{\infty}^{r}$$

$$= -\frac{G m_1 m_2}{r'}\Big|_{\infty}^{r}$$

$$= -\frac{G m_1 m_2}{r} - \cancel{-\frac{G m_1 m_2}{\infty}}^{\,0}$$

reference point $r \to \infty$

$U = 0$

$$\hookrightarrow U(r) = -\frac{G m_1 m_2}{r}$$

If instead we set $\underline{U=0}$ at $r=R$ ← radius of a planet

$$\int_R^r \frac{Gm_1m_2}{r}\,dr = -\frac{Gm_1m_2}{r}\Big|_R^r$$

$$= -\frac{Gm_1m_2}{r} + \frac{Gm_1m_2}{R}$$

$$U(r) = Gm_1m_2\left(\frac{1}{R} - \frac{1}{r}\right)$$

$r>R$

<u>Ex</u>: $r_1 \rightarrow r_2$ will require non-conservative work if $\Delta K = 0$

$$\cancel{K_i} + U_i + W_{NC} = \cancel{K_f} + U_f$$

choose $r\rightarrow\infty$ $U=0$

$$W_{NC} = \Delta U = U(r_2) - U(r_1)$$

$$= -\frac{Gm_1m_2}{r_2} + \frac{Gm_1m_2}{r_1}$$

$$= Gm_1m_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Spring Potential (1D)

$$\vec{F} = -kx\,\hat{x}$$

$dx\hat{x} + dy\hat{y} + \ldots$

$$U_s = -\int_0^x (-kx\hat{x}) \cdot d\vec{r}$$

unstretched spring

$$= \int_0^x kx' dx'$$

reference $x = 0$

$U = 0$

$$= k\frac{x'^2}{2}\Big|_0^x = \frac{1}{2}kx^2 = U_s$$

Potential Energy $\longrightarrow$ Force

$$-dU = đW = \vec{F} \cdot d\vec{r}$$

displacement

$\hookrightarrow$ dU depends on $d\vec{r}$

Cartesian $\quad d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

$\dfrac{\partial U}{\partial x} \leftarrow$ how much U changes in the x-direction w/ y + z fixed

$\dfrac{\partial U}{\partial y} \leftarrow$ same but x + z fixed

$$df = \frac{df}{dx}dx$$

<u>total differential</u>

$$\hookrightarrow dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$$

$$-\left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right) = F_x\,dx + F_y\,dy + F_z\,dz$$

$$-\frac{\partial U}{\partial x} = F_x \qquad -\frac{\partial U}{\partial y} = F_y \qquad -\frac{\partial U}{\partial z} = F_z$$

$$\vec{F} = \underbrace{-\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}}_{\text{this operation is called the gradient}}$$

$$\vec{F} = -\underbrace{\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)}_{\text{one particular differential operator}} U$$

"del" $\rightarrow \vec{\nabla}$

in Cartesian

$$\vec{F} = -\vec{\nabla}U$$

$$\vec{F} = -\vec{\nabla} U$$

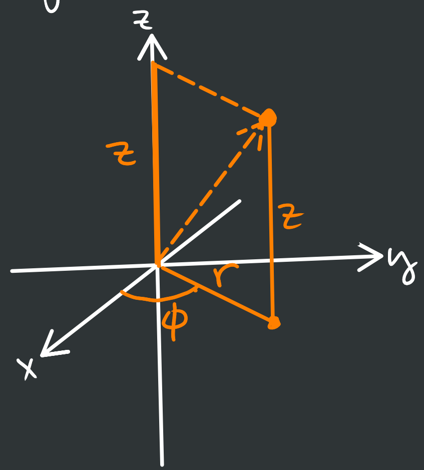
↳ more general than Cartesian

$$\vec{\nabla}_{xyz} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

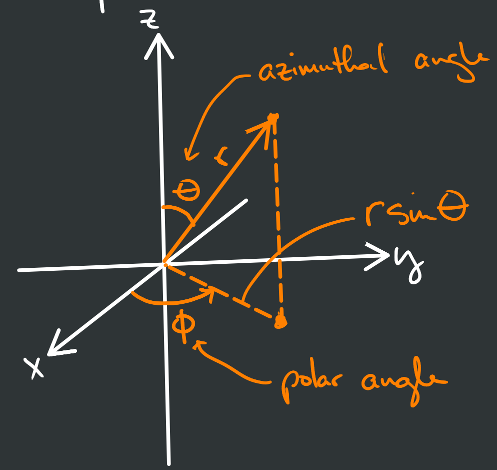
$$\vec{\nabla}_{r\phi z} = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{\nabla}_{r\theta\phi} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Cylindrical-Polar Coordinates



Spherical Coordinates



How do we know that a force will be path-independent?

$\vec{\nabla} \times \vec{F} = 0$

curl

$\vec{\nabla} \cdot \vec{F} \rightarrow$ scalar

$\vec{\nabla} \times \vec{F} \rightarrow$ vector

$\underbrace{\vec{\nabla} U}_{\text{gradient}} =$

$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$ $\vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{x} - (A_x B_z - A_z B_x)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

$$\vec{\nabla} \times \vec{F} \quad \text{if} \quad \vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z}$$

$$\vec{F} = \underset{F_x}{y\hat{x}} + \underset{F_y}{2x\hat{y}} + \underset{F_z}{0\hat{z}}$$

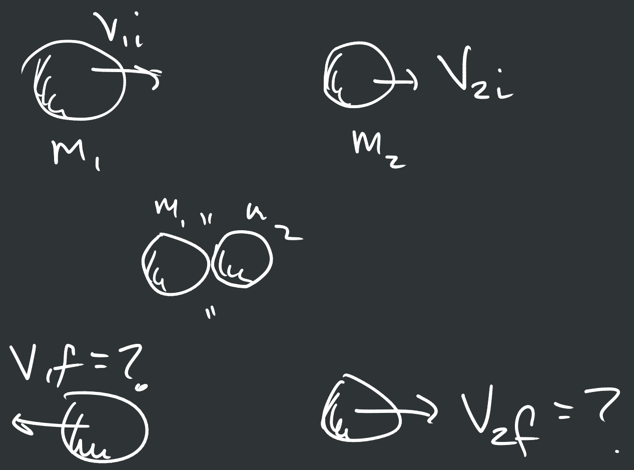
$$\vec{\nabla} \times \vec{F} = (2 - 1)\hat{z}$$

$$= 1\hat{z}$$

Elastic Collisions

① momentum is conserved

② kinetic energy is conserved



$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\cancel{\tfrac{1}{2}} m_1 v_{1i}^2 + \cancel{\tfrac{1}{2}} m_2 v_{2i}^2 = \cancel{\tfrac{1}{2}} m_1 v_{1f}^2 + \cancel{\tfrac{1}{2}} m_2 v_{2f}^2$$

Possible hints:

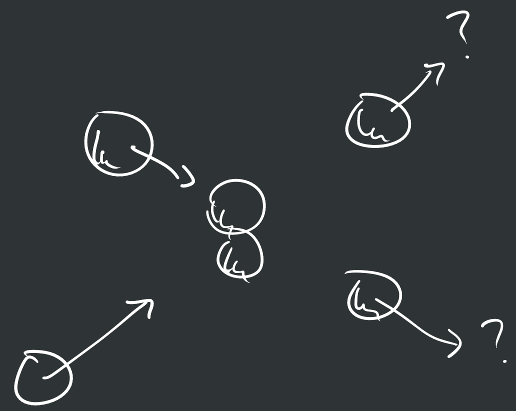
→ ① $x^2 - y^2 = (x+y)(x-y)$

→ ② use different reference frame

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} \cdot v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = v_{1f} + (v_{1i} - v_{2i})$$

$$v_{2f} = \frac{2m_2}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} \cdot v_{2i}$$



Central Forces

$$\vec{F}_G = -\frac{G m_1 m_2}{r^2}\hat{r} \leftarrow \text{spherical coordinates}$$

$$\vec{F}_e = \frac{k q_1 q_2}{r^2}\hat{r}$$

$$\underbrace{\vec{F}_G = -\frac{G m_1 m_2}{r^2}\hat{r}}_{\text{conservative ?}}$$

$$\vec{\nabla}\times\vec{F}_G = \underbrace{\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z}}_{\text{cartesian coordinates}}$$

$$\vec{F}_G = -\frac{G m_1 m_2}{r^2}\hat{r} \rightsquigarrow \text{in Cartesian}$$

$$r^2 = x^2 + y^2 + z^2 \qquad \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2+y^2+z^2}}$$

$$\vec{F}_G = \frac{-Gm_1m_2}{(x^2+y^2+z^2)^1}\left(\frac{x\hat{x}+y\hat{y}+z\hat{z}}{(x^2+y^2+z^2)^{1/2}}\right)$$

$$\frac{-Gm_1m_2}{(x^2+y^2+z^2)^{3/2}}\left(x\hat{x}+y\hat{y}+z\hat{z}\right)$$

$$F_{gx} = \frac{-Gm_1m_2x}{(x^2+y^2+z^2)^{3/2}} \qquad F_{gy} = \frac{-Gm_1m_2y}{(x^2+y^2+z^2)^{3/2}} \qquad F_{gz} = \frac{-Gm_1m_2z}{(x^2+y^2+z^2)^{3/2}}$$

$$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x}$$

$$-Gm_1m_2\left[\frac{\partial}{\partial y}\left(z\cdot(x^2+y^2+z^2)^{-3/2}\right) - \frac{\partial}{\partial z}\left(y\,(x^2+y^2+z^2)^{-3/2}\right)\right]$$

$$-Gm_1m_2\left[\underbrace{z\cdot\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{-3/2}}\right)-\underbrace{y\cdot\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{-3/2}}\Bigg)\right]$$

$$-\frac{3}{\cancel{2}}\left(x^2+y^2+z^2\right)^{-5/2}\cdot\cancel{2}y \qquad -\frac{3}{\cancel{2}}\left(x^2+y^2+z^2\right)^{-5/2}\cdot\cancel{2}z$$

$$-Gm_1m_2\underbrace{\left[-3yz\left(x^2+y^2+z^2\right)^{-5/2}+3yz\left(x^2+y^2+z^2\right)^{-5/2}\right]}_{=0!}$$

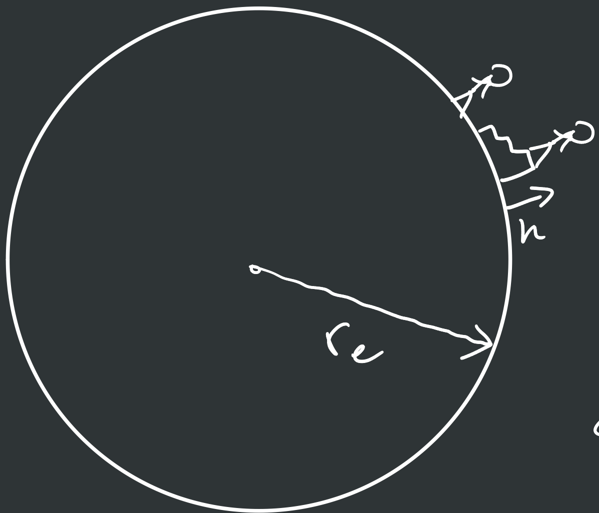
Use the curl in spherical coordinates to show the same thing!

$$\underbrace{U_G = \frac{G m_1 m_2}{r}}_{\text{Universal Gravitation}} \longleftrightarrow U_G = mgh$$

show

↳ near the surface of earth

↳ constant F_G



$g = 9.8 \frac{m}{s^2}$

$$\Delta U = U_f - U_i$$

$$= G m_e m \left(\frac{1}{r_e + h} - \frac{1}{r_e} \right)$$

$$= G m_e m \left(\frac{r_e}{(r_e + h) \cdot r_e} - \frac{r_e + h}{(r_e + h) r_e} \right)$$

$$= G m_e m \left(\frac{\cancel{r_e} - \cancel{r_e} - h}{(r_e + h) r_e} \right)$$

$$= -\frac{G m_e}{r_e} m \left(\frac{h}{r_e + h} \right) \left(\frac{r_e}{r_e} \right)$$

$$\underbrace{\frac{G m_e \cancel{m}}{r_e^2}}_{g} = \cancel{m} a = g$$

$r_e = 6371\,km$

$= 6.37 \cdot 10^6\, m$

$= 6.37 \cdot 10^6\, m + 1000\, m$

$= 6.371 \cdot 10^6\, m$

$$F = \underbrace{\frac{G m_e m}{r^2}}_{g} \Rightarrow mg$$

$$= -\underbrace{\frac{G m_e}{r_e^2}}_{g} \cdot m \cdot r_e \left(\frac{h}{r_e + h}\right)$$

$$= gm \cdot \cancel{r_e}\left(\frac{h}{r_e + h}\right)$$

$r_e + \cancel{h} \approx r_e$

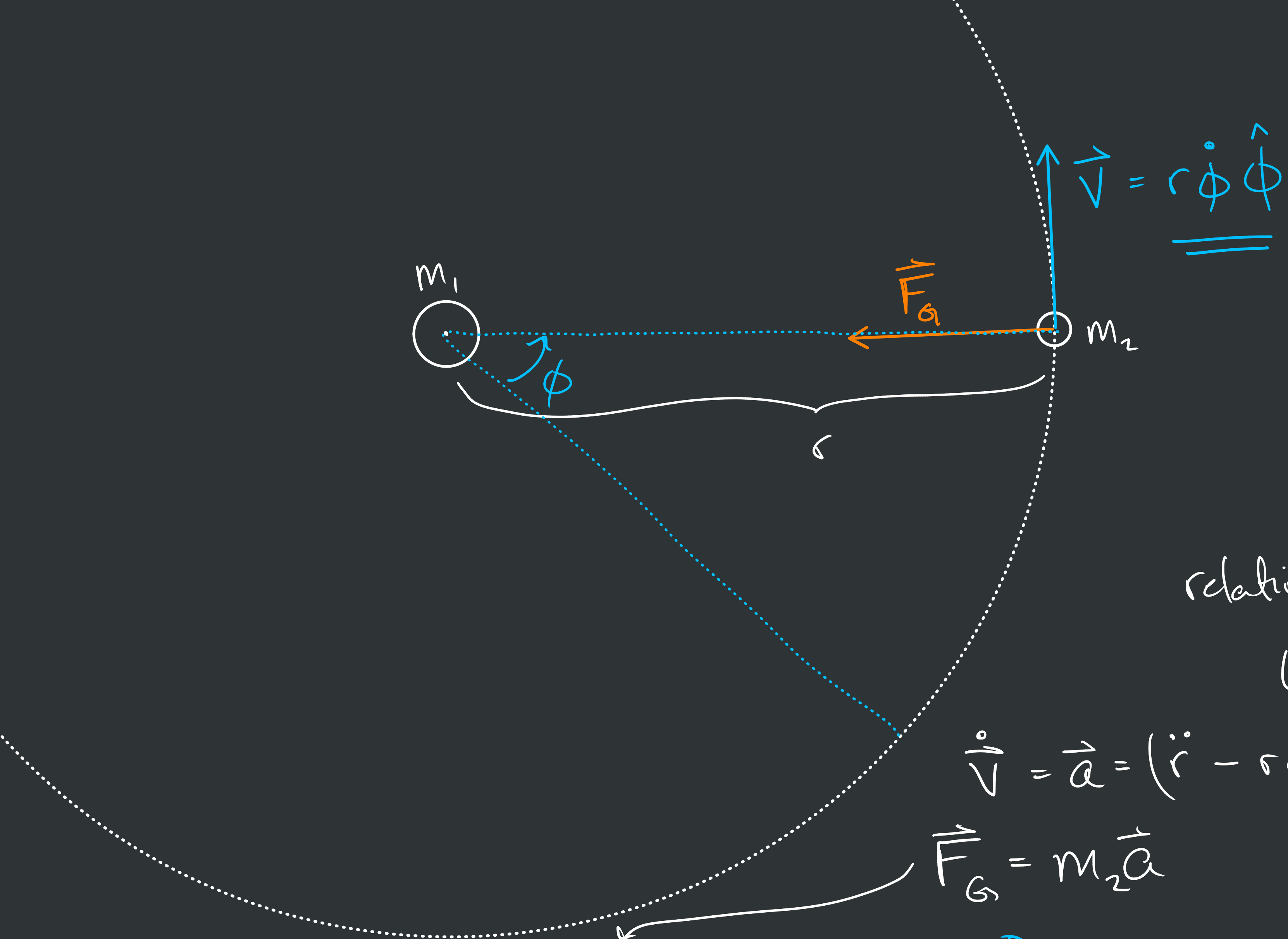
$= gmh = mgh$

→ Another approximation

$$\frac{1}{1+\epsilon} = (1+\epsilon)^{-1} \approx 1 - \epsilon$$

$\frac{h}{r_e}$

in general $\Rightarrow (1+\epsilon)^n \approx 1 + n\epsilon$



$$U = \frac{-G m_1 m_2}{r}$$

$$T = \frac{1}{2} m_1 v^2$$

relationship between

$$U + T$$

$$\dot{\vec{v}} = \vec{a} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi}$$

$$\vec{F}_G = m_2 \vec{a}$$

$$-\frac{G m_1 \cancel{m_2}}{r^2}\hat{r} = \cancel{m_2}\left[(\cancelto{0}{\ddot{r}} - r\dot{\phi}^2)\hat{r} + (\cancelto{0}{2\dot{r}}\dot{\phi} + r\ddot{\phi})\hat{\phi}\right]$$

$$\underbrace{\quad}\hat{r} + \underbrace{0}\hat{\phi} = \underbrace{\quad}\hat{r} + \underbrace{r\ddot{\phi}}_{=0}\hat{\phi}$$

$$\ddot{\phi} = 0$$

$$-\frac{Gm_1}{r^2}\hat{r} = -r\dot{\phi}^2\hat{r}$$

$$\frac{Gm_1}{r^2} = r\dot{\phi}^2 \longleftarrow \frac{v}{r} = \dot{\phi} = \omega$$

$$\frac{Gm_1}{r^{\cancel{2}}} = \frac{v^2}{\cancel{r}} \rightsquigarrow \underline{v = \left(\frac{Gm_1}{r}\right)^{1/2}}$$

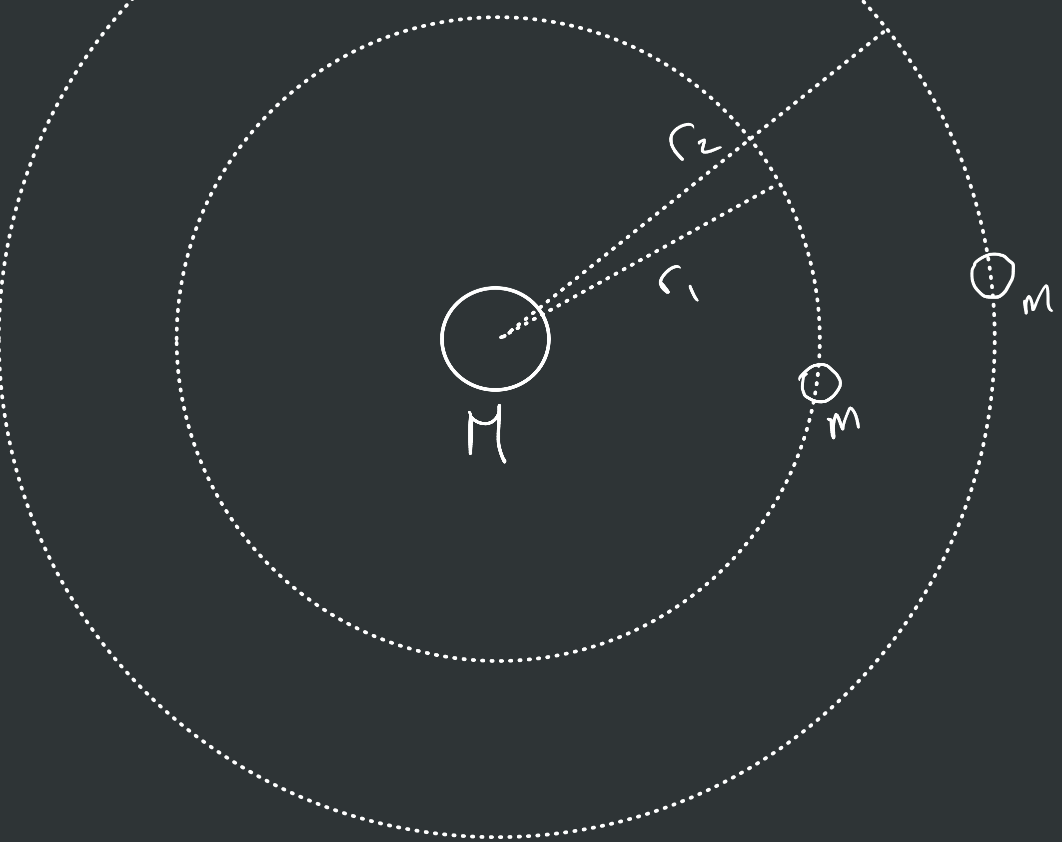
$$U = \frac{-Gm_1m_2}{r}$$

$$T = \frac{1}{2}m_2v^2 = \frac{1}{2}m_2\left(\frac{Gm_1}{r}\right) = \frac{Gm_1m_2}{2r}$$

$$T = \frac{-U}{2}$$

$$E = T + U$$

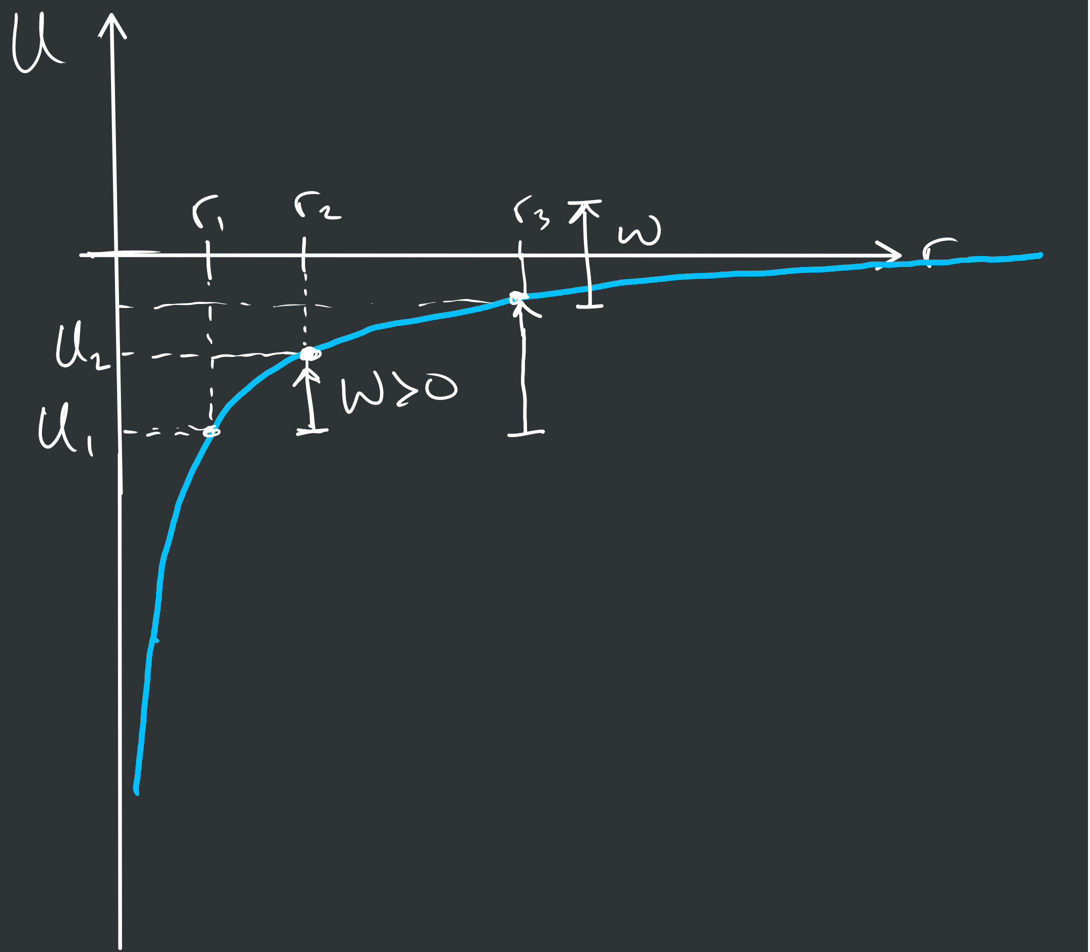
$$E = -\frac{U}{2} + U = \frac{U}{2}$$



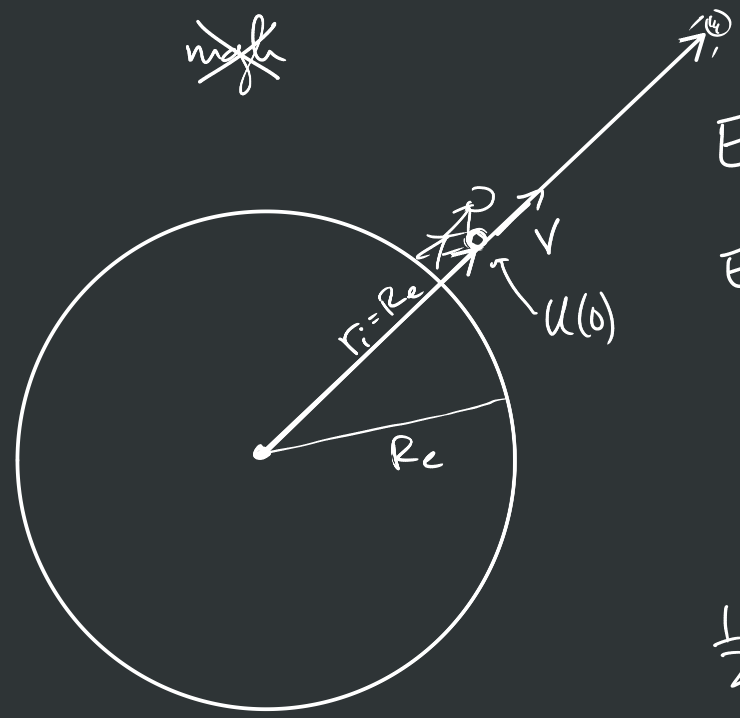
planet at r_1

$$U_1 = -\frac{GMm}{r_1}$$ potential energy relative to really far away

$$E = -\frac{GMm}{2r_1}$$



$\cancel{mgh}$



$$E_i = \frac{1}{2} m v_i^2 + 0 \leftarrow G m_1 m_2 \left(\frac{1}{R_e} - \frac{1}{r}\right) \quad r_i = R_E$$

$$E_f = \frac{1}{2} m v_f^2 + GMm \left(\frac{1}{R_e} - \frac{1}{r}\right)$$

$r > R_e$

$$\underline{E_i = E_f} \leftarrow \text{Conservation of energy}$$

$$\frac{1}{2} m v_i^2 = \frac{1}{2} m v_f^2 + GMm \left(\frac{1}{R_e} - \frac{1}{r}\right)$$

v_i: $\to v \to$ escape velocity

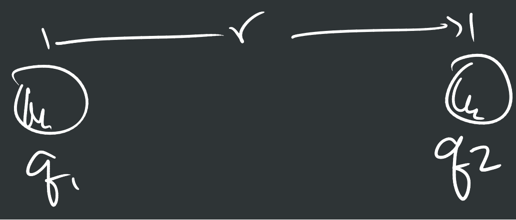
v_f: $\to\ \to 0$

r: $r \to \infty$

G: $\to 6.67 \cdot 10^{-11}$

$$\frac{1}{2} \cancel{m} v_e^2 = \frac{GM\cancel{m}}{R_e}$$

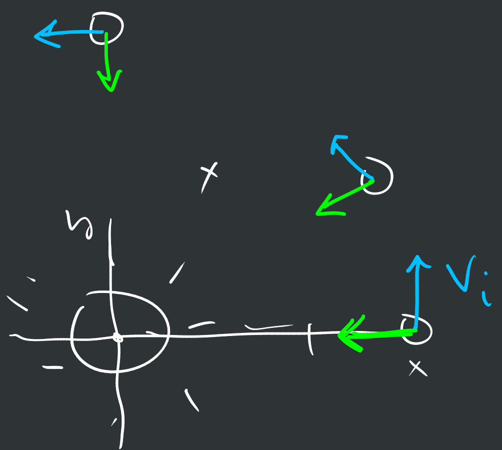
$$V_e = \left(\frac{2GM}{R_e}\right)^{1/2}$$



Emma's Project

$$\cancel{m_1}\ddot{\vec r}_1 = -\frac{G m_2 \cancel{m_1} |\vec r_1 - \vec r_2|}{|\vec r_1 - \vec r_2|^3} - \underbrace{G m_3 \cdots}$$

$$\vec F_g = -\frac{G \cancel{m_1} m_2 (x\hat x + y\hat y)}{(x^2+y^2)^{3/2}} = \cancel{m_1}(\underbrace{\ddot x \hat x + \ddot y \hat y}_{\ddot r})$$



fixed sun

$$\vec F_g = -\frac{G \cancel{m_2} m_1 (x\hat x + y\hat y)}{(x^2+y^2)^{3/2}} = \cancel{m_2}\ddot x \hat x + \ddot y \hat y$$

$$\boxed{-\frac{G m_2 x}{(x^2+y^2)^{3/2}}} = \ddot x \qquad \Bigg| \qquad \boxed{-\frac{G m_2 y}{(x^2+y^2)^{3/2}}} = \ddot y$$

$\vec{r}_1 = (0,0,0)$ $\quad$ $\vec{V}_1 = (-1,1,0)$ $\quad$ $m_1 = m_2 = m \quad m_3 = 10m$

$\vec{r}_2 = (1,1,0)$ $\quad$ $\vec{V}_2 = (-1,0,1)$ $\quad$ $R_x = \frac{0 + m + 10m}{12m} = \frac{11}{12} = 0.916 \;\Big|\; R_y = \frac{m - 10m}{12m} = -\frac{9}{12} = -0.75 \;\Big|\; R_z = \frac{-10m}{12m} = -\frac{5}{6}$

$\vec{r}_3 = (1,-1,-1)$ $\quad$ $\vec{V}_3 = (0,0,0)$ $\quad$ $\vec{R} = (0.916, -0.75, -0.833)$

$$\vec{V}_{cm} = \dot{\vec{R}} = \frac{1}{M}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha$$

$$V_{cm,x} = \frac{m\cdot(-1) + m(-1) + 10m(0)}{12m}$$

$$= \frac{-2m}{12m} = -\frac{1}{6}\ m/s$$

$$V_{cm,y} = \frac{1}{12}\ m/s$$

$$V_{cm,z} = \frac{1}{12}\ m/s$$

$$\vec{V}_{cm} = \left(-1/6,\ 1/12,\ 1/12\right)$$

$$\vec{p}_1 = m(-1,1,0)$$
$$= (-m, m, 0)$$
$$\vec{p}_2 = (-m, 0, m)$$
$$\vec{p}_3 = (0,0,0)$$

$$\sum_\alpha \vec{p}_\alpha = (-2m, m, m)$$

$$\vec{P}_{cm} = M\cdot\vec{V}_{cm} = 12m\left(-1/6,\ 1/12,\ 1/12\right)$$

$\vec{l} = \vec{r} \times \vec{p}$

$\longrightarrow = \vec{\mathbb{P}}_{cm} = (-2m, m, m)$ ✓

$l_1 = (0,0,0) \times (-m, m, 0) = 0$

$l_2 = (1,1,0) \times (-m, 0, m) = (m, -m, m)$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 0 \\ -m & 0 & m \end{vmatrix} = m\hat{x} - m\hat{y} + m\hat{z}$$

$l_3 = (1,-1,-1) \times (0,0,0) = 0$

$$\sum_\alpha l_\alpha = (m, -m, m)$$

1s later

$r_1 = (-1,1,0)$

$r_2 = (0,1,1)$

$r_3 = (0,1,1)$

r's changed

$\mathbb{R}_{cm}$ changed

p's not changed

$\mathbb{P}_{cm}$ not changed

l not changed

$\mathbb{L}$ not changed

$l_1 = (-1,1,0) \times (-m, m, 0) = 0$

$l_2 = (1,1,0) \times (-m, 0, m) = (m, -m, m)$

$l_3 = 0$

Reference System change

$$\vec{V}' = \vec{V} - \vec{V}^{S\to S'}$$

$$\left.\begin{aligned} V'_{1x} &= -1 - (-1/6) = -5/6 \\ V'_{1y} &= 1 - (1/12) = 11/12 \\ V'_{1z} &= 0 - (1/12) = -1/12 \end{aligned}\right\} \quad \vec{V}'_1 = \left(-5/6,\ 11/12,\ -1/12\right)$$

$$\vec{V}'_2 = \left(-5/6,\ -1/12,\ 11/12\right)$$

$$\vec{V}'_3 = \left(1/6,\ -1/12,\ -1/12\right)$$

$$p_{1x} + p_{2x} + p_{3x}$$

$$m(-5/6) + m(-5/6) + 10m(1/6) = 0$$

$$\mathbb{P} = 0$$

$$V'_{cm,x} = \frac{m(-5/6) + m(-5/6) + 10m(1/6)}{12m}$$

$$= \frac{-5/6 - 5/6 + 10/6}{12} = 0$$

$$\vec{V}_{cm} = 0$$

$$\vec{r}\,' = \vec{r} - \vec{r}^{\,s\to s'}$$

$$\vec{r}_1' = \left(0 - \tfrac{11}{12},\ 0 - \left(-\tfrac{3}{4}\right),\ 0 - \left(-\tfrac{5}{6}\right)\right)$$

$$\vec{r}_1' = \left(-\tfrac{11}{12},\ \tfrac{3}{4},\ \tfrac{5}{6}\right)$$

$$r_2' = \left(1 - \tfrac{11}{12},\ 1 + \tfrac{3}{4},\ 0 + \tfrac{5}{6}\right) = \left(\tfrac{1}{12},\ \tfrac{7}{4},\ \tfrac{5}{6}\right)$$

$$r_3' = \left(1 - \tfrac{11}{12},\ -1 + \tfrac{3}{4},\ -1 + \tfrac{5}{6}\right) = \left(\tfrac{1}{12},\ -\tfrac{1}{4},\ -\tfrac{1}{6}\right)$$

$$l_1' = r_1' \times p_1' = \left(-\tfrac{11}{12},\ \tfrac{3}{4},\ \tfrac{5}{6}\right) \times \left(-\tfrac{5}{6}m,\ \tfrac{11}{12}m,\ -\tfrac{1}{12}m\right)$$

$$= \left(-\frac{119m}{144},\ -\frac{37m}{48},\ -\frac{31m}{144}\right)$$

$$l_2' = \left(\tfrac{1}{12},\ \tfrac{7}{4},\ \tfrac{5}{6}\right) \times \left(-\tfrac{5}{6}m,\ -\tfrac{1}{12}m,\ \tfrac{11}{12}m\right) = \left(\frac{241m}{144},\ -\frac{37m}{48},\ \frac{209m}{144}\right)$$

$$l_3' = \left(\tfrac{1}{12},\ -\tfrac{1}{4},\ -\tfrac{1}{6}\right) \times \left(\tfrac{10}{6}m,\ -\tfrac{10}{12}m,\ -\tfrac{10}{12}m\right) = \left(\frac{5m}{72},\ -\frac{5}{24}m,\ \frac{25m}{72}\right)$$

$$\mathbb{L}' = \left(\frac{11m}{12},\ -\frac{7m}{4},\ \frac{19m}{12}\right)$$

$$\frac{Gm_1 \cancel{m_2}}{r^{\cancel{2}}} = \frac{\cancel{m_2} v^2}{\cancel{r}}$$

$$v = \sqrt{\frac{Gm_1}{r}} \qquad v = \sqrt{\frac{1 \cdot 100}{1}} = 10$$