Chapter 6 + 7 → Principle of Least Action and the Lagrangian

$\boxed{\text{Newton's Laws}}$ — $\boxed{\text{Lagrangian}}$ — $\boxed{\text{Hamiltonian}}$

→ "generalized coordinates"

→ Principle of Least Action

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$U = mgx$$

$$\underbrace{\int_{t_1}^{t_2} \Big(\tfrac{1}{2}m\underset{\substack{\uparrow \\ v(t)}}{\dot{x}^2} - mg\underset{\substack{\uparrow \\ x(t)}}{x} \Big)\, dt}_{\text{what is } x(t) \text{ and } \dot{x}(t) \text{ so this integral is smallest}}$$

Lagrangian

$$\mathcal{L} = T - U$$

Action

$$S = \int_{t_1}^{t_2} \mathcal{L}\big(x(t), \dot{x}(t), t\big)\, dt$$

$$= \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t)\, dt$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}}$$

Euler - Lagrange Equation

$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}mv^2$$

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 - U(x,y,z)$$

↳ 1D motion

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - U(x)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - U(x)\right)\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2 - U(x)\right)$$

$$\frac{d}{dt}(m\dot{x}) = -\frac{dU}{dx}$$

$$m\ddot{x} = -\frac{dU}{dx}$$

$\longrightarrow$ In 3D

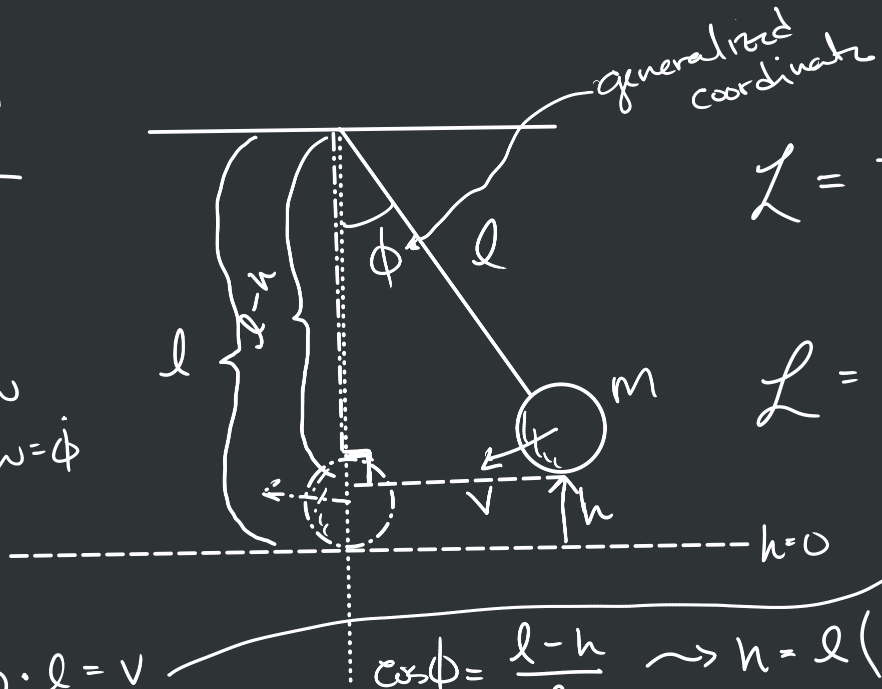
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x} \quad\Bigg|\quad y: \ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = \frac{\partial \mathcal{L}}{\partial y} \quad\Bigg|\quad z: \ \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) = \frac{\partial \mathcal{L}}{\partial z}$$

$$m\ddot{x} = -\frac{\partial U}{\partial x} \qquad m\ddot{y} = -\frac{\partial U}{\partial y} \qquad m\ddot{z} = -\frac{\partial U}{\partial z}$$

$$\underbrace{m\ddot{x}\,\hat{x} + m\ddot{y}\,\hat{y} + m\ddot{z}\,\hat{z}} = -\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}$$

$$\underbrace{\Sigma\vec{F} = m\ddot{\vec{r}}} = \underbrace{-\vec{\nabla}U = \Sigma\vec{F}}$$

Ex:



$v = r\omega$

$\omega = \dot{\phi}$

$\dot{\phi}\cdot l = v$

$\cos\phi = \frac{l-h}{l} \rightsquigarrow h = l(1-\cos\phi)$

$$\mathcal{L} = T - U$$

$T \leftarrow \dot{\phi}$, $U \leftarrow \phi$

$$\mathcal{L} = \frac{1}{2}mv^2 - mgh$$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1-\cos\phi)$$

$\hookrightarrow$ apply the Euler-Lagrange equation