Chapter 4 ⇝ Energy

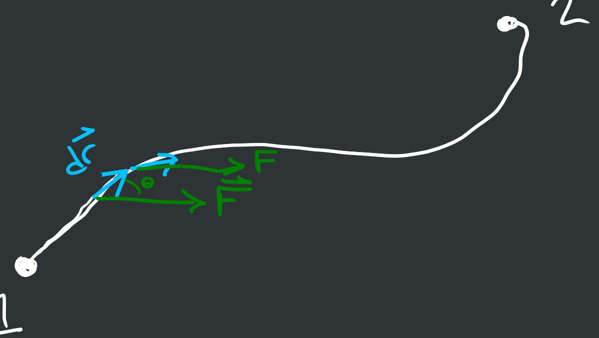
- Kinetic Energy

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{p^2}{2m}$$

- Work is a transfer of energy to the system

$$dW = \vec{F}\cdot d\vec{r} \Rightarrow W = \int_1^2 \vec{F}\cdot d\vec{r}$$

$|\vec{F}|\cdot|d\vec{r}|\cdot\cos\theta$



path integral means work depends on the path between points, unless it doesn't

- Amount of work done (total) is the same as the change in kinetic energy

$W = \Delta T$ ← Work-Kinetic energy theorem

- If the work done does not depend on the path taken, (it depends on the end points (positions)), that force is a conservative force

→ work done by a conservative force ⇒ potential energy

$$\Delta U = -W_{\substack{\text{conservative} \\ \text{force}}}$$

Conservative forces	nonconservative force
• gravitational force	• friction
• electric force	• drag
• spring force	• "applied force"

potential energy is defined by a reference point ($\vec{r}$ when $U=0$)

- Conservation of energy

$$W = \Delta K$$

$$W_{NC} = \underbrace{\Delta K + \Delta U}_{\text{mechanical energy}}$$

$$U_g = mgh$$

$$W_{NC} = T_f - T_i + U_f - U_i$$

$$T_i + U_i + W_{NC} = T_f + U_f$$

- if $\Delta U = -\int_1^2 \vec{F} \cdot d\vec{r} \Rightarrow$

$$dU = -W = -\vec{F} \cdot d\vec{r}$$

$$\cancel{\frac{dU}{d\vec{r}} = -\vec{F}}$$

$$\vec{F} = -\vec{\nabla} U = \frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}$$

gradient → cartesian

$$F = -\frac{dU}{dx}$$ ← one dimension

- What make a F conservative
  - F depends only on position (not v, t, a)
  - Work is path-independent
    - → $\vec{\nabla} \times \vec{F} = 0$
      - curl

$F_s = -kx$

→ $F_g = mg$

$F_G = \frac{G m_1 m_2}{r^2}$

$F_D = bv$

$= cv^2$

- everything above can be done in spherical coords. or cylindrical coords → often this will be easier central forces

$$\vec{F}(\vec{r}) = f(\vec{r})\,\hat{r}$$

$\hat{r}$ = radial force

$$\vec{F}_G(\vec{r}) = -\frac{Gm_1m_2}{r^2}\,\hat{r} \quad \leftarrow \text{in spherical coords.}$$

$$\vec{r} = x\hat{x} + y\hat{y} \qquad \hat{r} = \frac{\vec{r}}{|r|}$$

$$r = \sqrt{x^2 + y^2}$$

$$= -\left(\frac{Gm_1m_2}{x^2+y^2}\right) \cdot \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2+y^2}}$$

$$= -\frac{Gm_1m_2}{(x^2+y^2)^{3/2}} \cdot (x\hat{x} + y\hat{y})$$

$$\vec{\nabla} \times \vec{F}_G =$$

what is curl in spherical?

- All of this applies multiple particle

$$T = T_1 + T_2 + T_3 + \dots$$

$$T = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I \omega_{cm}^2$$

rot. inertia

Recall:

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

$$\frac{dT}{dt} = \frac{1}{2} m \cdot \frac{d}{dt}\left(\vec{v} \cdot \vec{v}\right)$$

$$= \frac{1}{2} m \left(\frac{d(\vec{v})}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d(\vec{v})}{dt} \right)$$

$$\Delta T = W$$

$$= \frac{1}{2} m \left(2 \dot{\vec{v}} \cdot \vec{v} \right)$$

$$= m \dot{\vec{v}} \cdot \vec{v}$$

$$\dot{\vec{p}} = \vec{F}$$

$$\frac{dT}{dt} = \vec{F} \cdot \vec{v}$$

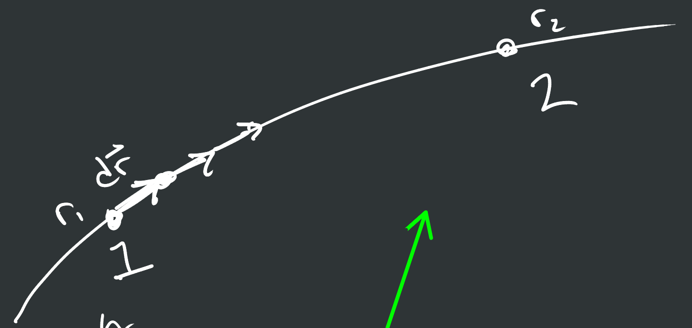
$$\frac{dT}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

$$dT = \underbrace{\vec{F} \cdot d\vec{r}}_{\text{define this as a small amount of work}}$$

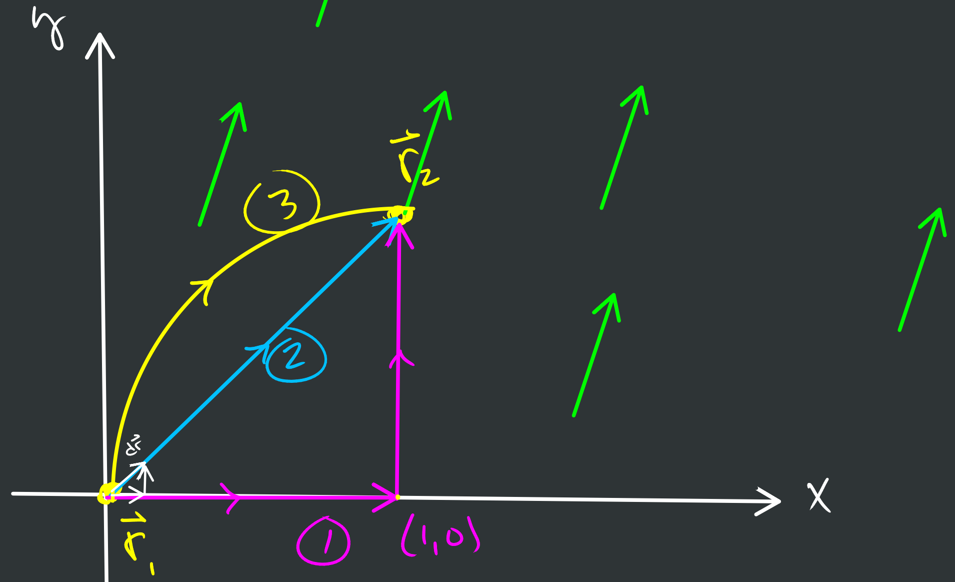
define this as a small amount of work

- dW does not make sense.
- $\rightarrow đW$ (in thermo)

$$\Delta T = T_2 - T_1 = \underbrace{\int_1^2 \vec{F} \cdot d\vec{r}}_{\text{line integral}}$$



$$\int f(x)\,dx$$



Ex: $\vec{F} = 1\hat{x} + 2\hat{y}$

$\vec{r}_1 = 0$

$\vec{r}_2 = 1\hat{x} + 1\hat{y}$

Path ①

$$\int_1^2 \vec{F}\cdot d\vec{r} = \int_{(0,0)}^{(1,0)} \vec{F}\cdot d\vec{r}_1 + \int_{(1,0)}^{(1,1)} \vec{F}\cdot d\vec{r}_2$$

$d\vec{r}_1 = d\vec{x} = dx\,\hat{x}$

$\vec{F}\cdot d\vec{x} = (1\hat{x} + 2\hat{y})\cdot(dx\,\hat{x} + 0\hat{y})$

$= 1\,dx$

$d\vec{r}_2 = dy\,\hat{y}$

$\vec{F}\cdot dy\,\hat{y} = (1\hat{x} + 2\hat{y})\cdot(0\hat{x} + dy\,\hat{y})$

$= 2\,dy$

$$\int_1^2 \vec{F}\cdot d\vec{r} = \int_{x=0}^{x=1} dx + \int_{y=0}^{y=1} 2\,dy$$

$$= x\Big|_0^1 + 2y\Big|_0^1$$

Work along path ①

$$= 1 + 2 = 3$$

Path ②

$$\int_1^2 \vec{F}\cdot d\vec{r} =$$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y}$$

$$\vec{F}\cdot d\vec{r} = (1\,\hat{x} + 2\hat{y})\cdot(dx\,\hat{x} + dy\,\hat{y})$$

$$= dx + 2dy$$

$$\int_1^2 (dx + 2dy) = \int_0^1 dx + \int_0^1 2\,dy = 3$$

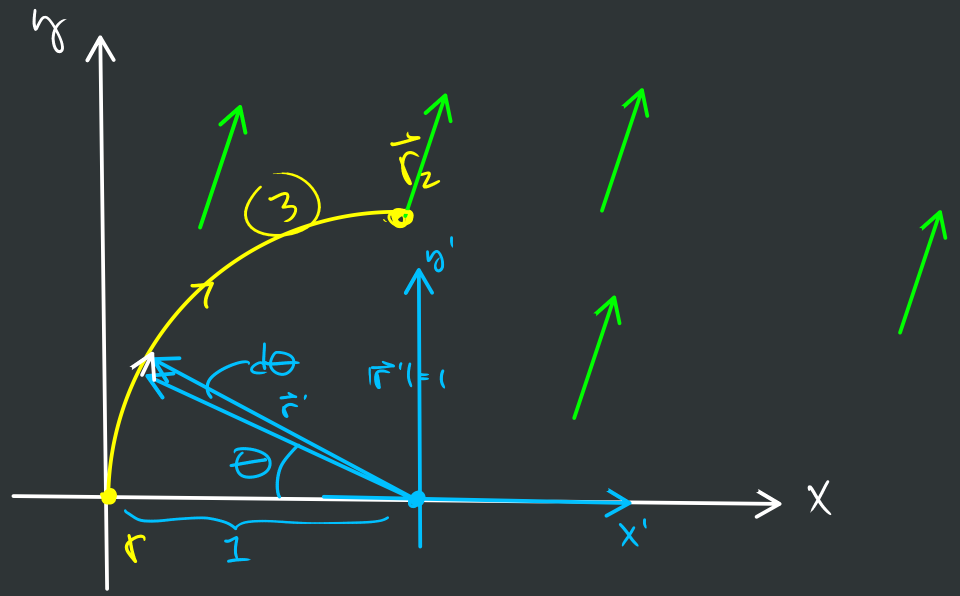
Path ③

$$\int_1^2 \underbrace{\vec{F}\cdot d\vec{r}}$$

$$\int_1^2 \left(1\hat{x} + 2\hat{y}\right)\cdot\left(\sin\theta\, d\theta\, \hat{x} + \cos\theta\, d\theta\, \hat{y}\right)$$

$$\int_1^2 \left(\sin\theta\, d\theta + 2\cos\theta\, d\theta\right)$$

$$\int_0^{\pi/2} \sin\theta\, d\theta + \int_0^{\pi/2} 2\cos\theta\, d\theta = 3$$



$\vec{r}\,' = -\cos\theta\, \hat{x}' + \sin\theta\, \hat{y}'$

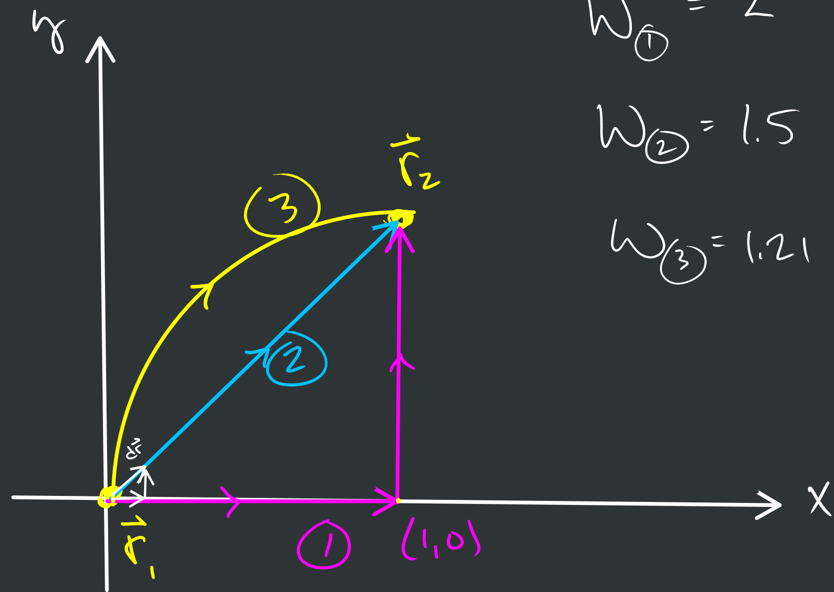
↓ change coordinate systems

$\vec{r} = (1-\cos\theta)\hat{x} + \sin\theta\, \hat{y}$

$d\vec{r} = \sin\theta\, d\theta\, \hat{x} + \cos\theta\, d\theta\, \hat{y}$

Homework: Same 3 paths

$$\vec{F} = y\hat{x} + 2x\hat{y}$$



$W_{1} = 2$

$W_{2} = 1.5$

$W_{3} = 1.21$

$$\vec{r} = (1-\cos\theta)\hat{x} + \sin\theta\,\hat{y}$$

$$d\vec{r} = \sin\theta\, d\theta\,\hat{x} + \cos\theta\, d\theta\,\hat{y}$$

$$W_{1} = \int_{(0,0)}^{(1,0)} \vec{F}\cdot \underbrace{d\vec{r}_1}_{dx\,\hat{x}} + \int_{(1,0)}^{(1,1)} \vec{F}\cdot \underbrace{d\vec{r}_2}_{dy\,\hat{y}}$$

$$(y\hat{x} + 2x\hat{y})\cdot(dx\,\hat{x} + 0\,\hat{y})$$

$$y\,dx + 0$$

$$= \int_{(0,0)}^{(1,0)} y\,dx + \int_{(1,0)}^{(1,1)} 2x\,dy$$

$$= \underbrace{\int_{x=0}^{x=1} 0\cdot dx}_{0} + \int_{y=0}^{y=1} 2\cdot(1)\,dy = 2y\Big|_0^1 = 2(1-0) = 2$$

$$W_{3} = \int_0^{\pi/2} y\sin\theta\, d\theta + \int_0^{\pi/2} 2x\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\, d\theta + 2\int_0^{\pi/2} (1-\cos\theta)\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\, d\theta + 2\int_0^{\pi/2} \cos\theta\, d\theta - 2\int_0^{\pi/2} \cos^2\theta\, d\theta$$

For <u>conservative</u> <u>force</u>

$$U(\vec{r}) = -W(\vec{r}_0 \to \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$$

Potential energy at $\vec{r}$ in reference to $\vec{r}_0$

conditions for conservative force

① $\vec{F}$ only depends on $\vec{r}$ (not $\vec{v}$, not t) — time

② Work done b/t any two points is independent of path taken

$(\vec{\nabla} \times \vec{F} = 0)$

What about $W(\vec{r}_1 \to \vec{r}_2)$ ← work to go between points not the reference point

$$W(\vec{r}_0 \to \vec{r}_2) = W(\vec{r}_0 \to \vec{r}_1) + W(\vec{r}_1 \to \vec{r}_2)$$

$$W(\vec{r}_1 \to \vec{r}_2) = \underbrace{W(\vec{r}_0 \to \vec{r}_2)}_{-U(\vec{r}_2)} - \underbrace{W(\vec{r}_0 \to \vec{r}_1)}_{-U(\vec{r}_1)}$$

$$= -\underbrace{\left(U(\vec{r}_2) - U(\vec{r}_1)\right)}_{\Delta U}$$

$$\underline{\underline{\Delta U}} = -W(\vec{r}_1 \to \vec{r}_2)$$

Now go back to work-kinetic energy theorem

$$W(\vec{r}_1 \to \vec{r}_2) = \Delta K$$

if only conservative forces are acting

$$-\Delta U = \Delta K$$

$$\Delta K + \Delta U = 0$$

$$E = K + U \quad \Big\} \text{ Mechanical Energy}$$

$$\Delta E = 0$$

Also true for multiple conservative forces.

→ multiple potential energy

$$U = U_g + U_s \; (+ U_e)$$

But, what if also non-conservative forces act?

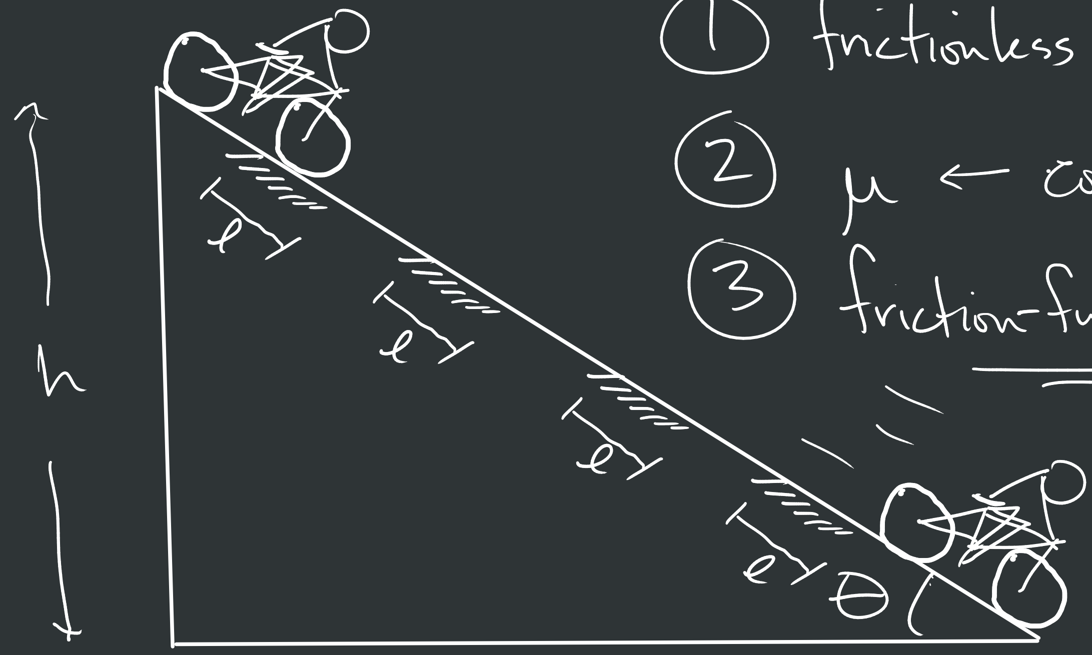
$$\Delta K + \Delta U \neq 0$$

Friction

$$\Delta K + \Delta U = W_{NC} \qquad W_f = \int \vec{F}_f \cdot d\vec{x}$$

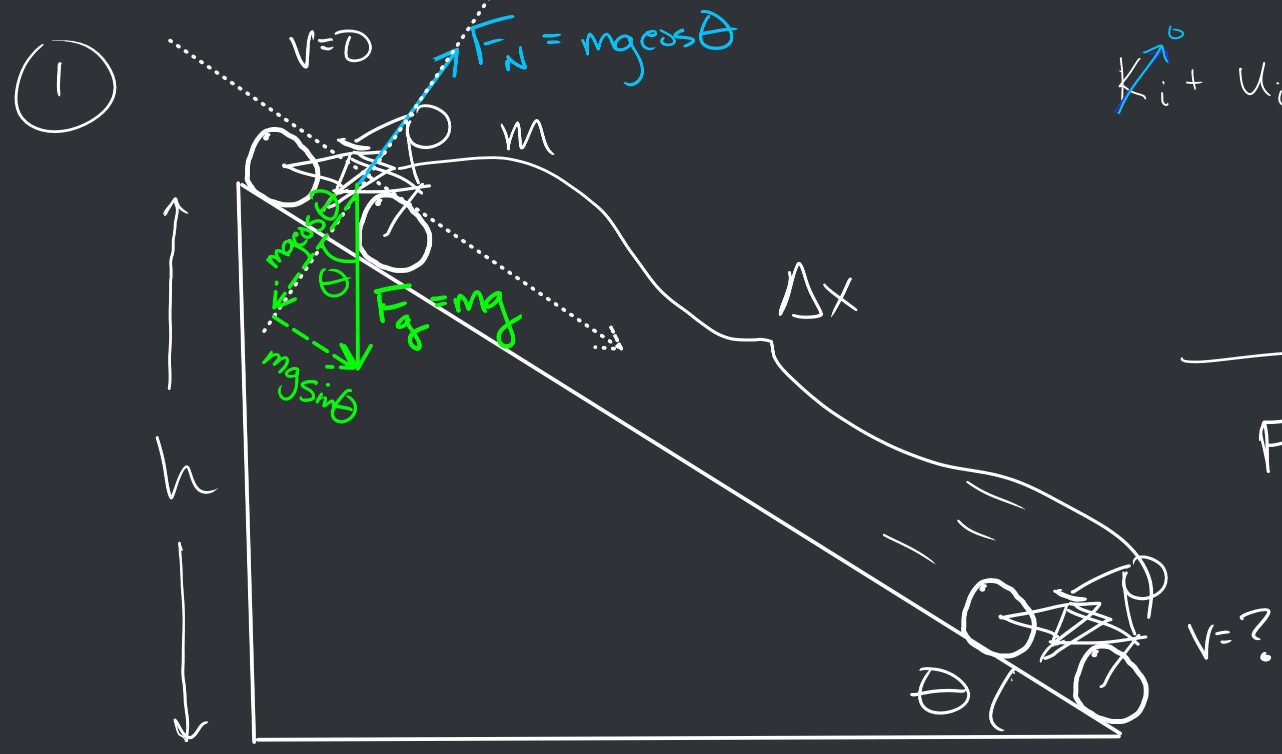
$$K_i + U_i + W_{NC} = K_f + U_f$$

Ex:



1. frictionless plane
2. $\mu \leftarrow$ constant
3. friction-full patches $\leftarrow$ no Newton's

Newton's Laws + Conservation of Energy



$$\cancel{K_i}^{\,0} + U_i + \cancel{W_{NC}}^{\,0} = K_f + \cancel{U_f}^{\,0}$$

$$\cancel{m}gh = \frac{1}{2}\cancel{m}v^2$$

$$v = \sqrt{2gh}$$

$$F_{NET,X} = \cancel{m}g\sin\theta = ma = \dot{p}$$

$$g\sin\theta = a$$

$$g\sin\theta = \frac{dv}{dt}$$

$$\int_0^t g\sin\theta\, dt = \int_{v_0=0}^{v} dv'$$

$$g\sin\theta \cdot t = v \qquad v_f = \cancel{v_i}^{\,0} + at$$

$$v = g\sin\theta\, t$$

$$\int_0^x dx = \int_0^t g\sin\theta\, t'\,dt'$$

$$x = \frac{g\sin\theta\, t^2}{2}$$

$$g\sin\theta = \frac{dv}{dt} = \frac{dv}{dx}\cdot\underbrace{\frac{dx}{dt}}_{v}$$

new DE:

$$\boxed{g\sin\theta = v\frac{dv}{dx}}$$

$$\int g\sin\theta\, dx = \int v\, dv$$

$$v_f^2 = v_i^2 + 2g\sin\theta\,\Delta x$$