Chapter 4 $\rightsquigarrow$ Energy

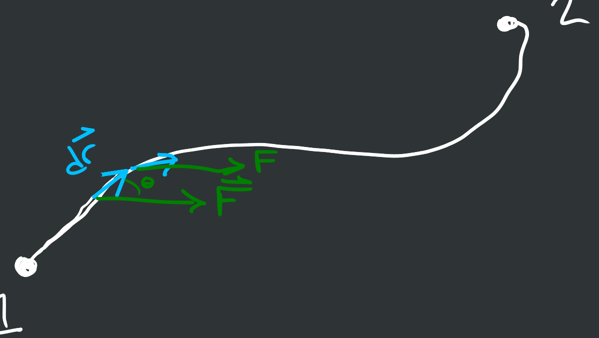
- Kinetic Energy

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{p^2}{2m}$$

- Work is a transfer of energy to the system

$$dW = \vec{F}\cdot d\vec{r} \Rightarrow W = \int_1^2 \vec{F}\cdot d\vec{r}$$

$|\vec{F}|\cdot|d\vec{r}|\cdot\cos\theta$



path integral means work depends on the path between points, unless it doesn't

- Amount of work done (total) is the same as the change in kinetic energy

$W = \Delta T$ ← Work-Kinetic energy theorem

- If the work done does not depend on the path taken, (it depends on the end points (positions)), that force is a conservative force

→ work done by a conservative force ⇒ potential energy

$$\Delta U = -W_{\substack{\text{conservative} \\ \text{force}}}$$

Conservative forces
- gravititational force
- electric force
- spring force

nonconservative force
- friction
- drag
- "applied force"

* potential energy is defined by a reference point ($\vec{r}$ when $U = 0$)

- Conservation of energy

$$W = \Delta K$$

$$W_{NC} = \underbrace{\Delta K + \Delta U}_{\text{mechanical energy}}$$

$$U_g = mgh$$

$$W_{NC} = T_f - T_i + U_f - U_i$$

$$T_i + U_i + W_{NC} = T_f + U_f$$

- if $\Delta U = -\int_1^2 \vec{F} \cdot d\vec{r} \Rightarrow$

$$dU = -W = -\vec{F} \cdot d\vec{r}$$

$$\cancel{\frac{dU}{d\vec{r}} = -\vec{F}}$$

$$\vec{F} = -\vec{\nabla} U = \frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}$$

gradient → cartesian

$$F = -\frac{dU}{dx}$$ ← one dimension

- What make a F conservative
  - F depends only on position (not v, t, a)
  - Work is path-independent

    ↳ $\vec{\nabla} \times \vec{F} = 0$

    ← curl

$F_s = -kx$

→ $F_g = mg$

$F_G = \frac{G m_1 m_2}{r^2}$

$F_D = bv$

$= cv^2$

- everything above can be done in spherical coords. or cylindrical coords → often this will be easier

Central forces

$$\vec{F}(\vec{r}) = f(\vec{r})\,\hat{r}$$

$\uparrow$ radial force

$$\vec{F}_G(\vec{r}) = -\frac{G m_1 m_2}{r^2}\,\hat{r} \quad \leftarrow \text{in spherical coords.}$$

$$\vec{r} = x\hat{x} + y\hat{y} \qquad \hat{r} = \frac{\vec{r}}{|r|}$$

$$r = \sqrt{x^2 + y^2}$$

$$= -\left(\frac{G m_1 m_2}{x^2 + y^2}\right) \cdot \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2+y^2}}$$

$$= -\frac{G m_1 m_2}{(x^2+y^2)^{3/2}} \cdot (x\hat{x} + y\hat{y})$$

$$\vec{\nabla} \times \vec{F}_G =$$

what is curl in spherical?

- All of this applies multiple particle

$$T = T_1 + T_2 + T_3 + \ldots$$

$$T = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I \omega_{cm}^2$$

rot. inertia

Recall:

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

$$\frac{dT}{dt} = \frac{1}{2} m \cdot \frac{d}{dt}\left(\vec{v} \cdot \vec{v}\right)$$

$$= \frac{1}{2} m \left(\frac{d}{dt}(\vec{v}) \cdot \vec{v} + \vec{v} \cdot \frac{d}{dt}(\vec{v}) \right)$$

$$\Delta T = W$$

$$= \frac{1}{2} m \left(2 \dot{\vec{v}} \cdot \vec{v} \right)$$

$$= m \dot{\vec{v}} \cdot \vec{v}$$

$$\dot{\vec{p}} = \vec{F}$$

$$\frac{dT}{dt} = \vec{F} \cdot \vec{v}$$

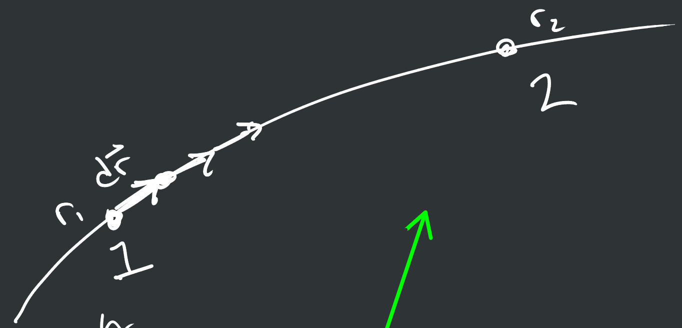
$$\frac{dT}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

$$dT = \underbrace{\vec{F} \cdot d\vec{r}}$$

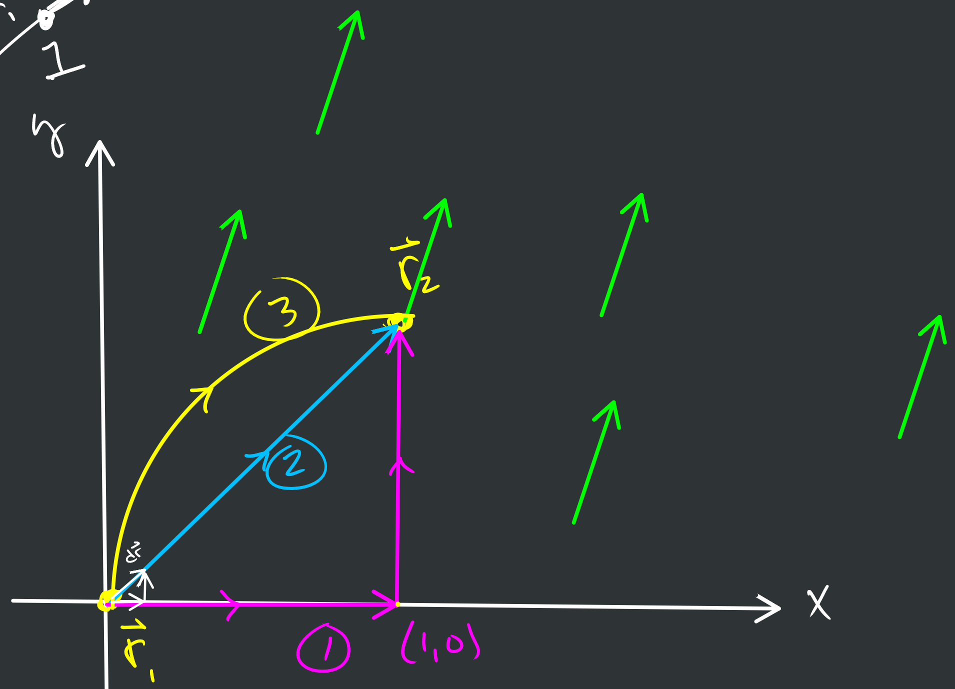
define this as a small amount of <u>work</u>

- dW does not make sense.
- $\rightarrow đW$ (in thermo)

$$\Delta T = T_2 - T_1 = \underbrace{\int_1^2 \vec{F} \cdot d\vec{r}}_{\text{line integral}}$$



$\int f(x)\,dx$



Ex: $\vec{F} = 1\hat{x} + 2\hat{y}$

$\vec{r}_1 = 0$

$\vec{r}_2 = 1\hat{x} + 1\hat{y}$

Path 1

$$\int_1^2 \vec{F}\cdot d\vec{r} = \int_{(0,0)}^{(1,0)} \vec{F}\cdot d\vec{r}_1 + \int_{(1,0)}^{(1,1)} \vec{F}\cdot d\vec{r}_2$$

$d\vec{r}_1 = d\vec{x} = dx\,\hat{x}$

$\vec{F}\cdot d\vec{x} = (1\hat{x} + 2\hat{y})\cdot(dx\,\hat{x} + 0\hat{y})$

$= 1\,dx$

$d\vec{r}_2 = dy\,\hat{y}$

$\vec{F}\cdot dy\hat{y} = (1\hat{x} + 2\hat{y})\cdot(0\hat{x} + dy\,\hat{y})$

$= 2dy$

$$\int_1^2 \vec{F}\cdot d\vec{r} = \int_{x=0}^{x=1} dx + \int_{y=0}^{y=1} 2\,dy$$

$$= x\Big|_0^1 + 2y\Big|_0^1$$

Work along path (1)

$$= 1 + 2 = 3$$

Path (2)

$$\int_1^2 \vec{F}\cdot d\vec{r} =$$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y}$$

$$\vec{F}\cdot d\vec{r} = (1\hat{x} + 2\hat{y})\cdot(dx\,\hat{x} + dy\,\hat{y})$$

$$= dx + 2dy$$

$$\int_1^2 (dx + 2dy) = \int_0^1 dx + \int_0^1 2\,dy = 3$$

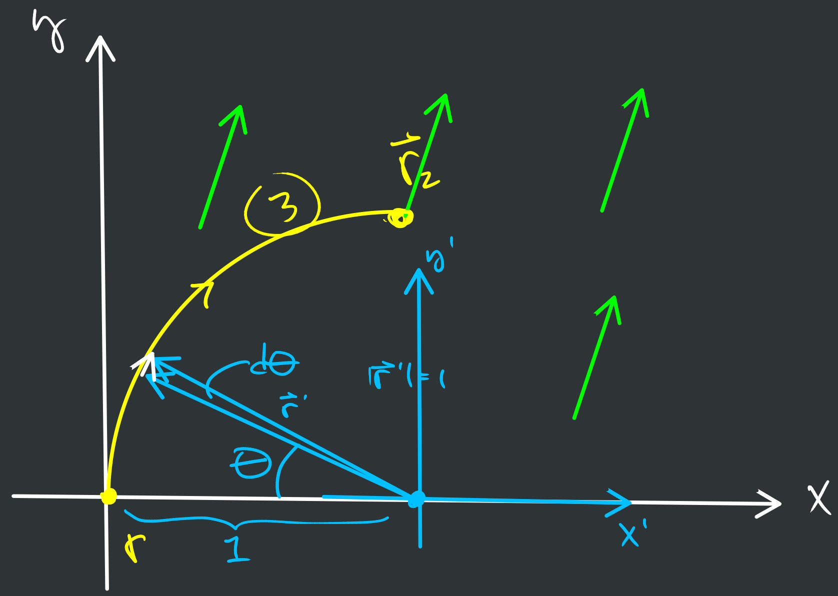
Path 3

$$\int_1^2 \vec{F}\cdot d\vec{r}$$

$$\int_1^2 (1\hat{x} + 2\hat{y})\cdot(\sin\theta\, d\theta\, \hat{x} + \cos\theta\, d\theta\, \hat{y})$$

$$\int_1^2 (\sin\theta\, d\theta + 2\cos\theta\, d\theta)$$

$$\int_0^{\pi/2} \sin\theta\, d\theta + \int_0^{\pi/2} 2\cos\theta\, d\theta = 3$$



$$\vec{r}\,' = -\cos\theta\, \hat{x}' + \sin\theta\, \hat{y}'$$

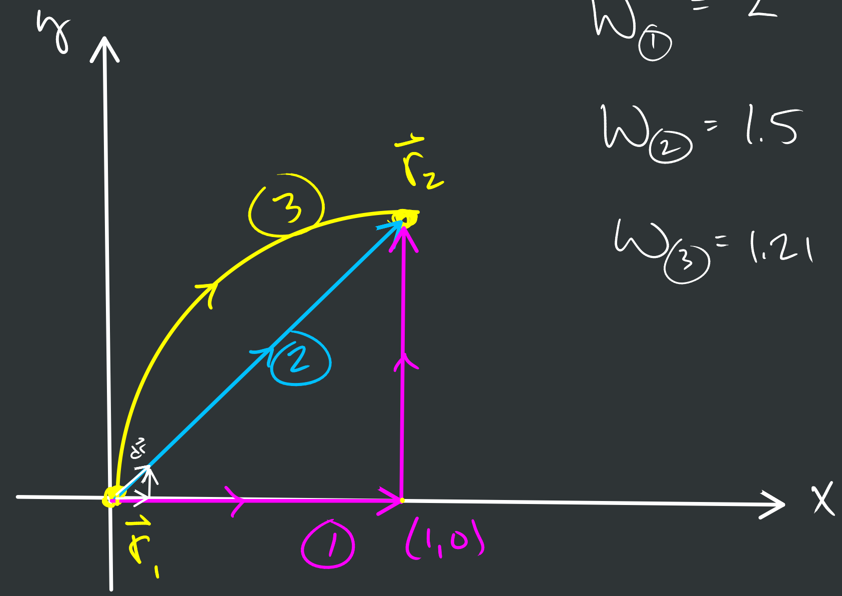
↓ change coordinate systems

$$\vec{r} = (1-\cos\theta)\hat{x} + \sin\theta\, \hat{y}$$

$$d\vec{r} = \sin\theta\, d\theta\, \hat{x} + \cos\theta\, d\theta\, \hat{y}$$

Homework: Same 3 paths

$$\vec{F} = y\hat{x} + 2x\hat{y}$$



$$W_{①} = 2$$

$$W_{②} = 1.5$$

$$W_{③} = 1.21$$

$$\vec{r} = (1-\cos\theta)\hat{x} + \sin\theta\,\hat{y}$$

$$d\vec{r} = \sin\theta\, d\theta\,\hat{x} + \cos\theta\, d\theta\,\hat{y}$$

$$W_{①} = \int_{(0,0)}^{(1,0)} \vec{F}\cdot \underbrace{d\vec{r}_1}_{dx\,\hat{x}} + \int_{(1,0)}^{(1,1)} \vec{F}\cdot \underbrace{d\vec{r}_2}_{dy\,\hat{y}}$$

$$(y\hat{x} + 2x\hat{y})\cdot(dx\,\hat{x} + 0\,\hat{y})$$

$$y\,dx + 0$$

$$= \int_{(0,0)}^{(1,0)} y\,dx + \int_{(1,0)}^{(1,1)} 2x\,dy$$

$$= \underbrace{\int_{x=0}^{x=1} 0\cdot dx}_{0} + \int_{y=0}^{y=1} 2\cdot(1)\,dy = 2y\Big|_0^1 = 2(1-0) = 2$$

$$W_{③} = \int_0^{\pi/2} y\sin\theta\, d\theta + \int_0^{\pi/2} 2x\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\, d\theta + 2\int_0^{\pi/2} (1-\cos\theta)\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^2\theta\, d\theta + 2\int_0^{\pi/2} \cos\theta\, d\theta - 2\int_0^{\pi/2} \cos^2\theta\, d\theta$$

For conservative force

$$U(\vec{r}) = -W(\vec{r}_0 \to \vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$$

Potential energy at $\vec{r}$ in reference to $\vec{r}_0$

conditions for conservative force

time

1. $\vec{F}$ only depends on $\vec{r}$ (not $\vec{v}$, not t)

2. Work done b/t any two points is independent of path taken
   ($\vec{\nabla} \times \vec{F} = 0$)

What about $W(\vec{r}_1 \to \vec{r}_2)$ ← work to go between points not the reference point

$$W(\vec{r}_0 \to \vec{r}_2) = W(\vec{r}_0 \to \vec{r}_1) + W(\vec{r}_1 \to \vec{r}_2)$$

$$W(\vec{r}_1 \to \vec{r}_2) = \underbrace{W(\vec{r}_0 \to \vec{r}_2)}_{-U(\vec{r}_2)} - \underbrace{W(\vec{r}_0 \to \vec{r}_1)}_{-U(\vec{r}_1)}$$

$$= -\underbrace{\left(U(\vec{r}_2) - U(\vec{r}_1)\right)}_{\Delta U}$$

$$\underline{\underline{\Delta U}} = -W(\vec{r}_1 \to \vec{r}_2)$$

Now go back to work-kinetic energy theorem

$$W(\vec{r}_1 \to \vec{r}_2) = \Delta K$$

if only conservative forces are acting

Conservation of Energy

$$-\Delta U = \Delta K$$

$$\Delta K + \Delta U = 0$$

$E = K + U$ } Mechanical Energy

$$\Delta E = 0$$

Also true for multiple conservative forces.

↳ multiple potential energy

$$U = U_g + U_s \; (+ U_e)$$

But, what if also non-conservative forces act?

$$\Delta K + \Delta U \neq 0$$

Conservation of Energy

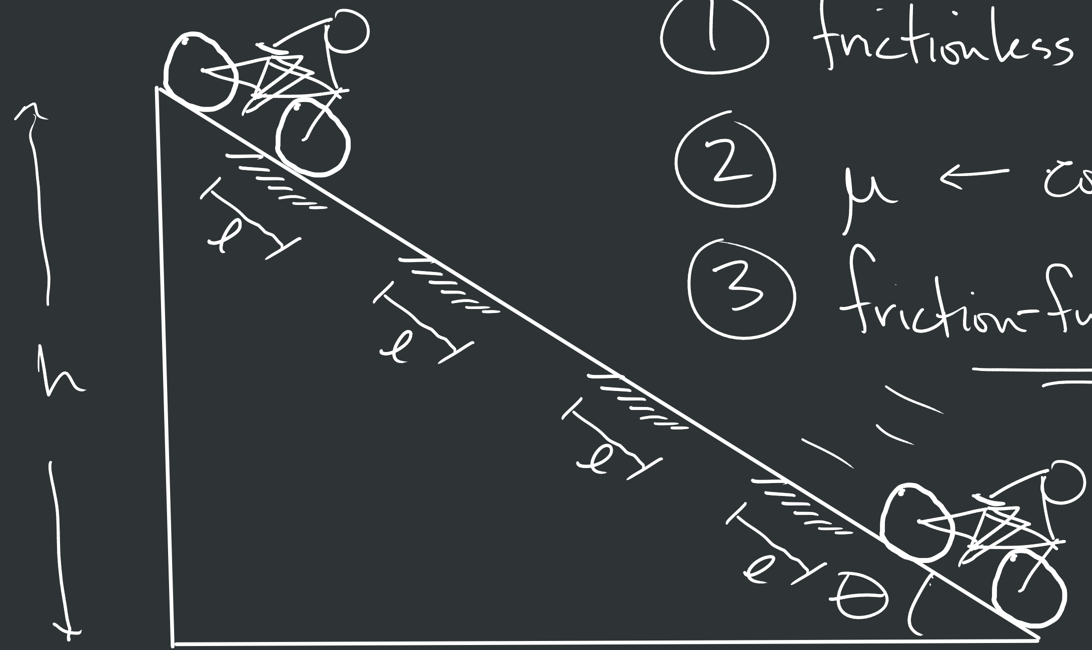
$$\Delta K + \Delta U = W_{NC}$$

$$K_i + U_i + W_{NC} = K_f + U_f$$

Friction

$$W_f = \int \vec{F}_f \cdot d\vec{x}$$

Ex:



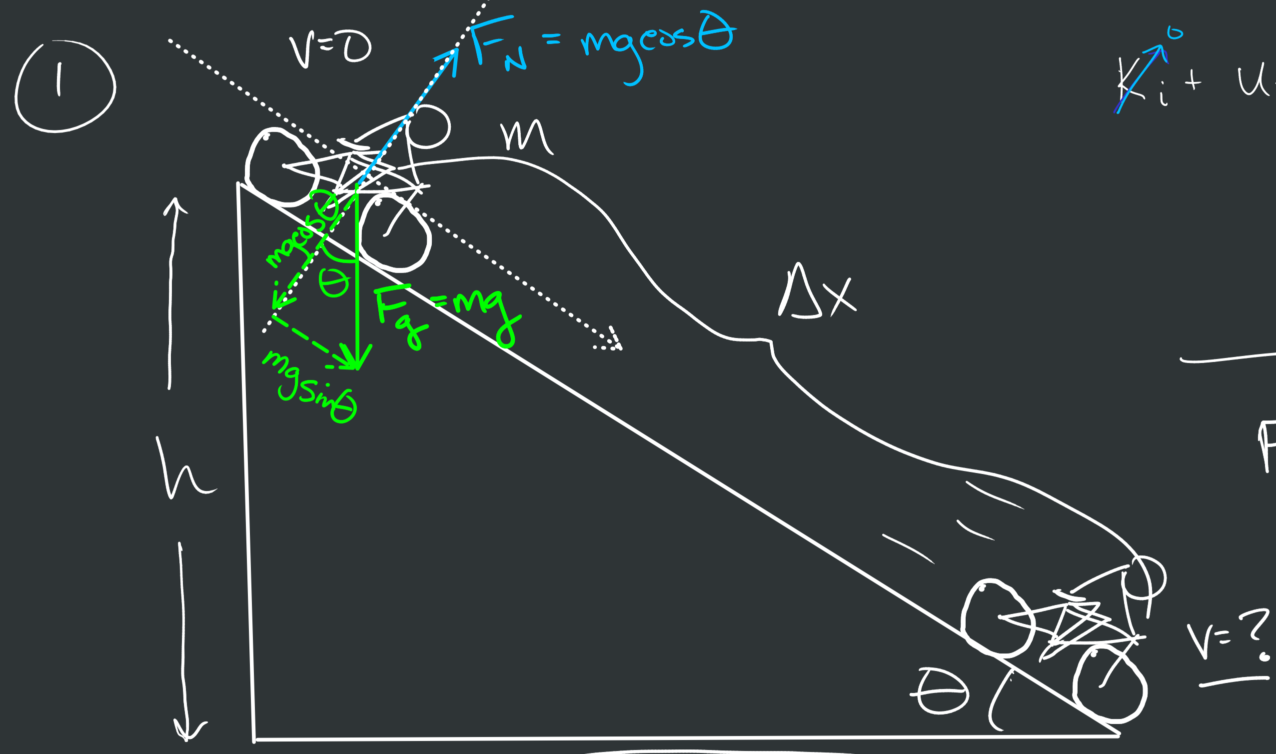
1. frictionless plane
2. $\mu \leftarrow$ constant
3. friction-full patches

$\leftarrow$ no Newton's

Newton's Laws

+

Conservation of Energy



$$\cancel{K_i}^0 + U_i + \cancel{W_{NC}}^0 = K_f + \cancel{U_f}^0$$

$$\cancel{m}gh = \frac{1}{2}\cancel{m}v^2$$

$$v = \sqrt{2gh}$$

$$F_{NET,x} = \cancel{m}g\sin\theta = ma = \dot{p}$$

$$g\sin\theta = a$$

$$g\sin\theta = \frac{dv}{dt}$$

$$\int_0^t g\sin\theta\,dt = \int_{v_0=0}^{v} dv'$$

$$g\sin\theta \cdot t = v$$

$$v_f = \cancel{v_i}^0 + at$$

$$v = g\sin\theta\, t$$

$$\int_0^x dx = \int_0^t g\sin\theta\, t'\,dt'$$

$$x = \frac{g\sin\theta\, t^2}{2}$$

$$g\sin\theta = \frac{dv}{dt} = \frac{dv}{dx}\cdot\underbrace{\frac{dx}{dt}}_{v}$$

$$g\sin\theta = v\frac{dv}{dx}$$

$$\int_{x=0}^{x} g\sin\theta\,dx' = \int_0^v v'\,dv'$$

$$g\sin\theta\, x'\Big|_0^x = \frac{v'^2}{2}\Big|_{0=v_i}^{v}$$

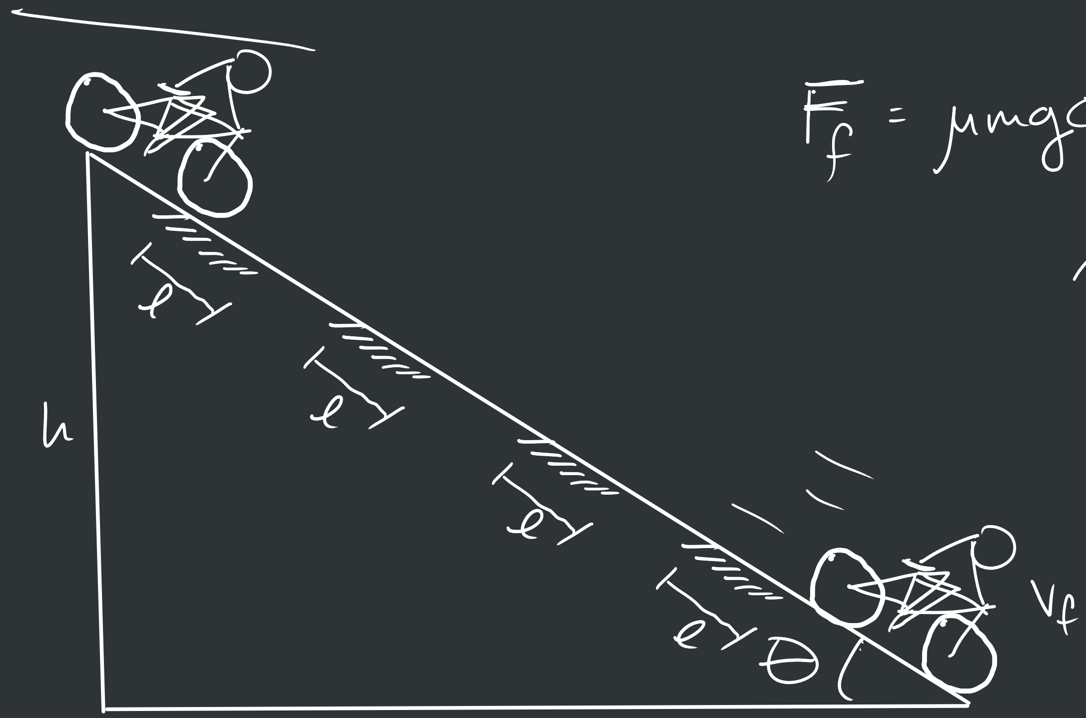
$$g\sin\theta \cdot x = \frac{v^2}{2}$$

$$\sqrt{2g\underbrace{\sin\theta\, x}_{h}} = V$$

$$V = \sqrt{2gh}$$

$$\sin\theta = \frac{h}{x} = \frac{opp}{hyp}$$

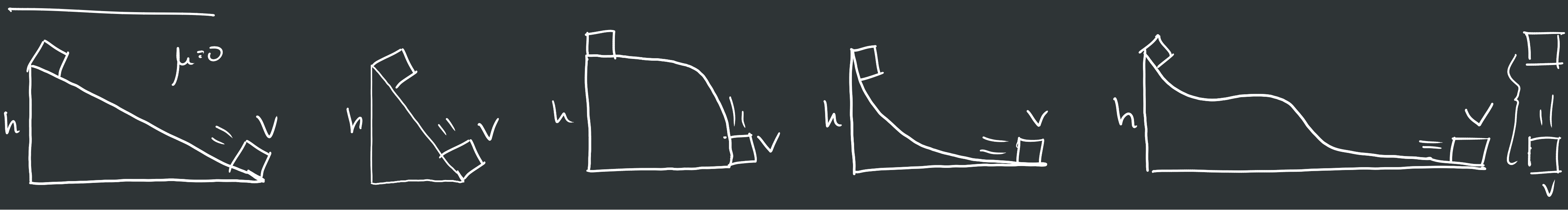
$$\sin\theta \cdot x = h$$



$$F_f = \mu m g\cos\theta$$

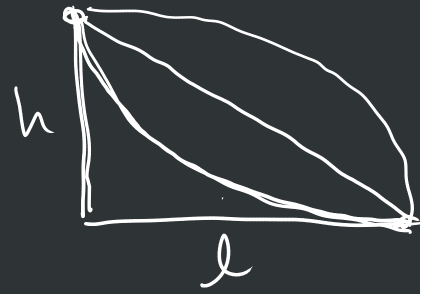
$$\cancel{K_i}^{0} + U_i + W_{NC} = K_f + \cancel{U_f}^{0}$$

$$\cancel{m}gh - 4\cdot\mu \cancel{m} g\cos\theta \cdot l = \frac{1}{2}\cancel{m} v_f^2$$



Potential Energy

$$U(\vec{r}) = -\int_{r_0}^{r} \vec{F}(\vec{r}\,')\cdot d\vec{r}\,'$$



Gravitational (constant)

$$\vec{F}(\vec{r}\,') = -mg\,\hat{y}$$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$U = -W(0 \to h\hat{y}) = +\int_0^h (+mg\hat{y})\cdot(dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z})$$

$$= \int_0^h mg\,dy$$

$$= mgy\Big|_0^h$$

$$U(h) = mgh$$

in relation to wherever we call $h=0 \to U=0$

Gravitation (universal)

$$\vec{F} = -\frac{G m_1 m_2}{r^2}\,\hat{r}$$

$$U(\vec{r}) = -W(0 \to \vec{r}) = -\int_{\infty}^{r} -\frac{G m_1 m_2}{r^2}\,\hat{r} \cdot dr\,\hat{r}$$

$$= \int_{\infty}^{r} \frac{G m_1 m_2}{r'^2}\,dr' = G m_1 m_2 \int_{\infty}^{r} r'^{-2}\,dr'$$

$$= G m_1 m_2 (-1) r'^{-1} \Big|_{\infty}^{r}$$

$$= -\frac{G m_1 m_2}{r'} \Big|_{\infty}^{r}$$

$$= \frac{-G m_1 m_2}{r} - \cancelto{0}{\frac{-G m_1 m_2}{\infty}}$$

reference point $r \to \infty$

$U = 0$

$$U(r) = -\frac{G m_1 m_2}{r}$$

If instead we set $U=0$ at $r=R$ ← radius of a planet

$$\int_R^r \frac{Gm_1m_2}{r}\,dr = -\frac{Gm_1m_2}{r}\Bigg|_R^r$$

$$= -\frac{Gm_1m_2}{r} + \frac{Gm_1m_2}{R}$$

$$U(r) = Gm_1m_2\left(\frac{1}{R} - \frac{1}{r}\right)$$

$$r > R$$

Ex: $r_1 \to r_2$ will require non-conservative work if $\Delta K = 0$

$$\cancel{K_i} + U_i + W_{NC} = \cancel{K_f} + U_f$$

choose $r \to \infty$, $U=0$

$$W_{NC} = \Delta U = U(r_2) - U(r_1)$$

$$= -\frac{Gm_1m_2}{r_2} + \frac{Gm_1m_2}{r_1}$$

$$= Gm_1m_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Spring Potential (1D)

$$\vec{F} = -kx\,\hat{x}$$

$dx\,\hat{x} + dy\,\hat{y} + \ldots$

$$U_s = -\int_0^x (-kx\hat{x}) \cdot d\vec{r}$$

unstretched spring

$$= \int_0^x kx'\,dx'$$

reference $x = 0$

$U = 0$

$$= k\frac{x'^2}{2}\Big|_0^x = \frac{1}{2}kx^2 = U_s$$

Potential Energy $\longrightarrow$ Force

$$-dU = đW = \vec{F} \cdot d\vec{r}$$

displacement

$\hookrightarrow dU$ depends on $d\vec{r}$

Cartesian $d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

$\dfrac{\partial U}{\partial x} \leftarrow$ how much U changes in the x-direction w/ y + z fixed

$\dfrac{\partial U}{\partial y} \leftarrow$ same but x + z fixed

total differential

$$df = \frac{df}{dx}dx$$

$$\hookrightarrow dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz$$

$$-\left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right) = F_x\,dx + F_y\,dy + F_z\,dz$$

$$-\frac{\partial U}{\partial x} = F_x \qquad -\frac{\partial U}{\partial y} = F_y \qquad -\frac{\partial U}{\partial z} = F_z$$

$$\vec{F} = \underbrace{-\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}}_{\text{this operation is called the } \underline{\text{gradient}}}$$

$$\vec{F} = -\underbrace{\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)}_{\text{one particular differential operator}} U$$

"del" $\rightarrow \vec{\nabla}$

<u>in</u> <u>Cartesian</u>

$$\vec{F} = -\vec{\nabla} U$$

$$\vec{F} = -\vec{\nabla} U$$

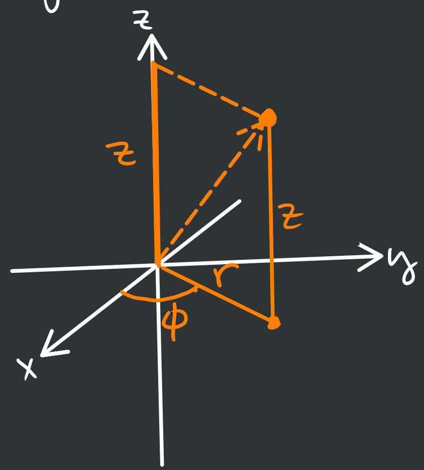
↳ more general than Cartesian

$$\vec{\nabla}_{xyz} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

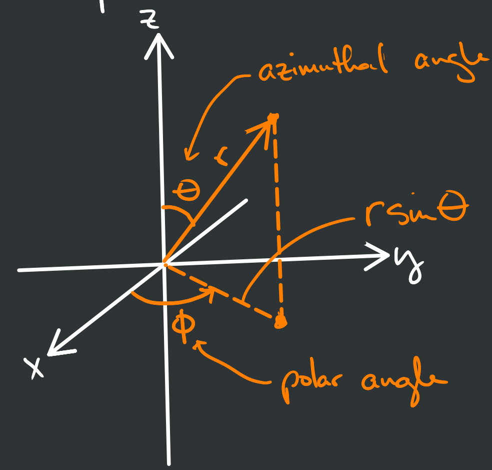
$$\vec{\nabla}_{r\phi z} = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{\nabla}_{r\theta\phi} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Cylindrical-Polar Coordinates



Spherical Coordinates



How do we know that a force will be path-independent?

$\vec{\nabla} \times \vec{F} = 0$

$\uparrow$ curl

$\vec{\nabla} \cdot \vec{F} \rightarrow$ scalar

$\vec{\nabla} \times \vec{F} \rightarrow$ vector

$\underbrace{\vec{\nabla} U}_{\text{gradient}} =$

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \qquad \vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{x} - (A_x B_z - A_z B_x)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

$\vec{\nabla} \times \vec{F}$ if $\vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z}$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) x - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z}$$

$$\vec{F} = \underset{F_x}{y\hat{x}} + \underset{F_y}{2x\hat{y}} + \underset{F_z}{0\hat{z}}$$

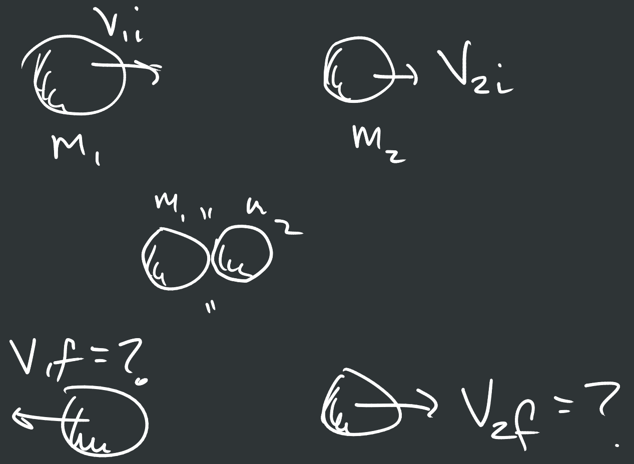
$$\vec{\nabla} \times \vec{F} = (2 - 1)\hat{z}$$

$$= 1\hat{z}$$

Elastic Collisions

① momentum is conserved

② kinetic energy is conserved



$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\cancel{\frac{1}{2}} m_1 v_{1i}^2 + \cancel{\frac{1}{2}} m_2 v_{2i}^2 = \cancel{\frac{1}{2}} m_1 v_{1f}^2 + \cancel{\frac{1}{2}} m_2 v_{2f}^2$$

Possible hints:

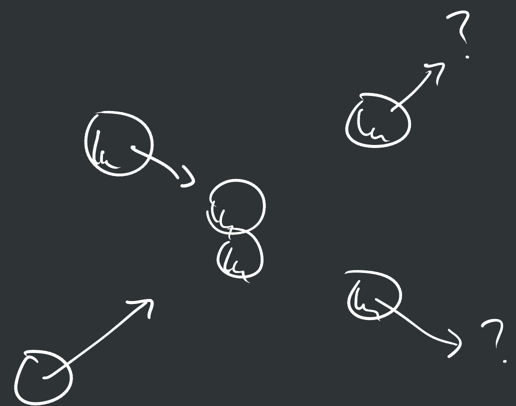
→ ① $x^2 - y^2 = (x+y)(x-y)$

→ ② use different reference frame

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} \cdot v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = v_{1f} + (v_{1i} - v_{2i})$$

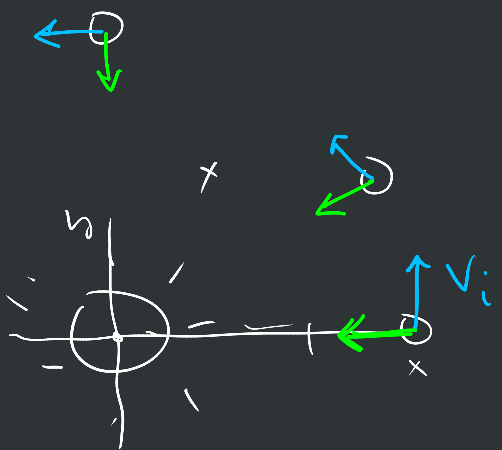
$$v_{2f} = \frac{2m_2}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_1 + m_2} \cdot v_{2i}$$



Emma's Project

$$\cancel{m_1}\ddot{\vec{r}}_1 = -\frac{G m_2 \cancel{m_1} |\vec{r}_1 - \vec{r}_2|}{|\vec{r}_1 - \vec{r}_2|^3} - \underbrace{G m_3 \cdots}$$

$$\vec{F}_g = -\frac{G \cancel{m_1} m_2 (x\hat{x} + y\hat{y})}{(x^2+y^2)^{3/2}} = \cancel{m_1}\underbrace{(\ddot{x}\hat{x} + \ddot{y}\hat{y})}_{\ddot{r}}$$



fixed sun

$$\vec{F}_g = -\frac{G \cancel{m_2} m_1 (x\hat{x} + y\hat{y})}{(x^2+y^2)^{3/2}} = \cancel{m_2}\ddot{x}\hat{x} + \ddot{y}\hat{y}$$

$$\boxed{\frac{-G m_2 x}{(x^2+y^2)^{3/2}}} = \ddot{x} \qquad \boxed{\frac{-G m_2 y}{(x^2+y^2)^{3/2}}} = \ddot{y}$$

$$\frac{Gm_1 \cancel{m_2}}{r^{\cancel{2}}} = \frac{\cancel{m_2} V^2}{\cancel{r}}$$

$$V = \sqrt{\frac{Gm_1}{r}} \qquad V = \sqrt{\frac{1 \cdot 100}{1}} = 10$$