

Chapter 4 \rightsquigarrow Energy

- Kinetic Energy

$$T = \frac{1}{2} m \vec{v} \cdot \vec{v} = \frac{1}{2} m \vec{v}^2 = \frac{\vec{p}^2}{2m}$$

- Work is a transfer of energy to the system

$$dW = \vec{F} \cdot d\vec{r} \Rightarrow W = \int_1^2 \vec{F} \cdot d\vec{r}$$

$|\vec{F}| \cdot |\vec{dr}| \cdot \cos \theta$

path integral means work
depends on the path between points,
unless it doesn't

- Amount of work done (total) is the same as the change in kinetic energy

$$\cancel{W = \Delta T} \quad \leftarrow \text{Work-Kinetic energy theorem}$$

- If the work done does not depend on the path taken, (it depends on the end points (positions)), that force is a conservative force

→ work done by a conservative force \Rightarrow potential energy

$$\Delta U = - W_{\text{conservative force}}$$

conservative forces

- gravitational force
- electric force
- spring force

non conservative force

- friction
- drag
- "applied force"

potential energy is defined by a reference point
 $(\vec{r} \text{ when } U=0)$

- Conservation of energy

$$W = \Delta K$$

$$W_{nc} = \underbrace{\Delta K + \Delta U}_{\text{mechanical energy}}$$

$$U_g = mgh$$

$$W_{nc} = T_f - T_i + U_f - U_i$$

$$T_i + U_i + W_{nc} = T_f + U_f$$

- if $\Delta U = - \int_1^2 \vec{F} \cdot d\vec{r} \Rightarrow dU = -W = -\vec{F} \cdot d\vec{r}$

$$\cancel{\frac{dU}{d\vec{r}} = -\vec{F}}$$

$$\vec{F} = -\vec{\nabla} U = \frac{\partial U}{\partial x} \hat{x} + \frac{\partial U}{\partial y} \hat{y} + \frac{\partial U}{\partial z} \hat{z}$$

gradient \rightarrow cartesian

$$F = -\frac{dU}{dx} \quad \leftarrow \text{one dimension}$$

• What makes a F conservative

- F depends only on position
(not v, t, a)

- Work is path-independent

$$\hookrightarrow \vec{\nabla} \times \vec{F} = 0$$

\curvearrowleft curl

- everything above can be done in spherical coords.
or cylindrical coords. \rightarrow often this will be easier

$$F_s = -kx$$

$$\rightarrow F_g = mg_j$$

$$F_G = \frac{G m_1 m_2}{r^2}$$

$$F_o = b v$$

$$= cv^2$$

Central forces

$$\vec{F}(\vec{r}) = f(\vec{r}) \hat{\vec{r}}$$

$\hat{\vec{r}}$ radial force

$$\vec{F}_G(\vec{r}) = -\frac{Gm_1 m_2}{r^2} \hat{r} \quad \leftarrow \text{in spherical coords.}$$

$$\vec{r} = x\hat{x} + y\hat{y} \quad \hat{r} = \frac{\vec{r}}{r}$$

$$r = \sqrt{x^2 + y^2}$$

$$= -\left(\frac{Gm_1 m_2}{x^2 + y^2}\right) \cdot \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2 + y^2}}$$

$$= -\frac{Gm_1 m_2}{(x^2 + y^2)^{3/2}} \cdot (x\hat{x} + y\hat{y}) \quad \leftarrow$$

$$\vec{\nabla} \times \vec{F}_G =$$

What is curl in spherical?

- All of this applies multiple particle

$$\overline{T} = T_1 + T_2 + T_3 + \dots$$

$$T = \frac{1}{2} M \overline{V_{cm}^2} + \frac{1}{2} \overline{I} \underbrace{\omega_{cm}^2}_{\text{rot. inertia}}$$

Recall:

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v} \cdot \vec{v}$$

$$\frac{dT}{dt} = \frac{1}{2}m \cdot \frac{d(\vec{v} \cdot \vec{v})}{dt}$$

$$= \frac{1}{2}m \left(\frac{d(\vec{v})}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d(\vec{v})}{dt} \right)$$

$$\Delta T = \omega$$

$$= \frac{1}{2}m(2\dot{\vec{v}} \cdot \vec{v})$$

$$= m\dot{\vec{v}} \cdot \vec{v}$$

$$\dot{\vec{P}} = \vec{F}$$

$$\frac{dT}{dt} = \vec{F} \cdot \vec{v}$$

$$\frac{dT}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

$$dT = \vec{F} \cdot d\vec{r}$$

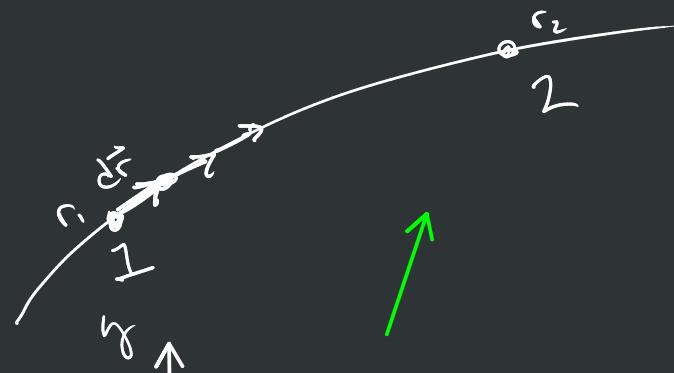
define this
as a small amount
of work

dW does not make
sense.
 $\Rightarrow dW$ (in thermo)

$$\Delta T = T_2 - T_1 = \int_1^2 \vec{F} \cdot d\vec{r}$$

line integral

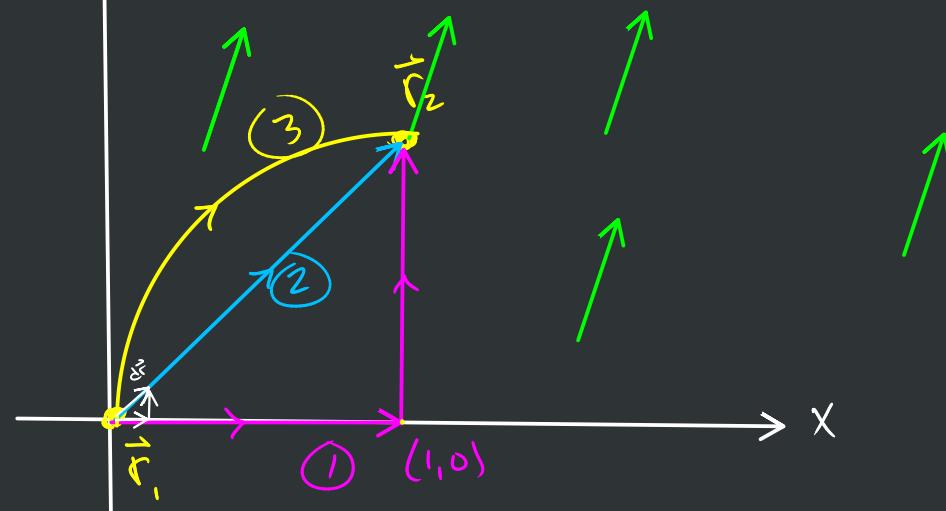
$$\int f(x) dx$$



Ex: $\vec{F} = 1\hat{x} + 2\hat{y}$

$$\vec{r}_1 = 0$$

$$\vec{r}_2 = 1\hat{x} + 1\hat{y}$$



Path ①

$$\int_1^2 \vec{F} \cdot d\vec{r} = \int_{(0,0)}^{(1,0)} \vec{F} \cdot d\vec{r}_1 + \int_{(1,0)}^{(1,1)} \vec{F} \cdot d\vec{r}_2$$

$$\begin{cases} d\vec{r}_1 = d\vec{x} = dx\hat{x} \\ \vec{F} \cdot d\vec{x} = (1\hat{x} + 2\hat{y}) \cdot (dx\hat{x} + 0\hat{y}) \\ = 1 dx \end{cases}$$

$$\begin{cases} d\vec{r}_2 = dy\hat{y} \\ \vec{F} \cdot dy\hat{y} = (1\hat{x} + 2\hat{y}) \cdot (0\hat{x} + dy\hat{y}) \\ = 2 dy \end{cases}$$

$$\int_1^2 \vec{F} \cdot d\vec{r} = \int_{x=0}^{x=1} dx + \int_{y=0}^{y=1} 2 dy$$

$$= x \Big|_0^1 + 2y \Big|_0^1 \quad \text{Work along path ①}$$

$$= 1 + 2 = 3$$

Path ②

$$\int_1^2 \vec{F} \cdot d\vec{r} =$$

↓

$$\begin{cases} d\vec{r} = dx\hat{x} + dy\hat{y} \\ \vec{F} \cdot d\vec{r} = (1\hat{x} + 2\hat{y}) \cdot (dx\hat{x} + dy\hat{y}) \\ = dx + 2dy \\ \int_1^2 (dx + 2dy) = \int_0^1 dx + \int_0^1 2dy = 3 \end{cases}$$

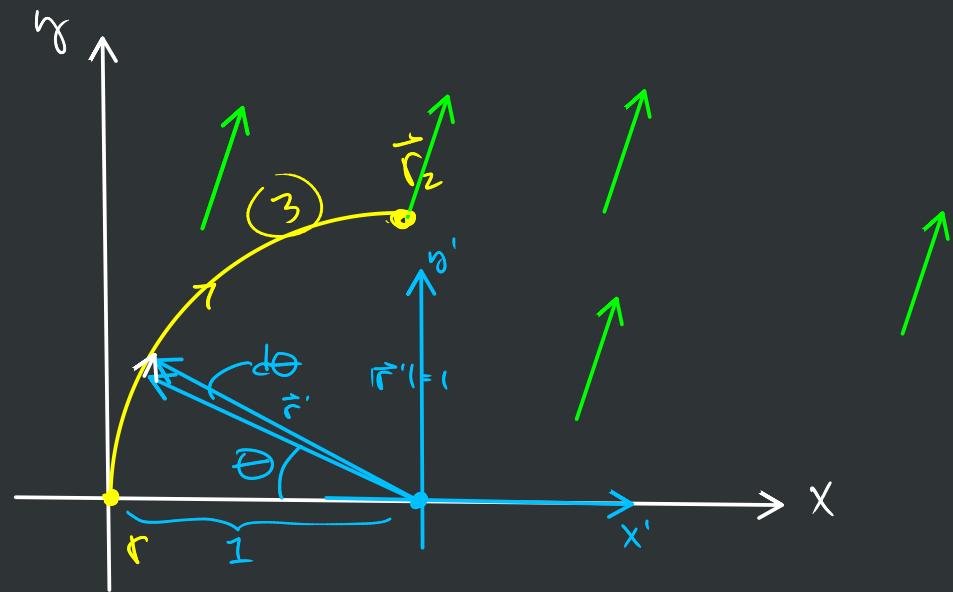
Path ③

$$\int_1^2 \vec{F} \cdot d\vec{r}$$

$$\int_1^2 ((1\hat{x} + 2\hat{y}) \cdot (\sin\theta d\theta \hat{x} + \cos\theta d\theta \hat{y}))$$

$$\int_1^2 (\sin\theta d\theta + 2\cos\theta d\theta)$$

$$\int_0^{\pi/2} \sin\theta d\theta + \int_0^{\pi/2} 2\cos\theta d\theta = 3$$



$$\vec{r}' = -\cos\theta \hat{x}' + \sin\theta \hat{y}'$$

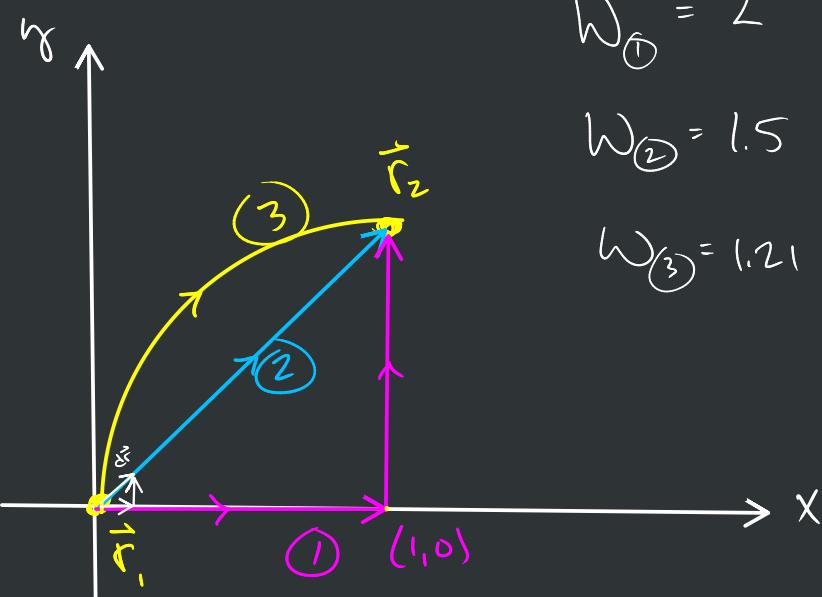
↓ change coordinate systems

$$\vec{r} = (1 - \cos\theta)\hat{x} + \sin\theta \hat{y}$$

$$d\vec{r} = \sin\theta d\theta \hat{x} + \cos\theta d\theta \hat{y}$$

Homework: Same 3 paths

$$\vec{F} = y\hat{x} + 2x\hat{y}$$



$$W_1 = 2$$

$$W_2 = 1.5$$

$$W_3 = 1.21$$

$$\vec{r} = (1 - \cos \theta)\hat{x} + \sin \theta \hat{y}$$

$$d\vec{r} = \sin \theta d\theta \hat{x} + \cos \theta d\theta \hat{y}$$

$$W_1 = \int_{(0,0)}^{(1,0)} \vec{F} \cdot d\vec{r}_1 + \int_{(1,0)}^{(1,1)} \vec{F} \cdot d\vec{r}_2$$

$$(y\hat{x} + 2x\hat{y}) \cdot (\hat{x} + 0\hat{y})$$

$$y_0 dx + 0$$

$$= \int_{(0,0)}^{(1,0)} y_0 dx + \int_{(0,0)}^{(1,1)} 2x dy$$

$$= \underbrace{\int_{x=0}^{x=1} 0 \cdot dx}_{0} + \underbrace{\int_{y=0}^{y=1} 2 \cdot (1) dy}_{2y \Big|_0^1} = 2(1 - 0) = 2$$

$$\left. \begin{aligned} W_2 &= \int_0^{\pi/2} y \sin \theta d\theta \hat{x} + \int_0^{\pi/2} 2x \cos \theta d\theta \hat{y} \\ &= \int_0^{\pi/2} \sin^2 \theta d\theta \hat{x} + 2 \int_0^{\pi/2} (1 - \cos \theta) \cos \theta d\theta \hat{y} \\ &= \int_0^{\pi/2} \sin^2 \theta d\theta \hat{x} + 2 \int_0^{\pi/2} \cos^2 \theta d\theta \hat{y} - 2 \int_0^{\pi/2} \cos^2 \theta d\theta \hat{x} \end{aligned} \right\}$$

For conservative force

$$U(\vec{r}) = -W(\vec{r}_0 \rightarrow \vec{r}) = - \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}') \cdot d\vec{r}'$$

↗ Potential energy
at \vec{r} in reference to \vec{r}_0

conditions for conservative force

- ① \vec{F} only depends on \vec{r} (not \vec{v} , not time)
- ② Work done b/t any two points is independent of path taken
 $(\vec{\nabla} \times \vec{F} = 0)$

What about $\omega(\vec{r}_1 \rightarrow \vec{r}_2)$ ← work to go between points not
the reference point

$$\omega(\vec{r}_0 \rightarrow \vec{r}_2) = \omega(\vec{r}_0 \rightarrow \vec{r}_1) + \omega(\vec{r}_1 \rightarrow \vec{r}_2)$$

$$\omega(\vec{r}_1 \rightarrow \vec{r}_2) = \underbrace{\omega(\vec{r}_0 \rightarrow \vec{r}_2)}_{-\mathcal{U}(\vec{r}_2)} - \underbrace{\omega(\vec{r}_0 \rightarrow \vec{r}_1)}_{-\mathcal{U}(\vec{r}_1)}$$

$$= - \underbrace{(\mathcal{U}(\vec{r}_2) - \mathcal{U}(\vec{r}_1))}_{\Delta \mathcal{U}}$$

$$\underline{\underline{\Delta \mathcal{U}}} = - \omega(\vec{r}_1 \rightarrow \vec{r}_2)$$

Now go back to work-kinetic energy theorem

$$W(\vec{r}_1 \rightarrow \vec{r}_2) = \Delta K$$

if only conservative forces are acting

Conservation } $\rightarrow -\Delta U = \Delta K$

of

Energy } $\rightarrow \Delta K + \Delta U = 0$

$E = K + U$ } Mechanical Energy

$\Delta E = 0$

Also true for multiple conservative forces.

↳ multiple potential energy

$$U = U_g + U_s (+ U_e)$$

But, what if also non-conservative forces act?

Conservation
of
Energy

$$\Delta K + \Delta U \neq 0$$

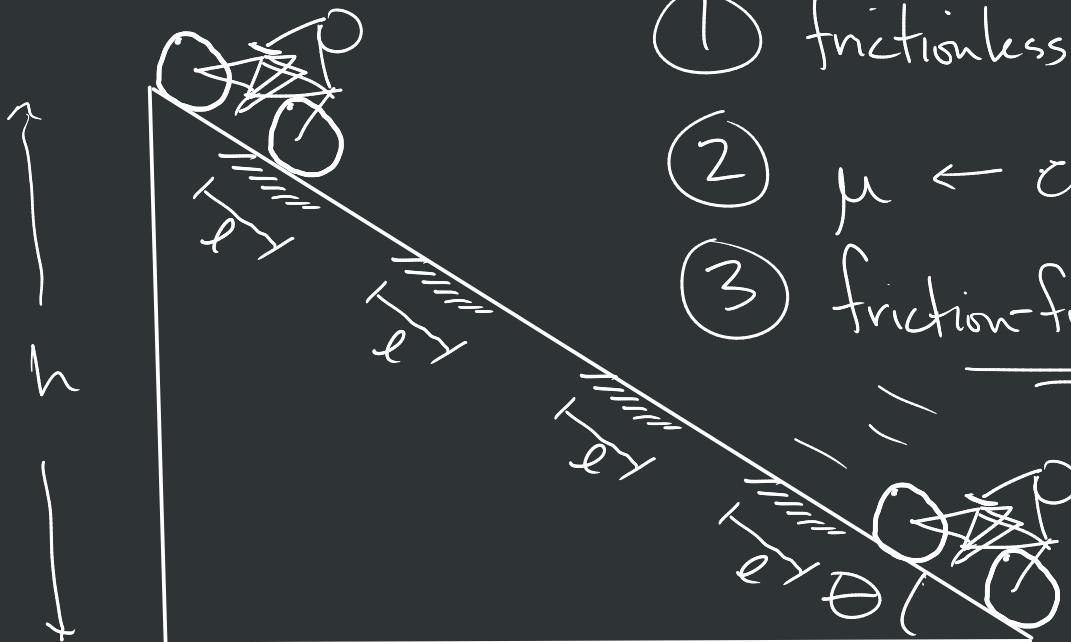
$$\Delta K + \Delta U = W_{NC}$$

$$K_i + U_i + W_{NC} = K_f + U_f$$

Friction

$$W_f = \int \vec{F}_f \cdot d\vec{x}$$

Ex:



- ① frictionless plane
- ② $\mu \leftarrow \text{constant}$
- ③ friction-full patches

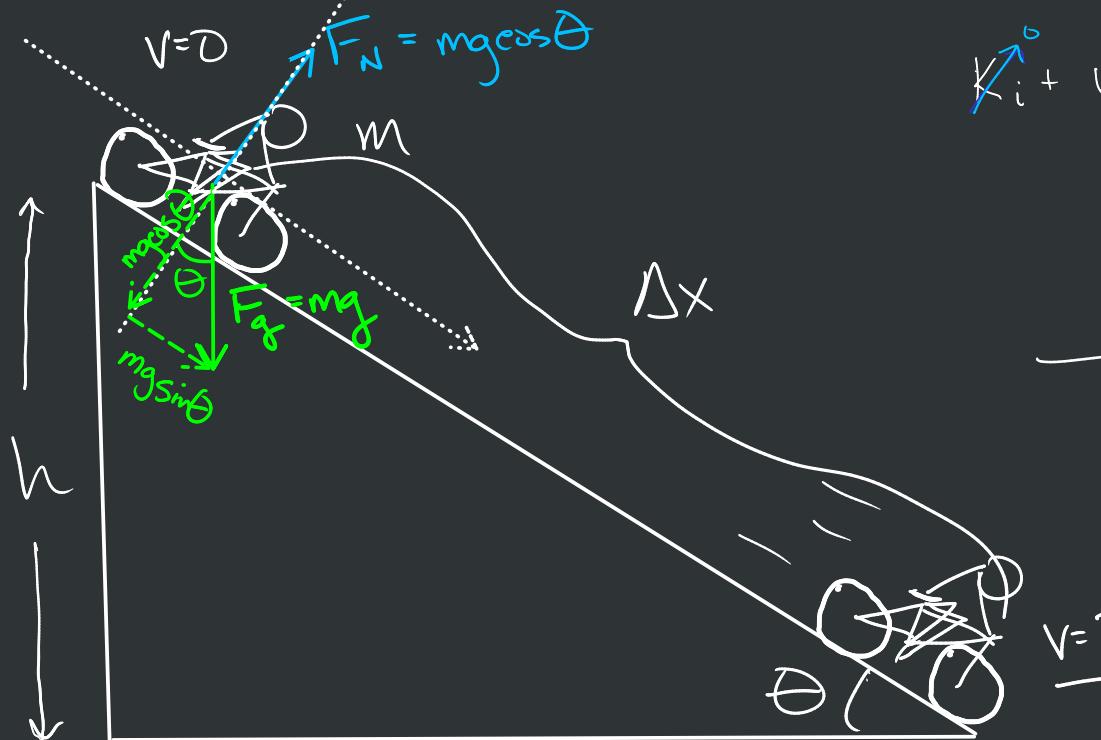
no
Newton's

Newton's Laws

+

Conservation
of
Energy

(1)



$$K_i + U_i + W_{NC}^o = K_f + U_f$$

$$mgh = \frac{1}{2}mv^2$$

$$V = \sqrt{2gh}$$

$$F_{NET,X} = mgs \sin \theta = ma = \dot{P}$$

$$g \sin \theta = a$$

$$g \sin \theta = \frac{dv}{dt}$$

$$\int_0^t g \sin \theta dt = \int_{V_0=0}^v dv'$$

$$g \sin \theta \cdot t = v$$

$$v = g \sin \theta t$$

$$\int_0^t dx' = \int_0^t g \sin \theta t' dt'$$

$$x = \frac{g \sin \theta t'^2}{2}$$

$$g \sin \theta = \frac{dv}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = v \frac{dv}{dx}$$

$$\int_0^x g \sin \theta dx' = \int_0^v v' dv'$$

$$g \sin \theta x' \Big|_0^x = \frac{v'^2}{2} \Big|_0^v = v_i^2$$

$$v_f = v_i + at$$

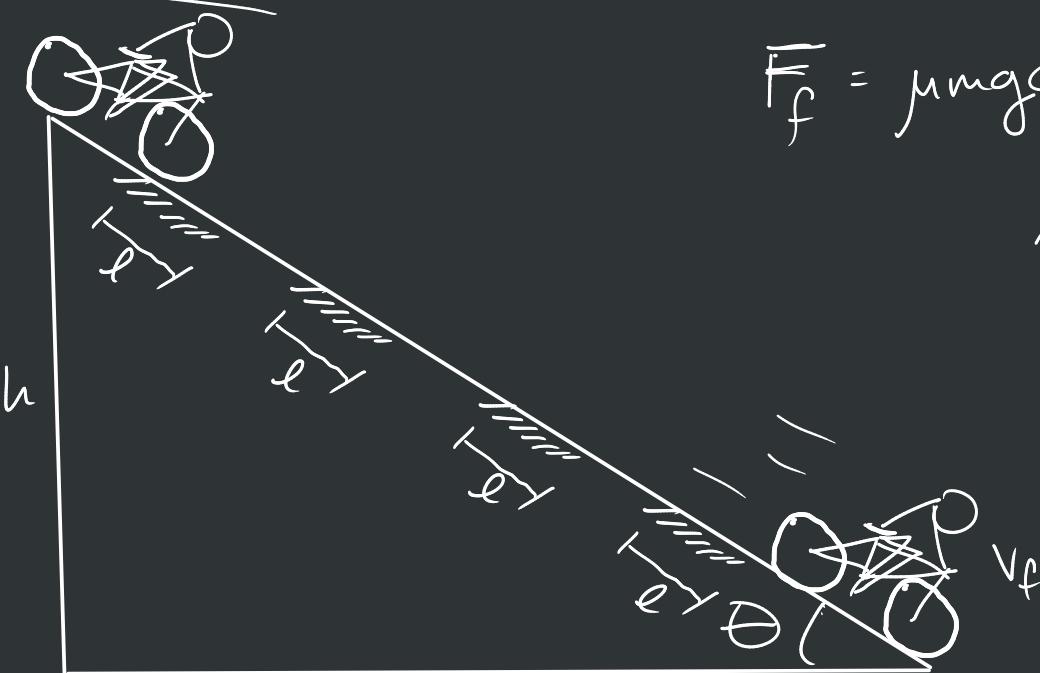
$$g \sin \theta \cdot x = \frac{v^2}{2}$$

$$\sqrt{2g \underbrace{\sin \theta x}_{h}} = v$$

$$\sin \theta = \frac{h}{x} = \frac{\text{opp}}{\text{hyp}}$$

$$\sin \theta \cdot x = h$$

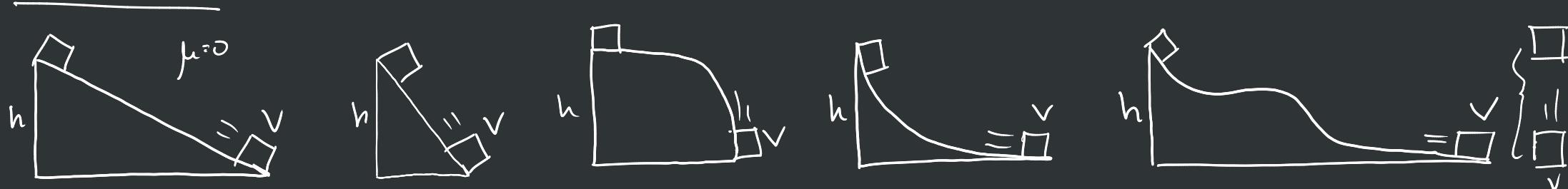
$$v = \sqrt{2gh}$$



$$F_f = \mu mg \cos \theta$$

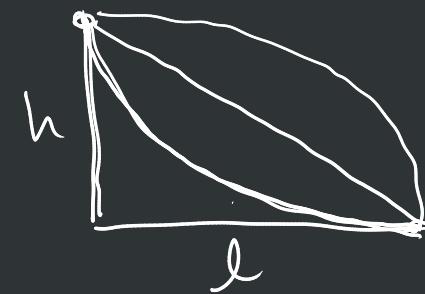
$$K_i^0 + U_i + \omega_{nc} = K_f + K_f^0$$

$$mgh - 4 \cdot \mu mg \cos \theta \cdot l = \frac{1}{2} m v_f^2$$



Potential Energy

$$U(\vec{r}) = - \int_{r_0}^r \vec{F}(\vec{r}') \cdot d\vec{r}'$$



Gravitational (constant)

$$\vec{F}(\vec{r}') = -mg\hat{\vec{y}}$$

$$d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

$$U = -W(0 \rightarrow h\hat{y}) = + \int_0^h (-mg\hat{y}) \cdot (dx\hat{x} + dy\hat{y} + dz\hat{z})$$

$$= \int_0^h mgdy$$

$$= mgy \Big|_0^h$$

in relation to
whenever we call $h=0 \rightarrow U=0$

$$U(h) = mgh$$

Gravitation (universal)

$$\vec{F} = -\frac{Gm_1 m_2}{r^2} \hat{r}$$



$$\begin{aligned} U(\vec{r}) &= -W(0 \rightarrow \vec{r}) = - \int_{\infty}^r -\frac{Gm_1 m_2}{r^2} \hat{r} \cdot dr \hat{r} \\ &= \int_{\infty}^r \frac{Gm_1 m_2}{r'^2} dr' = Gm_1 m_2 \int_{\infty}^r r'^{-2} dr' \\ &= Gm_1 m_2 (-1)r'^{-1} \Big|_{\infty}^r \\ &= -\frac{Gm_1 m_2}{r'} \Big|_{\infty}^r \end{aligned}$$

reference point
 $U=0$

$$U(r) = -\frac{Gm_1 m_2}{r}$$

$$= -\frac{Gm_1 m_2}{r} - \cancel{-\frac{Gm_1 m_2}{\infty}}$$

If instead we set $U=0$ at $r=R$

radius
of a
planet

$$\int_R^r \frac{Gm_1 m_2}{r} dr = -\frac{Gm_1 m_2}{r} \Big|_R^r$$

$$= -\frac{Gm_1 m_2}{r} + \frac{Gm_1 m_2}{R}$$

$$U(r) = Gm_1 m_2 \left(\frac{1}{R} - \frac{1}{r} \right)$$

$$r > R$$

Ex: $r_i \rightarrow r_f$ will require non-conservative work if $\Delta K = 0$

~~$K_i + U_i + \omega_{NC} = K_f + U_f$~~

choose $\overset{r \rightarrow \infty}{U=0}$

$$\omega_{NC} = \Delta U = U(r_f) - U(r_i)$$

$$= -\frac{Gm_1 m_2}{r_f} + \frac{Gm_1 m_2}{r_i}$$

$$= Gm_1 m_2 \left(\frac{1}{r_i} - \frac{1}{r_f} \right)$$

Spring Potential (1D)

$$\vec{F} = -kx \hat{x} \quad \downarrow dx \hat{x} + dy \hat{y} + \dots$$

$$U_s = - \int_0^x (-kx \hat{x}) \cdot d\vec{r}$$

$$= \int_0^x kx \hat{x} dx'$$

$$= \underline{k \frac{x^2}{2}} \Big|_0^x = \underline{\frac{1}{2} k x^2} = U_s$$

unstretched spring
reference $x=0$
 $U=0$

Potential Energy \rightarrow Force

$$-dU = dW = \vec{F} \cdot d\vec{r}$$

$\hookrightarrow dU$ depends on $d\vec{r}$

displacement

Cartesian $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\frac{\partial U}{\partial x}$ \leftarrow how much U changes in the x -direction
w/ y & z fixed

$\frac{\partial U}{\partial y}$ \leftarrow same but x & z fixed

total differential

$$\sum dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz$$

$$df = \frac{df}{dx} dx$$

$$-\left(\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz\right) = F_x dx + F_y dy + F_z dz$$

$$-\frac{\partial U}{\partial x} = F_x$$

$$-\frac{\partial U}{\partial y} = F_y$$

$$-\frac{\partial U}{\partial z} = F_z$$

$$\vec{F} = -\frac{\partial U}{\partial x} \hat{x} - \frac{\partial U}{\partial y} \hat{y} - \frac{\partial U}{\partial z} \hat{z}$$

this operation is called the gradient

$$\vec{F} = -\left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}\right) U$$

one particular differential operator
 "del" $\rightarrow \vec{\nabla}$

in Cartesian

$$\vec{F} = -\vec{\nabla} U$$

$$\vec{F} = -\vec{\nabla}U$$

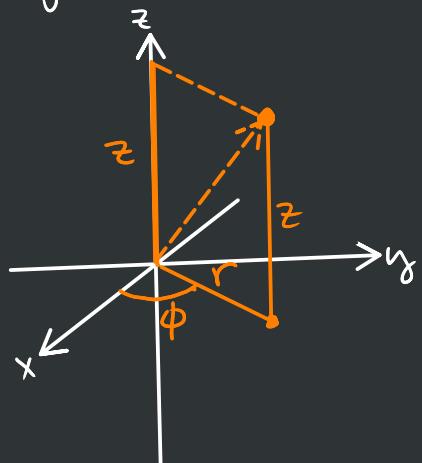
More general
than Cartesian

$$\vec{\nabla}_{xyz} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

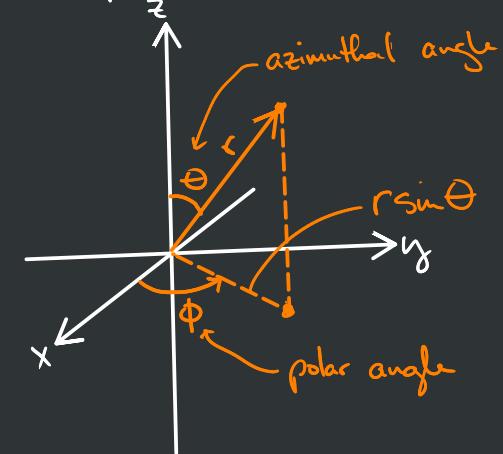
$$\vec{\nabla}_{r\phi z} = \hat{r} \frac{\partial}{\partial r} + \hat{\phi} \frac{1}{r} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z}$$

$$\vec{\nabla}_{r\theta\phi} = \hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\phi} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$

Cylindrical-Polar Coordinates



Spherical Coordinates



How do we know that a form will be path-independent?

$$\vec{V} \times \vec{F} = \vec{0}$$

\leftarrow curl

$\vec{\nabla} \cdot \vec{F}$ → scalar

$\vec{\nabla} \times \vec{F}$ vector

$$\vec{\nabla} u = \underbrace{}_{\text{gradient}}$$

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \quad \vec{B} = B_x \hat{x} + B_y \hat{y} + B_z \hat{z}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \hat{x} - (A_x B_z - A_z B_x) \hat{y} + (A_x B_y - A_y B_x) \hat{z}$$

$$\vec{\nabla} \times \vec{F} \quad \text{if} \quad \vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z}$$

$$\vec{\nabla} = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} + \frac{\partial}{\partial z} \hat{z}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\vec{\nabla} \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) \hat{y} + \left(\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} \right) \hat{z}$$

$$\vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z}$$

$$\vec{\nabla} \times \vec{F} = (2 - 1) \hat{z} \\ = 1 \hat{z}$$

Elastic Collisions

① momentum is conserved



② kinetic energy is conserved



$$V_{1f} = ?$$

$$V_{2f} = ?$$

$$m_1 V_{1i} + m_2 V_{2i} = \underline{m_1 V_{1f}} + \underline{m_2 V_{2f}}$$

$$\cancel{\frac{1}{2} m_1 V_{1i}^2} + \cancel{\frac{1}{2} m_2 V_{2i}^2} = \cancel{\frac{1}{2} m_1 V_{1f}^2} + \cancel{\frac{1}{2} m_2 V_{2f}^2}$$

Possible hints:

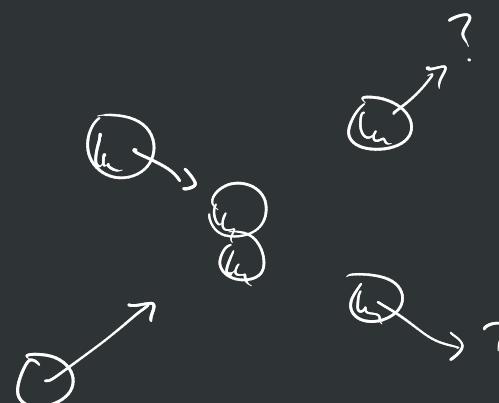
$$\rightarrow ① x^2 - y^2 = (x+y)(x-y)$$

$\rightarrow ②$ use different reference frame

$$V_f = \frac{m_1 - m_2}{m_1 + m_2} \cdot V_{1i} + \frac{2m_2}{m_1 + m_2} V_{2i}$$

$$V_{2f} = V_f + (V_{1i} - V_{2i})$$

$$V_{1f} = \frac{2m_1}{m_1 + m_2} V_{1i} + \frac{m_2 - m_1}{m_1 + m_2} \cdot V_{2i}$$



Central Forces

$$\vec{F}_G = -\frac{Gm_1m_2}{r^2} \hat{r} \quad \begin{matrix} \leftarrow \\ \text{spherical} \\ \text{coordinates} \end{matrix}$$

$$\vec{F}_e = \frac{kq_1q_2}{r^2} \hat{r}$$

Conservative?

$$\vec{\nabla} \times \vec{F}_G = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z} \right) \hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{z}$$

Cartesian coordinates

$$\vec{F}_G = -\frac{Gm_1m_2}{r^2} \hat{r} \quad \xrightarrow{\text{in Cartesian}}$$

$$r^2 = x^2 + y^2 + z^2 \quad \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}}$$

$$\vec{F}_G = -\frac{Gm_1m_2}{(x^2 + y^2 + z^2)^{3/2}} \left(\frac{\hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}}{(x^2 + y^2 + z^2)^{1/2}} \right)$$

$$-\frac{Gm_1m_2}{(x^2 + y^2 + z^2)^{3/2}} \left(\hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} \right)$$

$$F_{gx} = -\frac{Gm_1m_2 x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$F_{gy} = -\frac{Gm_1m_2 y}{(x^2 + y^2 + z^2)^{3/2}}$$

$$F_{gz} = -\frac{Gm_1m_2 z}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \hat{x}$$

$$-Gm_1m_2 \left[\frac{\partial}{\partial y} \left(z \cdot (x^2 + y^2 + z^2)^{-3/2} \right) - \frac{\partial}{\partial z} \left(y \cdot (x^2 + y^2 + z^2)^{-3/2} \right) \right]$$

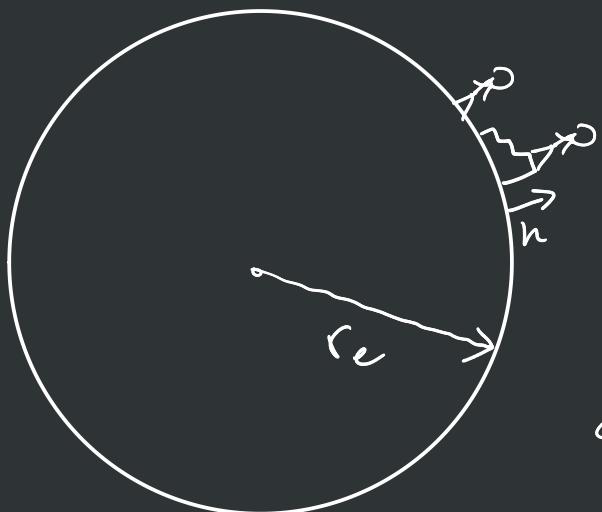
$$\begin{aligned}
 & -G_{m_1 m_2} \left[z \cdot \frac{\partial}{\partial y} \left(x^2 + y^2 + z^2 \right)^{-3/2} - y \cdot \frac{\partial}{\partial z} \left(x^2 + y^2 + z^2 \right)^{-3/2} \right] \\
 & \quad \downarrow \\
 & -\frac{3}{2} \left(x^2 + y^2 + z^2 \right)^{-5/2} \cdot \cancel{2yz} \quad -\frac{3}{2} \left(x^2 + y^2 + z^2 \right)^{-5/2} \cdot \cancel{2z} \\
 & -G_{m_1 m_2} \left[-3yz \left(x^2 + y^2 + z^2 \right)^{-5/2} + 3yz \left(x^2 + y^2 + z^2 \right)^{-5/2} \right] \\
 & \quad \underbrace{\qquad\qquad\qquad}_{=0!}
 \end{aligned}$$

Use the curl in spherical coordinates to show the same thing!

$$U_G = \frac{G m_1 m_2}{r} \quad \longleftrightarrow \quad U_G = mgh$$

Universal Gravitation shows

near the surface of earth
constant F_G



$$g = 9.8 \frac{\text{m}}{\text{s}^2}$$

$$\Delta U = U_f - U_i$$

$$= G m_e M \left(\frac{1}{r_e + h} - \frac{1}{r_e} \right)$$

$$= G m_e M \left(\frac{r_e}{(r_e + h) \cdot r_e} - \frac{r_e + h}{(r_e + h) r_e} \right)$$

$$= G m_e M \frac{\cancel{r_e} - \cancel{r_e} - h}{\cancel{(r_e + h)} r_e}$$

$$= - \frac{G m_e M}{r_e} \left(\frac{h}{r_e + h} \right) \left(\frac{r_e}{r_e} \right)$$

$$\frac{G m_e M}{r_e^2} = m a = g$$

$$= - \underbrace{\frac{Gm_e \cdot M \cdot r_e}{r_e^2}}_g \left(\frac{h}{r_e + h} \right)$$

$$= gm \cdot \cancel{r_e} \left(\frac{h}{r_e + h} \right)$$

~~$r_e + h \approx r_e$~~

$$= gm h = mgh$$

→ Another approximation

$$\frac{1}{1+\epsilon} = (1+\epsilon)^{-1} \approx 1-\epsilon$$

$$\frac{h}{r_e}$$

$$\text{in general } \Rightarrow (1+\epsilon)^n \approx 1+n\epsilon$$

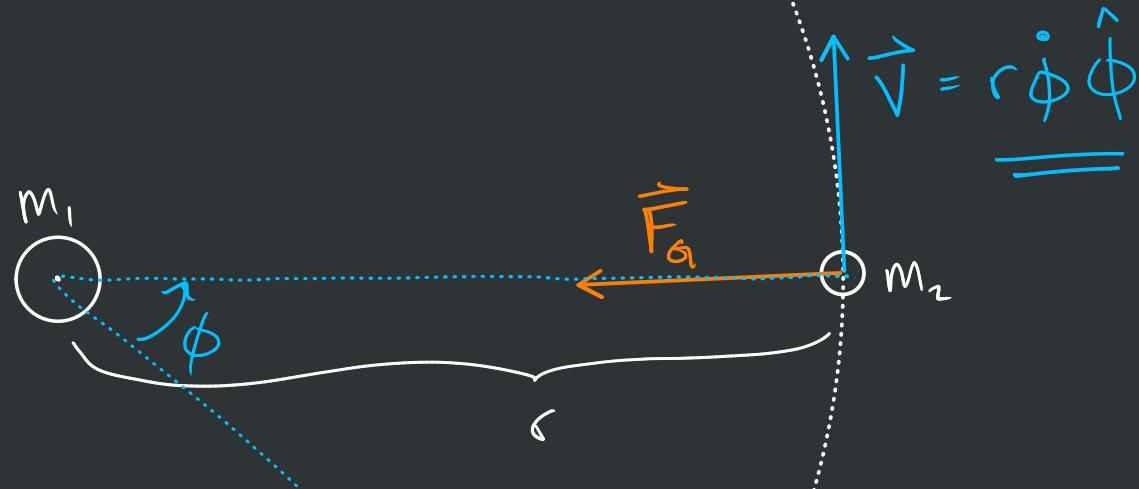
$$r_e = 6371 \text{ km}$$

$$= 6.37 \cdot 10^6 \text{ m}$$

$$= 6.37 \cdot 10^6 \text{ m} + 1000 \text{ m}$$

$$= 6.371 \cdot 10^6 \text{ m}$$

$$F = \underbrace{\frac{GMm}{r^2}}_{\text{ag}} \Rightarrow \text{mag} =$$



$$U = -\frac{Gm_1m_2}{r}$$

$$T = \frac{1}{2} m_1 v^2$$

relationship between

$$U + T$$

$$\ddot{v} = \vec{a} = (\ddot{r} - r\dot{\phi}^2)\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi}$$

$$\vec{F}_G = m_2 \vec{a}$$

$$-\frac{Gm_1m_2}{r^2} \hat{r} = m_2 [(\ddot{r} - r\dot{\phi}^2)\hat{r} + (2\dot{r}\dot{\phi} + r\ddot{\phi})\hat{\phi}]$$

$$\underbrace{\ddot{r}}_{=0} + \underbrace{2\dot{r}\dot{\phi}}_{\ddot{\phi}=0} = \underbrace{\ddot{r}}_{=0} + \underbrace{r\ddot{\phi}\hat{\phi}}_{\dot{\phi}=0}$$

$$-\frac{Gm_1}{r^2} \hat{r} = -r\dot{\phi}^2 \hat{r}$$

$$\frac{Gm_1}{r^2} = r\dot{\phi}^2 \quad \leftarrow \quad \frac{v}{r} = \dot{\phi} = \omega$$

$$\frac{Gm_1}{r^2} = \frac{v^2}{r} \quad \rightarrow \quad v = \underbrace{\left(\frac{Gm_1}{r}\right)^{1/2}}$$

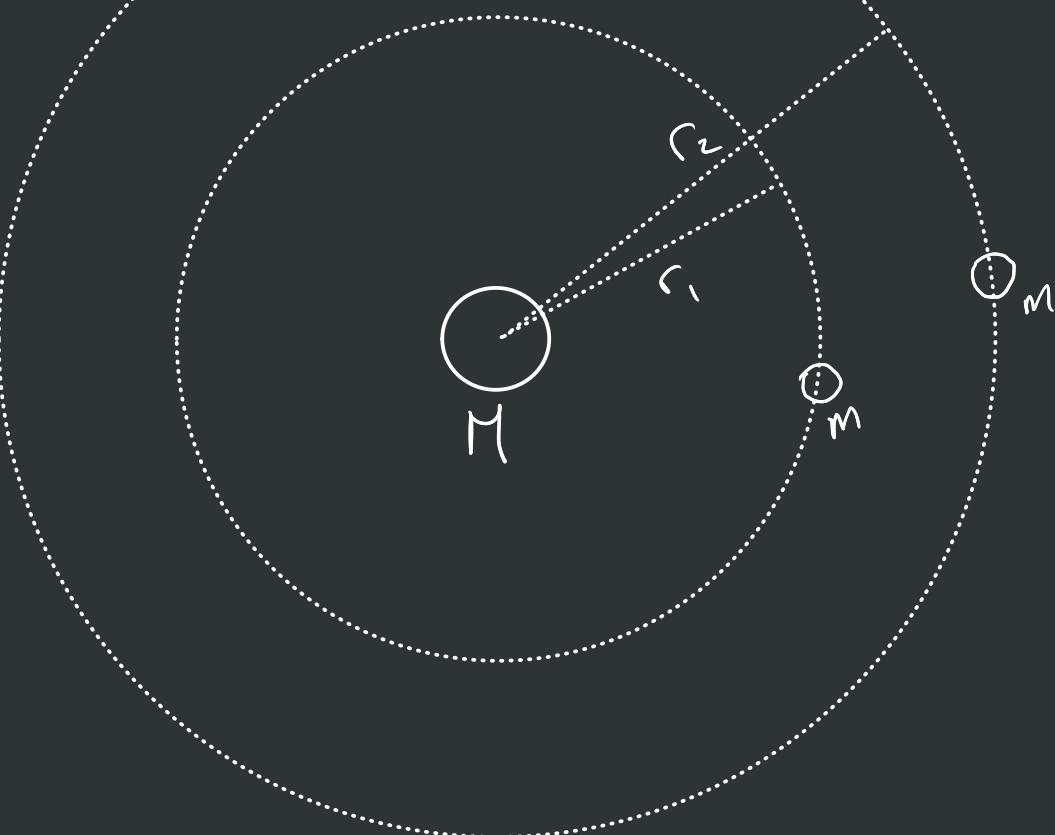
$$U = -\frac{Gm_1 m_2}{r}$$

$$T = \frac{1}{2} m_2 v^2 = \frac{1}{2} m_2 \left(\frac{Gm_1}{r} \right) = \frac{Gm_1 m_2}{2r}$$

$$T = -\frac{U}{2}$$

$$E = T + U$$

$$E = -\frac{U}{2} + U = \frac{U}{2}$$

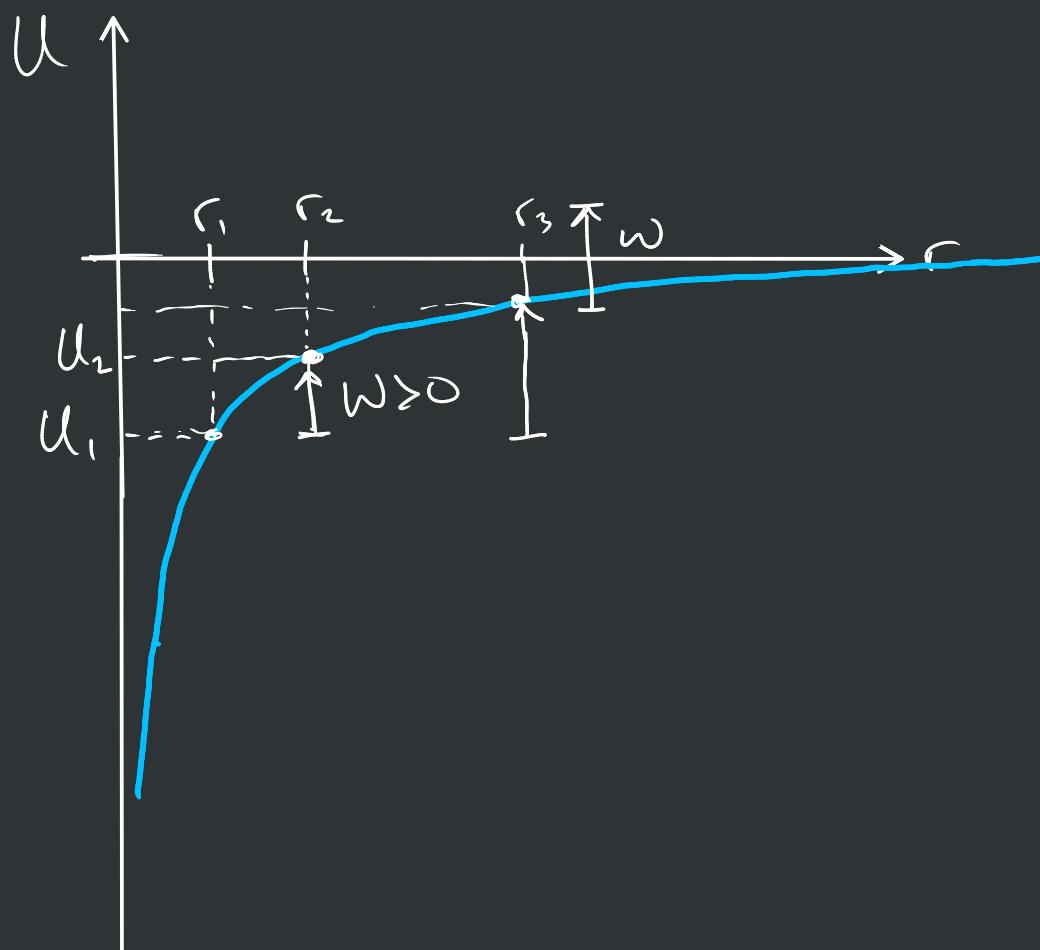


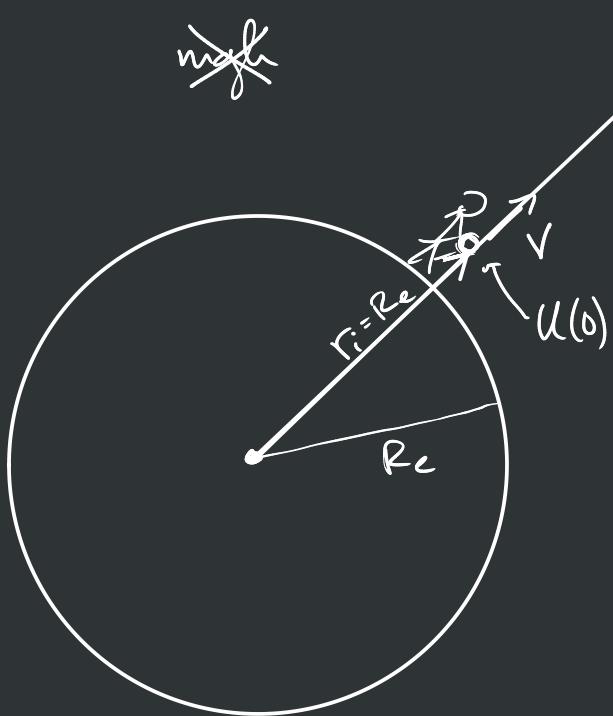
planet at r .

$$U = -\frac{GMm}{r}$$

$$E = -\frac{GMm}{2r}$$

potential energy
relative to
really far away





$$E_i = \frac{1}{2}mv_i^2 + 0 \xrightarrow{r_i=R_e} Gm_1m_2 \left(\frac{1}{R_e} - \frac{1}{r} \right) \quad r_i=R_e$$

$$E_f = \frac{1}{2}mv_f^2 + GMm \left(\frac{1}{R_e} - \frac{1}{r} \right) \quad r > R_e$$

$E_i = E_f \leftarrow$ Conservation of energy

$$\frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2 + GMm \left(\frac{1}{R_e} - \frac{1}{r} \right)$$

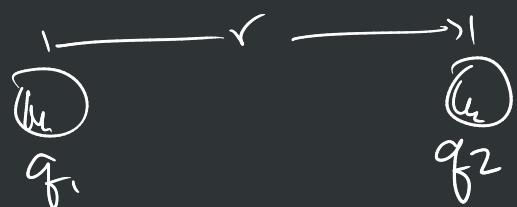
$\downarrow v \rightarrow \text{escape}$
 $\downarrow \text{velocity}$

$\uparrow r \rightarrow \infty$

$\Rightarrow 6.67 \cdot 10^{-11}$

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R_e}$$

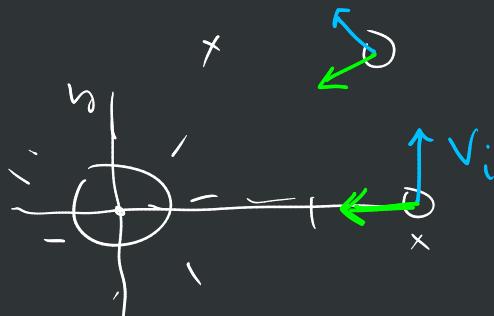
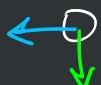
$$v_e = \sqrt{\frac{2GM}{R_e}}$$



Emma's Project

$$\ddot{\vec{r}}_1 = -\frac{Gm_2 \vec{v}_1}{|\vec{r}_1 - \vec{r}_2|^3} - \underbrace{Gm_3 \dots}_{\dots}$$

$$\vec{F}_g = -\frac{Gm_1 m_2}{(x^2 + y^2)^{3/2}} (x \hat{x} + y \hat{y}) = m_1 \underbrace{(x \ddot{x} + y \ddot{y})}_{\ddot{r}}$$



fixed
frame

$$\left\{ \begin{array}{l} \vec{F}_g = -\frac{Gm_1 m_2}{(x^2 + y^2)^{3/2}} (x \hat{x} + y \hat{y}) = m_2 \ddot{x} \hat{x} + \ddot{y} \hat{y} \\ \boxed{-\frac{Gm_2 x}{(x^2 + y^2)^{3/2}}} = \ddot{x} \\ \boxed{-\frac{Gm_2 y}{(x^2 + y^2)^{3/2}}} = \ddot{y} \end{array} \right.$$

$$\vec{r}_1 = (0, 0, 0)$$

$$\vec{r}_2 = (1, 1, 0)$$

$$\vec{r}_3 = (1, -1, -1)$$

$$\vec{v}_1 = (-1, 1, 0)$$

$$\vec{v}_2 = (-1, 0, 1)$$

$$\vec{v}_3 = (0, 0, 0)$$

$$m_1 = m_2 = m$$

$$m_3 = 10 \text{ m}$$

$$R_x = \frac{6 + m + 10m}{12m} = \frac{11}{12} = 0.916 \quad | \quad R_y = \frac{m - 10m}{12m} = -\frac{9}{12} = -0.75 \quad | \quad R_z = -\frac{10m}{12m} = -\frac{5}{6}$$

$$\vec{R} = (0.916, -0.75, -0.833)$$

$$\vec{v}_{cm} = \vec{R} = \frac{1}{M} \sum_m m_\alpha \vec{r}_\alpha$$

$$\vec{p}_1 = m(-1, 1, 0)$$

$$= (-m, m, 0)$$

$$\vec{p}_2 = (-m, 0, m)$$

$$\vec{p}_3 = (0, 0, 0)$$

$$\sum_\alpha \vec{p}_\alpha = (-2m, m, m)$$

$$\begin{aligned} v_{cm,x} &= \frac{m \cdot (-1) + m(-1) + 10m(0)}{12m} \\ &= -\frac{2m}{12m} = -\frac{1}{6} \text{ m/s} \end{aligned}$$

$$v_{cm,y} = \frac{1}{12} \text{ m/s}$$

$$v_{cm,z} = \frac{1}{12} \text{ m/s}$$

$$\vec{v}_{cm} = \left(-\frac{1}{6}, \frac{1}{12}, \frac{1}{12}\right)$$

$$\vec{p}_{cm} = M \cdot \vec{v}_{cm} = 12m \left(-\frac{1}{6}, \frac{1}{12}, \frac{1}{12}\right)$$

$$\vec{l} = \vec{r} \times \vec{p}$$

↗

$$= \overrightarrow{P_{cm}} = (-2m, m, m)$$

$$l_1 = (0, 0, 0) \times (-m, m, 0) = 0$$

$$l_2 = (1, 1, 0) \times (-m, 0, m) = (m, -m, m)$$

$$l_3 = (1, -1, -1) \times (0, 0, 0) = 0$$

$$\overline{\sum_{\alpha} l_{\alpha}} = (m, -m, m)$$

I's later

$$r_1 = (-1, 1, 0)$$

$$\begin{cases} r_2 = (0, 1, 1) \\ r_3 = (0, 1, 1) \end{cases}$$

r's changed

R_{cm} changed

p's not changed

P_{cm} not changed

l not changed

L not changed

$$l_1 = (-1, 1, 0) \times (-m, m, 0) = 0$$

$$l_2 = (1, 1, 0) \times (-m, 0, m) = (m, -m, m)$$

$$l_3 = 0$$

Reference System change

$$\vec{V}' = \vec{V} - \vec{V}_{S \rightarrow S'}$$

$$\left. \begin{aligned} V'_{1x} &= -1 - (-\frac{1}{6}) = -\frac{5}{6} \\ V'_{1y} &= 1 - (\frac{1}{12}) = \frac{11}{12} \\ V'_{1z} &= 0 - (\frac{1}{12}) = -\frac{1}{12} \end{aligned} \right\} \quad \vec{V}'_1 = \left(-\frac{5}{6}, \frac{11}{12}, -\frac{1}{12} \right)$$

$$\vec{V}'_2 = \left(-\frac{5}{6}, -\frac{1}{12}, \frac{11}{12} \right)$$

$$\vec{V}'_3 = \left(\frac{1}{6}, -\frac{1}{12}, -\frac{1}{12} \right)$$

$$\rho_{1x} + \rho_{2x} + \rho_{3x}$$

$$m(-\frac{5}{6}) + m(-\frac{5}{6}) + 10m(\frac{1}{6}) = 0$$

$$\bar{\rho} = 0$$

$$V'_{cm,x} = \frac{m(-\frac{5}{6}) + m(-\frac{5}{6}) + 10m(\frac{1}{6})}{12m}$$

$$= \frac{-\frac{5}{6} - \frac{5}{6} + \frac{10}{6}}{12} = 0$$

$$\vec{V}_{cm} = 0$$

$$\vec{l}_1 = \vec{r}_1' \times \vec{p}_1' = \left(-\frac{1}{12}, \frac{3}{4}, \frac{5}{6} \right) \times \left(-\frac{5}{6}m, \frac{11}{12}m, -\frac{1}{12}n \right)$$

$$= \left(-\frac{119m}{144}, -\frac{37m}{48}, -\frac{31m}{144} \right)$$

$$\vec{r}_1' = \vec{r} - \vec{r} \xrightarrow{\text{S} \rightarrow \text{S}}$$

$$\vec{r}_1' = \left(0 - \frac{1}{12}, 0 - \frac{3}{4}, 0 - \frac{5}{6} \right)$$

$$\vec{r}_1' = \left(-\frac{1}{12}, \frac{3}{4}, \frac{5}{6} \right)$$

$$\vec{r}_2' = \left(1 - \frac{1}{12}, 1 + \frac{3}{4}, 0 + \frac{5}{6} \right)$$

$$\vec{l}_2 = \left(\frac{1}{12}, \frac{7}{4}, \frac{5}{6} \right) \times \left(-\frac{5}{6}m, -\frac{1}{12}m, \frac{11}{12}m \right) = \left(\frac{241m}{144}, -\frac{37m}{48}, \frac{209m}{144} \right)$$

$$= \left(\frac{1}{12}, \frac{7}{4}, \frac{5}{6} \right)$$

$$\vec{r}_3' = \left(1 - \frac{1}{12}, -1 + \frac{3}{4}, -1 + \frac{5}{6} \right)$$

$$\vec{l}_3 = \left(\frac{1}{12}, -\frac{1}{4}, -\frac{1}{6} \right) \times \left(\frac{10}{6}m, -\frac{10}{12}m, -\frac{10}{12}n \right) = \left(\frac{5m}{72}, -\frac{5}{24}m, \frac{25m}{72} \right)$$

$$= \left(\frac{1}{12}, -\frac{1}{4}, -\frac{1}{6} \right)$$

$$\underline{L}' = \left(\frac{11m}{12}, -\frac{7}{4}n, \frac{19m}{12} \right)$$

$$\frac{Gm_1 m_2}{r^2} = \frac{m_2 v^2}{r}$$

$$v = \sqrt{\frac{Gm_1}{r}} \quad v = \sqrt{\frac{1 \cdot 100}{1}} = 10$$

