Chapter 2 - air resistance

$$\sum \vec{F} = \vec{F}_{NET} = m\ddot{\vec{r}}$$

$$\uparrow$$

$$\vec{F}_D = -f(v)\hat{v}$$

$$f(v) = \underbrace{a}_{0} + \underbrace{bv}_{\text{linear drag}} + \underbrace{cv^2}_{\text{quadratic drag}}$$

$$f_{lin} = bv \qquad b \propto D \qquad b = \beta D$$

related to viscosity

$$f_{quad} = cv^2 \qquad c \propto D^2 \qquad c = \gamma D^2$$

density

$$\frac{f_{quad}}{f_{lin}} \propto \frac{D^2 v^2}{D v} = D v$$

density

$$\frac{f_{quad}}{f_{lin}} \approx \text{Reynolds number} = \frac{\rho D v}{\eta}$$

viscosity

$$\vec{F}_{NET} = m\ddot{\vec{r}}$$

$$m\vec{g} - b\vec{v} = m\ddot{\vec{r}}$$

$$\underbrace{m\vec{g} - b\vec{v} = m\dot{\vec{v}}}_{\text{really two equations}}$$

$$-bv_x = m\dot{v}_x$$

$$\dot{v}_x = -\frac{b}{m} \cdot v_x$$

$$\boxed{\frac{dv_x}{dt} = -\frac{b}{m} \cdot v_x}$$

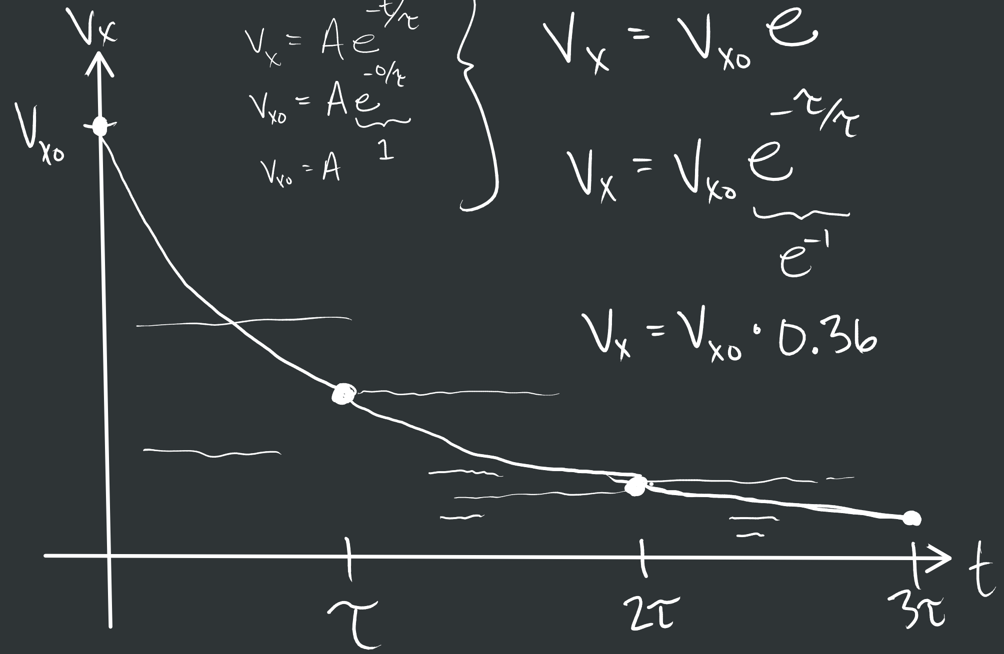
$$\frac{dv_x}{v_x} = -\frac{b}{m} dt$$

$$mg - bv_y = m\dot{v}_y$$

down ⊕ y-dir

$$V_x = Ae^{-t/\tau}$$
$$V_{x0} = A\underbrace{e^{-0/\tau}}_{1}$$
$$V_{x0} = A$$

$$V_x = V_{x0}\, e^{-t/\tau}$$
$$V_x = V_{x0}\,\underbrace{e^{-\tau/\tau}}_{e^{-1}}$$
$$V_x = V_{x0}\cdot 0.36$$



$$\int \frac{dv_x}{v_x} = -\frac{b}{m}\int dt$$

$$e^{\ln v_x} = e^{\left(-\frac{b}{m}t + C\right)} = e^{-\frac{b}{m}t}\cdot e^{C}$$

$$V_x = A\cdot e^{-\frac{b}{m}t}$$

$$\frac{b}{m} = \left[\frac{1}{s}\right] = \left[s^{-1}\right]$$

$$\tau = \frac{m}{b} = [s]$$ → time constant

$$V_x = Ae^{-t/\tau}$$

exponential decay

$$\dot{x} = V_{x0}\, e^{-t/\tau}$$

$$\frac{dx}{dt} = v_{x0} e^{-t/\tau}$$

$$\int dx = v_{x0} \int e^{-t/\tau} \, dt$$

$$\int_{x_0}^{x} dx' = \underbrace{v_{x0} \int_0^{t} e^{-t'/\tau} \, dt'}$$

function

$$x' \Big|_{x_0}^{x} = v_{x0}(-\tau) e^{-t'/\tau} \Big|_0^{t}$$

$$(x - x_0) = -v_{x0} \tau \cdot \left(e^{-t/\tau} - \cancelto{1}{e^{-0/\tau}} \right)$$

$$x - x_0 = v_{x0} \tau \left(1 - e^{-t/\tau} \right)$$

$$\boxed{x(t) = v_{x0} \tau \left(1 - e^{-t/\tau} \right) + x_0}$$

Linear Drag Vertically

$$mg - bv_y = m\dot{v}_y$$

if $\dot{v}_y = 0$

then $\underline{mg - bv_y = 0}$

$$v_y = \frac{mg}{b} \equiv v_t$$

velocity when $\dot{v} = 0$

$$m\dot{v}_y = mg - bv_y$$

$$= -b\left(-\frac{mg}{b} + v_y\right)$$

$$m\dot{v}_y = -b(-v_t + v_y)$$

$$u = (-v_t + v_y)$$

$$\dot{u} = \frac{du}{dt} = 0 + \dot{v}_y$$

$$\dot{u} = \dot{v}_y$$

$$m\dot{u} = -b \cdot u$$

$$\dot{u} = -\frac{b}{m} \cdot u$$

$$\tau = \frac{m}{b}$$

$$u = Ae^{-t/\tau}$$

$$-v_t + v_y = Ae^{-t/\tau}$$

$$V_y = Ae^{-t/\tau} + V_t$$

$$V_y(t=0) = V_{y0}$$

$$V_{y0} = A\underbrace{e^{-0/\tau}}_{1} + V_t$$

$$V_{y0} = A + V_t$$

$$A = V_{y0} - V_t$$

$$V_y = (V_{y0} - V_t)e^{-t/\tau} + V_t$$

$$\boxed{V_y = (V_{y0} - V_t)e^{-t/\tau} + V_t}$$

$$V_y = \underbrace{V_{y0}e^{-t/\tau}}_{t\to\infty,\ \to 0} + \underbrace{V_t\left(1 - e^{-t/\tau}\right)}_{t\to\infty}$$

$$\cancel{dt}\,\frac{dy}{\cancel{dt}} = \left((v_{y0} - v_t) e^{-t/\tau} + v_t \right) dt$$

$$\int_{y_0}^{y} dy' = \int_0^t \left((v_{y0} - v_t) e^{-t'/\tau} + v_t \right) dt'$$

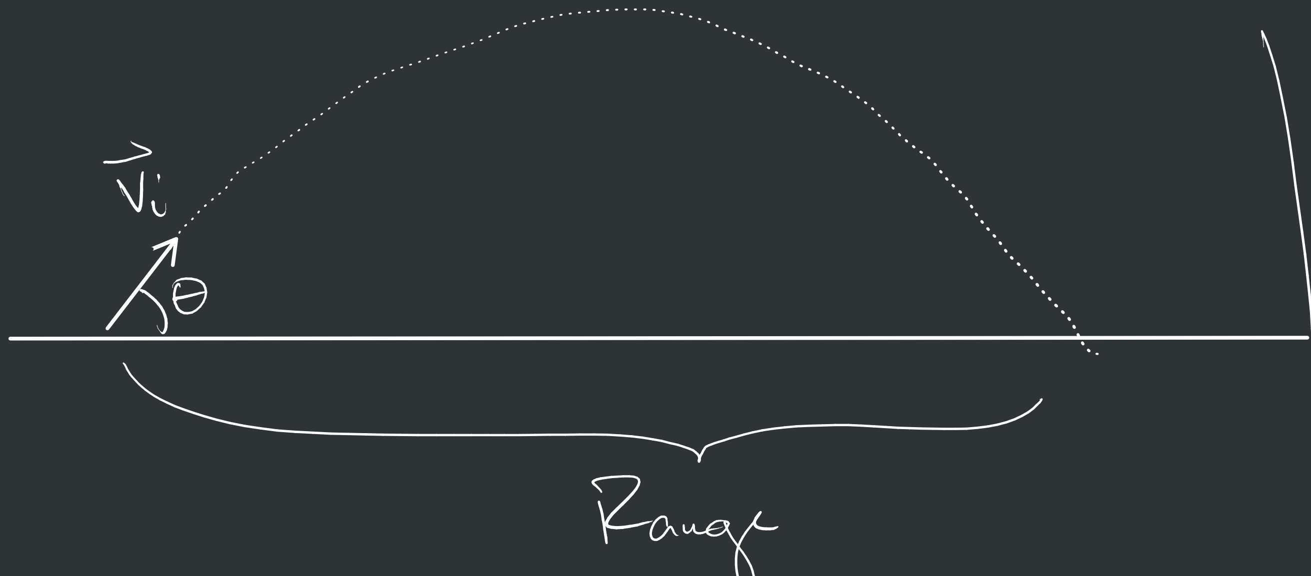
$$y' \Big|_{y_0}^{y} = \Big[(v_{y0} - v_t)(-\tau) e^{-t'/\tau} + v_t t' \Big]_0^t$$

$$y - y_0 = -(v_{y0} - v_t) \tau e^{-t/\tau} + v_t t - \Big[-(v_{y0} - v_t) \tau \cdot 1 + 0 \Big]$$

algebra

$$\boxed{y = (v_{y0} - v_t) \tau \left(1 - e^{-t/\tau} \right) + v_t t + y_0}$$

Range of Projectile Motion in vacuum



$v_x = v_i \cos\theta$

$x = v_x \cdot t = v_i \cos\theta \cdot t$

$v_{y0} = v_i \sin\theta$

$y = y_i + v_{y0} \cdot t + \frac{1}{2} a_y t^2$

$$\boxed{\begin{array}{l} \text{Range} \rightarrow \text{solve } y=0 \text{ for } t \\ \qquad\qquad \text{plug } t \text{ in for } x \end{array}}$$

$0 = v_{y0} t + \frac{1}{2} a_y t^2$

$0 = t\left(v_{y0} + \frac{1}{2} a_y t\right)$

$t = 0$ or $t = -\dfrac{2v_{y0}}{a_y}$

$\text{Range} = v_x \left(\dfrac{-2v_{y0}}{a_y}\right)$

$a_y = -g$

$R = \dfrac{2 v_x v_{y0}}{g}$

Range for Projectile Motion in Linear Drag

$$x = v_{x0}\tau\left(1 - e^{-t/\tau}\right) + \cancelto{0}{x_0} \qquad \bigg| \qquad y = (v_{y0} - v_t)\tau\left(1 - e^{-t/\tau}\right) + v_t t + \cancelto{0}{y_0}$$

$$x = v_{x0}\tau\left(1 - e^{-t/\tau}\right)$$

solve for t

$$y = (v_{y0} - v_t)\tau\left(1 - e^{-t/\tau}\right) + v_t t$$

$$= (v_{y0} - v_t)\tau\left(1 - e^{-t/\tau}\right) + v_t t$$

⋮

lots of algebra (to eliminate t)

⋮

$$y = \left(\frac{v_{y0} + v_t}{v_{x0}}\right)x + v_t\tau\ln\left(1 - \frac{x}{v_{x0}\tau}\right)$$

$$0 = \left(\frac{V_{y0} + V_t}{V_{x0}}\right) R + V_t \tau \ln\left(1 - \frac{R}{V_{x0}\tau}\right)$$

Taylor series approx

$$-\left(\frac{V_{y0} + V_t}{V_{x0}}\right)\frac{R}{V_t \tau} = \ln\left(1 - \frac{R}{V_{x0}\tau}\right) \leftarrow \ln(1-\epsilon) \hat{\approx}$$

$$\underbrace{-\left(\epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 + \dots\right)}_{\text{quadratic equation}}$$

$$e^{-\left(\frac{V_{y0} + V_t}{V_{x0}}\right)\frac{R}{V_t \tau}} = 1 - \frac{R}{V_{x0}\tau}$$

$$\boxed{\frac{R}{V_{x0}\tau} = 1 - e^{-\left(\frac{V_{y0} + V_t}{V_{x0}}\right)\frac{R}{V_t \tau}}}$$

$$\cancel{X = e^{X}}$$

$$\rightarrow X = e^{-X}$$

$$\rightarrow X = (1 - e^{-X})$$

Quadratic Drag

Horizontal

$$F_{NET} = m\dot{v}$$

$$-cv^2 = m\dot{v}$$

$$\vec{v} = v_x \hat{x}$$

$$m\frac{dv}{dt} = -cv^2$$

$$\frac{dv}{v^2} = -\frac{c}{m}\,dt$$

$$\int \frac{dv}{v^2} = -\frac{c}{m}\int dt$$

$$\int v^{-2}dv = -v^{-1} = -\frac{c}{m}t + D$$

$$x = e^{-x}$$

$$x = 0.567$$

$$x = (1 - e^{-x})$$

$$x = 0$$

$$m\frac{dv_x}{dt} = -c(v_x^2 + v_y^2)$$

$$m\frac{dv_y}{dt} = -c(v_x^2 + v_y^2)$$

coupled system of DE

$$\frac{1}{v} = \frac{c}{m}t + D$$

$$v(t) = \frac{1}{\frac{c}{m}t + D}$$

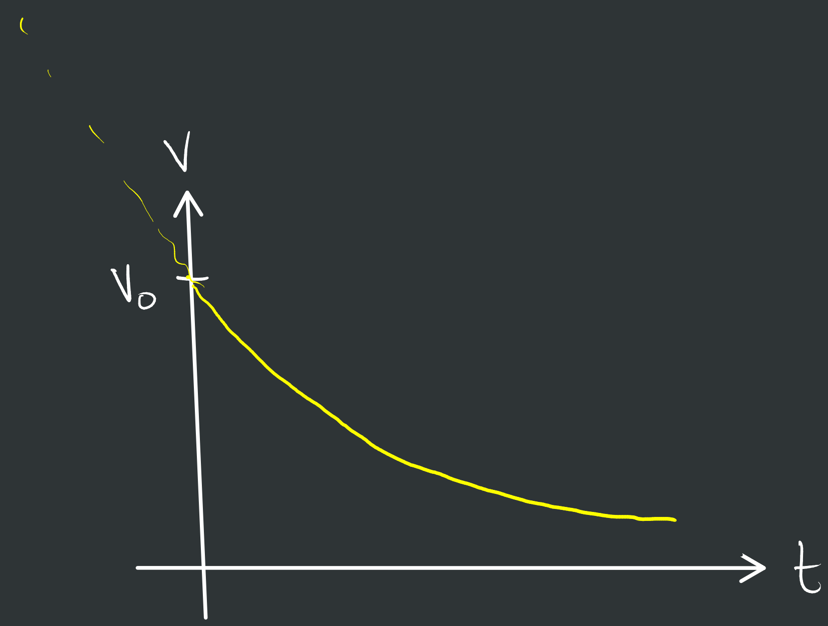
$$v_0 = v(0) = \frac{1}{D} \Rightarrow D = 1/v_0$$

$$v(t) = \frac{1}{\frac{c}{m}t + 1/v_0}$$

$$v(t) = \frac{v_0}{\frac{cv_0}{m}t + 1}$$

$$\tau = \frac{m}{cv_0}$$

$$\boxed{v(t) = \frac{v_0}{t/\tau + 1}}$$



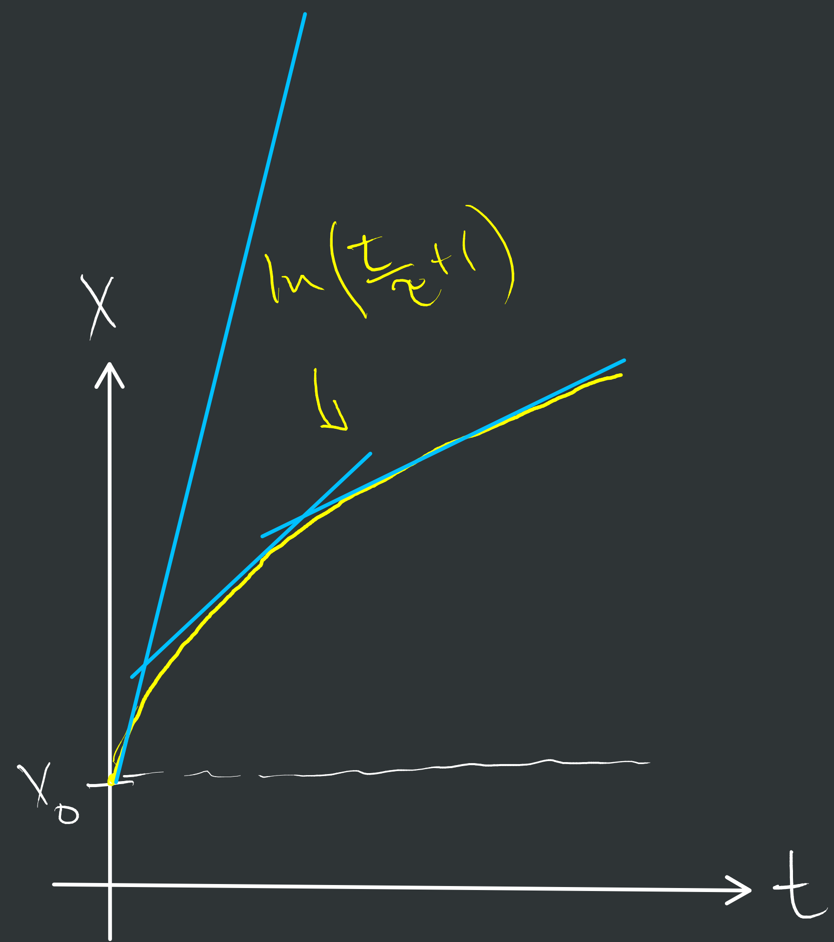
$$\frac{dx}{dt} = \frac{V_0}{t/\tau + 1}$$

$$\int dx = \int \frac{V_0}{t/\tau + 1}\, dt$$

$$x = V_0 \int \frac{1}{t/\tau + 1}\, dt$$

$$x = V_0 \tau \underbrace{\int \frac{1}{u} \cdot du}_{\ln(u)}$$

$$\left\{ \begin{aligned} u &= \frac{t}{\tau} + 1 \\ du &= \frac{du}{dt} dt \\ du &= \frac{1}{\tau} dt \\ \tau\, du &= dt \end{aligned} \right.$$



$$x(t) = V_0 \tau \ln\left(\frac{t}{\tau} + 1\right) + E$$

$$X_0 = x(0) = V_0 \tau \underbrace{\ln(1)}_{0} + E$$

$$X_0 = E$$

$$x(t) = V_0 \tau \ln\left(\frac{t}{\tau} + 1\right) + X_0$$

Vertical Direction

$$m\dot{v} = mg - cv^2$$

$$\dot{v} = 0 = mg - cv_t^2$$

$$v_t = \sqrt{\frac{mg}{c}}$$

$$\dot{v} = g - \frac{c}{m}\cdot\frac{g}{g}v^2$$

$$\dot{v} = g\left(1 - \frac{c}{mg}v^2\right)$$

$$\dot{v} = g\left(1 - \frac{v^2}{v_t^2}\right)$$