

# Chapter 6 & 7 $\rightarrow$ Principle of Least Action and the Lagrangian



$\rightarrow$  "generalized coordinates"  
 $\rightarrow$  Principle of Least Action

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$U = mgx$$

$$\int_{t_1}^{t_2} \left( \underbrace{\frac{1}{2}m\dot{x}^2}_{v(t)} - \underbrace{mgx}_{x(t)} \right) dt$$

what is  $x(t)$  and  $\dot{x}(t)$   
 $\hookrightarrow$  so this integral is smallest

Lagrangian

$$L = T - U$$

Action

$$S = \int_{t_1}^{t_2} L(x(t), \dot{x}(t), t) dt$$
$$= \int_{t_1}^{t_2} L(\overset{\downarrow}{x}, \overset{\downarrow}{\dot{x}}, t) dt$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\boxed{\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{\partial \mathcal{L}}{\partial x}}$$

Euler - Lagrange  
Equation

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m v^2$$

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 + \frac{1}{2} m \dot{z}^2 - U(x, y, z)$$

↳ 1D motion

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{d}{dt} \left( \frac{\partial}{\partial \dot{x}} \left( \frac{1}{2} m \dot{x}^2 - U(x) \right) \right) = \frac{\partial}{\partial x} \left( \frac{1}{2} m \dot{x}^2 - U(x) \right)$$

$$\frac{d}{dt} (m \dot{x}) = - \frac{dU}{dx}$$

$$m\ddot{x} = -\frac{dU}{dx}$$

→ In 3D

$$\left. \begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{\partial \mathcal{L}}{\partial x} \\ m\ddot{x} &= -\frac{\partial U}{\partial x} \end{aligned} \right|$$

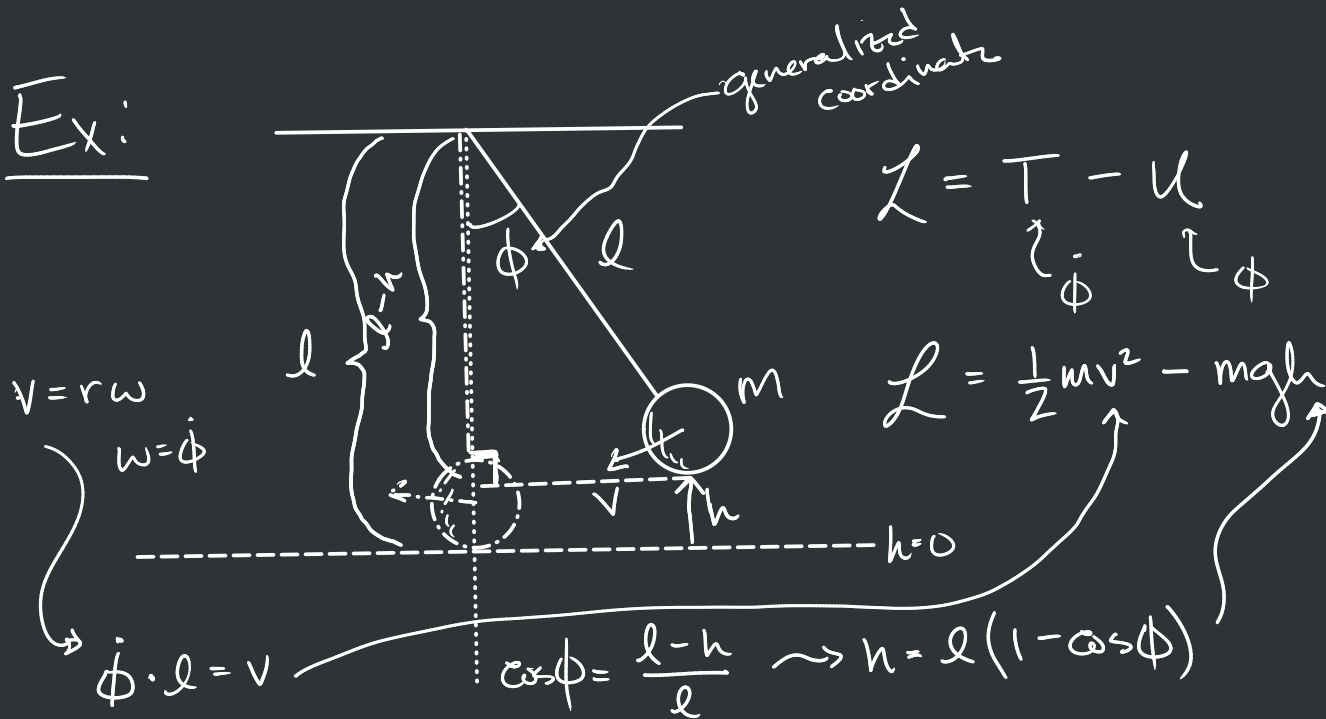
$$\left. \begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) &= \frac{\partial \mathcal{L}}{\partial y} \\ m\ddot{y} &= -\frac{\partial U}{\partial y} \end{aligned} \right|$$

$$\left. \begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) &= \frac{\partial \mathcal{L}}{\partial z} \\ m\ddot{z} &= -\frac{\partial U}{\partial z} \end{aligned} \right|$$

$$m\ddot{x}\hat{x} + m\ddot{y}\hat{y} + m\ddot{z}\hat{z} = -\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}$$

$$\underbrace{\sum \vec{F}} = m\ddot{\vec{r}} = -\underbrace{\vec{\nabla} U} = \sum \vec{F}$$

Ex:



$$\mathcal{L} = T - U$$

$\uparrow$        $\uparrow$   
 $\dot{\phi}$        $\phi$

$$\mathcal{L} = \frac{1}{2}mv^2 - mgh$$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1 - \cos \phi)$$

→ apply the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{2} m l^2 (2\dot{\phi}) = m l^2 \dot{\phi} \quad \left| \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m g l (-(-\sin \phi)) \right.$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = \frac{d}{dt} (m l^2 \dot{\phi}) = m l^2 \ddot{\phi} \quad \leftarrow \quad = -m g l \sin \phi$$

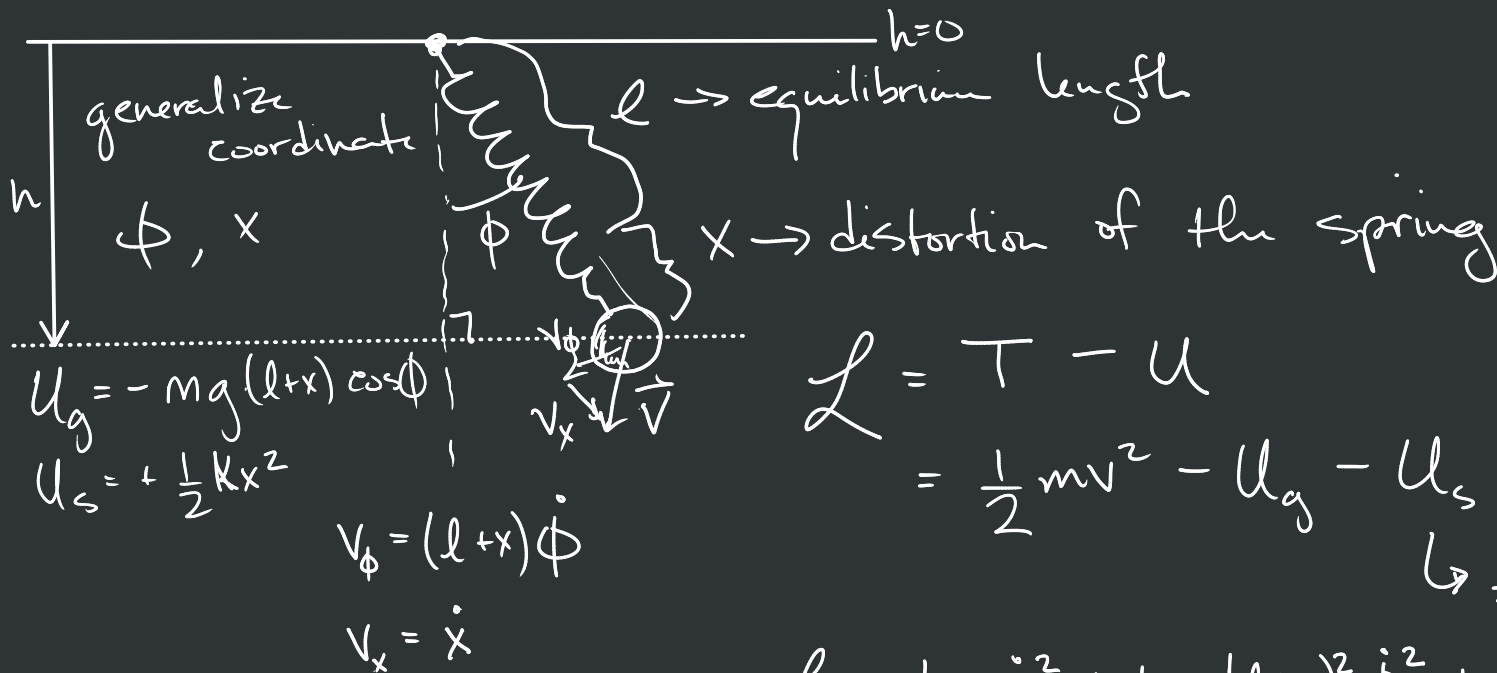
$$\ddot{\phi} = -\frac{g}{l} \sin \phi$$

$$m l^2 \ddot{\phi} = -m g l \sin \phi$$

$$\rightarrow \boxed{\ddot{\phi} = -\frac{g}{l} \sin \phi}$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{\partial \mathcal{L}}{\partial x}$$

Remember small angle  
approx  $\sin \phi \approx \phi$



$$\mathcal{L} = T - U$$

$$= \frac{1}{2} m v^2 - U_g - U_s$$

$\hookrightarrow \frac{1}{2} k x^2$

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m (l+x)^2 \dot{\phi}^2 + m g (l+x) \cos \phi - \frac{1}{2} k x^2$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m(l+x)^2 \dot{\phi}$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = m(l+x)^2 \ddot{\phi} + m2(l+x) \dot{x} \dot{\phi}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -mg(l+x) \sin \phi$$

Coriolis force

$$m(l+x)^2 \ddot{\phi} + m2(l+x) \dot{x} \dot{\phi} = -mg(l+x) \sin \phi$$

$$(l+x) \ddot{\phi} + 2\dot{x} \dot{\phi} + g \sin \phi = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = m\ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial x} = m(l+x) \dot{\phi}^2 + mg \cos \phi - kx$$

centrifugal

$$m\ddot{x} = m(l+x) \dot{\phi}^2 + mg \cos \phi - \frac{k}{m}x$$

$$\ddot{x} = (l+x) \dot{\phi}^2 + g \cos \phi - \frac{k}{m}x$$

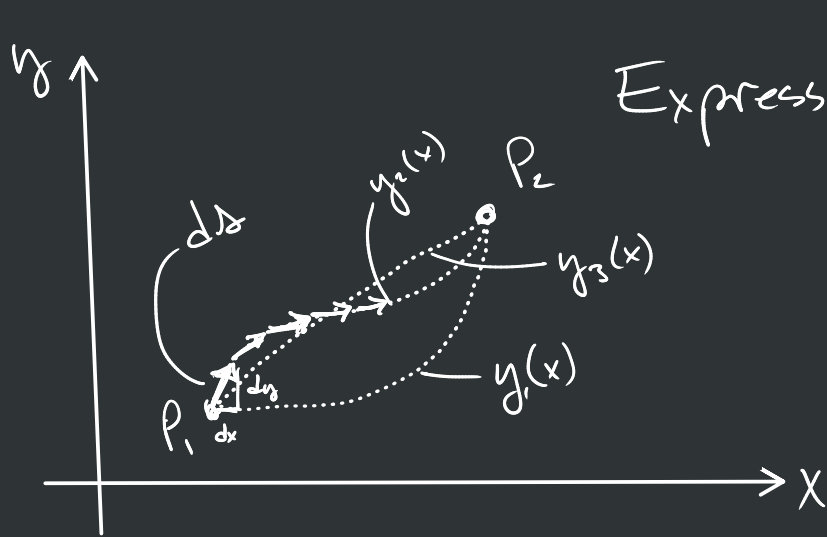
Calculus of Variations  $\rightarrow$  Find the min/max of a function expressible by an integral

minimize  $g = \int_{x_1}^{x_2} f(y, y', x) dx$

$$y' = \frac{dy}{dx}$$

$\rightarrow$  what is  $y$  that minimizes  $g$ ?

$$\frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = \frac{\partial f}{\partial y} \quad \leftarrow \text{E-L equation}$$



Express path length between 2 pts as an integral?

$$\Delta = \int_{P_1}^{P_2} ds$$

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$ds = \sqrt{dx^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)}$$

$$y' = \frac{dy}{dx}$$

$$= dx \cdot \sqrt{1 + y'^2}$$

total  
path  
length  $\rightarrow$

$$s = \int_{P_1}^{P_2} \underbrace{\sqrt{1 + y'^2}}_{\text{}} dx$$

$$f(y, y', x) = \sqrt{1 + y'^2}$$

$$\frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = \frac{\partial f}{\partial y} \leftarrow \underline{\underline{E-L}}$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2} (1 + y'^2)^{-1/2} \cdot (+2) y' \quad \Bigg| \quad \frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx} \left( \frac{1}{2} (1 + y'^2)^{-1/2} \cdot (+2) y' \right) = 0$$

$$\frac{1}{2} (1 + y'^2)^{-1/2} \cdot \cancel{(+2)} y' = \text{Constant} \quad \leftarrow$$

$$\frac{y'}{\sqrt{1+y'^2}} = C$$

$$\frac{y'^2}{1+y'^2} = C$$

$$y'^2 = C(1+y'^2)$$

$$y'^2 = C + Cy'^2$$

$$y'^2(1-C) = C$$

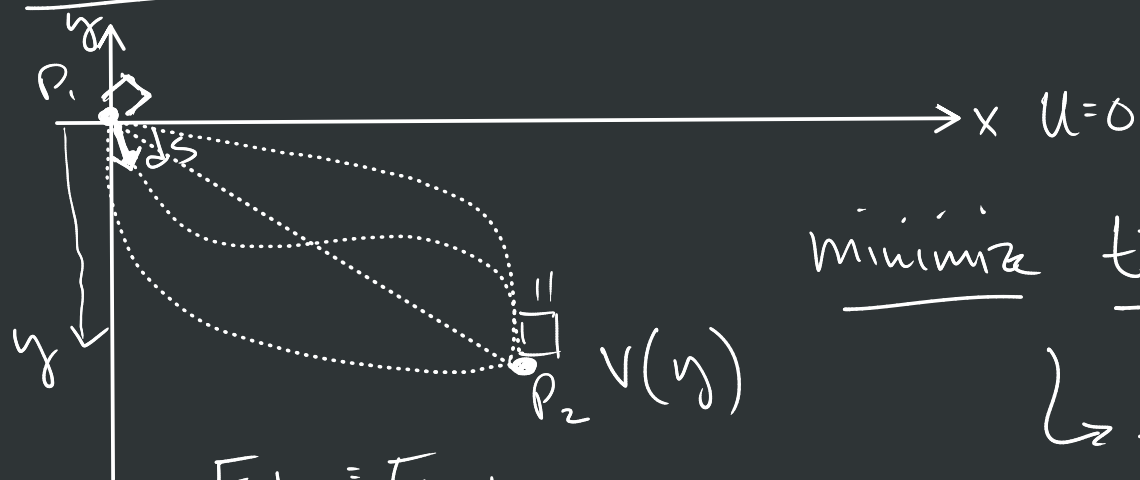
$$y'^2 = C$$

$$y' = C$$

$\frac{dy}{dx} = \underbrace{\text{Constant}}_{m \rightarrow \text{slope}} \quad \leftarrow$   
 integrate  $\rightarrow \underline{y = mx + b}$



# Brachistochrone



minimize time

$$\hookrightarrow t = \int \text{---}$$

$$E_{\text{top}} = E_{\text{bottom}}$$

$$0 = \frac{1}{2}mv^2 - mgy$$

$$mgy = \frac{1}{2}mv^2$$

$$v = \sqrt{2gy}$$

$$dt = \frac{ds}{v}$$

$$ds = dx \sqrt{1 + y'^2}$$

—OR—

$$ds = dy \sqrt{1 + x'^2}$$

$$x' = \frac{dx}{dy}$$

$$t = \frac{1}{\sqrt{2g}} \int \frac{\sqrt{1 + y'^2}}{\sqrt{y}} dx$$

$$\hookrightarrow \frac{d}{dx} \left( \frac{\partial f}{\partial y'} \right) = \frac{\partial f}{\partial y}$$

—OR—

$$t = \frac{1}{\sqrt{2g}} \int \frac{\sqrt{1 + x'^2}}{\sqrt{y}} dy$$

$$\hookrightarrow \frac{d}{dy} \left( \frac{\partial f}{\partial x'} \right) = \frac{\partial f}{\partial x}$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial f}{\partial x'} = C$$



