

## Chapter 3 - Momentum + Angular Momentum

Recall:  $\vec{p} = m\vec{v}$        $\sum \vec{F}_n = \dot{\vec{p}}$

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 \dots = \sum \vec{p}_n$$

$$\dot{\vec{P}} = \vec{F}_{\text{ext}}$$

If  $\vec{F}_{\text{ext}} = 0$

$$\dot{\vec{P}} = 0$$

$\rightarrow \vec{P}_i = \vec{P}_f$       Conservation of Momentum

If two objects,

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}$$

← really 2 equations

If collision is perfectly inelastic  
↳ stick together

then,

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_f + m_2 \vec{v}_f$$
$$= (m_1 + m_2) \vec{v}_f$$

mass add  
up!

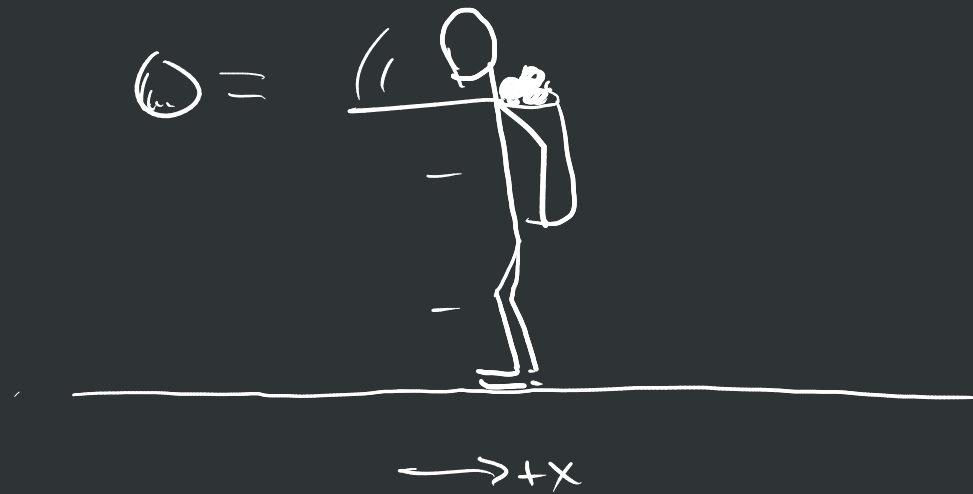
$$\vec{v}_f = \frac{m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i}}{m_1 + m_2}$$

✓ ← two equations!

# Rockets + Explosions

At the beginning

$$P = 0$$



Round 1

$$P_i = 0 = P_f = p_{ME} + p_{ball}$$

$$0 = \underbrace{(75 \text{ kg} + 99(0.5 \text{ kg}))}_ {124.5 \text{ kg}} v + (0.5 \text{ kg})(-15 \text{ m/s})$$

$$0 = 124.5 v - 7.5 \frac{\text{kg m}}{\text{s}}$$

$$v = \frac{7.5 \text{ kg m/s}}{124.5 \text{ kg}} = 0.0602 \text{ m/s}$$

Row 12

$$V' = V_{ice} - V_{s'}$$
$$-15 \text{ m/s} = V_{ice} - 0.0602 \text{ m/s}$$

$$-15 + 0.06 = V_{ice} = -14.94 \text{ m/s}$$

0 =



ice

$$P_i = P_f$$

$$(124.5)(0.0602 \text{ m/s}) + 0.5(-15) = 124 \cdot v + 0.5(-15) + 0.5(-14.94)$$

= 0

$$-0.5(-15 - 14.94) = 124 \cdot v$$

$$v = \frac{-0.5(-15 - 14.94)}{124} = \frac{-0.5(-15 - 15 + 0.0602)}{124}$$

$$= 0.1207 \text{ m/s}$$

but we need a pattern to do this numerically - this way is not giving that away

$$(m + \Delta m)(v + \Delta v) = mv + \Delta mv + m\Delta v + \Delta m\Delta v$$

$$\Delta P_{ME} = \Delta P_{ball}$$

$$124.5(0.0602) - 124v = 0.5(-14.94)$$

$m\Delta v \quad mv \quad \Delta mv$

$$0.5(15) + 0.5 \cdot 0.0602$$

$$\frac{0.5(14.94)}{124}$$

$$0.0602410$$

$$\frac{0.5(15 - \frac{0.5(15)}{124.5})}{124}$$

$$\frac{0.5(15)}{124.5}$$

$$0.0602406$$

$$\frac{0.5(15 - x) - x}{124} = ?$$

IGNORE

THESE

NOTES,  
REMINGTON  
AT WORK

$$v = \frac{-0.5(-15 - 14.94)}{124} = \frac{0.5(15 + 14.94)}{124}$$
$$= \frac{0.5(15)}{124} + \frac{0.5(14.94)}{124}$$

$0.06048 \quad 0.0602$

$$\frac{0.5(14.94)}{124} \stackrel{?}{=} \frac{0.5(15)}{124.5}$$

$0.0602 = 0.0602$

## Round 1 Redux - my frame

$$P_i = 0 = P_f = p_{ME} + p_{ball}$$

$$0 = \underbrace{(75 \text{ kg} + 99(0.5 \text{ kg}))}_ {124.5 \text{ kg}} v + (0.5 \text{ kg})(-15 \text{ m/s})$$

$$0 = 124.5 v - 7.5 \frac{\text{kg m}}{\text{s}}$$

$$v = \frac{7.5 \text{ kg m/s}}{124.5 \text{ kg}} = 0.0602 \text{ m/s}$$

## Round 2 Redux - my frame

$P_i'$

$$0 = 124.0 v + 0.5(-15)$$

$$v = \frac{0.5(15)}{124} = 0.06048 \text{ m/s}$$

$$\frac{0.5(15 - 0.602)}{124} =$$

$$\frac{\cancel{0.5}(15) - 0.5(\cancel{15})}{124} = \frac{\cancel{0.5}(\cancel{15})}{124.5}$$

$$\frac{1}{124} - \frac{0.5}{124.5} = \frac{1}{124.5}$$

Round 3 Redux - my frame

$P_i''$

$$D = 123.5 \text{ V} + 0.5(-15)$$

$$V = \frac{0.5(15)}{123.5} = 0.0607 \text{ m/s}$$

0.18

Generalize

$$dV = \frac{dP_{ex}}{m} \quad \leftarrow \text{Variable}$$

$$\int_{V_i}^{V_f} dV = \int_{m_0+m_f}^{m_0} \frac{V_{ex} dm}{m} = V_{ex} \int_{m_0+m_f}^{m_0} \frac{dm}{m}$$

$$V \Big|_{v_i}^{v_f} = V_{ex} \ln(m) \Big|_{m_0+m_f}^{m_0}$$

$$v_f - v_i = V_{ex} \left[ \ln(m_0) - \ln(m_0 + m_f) \right]$$

$$\underline{v_f = V_{ex} \ln\left(\frac{m_0}{m_0 + m_f}\right) + v_i}$$

$$\hookrightarrow \underline{\underline{V(t) = V_{ex} \ln\left(\frac{m(t)}{m_0 + m_f}\right) + v_i}}$$

test w berechnung

$$\begin{aligned} v_f &= -15 \ln\left(\frac{75\text{kg}}{125\text{kg}}\right) \\ &= \underline{\underline{+7.66 \text{ m/s}}} \end{aligned}$$

$$\rightarrow V(t) = V_{0x} \ln \left( \frac{m(t)}{m_0 + m_f} \right) + V_0$$

$$0 = +15 \ln \left( \frac{m_0}{m_0 + 0.5n} \right) + 10 \quad \leftarrow$$

How many  
baseballs to stop  
from 10 m/s

$$-10 = +15 \ln \left( \frac{m_0}{m_0 + 0.5n} \right)$$

$$\frac{-10}{+15} = \ln \left( \frac{m_0}{m_0 + 0.5n} \right)$$

$$e^{-\frac{10}{15}} = \frac{m_0}{m_0 + 0.5n}$$

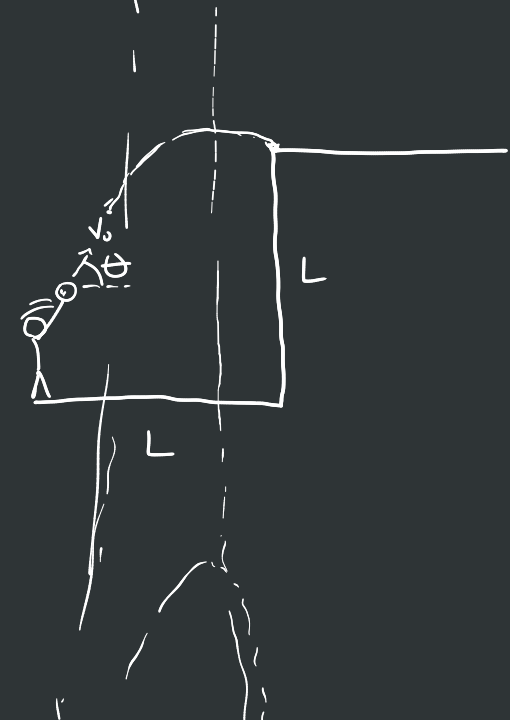
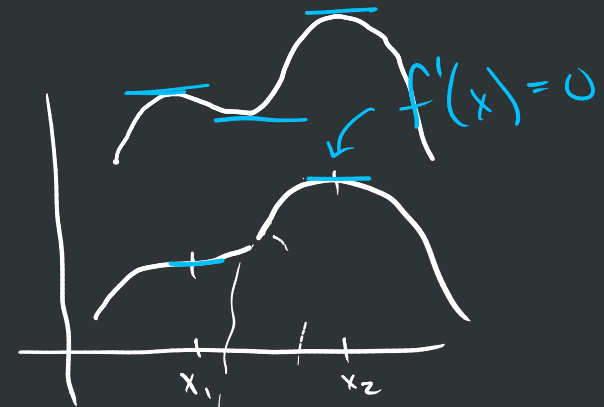
$$e^{-\frac{10}{15}} (m_0 + 0.5n) = m_0$$

$$m_0 + 0.5n = m_0 e^{\frac{10}{15}}$$

$$n = \frac{m_0 (e^{\frac{10}{15}} - 1)}{0.5}$$

$$m_0 = 75 \text{ kg}$$

$$n = 142.16 \quad \leftarrow \# \text{ of baseballs}$$





# Center of Mass



$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots}{m_1 + m_2 + \dots}$$

total, M

$$\vec{R} = \frac{1}{M} \sum_{\alpha=1}^N m_{\alpha} \vec{r}_{\alpha}$$

center of mass is a position

$$x = \frac{1}{M} \sum_{\alpha=1}^N m_{\alpha} x_{\alpha}$$

x-component (x position)

$$y = \frac{1}{M} \sum_{\alpha=1}^N m_{\alpha} y_{\alpha}$$

$$z = \dots$$

$$\dot{\vec{R}} = \frac{1}{M} \sum_{\alpha=1}^N m_{\alpha} \dot{\vec{r}}_{\alpha}$$

$$\vec{P} = M \dot{\vec{R}}$$

for continuous medium

$$m_{\alpha} \rightarrow dm$$

$$\vec{R} = \frac{1}{M} \int \vec{r} dm$$

volume

$$\vec{R} = \frac{1}{M} \int \vec{r} \rho dV$$

$$\rho = \frac{M}{V}$$

$$\vec{P} = \cancel{M} \left( \frac{1}{\cancel{M}} \sum_{\alpha} m_{\alpha} \dot{\vec{r}}_{\alpha} \right)$$

$$= \sum_{\alpha} m_{\alpha} \dot{\vec{r}}_{\alpha}$$

$$\vec{P} = \sum_{\alpha} \vec{p}_{\alpha} \quad \leftarrow \text{already knew from Ch. 1.}$$

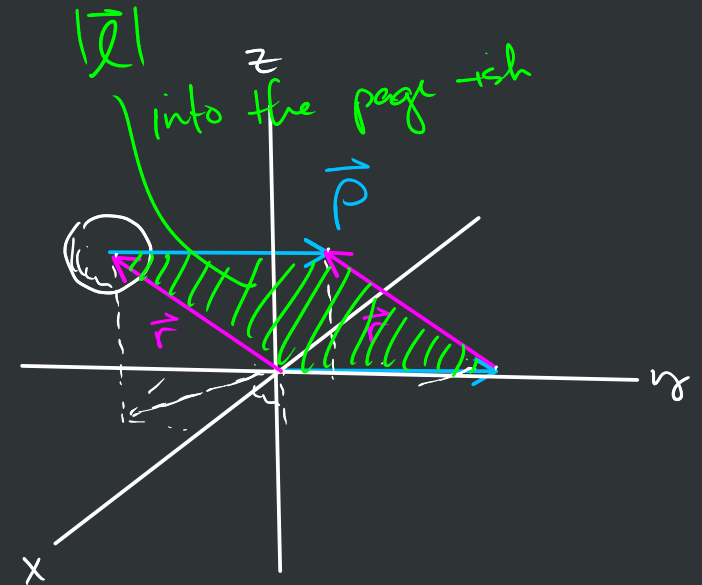
$$\frac{d}{dt} \vec{P} = M \ddot{\vec{R}} = \vec{F}_{\text{ext}} \quad \leftarrow \text{can treat a collection as a single particle at the center of mass}$$

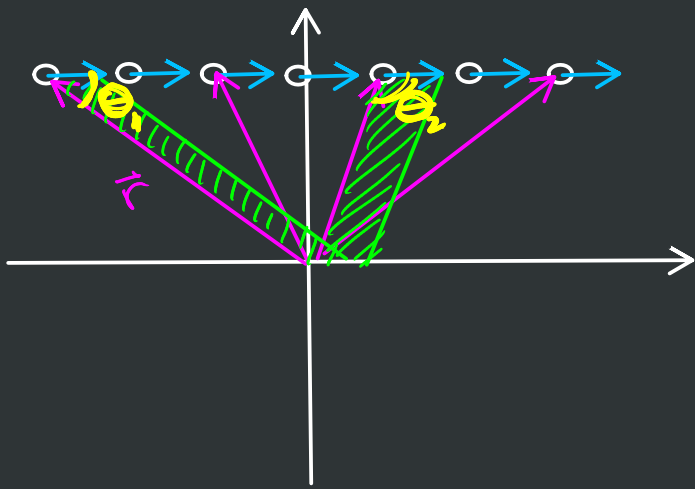
Angular Momentum - conserved quantity      angle b/w  $\vec{r}$  &  $\vec{p}$

$$\vec{L} = \vec{r} \times \vec{p}$$

angular momentum

$$|\vec{L}| = |\vec{r}| |\vec{p}| \sin \theta$$





$$\begin{aligned}\dot{\vec{l}} &= \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} \\ &= \dot{\vec{r}} \times m\vec{v} \\ &= m \underbrace{\dot{\vec{r}} \times \vec{v}}_{=0} + \vec{r} \times \dot{\vec{p}}\end{aligned}$$

$$\dot{\vec{l}} = \vec{r} \times \dot{\vec{p}}$$

$$\dot{\vec{l}} = \underbrace{\vec{r} \times \vec{F}}_{\text{torque}} \begin{cases} \vec{\tau} \leftarrow \text{tau} \\ \vec{\Gamma} \leftarrow \text{Gamma} \\ \vec{N} \leftarrow \text{en} \end{cases}$$

$$\dot{\vec{l}} = \vec{\Gamma}$$

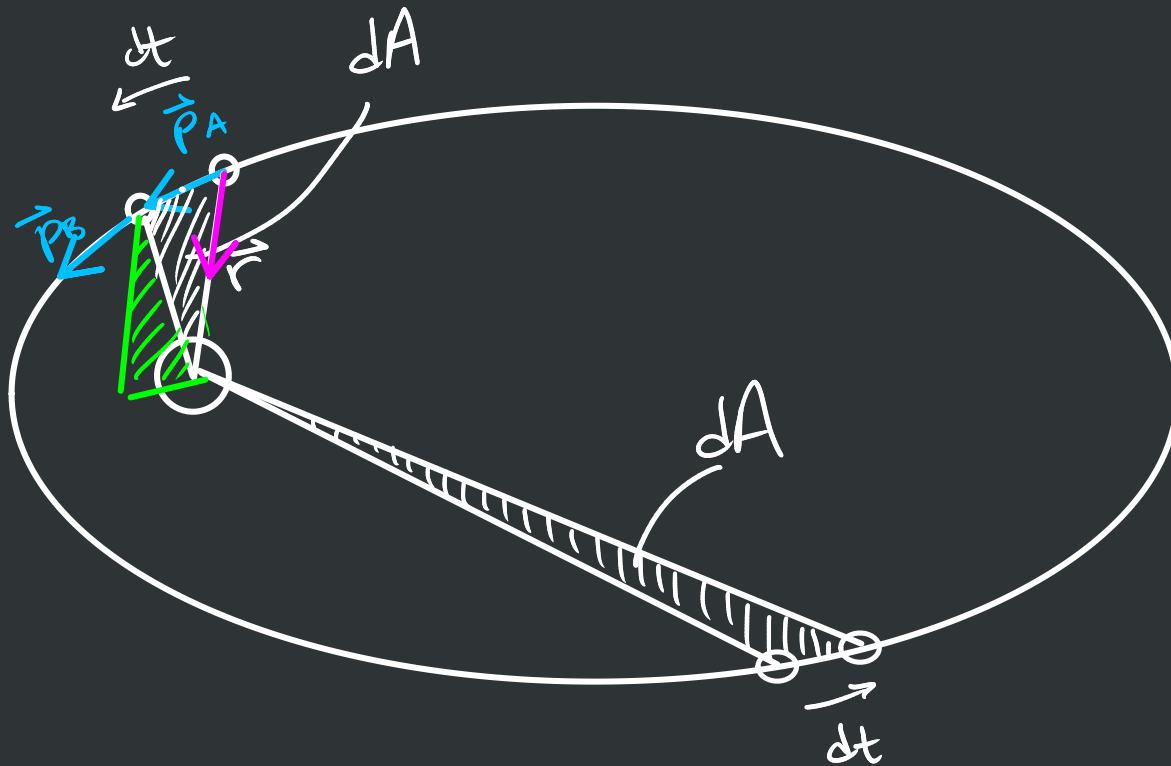
$$\vec{\Gamma} = \dot{\vec{l}}$$

if  $\vec{F} = 0$ ,  $\dot{\vec{l}} = 0$ ,  $\vec{l} = \text{constant}$

Newton's 2<sup>nd</sup>  
for rotation

net torque = rate of change of angular mom.

# Kepler's 2<sup>nd</sup> Law



$$d\vec{s} = \vec{v} dt$$

$$dA = \frac{1}{2} (\vec{r} \times \vec{v} dt)$$

$$dA = \frac{1}{2} (\vec{r} \times \frac{\vec{p}}{m} dt)$$

$$\frac{dA}{dt} = \frac{1}{2m} (\underbrace{\vec{r} \times \vec{p}}_{\vec{l}})$$

$$\frac{dA}{dt} = \frac{\vec{l}}{2m} \leftarrow \text{constant}$$

$$=$$

## Polar Coordinates

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{p} = \vec{v} \cdot m$$

$\omega \rightarrow$  angular velocity

$$\vec{v} = \dot{r} \hat{r} + r \dot{\phi} \hat{\phi}$$

$$= m \cdot r \hat{r} \times (\dot{r} \hat{r} + r \dot{\phi} \hat{\phi})$$

$$= mr (\hat{r} \times \dot{r} \hat{r} + \hat{r} \times r \dot{\phi} \hat{\phi})$$

$$= mr (\underbrace{\dot{r} \hat{r} \times \hat{r}}_{=0} + r \dot{\phi} \underbrace{\hat{r} \times \hat{\phi}}_{\hat{z}})$$

$$\vec{L} = mr^2 \dot{\phi} \hat{z}$$

$$|\vec{L}| = mr^2 \omega$$

$\omega$  may or may not be constant

$$\vec{L} = m r^2 \cdot \vec{\omega}$$

$\underbrace{\quad}_{\text{moment of inertia}} \quad \underbrace{\quad}_{\text{angular velocity}}$

$\rightarrow$  moment of inertia

$I = m r^2 \leftarrow$  only single point mass

$$\vec{p} = m \cdot \vec{v}$$

$\uparrow$  translational velocity

What about multiple particles?

$$\vec{L} = \sum_{\alpha=1}^N \vec{L}_{\alpha} = \sum_{\alpha=1}^N \vec{r}_{\alpha} \times \vec{p}_{\alpha}$$

$$\dot{\vec{L}} = \sum_{\alpha=1}^N \dot{\vec{r}}_{\alpha} \times \vec{p}_{\alpha} + \sum_{\alpha=1}^N \vec{r}_{\alpha} \times \dot{\vec{p}}_{\alpha}$$

$$\underbrace{\vec{r}_{\alpha} \times m \dot{\vec{v}}_{\alpha}}_{=0} = \vec{r}_{\alpha} \times \dot{\vec{p}}_{\alpha}$$

$$\dot{\vec{L}} = \sum_{\alpha=1}^N \vec{r}_{\alpha} \times \vec{F}_{\alpha} \quad \leftarrow \dot{\vec{p}}_{\alpha}$$

$$\vec{F}_\alpha = \vec{F}_\alpha^{\text{ext}} + \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta}$$

$$\dot{\vec{L}} = \sum_{\alpha=1}^N \left[ \vec{r}_\alpha \times \left( \vec{F}_\alpha^{\text{ext}} + \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta} \right) \right]$$

$$\dot{\vec{L}} = \underbrace{\sum_{\alpha=1}^N \vec{r}_\alpha \times \vec{F}_\alpha^{\text{ext}}}_{\vec{\Gamma}^{\text{ext}}} + \sum_{\alpha=1}^N \vec{r}_\alpha \times \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta}$$

$$\underbrace{\sum_{\alpha=1}^N \sum_{\beta \neq \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta}}_{=0}$$

$$\dot{\vec{L}} = \vec{\Gamma}^{\text{ext}}$$

$$\vec{L} = \sum_{\alpha=1}^N \vec{r}_{\alpha} \times \vec{p}_{\alpha}$$

go back to polar coordinates  $\rightarrow \hat{\phi}$  and  $\hat{r}$

$$\vec{L} = \sum_{\alpha} m_{\alpha} r_{\alpha}^2 \dot{\phi}_{\alpha} \hat{z}$$

all  $m_{\alpha}$  have same  $\dot{\phi}_{\alpha} = \omega$   $\leftarrow$  solid object spinning

$$\vec{L} = \vec{\omega} \cdot \underbrace{\sum_{\alpha} m_{\alpha} r_{\alpha}^2}_{I}$$

$$I = \sum_{\alpha} m_{\alpha} r_{\alpha}^2$$

$\downarrow$  continuous medium

$$\vec{L} = \underbrace{I}_{\text{tensor}} \cdot \vec{\omega}$$

in general, a tensor

$$I = \int r^2 dm$$

$$I = \int r^2 \rho dV$$

$\leftarrow$  very similar integral to the CM integral



Solid sphere rotating about its center

$$I = \frac{2}{5} MR^2$$

disk rotating about its center

$$I = \frac{1}{2} MR^2$$

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Rotations about the center of mass frame

$$\underbrace{\dot{\vec{L}}}_{\text{CM}} = \vec{\tau}_{\text{ext}}$$

← still true measured  
about the center of  
mass frame

↳ inertial reference  
frame

still works in CM frame  
even if CM frame is  
not inertial





















