Newton's Laws

1. Obj at rest remains at rest.....
2. $\vec{F} = \frac{d\vec{p}}{dt} = m\vec{a}$
3. Action – Reaction

$\vec{F}_{NET}$

$\vec{p} = m\vec{v}$

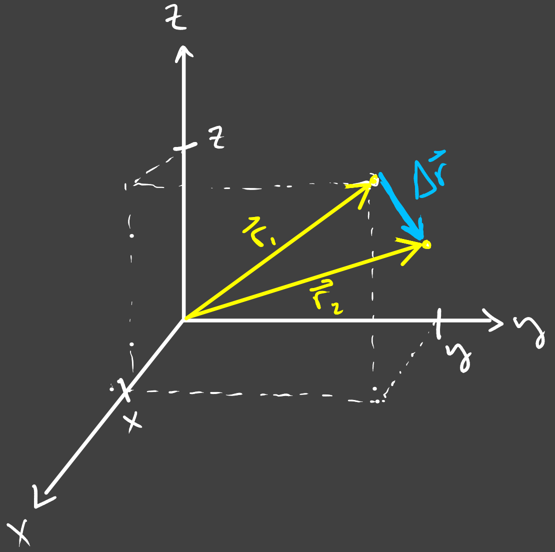
velocity

$\vec{a} = \frac{d\vec{v}}{dt} \approx \frac{\Delta \vec{v}}{\Delta t}$

$\frac{d\vec{x}}{dt} \approx \frac{\Delta \vec{x}}{\Delta t}$

x => position

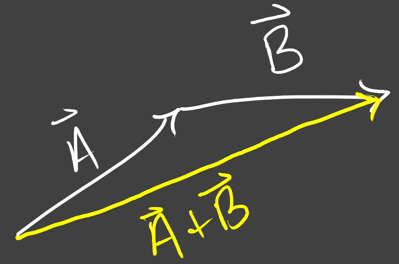
jerk $= \frac{da}{dt}$

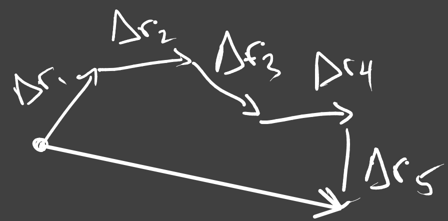


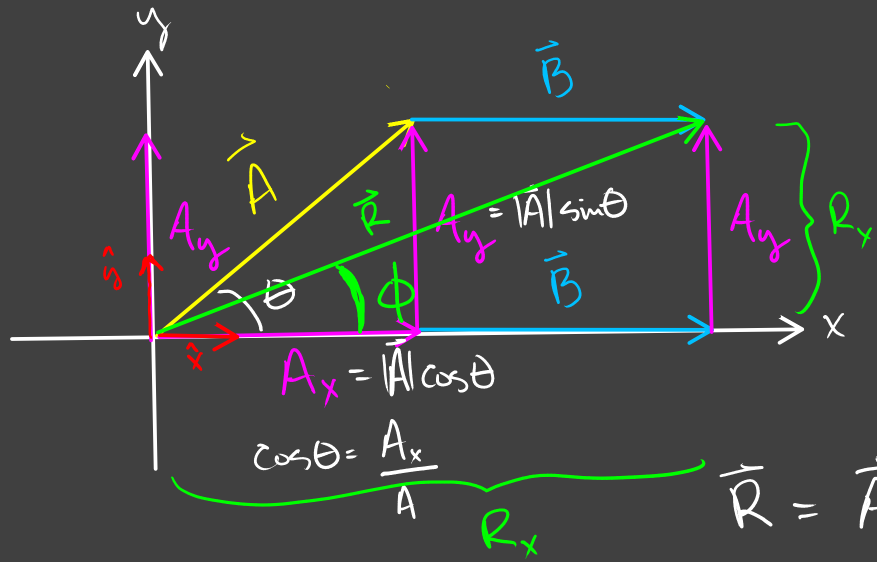
$$\vec{r}_1 + \Delta\vec{r} = \vec{r}_2$$

position

displacement







$$\vec{A} = A_x \hat{x} + A_y \hat{y}$$

$$\vec{B} = B_x \hat{x} + B_y \hat{y}$$

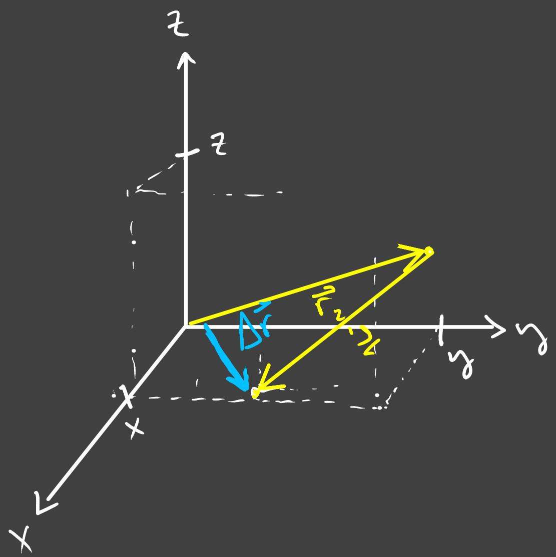
$$\vec{R} = \vec{A} + \vec{B} = \overbrace{(A_x + B_x)}^{R_x} \hat{x} + \overbrace{(A_y + B_y)}^{R_x} \hat{y}$$

$$|\vec{R}| = \sqrt{R_x^2 + R_y^2}$$

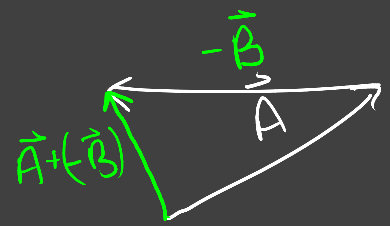
$$\tan\phi = \frac{R_y}{R_x} \sim \phi = \tan^{-1}\left(\frac{R_y}{R_x}\right)$$

	x	y
A:	A_x	A_y
B:	B_x	B_y
R:	R_x	R_y

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$



$$\vec{A} - \vec{B}$$



$$\vec{A}-\vec{B} = (A_x - B_x)\hat{x} + (A_y - B_y)\hat{y}$$

- scalar multiplication

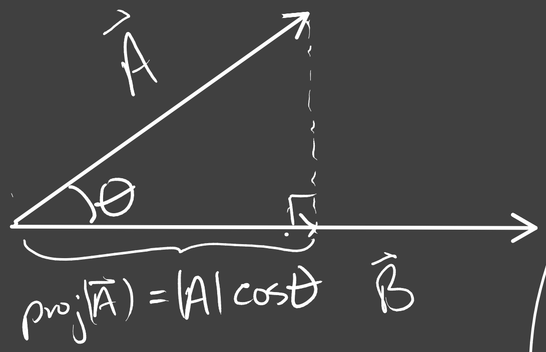
$$c\vec{A} = cA_x\hat{x} + cA_y\hat{y} + cA_z\hat{z}$$

HW: 9, 23, 24, 26, 27, 30, 46, 48

Friday

- dot product / scalar product

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$



$$\vec{A}\cdot\vec{B} = |\vec{A}|\cos\theta \cdot |\vec{B}|$$

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$\vec{A}\cdot\vec{B} = A_xB_x + A_y\cdot B_y + A_z\cdot B_z = \vec{B}\cdot\vec{A}$$

$$|\vec{r}| = \sqrt{\vec{r}\cdot\vec{r}} \qquad \vec{r}^2 = \vec{r}\cdot\vec{r}$$

- cross product

$$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$$

$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_yB_z - B_yA_z)\hat{x} - (A_xB_z - B_xA_z)\hat{y} + (A_xB_y - B_xA_y)\hat{z}$$

Derivative of Vectors

$$\frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$\frac{d(\vec{r}+\vec{s})}{dt} = \frac{d\vec{r}}{dt} + \frac{d\vec{s}}{dt}$$

$$\frac{d(f\vec{r})}{dt} = \vec{r}\,\frac{df}{dt} + f \cdot \frac{d\vec{r}}{dt}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} + x\cancelto{0}{\frac{d\hat{x}}{dt}} + y\cancelto{0}{\frac{d\hat{y}}{dt}} + z\cancelto{0}{\frac{d\hat{z}}{dt}}$$

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

$$\vec{V} = V_x\hat{x} + V_y\hat{y} + V_z\hat{z}$$

$$V_x = \frac{dx}{dt} \qquad V_y = \frac{dy}{dt} \qquad V_z = \frac{dz}{dt}$$

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1. If $\vec{F} = 0$, then $\vec{v}$ is constant
2. $\vec{F} = \dfrac{d\vec{p}}{dt}$ ⟵ $\vec{p}$ = momentum
3. $\vec{F}_{12} = -\vec{F}_{21}$

1st - Newton's Law hold in inertial refirance frames

2nd - $\vec{F} = \frac{d\vec{p}}{dt}$ —— $\vec{p} = m\vec{v}$

mass does not change

$$\frac{d\vec{p}}{dt} = m\underbrace{\frac{d\vec{v}}{dt}}$$

$$= m\vec{a}$$

$$\frac{d\vec{r}}{dt} = \dot{\vec{r}} = \vec{v}$$

$$\frac{d\vec{v}}{dt} = \dot{\vec{v}} = \ddot{\vec{r}} = \vec{a}$$

$$\frac{d\vec{p}}{dt} = \dot{\vec{p}} = \vec{F} = m\dot{\vec{v}} = m\ddot{\vec{r}}$$

$$\vec{F} = m\ddot{\vec{r}}$$

Ex:

$$\vec{F} = F_x \hat{x}$$

$$\ddot{\vec{r}} = \frac{F_x \hat{x}}{m}$$

$$\frac{d^2x}{dt^2} = \frac{F_x}{m}$$

$$\ddot{x} = \frac{F_x}{m}$$

$$\dot{x} = \int \ddot{x}\,dt = \int \frac{F_x}{m}\,dt = \frac{F_x}{m}\underbrace{\int 1\,dt}_{t} = \frac{F_x}{m}t + C$$

$$\dot{x} = \frac{F_x}{m}t + C$$

$$\dot{x}(t=0) = V_{x0}$$

$$F_y = 0$$

$$F_z = 0$$

$$\ddot{y} = 0$$

$$\dot{y} = \int \ddot{y}\,dt = \int 0\,dt$$

$$\dot{y} = V_{y0}$$ y vel is constant

$$y = \int \dot{y}\,dt = \int V_{y0}\,dt$$

$$y = V_{y0} t + y_0$$

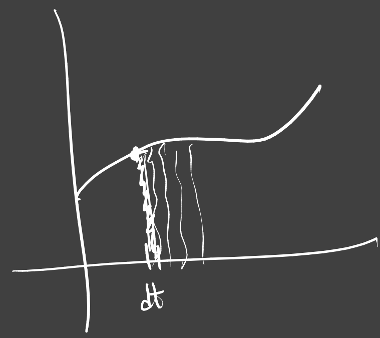
$$\ddot{z} = 0$$

$$z = V_{z0} t + z_0$$

$$\boxed{\dot{x} = \frac{F_x}{m} t + V_{x0}}$$

$$\underset{\sim\sim\sim}{x = \int \frac{dx}{dt} dt}$$

$$x = \int \dot{x}\, dt = \int \left(\frac{F_x}{m} t + V_{x0} \right) dt$$



$$\rightarrow x = \frac{F_x}{m} \int t\, dt + V_{x0} \int dt$$

$$x = \frac{1}{2} \cdot \frac{F_x}{m} \cdot t^2 + V_{x0} t + D$$

$$\hookleftarrow X_0$$

$$\rightarrow \boxed{x(t) = \frac{1}{2} \frac{F_x}{m} t^2 + V_{x0} \cdot t + X_0}$$

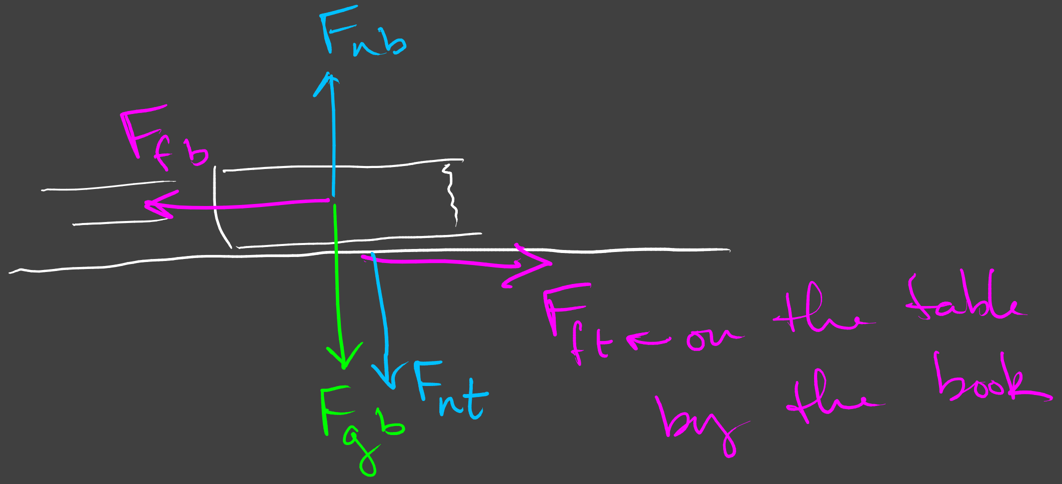
$$X_0 + V_i t + \frac{1}{2} a t^2$$

$$\frac{\vec{F}}{m} = \ddot{\vec{r}} \quad \begin{cases} \frac{F_x}{m} = \ddot{x} \\ \frac{F_y}{m} = \ddot{y} \\ \frac{F_z}{m} = \ddot{z} \end{cases}$$

$$\ddot{\vec{r}} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}$$

3rd

$\vec{F}_{12} = -\vec{F}_{21}$; force occur in pairs



$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_1^{ext} = \dot{p}_1 \qquad \vec{F}_2 = \vec{F}_{21} + \vec{F}_2^{ext} = \dot{p}_2$$

$$\vec{P} = \vec{p}_1 + \vec{p}_2$$

$$\dot{\vec{P}} = \dot{\vec{p}}_1 + \dot{\vec{p}}_2$$

$$\dot{\vec{P}} = \underbrace{\vec{F}_{12}}_{-\vec{F}_{21}} + \vec{F}_1^{ext} + \vec{F}_{21} + \vec{F}_2^{ext}$$

$$\dot{\vec{P}} = \vec{F}_1^{ext} + \vec{F}_2^{ext}$$

$$\dot{\vec{P}} = \vec{F}^{ext}$$

so, if $\vec{F}^{ext} = 0$

$$\dot{\vec{P}} = 0$$

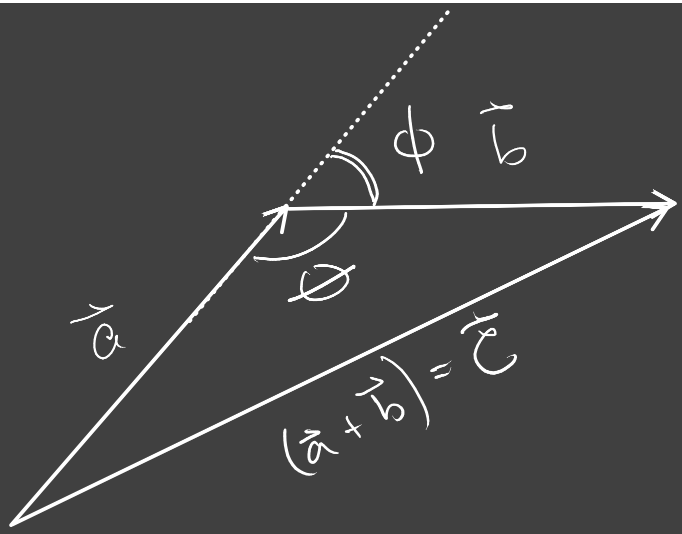
$$\vec{P} = \text{constant}$$

$$\vec{p}_1(t_1) + \vec{p}_2(t_1) = \vec{p}_1(t_2) + \vec{p}_2(t_2)$$

HW: 9, 23, 24, 26, 27, 30, 46, 48

9\]

1.9 ★ In elementary trigonometry, you probably learned the law of cosines for a triangle of sides a, b, and c, that $c^2 = a^2 + b^2 - 2ab\cos\theta$, where θ is the angle between the sides a and b. Show that the law of cosines is an immediate consequence of the identity $(\mathbf{a} + \mathbf{b})^2 = a^2 + b^2 + 2\mathbf{a} \cdot \mathbf{b}$.



$$(\vec{a} + \vec{b})^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$$

$$= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} + 2\vec{a} \cdot \vec{b}$$

$$c^2 = a^2 + b^2 + 2\vec{a} \cdot \vec{b}$$

$$= a^2 + b^2 + 2|\vec{a}||\vec{b}|\cos\phi$$

$$\phi = \pi - \theta$$

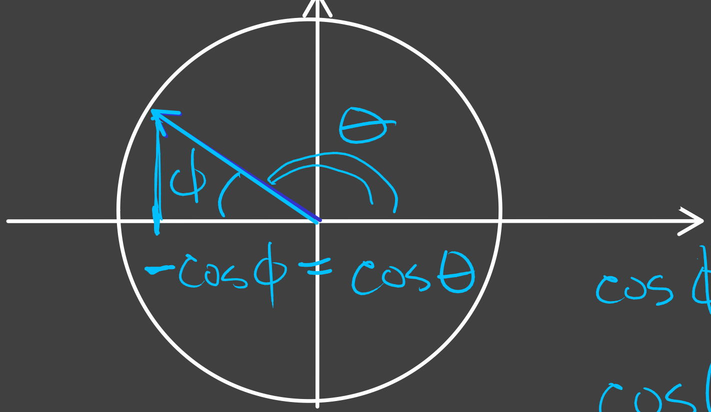
$$= a^2 + b^2 + 2|\vec{a}||\vec{b}|\cos(\pi - \theta)$$

$\rightarrow \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\alpha = \pi \quad \beta = \theta$

$$\cos(\pi - \theta) = \underbrace{\cos\pi}_{-1} \cos\theta + \underbrace{\sin\pi}_{0} \sin\theta$$

$$= -\cos\theta \checkmark$$



$\cos\phi = -\cos\theta$

$\cos(\pi-\theta) = -\cos\theta$

23

1.23 ★★ The unknown vector **v** satisfies **b** · **v** = λ and **b** × **v** = **c**, where λ, **b**, and **c** are fixed and known. Find **v** in terms of λ, **b**, and **c**.

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

$A\times \quad B\times C$

$\vec{b}\cdot\vec{v} = \lambda$

$$\vec{b}\times(\vec{b}\times\vec{v}) = \underbrace{(\vec{b}\cdot\vec{v})}_{\lambda}\vec{b} - \underbrace{(\vec{b}\cdot\vec{b})}_{b^2}\vec{v} = \vec{b}\times\vec{c}$$

$$b^2\vec{v} = \lambda\vec{b} - \vec{b}\times\vec{c}$$

$$\vec{v} = \frac{\lambda\vec{b} - \vec{b}\times\vec{c}}{b^2}$$

24

1.24 ★ In case you haven't studied any differential equations before, I shall be introducing the necessary ideas as needed. Here is a simple excercise to get you started: Find the general solution of the first-order equation $df/dt = f$ for an unknown function $f(t)$. [There are several ways to do this. One is to rewrite the equation as $df/f = dt$ and then integrate both sides.] How many arbitrary constants does the general solution contain? [Your answer should illustrate the important general theorem that the solution to any nth-order differential equation (in a very large class of "reasonable" equations) contains n arbitrary constants.]

1.25 ★ Answer the same questions as in Problem 1.24, but for the differential equation $\boxed{df/dt = -3f}$.

$$\boxed{\frac{df}{dt} = f}$$

$$\int \frac{1}{f} df = \int dt$$

$$\ln f = t + C$$

$$e^{\ln f} = e^{(t+C)}$$

$$f = e^{(t+C)}$$

$$f = e^{t} \cdot e^{C}$$

$$\boxed{f = Ae^{t}}$$

$$f(t=0) = A$$

initial condition

$$\frac{df}{f} = -3dt$$

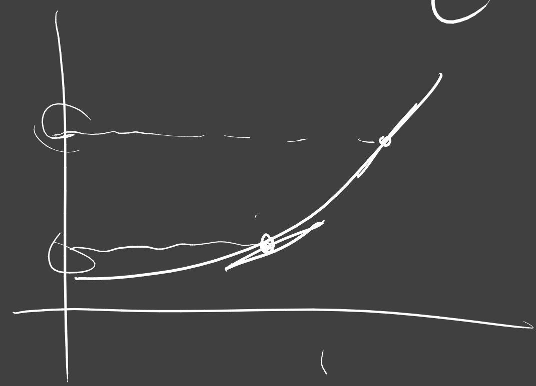
$$= -3t + C$$

$$e^{-3t+C}$$

$$f = Ae^{-3t}$$

$$\boxed{\frac{df}{dt} = At} \longrightarrow f = \frac{A}{2}t^2 + C$$

a^t e^t



$$\to \frac{df}{dt} = f$$

$$\frac{d(e^t)}{dt} = e^t$$

$$\Delta f = \frac{df}{dt} \cdot \Delta t$$

$$df = \frac{df}{dt} \cdot dt$$

$$f_1 = f_0 + \Delta f$$

$$\frac{d^2 f}{dt^2} = c \cdot f$$

t

$$f = Ae^{1t} \to Ae^{5t}$$

$$\ddot{X} = \frac{F}{m}$$

24) S | S' | S''

S

$x = 0$

$y = u_y \cdot t$

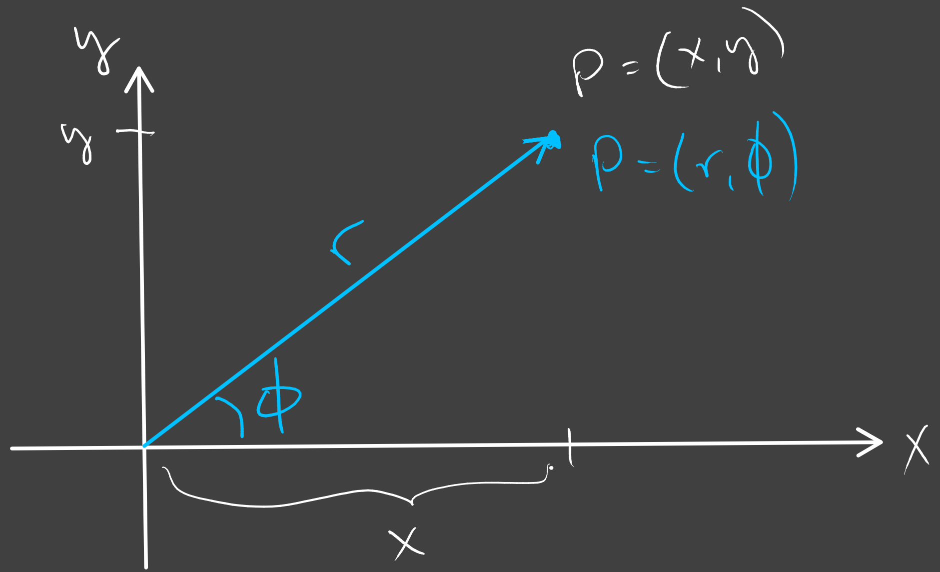
S'

$x' = \cancel{x}^{\to 0} - v_x \cdot t$

$y' = y - \cancel{v_y \cdot t}^{\to 0}$

$S' = S$ – change between frame

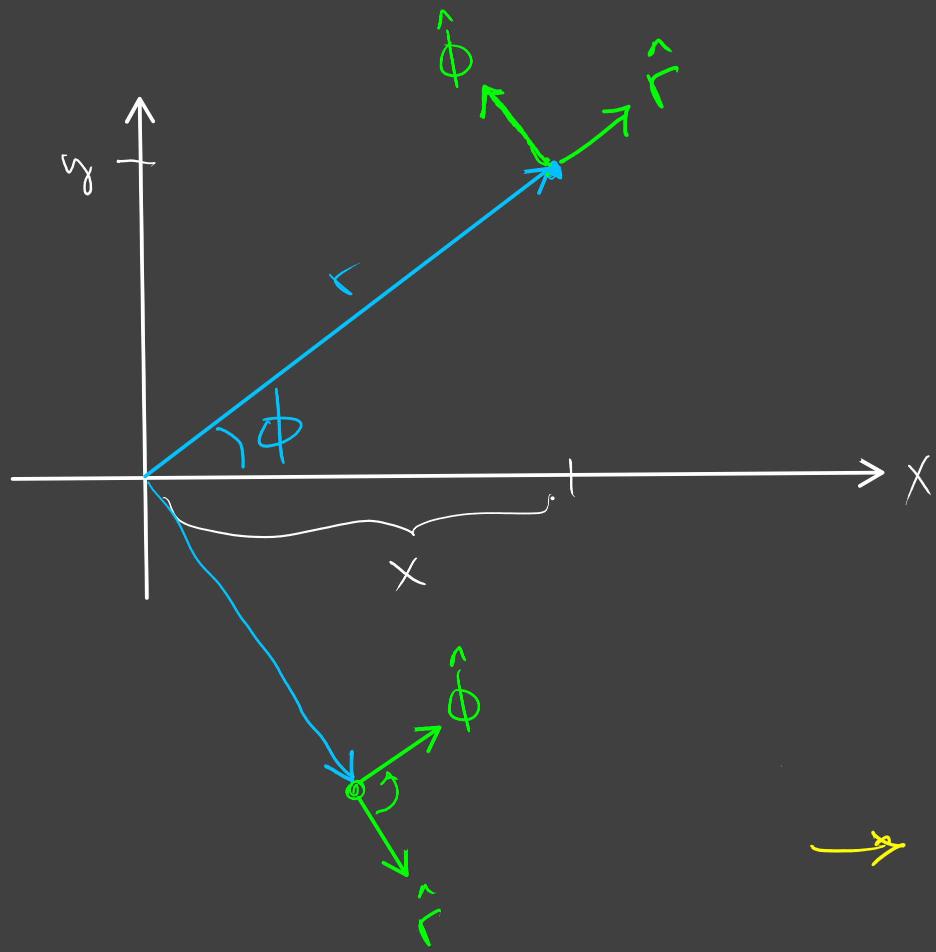
Polar Coordinates



$$\left.\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}\right\} \longleftrightarrow \left\{\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right.$$

$$\vec{F} = F_x \hat{x} + F_y \hat{y}$$

$$\vec{F} = F_r \hat{r} + F_\phi \hat{\phi}$$



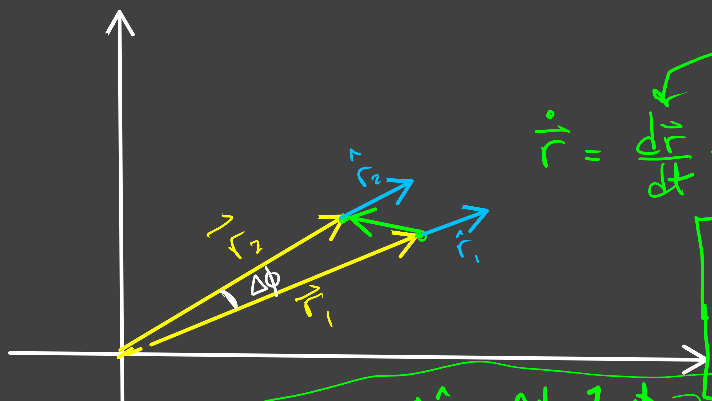
$$\hat r = \frac{\vec r}{|\vec r|}$$

$$\hat a = \frac{\vec a}{|\vec a|}$$

in cartesian

$$\vec r = x\hat x + y\hat y + z\hat z$$

$$\rightarrow \vec r = \underbrace{r\hat r}$$



$$\dot{\vec r} = \frac{d\vec r}{dt} = \frac{d(r\hat r)}{dt} = \hat r\frac{dr}{dt} + r\frac{d\hat r}{dt}$$

$$\boxed{\begin{aligned}\vec V &= \dot{\vec r} = \dot r\hat r + r\dot\phi\hat\phi \\ \vec V &= V_r\hat r + V_\phi\hat\phi\end{aligned}}$$

ω angular velocity

$$\Delta\hat r = \Delta\phi\cdot 1\,\hat\phi = \Delta\phi\,\hat\phi$$

$$\frac{\Delta\hat r}{\Delta t} = \frac{\Delta\phi}{\Delta t}\hat\phi \rightsquigarrow \frac{d\hat r}{dt} = \dot{\hat r} = \frac{d\phi}{dt}\hat\phi = \dot\phi\hat\phi$$

$s = \phi \cdot r$

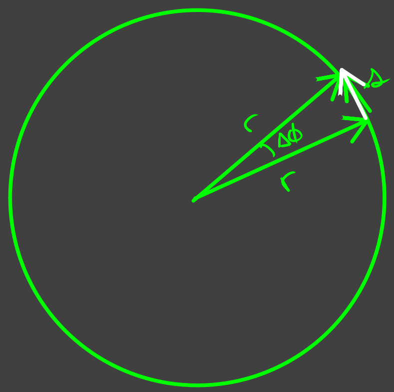
$v = \omega \cdot r$

$a = \alpha \cdot r$

circle

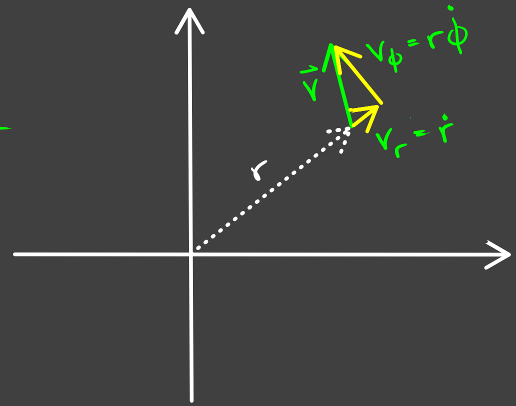
constant r

$\dot r = 0$



$\Delta\phi = \frac{s}{r}$

$\hookrightarrow s = \Delta\phi\, r$



Now for acceleration

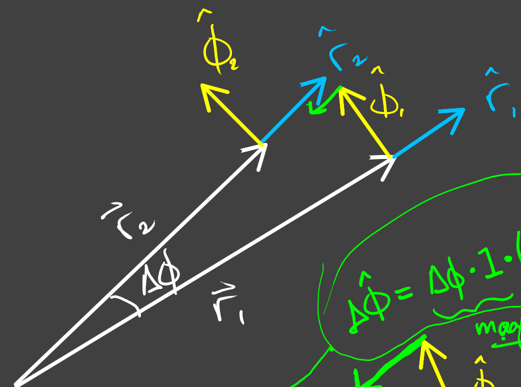
$$\vec a = \ddot{\vec r} = \frac{d}{dt}\left(\dot{\vec r}\right) = \frac{d}{dt}\left(\dot r \hat r + r \dot\phi \hat\phi\right)$$

$$= \frac{d}{dt}\left(\dot r \hat r\right) + \frac{d}{dt}\left(r \dot\phi \hat\phi\right)$$

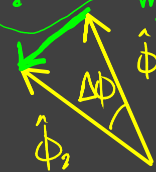
$$= \ddot r \hat r + \boxed{\dot r \dot{\hat r}} + \boxed{\dot r \dot\phi \hat\phi} + r \ddot\phi \hat\phi + r \dot\phi \dot{\hat\phi}$$

$\dot{\hat r} = \dot\phi \hat\phi$

$\dot{\hat\phi} = -\dot\phi \hat r$



$\Delta\hat\phi = \underbrace{\Delta\phi \cdot 1}_{\text{magnitude}} \cdot (-\hat r)$



$\frac{\Delta\hat\phi}{\Delta t} \approx \dot{\hat\phi} = -\frac{\Delta\phi}{\Delta t}\hat r$

$\downarrow$

$\dot{\hat\phi} = -\frac{d\phi}{dt}\hat r = -\dot\phi \hat r$

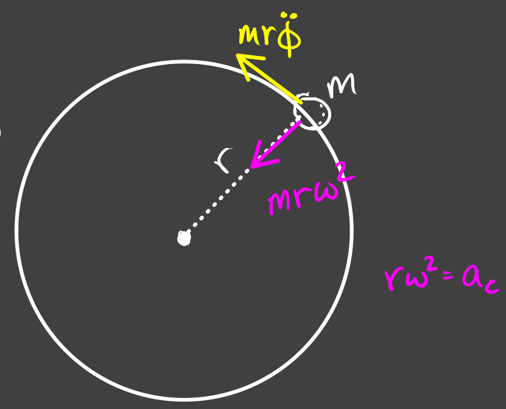
$$\boxed{\vec a = \ddot{\vec r} = \left(\ddot r - r\dot\phi^2\right)\hat r + \left(2\dot r \dot\phi + r\ddot\phi\right)\hat\phi}$$

special case of constant r, $\dot r = 0$, $\ddot r = 0$

$$\ddot{\vec r} = \underbrace{-r\dot\phi^2\hat r}_{} + r\ddot\phi\,\hat\phi$$

$\dot r = \ddot r = 0$

$$\boxed{\vec F = m\ddot{\vec r} = \underbrace{-mr\omega^2}_{\text{centripetal force}}\hat r + \underbrace{mr\alpha}_{\text{torque}}\hat\phi}$$



<u>Newton's 2^{nd}</u> (polar)

$$\vec F = m(\ddot r - r\dot\phi^2)\hat r + m(2\dot r\dot\phi + r\ddot\phi)\hat\phi$$

$$F_r = m(\ddot r - r\dot\phi^2)$$

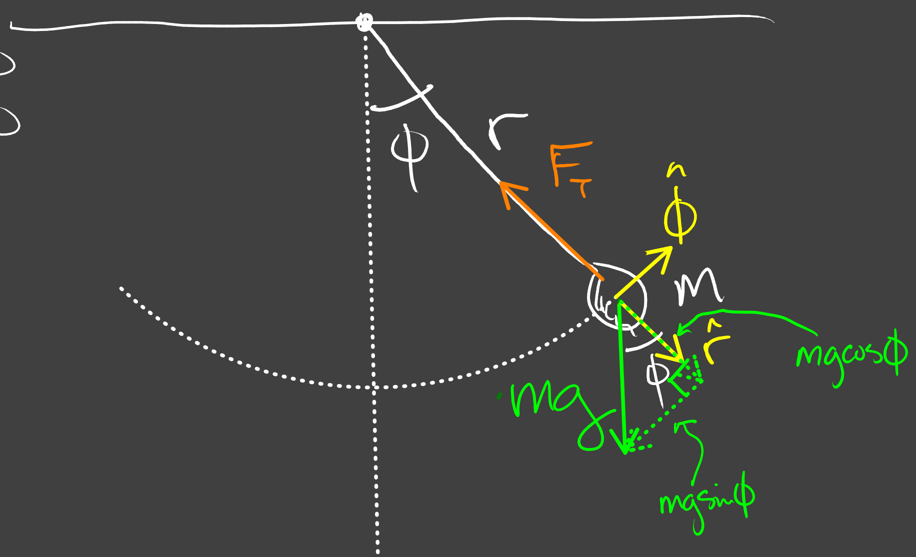
$$F_\phi = m(2\dot r\dot\phi + r\ddot\phi)$$

Ex.

time for one complete trip → period

$\dot{r} = 0$

$\ddot{r} = 0$



$$\underbrace{\vec{F}}_{\text{net force}} = m\ddot{\vec{r}} = \underbrace{-mr\dot{\phi}^2\hat{r} + mr\ddot{\phi}\hat{\phi}}$$

$$\vec{F} = \Sigma \vec{F}$$

$$F_{NET,R} = -F_T + mg\cos\phi$$

$$F_{NET,\phi} = -mg\sin\phi$$

$$(-F_T + mg\cos\phi)\hat{r} + (-mg\sin\phi)\hat{\phi} = -mr\dot{\phi}^2\hat{r} + mr\ddot{\phi}\hat{\phi}$$

$$-F_T + mg\cos\phi = -mr\dot{\phi}^2$$

$$-\cancel{m}g\sin\phi = \cancel{m}r\ddot{\phi}$$

$$\ddot{\phi} = -\frac{g}{r}\sin\phi \qquad \dot{\phi} = +\frac{g}{r}\int \sin\phi(t)\,dt$$

$$\phi_2 = \phi_1 + \dot{\phi}\cdot\Delta t \qquad \dot{\phi}_2 = \dot{\phi}_1 + \ddot{\phi}\cdot\Delta t$$

$$\sin\phi = \phi - \cancel{\frac{\phi^3}{3!} + \cdots}$$

$\underline{\sin\phi \approx \phi}$ for small ϕ

$$\boxed{\ddot{\phi} = -\frac{g}{r}\phi}$$

<u>Guess</u>

$$\phi = A\cos(Bt) + C\sin(Bt)$$

$$\dot{\phi} = -AB\sin Bt + BC\cos(Bt)$$

$$\ddot{\phi} = -AB^2\cos Bt - B^2C\sin(Bt)$$

$$-AB^2\cos Bt - B^2C\sin(Bt) = -\frac{g}{r}\left(A\cos(Bt) + C\sin(Bt)\right)$$

$$+B^2\left(\cancel{A\cos(Bt) + C\sin(Bt)}\right) = +\frac{g}{r}\left(\cancel{A\cos(Bt) + C\sin(Bt)}\right) \Rightarrow \underline{B = \sqrt{\frac{g}{r}}}$$

A & C in terms of $\underbrace{\phi_0 \;\&\; \dot{\phi}_0}_{\text{initial condition}}$

$$\phi = A\cos(Bt) + C\sin(Bt)$$

rad/~~seconds~~ ~~seconds~~

$\dfrac{\text{rad}}{\text{sec}}$ ← angular frequency

$$\sqrt{\frac{g}{r}} = \omega = 2\pi f = \frac{2\pi}{T} \quad f = \frac{1}{T}$$

$$\phi_0 = \phi(t=0) = A$$

$$\phi_0 = A$$

$\sqrt{\frac{g}{r}}$

$$\dot{\phi}_0 = \dot{\phi}(t=0) = +BC$$

$$\dot{\phi}_0 = \sqrt{\frac{g}{r}} \cdot C$$

$$C = \dot{\phi}_0 \sqrt{\frac{r}{g}}$$

only applies to small ϕ

$$\boxed{\phi(t) = \phi_0 \cos\left(\sqrt{\frac{g}{r}}\, t\right) + \dot{\phi}_0 \sqrt{\frac{r}{g}} \sin\left(\sqrt{\frac{g}{r}}\, t\right)}$$

if $\dot{\phi}_0 = 0$

$$\phi(t) = \phi_0 \cos\left(\sqrt{\frac{g}{r}} \cdot t\right)$$

$$T = 2\pi\sqrt{\frac{r}{g}}$$