Chapter 3 - Momentum + Angular Momentum

Recall: $\vec{p} = m\vec{v}$ $\qquad \sum_n \vec{F}_n = \dot{\vec{p}}$

$$\mathbb{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 \ldots = \sum \vec{p}_n$$

$$\dot{\vec{\mathbb{P}}} = \vec{F}^{ext}$$

If $\vec{F}^{ext} = 0$

$$\dot{\vec{\mathbb{P}}} = 0$$

$$\mathbb{P}_i = \mathbb{P}_f$$ Conservation of Momentum

If two objects,

really 2 equations

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1 \underset{?}{\vec{v}_{1f}} + m_2 \underset{?}{\vec{v}_{2f}}$$

If collision is perfectly inelastic
↳ stick together

then,

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_f + m_2\vec{v}_f$$

$$= \underbrace{(m_1 + m_2)}_{\text{mass add up!}}\vec{v}_f$$

$$\vec{v}_f = \frac{m_1\vec{v}_{1i} + m_2\vec{v}_{2i}}{m_1 + m_2}$$ ✓ ← two equations!

Rockets + Explosions

At the beginning

$\mathbb{P} = 0$

Round 1

$\mathbb{P}_i = 0 = \mathbb{P}_f = p_{ME} + p_{ball}$

$0 = \underbrace{(75\,kg + 99(0.5\,kg))}_{124.5\,kg} v + (0.5\,kg)(-15\,m/s)$

$0 = 124.5 v - 7.5\,kg\,m/s$

$v = \dfrac{7.5\,kg\,m/s}{124.5\,kg} = 0.0602\,m/s$

Round 2

$$v' = v_{ice} - v_{s'}$$

$$-15\,\text{m/s} = v_{ice} - 0.0602\,\text{m/s}$$

$$-15 + 0.06 = v_{ice} = -14.94\,\text{m/s}$$

$$\mathbb{P}_i = \mathbb{P}_f$$

$$\underbrace{(124.5)(0.0602\,\text{m/s}) + 0.5(-15)}_{=0} = 124 \cdot v + \underbrace{0.5(-15) + 0.5(-14.94)}$$

$$-0.5(-15 - 14.94) = 124 \cdot v$$

$$v = \frac{-0.5(-15 - 14.94)}{124} = \frac{-0.5(-15 - 15 + 0.0602)}{124}$$

$-15 + 0.0602$

$$= 0.1207\,\text{m/s}$$

but we need a pattern to do this numerically - this way is not giving that away

Round 1 Redux – my frame

$\mathbb{P}_i = 0 = \mathbb{P}_f = p_{ME} + p_{ball}$

$$0 = \underbrace{(75\,kg + 99(0.5\,kg))}_{124.5\,kg}v + (0.5\,kg)(-15\,m/s)$$

$$0 = 124.5\,v - 7.5\,\frac{kg\,m}{s}$$

$$v = \frac{7.5\,kg\,m/s}{124.5\,kg} = 0.0602\,m/s$$

Round 2 Redux – my frame

$$\overset{\mathbb{P}_i'}{0} = 124.0 \cdot v + 0.5(-15)$$

$$v = \frac{0.5(15)}{124} = 0.06048\,m/s$$

Round 3 Redux – my frame

$$\overset{P_i''}{0} = 123.5V + 0.5(-15)$$

$$V = \frac{0.5(15)}{123.5} = 0.0607 \text{ m/s}$$

0.18

Generalize

$$dV = \frac{dP_{ex}}{m}$$

variable

$$\int_{V_i}^{V_f} dV = \int_{m_o+m_f}^{m_o} \frac{V_{ex}\, dm}{m} = V_{ex} \int_{m_o+m_f}^{m_o} \frac{dm}{m}$$

$$v\Big|_{v_i}^{v_f} = v_{ex}\ln(m)\Big|_{m_0+m_f}^{m_0}$$

$$v_f - v_i = v_{ex}\left[\ln(m_0) - \ln(m_0+m_f)\right]$$

$$v_f = v_{ex}\ln\left(\frac{m_0}{m_0+m_f}\right) + v_i$$

$$\hookrightarrow \underline{\underline{v(t)}} = v_{ex}\ln\left(\frac{m(t)}{m_0+m_f}\right) + v_i$$

test w baseballs

$$v_f = -15\ln\left(\frac{75\text{kg}}{125\text{kg}}\right)$$

$$= \underline{\underline{+7.66\ \text{m/s}}}$$

$$\rightarrow V(t) = V_{ex} \ln\left(\frac{m(t)}{m_0 + m_f}\right) + V_0$$

$$0 = +15 \ln\left(\frac{m_0}{m_0 + 0.5n}\right) + 10$$

How many baseballs to stop from 10 m/s

$$-10 = +15 \ln\left(\frac{m_0}{m_0 + 0.5n}\right)$$

$$\frac{-10}{+15} = \ln\left(\frac{m_0}{m_0 + 0.5n}\right)$$

$$e^{\frac{-10}{15}} = \frac{m_0}{m_0 + 0.5n}$$

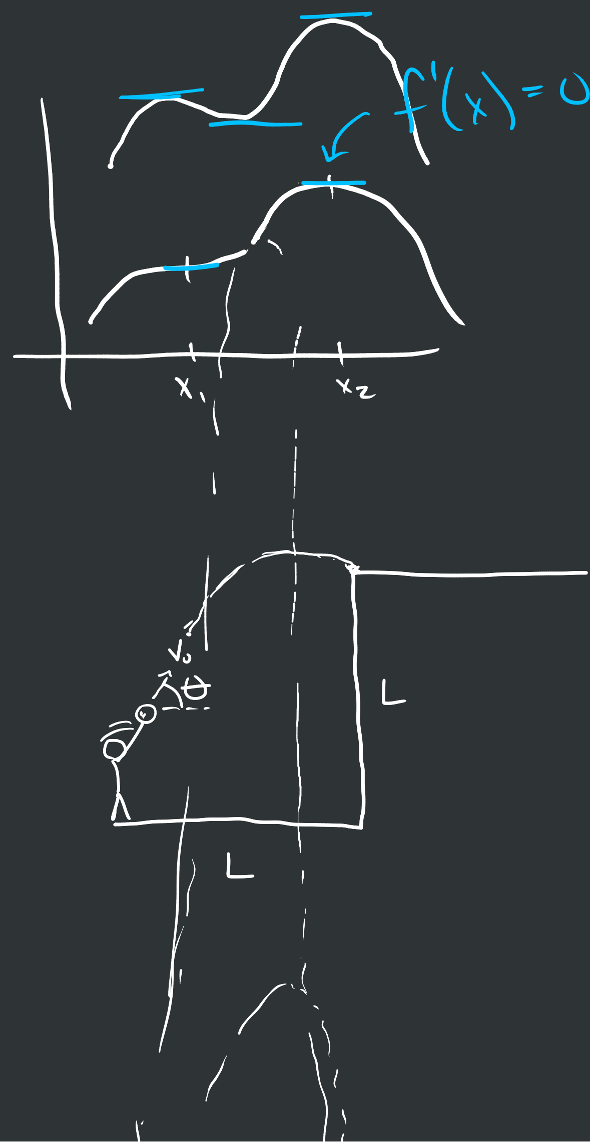
$$e^{\frac{-10}{15}}(m_0 + 0.5n) = m_0$$

$$m_0 + 0.5n = m_0 e^{10/15}$$

$$n = \frac{m_0(e^{10/15} - 1)}{0.5}$$

$m_0 = 75\text{kg}$

$n = 142.16$ ← # of baseballs



Center of Mass

$$\vec{\mathbb{R}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \cdots}{m_1 + m_2 + \cdots}$$

total, M

$$\boxed{\vec{\mathbb{R}} = \frac{1}{M} \sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha}$$

center of mass is a position

$$\mathbb{X} = \frac{1}{M} \sum_{\alpha=1}^{N} m_\alpha x_\alpha$$

x-component (x position)

$$\mathbb{Y} = \frac{1}{M} \sum_{\alpha=1}^{N} m_\alpha y_\alpha$$

$$\mathbb{Z} = \vdots$$

for continuous medium

$$m_\alpha \rightarrow dm$$

$$\vec{\mathbb{R}} = \frac{1}{M} \int \vec{r}\, dm$$

volume

$$\boxed{\vec{\mathbb{R}} = \frac{1}{M} \int \vec{r} \rho \, dV}$$

$$\rho = \frac{M}{V}$$

$$\dot{\vec{\mathbb{R}}} = \frac{1}{M} \sum_{\alpha=1}^{N} m_\alpha \dot{\vec{r}}_\alpha$$

$$\vec{\mathbb{P}} = M \dot{\vec{\mathbb{R}}}$$

$$\vec{\mathbb{P}} = \cancel{M}\left(\frac{1}{\cancel{M}}\sum_\alpha m_\alpha \dot{\vec{r}}_\alpha\right)$$

$$= \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha$$

$$\vec{\mathbb{P}} = \sum_\alpha \vec{p}_\alpha$$ ← already knew from Ch. 1.

$\frac{d}{dt}$

$$\dot{\vec{\mathbb{P}}} = M\ddot{\vec{R}} = \vec{F}^{ext}$$ ← can treat a collection as a single particle at the center of mass

$$\dot{\vec{\mathbb{P}}} = \vec{F}^{ext}$$

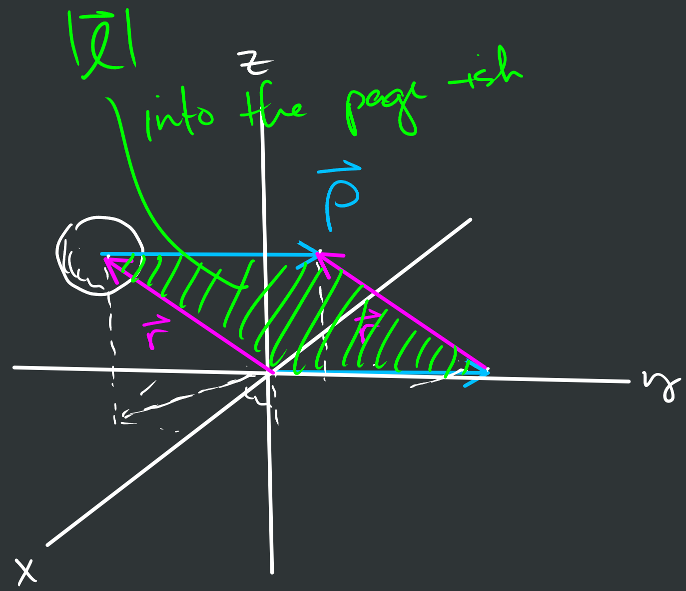
Angular Momentum - conserved quantity

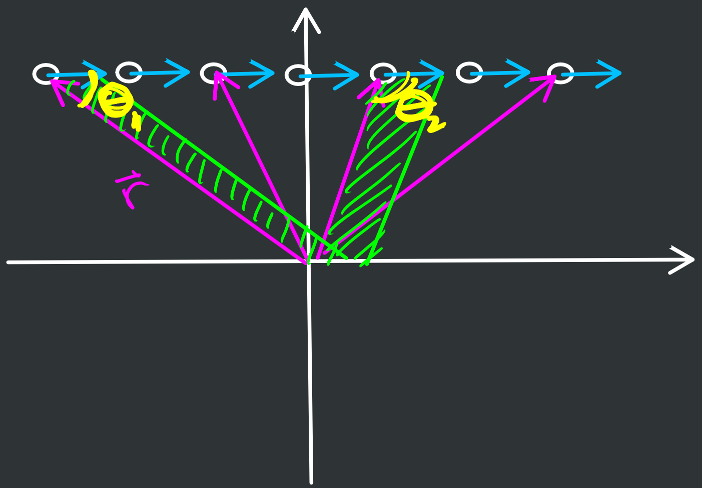
$$\vec{l} = \vec{r} \times \vec{p}$$

angular momentum

angle b/t $\vec{r}$ + $\vec{p}$

$$|\vec{l}| = |\vec{r}||\vec{p}|\sin\theta$$





$$\dot{\vec{l}} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}}$$

$$= \dot{\vec{r}} \times m\vec{v}$$

$$= m\underbrace{\dot{\vec{r}} \times \dot{\vec{r}}}_{=0} + \vec{r} \times \dot{\vec{p}}$$

$$\dot{\vec{l}} = \vec{r} \times \dot{\vec{p}}$$

$$\dot{\vec{l}} = \underbrace{\vec{r} \times \vec{F}}_{\text{torque}}$$

torque → $\vec{\tau}$ ← tau

torque → $\vec{\Gamma}$ ← Gamma

torque → $\vec{N}$ ← en

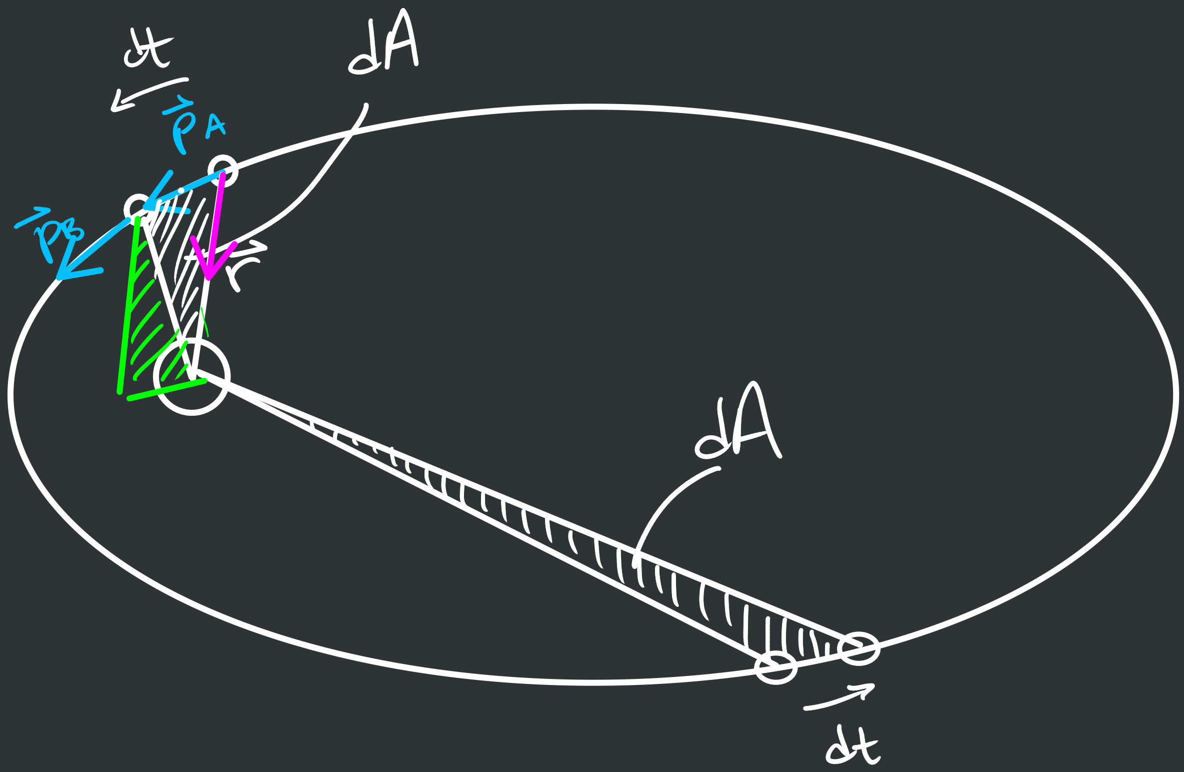
$$\dot{\vec{l}} = \vec{\Gamma}$$

if $\vec{\Gamma} = 0$, $\dot{\vec{l}} = 0$, $\vec{l} = \text{constant}$

$$\vec{\Gamma} = \dot{\vec{l}}$$

Newton's 2nd for rotation { net torque = rate of change of angular mome.

Kepler's 2^{nd} Law



$$d\vec{s} = \vec{v}\,dt$$

$$dA = \frac{1}{2}\left(\vec{r} \times \vec{v}\,dt\right)$$

$$dA = \frac{1}{2}\left(\vec{r} \times \frac{\vec{p}}{m}\,dt\right)$$

$$\frac{dA}{dt} = \frac{1}{2m}\underbrace{\left(\vec{r} \times \vec{p}\right)}_{\vec{\ell}}$$

$$\frac{dA}{dt} = \frac{\vec{\ell}}{2m}$$

constant

constant

Polar Coordinates

$$\vec{l} = \vec{r} \times \underbrace{\vec{p}}_{\vec{v}\cdot m}$$

$\omega \rightarrow$ angular velocity

$$\vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$$

$$= m \cdot r\hat{r} \times (\dot{r}\hat{r} + r\dot{\phi}\hat{\phi})$$

$$= mr\left(\hat{r} \times \dot{r}\hat{r} + \hat{r} \times r\dot{\phi}\hat{\phi}\right)$$

$$= mr(\dot{r}\underbrace{\hat{r}\times\hat{r}}_{=0} + r\dot{\phi}\underbrace{\hat{r}\times\hat{\phi}}_{\hat{z}})$$

$$\vec{l} = mr^2\dot{\phi}\hat{z}$$

$$|\vec{l}| = mr^2\omega$$

May or may not be constant

$$\vec{l} = \underbrace{mr^2}_{\text{moment of inertia}} \cdot \vec{\omega}$$

angular velocity

$$I = mr^2 \leftarrow \text{only single point mass}$$

$$\vec{p} = m \cdot \vec{v}$$

translational velocity

What about multiple particles?

$$\vec{\mathbb{L}} = \sum_{\alpha=1}^{N} \vec{l}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{p}_\alpha$$

$$\dot{\vec{\mathbb{L}}} = \sum_{\alpha=1}^{N} \underbrace{\dot{\vec{r}}_\alpha \times \vec{p}_\alpha}_{=0} + \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \dot{\vec{p}}_\alpha$$

$$\dot{\vec{r}}_\alpha \times m\vec{v}, \quad \vec{v} = \dot{\vec{r}}_\alpha$$

$$\dot{\vec{\mathbb{L}}} = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha \leftarrow \dot{\vec{p}}_\alpha$$

$$\vec{F}_\alpha = \vec{F}_\alpha^{ext} + \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta}$$

$$\dot{\vec{\mathbb{L}}} = \sum_{\alpha=1}^{N} \left[\dot{\vec{r}}_\alpha \times \left(\vec{F}_\alpha^{ext} + \sum_{\beta \neq \alpha}^{N} F_{\alpha\beta} \right) \right]$$

$$\dot{\vec{\mathbb{L}}} = \underbrace{\sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha^{ext}}_{\vec{\Gamma}^{ext}} + \sum_{\alpha=1}^{N} \dot{\vec{r}}_\alpha \times \sum_{\beta \neq \alpha}^{N} F_{\alpha\beta}$$

$$\underbrace{\sum_{\alpha=1}^{N} \sum_{\beta \neq \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta}}_{=0}$$

$$\underbrace{\dot{\vec{\mathbb{L}}} = \vec{\Gamma}^{ext}}$$

$$\vec{\mathbb{L}} = \sum_{\alpha=1}^{N} \vec{r} \times \vec{p}$$

go back to polar coordinates $\rightarrow \hat{\phi}$ and $\hat{r}$

$$\vec{\mathbb{L}} = \sum_{\alpha} m_\alpha r_\alpha^2 \dot{\phi}_\alpha \hat{z}$$

all m_α have same $\dot{\phi}_\alpha = \omega$ ← solid object spinning

$$\vec{\mathbb{L}} = \vec{\omega} \cdot \underbrace{\sum_{\alpha} m_\alpha r_\alpha^2}$$

$$\hookrightarrow I = \sum_{\alpha} m_\alpha r_\alpha^2$$

$\downarrow$ continuous medium

$$\vec{\mathbb{L}} = \underbrace{I}_{\text{in general, a \underline{tensor}}} \cdot \vec{\omega}$$

$$I = \int r^2 dm$$

$$\boxed{I = \int r^2 \rho \, dV}$$

← very similar integral to the CM integral solid sphere rotating about its center

$$I = \frac{2}{5} M R^2$$

disk rotating about its center

$$I = \frac{1}{2} M R^2$$

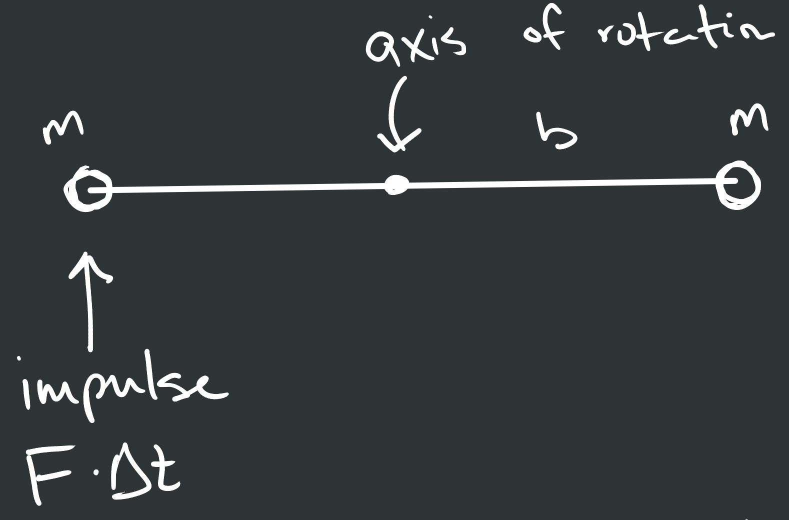
Rotations about the center of mass frame

$$\underbrace{\dot{\vec{L}} = \vec{\Gamma}^{ext}}$$

← still true measured about the center of mass frame

↳ inertial reference frame

still works in CM frame even if CM frame is not inertial



$$\vec{\Gamma} = \vec{F} \times \vec{b}$$
$$= |F|\cdot|b| \underbrace{\sin 90^\circ}_{1}$$

$$\dot{\vec{P}} = \vec{F}^{ext} \quad \Big\}\ \text{Newton's } 2^{nd}$$

$$M\dot{V}_{cm}$$

$$\frac{M\Delta V_{cm}}{\Delta t}$$

$$MV_{cm} = \underbrace{F^{ext}\cdot\Delta t}$$

$$V_{cm} = \frac{F\Delta t}{M}$$

$$V_{cm} = \frac{F\Delta t}{2m}$$

$$\dot{\vec{\mathbb{L}}} = \vec{\Gamma}^{ext}$$

choose an axis of rotation

CM

$$\vec{\mathbb{L}} = \vec{\Gamma}^{ext}\cdot\Delta t$$

$$I\cdot\omega_{cm} = (F\cdot b)\,\Delta t$$

$$\sum_{\alpha}^{N} m_\alpha r_\alpha^2 = \sum_{\alpha}^{2} m_\alpha r_\alpha^2$$
$$= mb^2 + mb^2$$
$$= 2mb^2$$

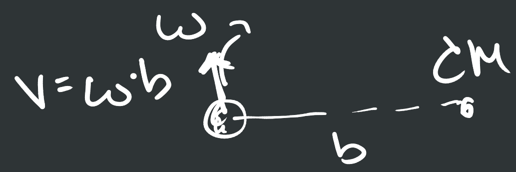
• What is the speed of the right and left masses from the "lab" frame

$$2mb^2 \cdot \omega_{CM} = Fb\Delta t$$

$$\omega_{cm} = \frac{Fb\Delta t}{2mb^2}$$

$$\omega_{cm} = \frac{F\Delta t}{2mb}$$

$$V^{s'} = V^{s} - V^{s'\to s}$$



object's velocity in another frame

object's velocity in one frame

velocity between frame

left mass speed in the lab frame

$$V_\ell^{s'} = \omega \cdot b - (-V_{cm})$$

$$V_\ell^{s'} = \frac{F\Delta t}{2mb}\cdot b + \frac{F\Delta t}{2m} = \frac{F\Delta t}{m}$$

right mass speed in the lab frame

$$V_r^{s'} = -\omega b - (-V_{cm})$$

$$= -\frac{F\Delta t}{2mb} \cdot b + \frac{F\Delta t}{2m}$$

$$V_r^{s'} = 0$$

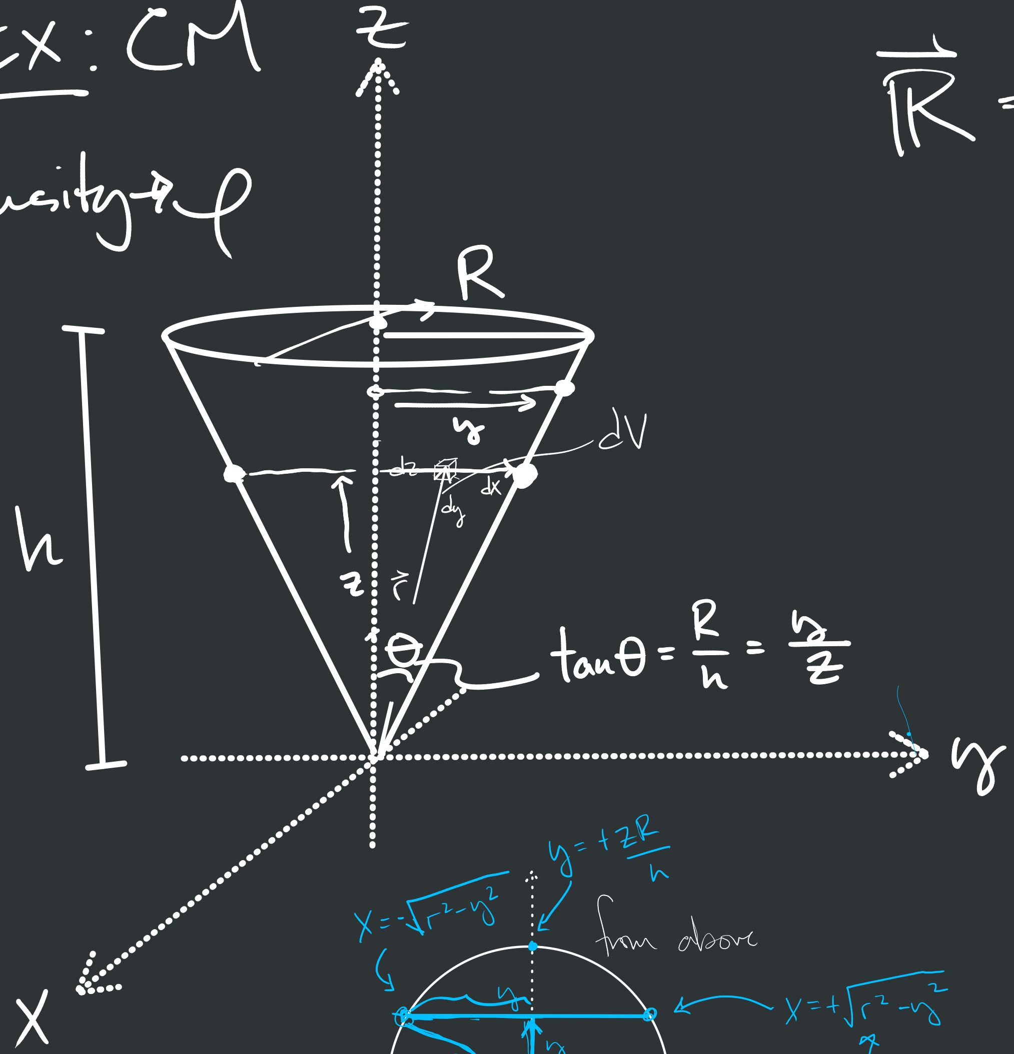
$$\overset{\text{CM}}{V^{s'}} = \overset{\text{lab}}{V^{s}} - \underbrace{V^{s' \to s}}$$

$$\omega b = \underline{\underline{V^{s}}} - V_{cm}$$

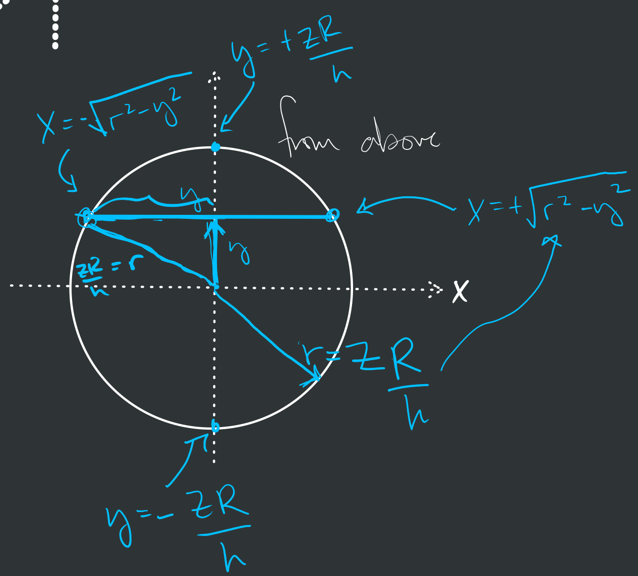
$$V^{lab} = \omega b + V_{cm}$$

Ex: CM

density → ρ



$$\tan\theta = \frac{R}{h} = \frac{y}{z}$$



$$\vec{R} = \frac{1}{M}\int \vec{r}\,\rho\, dV$$

$$V_{cone} = \frac{\pi R^2 h}{3}$$

$$\rho = \frac{M}{V_{cone}} = \frac{3\cdot M}{\pi R^2 h}$$

$$= \frac{\rho}{M}\int \vec{r}\, dV$$

$$= \frac{3\cancel{M}}{\cancel{M}\cdot \pi R^2 h}\int \vec{r}\, dV$$

$$dV \rightarrow dx\,dy\,dz$$

$$\iiint_{z\,y\,x} \vec{r}\, dx\,dy\,dz$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\iiint_{z\,y\,x} \left(x\hat{x} + y\hat{y} + z\hat{z}\right) dx\,dy\,dz$$

$$\hat{x}\iiint_{z\,y\,x} x\,dx\,dy\,dz + \hat{y}\iiint_{z\,y\,x} y\,dx\,dy\,dz + \hat{z}\iiint_{z\,y\,x} z\,dx\,dy\,dz$$

$$\underbrace{\hat{x}\int_0^h\int_{-\frac{zR}{h}}^{+\frac{zR}{h}}\int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} x\,dx\,dy\,dz}_{=0} + \underbrace{\hat{y}\int_0^h\int_{-\frac{zR}{h}}^{+\frac{zR}{h}}\int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} y\,dx\,dy\,dz}_{=0} + \hat{z}\int_0^h\int_{-\frac{zR}{h}}^{+\frac{zR}{h}}\int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} z\,dx\,dy\,dz$$

<u>proof:</u>

$$\hat{x}\int_0^h\int_{-\frac{zR}{h}}^{+\frac{zR}{h}}\underbrace{\int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} x\,dx}\,dy\,dz$$

$$\left.\frac{x^2}{2}\right|_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} = \left(\frac{r^2-y^2}{2}\right) - \left(\frac{r^2-y^2}{2}\right) = 0$$

so that was not too bad

$$\int_0^h\int_{\frac{zR}{h}}^{\frac{zR}{h}} y\left(\underbrace{\int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} dx}\right) dy\,dz$$

$$\left. x\right|_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} = \sqrt{r^2-y^2} - \left(-\sqrt{r^2-y^2}\right) = 2\sqrt{r^2+y^2}$$

$$\underbrace{\int_0^h\int_{-\frac{zR}{h}}^{\frac{zR}{h}} y\left(2\sqrt{r^2+y^2}\right) dy\,dz}$$

$$\left.\begin{aligned} r^2+y^2 &= u \\ 2y\,dy &= du \end{aligned}\right\} \text{u-substitution}$$

$$\int u^{1/2}\,du \rightarrow \frac{2}{3}u^{3/2} \rightarrow \left.\frac{2}{3}\left(r^2+y^2\right)^{3/2}\right|_{-\frac{zR}{h}}^{\frac{zR}{h}} = \frac{2}{3}\underbrace{\left[\left(r^2+\left(\frac{zR}{h}\right)^2\right)^{3/2} - \left(r^2+\left(-\frac{zR}{h}\right)^2\right)^{3/2}\right]}_{=0}$$

$(-1)^2=+1$

$$\int_0^h z\int_{-\frac{zR}{h}}^{\frac{zR}{h}}\underbrace{\int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} dx}_{=2\sqrt{r^2-y^2}}\,dy\,dz$$

$$\int_{-\frac{zR}{h}}^{\frac{zR}{h}} 2\sqrt{r^2-y^2}\,dy$$

look up in Schaum's

$$= \cancel{2}\left.\left(\frac{y\sqrt{r^2-y^2}}{\cancel{2}} + \frac{r^2}{\cancel{2}}\sin^{-1}\left(\frac{y}{r}\right)\right)\right|_{-\frac{zR}{h}}^{+\frac{zR}{h}}$$

recall $r = \frac{zR}{h}$

$$= \underbrace{\frac{zR}{h}\sqrt{\left(\frac{zR}{h}\right)^2-\left(\frac{zR}{h}\right)^2}}_{=0} + \left(\frac{zR}{h}\right)^2\underbrace{\sin^{-1}\left(\frac{zR\cdot h}{h\cdot zR}\right)}_{\sin^{-1}(1)=\pi/2} - \underbrace{\left(-\frac{zR}{h}\right)\sqrt{\left(\frac{zR}{h}\right)^2-\left(\frac{zR}{h}\right)^2}}_{=0} + \left(\frac{zR}{h}\right)^2\underbrace{\sin^{-1}(-1)}_{+\pi/2}$$

$$= \pi\left(\frac{zR}{h}\right)^2$$

$$\int_0^h z\left(\pi\frac{z^2R^2}{h^2}\right)dz$$

$$= \pi\frac{R^2}{h^2}\int_0^h z^3\,dz$$

$$= \pi\frac{R^2}{h^2}\left.\left(\frac{z^4}{4}\right)\right|_0^h$$

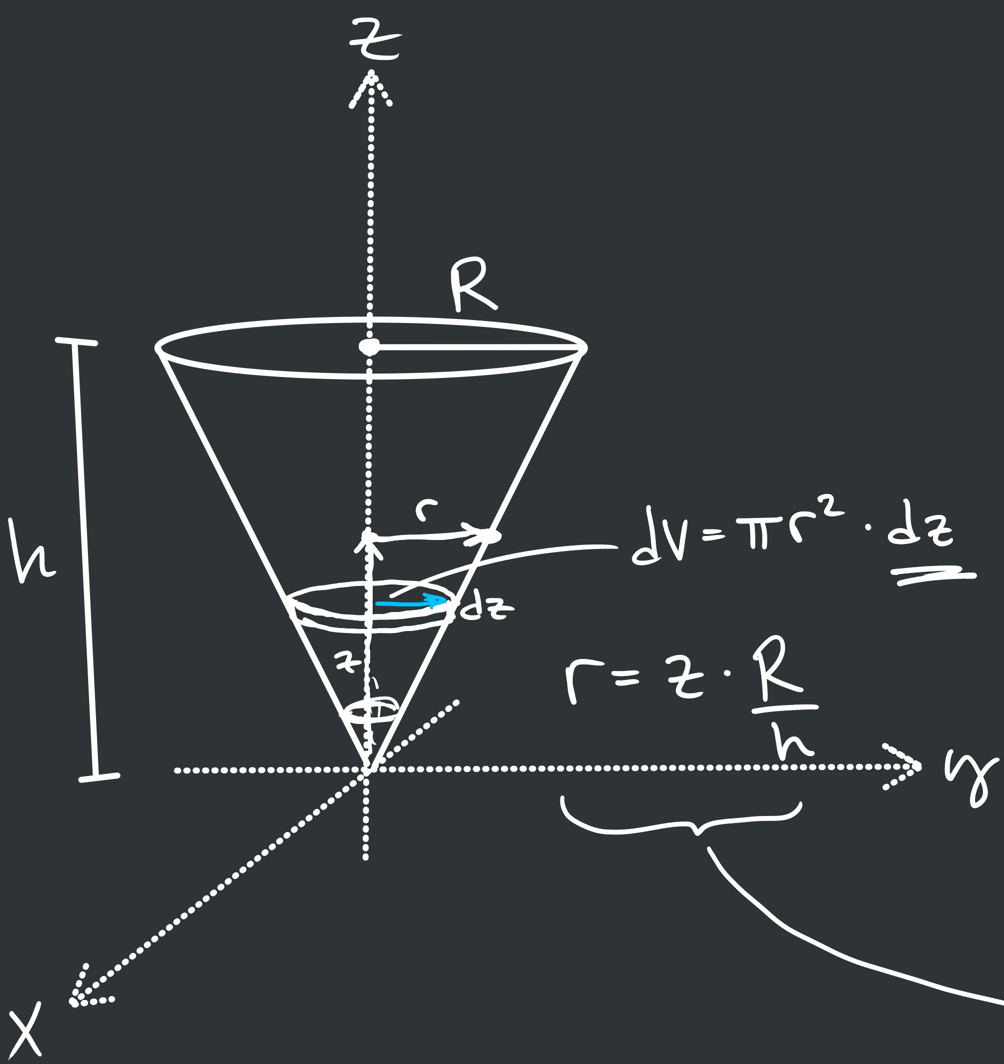
$$= \pi\frac{R^2}{\cancel{h^2}}\cdot\frac{h^{\cancel{4}\,2}}{4}$$

$$= \boxed{\frac{\pi R^2 h^2}{4}}$$

$$\mathbb{R} = \frac{3\cancel{M}}{\cancel{M}\cdot\cancel{\pi}\cancel{R^2}\cancel{h}}\cdot\frac{\cancel{\pi}\cancel{R^2}h^{\cancel{2}}}{4}$$

$$= \frac{3h}{4}$$

so I am off by a factor of 1/2



$dV = \pi r^2 \cdot dz$

$r = z \cdot \frac{R}{h}$

$\vec{R}$

$\int \vec{r}\, dV$

$\int z\, \pi r^2\, dz$

$\vec{r}$ in cylindrical

$\cancel{r\hat{r}} + z\hat{z}$

$\to 0$ by symmetry

$\pi \int z\, r^2\, dz$

$\pi \int_0^h z \left(z\frac{R}{h}\right)^2 dz$

$\pi \left(\frac{R}{h}\right)^2 \int_0^h z^3\, dz$

$\pi \left(\frac{R}{h}\right)^2 \left(\frac{z^4}{4}\right)\Big|_0^h$

put the constants back

$$R = \frac{3\cancel{M}}{\cancel{M}\cdot\cancel{\pi}\cancel{R^2}\cancel{h}}\,\cancel{\pi}\,\frac{\cancel{R^2}}{\cancel{h^2}}\left(\frac{h^{\cancel{4}}}{4}\right)$$

$$\boxed{R = \frac{3}{4}h}$$ ← in the z direction