

Chapter 4 \leadsto Energy

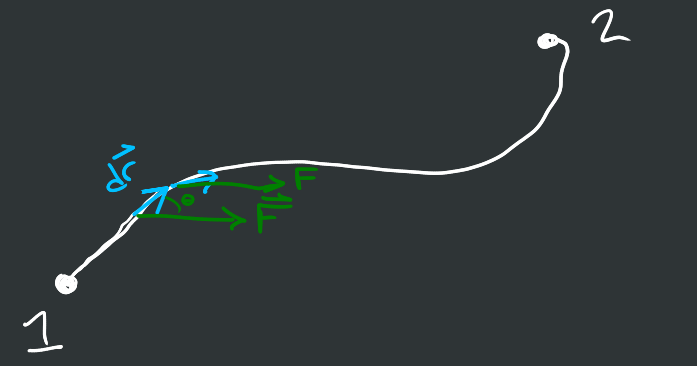
- Kinetic Energy

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v} \cdot \vec{v} = \frac{p^2}{2m}$$

- Work is a transfer of energy to the system

$$dW = \vec{F} \cdot d\vec{r} \Rightarrow W = \int_1^2 \vec{F} \cdot d\vec{r}$$

$|\vec{F}| \cdot |d\vec{r}| \cdot \cos\theta$



The diagram shows a curved path starting at point 1 and ending at point 2. A small segment of the path is highlighted in blue, with a blue vector $d\vec{r}$ tangent to it. A green vector \vec{F} is shown at point 1, pointing towards point 2. The angle between \vec{F} and $d\vec{r}$ is labeled θ .

path integral means work depends on the path between points, unless it doesn't

- Amount of work done (total) is the same as the change in kinetic energy

$$W = \Delta T$$

← Work-Kinetic energy theorem

- If the work done does not depend on the path taken, (it depends on the end points (positions)), that force is a conservative force

→ work done by a conservative force \Rightarrow potential energy

$$\Delta U = - W_{\text{conservative force}}$$

Conservative forces

- gravitational force
- electric force
- spring force

non conservative force

- friction
- drag
- "applied force"

↓ potential energy is defined by a reference point
(\vec{r} when $U=0$)

• conservation of energy

$$W = \Delta K$$

$$\hookrightarrow W_{nc} = \underbrace{\Delta K + \Delta U}_{\text{mechanical energy}}$$

$$U_g = mgh$$

$$W_{nc} = T_f - T_i + U_f - U_i$$

$$T_i + U_i + W_{nc} = T_f + U_f$$

$$\bullet \text{ if } \Delta U = - \int_i^f \vec{F} \cdot d\vec{r} \Rightarrow$$

$$dU = -W = -\vec{F} \cdot d\vec{r}$$

~~$$\frac{dU}{d\vec{r}} = -\vec{F}$$~~

$$\vec{F} = -\vec{\nabla} U = \frac{\partial U}{\partial x} \hat{x} + \frac{\partial U}{\partial y} \hat{y} + \frac{\partial U}{\partial z} \hat{z}$$

\uparrow gradient \rightarrow cartesian

$$F = -\frac{dU}{dx} \leftarrow \text{one dimension}$$

• What make a F conservative

- F depends only on position
(not v, t, a)

• Work is path-independent

$$\hookrightarrow \vec{\nabla} \times \vec{F} = 0$$

\uparrow curl

$$F_s = -kx$$

$$\rightarrow F_g = mg$$

$$F_G = \frac{G m_1 m_2}{r^2}$$

$$F_0 = b \cdot v$$

$$= c v^2$$

• everything above can be done in spherical coords,
or cylindrical coords \rightarrow often this will be easier

central forces

$$\vec{F}(\vec{r}) = f(r) \hat{r}$$

\hat{r} radial force

$$\vec{F}_G(\vec{r}) = -\frac{Gm_1 m_2}{r^2} \hat{r} \quad \leftarrow \text{in spherical coords.}$$

$$\vec{r} = x\hat{x} + y\hat{y} \quad \hat{r} = \frac{\vec{r}}{r}$$

$$r = \sqrt{x^2 + y^2}$$

$$= -\left(\frac{Gm_1 m_2}{x^2 + y^2}\right) \cdot \frac{x\hat{x} + y\hat{y}}{\sqrt{x^2 + y^2}}$$

$$= -\frac{Gm_1 m_2}{(x^2 + y^2)^{3/2}} \cdot (x\hat{x} + y\hat{y}) \quad \leftarrow$$

$$\vec{\nabla} \times \vec{F}_G =$$

\curvearrowright what is curl in spherical?

- All of this applies multiple particle

$$T = T_1 + T_2 + T_3 + \dots$$

$$T = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} \underset{\substack{\uparrow \\ \text{rot. inertia}}}{I} \omega_{cm}^2$$

Recall: $T = \frac{1}{2} m v^2 = \frac{1}{2} m \vec{v} \cdot \vec{v}$

$$\frac{dT}{dt} = \frac{1}{2} m \cdot \frac{d(\vec{v} \cdot \vec{v})}{dt}$$

$$= \frac{1}{2} m \left(\frac{d(\vec{v})}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d(\vec{v})}{dt} \right)$$

$$\Delta T = W$$

$$= \frac{1}{2} m (2 \dot{\vec{v}} \cdot \vec{v})$$

$$= m \dot{\vec{v}} \cdot \vec{v}$$

$$\vec{p} = \vec{F}$$

$$\frac{dT}{dt} = \vec{F} \cdot \vec{v}$$

$$\frac{dT}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt}$$

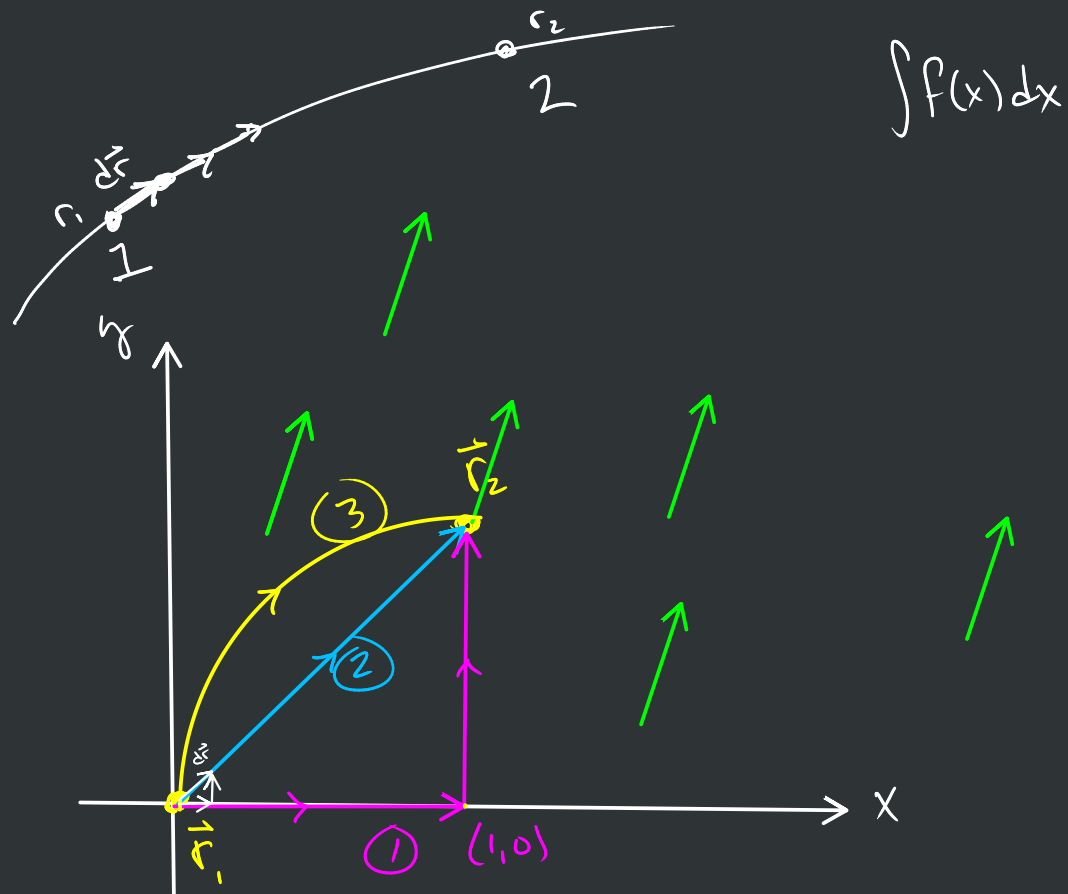
$$\rightarrow dT = \vec{F} \cdot d\vec{r}$$

define this
as a small amount
of work

dW does not make
sense.
 $\rightarrow dW$ (in thermo)

$$\Delta T = T_2 - T_1 = \underbrace{\int_1^2 \vec{F} \cdot d\vec{r}}_{\text{line integral}}$$

Ex: $\vec{F} = 1\hat{x} + 2\hat{y}$
 $\vec{r}_1 = 0$
 $\vec{r}_2 = 1\hat{x} + 1\hat{y}$



Path (1)

$$\int_1^2 \vec{F} \cdot d\vec{r} = \underbrace{\int_{(0,0)}^{(1,0)} \vec{F} \cdot d\vec{r}_1}_{\text{Path 1}} + \underbrace{\int_{(1,0)}^{(1,1)} \vec{F} \cdot d\vec{r}_2}_{\text{Path 2}}$$

$$\begin{aligned} d\vec{r}_1 &= d\vec{x} = dx \hat{x} \\ \vec{F} \cdot d\vec{x} &= (1\hat{x} + 2\hat{y}) \cdot (dx \hat{x} + 0\hat{y}) \\ &= 1 dx \end{aligned}$$

$$\begin{aligned} d\vec{r}_2 &= dy \hat{y} \\ \vec{F} \cdot dy \hat{y} &= (1\hat{x} + 2\hat{y}) \cdot (0\hat{x} + dy \hat{y}) \\ &= 2 dy \end{aligned}$$

$$\int_1^2 \vec{F} \cdot d\vec{r} = \int_{x=0}^{x=1} dx + \int_{y=0}^{y=1} 2 dy$$

$$= x|_0^1 + 2y|_0^1$$

$$= 1 + 2 = 3$$

Work along path ①

Path ②

$$\int_1^2 \vec{F} \cdot d\vec{r} =$$

$$d\vec{r} = dx \hat{x} + dy \hat{y}$$

$$\vec{F} \cdot d\vec{r} = (1\hat{x} + 2\hat{y}) \cdot (dx\hat{x} + dy\hat{y})$$

$$= dx + 2dy$$

$$\int_1^2 (dx + 2dy) = \int_0^1 dx + \int_0^1 2dy = 3$$

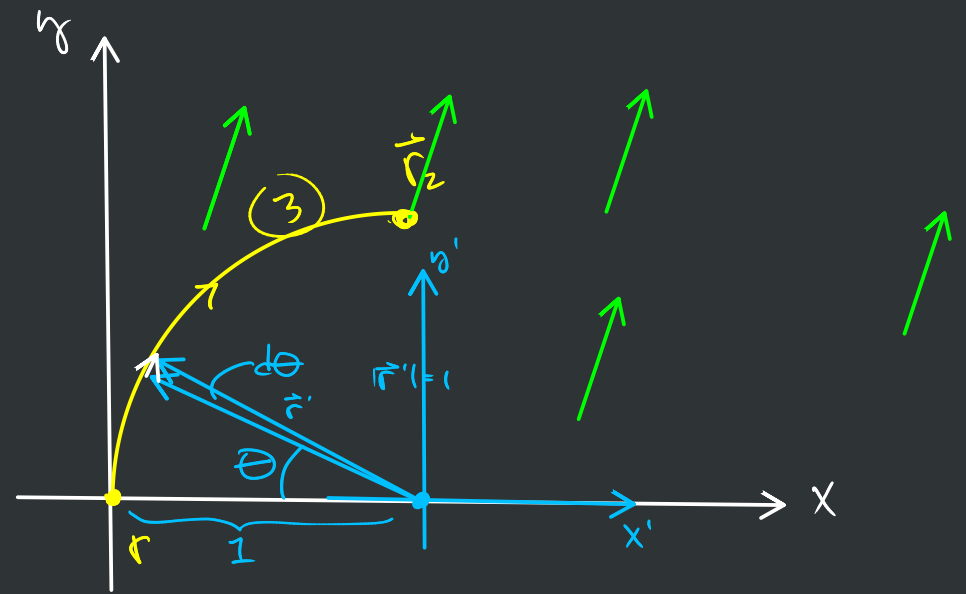
Path ③

$$\int_1^2 \vec{F} \cdot d\vec{r}$$

$$\int_1^2 (1\hat{x} + 2\hat{y}) \cdot (\sin\theta d\theta\hat{x} + \cos\theta d\theta\hat{y})$$

$$\int_1^2 (\sin \theta d\theta + 2 \cos \theta d\theta)$$

$$\int_0^{\pi/2} \sin \theta \, d\theta + \int_0^{\pi/2} 2 \cos \theta \, d\theta = 3$$



$$\vec{r}' = -\cos\theta \hat{x}' + \sin\theta \hat{y}'$$

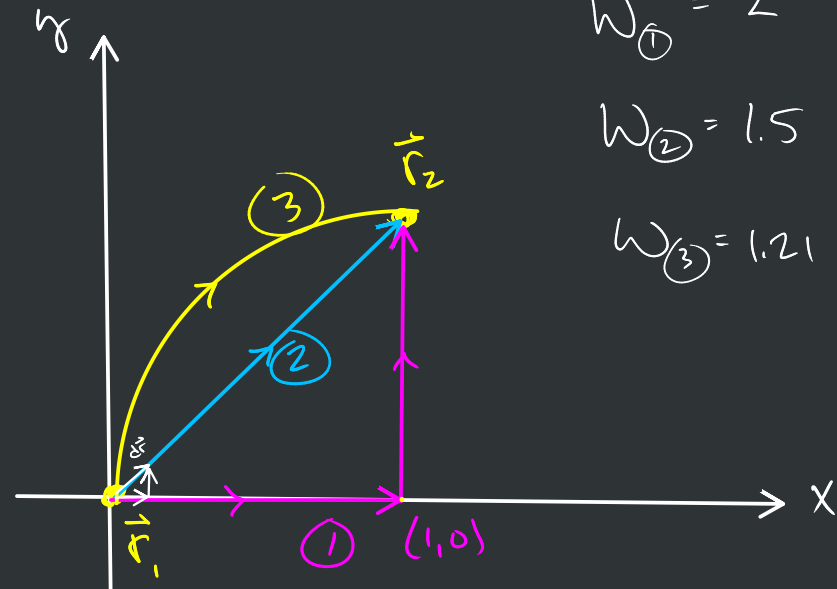
↓ change coordinate systems

$$\vec{r} = (1 - \cos\theta)\hat{x} + \sin\theta\hat{y}$$

$$d\vec{r} = \sin\Theta d\Theta \hat{x} + \cos\Theta d\Theta \hat{y}$$

Homework: Same 3 paths

$$\vec{F} = y\hat{x} + 2x\hat{y}$$



$$W_{(1)} = 2$$

$$W_{(2)} = 1.5$$

$$W_{(3)} = 1.21$$

$$\vec{r} = (1 - \cos\theta)\hat{x} + \sin\theta\hat{y}$$

$$d\vec{r} = \sin\theta d\theta\hat{x} + \cos\theta d\theta\hat{y}$$

$$\begin{aligned} W_0 &= \int_{(0,0)}^{(1,0)} \vec{F} \cdot d\vec{r}_1 + \int_{(1,0)}^{(1,1)} \vec{F} \cdot d\vec{r}_2 \\ &\quad \downarrow \quad \downarrow \\ &\quad dx\hat{x} \quad dy\hat{y} \\ &= (y\hat{x} + 2x\hat{y}) \cdot (dx\hat{x} + 0\hat{y}) \\ &\quad y dy + 0 \\ &= \int_{(0,0)}^{(1,0)} y dx + \int_{(1,0)}^{(1,1)} 2x dy \\ &= \underbrace{\int_{x=0}^{x=1} 0 \cdot dx}_0 + \int_{y=0}^{y=1} 2 \cdot (1) dy = 2y \Big|_0^1 = 2(1-0) = 2 \end{aligned}$$

$$\begin{aligned} W_3 &= \int_0^{\pi/2} y \sin\theta d\theta + \int_0^{\pi/2} 2x \cos\theta d\theta \\ &\quad \uparrow \\ &= \int_0^{\pi/2} \sin^2\theta d\theta + 2 \int_0^{\pi/2} (1 - \cos\theta) \cos\theta d\theta \\ &= \int_0^{\pi/2} \sin^2\theta d\theta + 2 \int_0^{\pi/2} \cos\theta d\theta - 2 \int_0^{\pi/2} \cos^2\theta d\theta \end{aligned}$$

For conservative force

$$U(\vec{r}) = -W(\vec{r}_0 \rightarrow \vec{r}) = - \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}') \cdot d\vec{r}'$$

Potential energy
at \vec{r} in reference to \vec{r}_0

conditions for conservative force

① \vec{F} only depends on \vec{r} (not \vec{v} , not t)

② Work done b/t any two points
is independent of path taken
($\vec{\nabla} \times \vec{F} = 0$)

time
↓

What about $W(\vec{r}_1 \rightarrow \vec{r}_2)$ \leftarrow work to go between points not the reference point

$$W(\vec{r}_0 \rightarrow \vec{r}_2) = W(\vec{r}_0 \rightarrow \vec{r}_1) + W(\vec{r}_1 \rightarrow \vec{r}_2)$$

$$W(\vec{r}_1 \rightarrow \vec{r}_2) = \underbrace{W(\vec{r}_0 \rightarrow \vec{r}_2)}_{-U(\vec{r}_2)} - \underbrace{W(\vec{r}_0 \rightarrow \vec{r}_1)}_{-U(\vec{r}_1)}$$

$$= - \underbrace{(U(\vec{r}_2) - U(\vec{r}_1))}_{\Delta U}$$

$$\underline{\underline{\Delta U}} = -W(\vec{r}_1 \rightarrow \vec{r}_2)$$

Now go back to work-kinetic energy theorem

$$W(\vec{r}_1 \rightarrow \vec{r}_2) = \Delta K$$

if only conservative forces are acting

Conservation
of
Energy

$$\rightarrow -\Delta U = \Delta K$$

$$\rightarrow \Delta K + \Delta U = 0$$

$$E = K + U \quad \left. \vphantom{E = K + U} \right\} \text{Mechanical Energy}$$

$$\Delta E = 0$$

Also true for multiple conservative forces.

↳ multiple potential energy

$$U = U_g + U_s (+ U_e)$$

But, what if also non-conservative forces act?

Conservation
of
Energy

$$\Delta K + \Delta U \neq 0$$

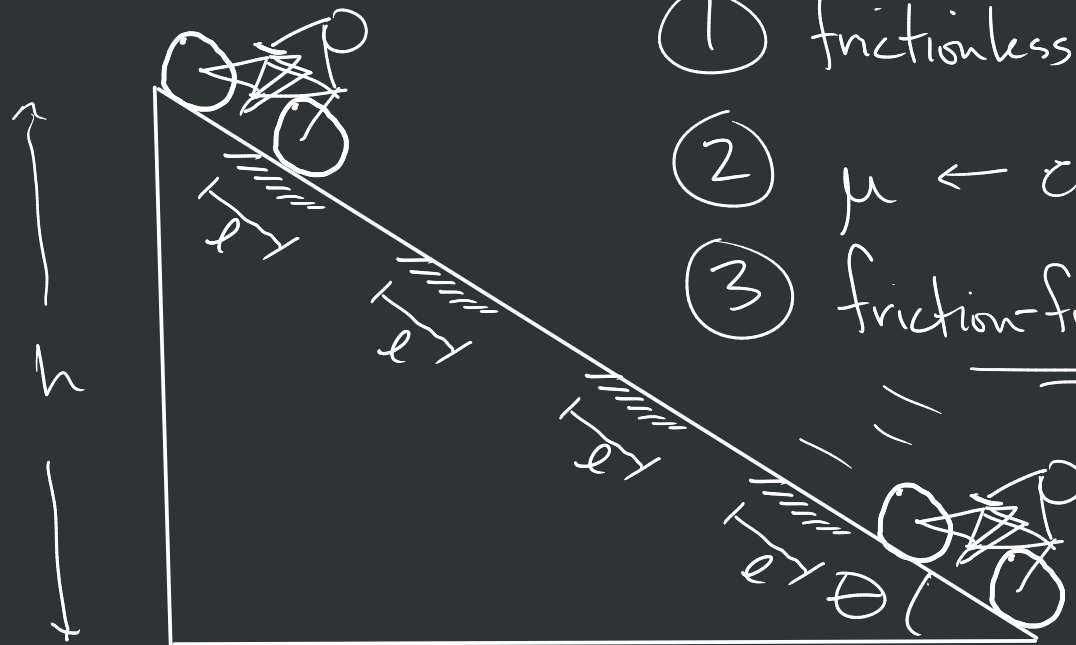
$$\Delta K + \Delta U = W_{nc}$$

$$K_i + U_i + W_{nc} = K_f + U_f$$

Friction

$$W_f = \int \vec{F}_f \cdot d\vec{x}$$

Ex:

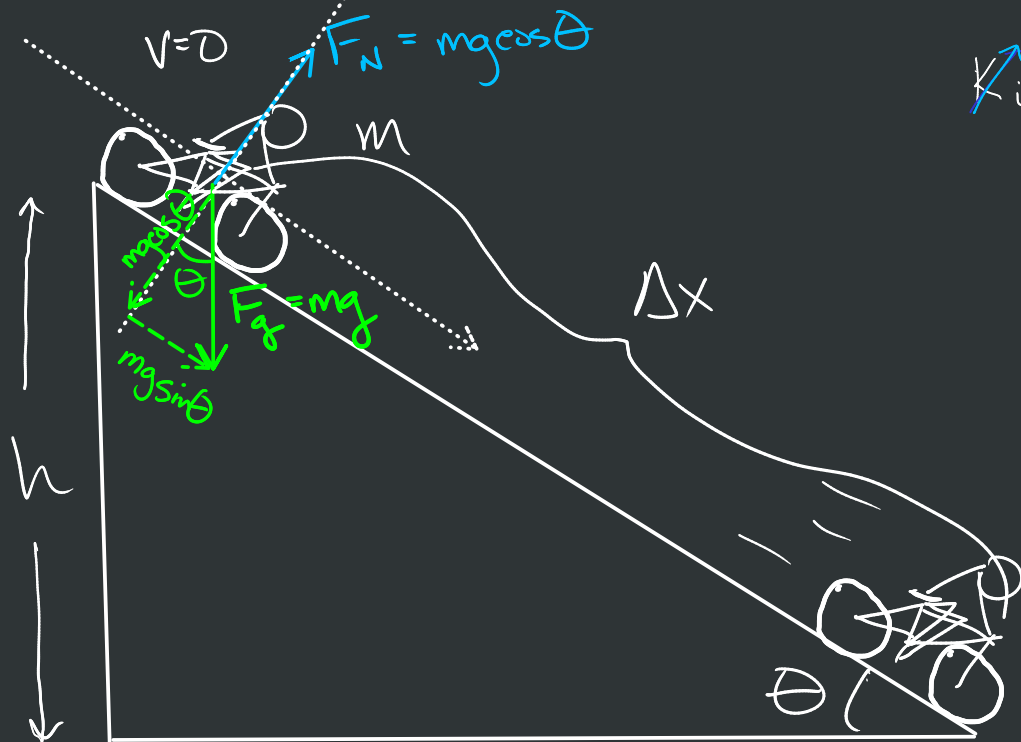


- ① frictionless plane
- ② $\mu \leftarrow$ constant
- ③ friction-full patches

\nwarrow no
Newton's

Newton's Laws
+
Conservation
of
Energy

1



$$\cancel{K_i} + \cancel{U_i} + \cancel{W_{NC}} = \cancel{K_f} + \cancel{U_f}$$

$$\cancel{mgh} = \frac{1}{2} \cancel{mv^2}$$

$$v = \sqrt{2gh}$$

$$F_{NET,X} = mg \sin \theta = ma = \dot{p}$$

$$g \sin \theta = a$$

$$g \sin \theta = \frac{dv}{dt}$$

$$\int_0^t g \sin \theta dt = \int_{v_0=0}^v dv'$$

$$g \sin \theta \cdot t = v \quad v_f = v_i + at$$

$$v = g \sin \theta t$$

$$\int_0^x dx = \int_0^t g \sin \theta t' dt'$$

$$x = \frac{g \sin \theta t^2}{2}$$

$$g \sin \theta = \frac{dv}{dt} = \frac{dv}{dx} \cdot \underbrace{\frac{dx}{dt}}_v$$

$$g \sin \theta = v \frac{dv}{dx}$$

$$\int_{x=0}^x g \sin \theta dx' = \int_0^v v' dv'$$

$$g \sin \theta x' \Big|_0^x = \frac{v'^2}{2} \Big|_0^v = \frac{v^2}{2}$$

$$g \sin \theta \cdot x = \frac{v^2}{2}$$

$$\sqrt{2g \underbrace{\sin \theta \cdot x}_h} = v$$

$$v = \sqrt{2gh}$$

$$\sin \theta = \frac{h}{x} = \frac{\text{opp}}{\text{hyp}}$$

$$\sin \theta \cdot x = h$$

Potential Energy

$$U(\vec{r}) = - \int_{\vec{r}_0}^{\vec{r}} \underline{\vec{F}(\vec{r}') \cdot d\vec{r}'}$$

Gravitational (constant)

$$\vec{F}(\vec{r}') = -mg \hat{y}$$

$$d\vec{r} = dx \hat{x} + dy \hat{y} + dz \hat{z}$$

$$U = -W(0 \rightarrow h \hat{y}) = + \int_0^h (+mg \hat{y}) \cdot (dx \hat{x} + dy \hat{y} + dz \hat{z})$$

$$= \int_0^h mg dy$$

$$= mgy \Big|_0^h$$

$U(h) = mgh$ ← in relation to
wherever we call $h=0 \rightarrow U=0$

Gravitation (universal)

$$\vec{F} = - \frac{G m_1 m_2}{r^2} \hat{r}$$



$$U(\vec{r}) = -W(0 \rightarrow \vec{r}) = - \int_{\infty}^{\vec{r}} - \frac{G m_1 m_2}{r^2} \hat{r} \cdot d\vec{r} \hat{r}$$

$$= \int_{\infty}^{\vec{r}} \frac{G m_1 m_2}{r'^2} dr' = G m_1 m_2 \int_{\infty}^{\vec{r}} r'^{-2} dr'$$

$$= G m_1 m_2 (-1) r'^{-1} \Big|_{\infty}^{\vec{r}}$$

$$= - \frac{G m_1 m_2}{r'} \Big|_{\infty}^{\vec{r}}$$

$$= - \frac{G m_1 m_2}{r} - \cancel{\frac{-G m_1 m_2}{\infty}} \xrightarrow{0}$$

reference point
 $U=0$ at $r \rightarrow \infty$

$$\rightarrow U(r) = - \frac{G m_1 m_2}{r}$$

If instead we set $U=0$ at $r=R$ ↖ radius
of a
planet

$$\int_R^r \frac{Gm_1 m_2}{r} dr = - \frac{Gm_1 m_2}{r} \Big|_R^r$$
$$= - \frac{Gm_1 m_2}{r} + \frac{Gm_1 m_2}{R}$$

$$U(r) = Gm_1 m_2 \left(\frac{1}{R} - \frac{1}{r} \right)$$

$r > R$

Ex: $r_1 \rightarrow r_2$ will require non-conservative work if $\Delta K = 0$

$$\cancel{K}_i + U_i + W_{nc} = \cancel{K}_f + U_f$$

↙ choose $r \rightarrow \infty$
 $U=0$

$$W_{nc} = \Delta U = U(r_2) - U(r_1)$$

$$= - \frac{Gm_1 m_2}{r_2} + \frac{Gm_1 m_2}{r_1}$$

$$= Gm_1 m_2 \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

Spring Potential (1D)

$$\vec{F} = -kx \hat{x}$$

$$\downarrow dx \hat{x} + dy \hat{y} + \dots$$

$$U_s = - \int_0^x (-kx \hat{x}) \cdot d\vec{r}$$

$$= \int_0^x kx' dx'$$

$$= \left. k \frac{x'^2}{2} \right|_0^x = \frac{1}{2} k x^2 = U_s$$

reference

$$x=0 \\ U=0$$

unstretched spring

Potential Energy \rightarrow Force

$$-dU = dW = \vec{F} \cdot d\vec{r}$$

$\hookrightarrow dU$ depends on $d\vec{r}$ \nwarrow displacement

Cartesian $d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$

$\frac{\partial U}{\partial x}$ \leftarrow how much U changes in the x -direction
w/ y & z fixed

$\frac{\partial U}{\partial y}$ \leftarrow same but x & z fixed

total differential

$$\hookrightarrow dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz$$

$$df = \frac{df}{dx} dx$$



$$-\left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz\right) = F_x dx + F_y dy + F_z dz$$

$$-\frac{\partial u}{\partial x} = F_x \quad -\frac{\partial u}{\partial y} = F_y \quad -\frac{\partial u}{\partial z} = F_z$$

$$\vec{F} = -\frac{\partial u}{\partial x} \hat{x} - \frac{\partial u}{\partial y} \hat{y} - \frac{\partial u}{\partial z} \hat{z}$$

this operation is called the gradient

$$= -\left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}\right) u$$

one particular differential operator

"del" $\rightarrow \vec{\nabla}$

in Cartesian

$$F = -\vec{\nabla} u$$

