Chapter 6 + 7 → Principle of Least Action and the Lagrangian

$$\boxed{\text{Newton's Laws}} \text{ — } \boxed{\text{Lagrangian}} \text{ — } \boxed{\text{Hamiltonian}}$$

→ "generalized coordinates"

→ Principle of Least Action

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$U = mgx$$

$$\underbrace{\int_{t_1}^{t_2} \Big(\frac{1}{2}m\underset{v(t)}{\dot{x}^2} - mg\underset{x(t)}{x}\Big)dt}_{\text{what is } x(t) \text{ and } \dot{x}(t) \text{ so this integral is smallest}}$$

Lagrangian

$$\mathcal{L} = T - U$$

Action

$$S = \int_{t_1}^{t_2} \mathcal{L}\big(x(t), \dot{x}(t), t\big)dt$$

$$= \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t)dt$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}}$$

Euler - Lagrange Equation

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}mv^2$$

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 - U(x,y,z)$$

↳ 1D motion

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - U(x)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - U(x)\right)\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2 - U(x)\right)$$

$$\frac{d}{dt}(m\dot{x}) = -\frac{dU}{dx}$$

$$m\ddot{x} = -\frac{dU}{dx}$$

$\longrightarrow$ In 3D

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x} \qquad m\ddot{x} = -\frac{\partial U}{\partial x}$$

y

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = \frac{\partial \mathcal{L}}{\partial y} \qquad m\ddot{y} = -\frac{\partial U}{\partial y}$$

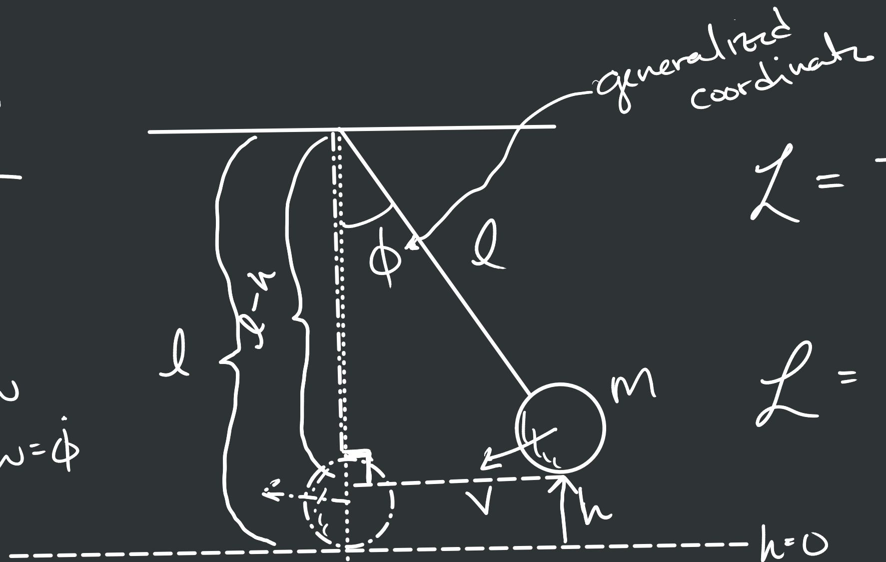
z

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) = \frac{\partial \mathcal{L}}{\partial z} \qquad m\ddot{z} = -\frac{\partial U}{\partial z}$$

$$\underbrace{m\ddot{x}\,\hat{x} + m\ddot{y}\,\hat{y} + m\ddot{z}\,\hat{z}} = -\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}$$

$$\underbrace{\Sigma \vec{F} = m\ddot{\vec{r}}} = \underbrace{-\vec{\nabla} U = \Sigma \vec{F}}$$

Ex:



$$\mathcal{L} = \underset{\dot{\phi}}{T} - \underset{\phi}{U}$$

$$\mathcal{L} = \frac{1}{2}mv^2 - mgh$$

$v = r\omega$

$\omega = \dot{\phi}$

$\dot{\phi} \cdot l = v$

$\cos\phi = \frac{l-h}{l} \rightsquigarrow h = l(1-\cos\phi)$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1-\cos\phi)$$

$\hookrightarrow$ apply the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \dot\phi} = \frac{1}{2} m l^2 (2\dot\phi) = m l^2 \dot\phi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) = \frac{d}{dt}(m l^2 \dot\phi) = m l^2 \ddot\phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -mgl(-(-\sin\phi))$$

$$= -mgl\sin\phi$$

$$\cancel{m}\cancel{l^2}\ddot\phi = -\cancel{m}g\cancel{l}\sin\phi$$

$$\boxed{\ddot\phi = -\frac{g}{l}\sin\phi}$$

$$\ddot\phi = -\frac{g}{l}\sin\phi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot x}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

Remember small angle approx $\underline{\sin\phi \approx \phi}$

$h=0$

generalize coordinate

h

ϕ, x

$l \to$ equilibrium length

$x \to$ distortion of the spring

$U_g = -mg(l+x)\cos\phi$

$U_s = +\frac{1}{2}kx^2$

$v_\phi = (l+x)\dot\phi$

$v_x = \dot x$

$$\mathcal{L} = T - U$$

$$= \frac{1}{2}mv^2 - U_g - U_s$$

$$U_s \hookrightarrow \frac{1}{2}kx^2$$

$$\mathcal{L} = \frac{1}{2}m\dot x^2 + \frac{1}{2}m(l+x)^2\dot\phi^2 + mg(l+x)\cos\phi - \frac{1}{2}kx^2$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) = \frac{\partial \mathcal{L}}{\partial \phi}}$$

$$\frac{\partial \mathcal{L}}{\partial \dot\phi} = m(l+x)^2\dot\phi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) = m(l+x)^2\ddot\phi + m2(l+x)\dot x\dot\phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -mg(l+x)\sin\phi$$

$$\cancel{m}(l+x)^{\cancel{2}}\ddot\phi + \underbrace{\cancel{m}2\cancel{(l+x)}\dot x\dot\phi}_{\text{Coriolis force}} = -\cancel{m}g\cancel{(l+x)}\sin\phi$$

$$\boxed{(l+x)\ddot\phi + 2\dot x\dot\phi + g\sin\phi = 0}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot x}\right) = \frac{\partial \mathcal{L}}{\partial x}}$$

$$\frac{\partial \mathcal{L}}{\partial \dot x} = m\dot x$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot x}\right) = m\ddot x$$

$$\frac{\partial \mathcal{L}}{\partial x} = \underbrace{m(l+x)\dot\phi^2}_{\text{centrifugal}} + mg\cos\phi - kx$$

$$\cancel{m}\ddot x = \cancel{m}(l+x)\dot\phi^2 + \cancel{m}g\cos\phi - \frac{kx}{m}$$

$$\boxed{\ddot x = (l+x)\dot\phi^2 + g\cos\phi - \frac{kx}{m}}$$

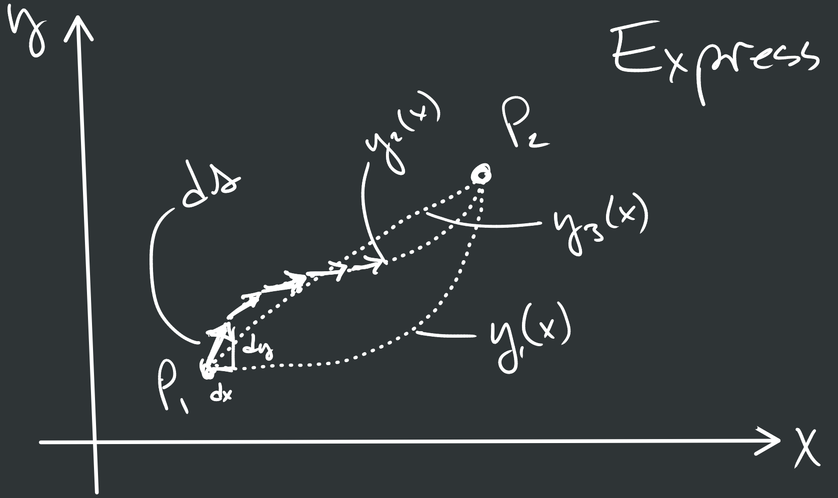
Calculus of Variations $\rightarrow$ find the min/max of a function expressible by an integral

$$g = \int_{x_1}^{x_2} f(y, y', x)\,dx \qquad y' = \frac{dy}{dx}$$

minimize

$\hookrightarrow$ what is y that minimizes g?

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y} \quad \longleftarrow \text{E-L equation}$$



Express path length between 2 pts as an integral?

$$s = \int_{P_1}^{P_2} ds$$

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$ds = \sqrt{dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$$

$$y' = \frac{dy}{dx}$$

$$= dx \cdot \sqrt{1 + y'^2}$$

total path length $\rightarrow$ $$s = \int_{P_1}^{P_2} \underbrace{\sqrt{1+y'^2}}\, dx$$

$$f(y, y', x) = \sqrt{1+y'^2}$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y} \leftarrow \underline{\underline{\text{E-L}}}$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2}\left(1+y'^2\right)^{-1/2} \cdot (+2)\, y' \quad \Bigg| \quad \frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{1}{2}\left(1+y'^2\right)^{-1/2} \cdot (+2)\, y'\right) = 0$$

$$\frac{1}{\cancel{2}}\left(1+y'^2\right)^{-1/2} \cdot (\cancel{+2})\, y' = \text{Constant} \leftarrow$$

$$\frac{y'}{\sqrt{1+y'^2}} = C$$

$$\frac{y'^2}{1+y'^2} = C$$

$$y'^2 = C\left(1+y'^2\right)$$

$$y'^2 = C + Cy'^2$$

$$y'^2(1-C) = C$$

$$y'^2 = C$$

$$y' = C$$

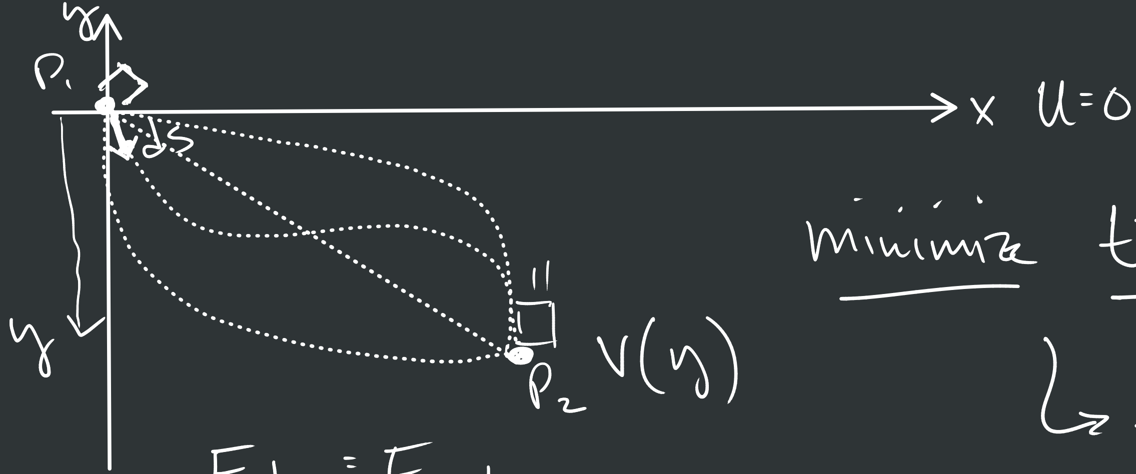
$$\frac{dy}{dx} = \underbrace{\text{Constant}}_{m}$$

m → slope

integrate

$$\underline{y = mx + b}$$

Brachistochrone



$E_{top} = E_{bottom}$

$0 = \frac{1}{2}mv^2 - mgy$

$\cancel{m}gy = \frac{1}{2}\cancel{m}v^2$

$v = \sqrt{2gy}$

minimize time

$\hookrightarrow t = \int dt$

$dt = \frac{ds}{v}$

$ds^2 = dx^2 + dy^2$

$ds = \sqrt{dx^2 + dy^2}$

$y' = \frac{dy}{dx}$

$ds = dx\sqrt{1 + y'^2}$

—OR—

$ds = dy\sqrt{1 + x'^2}$

$x' = \frac{dx}{dy}$

$$t = \frac{1}{\sqrt{2g}} \int \overbrace{\frac{\sqrt{1 + y'^2}}{\sqrt{y}}}^{f} dx \quad \text{— OR —} \quad t = \frac{1}{\sqrt{2g}} \int \overbrace{\frac{\sqrt{1 + x'^2}}{\sqrt{y}}}^{f} dy$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y} \qquad \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = \frac{\partial f}{\partial x}$$

$$\frac{\partial f}{\partial y} \neq 0$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$$

$$\rightarrow \frac{\partial f}{\partial x'} = C$$

$$\frac{\partial f}{\partial x'} = \frac{\partial}{\partial x'}\left(\frac{(1+x'^2)^{1/2}}{(y)^{1/2}}\right) = \frac{1}{\cancel{2}}(1+x'^2)^{-1/2} \cdot \cancel{2}x' \cdot \frac{1}{\sqrt{y}} = C$$

$$\frac{x'}{\sqrt{y(1+x'^2)}} = C$$

Convenient b/c I know already

$$\frac{x'^2}{y(1+x'^2)} = C = \frac{1}{2a}$$

solve for x'

algebra goes here

$$x' = \sqrt{\frac{y}{2a-y}}$$

$$\frac{dx}{dy} = \sqrt{\frac{y}{2a-y}} \quad \leftarrow \text{differential equation}$$

$$\int dx = \int \sqrt{\frac{y}{2a-y}}\, dy$$

$$x = \int \sqrt{\frac{y}{2a-y}}\, dy$$

$$y = 2a\sin^2\phi$$

$$\underline{\underline{y = a(1-\cos\theta)}}$$

answer

parametric equations
$$\begin{cases} x(\theta) = a(\theta - \sin\theta) + \text{const} \\ y(\theta) = a(1-\cos\theta) \end{cases}$$

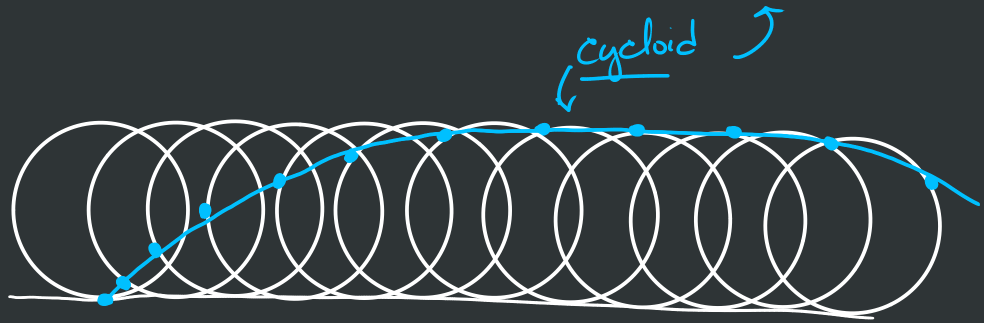
My problem $P_1 = (0,0)$.

$$\left| \begin{array}{l} x = a(\theta - \sin\theta) \\ y = a(1-\cos\theta) \end{array} \right.$$

$$y(\theta?) = 0 = a(1-\cos\theta)$$

$$\theta_i = 0$$

$$x(\theta=0) = 0 = a(0 - \sin 0) + \underline{\underline{\text{const}}}_{=0}$$



HW: What is the amount of time to reach the bottom of the brachistochrone?

In[23]:=
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ParametricPlot[{2 (th - Sin[th]), -2 (1 - Cos[th])}, {th, 0, Pi}]
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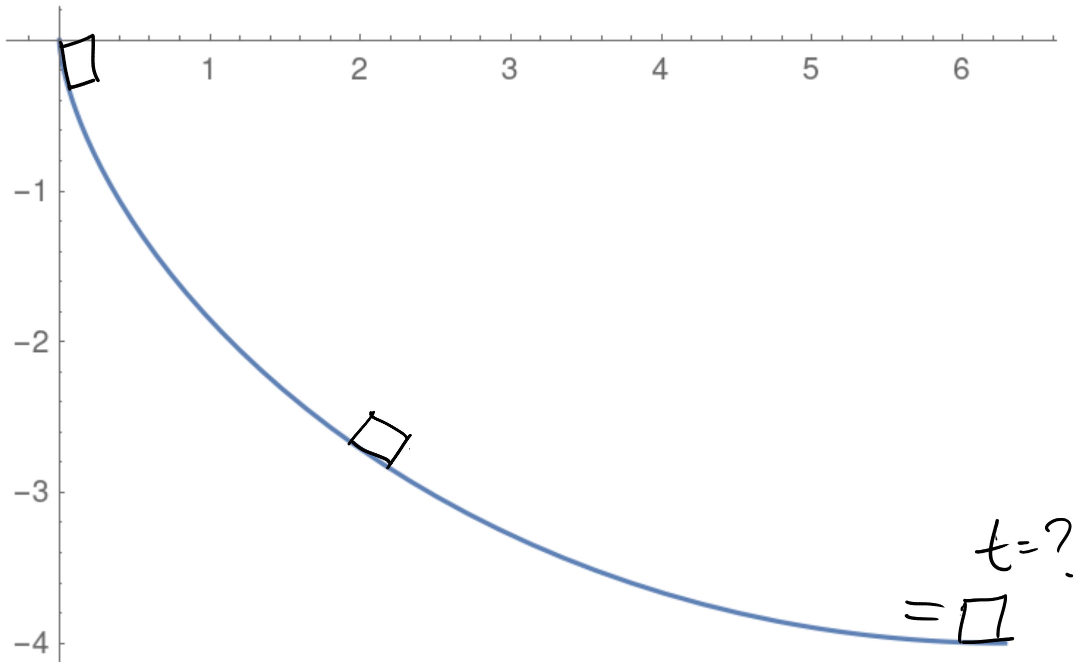
Out[23]=



Back to the Lagrangian $\mathcal{L} = T - U$

$q_1 = x \quad \dot{q}_1 = \dot{x}$

$q_2 = \phi \quad \dot{q}_2 = \dot{\phi}$

generalized coordinates, q_i

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

if $\dfrac{\partial \mathcal{L}}{\partial q_i} = 0$, then $\dfrac{\partial \mathcal{L}}{\partial \dot{q}_i} = C$

$q_i \rightarrow$ cyclic (ignorable) coordinate

$\dfrac{\partial \mathcal{L}}{\partial q_i} \rightarrow$ generalized force

$\dfrac{\partial \mathcal{L}}{\partial \dot{q}_i} \rightarrow$ generalized momentum

$\hookrightarrow$ conserved quantity