Chapter 6 + 7 → Principle of Least Action and the Lagrangian

Newton's Laws — Lagrangian — Hamiltonian

→ "generalized coordinates"

→ Principle of Least Action

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$U = mgx$$

$$\underbrace{\int_{t_1}^{t_2}\Big(\frac{1}{2}m\underset{\substack{\uparrow \\ v(t)}}{\dot{x}^2} - mg\underset{\substack{\uparrow \\ x(t)}}{x}\Big)dt}_{\text{what is } x(t) \text{ and } \dot{x}(t) \text{ so this integral is smallest}}$$

Lagrangian

$$\mathcal{L} = T - U$$

Action

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}\big(x(t), \dot{x}(t), t\big)dt$$

$$= \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t)dt$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}}$$

<u>Euler</u> - <u>Lagrange</u>
<u>Equation</u>

$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}mv^2$$

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 - U(x,y,z)$$

↳ <u>1D motion</u>

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - U(x)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - U(x)\right)\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2 - U(x)\right)$$

$$\frac{d}{dt}(m\dot{x}) = -\frac{dU}{dx}$$

$$m\ddot{x} = -\frac{dU}{dx}$$

$\longrightarrow$ In 3D

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x} \qquad m\ddot{x} = -\frac{\partial U}{\partial x}$$

y

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = \frac{\partial \mathcal{L}}{\partial y} \qquad m\ddot{y} = -\frac{\partial U}{\partial y}$$

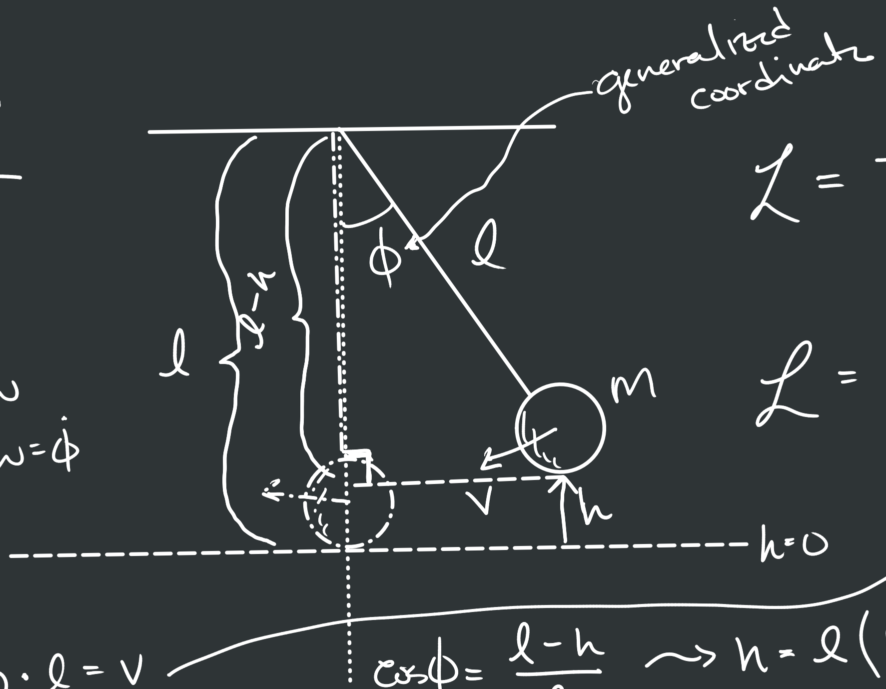
z

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) = \frac{\partial \mathcal{L}}{\partial z} \qquad m\ddot{z} = -\frac{\partial U}{\partial z}$$

$$\underbrace{m\ddot{x}\,\hat{x} + m\ddot{y}\,\hat{y} + m\ddot{z}\,\hat{z}}_{} = -\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}$$

$$\underbrace{\Sigma\vec{F}}_{} = m\ddot{\vec{r}} = \underbrace{-\vec{\nabla}U}_{} = \Sigma\vec{F}$$

Ex:



$$\mathcal{L} = T - U$$

$T \leftarrow \dot{\phi}$, $U \leftarrow \phi$

$$\mathcal{L} = \frac{1}{2}mv^2 - mgh$$

$v = r\omega$

$\omega = \dot{\phi}$

$$\dot{\phi}\cdot l = v \qquad \cos\phi = \frac{l-h}{l} \rightsquigarrow h = l(1-\cos\phi)$$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1-\cos\phi)$$

$\hookrightarrow$ apply the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \dot\phi} = \frac{1}{2} m l^2 (2\dot\phi) = m l^2 \dot\phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -mgl(-(-\sin\phi))$$

$$= -mgl\sin\phi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) = \frac{d}{dt}(m l^2 \dot\phi) = m l^2 \ddot\phi$$

$$\cancel{m}\cancel{l^2}\ddot\phi = -\cancel{m}g\cancel{l}\sin\phi$$

$$\boxed{\ddot\phi = -\frac{g}{l}\sin\phi}$$

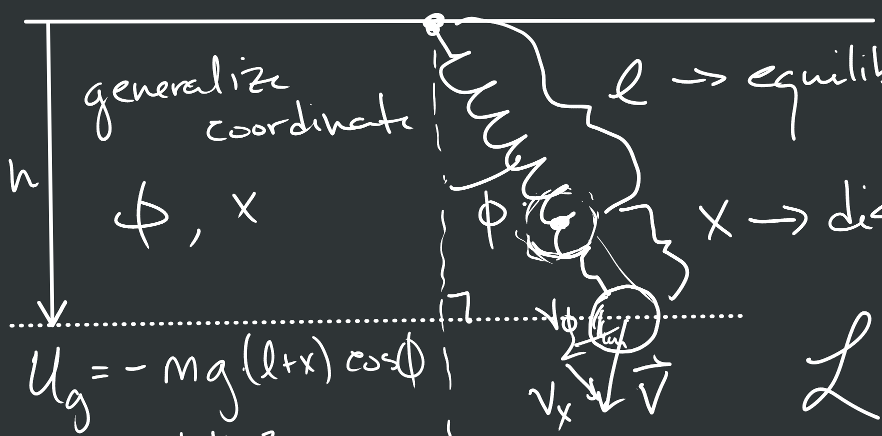
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot x}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

Remember small angle approx $\sin\phi \approx \phi$

$$\ddot\phi = -\frac{g}{l}\sin\phi$$

generalize coordinate

ϕ, x



$l \rightarrow$ equilibrium length

$x \rightarrow$ distortion of the spring

$U_g = -mg(l+x)\cos\phi$

$U_s = +\frac{1}{2}kx^2$

$V_\phi = (l+x)\dot\phi$

$V_x = \dot x$

$$\mathcal{L} = T - U$$

$$= \frac{1}{2}mv^2 - U_g - U_s$$

$\hookrightarrow \frac{1}{2}kx^2$

$$\mathcal{L} = \frac{1}{2}m\dot x^2 + \frac{1}{2}m(l+x)^2\dot\phi^2 + mg(l+x)\cos\phi - \frac{1}{2}kx^2$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m(l+x)^2 \dot{\phi}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = m(l+x)^2 \ddot{\phi} + m2(l+x)\dot{x}\dot{\phi}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -mg(l+x)\sin\phi$$

$$\cancel{m}(l+x)^{\cancel{2}}\ddot{\phi} + \underbrace{\cancel{m}2\cancel{(l+x)}\dot{x}\dot{\phi}}_{\text{Coriolis force}} = -\cancel{m}g\cancel{(l+x)}\sin\phi$$

$$\boxed{(l+x)\ddot{\phi} + 2\dot{x}\dot{\phi} + g\sin\phi = 0}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial x} = \underbrace{m(l+x)\dot{\phi}^2}_{\text{centrifugal}} + mg\cos\phi - kx$$

$$\cancel{m}\ddot{x} = \cancel{m}(l+x)\dot{\phi}^2 + \cancel{m}g\cos\phi - \frac{kx}{m}$$

$$\boxed{\ddot{x} = (l+x)\dot{\phi}^2 + g\cos\phi - \frac{kx}{m}}$$

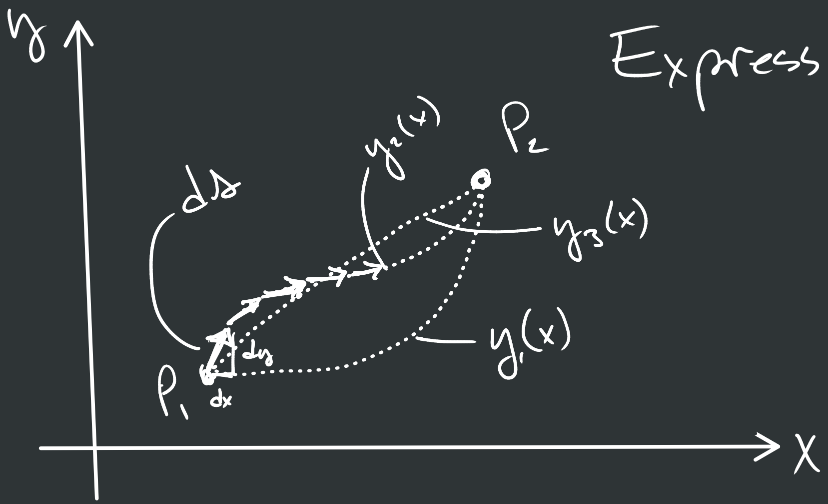
Calculus of Variations → find the min/max of a function expressible by an integral

$$g = \int_{x_1}^{x_2} f(y, y', x)\,dx \qquad y' = \frac{dy}{dx}$$

minimize

→ what is y that minimizes g?

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y} \quad \leftarrow \text{E-L equation}$$



Express path length between 2 pts as an integral?

$$\Delta = \int_{P_1}^{P_2} ds$$

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$ds = \sqrt{dx^2\left(1 + \left(\frac{dy}{dx}\right)^2\right)}$$

$$y' = \frac{dy}{dx}$$

$$= dx \cdot \sqrt{1 + y'^2}$$

total path length $\rightarrow$ $$s = \int_{P_1}^{P_2} \underbrace{\sqrt{1+y'^2}}\, dx$$

$$f(y, y', x) = \sqrt{1+y'^2}$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y} \leftarrow \text{E-L}$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2}\left(1+y'^2\right)^{-1/2} \cdot (+2)\, y' \qquad \Bigg| \qquad \frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx}\left(\frac{1}{2}\left(1+y'^2\right)^{-1/2} \cdot (+2)\, y'\right) = 0$$

$$\frac{1}{\cancel{2}}\left(1+y'^2\right)^{-1/2} \cdot (\cancel{+2})\, y' = \text{Constant} \leftarrow$$

$$\frac{y'}{\sqrt{1+y'^2}} = C$$

$$\frac{y'^2}{1+y'^2} = C$$

$$y'^2 = C\left(1+y'^2\right)$$

$$y'^2 = C + Cy'^2$$

$$y'^2(1-C) = C$$

$$y'^2 = C$$

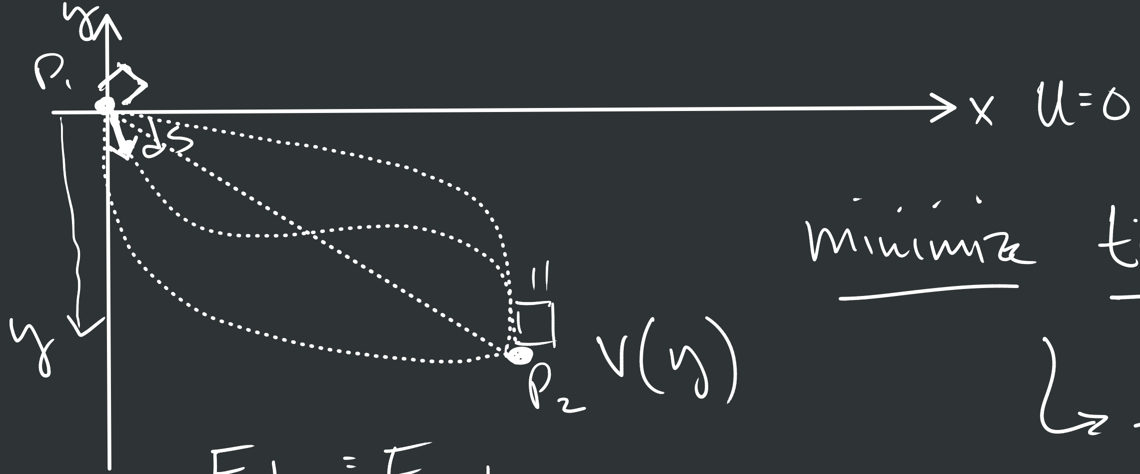
$$y' = C$$

$$\frac{dy}{dx} = \underbrace{\text{Constant}}_{m \rightarrow \text{slope}}$$

integrate

$$\underline{y = mx + b}$$

Brachistochrone



minimize time

$\hookrightarrow t = \int dt$

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

$y' = \frac{dy}{dx}$

$$ds = dx\sqrt{1 + y'^2}$$

—OR—

$$ds = dy\sqrt{1 + x'^2}$$

$x' = \frac{dx}{dy}$

$E_{top} = E_{bottom}$

$$0 = \frac{1}{2}mv^2 - mgy$$

$$\cancel{m}gy = \frac{1}{2}\cancel{m}v^2$$

$$v = \sqrt{2gy}$$

$$dt = \frac{ds}{v}$$

$$t = \frac{1}{\sqrt{2g}}\int \overbrace{\frac{\sqrt{1+y'^2}}{\sqrt{y}}}^{f}\, dx \quad \text{—OR—} \quad t = \frac{1}{\sqrt{2g}}\int \overbrace{\frac{\sqrt{1+x'^2}}{\sqrt{y}}}^{f}\, dy$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y} \qquad\qquad \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = \frac{\partial f}{\partial x}$$

$$\frac{\partial f}{\partial y} \neq 0$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$$

$$\rightarrow \frac{\partial f}{\partial x'} = C$$

$$\frac{\partial f}{\partial x'} = \frac{\partial}{\partial x'}\left(\frac{(1+x'^2)^{1/2}}{(y)^{1/2}}\right) = \frac{1}{\cancel{2}}(1+x'^2)^{-1/2} \cdot \cancel{2}x' \cdot \frac{1}{\sqrt{y}} = C$$

$$\frac{x'}{\sqrt{y(1+x'^2)}} = C$$

Convenient b/c I know already

$$\frac{x'^2}{y(1+x'^2)} = C = \frac{1}{2a}$$

solve for x'

algebra goes here

$$x' = \sqrt{\frac{y}{2a-y}}$$

$$\frac{dx}{dy} = \sqrt{\frac{y}{2a-y}}$$ ← differential equation

$$\int dx = \int \sqrt{\frac{y}{2a-y}}\, dy$$

$$x = \int \sqrt{\frac{y}{2a-y}}\, dy$$

$y = 2a\sin^2\phi$

$\underline{\underline{y = a(1-\cos\theta)}}$

answer

parametric equations

$$\begin{cases} x(\theta) = a(\theta - \sin\theta) + \text{const} \\ y(\theta) = a(1-\cos\theta) \end{cases}$$

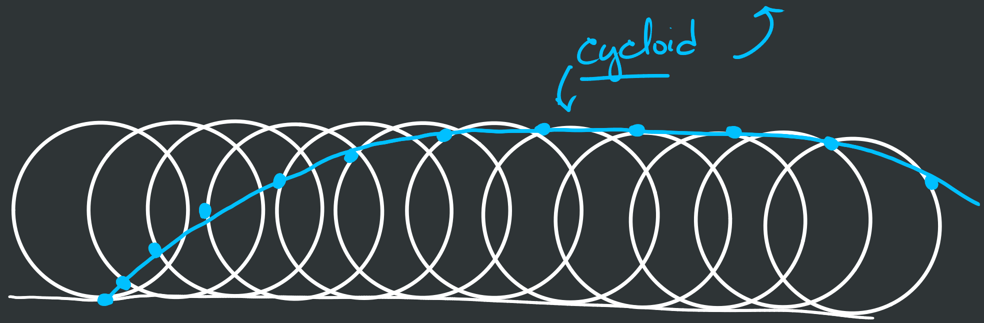
My problem $P_1 = (0,0)$.

$$\left| \begin{array}{l} x = a(\theta - \sin\theta) \\ y = a(1-\cos\theta) \end{array} \right.$$

$y(\theta?) = 0 = a(1-\cos\theta)$

$\theta_i = 0$

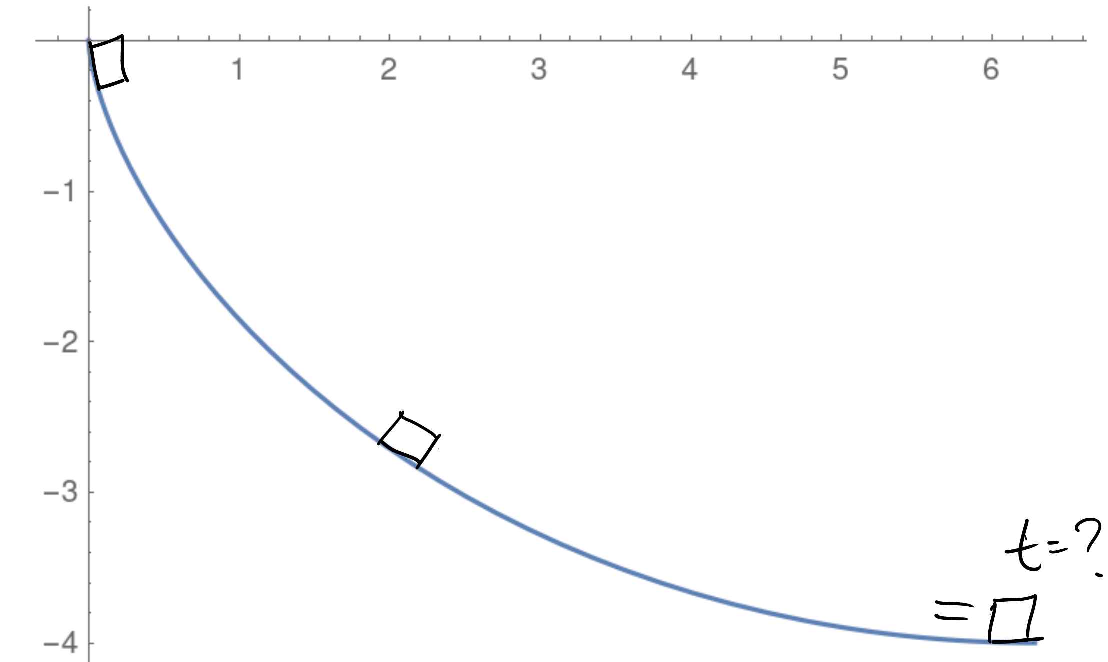
$x(\theta=0) = 0 = a(0 - \sin 0) + \underline{\underline{\overset{=0}{\text{const}}}}$



HW: What is the amount of time to reach the bottom of the brachistochrone?

In[23]:=
```
ParametricPlot[{2 (th - Sin[th]), -2 (1 - Cos[th])}, {th, 0, Pi}]
```

Out[23]=



Back to the Lagrangian $\mathcal{L} = T - U$

$q_1 = x \quad \dot{q}_1 = \dot{x}$

$q_2 = \phi \quad \dot{q}_2 = \dot{\phi}$

generalized coordinates, q_i

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

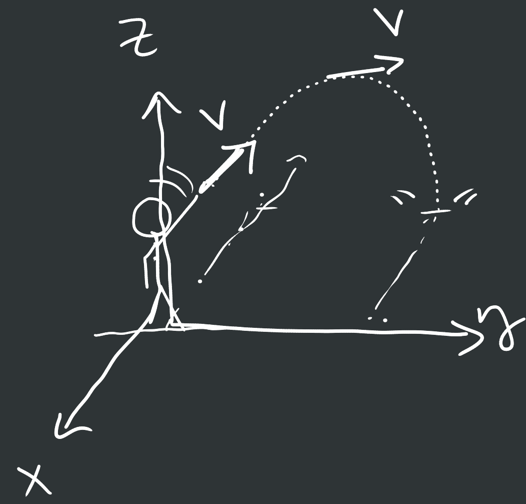
if $\frac{\partial \mathcal{L}}{\partial q_i} = 0$, then $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = C \leftarrow$ constant in time (conserved quantity)

$q_i \rightarrow$ cyclic (ignorable) coordinate

$\frac{\partial \mathcal{L}}{\partial q_i} \rightarrow$ generalized force

$\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \rightarrow$ generalized momentum

Ex: ball thrown through the air



$$\vec{V} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$$

$$= \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

$$= \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$$

$$V^2 = \vec{V}\cdot\vec{V} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$\frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\mathcal{L} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

↳ does not explicitly depend on t, x, y

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}} = C_1 \qquad \Bigg| \qquad \frac{\partial \mathcal{L}}{\partial y} = 0 \rightarrow \frac{\partial \mathcal{L}}{\partial \dot{y}} = C_2$$

linear momentum → $m\dot{x} = C_1$ $\qquad$ $m\dot{y} = C_2$

mv_x $\qquad$ mv_y

$$\frac{\partial \mathcal{L}}{\partial z} = -mg \qquad \frac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{z}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) = m\ddot{z} + \cancel{\dot{m}}^{\,0}\dot{z}$$

$$-\cancel{m}g = \cancel{m}\ddot{z}$$

$$\ddot{z} = -g$$

$$\rightarrow \quad \dot{z} = -gt + D$$

$$\rightarrow \quad \underbrace{z = -\frac{1}{2}gt^2 + Dt + F}_{\text{now apply initial conditions}}$$

$$\dot{z} = \frac{dz}{dt}$$

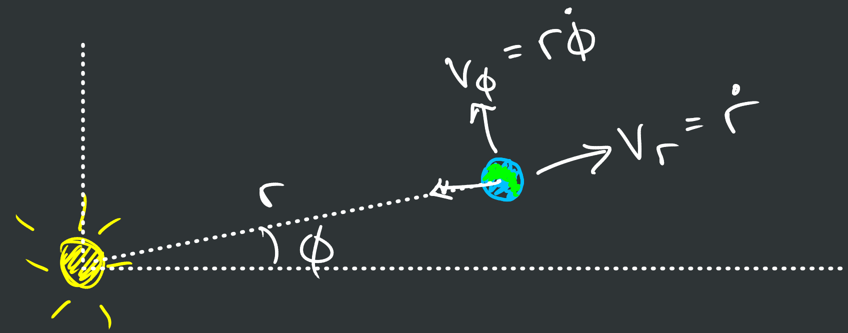
$$\frac{d}{dt}\left(\frac{dz}{dt}\right)$$

$$\frac{d^2 z}{dt^2}$$

$$\ddot{z}$$

Ex: central force

generalized coord: r, ϕ



$$\mathcal{L} = \frac{1}{2} m\left(\dot{r}^2 + (r\dot{\phi})^2\right) - U(r)$$

↳ does depend on t, ϕ

$$\underbrace{\frac{\partial \mathcal{L}}{\partial \phi}}_{\text{generalized force}} = 0 \Rightarrow \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}}}_{\text{generalized momentum}} = C = \underbrace{m r^2 \dot{\phi}}_{\text{angular momentum}}$$

$I \cdot \omega$

$(mr^2) \cdot \omega$

$m \underbrace{r\omega}_{v_\phi} \cdot r$

$m v_\phi \cdot r$

$\vec{r} \times \vec{p}$

What about conservation of energy?

What is the change in the $\mathcal{L}$ over time?

$$\mathcal{L}(q_1, q_2, \dots q_N, \dot{q}_1, \dot{q}_2, \dots \dot{q}_N, t)$$

$$\frac{d\mathcal{L}(q_1, q_2, \dots q_N, \dot{q}_1, \dot{q}_2, \dots \dot{q}_N, t)}{dt} = \frac{\partial \mathcal{L}}{\partial q_1}\dot{q}_1 + \frac{\partial \mathcal{L}}{\partial q_2}\cdot\dot{q}_2 + \dots \frac{\partial \mathcal{L}}{\partial q_N}\cdot\dot{q}_N$$

$$+ \frac{\partial \mathcal{L}}{\partial \dot{q}_1}\cdot\ddot{q}_1 + \frac{\partial \mathcal{L}}{\partial \dot{q}_2}\ddot{q}_2 + \dots \frac{\partial \mathcal{L}}{\partial \dot{q}_N}\ddot{q}_N$$

$$+ \frac{\partial \mathcal{L}}{\partial t}$$

terms that explicitly depend on time

Remember, E-L equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}$$

generalized momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$p_i$$

$$\frac{d}{dt}(p_i) = \dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i}$$

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \sum_{i=1}^{N} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t}$$

$$= \sum_{i=1}^{N} \dot{p}_i \dot{q}_i + \sum_i p_i \ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t}$$

$$= \sum_i \big(\underbrace{\dot{p}_i \dot{q}_i + p_i \ddot{q}_i}_{\text{derivative product rule!}} \big) + \frac{\partial \mathcal{L}}{\partial t}$$

$$\frac{d\mathcal{L}}{dt} = \sum_i \frac{d}{dt}\left(p_i \dot{q}_i\right) + \underbrace{\frac{\partial \mathcal{L}}{\partial t}}_{=0}$$

$=0$ most of the time, for most important systems

$$0 = \sum_i \frac{d}{dt}\left(p_i \dot{q}_i\right) - \frac{d\mathcal{L}}{dt}$$

$$0 = \frac{d}{dt}\sum_i (p_i \dot{q}_i) - \frac{d\mathcal{L}}{dt}$$

$$0 = \frac{d}{dt}\Big(\underbrace{\sum_i (p_i \dot{q}_i) - \mathcal{L}}_{\text{constant!}}\Big)$$

when $\frac{\partial \mathcal{L}}{\partial t} = 0$

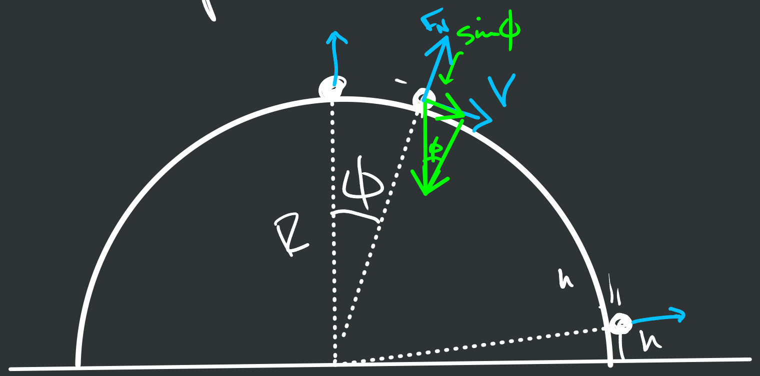
$$\mathcal{H} = \sum_i (p_i \dot{q}_i) - \mathcal{L} \quad \leftarrow \underline{\text{Hamiltonian}} \ (\text{conserved quantity})$$

↳ usually the total energy (in an inertial rf)

$$\mathcal{H} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \cdot \dot{q}_i\right) - \mathcal{L}$$

$\mathcal{H} = T + U$ (usually)

An example:



$v = \sqrt{2gR}$

$$\mathcal{L} = \frac{1}{2} m (R\dot{\phi})^2 - mgR\cos\phi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mR^2\dot{\phi} \qquad \frac{\partial \mathcal{L}}{\partial \phi} = +mgR\sin\phi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = mR^2\ddot{\phi}$$

$$\cancel{m}R^{\cancel{2}}\ddot{\phi} = \cancel{m}g\cancel{R}\sin\phi$$

$$\boxed{\ddot{\phi} = \frac{g}{R}\sin\phi} \leftarrow \text{pendulum}$$

small ang approx

$$\boxed{\ddot{\phi} = \frac{g}{R}\phi}$$

important

$$\ddot{\phi} = \frac{g}{R}\phi$$

$\sin\phi \approx \phi$

guess $\longrightarrow$ $\phi(t) = A\cos(\omega t) + B\sin(\omega t)$

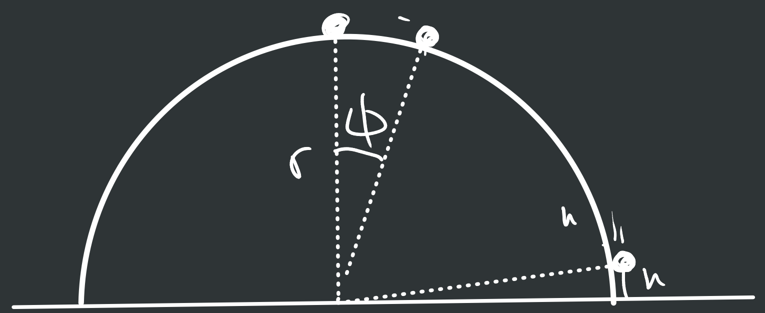
$$\omega = \sqrt{\frac{g}{R}}$$

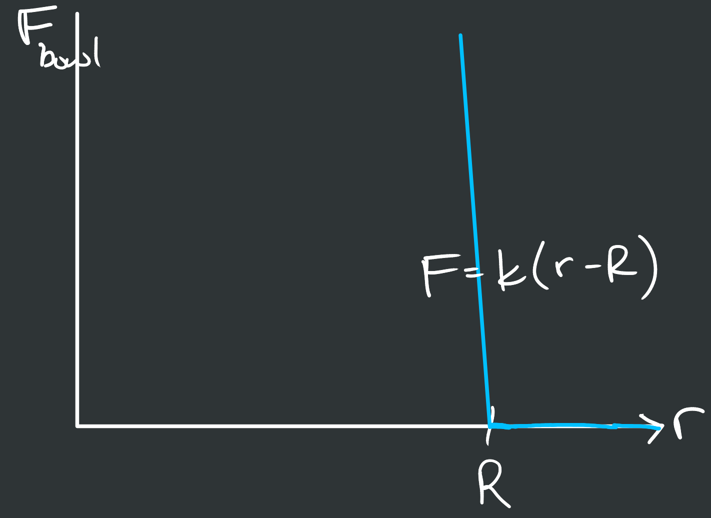
use initial conditions to find A & B.

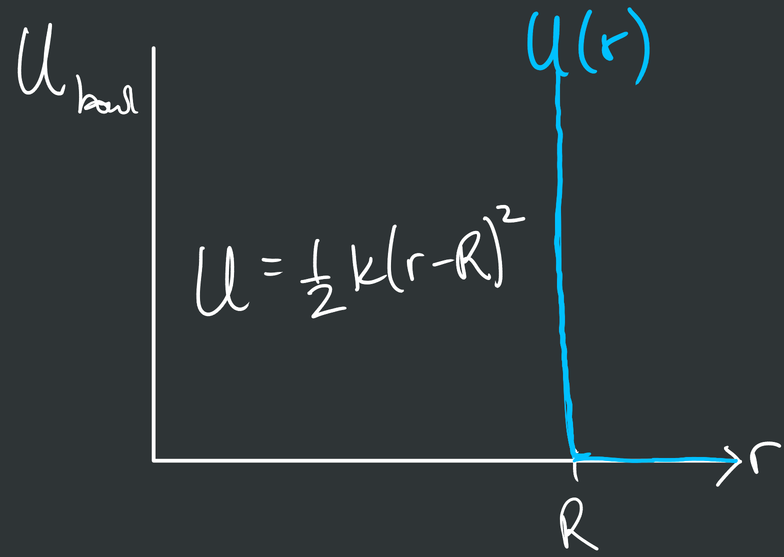
$$\underbrace{mR^2\ddot{\phi}}_{a_R} = \underbrace{mg\sin\phi \cdot R}_{\text{torque}}$$

What about the normal force?

$\hookrightarrow$ force of constraint







$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + (r\dot{\phi})^2\right) - mgr\cos\phi - U(r)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r}$$