

Chapter 6 & 7 \rightarrow Principle of Least Action and the Lagrangian



\rightarrow "generalized coordinates"
 \rightarrow Principle of Least Action

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$U = mgx$$

$$\int_{t_1}^{t_2} \left(\underbrace{\frac{1}{2}m\dot{x}^2}_{v(t)} - \underbrace{mgx}_{x(t)} \right) dt$$

what is $x(t)$ and $\dot{x}(t)$
 \hookrightarrow so this integral is smallest

Lagrangian

$$L = T - U$$

Action

$$S = \int_{t_1}^{t_2} L(x(t), \dot{x}(t), t) dt$$
$$= \int_{t_1}^{t_2} L(\overset{\downarrow}{x}, \overset{\downarrow}{\dot{x}}, t) dt$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\boxed{\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{\partial \mathcal{L}}{\partial x}}$$

Euler - Lagrange
Equation

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2} m v^2$$

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 + \frac{1}{2} m \dot{z}^2 - U(x, y, z)$$

↳ 1D motion

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - U(x)$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}} \left(\frac{1}{2} m \dot{x}^2 - U(x) \right) \right) = \frac{\partial}{\partial x} \left(\frac{1}{2} m \dot{x}^2 - U(x) \right)$$

$$\frac{d}{dt} (m \dot{x}) = - \frac{dU}{dx}$$

$$m\ddot{x} = -\frac{dU}{dx}$$

→ In 3D

$$\left. \begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{\partial \mathcal{L}}{\partial x} \\ m\ddot{x} &= -\frac{\partial U}{\partial x} \end{aligned} \right|$$

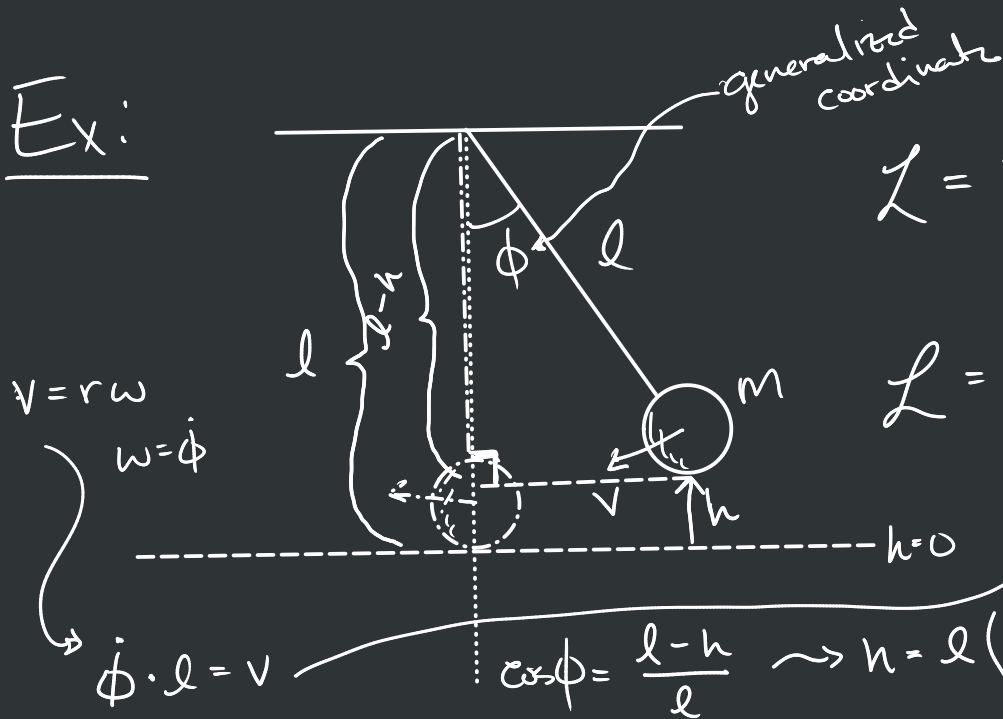
$$\left. \begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) &= \frac{\partial \mathcal{L}}{\partial y} \\ m\ddot{y} &= -\frac{\partial U}{\partial y} \end{aligned} \right|$$

$$\left. \begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) &= \frac{\partial \mathcal{L}}{\partial z} \\ m\ddot{z} &= -\frac{\partial U}{\partial z} \end{aligned} \right|$$

$$m\ddot{x}\hat{x} + m\ddot{y}\hat{y} + m\ddot{z}\hat{z} = -\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}$$

$$\underbrace{\sum \vec{F}} = m\ddot{\vec{r}} = -\underbrace{\vec{\nabla} U} = \sum \vec{F}$$

Ex:



$$\mathcal{L} = T - U$$

\uparrow \uparrow
 $\dot{\phi}$ ϕ

$$\mathcal{L} = \frac{1}{2}mv^2 - mgh$$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1 - \cos\phi)$$

→ apply the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{2} m l^2 (2\dot{\phi}) = m l^2 \dot{\phi} \quad \left| \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m g l (-(-\sin \phi)) \right.$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = \frac{d}{dt} (m l^2 \dot{\phi}) = m l^2 \ddot{\phi} \quad \left. = -m g l \sin \phi \right.$$

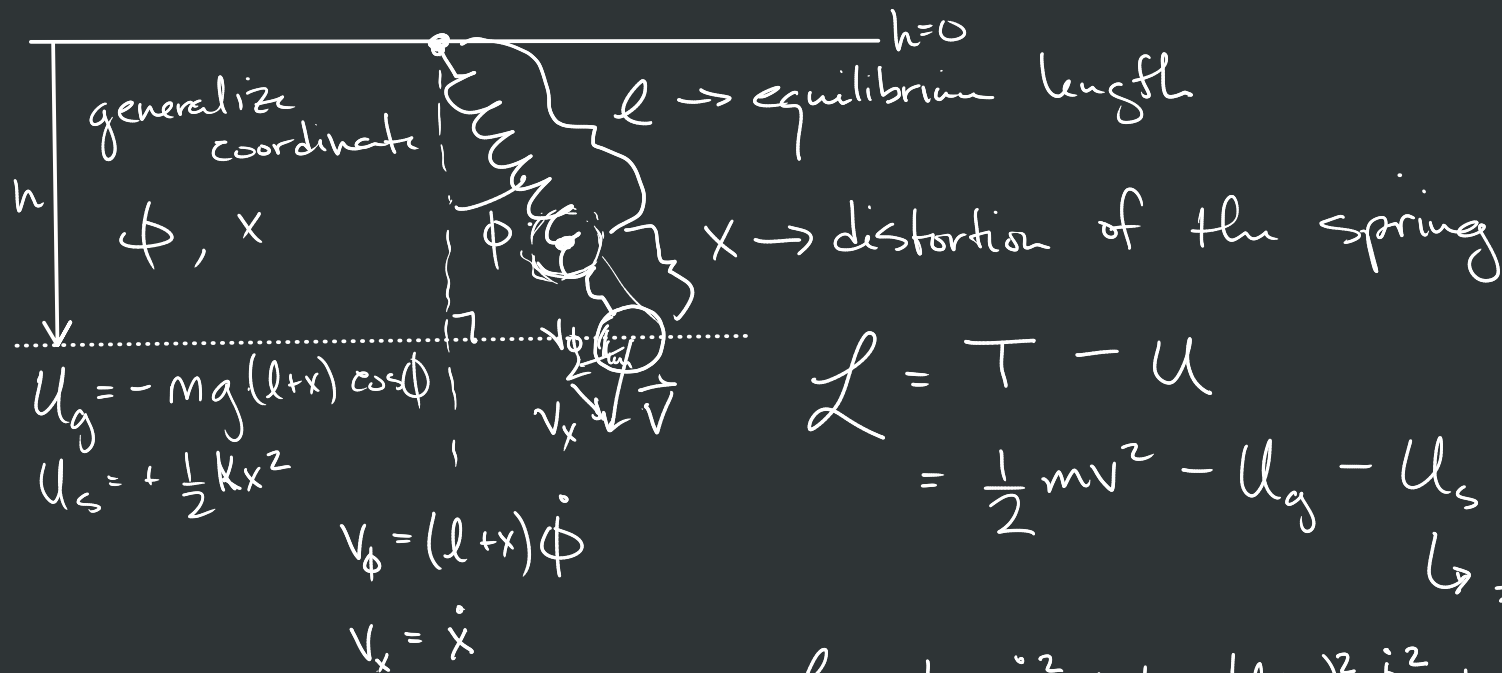
$$m l^2 \ddot{\phi} = -m g l \sin \phi$$

$$\ddot{\phi} = -\frac{g}{l} \sin \phi$$

$$\rightarrow \boxed{\ddot{\phi} = -\frac{g}{l} \sin \phi}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{\partial \mathcal{L}}{\partial x}$$

Remember small angle
approx $\sin \phi \approx \phi$



$$\mathcal{L} = T - U$$

$$= \frac{1}{2} m v^2 - U_g - U_s$$

$\hookrightarrow \frac{1}{2} k x^2$

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m (l+x)^2 \dot{\phi}^2 + m g (l+x) \cos \phi - \frac{1}{2} k x^2$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m(l+x)^2 \dot{\phi}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) = m(l+x)^2 \ddot{\phi} + m2(l+x) \dot{x} \dot{\phi}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -mg(l+x) \sin \phi$$

Coriolis force

$$m(l+x)^2 \ddot{\phi} + m2(l+x) \dot{x} \dot{\phi} = -mg(l+x) \sin \phi$$

$$(l+x) \ddot{\phi} + 2\dot{x} \dot{\phi} + g \sin \phi = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = m\ddot{x}$$

$$\frac{\partial \mathcal{L}}{\partial x} = m(l+x) \dot{\phi}^2 + mg \cos \phi - kx$$

centrifugal

$$m\ddot{x} = m(l+x) \dot{\phi}^2 + mg \cos \phi - \frac{k}{m}x$$

$$\ddot{x} = (l+x) \dot{\phi}^2 + g \cos \phi - \frac{k}{m}x$$

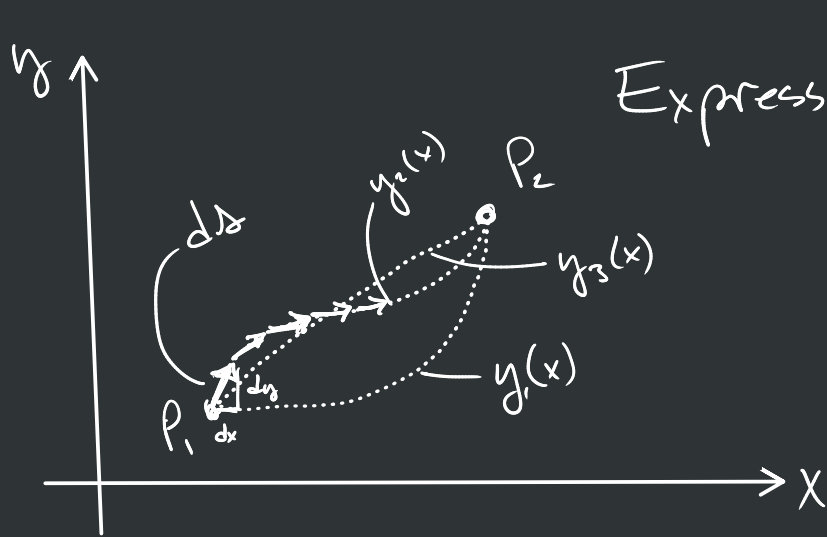
Calculus of Variations \rightarrow Find the min/max of a function expressible by an integral

minimize $g = \int_{x_1}^{x_2} f(y, y', x) dx$

$$y' = \frac{dy}{dx}$$

\rightarrow what is y that minimizes g ?

$$\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = \frac{\partial f}{\partial y} \quad \leftarrow \text{E-L equation}$$



Express path length between 2 pts as an integral?

$$\Delta = \int_{P_1}^{P_2} ds$$

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$ds = \sqrt{dx^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)}$$

$$y' = \frac{dy}{dx}$$

$$= dx \cdot \sqrt{1 + y'^2}$$

total
path
length \rightarrow

$$s = \int_{P_1}^{P_2} \underbrace{\sqrt{1 + y'^2}}_{\text{}} dx$$

$$f(y, y', x) = \sqrt{1 + y'^2}$$

$$\frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = \frac{\partial f}{\partial y} \leftarrow \underline{\underline{E-L}}$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2} (1 + y'^2)^{-1/2} \cdot (+2) y' \quad \Bigg| \quad \frac{\partial f}{\partial y} = 0$$

$$\frac{d}{dx} \left(\frac{1}{2} (1 + y'^2)^{-1/2} \cdot (+2) y' \right) = 0$$

$$\frac{1}{2} (1 + y'^2)^{-1/2} \cdot (+2) y' = \text{Constant} \quad \leftarrow$$

$$\frac{y'}{\sqrt{1+y'^2}} = C$$

$$\frac{y'^2}{1+y'^2} = C$$

$$y'^2 = C(1+y'^2)$$

$$y'^2 = C + Cy'^2$$

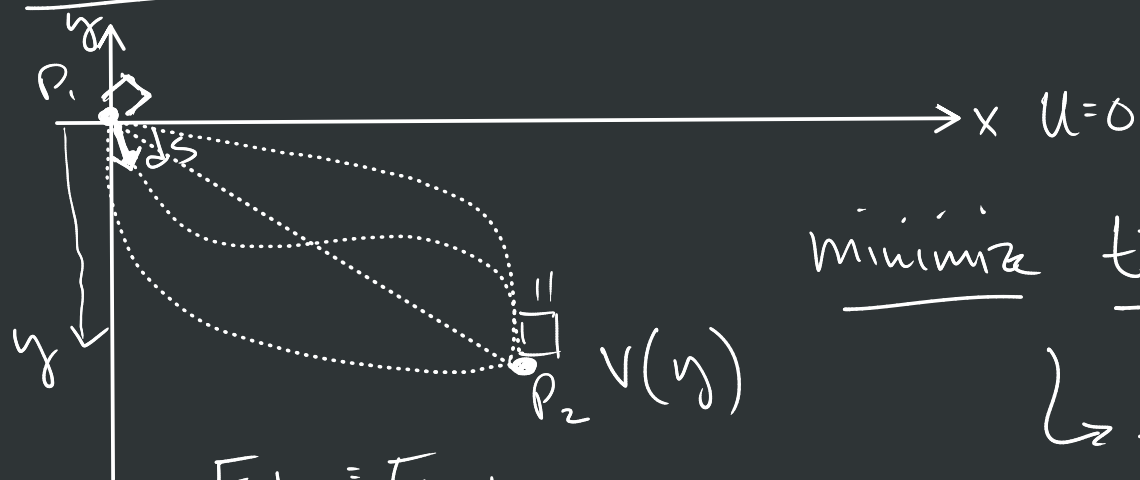
$$y'^2(1-C) = C$$

$$y'^2 = C$$

$$y' = C$$

$\frac{dy}{dx} = \underbrace{\text{Constant}}_{m \rightarrow \text{slope}} \quad \leftarrow$
 integrate $\rightarrow \underline{y = mx + b}$

Brachistochrone



minimize time

$$\hookrightarrow t = \int dt$$

$$E_{\text{top}} = E_{\text{bottom}}$$

$$0 = \frac{1}{2}mv^2 - mgy$$

$$mgy = \frac{1}{2}mv^2$$

$$v = \sqrt{2gy}$$

$$dt = \frac{ds}{v}$$

$$ds^2 = dx^2 + dy^2$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$y' = \frac{dy}{dx}$$

$$ds = dx \sqrt{1 + y'^2}$$

—OR—

$$ds = dy \sqrt{1 + x'^2}$$

$$x' = \frac{dx}{dy}$$

$$t = \frac{1}{\sqrt{2g}} \int \frac{\overbrace{\sqrt{1 + y'^2}}^f}{\sqrt{y}} dx$$

$$\rightarrow \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = \frac{\partial f}{\partial y}$$

—OR—

$$t = \frac{1}{\sqrt{2g}} \int \frac{\overbrace{\sqrt{1 + x'^2}}^f}{\sqrt{y}} dy$$

$$\rightarrow \frac{d}{dy} \left(\frac{\partial f}{\partial x'} \right) = \frac{\partial f}{\partial x}$$

$$\frac{\partial f}{\partial y} \neq 0$$

$$\frac{\partial f}{\partial x} = 0$$

$$\frac{d}{dy} \left(\frac{\partial f}{\partial x'} \right) = 0$$

$$\rightarrow \frac{\partial f}{\partial x'} = C$$

$$\frac{\partial f}{\partial x'} = \frac{\partial}{\partial x'} \left(\frac{(1+x'^2)^{1/2}}{(y)^{1/2}} \right) = \frac{1}{2} (1+x'^2)^{-1/2} \cdot 2x' \cdot \frac{1}{\sqrt{y}} = C$$

$$\frac{x'}{\sqrt{y(1+x'^2)}} = C$$

Convenient b/c I know already

$$\frac{x'^2}{y(1+x'^2)} = C = \frac{1}{2a}$$

↓ solve for x'
↓ algebra goes here

$$x' = \sqrt{\frac{y}{2a-y}}$$

$$\frac{dx}{dy} = \sqrt{\frac{y}{2a-y}} \quad \leftarrow \text{differential equation}$$

$$\int dx = \int \sqrt{\frac{y}{2a-y}} dy$$

$$X = \int \sqrt{\frac{y}{2a-y}} dy$$

$$y = 2a \sin^2 \phi$$

$$\underline{\underline{y = a(1 - \cos \theta)}}$$

parametric
equations

answer

$$\begin{cases} x(\theta) = a(\theta - \sin \theta) + \text{const} \\ y(\theta) = a(1 - \cos \theta) \end{cases}$$

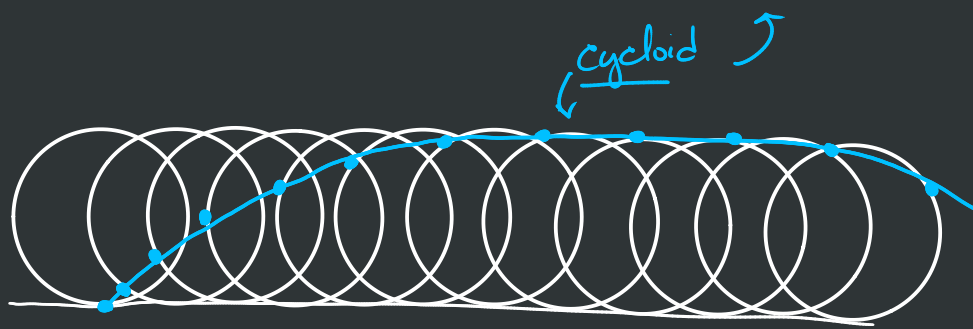
My problem $P_i = (0,0)$.

$$\begin{cases} x = a(\theta - \sin \theta) \\ y = a(1 - \cos \theta) \end{cases}$$

$$y(\theta?) = 0 = a(1 - \cos \theta)$$

$$\theta_i = 0$$

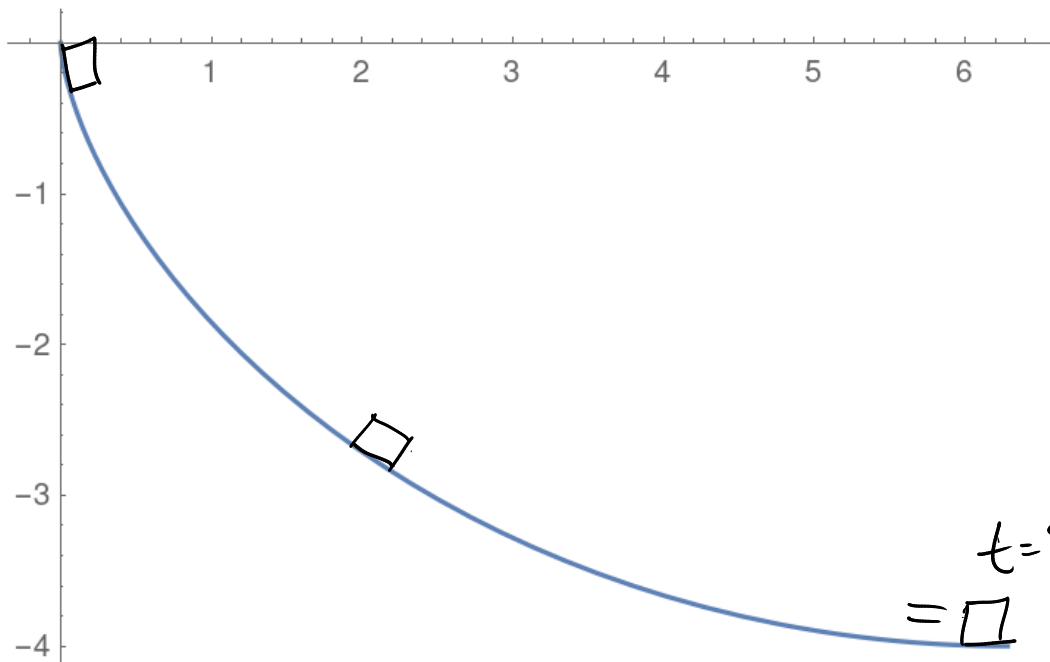
$$x(\theta=0) = 0 = a(0 - \sin 0) + \underline{\underline{\text{const}}} = 0$$



HW: What is the amount of time to reach the bottom of the brachistochrone?

In[23]:= ParametricPlot[{2 (th - Sin[th]), -2 (1 - Cos[th])}, {th, 0, Pi}]

Out[23]=



Back to the Lagrangian $L = T - U$

$$q_1 = x \quad \dot{q}_1 = \dot{x}$$
$$q_2 = \phi \quad \dot{q}_2 = \dot{\phi}$$

generalized coordinates, q_i

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$$

if $\frac{\partial L}{\partial q_i} = 0$, then $\frac{\partial L}{\partial \dot{q}_i} = C \leftarrow \text{constant in time}$
(conserved quantity)

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = \frac{\partial L}{\partial x}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi}$$

$q_i \rightarrow$ cyclic (ignorable) coordinate

$\frac{\partial L}{\partial \dot{q}_i} \rightarrow$ generalized momentum

$\frac{\partial L}{\partial q_i} \rightarrow$ generalized force

Ex: ball thrown through the air

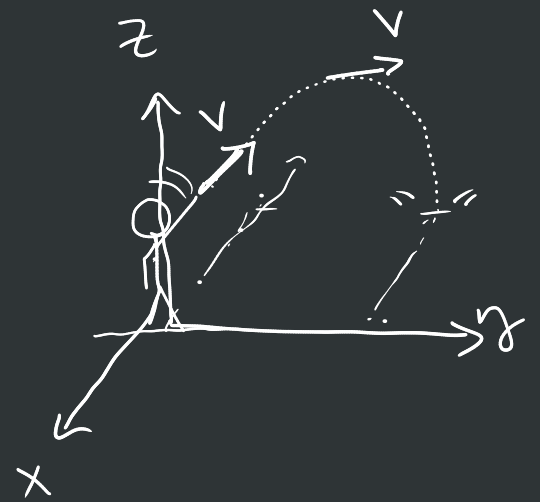
$$\vec{V} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$$

$$= \frac{dx}{dt} \hat{x} + \frac{dy}{dt} \hat{y} + \frac{dz}{dt} \hat{z}$$

$$= \dot{x} \hat{x} + \dot{y} \hat{y} + \dot{z} \hat{z}$$

$$v^2 = \vec{V} \cdot \vec{V} = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$\frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$



$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

↳ does not explicitly depend on t, x, y

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}} = C_1$$

linear
momentum $\rightarrow m \dot{x} = C_1$
 $m v_x$

$$\frac{\partial \mathcal{L}}{\partial y} = 0 \rightarrow \frac{\partial \mathcal{L}}{\partial \dot{y}} = C_2$$

$m \dot{y} = C_2$
 $m v_y$

$$\frac{\partial \mathcal{L}}{\partial z} = -mg$$

$$\frac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{z}$$

$$\dot{z} = \frac{dz}{dt}$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{z}} \right) = m\ddot{z} + \cancel{m\dot{z}}$$

$$\frac{d}{dt} \left(\frac{dz}{dt} \right)$$

$$-mg = m\ddot{z}$$

$$\frac{d^2 z}{dt^2}$$

$$\ddot{z} = -g$$

$$\ddot{z}$$

$$\rightarrow \dot{z} = -gt + D$$

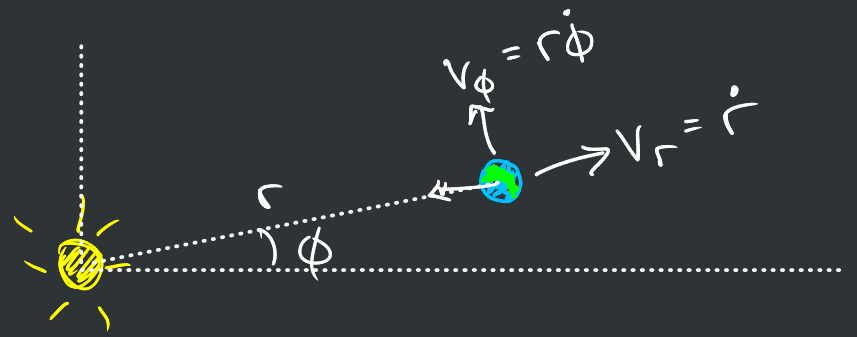
$$\rightarrow z = -\frac{1}{2}gt^2 + Dt + F$$

now apply initial conditions

Ex: central force
generalized coord: r, ϕ

$$\mathcal{L} = \frac{1}{2} m (\dot{r}^2 + (r\dot{\phi})^2) - U(r)$$

\mathcal{L} does depend on t, ϕ



$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = C = \underbrace{mr^2\dot{\phi}}_{\text{angular momentum}}$$

\uparrow generalized force \uparrow generalized momentum

$$\begin{aligned}
 &I \cdot \omega \\
 &(mr^2) \cdot \omega \\
 &\underbrace{mr\omega} \cdot r \\
 &\quad v_{\phi} \\
 &m v_{\phi} \cdot r \\
 &\quad \vec{r} \times \vec{p}
 \end{aligned}$$

What about conservation of energy?

What is the change in the L over time?

$$L(q_1, q_2, \dots, q_N, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_N, t)$$

$$\frac{dL}{dt}(q_1, q_2, \dots, q_N, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_N, t) = \frac{\partial L}{\partial q_1} \dot{q}_1 + \frac{\partial L}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial L}{\partial q_N} \dot{q}_N$$

Remember, E-L equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad \uparrow \text{generalized momentum}$$

$$\frac{d}{dt}(p_i) = \dot{p}_i = \frac{\partial L}{\partial q_i}$$

$$+ \frac{\partial L}{\partial \dot{q}_1} \ddot{q}_1 + \frac{\partial L}{\partial \dot{q}_2} \ddot{q}_2 + \dots + \frac{\partial L}{\partial \dot{q}_N} \ddot{q}_N$$

$$+ \frac{\partial L}{\partial t}$$

terms that explicitly depend on time

$$\frac{d\mathcal{L}}{dt} = \sum_{i=1}^N \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i + \sum_{i=1}^N \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t}$$

$$= \sum_{i=1}^N \dot{p}_i \dot{q}_i + \sum_i p_i \ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t}$$

$$= \sum_i \left(\dot{p}_i \dot{q}_i + p_i \ddot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t}$$

derivative product
rule!

$$\frac{d\mathcal{L}}{dt} = \sum_i \frac{d}{dt} (p_i \dot{q}_i) + \underbrace{\frac{\partial \mathcal{L}}{\partial t}}_{=0}$$

most of the time, for
most important systems

$$0 = \sum_i \frac{d}{dt} (p_i \dot{q}_i) - \frac{d\mathcal{L}}{dt}$$

$$0 = \frac{d}{dt} \sum_i (p_i \dot{q}_i) - \frac{dL}{dt}$$

$$0 = \frac{d}{dt} \left(\underbrace{\sum_i (p_i \dot{q}_i) - L}_{\text{constant!}} \right)$$

when $\frac{\partial L}{\partial t} = 0$

$$H = \sum_i (p_i \dot{q}_i) - L \quad \leftarrow \text{Hamiltonian (conserved quantity)}$$

↳ usually the total energy (in an inertial ref)

$$H = \sum_i \left(\frac{\partial L}{\partial \dot{q}_i} \cdot \dot{q}_i \right) - L$$

$$H = T + U \quad (\text{usually})$$

