Chapter 6 + 7 → Principle of Least Action and the Lagrangian

$\boxed{\text{Newton's Laws}}$ — $\boxed{\text{Lagrangian}}$ — $\boxed{\text{Hamiltonian}}$

→ "generalized coordinates"

→ Principle of Least Action

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2$$

$$U = mgx$$

$$\underbrace{\int_{t_1}^{t_2} \Big(\frac{1}{2}m\underset{\uparrow\, v(t)}{\dot{x}^2} - mg\underset{\uparrow\, x(t)}{x}\Big)dt}_{\text{what is } x(t) \text{ and } \dot{x}(t) \text{ so this integral is smallest}}$$

<u>Lagrangian</u>

$$\mathcal{L} = T - U$$

<u>Action</u>

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(x(t), \dot{x}(t), t)\,dt$$

$$= \int_{t_1}^{t_2} \mathcal{L}(x, \dot{x}, t)\,dt$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}}$$

Euler - Lagrange Equation

$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}mv^2$$

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m\dot{z}^2 - U(x,y,z)$$

↳ 1D motion

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - U(x)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2 - U(x)\right)\right) = \frac{\partial}{\partial x}\left(\frac{1}{2}m\dot{x}^2 - U(x)\right)$$

$$\frac{d}{dt}(m\dot{x}) = -\frac{dU}{dx}$$

$$m\ddot{x} = -\frac{dU}{dx}$$

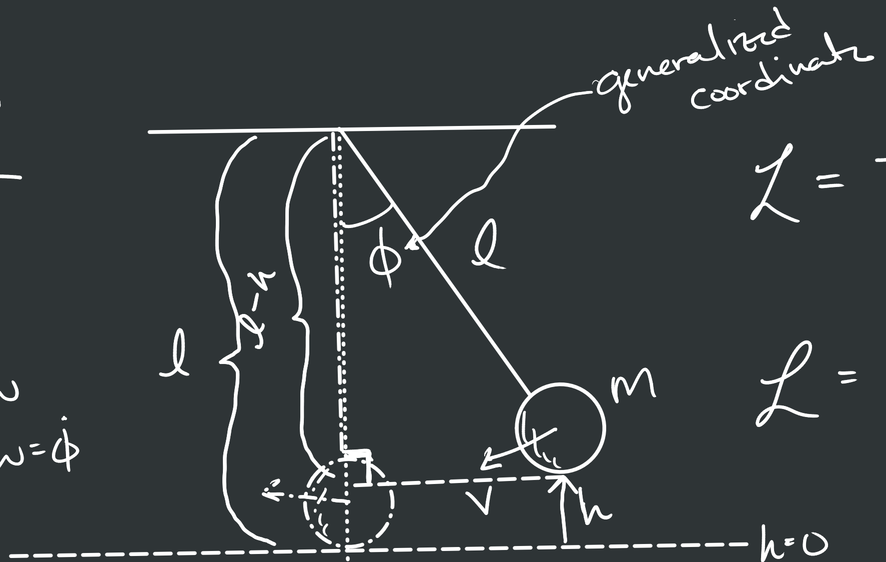
$\longrightarrow$ In 3D

	y	z
$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$	$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = \frac{\partial \mathcal{L}}{\partial y}$	$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}}\right) = \frac{\partial \mathcal{L}}{\partial z}$
$m\ddot{x} = -\frac{\partial U}{\partial x}$	$m\ddot{y} = -\frac{\partial U}{\partial y}$	$m\ddot{z} = -\frac{\partial U}{\partial z}$

$$\underbrace{m\ddot{x}\,\hat{x} + m\ddot{y}\,\hat{y} + m\ddot{z}\,\hat{z}} = -\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}$$

$$\underbrace{\Sigma\vec{F} = m\ddot{\vec{r}}} = \underbrace{-\vec{\nabla}U = \Sigma\vec{F}}$$

Ex:



$v = r\omega$

$\omega = \dot{\phi}$

$\dot{\phi}\cdot l = v$

$$\mathcal{L} = T - U$$

$T \leftarrow \dot{\phi}$, $\quad U \leftarrow \phi$

$$\mathcal{L} = \frac{1}{2}mv^2 - mgh$$

$$\cos\phi = \frac{l-h}{l} \rightsquigarrow h = l(1-\cos\phi)$$

$$\mathcal{L} = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1-\cos\phi)$$

$\hookrightarrow$ apply the Euler-Lagrange equation

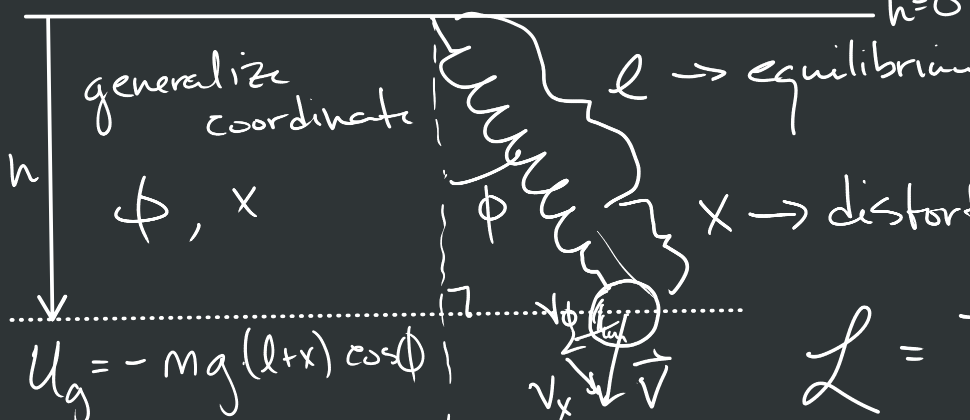
$$\frac{\partial \mathcal{L}}{\partial \dot\phi} = \frac{1}{2} m\ell^2 (2\dot\phi) = m\ell^2 \dot\phi \qquad \Bigg| \qquad \frac{\partial \mathcal{L}}{\partial \phi} = -mg\ell\big(-(-\sin\phi)\big)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot\phi}\right) = \frac{d}{dt}\left(m\ell^2\dot\phi\right) = m\ell^2\ddot\phi \qquad \qquad = -mg\ell\sin\phi$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot x}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\cancel{m}\cancel{\ell^2}\ddot\phi = -\cancel{m}g\cancel{\ell}\sin\phi$$

$$\ddot\phi = -\frac{g}{\ell}\sin\phi \qquad \longrightarrow \qquad \ddot\phi = -\frac{g}{\ell}\sin\phi$$



$\ell \to$ equilibrium length

$x \to$ distortion of the spring

$$U_g = -mg(\ell+x)\cos\phi$$

$$U_s = +\frac{1}{2}kx^2$$

$$v_\phi = (\ell+x)\dot\phi$$

$$v_x = \dot x$$

$$\mathcal{L} = T - U$$

$$= \frac{1}{2}mv^2 - U_g - U_s$$

$$\hookrightarrow \frac{1}{2}kx^2$$

$$\mathcal{L} = \frac{1}{2}m\dot x^2 + \frac{1}{2}m(\ell+x)^2\dot\phi^2 + mg(\ell+x)\cos\phi - \frac{1}{2}kx^2$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$