Chapter 2 - air resistance

$$\sum \vec{F} = \vec{F}_{NET} = m\ddot{\vec{r}}$$

$$\uparrow$$

$$\vec{F}_D = -f(v)\hat{v}$$

$$f(v) = \underset{\substack{\uparrow \\ 0}}{a} + \underset{\substack{\uparrow \\ \text{linear drag}}}{bv} + \underset{\substack{\uparrow \\ \text{quadratic drag}}}{cv^2}$$

$f_{lin} = bv$ $\qquad b \propto D \qquad b = \beta D$ ← related to viscocity

$f_{quad} = cv^2$ $\qquad c \propto D^2 \qquad c = \gamma D^2$ ← density

$$\frac{f_{quad}}{f_{lin}} \propto \frac{D^2 v^2}{D v} = D v$$

$$\frac{f_{quad}}{f_{lin}} \approx \text{Reynolds number} = \frac{\rho D v}{\eta}$$

density

viscosity

$$\vec{F}_{NET} = m\ddot{\vec{r}}$$

$$m\vec{g} - b\vec{v} = m\ddot{\vec{r}}$$

$$\underbrace{m\vec{g} - b\vec{v} = m\dot{\vec{v}}}_{\text{really two equations}}$$

$$-bv_x = m\dot{v}_x \qquad \Big| \qquad mg - bv_y = m\dot{v}_y$$

down ⊕ y-dir

$$\dot{v}_x = \frac{-b}{m} \cdot v_x$$

$$\boxed{\frac{dv_x}{dt} = -\frac{b}{m} \cdot v_x}$$

$$\frac{dv_x}{v_x} = -\frac{b}{m} dt$$

$$\int \frac{dv_x}{v_x} = -\frac{b}{m}\int dt$$

$$e^{\ln v_x} = e^{\left(-\frac{b}{m}t + C\right)} = e^{-\frac{b}{m}t} \cdot e^{C}$$

$$v_x = A \cdot e^{-\frac{b}{m}t}$$

$$\frac{b}{m} = \left[\frac{1}{s}\right] = \left[s^{-1}\right]$$

$$\tau = \frac{m}{b} = [s] \rightarrow \text{time constant}$$

$$v_x = A e^{-t/\tau}$$

exponential decay

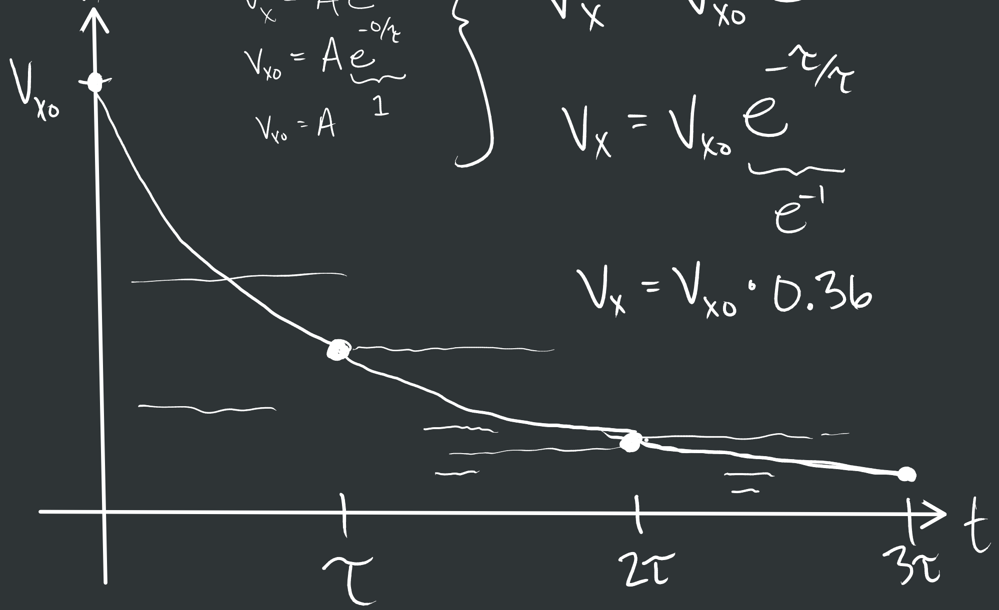
$$\dot{x} = v_{x0} e^{-t/\tau}$$

$$v_x = A e^{-t/\tau}$$
$$v_{x0} = A \underbrace{e^{-0/\tau}}_{1}$$
$$v_{x0} = A$$

$$v_x = v_{x0} e^{-t/\tau}$$

$$v_x = v_{x0} \underbrace{e^{-\tau/\tau}}_{e^{-1}}$$

$$v_x = v_{x0} \cdot 0.36$$



$$\frac{dx}{dt} = v_{x0}\, e^{-t/\tau}$$

$$\int dx = v_{x0} \int e^{-t/\tau}\, dt$$

$$\int_{x_0}^{x} dx' = v_{x0} \underbrace{\int_0^{t} e^{-t'/\tau}\, dt'}$$

$$x' \Big|_{x_0}^{\overset{\text{function}}{x}} = v_{x0}(-\tau)\, e^{-t'/\tau} \Big|_0^{t}$$

$$(x - x_0) = -v_{x0}\, \tau \cdot \left(e^{-t/\tau} - \cancelto{1}{e^{-0/\tau}} \right)$$

$$x - x_0 = v_{x0}\, \tau \left(1 - e^{-t/\tau} \right)$$

$$\boxed{x(t) = v_{x0}\, \tau \left(1 - e^{-t/\tau} \right) + x_0}$$

Linear Drag Vertically

$$mg - bv_y = m\dot{v}_y$$

if $\dot{v}_y = 0$

then $mg - bv_y = 0$

$$v_y = \frac{mg}{b} \equiv v_t$$

velocity when $\dot{v} = 0$

$$m\dot{v}_y = mg - bv_y$$

$$= -b\left(-\frac{mg}{b} + v_y\right)$$

$$m\dot{v}_y = -b(-v_t + v_y)$$

$$u = (-v_t + v_y)$$

$$\dot{u} = \frac{du}{dt} = 0 + \dot{v}_y$$

$$\dot{u} = \dot{v}_y$$

$$m\dot{u} = -b \cdot u$$

$$\dot{u} = -\frac{b}{m} \cdot u$$

$$\tau = \frac{m}{b}$$

$$u = Ae^{-t/\tau}$$

$$-v_t + v_y = Ae^{-t/\tau}$$

$$V_y = Ae^{-t/\tau} + V_t$$

$$V_y(t=0) = V_{y0}$$

$$V_{y0} = A\underbrace{e^{-0/\tau}}_{1} + V_t$$

$$V_{y0} = A + V_t$$

$$A = V_{y0} - V_t$$

$$V_y = (V_{y0} - V_t)e^{-t/\tau} + V_t$$

$$\boxed{V_y = (V_{y0} - V_t)e^{-t/\tau} + V_t}$$

$$V_y = \underbrace{V_{y0}e^{-t/\tau}}_{t\to\infty \;\to 0} + \underbrace{V_t(1-e^{-t/\tau})}_{t\to\infty}$$

$$\cancel{dt}\,\frac{dy}{\cancel{dt}} = \left((v_{y0} - v_t) e^{-t/\tau} + v_t \right) dt$$

$$\int_{y_0}^{y} dy' = \int_0^t \left((v_{y0} - v_t) e^{-t'/\tau} + v_t \right) dt'$$

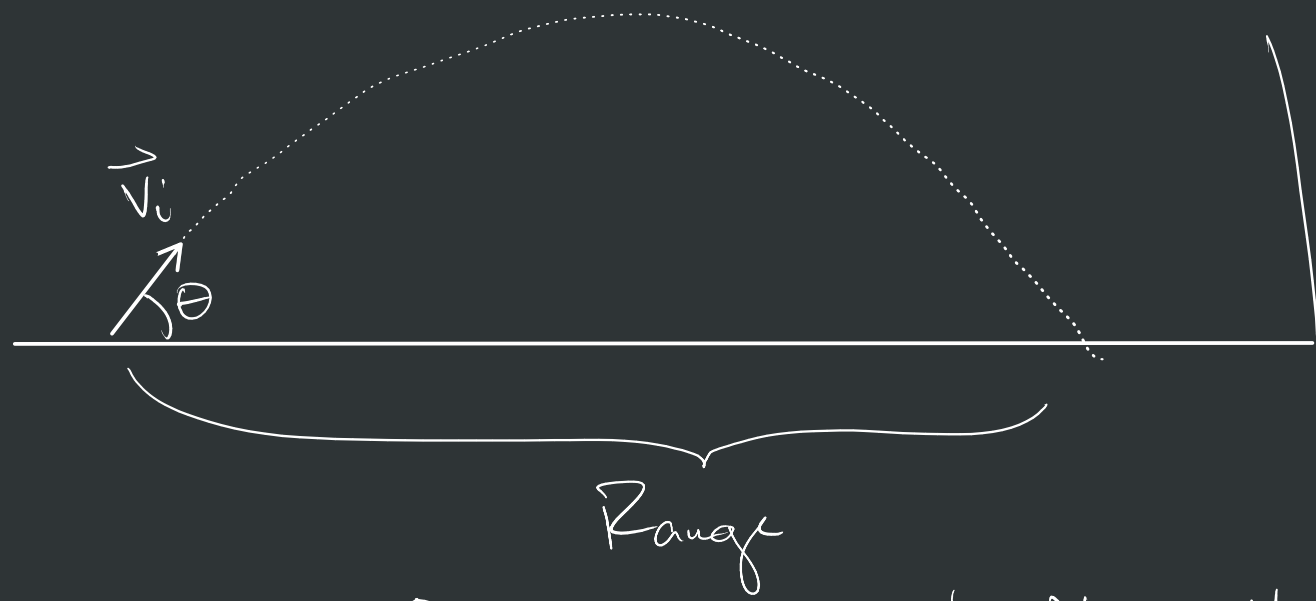
$$y' \Big|_{y_0}^{y} = \Big[(v_{y0} - v_t)(-\tau) e^{-t'/\tau} + v_t t' \Big]_0^t$$

$$y - y_0 = -(v_{y0} - v_t)\tau e^{-t/\tau} + v_t t - \left[-(v_{y0} - v_t)\tau \cdot 1 + 0 \right]$$

algebra

$$\boxed{y = (v_{y0} - v_t)\tau \left(1 - e^{-t/\tau}\right) + v_t t + y_0}$$

Range of Projectile Motion in vacuum



$V_x = V_i \cos\theta$

$X = V_x \cdot t = V_i \cos\theta \cdot t$

$V_{y0} = V_i \sin\theta$

$y = y_i + V_{y0} \cdot t + \frac{1}{2} a_y t^2$

$$\boxed{\begin{array}{l}\text{Range} \rightarrow \text{solve } y=0 \text{ for } t \\ \qquad\qquad \text{plug } t \text{ in for } x\end{array}}$$

$0 = V_{y0} t + \frac{1}{2} a_y t^2$

$0 = t\left(V_{y0} + \frac{1}{2} a_y t\right)$

$t = 0$ or $t = -\dfrac{2V_{y0}}{a_y}$

$\text{Range} = V_x \left(\dfrac{-2V_{y0}}{a_y} \right)$

$a_y = -g$

$R = \dfrac{2 V_x V_{y0}}{g}$

Range for Projectile Motion in Linear Drag

$$x = v_{x0}\tau\left(1 - e^{-t/\tau}\right) + \cancelto{0}{x_0} \quad\Bigg|\quad y = (v_{y0} - v_t)\tau\left(1 - e^{-t/\tau}\right) + v_t t + \cancelto{0}{y_0}$$

$$x = v_{x0}\tau\left(1 - e^{-t/\tau}\right)$$

solve for t

$$y = (v_{y0} - v_t)\tau\left(1 - e^{-t/\tau}\right) + v_t t$$

$$= (v_{y0} - v_t)\tau\left(1 - e^{-t/\tau}\right) + v_t t$$

⋮

lots of algebra (to eliminate t)

⋮

$$y = \left(\frac{v_{y0} + v_t}{v_{x0}}\right)x + v_t\tau\ln\left(1 - \frac{x}{v_{x0}\tau}\right)$$

$$0 = \left(\frac{V_{y0} + V_t}{V_{x0}}\right) R + V_t \tau \ln\left(1 - \frac{R}{V_{x0}\tau}\right)$$

$$-\left(\frac{V_{y0} + V_t}{V_{x0}}\right)\frac{R}{V_t \tau} = \ln\left(1 - \frac{R}{V_{x0}\tau}\right) \leftarrow$$

Taylor series approx

$$\underbrace{\ln(1-\epsilon) \approx -\left(\epsilon + \frac{1}{2}\epsilon^2 + \frac{1}{3}\epsilon^3 + \dots\right)}_{\text{quadratic equation}}$$

$$e^{-\left(\frac{V_{y0} + V_t}{V_{x0}}\right)\frac{R}{V_t \tau}} = 1 - \frac{R}{V_{x0}\tau}$$

$$\boxed{\frac{R}{V_{x0}\tau} = 1 - e^{-\left(\frac{V_{y0} + V_t}{V_{x0}}\right)\frac{R}{V_t \tau}}}$$

$$\cancel{X = e^{X}}$$

$$\rightarrow X = e^{-X}$$

$$\rightarrow X = (1 - e^{-X})$$

Quadratic Drag

Horizontal

$$F_{NET} = m\dot{v}$$

$$-cv^2 = m\dot{v}$$

$$\vec{v} = v_x \hat{x}$$

$$m\frac{dv}{dt} = -cv^2$$

$$\frac{dv}{v^2} = -\frac{c}{m}dt$$

$$\int \frac{dv}{v^2} = -\frac{c}{m}\int dt$$

$$\int v^{-2}dv = -v^{-1} = -\frac{c}{m}t + D$$

$$x = e^{-x}$$

$$\underline{x = 0.567}$$

$$x = (1 - e^{-x})$$

$$x = 0$$

$$\left.\begin{aligned} m\frac{dv_x}{dt} &= -c(v_x^2 + v_y^2) \\ m\frac{dv_y}{dt} &= -c(v_x^2 + v_y^2) \end{aligned}\right\} \text{coupled system of DE}$$

$$\frac{1}{v} = \frac{c}{m}t + D$$

$$v(t) = \frac{1}{\frac{c}{m}t + D}$$

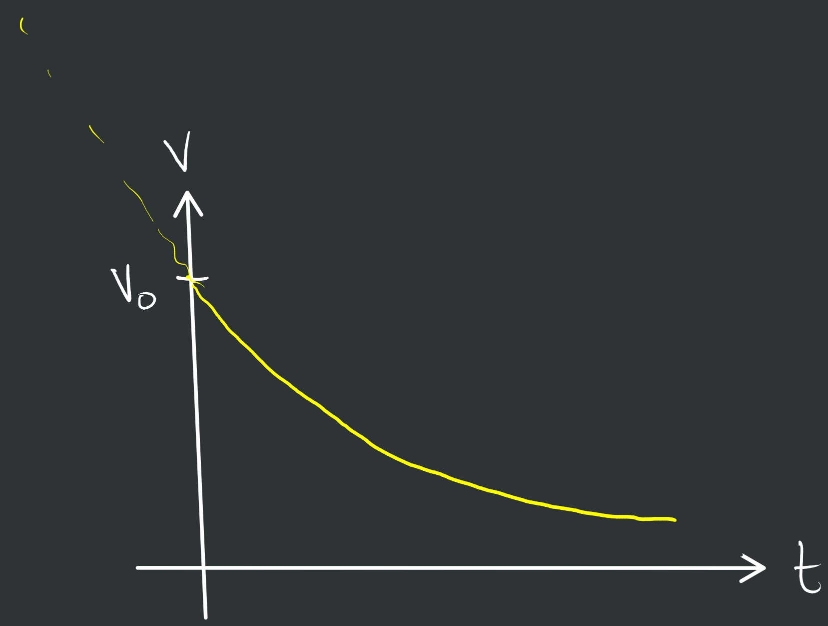
$$v_0 = v(0) = \frac{1}{D} \Rightarrow D = 1/v_0$$

$$v(t) = \frac{1}{\frac{c}{m}t + 1/v_0}$$

$$v(t) = \frac{v_0}{\frac{cv_0}{m}t + 1}$$

$$\tau = \frac{m}{cv_0}$$

$$\boxed{v(t) = \frac{v_0}{t/\tau + 1}}$$



$$\frac{dx}{dt} = \frac{V_0}{t/\tau + 1}$$

$$\int dx = \int \frac{V_0}{t/\tau + 1}\, dt$$

$$\left\{ \begin{aligned} u &= \frac{t}{\tau} + 1 \\ du &= \frac{du}{dt}\, dt \\ du &= \frac{1}{\tau}\, dt \\ \tau\, du &= dt \end{aligned} \right.$$

$$x = V_0 \int \frac{1}{t/\tau + 1}\, dt$$

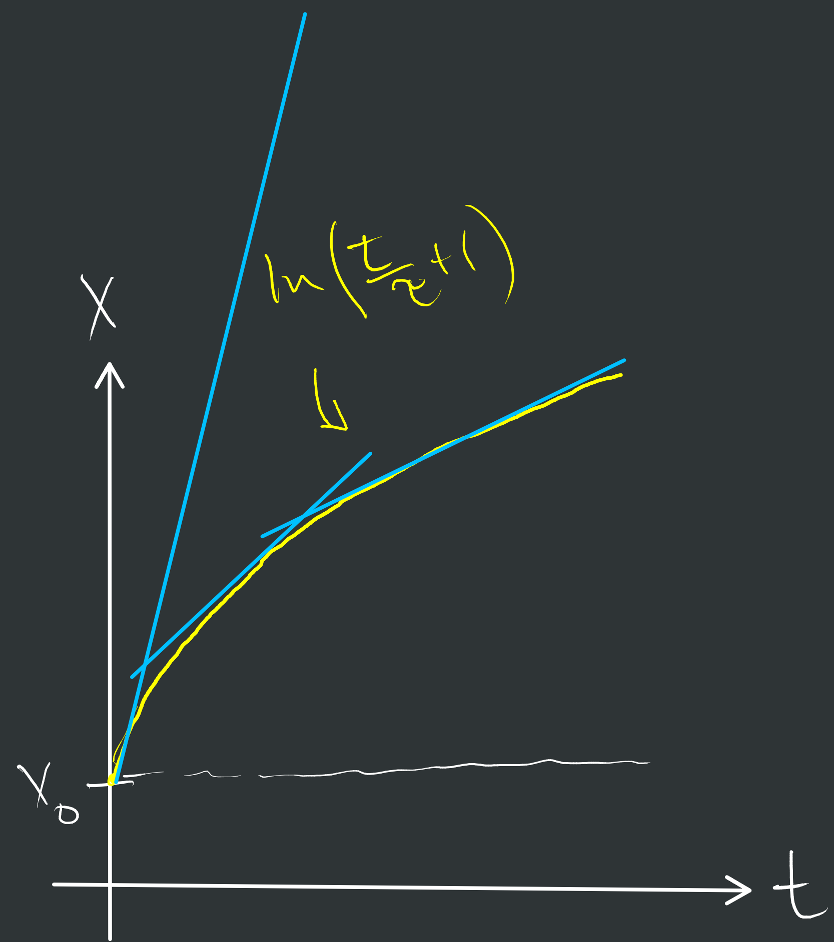
$$x = V_0 \tau \underbrace{\int \frac{1}{u} \cdot du}_{\ln(u)}$$

$$x(t) = V_0 \tau \ln\left(\frac{t}{\tau} + 1\right) + E$$

$$X_0 = x(0) = V_0 \tau \underbrace{\ln(1)}_{0} + E$$

$$X_0 = E$$

$$x(t) = V_0 \tau \ln\left(\frac{t}{\tau} + 1\right) + X_0$$



Vertical Direction

$$m\dot{v} = mg - cv^2$$

$$\dot{v} = 0 = mg - cv_t^2$$

$$v_t = \sqrt{\frac{mg}{c}} \quad \leftarrow \text{terminal velocity}$$

$$\dot{v} = g - \frac{c}{m}\cdot\frac{g}{g}v^2$$

$$\dot{v} = g\left(1 - \frac{c}{mg}v^2\right)$$

$$\dot{v} = g\left(1 - \frac{v^2}{v_t^2}\right)$$

$$\boxed{\frac{dv}{dt} = g\left(1 - \frac{v^2}{v_t^2}\right)}$$

solve for v

$$\int \frac{dv}{(1 - v^2/v_t^2)} = \int g\,dt$$

$$\int \frac{dv}{\left(1 - v^2/v_t^2\right)} = \int g\,dt$$

$$= g\,t + C$$

$$\tanh(u) = \frac{v}{v_t}$$

$$\operatorname{sech}^2(u)\,du = \frac{1}{v_t}\,dv$$

$$\operatorname{sech}^2(u) + \tanh^2(u) = 1$$

$$\operatorname{sech}^2(u) = 1 - \tanh^2 u$$

$$\int \frac{v_t \operatorname{sech}^2(u)\,du}{1 - \tanh^2(u)}$$

$$\int \frac{v_t\, \cancel{\operatorname{sech}^2(u)}\,du}{\cancel{\operatorname{sech}^2(u)}}$$

$$v_t \int du = v_t \cdot u = gt + C$$

$$\boxed{\begin{aligned} \sinh\theta &= \frac{e^{\theta} - e^{-\theta}}{2} \\ \cosh\theta &= \frac{e^{\theta} + e^{-\theta}}{2} \\ \tanh\theta &= \frac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}} \end{aligned}}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\tan\theta = \frac{\qquad}{\qquad}$$

$$e^{x} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{i\theta^3}{3!} + \cdots$$

$$i = \sqrt{-1}$$

$$i^2 = \sqrt{-1}^{\,2}$$

$$i^2 = -1$$

$$e^{i\theta} = 1 + \underline{i\theta} - \frac{\theta^2}{2!} - \underline{\frac{i\theta^3}{3!}} + \frac{\theta^4}{4!} + \underline{\frac{i\theta^5}{5!}} - \frac{\theta^6}{6!}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\tanh(u) = \frac{V}{V_t}$$

$$u = \tanh^{-1}\left(\frac{V}{V_t}\right)$$

$$V_t \cdot \tanh^{-1}\left(\frac{V}{V_t}\right) = gt + C$$

$$\tanh^{-1}\left(\frac{V}{V_t}\right) = \frac{gt}{V_t} + C$$

$$\frac{V}{V_t} = \tanh\left(\frac{gt}{V_t} + C\right)$$

$$V = V_t \cdot \tanh\left(\frac{gt}{V_t} + C\right)$$

$$V(t=0) = V_0$$

$$V_0 = V_t \tanh(C)$$

$$\tanh^{-1}\left(\frac{V_0}{V_t}\right) = C$$

$$e^{i\theta} = \underbrace{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots}_{\cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}_{\sin\theta}$$

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\cosh^2\theta - \sinh^2\theta = 1$$

$$\operatorname{sech}^2\theta + \tanh^2\theta = 1$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\frac{d}{d\theta}(\sinh\theta) = \cosh\theta$$

$$\frac{d}{d\theta}\tanh\theta = \operatorname{sech}^2\theta$$

$$\frac{d}{d\theta}(\sin\theta) = \cos\theta$$

$$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

$$\frac{d}{d\theta}(\tan\theta) = -\sec^2\theta = -\frac{1}{\cos^2\theta}$$

$$V(t) = V_t \cdot \tanh\left(\frac{gt}{V_t} + \tanh^{-1}\left(\frac{V_0}{V_t}\right)\right)$$

$$\frac{dy}{dt} = V_t \cdot \tanh\left(\frac{gt}{V_t} + \tanh^{-1}\left(\frac{V_0}{V_t}\right)\right)$$

$$dy = V_t \cdot \tanh\left(\frac{gt}{V_t} + \tanh^{-1}\left(\frac{V_0}{V_t}\right)\right) \cdot dt$$

$$\int dy = V_t \int \tanh\underbrace{\left(\frac{gt}{V_t} + \tanh^{-1}\left(\frac{V_0}{V_t}\right)\right)}_{u} \cdot dt$$

$$u \Rightarrow du = \frac{g}{V_t}\,dt \qquad dt = \frac{V_t}{g}\,du$$

$$y = V_t \int \frac{V_t}{g} \cdot \tanh(u)\,du$$

$$y = \frac{v_t^2}{g} \ln\left(\cosh(u)\right) + C$$

$$\rightarrow \underbrace{y = \frac{v_t^2}{g} \ln\left(\cosh\left(\frac{gt}{v_t} + \tanh^{-1}\left(\frac{v_0}{v_t}\right)\right)\right) + C}_{}$$

$$m\dot{\vec{V}} = -CV^2\hat{v} + mg\hat{y}$$

$$= -CV \cdot \underbrace{V\hat{v}}_{\vec{V}}$$

$$\vec{V} = V_x\hat{x} + V_y\hat{y}$$

$$V = \sqrt{V_x^2 + V_y^2}$$

$$m\dot{\vec{V}} = -CV\vec{V} + mg\hat{y}$$

$$m\dot{\vec{V}} = -C\sqrt{V_x^2 + V_y^2} \cdot \vec{V} + mg\hat{y}$$

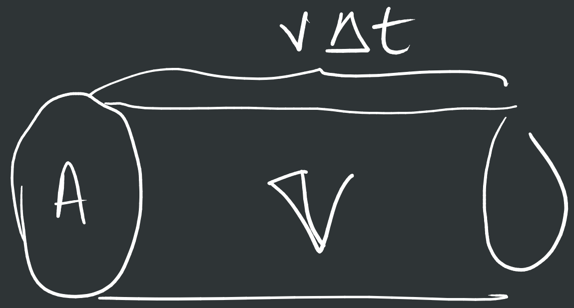
$$\dot{\vec{V}} = \dot{V}_x\hat{x} + \dot{V}_y\hat{y}$$

$$\left.\begin{aligned} &\bullet\ m\dot{V}_x = -C\sqrt{V_x^2 + V_y^2} \cdot V_x \\ &\bullet\ m\dot{V}_y = -C\sqrt{V_x^2 + V_y^2} \cdot V_y + mg\hat{y} \end{aligned}\right\}$$ Coupled system of DE

$$m\frac{dv}{dt} = -cv^2$$

$$\frac{dv}{dt} = -\frac{c}{m}v^2$$

$$F_{quad} = \underbrace{\underbrace{K}_{1/4}\,\rho_{fl}\,A \cdot v^2}_{cv^2}$$



$$m = \rho V = \rho A \cdot v \cdot \Delta t$$

$$\rightarrow \frac{m}{\Delta t} = \rho A \cdot v$$

$$\frac{\Delta V}{\Delta t} = A \cdot v$$

#10

density volume

$$F_B = \underbrace{\rho_{fl} \mathbb{V}_{obj}}_{\text{mass of displaced fluid}} \cdot g$$

$$m\dot{v} = mg - bv - \rho_{fl} \mathbb{V} \cdot g$$

$\equiv$?

$$0 = mg - bv_{ter} - \rho_{fl} \mathbb{V} g$$