

Chapter 3 - Momentum + Angular Momentum

Recall: $\vec{p} = m\vec{v}$ $\sum \vec{F}_n = \dot{\vec{p}}$

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 \dots = \sum \vec{p}_n$$

$$\dot{\vec{P}} = \vec{F}_{\text{ext}}$$

If $\vec{F}_{\text{ext}} = 0$

$$\dot{\vec{P}} = 0$$

$\rightarrow \vec{P}_i = \vec{P}_f$ Conservation of Momentum

If two objects,

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}$$

← really 2 equations

If collision is perfectly inelastic
↳ stick together

then,

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_f + m_2 \vec{v}_f$$
$$= (m_1 + m_2) \vec{v}_f$$

mass add
up!

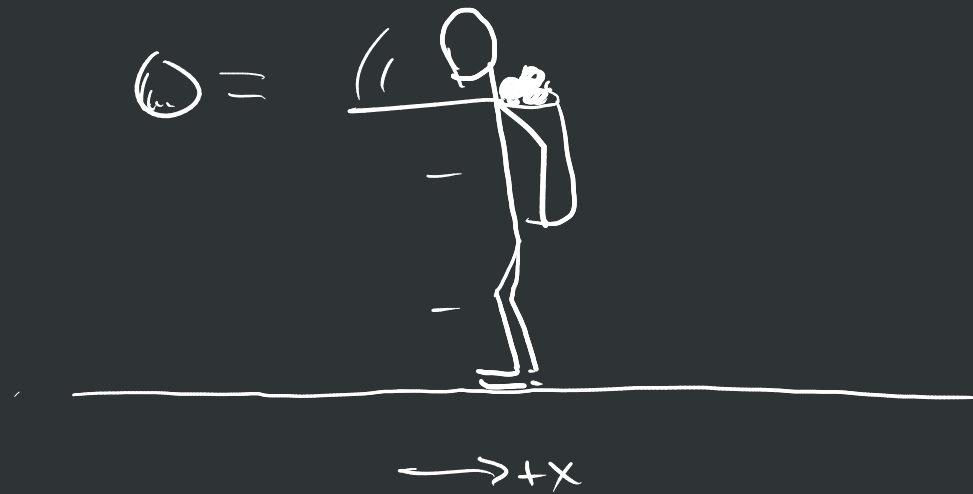
$$\vec{v}_f = \frac{m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i}}{m_1 + m_2}$$

✓ ← two equations!

Rockets + Explosions

At the beginning

$$P = 0$$



Round 1

$$P_i = 0 = P_f = p_{ME} + p_{ball}$$

$$0 = \underbrace{(75 \text{ kg} + 99(0.5 \text{ kg}))}_ {124.5 \text{ kg}} v + (0.5 \text{ kg})(-15 \text{ m/s})$$

$$0 = 124.5 v - 7.5 \frac{\text{kg m}}{\text{s}}$$

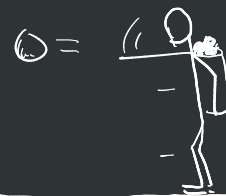
$$v = \frac{7.5 \text{ kg m/s}}{124.5 \text{ kg}} = 0.0602 \text{ m/s}$$

Row 12

$$V' = V_{ice} - V_{s'}$$
$$-15 \text{ m/s} = V_{ice} - 0.0602 \text{ m/s}$$

$$-15 + 0.06 = V_{ice} = -14.94 \text{ m/s}$$

0 =



ice

$$P_i = P_f$$

$$(124.5)(0.0602 \text{ m/s}) + 0.5(-15) = 124 \cdot v + 0.5(-15) + 0.5(-14.94)$$

= 0

$$-0.5(-15 - 14.94) = 124 \cdot v$$

$$v = \frac{-0.5(-15 - 14.94)}{124} = \frac{-0.5(-15 - 15 + 0.0602)}{124}$$

$$= 0.1207 \text{ m/s}$$

but we need a pattern to do this numerically - this way is not giving that away

$$(m + \Delta m)(v + \Delta v) = mv + \Delta mv + m\Delta v + \Delta m\Delta v$$

$$\Delta P_{ME} = \Delta P_{ball}$$

$$124.5(0.0602) - 124v = 0.5(-14.94)$$

$m\Delta v \quad mv \quad \Delta mv$

$$0.5(15) + 0.5 \cdot 0.0602$$

$$\frac{0.5(14.94)}{124}$$

$$0.0602410$$

$$\frac{0.5(15) - 0.5(15)}{124}$$

$$\frac{0.5(15)}{124.5}$$

$$0.0602406$$

$$\frac{0.5(15-x) - x}{124} = ?$$

IGNORE

THESE

NOTES,
REMINGTON
AT WORK

$$v = \frac{-0.5(-15 - 14.94)}{124} = \frac{0.5(15 + 14.94)}{124}$$
$$= \frac{0.5(15)}{124} + \frac{0.5(14.94)}{124}$$

$0.06048 \quad 0.0602$

$$\frac{0.5(14.94)}{124} \stackrel{?}{=} \frac{0.5(15)}{124.5}$$

$0.0602 = 0.0602$

Round 1 Redux - my frame

$$P_i = 0 = P_f = p_{ME} + p_{ball}$$

$$0 = \underbrace{(75 \text{ kg} + 99(0.5 \text{ kg}))}_ {124.5 \text{ kg}} v + (0.5 \text{ kg})(-15 \text{ m/s})$$

$$0 = 124.5 v - 7.5 \frac{\text{kg m}}{\text{s}}$$

$$v = \frac{7.5 \text{ kg m/s}}{124.5 \text{ kg}} = 0.0602 \text{ m/s}$$

Round 2 Redux - my frame

P_i'

$$0 = 124.0 v + 0.5(-15)$$

$$v = \frac{0.5(15)}{124} = 0.06048 \text{ m/s}$$

$$\frac{0.5(15 - 0.602)}{124} =$$

$$\frac{0.5(15) - 0.5(15)}{124} = \frac{0.5(15)}{124.5}$$

$$\frac{1}{124} - \frac{0.5}{124.5} = \frac{1}{124.5}$$

Round 3 Redux - my frame

P_i''

$$D = 123.5 \text{ V} + 0.5(-15)$$

$$V = \frac{0.5(15)}{123.5} = 0.0607 \text{ m/s}$$

0.18

Generalize

$$dV = \frac{dP_{ex}}{m} \quad \leftarrow \text{Variable}$$

$$\int_{V_i}^{V_f} dV = \int_{m_0 + m_f}^{m_0} \frac{V_{ex} dm}{m} = V_{ex} \int_{m_0 + m_f}^{m_0} \frac{dm}{m}$$

$$V \Big|_{v_i}^{v_f} = V_{ex} \ln(m) \Big|_{m_0+m_f}^{m_0}$$

$$v_f - v_i = V_{ex} \left[\ln(m_0) - \ln(m_0 + m_f) \right]$$

$$\underline{v_f = V_{ex} \ln\left(\frac{m_0}{m_0 + m_f}\right) + v_i}$$

$$\hookrightarrow \underline{\underline{V(t) = V_{ex} \ln\left(\frac{m(t)}{m_0 + m_f}\right) + v_i}}$$

test w berechnung

$$\begin{aligned} v_f &= -15 \ln\left(\frac{75\text{kg}}{125\text{kg}}\right) \\ &= \underline{\underline{+7.66 \text{ m/s}}} \end{aligned}$$

$$\rightarrow V(t) = V_{0x} \ln \left(\frac{m(t)}{m_0 + m_f} \right) + V_0$$

$$0 = +15 \ln \left(\frac{m_0}{m_0 + 0.5n} \right) + 10 \quad \leftarrow$$

How many
baseballs to stop
from 10 m/s

$$-10 = +15 \ln \left(\frac{m_0}{m_0 + 0.5n} \right)$$

$$\frac{-10}{+15} = \ln \left(\frac{m_0}{m_0 + 0.5n} \right)$$

$$e^{\frac{-10}{15}} = \frac{m_0}{m_0 + 0.5n}$$

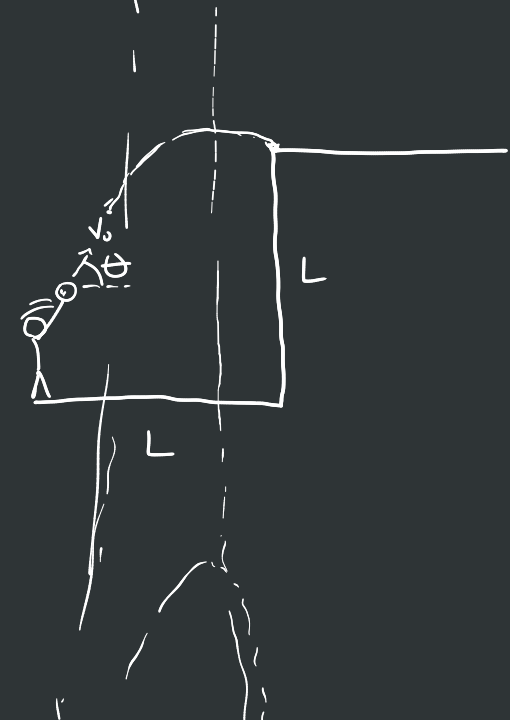
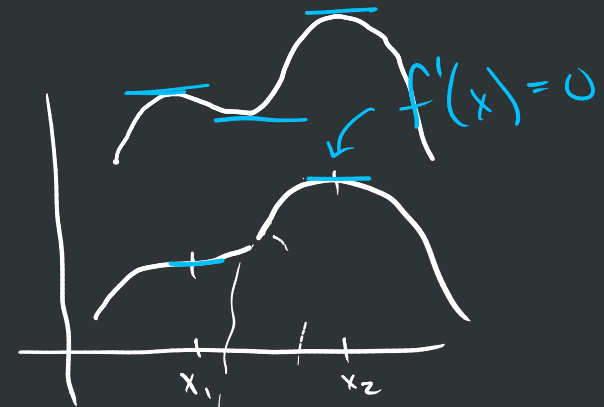
$$e^{\frac{-10}{15}} (m_0 + 0.5n) = m_0$$

$$m_0 + 0.5n = m_0 e^{\frac{10}{15}}$$

$$n = \frac{m_0 (e^{\frac{10}{15}} - 1)}{0.5}$$

$$m_0 = 75 \text{ kg}$$

$$n = 142.16 \quad \leftarrow \# \text{ of baseballs}$$



Center of Mass

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots}{m_1 + m_2 + \dots}$$

total, M



for continuous medium
 $m_\alpha \rightarrow dm$

$$\vec{R} = \frac{1}{M} \sum_{\alpha=1}^N m_\alpha \vec{r}_\alpha$$

center of mass is a position

$$X = \frac{1}{M} \sum_{\alpha=1}^N m_\alpha x_\alpha$$

x-component (x position)

$$Y = \frac{1}{M} \sum_{\alpha=1}^N m_\alpha y_\alpha$$

$$Z = \dots$$

$$\dot{\vec{R}} = \frac{1}{M} \sum_{\alpha=1}^N m_\alpha \dot{\vec{r}}_\alpha$$

$$\vec{P} = M \dot{\vec{R}}$$

$$\vec{R} = \frac{1}{M} \int \vec{r} dm$$

volume

$$\vec{R} = \frac{1}{M} \int \vec{r} \rho dV$$

$$\rho = \frac{M}{V}$$

$$\vec{P} = \cancel{M} \left(\frac{1}{\cancel{M}} \sum_{\alpha} m_{\alpha} \dot{\vec{r}}_{\alpha} \right)$$

$$= \sum_{\alpha} m_{\alpha} \dot{\vec{r}}_{\alpha}$$

$$\vec{P} = \sum_{\alpha} \vec{p}_{\alpha} \quad \leftarrow \text{already knew from Ch. 1.}$$

$$\frac{d}{dt} \vec{P} = M \ddot{\vec{R}} = \vec{F}_{\text{ext}} \quad \leftarrow \text{can treat a collection as a single particle at the center of mass}$$

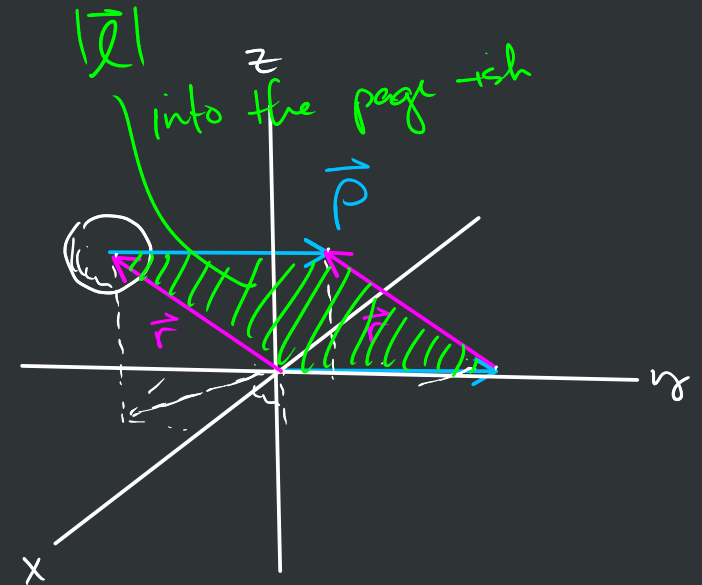
$$\dot{\vec{P}} = \vec{F}_{\text{ext}}$$

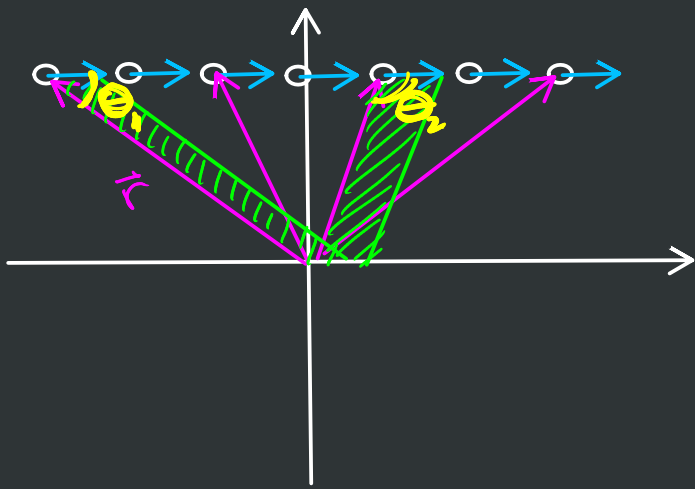
Angular Momentum - conserved quantity angle b/w \vec{r} & \vec{p}

$$\vec{L} = \vec{r} \times \vec{p}$$

angular momentum

$$|\vec{L}| = |\vec{r}| |\vec{p}| \sin \theta$$





$$\begin{aligned}\dot{\vec{l}} &= \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} \\ &= \dot{\vec{r}} \times m\vec{v} \\ &= m \underbrace{\dot{\vec{r}} \times \vec{v}}_{=0} + \vec{r} \times \dot{\vec{p}}\end{aligned}$$

$$\dot{\vec{l}} = \vec{r} \times \dot{\vec{p}}$$

$$\dot{\vec{l}} = \underbrace{\vec{r} \times \vec{F}}_{\text{torque}} \begin{cases} \vec{\tau} \leftarrow \text{tau} \\ \vec{\Gamma} \leftarrow \text{Gamma} \\ \vec{N} \leftarrow \text{en} \end{cases}$$

$$\dot{\vec{l}} = \vec{\Gamma}$$

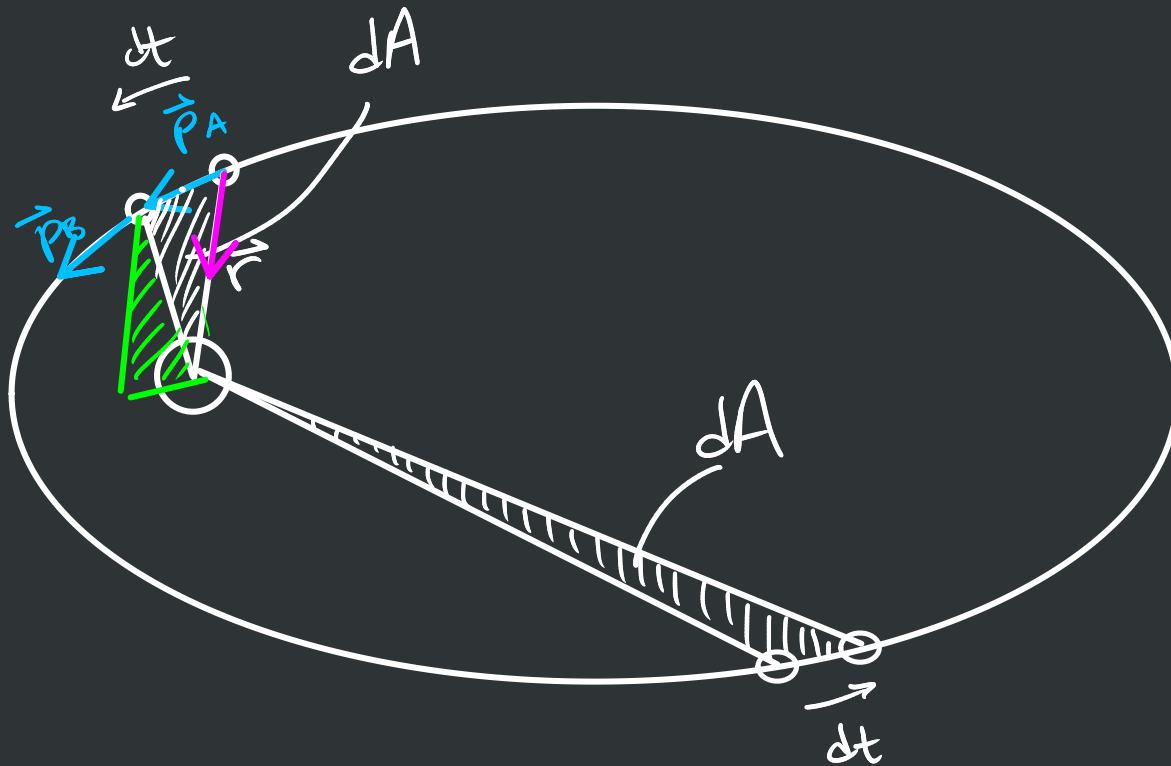
$$\vec{\Gamma} = \dot{\vec{l}}$$

if $\vec{F} = 0$, $\dot{\vec{l}} = 0$, $\vec{l} = \text{constant}$

Newton's 2nd
for rotation

net torque = rate of change of angular mom.

Kepler's 2nd Law



$$d\vec{s} = \vec{v} dt$$

$$dA = \frac{1}{2} (\vec{r} \times \vec{v} dt)$$

$$dA = \frac{1}{2} (\vec{r} \times \frac{\vec{p}}{m} dt)$$

$$\frac{dA}{dt} = \frac{1}{2m} (\underbrace{\vec{r} \times \vec{p}}_{\vec{l}})$$

$$\frac{dA}{dt} = \frac{\vec{l}}{2m} \leftarrow \text{constant}$$

$$=$$

Polar Coordinates

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{p} = \vec{v} \cdot m$$

$\omega \rightarrow$ angular velocity

$$\vec{v} = \dot{r} \hat{r} + r \dot{\phi} \hat{\phi}$$

$$= m \cdot r \hat{r} \times (\dot{r} \hat{r} + r \dot{\phi} \hat{\phi})$$

$$= mr (\hat{r} \times \dot{r} \hat{r} + \hat{r} \times r \dot{\phi} \hat{\phi})$$

$$= mr (\underbrace{\dot{r} \hat{r} \times \hat{r}}_{=0} + r \dot{\phi} \underbrace{\hat{r} \times \hat{\phi}}_{\hat{z}})$$

$$\vec{L} = mr^2 \dot{\phi} \hat{z}$$

$$|\vec{L}| = mr^2 \omega$$

ω may or may not be constant

$$\vec{L} = m r^2 \cdot \vec{\omega}$$

$\underbrace{\quad}_{\text{moment of inertia}} \quad \underbrace{\quad}_{\text{angular velocity}}$

\rightarrow moment of inertia

$I = m r^2 \leftarrow$ only single point mass

$$\vec{p} = m \cdot \vec{v}$$

\uparrow translational velocity

What about multiple particles?

$$\vec{L} = \sum_{\alpha=1}^N \vec{L}_{\alpha} = \sum_{\alpha=1}^N \vec{r}_{\alpha} \times \vec{p}_{\alpha}$$

$$\dot{\vec{L}} = \sum_{\alpha=1}^N \dot{\vec{r}}_{\alpha} \times \vec{p}_{\alpha} + \sum_{\alpha=1}^N \vec{r}_{\alpha} \times \dot{\vec{p}}_{\alpha}$$

$\underbrace{\vec{r}_{\alpha} \times m \dot{\vec{v}}_{\alpha}}_{=0} = \dot{\vec{p}}_{\alpha}$

$$\dot{\vec{L}} = \sum_{\alpha=1}^N \vec{r}_{\alpha} \times \vec{F}_{\alpha} \quad \leftarrow \dot{\vec{p}}_{\alpha}$$

$$\vec{F}_\alpha = \vec{F}_\alpha^{\text{ext}} + \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta}$$

$$\dot{\vec{L}} = \sum_{\alpha=1}^N \left[\vec{r}_\alpha \times \left(\vec{F}_\alpha^{\text{ext}} + \sum_{\beta \neq \alpha} \vec{F}_{\alpha\beta} \right) \right]$$

$$\dot{\vec{L}} = \underbrace{\sum_{\alpha=1}^N \vec{r}_\alpha \times \vec{F}_\alpha^{\text{ext}}}_{\vec{\Gamma}^{\text{ext}}} + \sum_{\alpha=1}^N \vec{r}_\alpha \times \sum_{\beta \neq \alpha}^N \vec{F}_{\alpha\beta}$$

$$\underbrace{\sum_{\alpha=1}^N \sum_{\beta \neq \alpha} \vec{r}_\alpha \times \vec{F}_{\alpha\beta}}_{=0}$$

$$\dot{\vec{L}} = \vec{\Gamma}^{\text{ext}}$$

$$\vec{L} = \sum_{\alpha=1}^N \vec{r}_{\alpha} \times \vec{p}_{\alpha}$$

go back to polar coordinates $\rightarrow \hat{\phi}$ and \hat{r}

$$\vec{L} = \sum_{\alpha} m_{\alpha} r_{\alpha}^2 \dot{\phi}_{\alpha} \hat{z}$$

all m_{α} have same $\dot{\phi}_{\alpha} = \omega$ \leftarrow solid object spinning

$$\vec{L} = \vec{\omega} \cdot \underbrace{\sum_{\alpha} m_{\alpha} r_{\alpha}^2}_{I}$$

$$I = \sum_{\alpha} m_{\alpha} r_{\alpha}^2$$

\downarrow continuous medium

$$I = \int r^2 dm$$

very similar integral to the CM integral

$$I = \int r^2 \rho dV$$

in general, a tensor

Solid sphere rotating about its center

$$I = \frac{2}{5} MR^2$$

disk rotating about its center

$$I = \frac{1}{2} MR^2$$

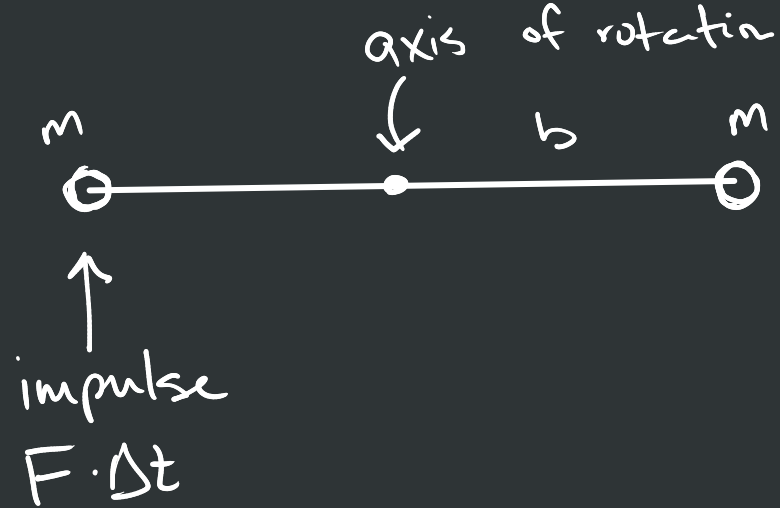
Rotations about the center of mass frame

$$\underbrace{\dot{\vec{L}}}_{\text{CM}} = \vec{\tau}_{\text{ext}}$$

← still true measured
about the center of
mass frame

↳ inertial reference
frame

still works in CM frame
even if CM frame is
not inertial



$$\dot{\vec{P}} = \vec{F}_{\text{ext}} \quad \left. \vphantom{\dot{\vec{P}} = \vec{F}_{\text{ext}}} \right\} \text{Newton's 2nd}$$

$$M \dot{V}_{\text{cm}} \rightarrow \frac{M \Delta V_{\text{cm}}}{\Delta t}$$

$$M \Delta V_{\text{cm}} = \underbrace{F_{\text{ext}} \cdot \Delta t}$$

$$V_{\text{cm}} = \frac{F \Delta t}{M}$$

$$V_{\text{cm}} = \frac{F \Delta t}{2m}$$

$$\begin{aligned} \vec{\Gamma} &= \vec{F} \times \vec{b} \\ &= |\vec{F}| |\vec{b}| \underbrace{\sin 90^\circ}_1 \end{aligned}$$

$$\dot{\vec{L}} = \vec{\Gamma}_{\text{ext}}$$

$$\vec{L} = \vec{\Gamma}_{\text{ext}} \cdot \Delta t$$

choose an axis
of rotation
CM

$$I \cdot \omega_{\text{cm}} = (F \cdot b) \Delta t$$

$$\begin{aligned} \sum \sum_{\alpha}^N m_{\alpha} r_{\alpha}^2 &= \sum_{\alpha}^2 m_{\alpha} r_{\alpha}^2 \\ &= m b^2 + m b^2 \\ &= 2 m b^2 \end{aligned}$$

• What is the speed
of the right and
left masses from
the "lab" frame

$$2mb^2 \cdot \omega_{cm} = Fb\Delta t$$

$$\omega_{cm} = \frac{Fb\Delta t}{2mb^2}$$

$$\omega_{cm} = \frac{F\Delta t}{2mb}$$

$$V^{S'} = V^S - V^{S' \rightarrow S}$$

↑
object's
velocity
in another
frame

↑
object's velocity
in one frame

velocity between
frame



left mass speed in the lab frame

$$V_l^{S'} = \omega \cdot b - (-V_{cm})$$

$$V_l^{S'} = \frac{F\Delta t}{2mb} \cdot b + \frac{F\Delta t}{2m} = \frac{F\Delta t}{m}$$

right mass spread in the lab frame

$$v_r^{s'} = -\omega b - (-v_{cm})$$

$$= -\frac{F\Delta t}{2mb} \cdot b + \frac{F\Delta t}{2m}$$

$$v_r^{s'} = 0$$

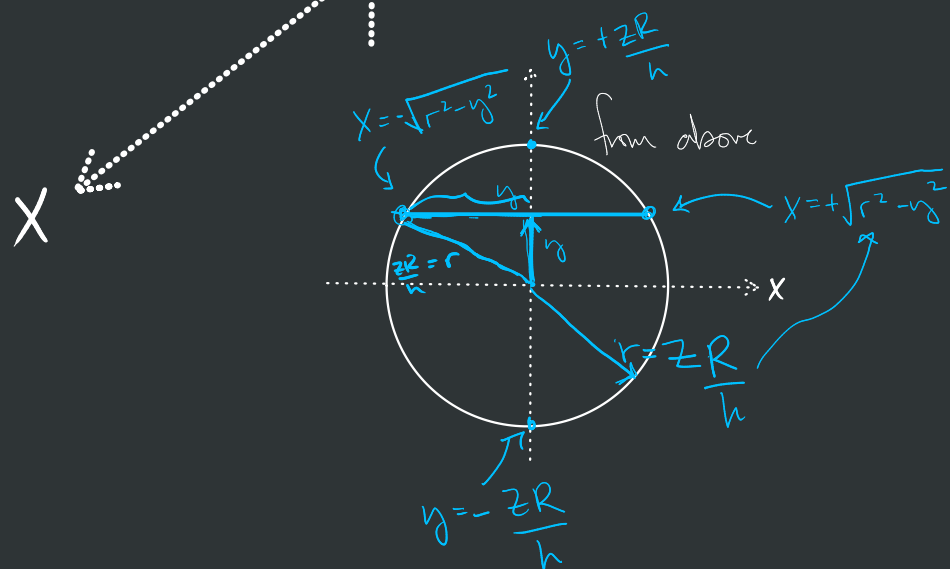
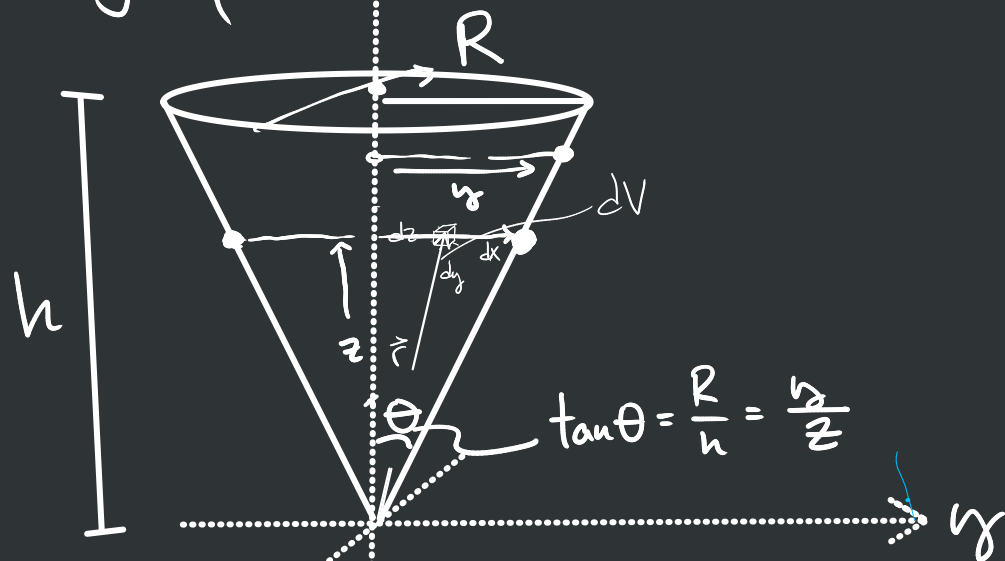
$$V^{s'} = V^s - \underbrace{V^{s' \rightarrow s}}_{\text{cm}}$$

$$\omega_b = \underbrace{V^s}_{\text{lab}} - V_{cm}$$

$$V^{lab} = \omega_b + V_{cm}$$

Ex: CM

density ρ



$$\vec{R} = \frac{1}{M} \int \vec{r} \rho dV$$

$$= \frac{\rho}{M} \int \vec{r} dV$$

$$= \frac{3M}{M \cdot \pi R^2 h} \int \vec{r} dV$$

$$\downarrow$$

$$dx dy dz$$

$$\iiint \vec{r} dx dy dz$$

$$\vec{r} = x \hat{x} + y \hat{y} + z \hat{z}$$

$$V_{\text{cone}} = \frac{\pi R^2 h}{3}$$

$$\rho = \frac{M}{V_{\text{cone}}} = \frac{3M}{\pi R^2 h}$$

$$\iiint_{\substack{z \\ y \\ x}} (x \hat{x} + y \hat{y} + z \hat{z}) dx dy dz$$

$$\hat{x} \iiint_{\substack{z \\ y \\ x}} x dx dy dz + \hat{y} \iiint_{\substack{z \\ y \\ x}} y dx dy dz + \hat{z} \iiint_{\substack{z \\ y \\ x}} z dx dy dz$$

$$\hat{x} \iiint_{\substack{0 \leq z \leq \frac{h+2R}{h} \\ -\frac{2R}{h} \leq y \leq \sqrt{r^2-y^2} \\ -\sqrt{r^2-y^2} \leq x \leq \sqrt{r^2-y^2}}} x dx dy dz + \hat{y} \iiint_{\substack{0 \leq z \leq \frac{h+2R}{h} \\ -\frac{2R}{h} \leq y \leq \sqrt{r^2-y^2} \\ -\sqrt{r^2-y^2} \leq x \leq \sqrt{r^2-y^2}}} y dx dy dz$$

$$+ \hat{z} \iiint_{\substack{0 \leq z \leq \frac{h+2R}{h} \\ -\frac{2R}{h} \leq y \leq \sqrt{r^2-y^2} \\ -\sqrt{r^2-y^2} \leq x \leq \sqrt{r^2-y^2}}} z dx dy dz$$

proof:

$$\hat{x} \iiint_{\substack{0 \leq z \leq \frac{h+2R}{h} \\ -\frac{2R}{h} \leq y \leq \sqrt{r^2-y^2} \\ -\sqrt{r^2-y^2} \leq x \leq \sqrt{r^2-y^2}}} x dx dy dz$$

$$\frac{x^2}{2} \Big|_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} = \left(\frac{r^2-y^2}{2} \right) - \left(-\frac{r^2-y^2}{2} \right) = 0$$

so that was not too bad

$$\int_0^h \int_{-\frac{2R}{h}}^{\frac{2R}{h}} y \left(\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx \right) dy dz = 0$$

$$x \Big|_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} = \sqrt{r^2-y^2} - (-\sqrt{r^2-y^2}) = 2\sqrt{r^2-y^2}$$

$$\int_0^h \int_{-\frac{2R}{h}}^{\frac{2R}{h}} y (2\sqrt{r^2-y^2}) dy dz$$

$r^2 + y^2 = u$ $\left\{ \begin{array}{l} u\text{-substitution} \\ 2y dy = du \end{array} \right.$

$$\int u^{\frac{1}{2}} du \rightarrow \frac{2}{3} u^{\frac{3}{2}} \rightarrow \frac{2}{3} (r^2 + y^2)^{\frac{3}{2}} \Big|_{-\frac{2R}{h}}^{\frac{2R}{h}} = \frac{2}{3} \left[(r^2 + (\frac{2R}{h})^2)^{\frac{3}{2}} - (r^2 + (-\frac{2R}{h})^2)^{\frac{3}{2}} \right] = 0$$

$$\int_0^h z \int_{-\frac{2R}{h}}^{\frac{2R}{h}} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx dy dz$$

look up in Schaum's

$$\int_{-\frac{2R}{h}}^{\frac{2R}{h}} 2\sqrt{r^2-y^2} dy \leftarrow \text{recall } r = \frac{2R}{h}$$

$$= 2 \left(y \frac{\sqrt{r^2-y^2}}{2} + \frac{r^2}{2} \sin^{-1} \left(\frac{y}{r} \right) \right) \Big|_{-\frac{2R}{h}}^{\frac{2R}{h}}$$

$$= \frac{2R}{h} \left[\frac{(\frac{2R}{h})^2 - (\frac{2R}{h})^2}{2} \right] + \left(\frac{2R}{h} \right)^2 \left[\sin^{-1} \left(\frac{2R-h}{h+2R} \right) - \left(-\frac{2R}{h} \right) \sqrt{\left(\frac{2R}{h} \right)^2 - \left(\frac{2R}{h} \right)^2} \right] + \left(\frac{2R}{h} \right)^2 \sin^{-1}(-1)$$

$$= 0 + \frac{2R}{h} \left(\frac{2R}{h} \right)^2 \left[\sin^{-1}(1) - \sin^{-1}(-1) \right] = \pi \left(\frac{2R}{h} \right)^2$$

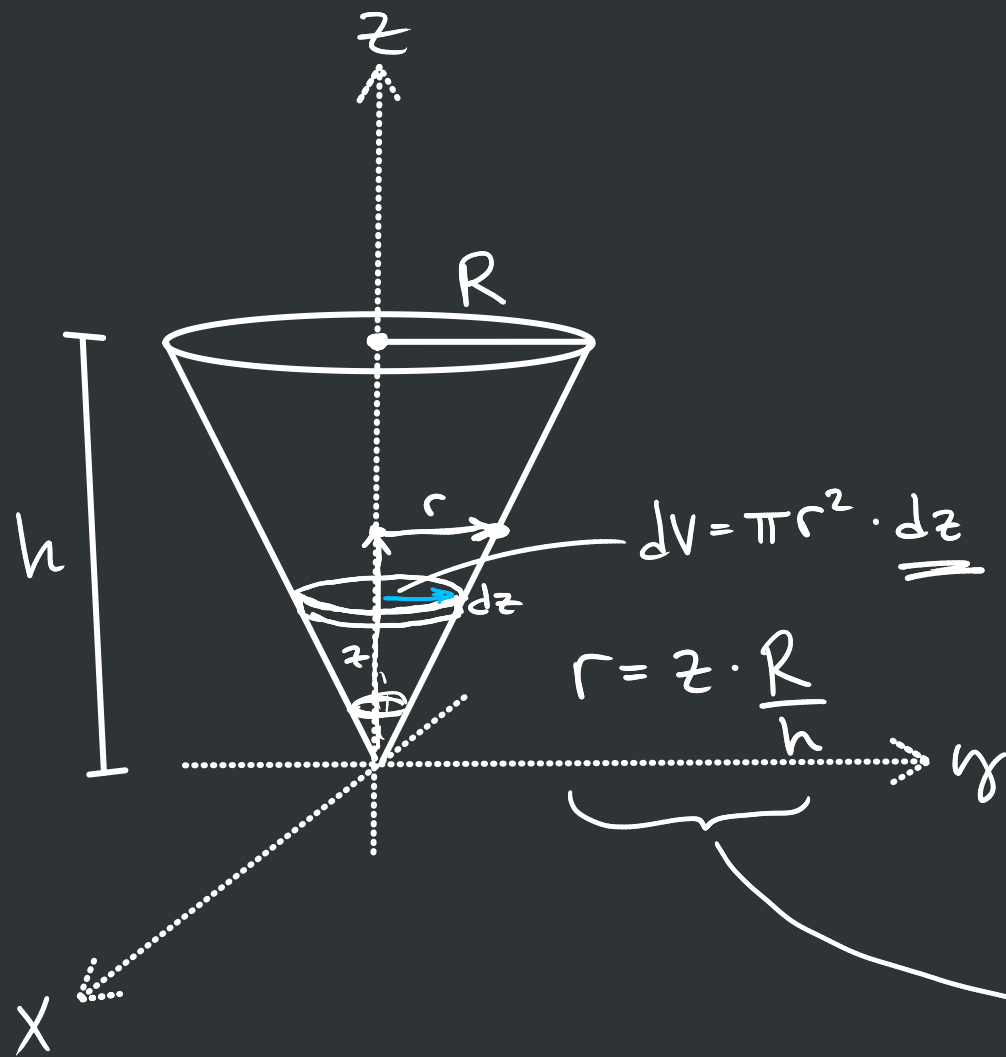
$$R = \frac{3}{4} \cdot \frac{\pi R^2 h}{4}$$

so I am off by a factor of $\frac{1}{2}$

$$\int_0^h z \left(\pi \frac{R^2}{h^2} \right) dz$$

$$= \pi \frac{R^2}{h^2} \int_0^h z^2 dz$$

$$= \pi \frac{R^2}{h^2} \left(\frac{z^3}{3} \right) \Big|_0^h = \pi \frac{R^2}{h^2} \cdot \frac{h^3}{3} = \frac{\pi R^2 h}{3}$$



\vec{R}

$$\int \vec{r} dV$$

$$\int z \pi r^2 dz$$

\vec{r} in cylindrical by symmetry
 $\cancel{\vec{r} \hat{r}} + z \hat{z}$

$$\pi \int z r^2 dz$$

$$\pi \int_0^h z \left(z \frac{R}{h} \right)^2 dz$$

$$\pi \left(\frac{R}{h} \right)^2 \int_0^h z^3 dz$$

$$\pi \left(\frac{R}{h} \right)^2 \left(\frac{z^4}{4} \right) \Big|_0^h$$

put the constants
back \rightarrow

$$R = \frac{3 \cancel{M}}{\cancel{M} \cdot \cancel{\pi} R^4} \cdot \pi \frac{\cancel{R}}{\cancel{R}} \left(\frac{h^4}{4} \right)$$

$$\boxed{R = \frac{3}{4} h} \leftarrow \text{in the } z \text{ direction}$$

