Chapter 3 - special Remington section

$$f''(x) = -k \cdot f(x)$$

$$f(x) = A\sin(\omega x)$$

$$\nabla^2 \phi = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \cancelto{0}{\frac{\partial^2 \phi}{\partial z^2}} = 0$$

Let's stick w/ two dimensions, no dependence in the z-direction

Assume $\phi = f(x) \cdot g(y)$

$$\frac{\partial^2 (f \cdot g)}{\partial x^2} + \frac{\partial^2 (f \cdot g)}{\partial y^2} = 0$$

$$g \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2} = 0$$ } divide this equation by $f \cdot g$

$$\frac{1}{f} \cdot \frac{\partial^2 f}{\partial x^2} + \frac{1}{g} \frac{\partial^2 g}{\partial y^2} = 0$$

$$\frac{1}{f}\cdot\frac{\partial^2 f}{\partial x^2} = k^2$$

$$\frac{1}{g}\frac{\partial^2 g}{\partial y^2} = -k^2$$

$$\frac{\partial^2 f}{\partial x^2} = k^2 \cdot f$$

$$\frac{\partial^2 g}{\partial y^2} = -k^2 g$$

$$f(x) = Ae^{kx}$$

or

$$Be^{-kx}$$

$$g(y) = C\sin(ky)$$

or

$$= D\cos(ky)$$

linear combination

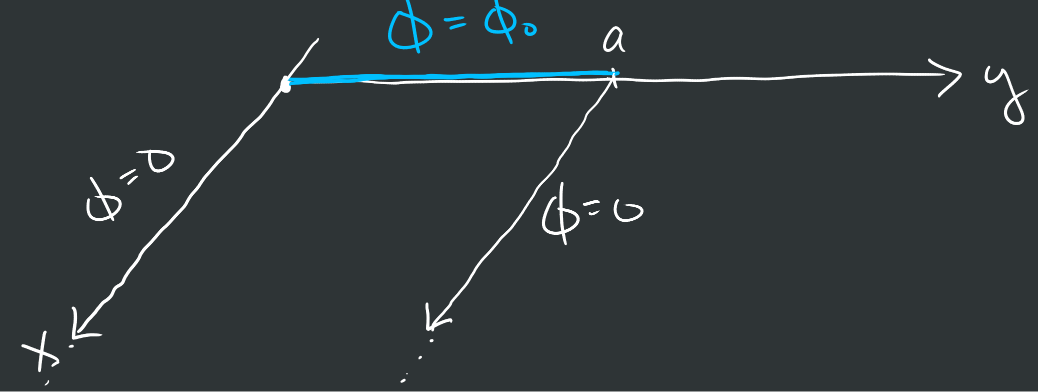
So

$$f(x) = Ae^{kx} + Be^{-kx}$$

So

$$g(y) = C\sin(ky) + D\cos(ky)$$

So now we need Boundary Conditions



last B.C. to use

$x = 0,\ \phi = \phi_0$

✓ $x \to \infty,\ \phi \to 0$

$\underbrace{\hphantom{x \to \infty,\ \phi \to 0}}_{}$ Ae^{kx} blows up w/ $x \to \infty$, $\therefore A = 0$

+

✓ $y = 0,\ \phi = 0$

✓ $y = a,\ \phi = 0$

$\underbrace{\hphantom{y = a,\ \phi = 0}}_{}$ $\cos(ky) \neq 0$, when $y = 0$, so $D = 0$

so $\phi = f(x) \cdot g(y) = \underbrace{B e^{-kx} \cdot C}_{\text{absorb constants}} \sin(ky)$

$$= B e^{-kx} \cdot \sin(ky)$$

<u>now</u> $y = a,\ \phi = 0$ ✓

$$0 = \sin(k \cdot a)$$

$$k \cdot a = 0, \pi, 2\pi, 3\pi, \ldots$$

$$= n\pi \qquad n = 0, 1, 2 \ldots$$

$$k = \frac{n\pi}{a}$$

now $\phi(x,y) = B e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a} y\right)$

Since this is a valid solution for infinitely many n, we need a <u>weighted</u> <u>sum</u> of the individual solutions

↳ <u>linear</u> <u>combination</u>

$$\phi = \alpha_1 \phi_1 + \alpha_2 \phi_2 + \alpha_3 \phi_3 + \ldots\ldots$$

So that:

$$\nabla^2 \phi = \nabla^2 (\alpha_1 \phi_1 + \alpha_2 \phi_2 + \alpha_3 \phi_3 + \ldots\ldots) = \alpha_1 \nabla^2 \phi_1 + \alpha_2 \nabla^2 \phi_2 + \ldots = 0$$

I'll absorb each α into $B \rightarrow B_n$

$$\phi(x,y) = \sum_{n=1}^{\infty} B_n e^{-\frac{n\pi}{a} x} \cdot \sin\left(\frac{n\pi}{a} y\right)$$

So now we apply the last B.C. $(x=0,\ \phi = \phi_0)$

$$\phi(0,y) = \phi_0(y) = \sum_{n=1}^{\infty} B_n \underbrace{e^{-\frac{n\pi}{a} \cdot 0}}_{1} \cdot \sin\left(\frac{n\pi}{a} y\right)$$

$$\phi_0 = \underbrace{\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a} \cdot y\right)}_{\text{Fourier series!}}$$

Multiply both sides of the equation by $\sin\left(\frac{n'\pi}{a} y\right)$ where n' is another constant. And then integrate!

$$\int_0^a \phi_0 \sin\left(\frac{n'\pi}{a} y\right) dy = \sum_{n=1}^{\infty} B_n \underbrace{\int_0^a \sin\left(\frac{n\pi}{a} y\right) \cdot \sin\left(\frac{n'\pi}{a} y\right) dy}_{= \frac{a}{2} \text{ only when } n = n'}$$

$$\int_0^a \phi_0 \sin\left(\frac{n'\pi}{a} y\right) dy = \sum_{n=1}^{\infty} B_n \frac{a}{2} \delta_{nn'} = B_{n'} \frac{a}{2}$$

$$B_{n'} = \frac{2}{a} \int_0^a \phi_0 \sin\left(\frac{n'\pi}{a} y\right) dy$$

$$B_n = \frac{2}{a} \int_0^a \phi_0 \sin\left(\frac{n\pi}{a} y\right) dy$$

$$B_n = \frac{2\phi_0}{a} \int_0^a \sin\left(\frac{n\pi}{a} y\right) dy$$

$$= \frac{2\phi_0}{a} \cdot \left(\frac{a}{n\pi}\right) \left(-\cos\left(\frac{n\pi}{a} y\right)\right)\Bigg|_0^a$$

So $B_n = \frac{2\phi_0}{\cancel{a}} \cdot \left(\frac{\cancel{a}}{n\pi}\right) \cdot 2$

for odd values of n

$$B_n = \frac{4\phi_0}{n\pi} \quad \text{where } n \text{ is odd.}$$

$$-\left[\cos\left(\frac{n\pi}{\cancel{a}}\cancel{a}\right) - \cos(0)\right]$$

$$-\left[\cos(n\pi) - 1\right]$$

$$\underbrace{\underbrace{-\cos(n\pi)}_{\substack{-1 \text{ for odd } n \\ +1 \text{ for even } n}} + 1}_{\substack{+2 \text{ for odd } n \\ 0 \text{ for even } n}}$$

$$\phi(x,y) = \sum_{n=1}^{\infty} B_n e^{-\frac{n\pi}{a}x} \cdot \sin\left(\frac{n\pi}{a} y\right)$$

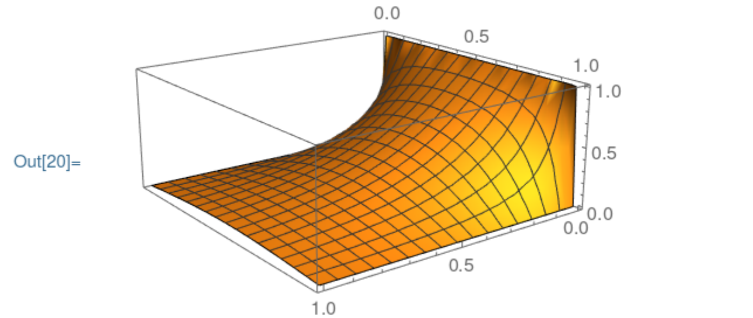
$$\phi(x,y) = \frac{4\phi_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \cdot e^{-\frac{n\pi}{a}x} \cdot \sin\left(\frac{n\pi}{a} \cdot y\right)$$

↳ lets plot in mathematica!

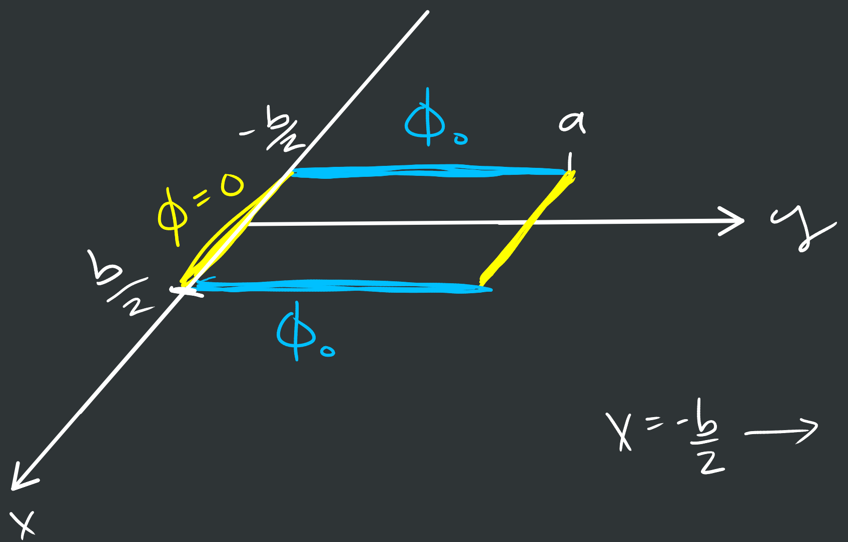
$$n \to 2n+1$$

```
In[20]:= Plot3D[
  4 * 1 / Pi *
   Sum[1 / (2 n + 1) * Exp[- (2 n + 1) * Pi / 1 * x] *
     Sin[(2 n + 1) * Pi / 1 * y], {n, 0, 1000}],
  {x, 0, 1}, {y, 0, 1}]
```

Out[20]=



Now for another example:



$$\frac{\partial^2 f}{\partial x^2} = k^2 f \qquad \frac{\partial^2 g}{\partial y^2} = -k^2 g$$

$$f = Ae^{kx} + Be^{-kx} \qquad g = C\sin(ky) + \underbrace{D\cos(ky)}_{D=0}$$

$D = 0$ since $\phi = 0$ at $y = 0$

$$x = -\frac{b}{2} \longrightarrow \phi_0 = Ae^{-kb/2} + Be^{+k \cdot b/2}$$

and

$$x = +\frac{b}{2} \qquad \underbrace{\phi_0 = Ae^{+kb/2} + Be^{-k\frac{b}{2}}}$$

so this can only be true if $A = B$

$$f = A\left(e^{kx} + e^{-kx}\right)$$

$$g = C\sin ky$$

$\phi = 0,\ y = a$

so $0 = C\sin(ka)$

$0 = \sin(ka)$

$ka = 0\pi, 1\pi, 2\pi, 3\pi$

$ka = n\pi \qquad n = 1, 2, 3 \ldots$

$$k = \frac{n\pi}{a}$$

So...

$$\phi(x,y) = \sum_{n=1}^{\infty} \underbrace{A_n\left(e^{kx} + e^{-kx}\right)}_{A} \sin(ky)$$

$$2 \cdot \cosh(kx) \longrightarrow \cosh(kx) = \frac{e^{kx} + e^{-kx}}{2}$$

$$\sinh(kx) = \frac{e^{kx} - e^{-kx}}{2}$$

$$\cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2}$$

$$\sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i}$$

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \cosh(kx) \sin(ky)$$

So now, last BC.

$\rightarrow \phi = \phi_0$ at $x = b$

$$\phi_0 = \sum_{n=1}^{\infty} A_n \cosh(kb) \cdot \sin(ky)$$

?

now we apply Fourier's trick

$$\underbrace{\int_0^a \phi_0 \sin\left(\frac{n'\pi}{a} y\right) dy}_{\phi_0\left(\frac{a}{n'\pi}\right)\underbrace{\left(-\cos\left(\frac{n'\pi}{a}y\right)\right)\Big|_0^a}_{\substack{=2 \text{ odd } n' \\ =0 \text{ even } n'}}} = \underbrace{\sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi}{a} b\right) \int_0^a \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{n'\pi}{a} y\right) dy}_{\sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi}{a} b\right) \cdot \frac{a}{2} \delta_{nn'}}$$

$$= A_{n'} \cosh\left(\frac{n'\pi}{a} b\right) \cdot \frac{a}{2}$$

$$\phi_0 \cdot \frac{2\cancel{a}}{n'\pi} = A_{n'} \cosh\left(\frac{n'\pi}{a} b\right) \cdot \frac{\cancel{a}}{2}$$

drop primes

solve for A_n

$$\underbrace{A_n = \frac{4\phi_0}{n\pi \cosh\left(\frac{n\pi}{a} b\right)}}$$ for odd n

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \cosh(kx) \sin(ky)$$

$$\phi(x,y) = \frac{4\phi_0}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n} \frac{\cosh\left(\frac{n\pi}{a} x\right)}{\cosh\left(\frac{n\pi b}{a}\right)} \cdot \sin\left(\frac{n\pi}{a} y\right)$$