Electrostatics

- what forces are on charges?
- what gives rise to those forces?
- how much energy can be stored / released?

Charge → conservation of charge → in an isolated system the total electric charge never changes

→ quantization of charge

charge of an electron / proton is the smallest isolated charge → e (elementary charge)

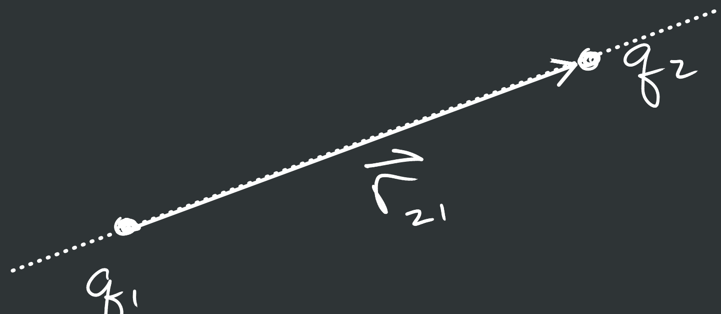
q is variable for charge

SI unit for charge is Coulomb

$$1e \equiv 1.602\ldots \times 10^{-19}\ C$$

$$1C = \frac{1}{1.602 \cdot 10^{-19}} e = 6.242 \cdot 10^{18}\ e$$

So now to electrostatics:



The force between q_1 + q_2 is directly proportional to the product of charge and inversely proportional to the square of the distance between them.

$$\vec{F}_{21} = k \frac{q_1 q_2}{r_{21}^2} \hat{r}_{21}$$

} product of charges

} separation distance

k → constant of proportionality

→ unit vector $\hat{r}_{21} = \frac{\vec{r}_{21}}{|\vec{r}_{21}|}$

Ex. $\vec{r}_{21} = \langle 3, 1 \rangle$

$\vec{r}_{21} = 3\hat{x} + 1\hat{y}$

$|\vec{r}_{21}| = \sqrt{3^2 + 1^2}$

$= \sqrt{10}$

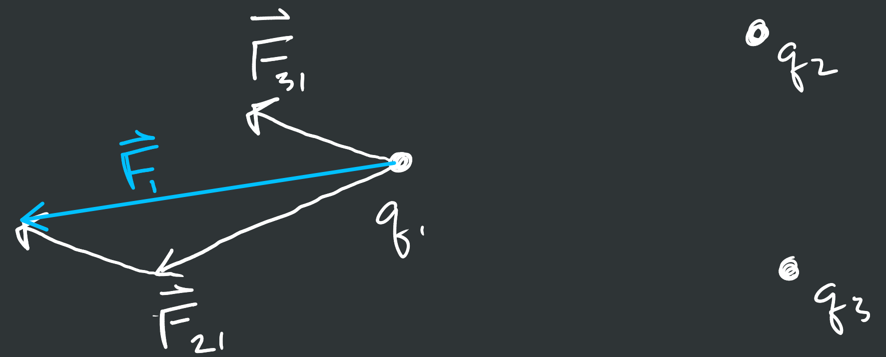
$\hat{r}_{21} = \frac{3\hat{x} + 1\hat{y}}{\sqrt{10}}$

→ SI system

$k = 8.988 \cdot 10^9 \frac{N m^2}{C^2}$

$k = \frac{1}{4\pi\epsilon_0} \rightarrow \epsilon_0 = 8.85 \cdot 10^{-12} \frac{C^2}{N m^2}$

What about more than two charges?



$\vec{F}_1 = \vec{F}_{21} + \vec{F}_{31}$

How do we do work?

$$W = \int \vec{\text{Force}} \cdot \vec{\text{displacement}}$$

$$W = -\int \vec{F} \cdot d\vec{r}$$

dot product mean relative directions of force and displacement matter

$$W = \int_{\infty}^{r_{12}} -\frac{kq_1q_2}{r^2}\,\hat{r} \cdot \underbrace{d\vec{r}}_{dr\cdot\hat{r}}$$

work I do

where force is zero

I have to push against the Coulomb force (equal & opposite)

I do this in the $-\hat{r}$ direction

$$W = -kq_1q_2 \int_{\infty}^{r_{12}} \frac{1}{r^2}\hat{r}\cdot\hat{r}\,dr \qquad \hat{r}\cdot\hat{r} = 1$$

$$W = -kq_1q_2 \int_{\infty}^{r_{12}} r^{-2}\,dr$$

$$W = -kq_1q_2(-1)\,r^{-1}\Big|_{\infty}^{r_{12}}$$

$$W = \frac{kq_1q_2}{r}\Big|_{\infty}^{r_{12}}$$

$$= \frac{kq_1q_2}{r_{12}} - \cancel{\frac{kq_1q_2}{\infty}} \to 0$$

$$\boxed{W_{ME} = \frac{kq_1q_2}{r_{12}}}$$ from ∞ far away

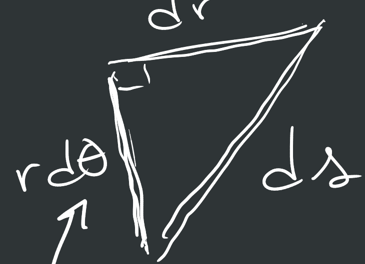
Now, is this electric force conservative?

→ what does this mean

→ path independent

→ only depends on position

→ $\underbrace{\vec{\nabla} \times}_{\text{curl}} \vec{F} = 0$ ← save for later



back and forth are the same amount of work

this is perpendicular to the force so the dot product would be zero

only the radial part of the motion contributes to the work so only radial position matters

How do we handle multiple charges

$$\vec{F}_3 = \vec{F}_{31} + \vec{F}_{32}$$

$$W_3 = \int \vec{F}_3 \cdot d\vec{r}$$

$$W_3 = \int \left(\vec{F}_{32} + \vec{F}_{31}\right) \cdot d\vec{r}$$

$$W_3 = \int \vec{F}_{32} \cdot d\vec{r} + \int \vec{F}_{31} \cdot d\vec{r}$$

$$W_3 = W_{32} + W_{31}$$

So the total work is:

$$W_T = W_{21} + W_{31} + W_{32}$$

$$W_T = \frac{kq_1q_2}{r_{21}} + \frac{kq_1q_3}{r_{31}} + \frac{kq_2q_3}{r_{32}}$$

So I have done work to build this charge distribution.

Potential Energy is in the system.

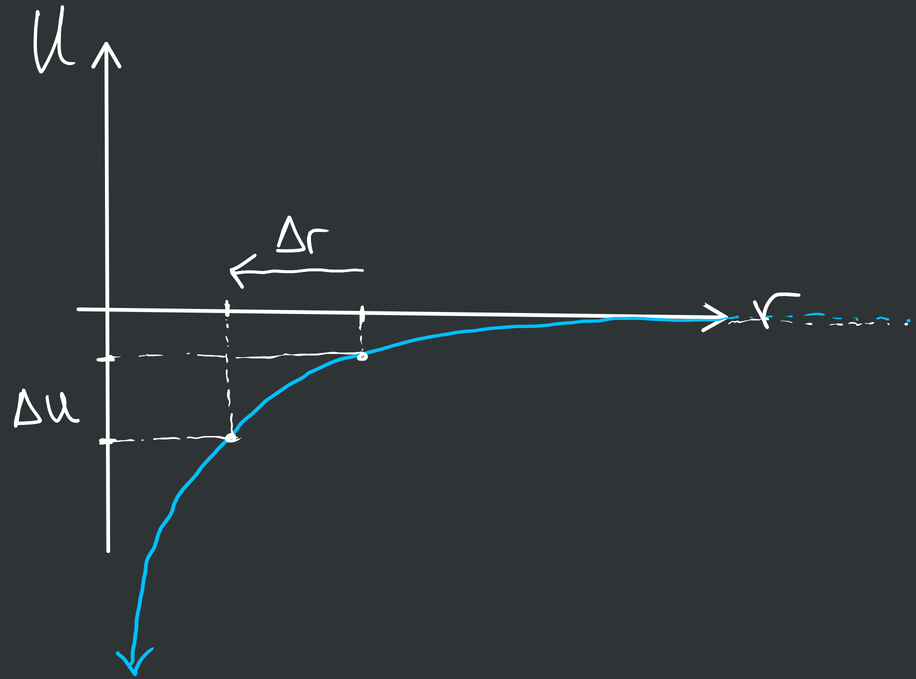
$U = W_{ext} = -W_{charges}$

independent of the order that the charges were assembled

Back to two charges:

$U = -k\dfrac{q_1 q_2}{r_{12}}$ ⟶ plot this ⟶

$U = 0,\ r_{12} \to \infty$



Force → Electric Field

[N] [N/C]

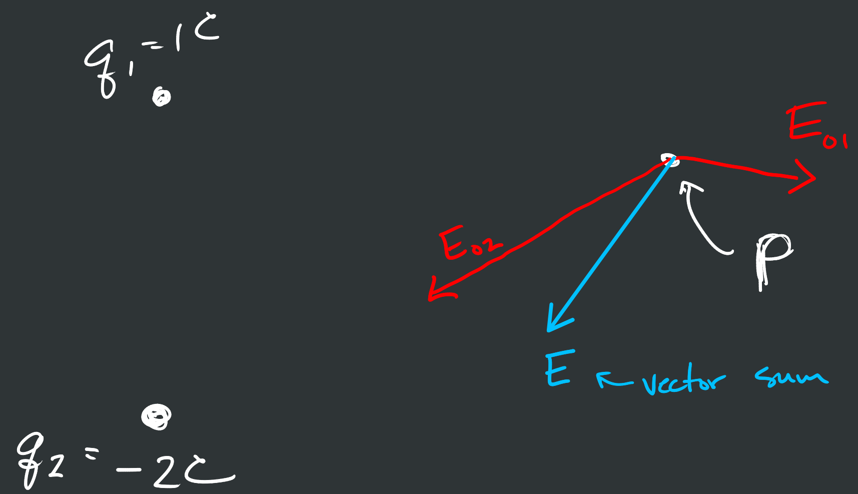
Electric Potential Energy → Electric Potential (Voltage)

[J] [J/C = Volt]

Electric Field → force per unit of charge

$$\vec{E} \equiv \frac{\vec{F}}{q_2} \Rightarrow \boxed{\vec{F} = q_2 \vec{E}}$$

$$\boxed{\vec{E}(x,y,z) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{N} \frac{q_j \hat{r}_{oj}}{r_{oj}^2}}$$



How do we generalize to a continuous distribution of charge?

charge density $\longrightarrow$ $dq = \rho\, dV$

$\hookrightarrow$ "rho" ρ

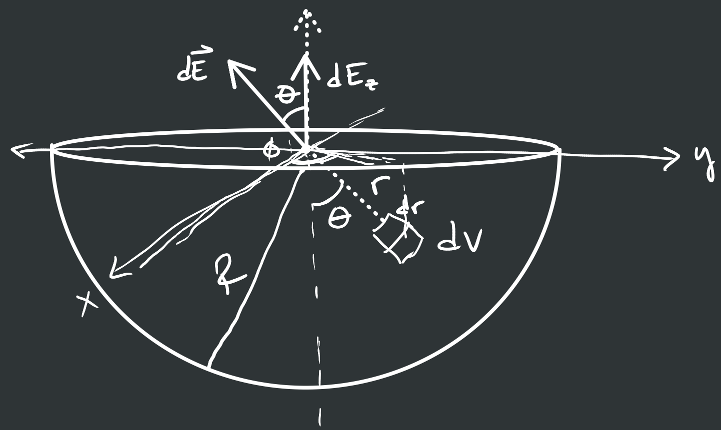
$$\vec{E}(x,y,z) = \frac{1}{4\pi\epsilon_0}\sum_{j=1}^{N} \frac{q_j \hat{r}_{o_j}}{r_{o_j}^2} \Longrightarrow \vec{E}(x,y,z) = \frac{1}{4\pi\epsilon_0}\int_Q \frac{dq}{r^2}\hat{r}$$

$$\Downarrow$$

$$\vec{E}(x,y,z) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\vec{r})}{r^2}\, dV\, \hat{r}$$

Example:

ρ is uniform



$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\, dV}{r^2}\hat{r} \rightarrow \vec{E} = \cancel{}^{\,0}\hat{x} + \cancel{}^{\,0}\hat{y} + \_\_\,\hat{z}$$

$\hookrightarrow$ simplify to a problem in the $\hat{z}$ direction!

$$dE_z = \frac{\rho\, dV \cos\theta}{4\pi\epsilon_0 r^2} = \frac{\rho \cos\theta}{4\pi\epsilon_0 \cancel{r^2}} \cdot \cancel{r^2} \sin\theta\, dr\, d\theta\, d\phi$$

$$E_z = \frac{\rho}{4\pi\epsilon_0} \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\int_{r=0}^{R} \cos\theta \sin\theta \, dr \, d\theta \, d\phi$$

$$E_z = \frac{\rho}{4\pi\epsilon_0} \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \, d\theta \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^{R} dr}_{R}$$

$$E_z = \frac{\rho R}{2\epsilon_0} \int_{\theta=0}^{\pi/2} \underbrace{\cos\theta \sin\theta}\, d\theta$$

$$\cos\theta \sin\theta = \frac{1}{2}\sin 2\theta$$

$$\frac{1}{2}\int_0^{\pi/2} \sin 2\theta \, d\theta$$

guess: $-2\cos 2\theta$

take derivative

$2\sin 2\theta \cdot 2$

$4\sin 2\theta$

$-\cos 2\theta$

$\downarrow$

$2\cos 2\theta$

$-\frac{1}{2}\cos 2\theta$

$\sin 2\theta$ ✓

$$E_z = \frac{\rho R}{2\epsilon_0}\left[\frac{1}{2}\left(-\frac{1}{2}\cos 2\theta\right)\right]_0^{\pi/2}$$

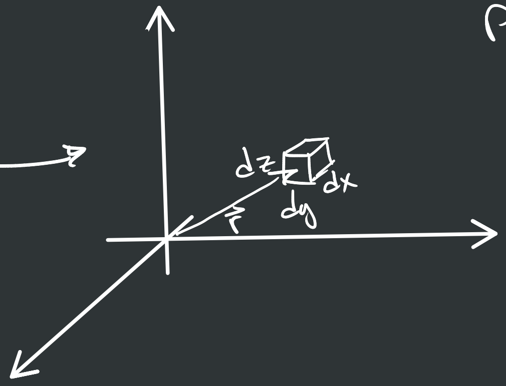
$\cos \pi = -1$

$\cos 0 = 1$

$$E_z = \frac{\rho R}{4\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\rho R}{4\epsilon_0}\hat{z}}$$

Side bar:

Cartesian →

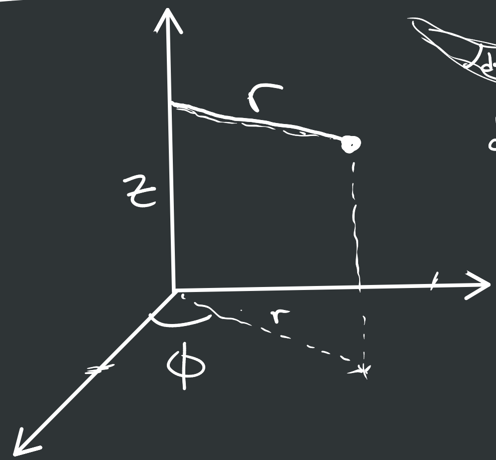


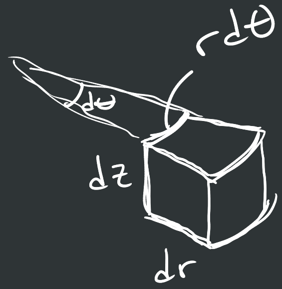
position vector $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$\boxed{dV = dx\,dy\,dz}$$

cylindrical

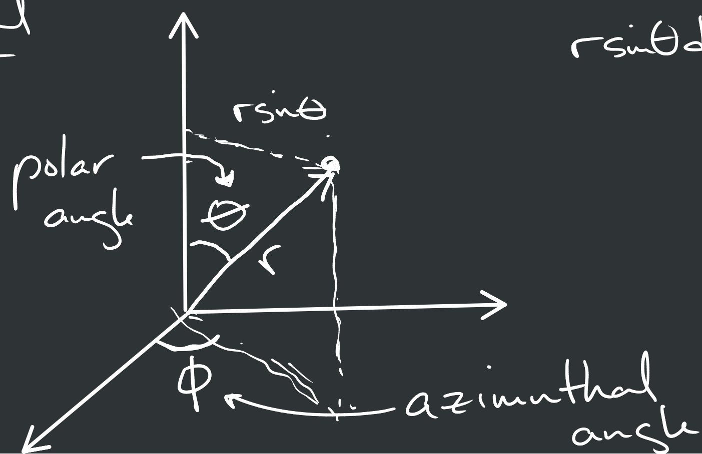


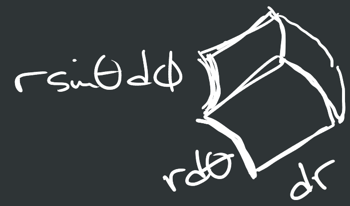


$$dV = dr\,(r\,d\theta)\,dz$$

$$\boxed{dV = r\,dr\,d\phi\,dz}$$

spherical





$$dV = dr\,(r\,d\theta)(r\sin\theta\,d\phi)$$

$$\boxed{dV = r^2\sin\theta\,dr\,d\theta\,d\phi}$$

HW: 35, 36, 40, 42, 49, 63a, 65, 77

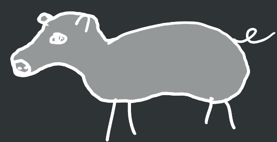
35| $m_e = 9 \cdot 10^{-31}\,\text{kg}$

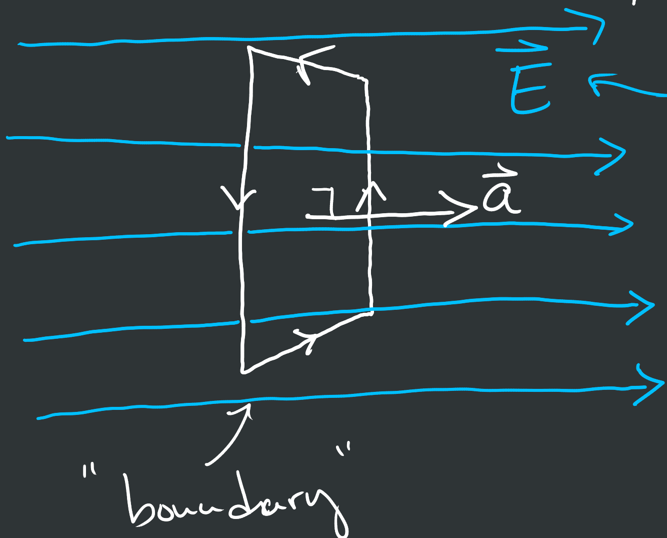
$F_g = m \cdot g$ ← weight

$F_g = 9 \cdot 10^{-31}\,\text{kg} \cdot 9.8\,\frac{\text{N}}{\text{kg}} =$

$$F = \frac{k q_1 q_2}{r^2}$$

Flux

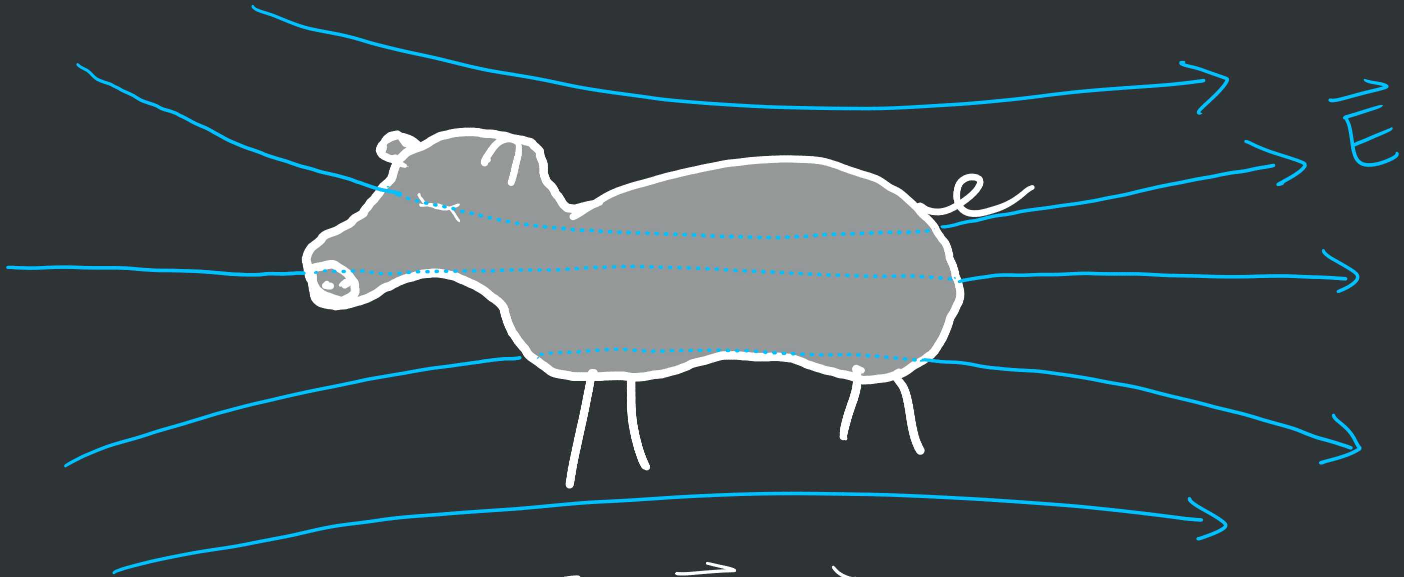




$\vec{E}$ ← vector field - function that outputs a vector

$$\Phi = \underbrace{\vec{E} \cdot \vec{a}}_{\text{dot product}}$$

flux



$$d\Phi = \vec{E} \cdot d\vec{a}$$

$$\Phi = \int_S \vec{E} \cdot d\vec{a}$$

surface integral

Gauss' Law

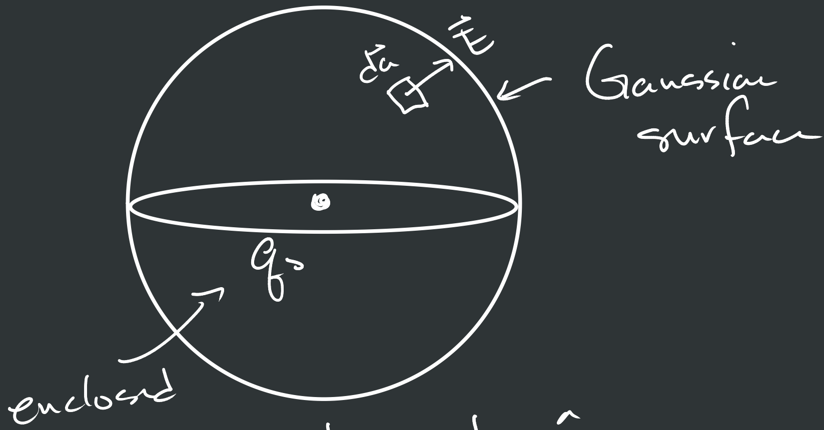
Net flux through a closed surface (like the pig) is proportional to the charge inside the surface.

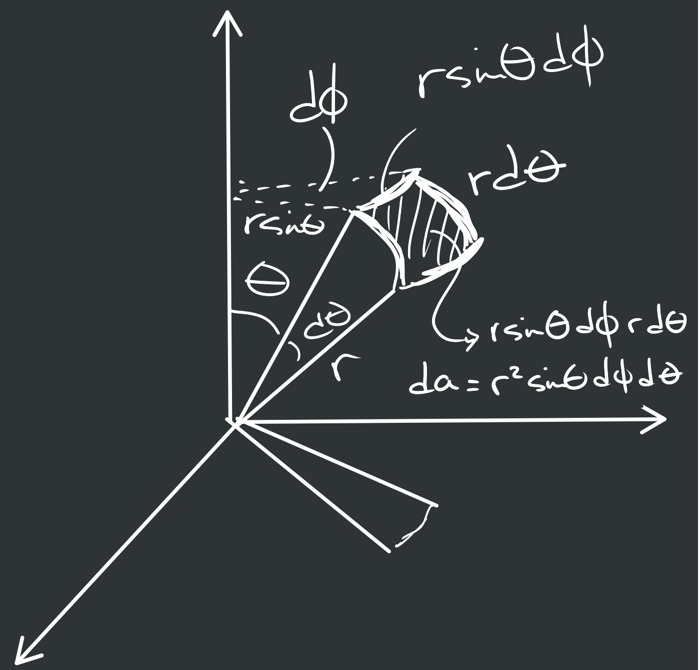
$$\Phi \propto q_{enclosed}$$

$$\boxed{\Phi = \frac{q_{enclosed}}{\epsilon_0}}$$

Lets check! w/ a point charge

$$\vec{E} = \frac{q_0}{4\pi\epsilon_0 r^2}\hat{r}$$ ← spherical coordinates w/ q_0 at the origin





$$d\vec{a} = da\,\hat{r}$$

$$\vec{E}\cdot d\vec{a} = \frac{q_0}{4\pi\epsilon_0 r^2}\hat{r} \cdot da\,\hat{r}$$

$$= \frac{q_0\, da}{4\pi\epsilon_0 r^2}$$

$$\Phi = \int_S \vec{E}\cdot d\vec{a}$$

$$\Phi = \int \frac{q_0\, da}{4\pi\epsilon_0 r^2}$$

$$\Phi = \frac{q_0}{4\pi\epsilon_0} \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \frac{\cancel{r^2}\sin\theta\, d\phi\, d\theta}{\cancel{r^2}}$$

$$\Phi = \frac{q_0}{4\pi\epsilon_0} \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \sin\theta\, d\phi\, d\theta$$

α
β
γ
δ
ϵ
η
θ ζ
κ
λ
μ
ν
o
π
ρ
σ
τ
u
ϕ
ψ
χ
ω

$$\Phi = \frac{q_0}{4\pi\epsilon_0} \underbrace{\int_{\theta=0}^{\pi} \sin\theta \, d\theta}_{-\cos\theta\Big|_0^{\pi}} \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{=2\pi}$$

$$-\cos\pi - -\cos 0$$

$$-(-1) - -(+1)$$

$$1 + 1 = 2$$

$$\Phi = \frac{q_0}{\cancel{4\pi}\epsilon_0} \cdot \cancel{2 \cdot 2\pi}$$

$$\boxed{\Phi = \frac{q_0}{\epsilon_0}}$$

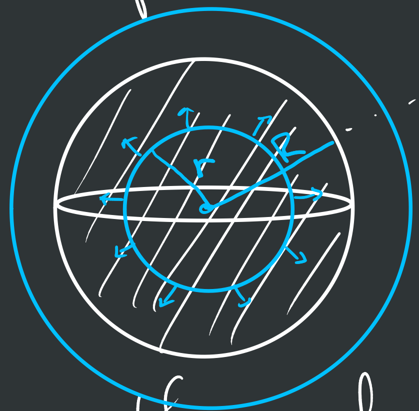
This is <u>not</u> limited to point charges

$$\underset{\text{charge enclosed by Gaussian surface}}{q_{encl}} = \int_{\text{enclosed volume}} \rho \, dV$$

$$\boxed{\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\int_V \rho\, dv}$$

Gauss's Law

Ex. we have a sphere of uniform charge density. What is the electric field inside + outside the sphere



outside the sphere: $r > R$

$$E \cdot 4\pi r^2 = \underbrace{\frac{\rho}{\epsilon_0}\int dV_{encl}}_{Q_{encl}\ =\ \rho V}$$

$$\underbrace{\int dV_{encl}}_{\frac{4}{3}\pi R^3}$$

$$E\cdot \cancel{4\pi} r^2 = \frac{\rho}{\epsilon_0}\cdot \frac{\cancel{4}}{3}\cancel{\pi} R^3$$

$$\boxed{E = \frac{\rho R^3}{3\epsilon_0 r^2}}$$

$$E\cdot 4\pi r^2 = \frac{Q_{encl}}{\epsilon_0}$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

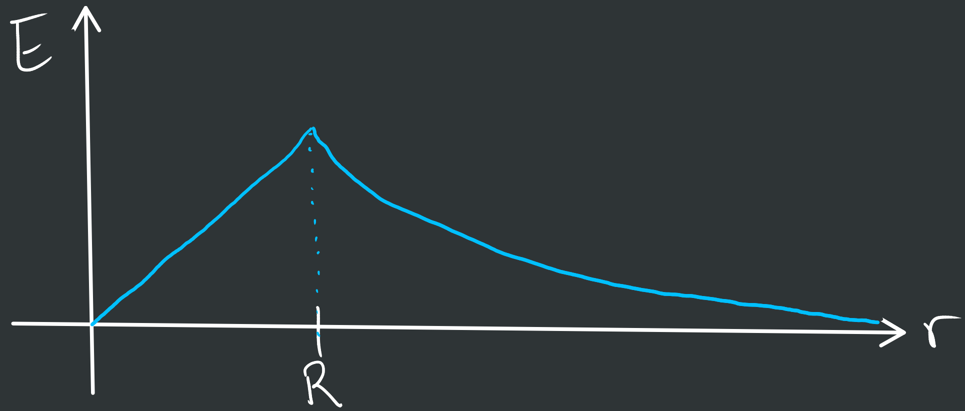
inside the sphere: $(r<R)$

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\int \rho\, dV_{encl}$$

$$E\cdot \cancel{4\pi}\cancel{r^2} = \frac{\rho}{\epsilon_0}\frac{\cancel{4}}{3}\cancel{\pi} r^{\cancel{3}}$$

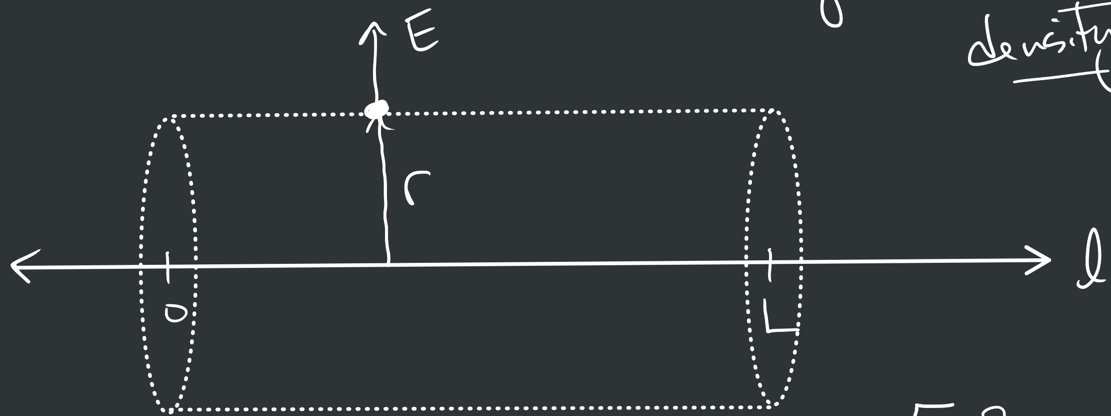
$$E = \frac{\rho r}{3\epsilon_0}$$

plot!



$$\rho = \frac{dq}{dV}$$

Ex: infinite line of charge

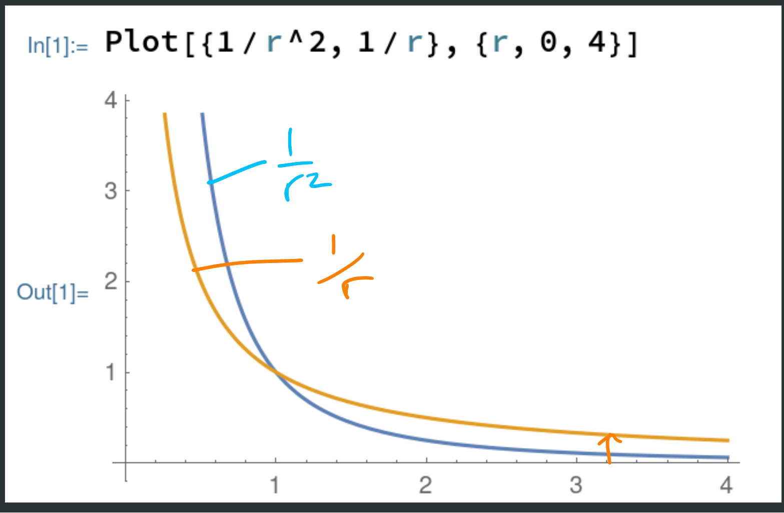


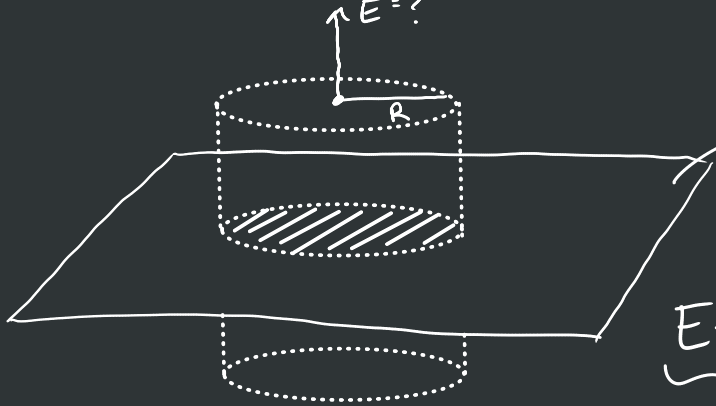
linear charge density $\rightarrow$ $\lambda = \dfrac{dq}{dl}$ $\qquad dq_{encl} = \lambda \cdot dl$

$$\oint_s \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int_0^L \lambda \cdot dl_{encl}$$

$$\underbrace{E \cdot 2\pi r \cdot \cancel{L}}_{\text{flux through sides assuming } E \text{ is } \perp \text{ to the sides}} + \underbrace{0}_{\text{ends}} = \frac{\lambda \cancel{L}}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$





area charge density

σ

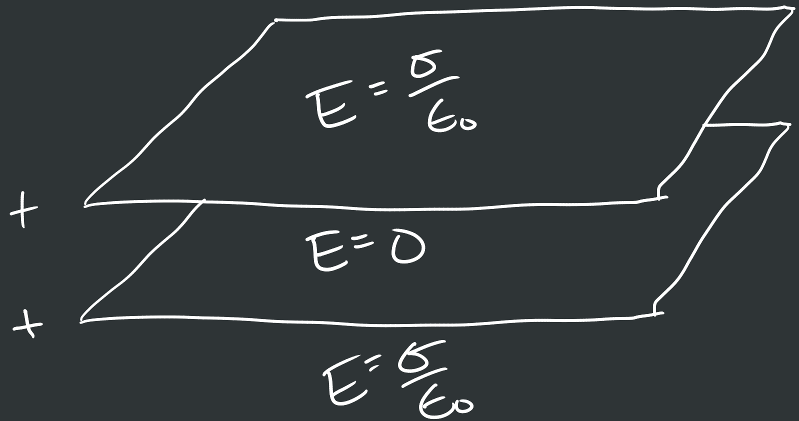
$$\int \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\int \sigma \, da_{encl}$$

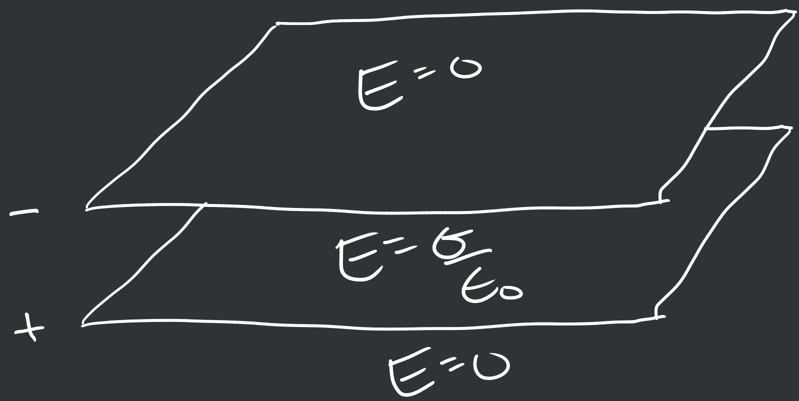
$$\underbrace{E\cdot\pi R^2}_{\text{bottom}} + \underbrace{E\cdot\pi R^2}_{\text{top}} + \underbrace{0}_{\text{sides}} = \frac{\sigma}{\epsilon_0}\pi R^2$$

$$E\cdot 2\cancel{\pi R^2} = \frac{\sigma}{\epsilon_0}\cancel{\pi R^2}$$

$$\boxed{E = \frac{\sigma}{2\epsilon_0}}$$

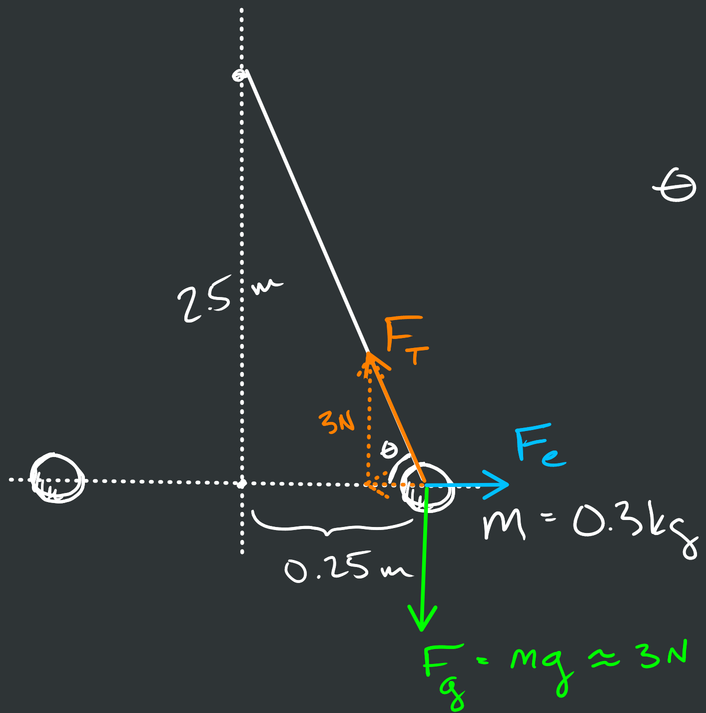
independent of r!





Homework

36)



$$\theta = \tan^{-1}\left(\frac{2.5}{0.25}\right)$$

$$= 84 \text{ ish}^\circ$$

$$\tan 84\text{ish} = \frac{3N}{F_{Tx}}$$

$$F_{Tx} = \frac{3N}{\tan 84^\circ}$$

$$F_{Tx} = 0.32\,N$$

$$F_e = \frac{kq^2}{r^2}$$

$$0.32N = \frac{kq^2}{(0.5\,m)^2}$$

$$q = \sqrt{\frac{0.32N\,(0.5)^2}{9\cdot 10^9}}$$

$$q = 3\cdot 10^{-6}\,C$$

$$\underline{\underline{3\mu C}}$$

40)

$$-\frac{\cancel{ke^2}}{r_1} - \frac{\cancel{ke^2}}{r_2} + \frac{\cancel{ke^2}}{r_2 - r_1} = 0$$

$$-\frac{1}{r_1}-\frac{1}{r_2}+\frac{1}{r_2-r_1}=0$$

Fibonacci numbers

1, 1, 2, 3, 5, 8, 13 .....

$$\frac{1}{r_2-r_1}=\frac{1}{r_1}\left(\frac{r_2}{r_2}\right)+\frac{1}{r_2}\left(\frac{r_1}{r_1}\right)$$

$$\frac{1}{r_2-r_1}=\frac{r_2}{r_1r_2}+\frac{r_1}{r_1r_2}=\frac{r_2+r_1}{r_1r_2}$$

$$\frac{1}{r_2-r_1}=\frac{r_2+r_1}{r_1r_2}$$

$$r_1r_2=(r_2+r_1)(r_2-r_1)$$

$$\frac{r_1r_2}{r_1^2}=\frac{r_2^2-r_1^2}{r_1^2}$$

$$\frac{r_2}{r_1}=\frac{r_2^2}{r_1^2}-1$$

$$x=x^2-1$$

$$0=x^2-x-1$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{1\pm\sqrt{1-4(1)(-1)}}{2}$$

$$=\frac{1\pm\sqrt{5}}{2}$$

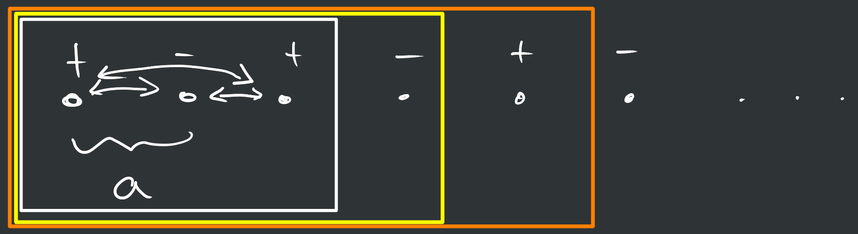
$\rightarrow$ 1.618....

$\rightarrow$ −0.618....

$$\frac{r_2}{r_1}=1.618\ ✓$$

$$\cancel{\frac{r_2}{r_1}=-0.618?}$$

42



HINT: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

$$-\frac{ke^2}{a} + \frac{ke^2}{2a} - \frac{ke^2}{a}$$

$$\boxed{-\frac{2ke^2}{a} + \frac{ke^2}{2a} - \frac{ke^2}{3a} + \frac{ke^2}{2a} - \frac{ke^2}{a}}$$

$$-\frac{3ke^2}{a} + \frac{2ke^2}{2a} - \frac{ke^2}{3a} + \frac{ke^2}{4a} - \frac{ke^2}{3a} + \frac{ke^2}{2a} - \frac{ke^2}{a}$$

$$-\frac{4ke^2}{a} + \frac{3ke^2}{2a} - \frac{2ke^2}{3a} + \frac{ke^2}{4a}$$

$$\frac{U}{N} = -\frac{ke^2}{Na}\left(\frac{N-1}{1} - \frac{N-2}{2} + \frac{N-3}{3} - \frac{N-4}{4} + \dots\right)$$

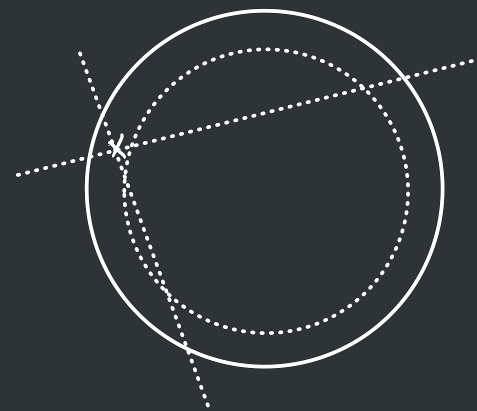
$$= -\frac{ke^2}{Na}\left(\frac{N}{1} - \cancel{\frac{1}{1}} - \frac{N}{2} + \cancel{\frac{2}{2}} + \frac{N}{3} - \cancel{\frac{3}{3}} - \frac{N}{4} + \cancel{\frac{4}{4}} + \dots\right)$$

$$= -\frac{ke^2}{a}\underbrace{\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right)}_{\ln(1+1) = \ln(2)}$$

HINT: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

$$\frac{U}{N} = -\frac{ke^2}{a}\ln(2)$$

63a) q, m



$$E_{outside} = \frac{k\sigma A}{r^2}$$

$$E_{inside} = 0$$

$$\int E \cdot da = \frac{q_{encl}}{\epsilon_0}$$

$$F_{NET} = ma$$

$$-\frac{k\sigma A q}{r^2} = ma$$

$$-\frac{k\sigma A q}{mr^2} = a$$

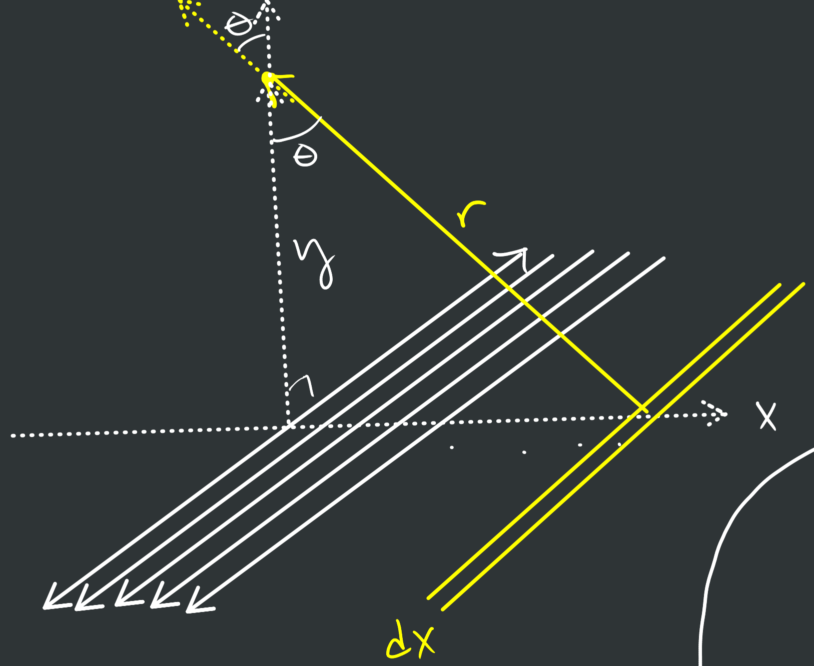
$$a = \frac{dv}{dt}$$

$$a = \frac{dv}{dr}\underbrace{\frac{dr}{dt}}_{v}$$

$$\frac{a}{v} = \frac{dv}{dr}$$

$$a\,dr = v\,dv$$

65



$$\sigma = \frac{Q}{A} = \frac{Q}{l \cdot w} = \frac{\lambda}{W} \rightarrow dx$$

$$\sigma \cdot dx = \lambda$$

$$dE = \frac{\lambda}{2\pi\varepsilon_0 r}\cos\theta$$

$$dE = \frac{\sigma\, dx}{2\pi\varepsilon_0 r}\cos\theta$$

$$r = \sqrt{x^2+y^2}$$

$$\cos\theta = \frac{y}{r}$$

$$\cos\theta = \frac{y}{\sqrt{x^2+y^2}}$$

$$\int_{\infty}^{R} a\,dr = \int_{\infty}^{R} v\,dv$$

$$= \frac{1}{2}v^2\Big|_{\infty}^{R}$$

$$= \frac{1}{2}v_R^2$$

$\boxed{77}$