

Electrostatics

- what forces are on charges?
- what gives rise to those forces?
- how much energy can be stored / released?

Charge \rightarrow conservation of charge \rightarrow in an isolated system the total electric charge never changes

\rightarrow quantization of charge

charge of an electron/proton is the smallest isolated charge $\rightarrow e$ (elementary charge)

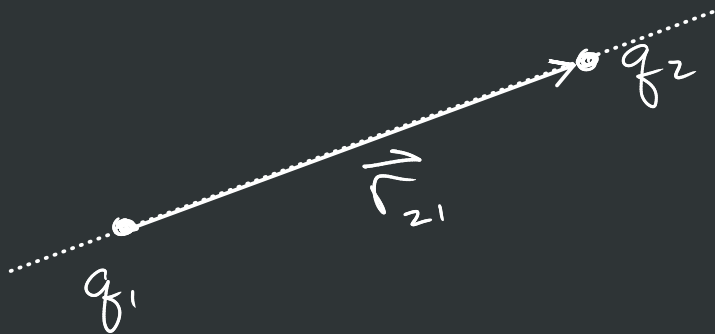
q is variable for charge

SI unit for charge is Coulomb

$$|e| = 1.602 \dots \times 10^{-19} \text{ C}$$

$$1 \text{ C} = \frac{1}{1.602 \cdot 10^{-19}} e = 6.242 \cdot 10^{18} e$$

So now to electrostatics:



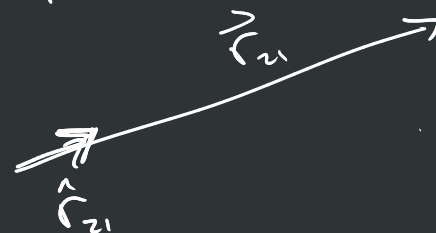
The force between $q_1 + q_2$ is directly proportional to the product of charge and inversely proportional to the square of the distance between them.

$$\vec{F}_{21} = k \frac{q_1 q_2}{r_{21}^2} \hat{r}_{21} \quad \left\{ \begin{array}{l} \text{product of charges} \\ \text{separation distance} \end{array} \right.$$

constant of proportionality

unit vector

$$\hat{r}_{21} = \frac{\vec{r}_{21}}{|\vec{r}_{21}|}$$



$$\text{Ex. } \vec{r}_{21} = \langle 3, 1 \rangle$$

$$\vec{r}_{21} = 3\hat{x} + 1\hat{y}$$

$$|\vec{r}_{21}| = \sqrt{3^2 + 1^2}$$

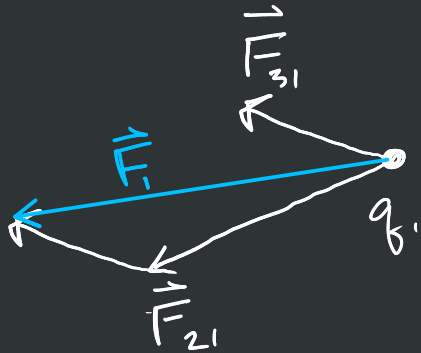
$$\hat{r}_{21} = \frac{3\hat{x} + 1\hat{y}}{\sqrt{10}}$$

SI system

$$k = 8.988 \cdot 10^9 \frac{\text{Nm}^2}{\text{C}^2}$$

$$k = \frac{1}{4\pi\epsilon_0} \rightarrow \epsilon_0 = 8.85 \cdot 10^{-12} \frac{\text{C}^2}{\text{Nm}^2}$$

What about more than two charges?



q_2

q_3

$$\vec{F}_1 = \vec{F}_{21} + \vec{F}_{31}$$

How do we do work?

$$W = \int \vec{F}_{\text{force}} \cdot \vec{\text{displacement}}$$

$$W = - \int \vec{F} \cdot d\vec{r}$$

dot product means relative directions of force and displacement matter



$$W = \int_{\infty}^{r_{12}} -k \frac{q_1 q_2}{r^2} \hat{r} \cdot d\vec{r}$$

work I do

where force is zero

$\hat{r} \cdot d\vec{r}$

I have to push against the Coulomb force (equal & opposite)

I do this in the $-\hat{r}$ direction

$$W = -kq_1q_2 \int_{\infty}^{r_{12}} \frac{1}{r^2} \hat{r} \cdot \hat{r} dr \quad \hat{r} \cdot \hat{r} = 1$$

$$W = -kq_1q_2 \int_{\infty}^{r_{12}} r^{-2} dr$$

$$W = -kq_1q_2 (-1) r^{-1} \Big|_{\infty}^{r_{12}}$$

$$W = \frac{kq_1q_2}{r} \Big|_{\infty}^{r_{12}}$$

$$= \frac{kq_1q_2}{r_{12}} - \cancel{\frac{kq_1q_2}{\infty}} \rightarrow 0$$

$$\boxed{W_{ME} = \frac{kq_1q_2}{r_{12}}} \text{ from } \infty \text{ far away}$$

Now, is this electric force conservative?

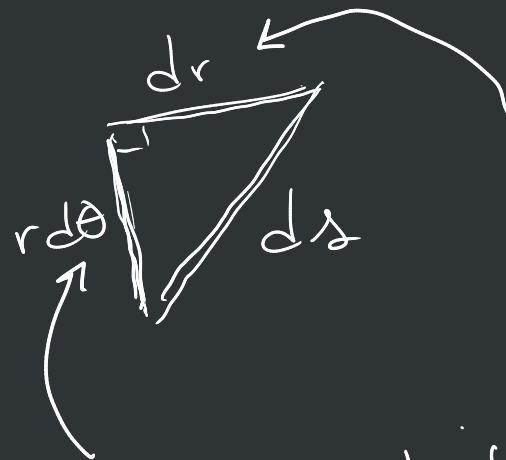
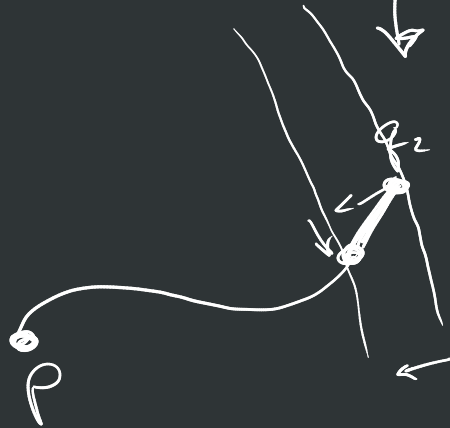
↳ what does this mean

→ path independent

↳ only depends
on position

→ $\underbrace{\vec{\nabla} \times \vec{F}}_{\text{curl}} = 0$ ← save for later

q_1



back and forth
are the same
amount of
work

only the radial
part of the motion
contributes to the work
so only radial position matters

this is perpendicular
to the force so the
dot product would be zero

How do we handle multiple charges

q_1 q_3

$$\vec{F}_3 = \vec{F}_{31} + \vec{F}_{32}$$

q_2

$$W_3 = \int \vec{F}_3 \cdot d\vec{r}$$

$$W_3 = \int (\vec{F}_{32} + \vec{F}_{31}) \cdot d\vec{r}$$

$$W_3 = \int \vec{F}_{32} \cdot d\vec{r} + \int \vec{F}_{31} \cdot d\vec{r}$$

So the total work is:

$$W_T = W_{21} + W_{31} + W_{32}$$

$$W_T = \frac{kq_1q_2}{r_{21}} + \frac{kq_1q_3}{r_{31}} + \frac{kq_2q_3}{r_{32}}$$

So I have done work to build this charge distribution.

$$W_3 = W_{32} + W_{31}$$

Potential Energy is in the system.

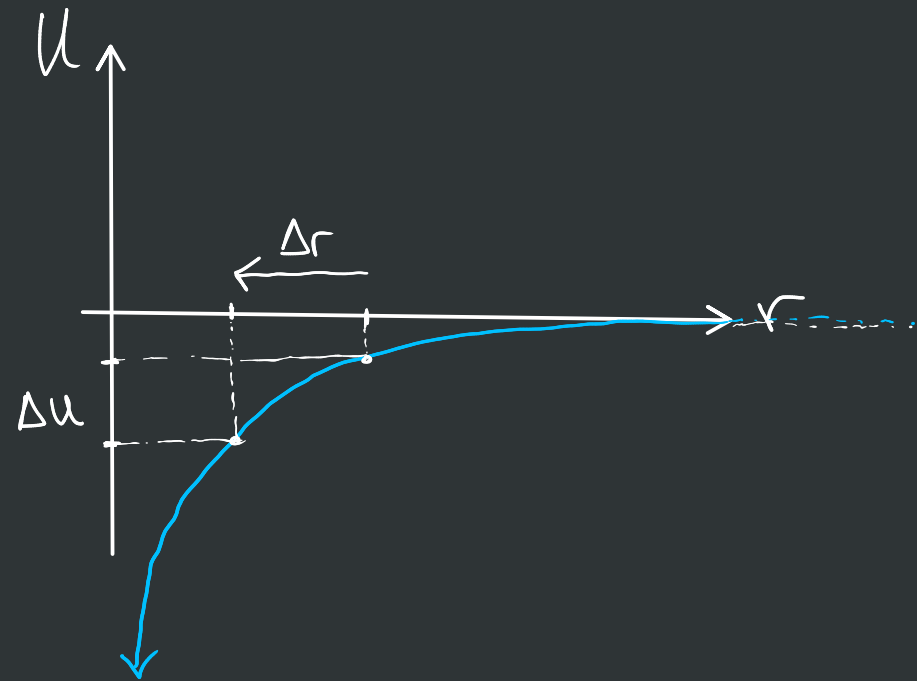
→ $U = W_{\text{ext}} = -W_{\text{charges}}$

→ independent of the order that the charges were assembled

Back to two charges:

$U = -k \frac{q_1 q_2}{r_{12}}$ → plot this →

$U = 0, r_{12} \rightarrow \infty$



$$\underline{\text{Force}} \rightarrow \underline{\text{Electric Field}}$$

$$[N] \quad [N/C]$$

$$\underline{\text{Electric Potential Energy}} \rightarrow \underline{\text{Electric Potential}} \text{ (Voltage)}$$

$$[J] \quad [J/C = \text{Volt}]$$

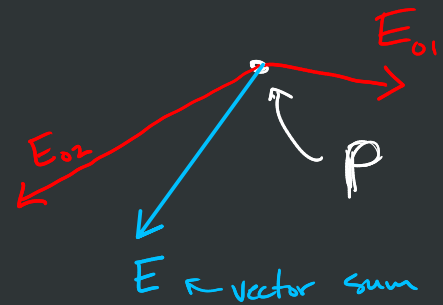
Electric Field \rightarrow force per unit of charge

$$\vec{E} \equiv \frac{\vec{F}}{q_2} \Rightarrow \boxed{\vec{F} = q_2 \vec{E}}$$

$$\boxed{\vec{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^N \frac{q_j \hat{r}_{oj}}{r_{oj}^2}}$$

$$q_1 = 1C$$

$$q_2 = -2C$$



How do we generalize to a continuous distribution of charge?

charge density $\rightarrow dq = \rho dv$
"rho" $\rightarrow \rho$

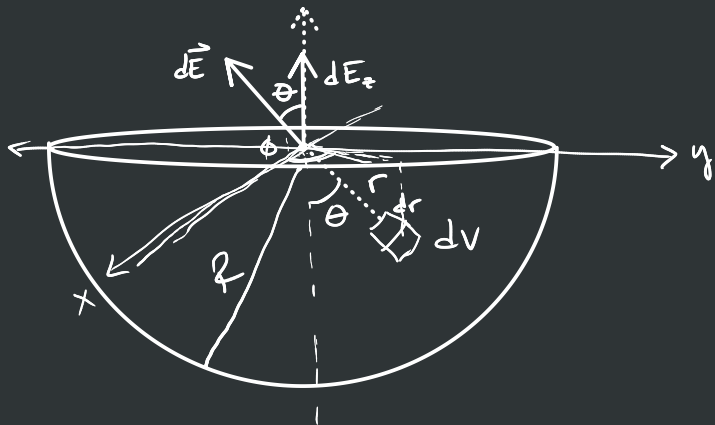
$$\vec{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^N \frac{q_j \hat{r}_{oj}}{r_{oj}^2} \Rightarrow \vec{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \int_Q \frac{dq}{r^2} \hat{r}$$



$$\vec{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{r})}{r^2} dv \hat{r}$$

Example:

ρ is uniform



$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho dv}{r^2} \hat{r} \rightarrow \vec{E} = \cancel{\hat{x}} + \cancel{\hat{y}} + \hat{z}$$

\uparrow simplify to a problem in the \hat{z} direction!

$$dE_z = \frac{\rho dv \cos\theta}{4\pi\epsilon_0 r^2} = \frac{\rho \cos\theta}{4\pi\epsilon_0} \cancel{r^2} \sin\theta dr d\theta d\phi$$

$$E_z = \frac{\rho}{4\pi\epsilon_0} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \int_{r=0}^R \cos\theta \sin\theta \, dr \, d\theta \, d\phi$$

$$E_z = \frac{\rho}{4\pi\epsilon_0} \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \, d\theta \underbrace{\int_0^{2\pi} d\phi}_{2\pi} \underbrace{\int_0^R dr}_R$$

$$E_z = \frac{\rho R}{2\epsilon_0} \int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$\searrow \cos\theta \sin\theta = \frac{1}{2} \sin 2\theta$

$$\frac{1}{2} \int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$E_z = \frac{\rho R}{2\epsilon_0} \left[\frac{1}{2} \left(-\frac{1}{2} \cos 2\theta \right) \right]_0^{\pi/2}$$

$$\begin{array}{|l} \cos \pi = -1 \\ \cos 0 = 1 \end{array}$$

$$E_z = \frac{\rho R}{4\epsilon_0} \Rightarrow \boxed{\vec{E} = \frac{\rho R}{4\epsilon_0} \hat{z}}$$

guess: $-2\cos 2\theta$

↓ take derivative

$$\begin{array}{l} 2\sin 2\theta \cdot 2 \\ 4\sin 2\theta \end{array}$$

$$-\frac{1}{2} \cos 2\theta$$

$\sin 2\theta \checkmark$

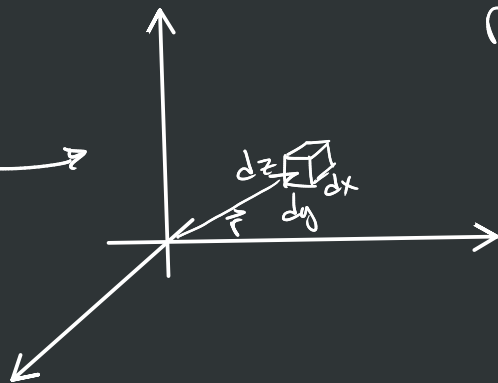
$$-\cos 2\theta$$

↓

$$2\cos 2\theta$$

Side bar:

Cartesian →

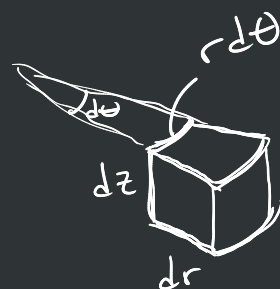
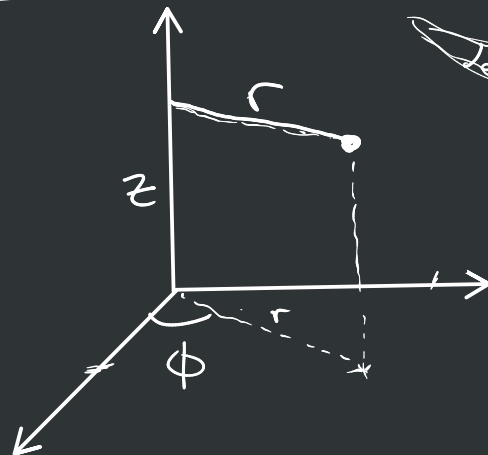


Position vector $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

$$\boxed{dV = dx dy dz}$$

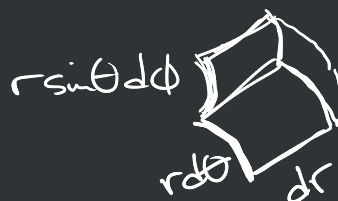
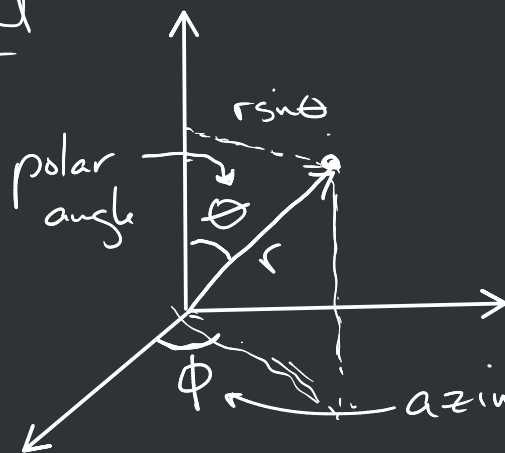
Cylindrical



$$dV = dr (r d\theta) dz$$

$$\boxed{dV = r dr d\phi dz}$$

Spherical



$$dV = dr (r d\theta) (r \sin\theta d\phi)$$

$$\boxed{dV = r^2 \sin\theta dr d\theta d\phi}$$

HW: 35, 36, 40, 42, 49, 63a, 65, 77

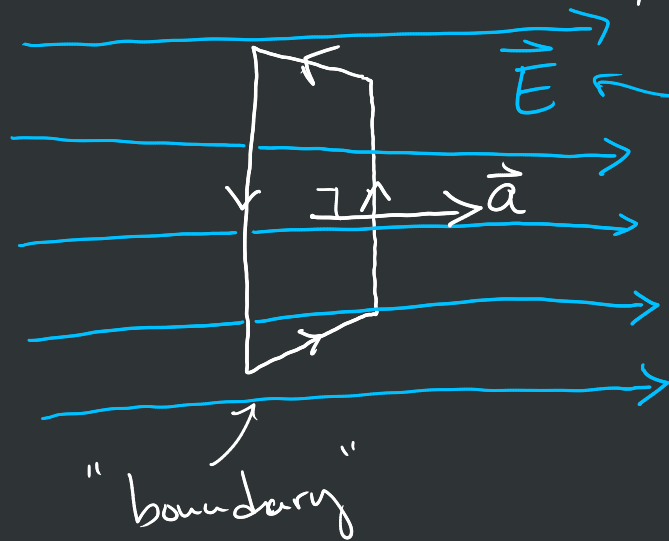
35 | $m_e = 9 \cdot 10^{-31} \text{ kg}$

$$F_g = m \cdot g \leftarrow \text{weight}$$

$$F_g = 9 \cdot 10^{-31} \text{ kg} \cdot 9.8 \frac{\text{N}}{\text{kg}} =$$

$$F = k \frac{q_1 q_2}{r^2}$$

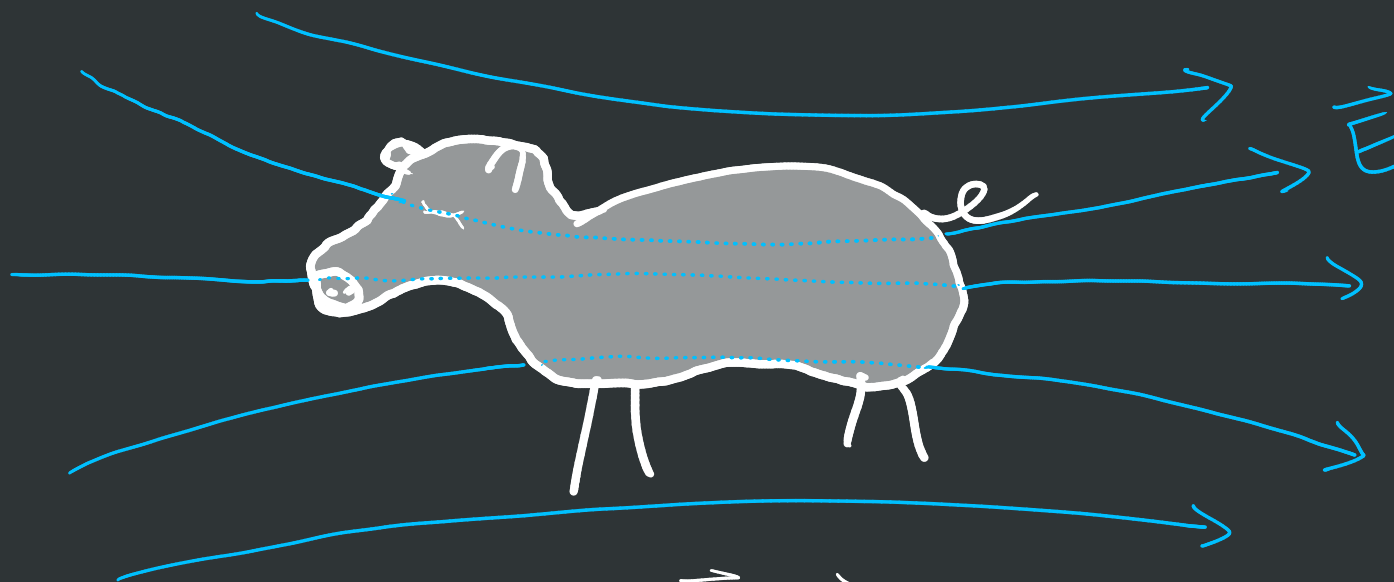
Flux



vector field - function that outputs a vector

$$\Phi = \underbrace{\vec{E} \cdot \vec{a}}_{\text{dot product}}$$

flux



$$d\Phi = \vec{E} \cdot d\vec{a}$$

$$\Phi = \int_S \vec{E} \cdot d\vec{a}$$

↖ surface integral

Gauss' Law

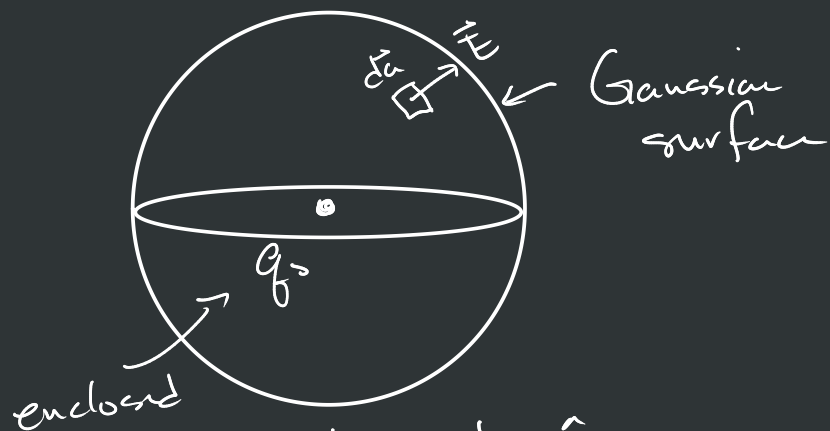
Net flux through a closed surface (like the pig) is proportional to the charge inside the surface.

$$\Phi \propto q_{\text{enclosed}}$$

$$\left| \Phi = \frac{q_{\text{enclosed}}}{\epsilon_0} \right|$$

Lets check! w/ a point charge

$$\vec{E} = \frac{q_0}{4\pi\epsilon_0 r^2} \hat{r} \quad \leftarrow \text{spherical coordinates w/ } q_0 \text{ at the origin}$$



$$d\vec{a} = da \hat{r}$$

$$\vec{E} \cdot d\vec{a} = \frac{q_0}{4\pi\epsilon_0 r^2} \hat{r} \cdot da \hat{r}$$

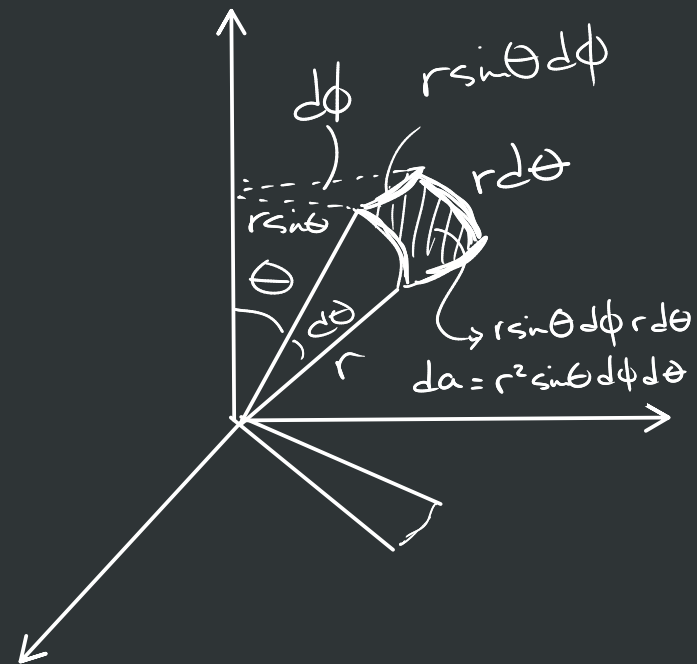
$$= \frac{q_0 da}{4\pi\epsilon_0 r^2}$$

$$\Phi = \int_S \vec{E} \cdot d\vec{a}$$

$$\Phi = \int \frac{q_0 \cancel{da}}{4\pi\epsilon_0 r^2}$$

$$\Phi = \frac{q_0}{4\pi\epsilon_0} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\cancel{r^2} \sin\theta d\phi d\theta}{\cancel{r^2}}$$

$$\Phi = \frac{q_0}{4\pi\epsilon_0} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \sin\theta d\phi d\theta$$



α
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$$\Phi = \frac{q_0}{4\pi\epsilon_0} \int_{\theta=0}^{\pi} \sin\theta d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$\underbrace{\int_{\theta=0}^{\pi} \sin\theta d\theta}_{-\cos\theta \Big|_0^{\pi} = 2} \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{=2\pi}$$

$$-\cos\pi - -\cos 0$$

$$-(-1) - -(+1)$$

$$1 + 1 = 2$$

$$\Phi = \frac{q_0}{4\pi\epsilon_0} \cdot 2 \cdot 2\pi$$

$$\boxed{\Phi = \frac{q_0}{\epsilon_0}}$$

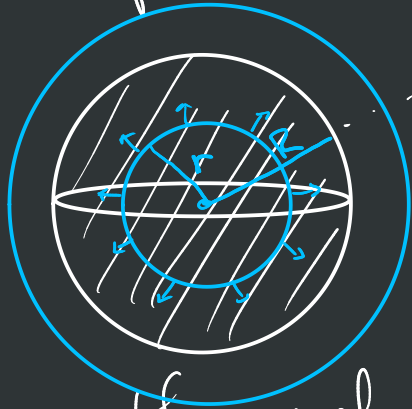
This is not limited to point charges

charge enclosed by Gaussian surface $\rightarrow q_{\text{encl}} = \int \rho dv$

enclosed volume

$$\left| \int_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int_V \rho dv \right| \quad \underline{\text{Gauss's Law}}$$

Ex. we have a sphere of uniform charge density
what is the electric field inside + outside the sphere



inside the sphere: ($r < R$)

$$\int_S \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int_V \rho dv_{\text{encl}}$$

$$E \cdot \cancel{4\pi r^2} = \frac{\rho}{\epsilon_0} \cdot \cancel{\frac{4}{3}\pi} r^3$$

$$E = \frac{\rho r}{3\epsilon_0}$$

$$E \cdot \cancel{4\pi} r^2 = \frac{\rho}{\epsilon_0} \cdot \cancel{\frac{4}{3}\pi} R^3$$

$$E = \frac{\rho R^3}{3\epsilon_0 r^2}$$

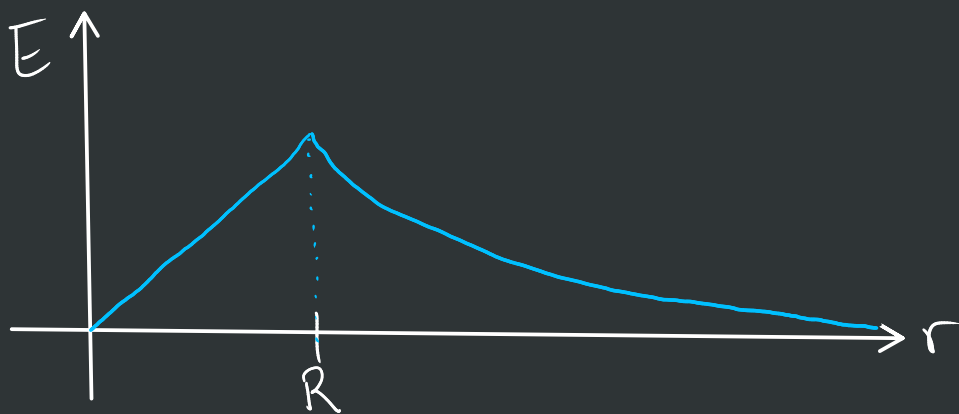
outside the sphere: ($r > R$)

$$E \cdot 4\pi r^2 = \frac{\rho}{\epsilon_0} \underbrace{\int_V dv_{\text{encl}}}_{\frac{4}{3}\pi R^3} \quad \text{--- } Q_{\text{encl}} = \rho V$$

$$E \cdot 4\pi r^2 = \frac{Q_{\text{encl}}}{\epsilon_0}$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2}$$

plot!

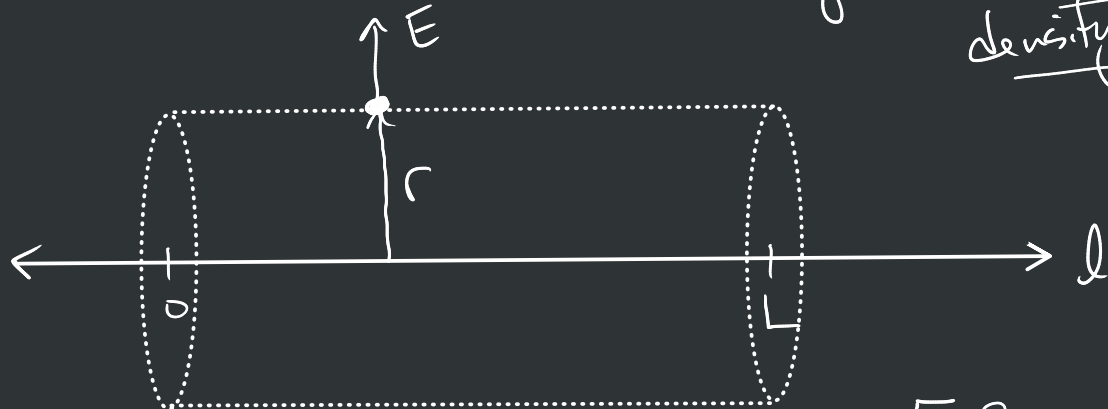


$$\rho = \frac{dq}{dV}$$

Ex: infinite line of charge

linear
charge
density \rightarrow

$$\lambda = \frac{dq}{dl} \quad dq_{\text{enc}} = \lambda \cdot dl$$

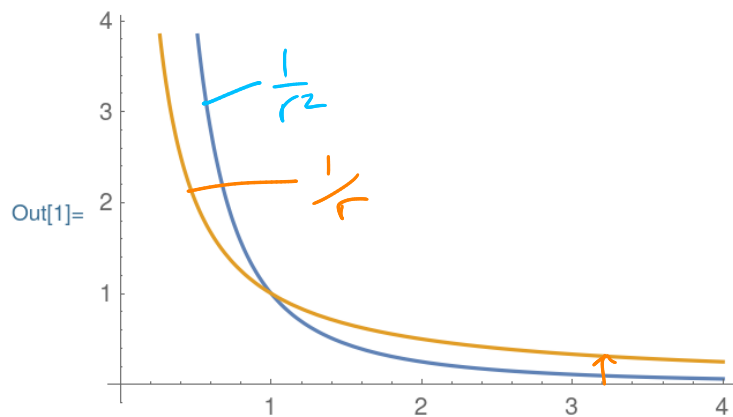


$$\int_s \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int_0^L \lambda \cdot dl_{\text{enc}}$$

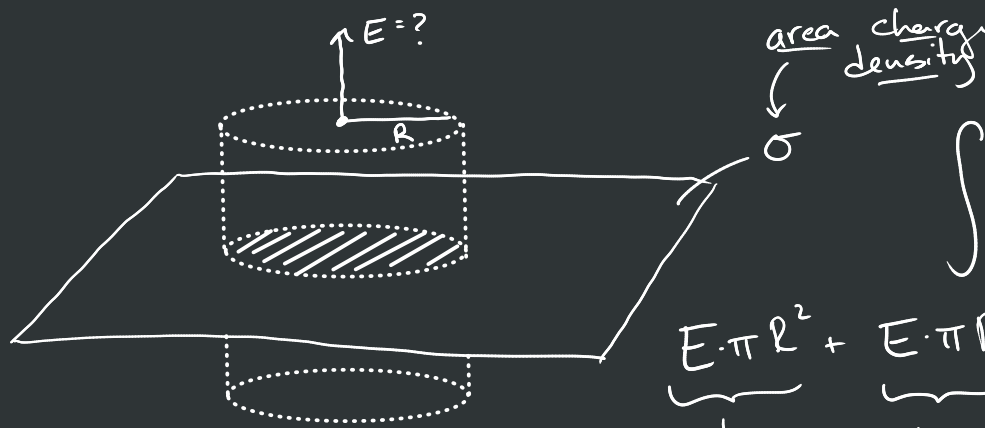
$$\underbrace{E \cdot 2\pi r \cdot L}_{\text{flux through sides assuming } E \text{ is } \perp \text{ to the sides}} + \underbrace{0}_{\text{ends}} = \frac{\lambda L}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

In[1]:= Plot[{1/r^2, 1/r}, {r, 0, 4}]



flux through sides
assuming E is
 \perp to the sides



area charge density

σ

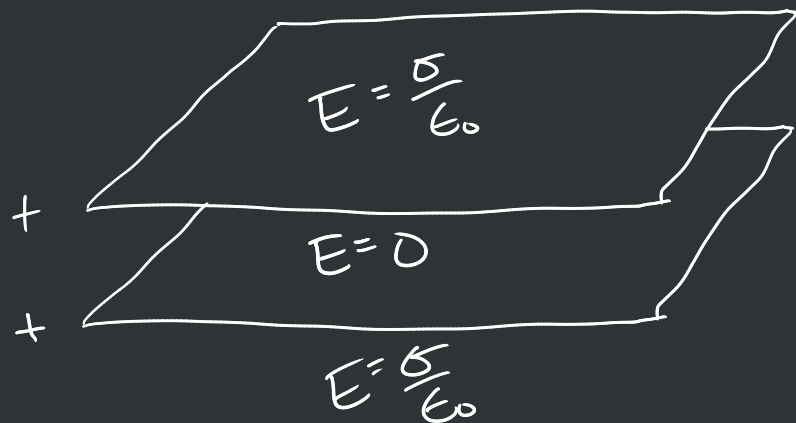
$$\int \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int \sigma da_{\text{encl}}$$

$$\underbrace{E \cdot \pi R^2}_{\text{bottom}} + \underbrace{E \cdot \pi R^2}_{\text{top}} + \underbrace{0}_{\text{sides}} = \frac{\sigma}{\epsilon_0} \pi R^2$$

$$E \cdot 2\pi R^2 = \frac{\sigma \pi R^2}{\epsilon_0}$$

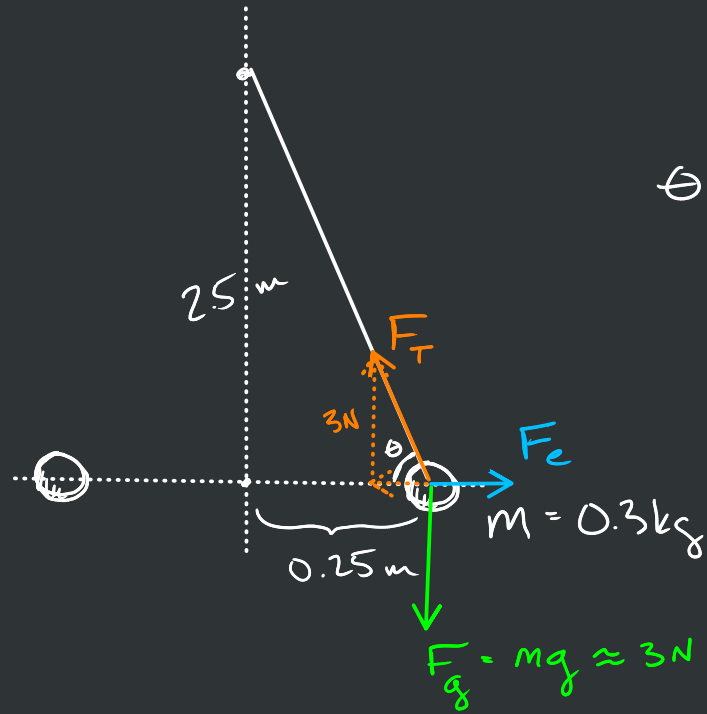
$$E = \frac{\sigma}{2\epsilon_0}$$

← independent of r !



Homework

36



$$\theta = \tan^{-1}\left(\frac{2.5}{0.25}\right)$$

$$= 84.1^\circ$$

$$\tan 84.1^\circ = \frac{3 \text{ N}}{F_{Tx}}$$

$$F_{Tx} = \frac{3 \text{ N}}{\tan 84^\circ}$$

$$F_{Tx} = 0.32 \text{ N}$$

$$F_e = \frac{kq^2}{r^2}$$

$$0.32 \text{ N} = \frac{kq^2}{(0.5 \text{ m})^2}$$

$$q = \sqrt{\frac{0.32 \text{ N} (0.5)^2}{9 \cdot 10^9}}$$

$$q = 3 \cdot 10^{-6} \text{ C}$$

$$\underline{\underline{3 \mu\text{C}}}$$

40



$$\frac{-\cancel{k}e^2}{r_1} - \frac{\cancel{k}e^2}{r_2} + \frac{\cancel{k}e^2}{r_2 - r_1} = 0$$

$$-\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_2 - r_1} = 0$$

Fibonacci numbers
1, 1, 2, 3, 5, 8, 13, ...

$$\frac{1}{r_2 - r_1} = \frac{1}{r_1 \left(\frac{r_2}{r_1} \right)} + \frac{1}{r_2 \left(\frac{r_1}{r_2} \right)}$$

$$\frac{1}{r_2 - r_1} = \frac{r_2}{r_1 r_2} + \frac{r_1}{r_1 r_2} = \frac{r_2 + r_1}{r_1 r_2}$$

$$\frac{1}{r_2 - r_1} = \frac{r_2 + r_1}{r_1 r_2}$$

$$r_1 r_2 = (r_2 + r_1)(r_2 - r_1)$$

$$\frac{r_1 r_2}{r_1^2} = \frac{r_2^2 - r_1^2}{r_1^2}$$

$$\frac{r_2}{r_1} = \frac{r_2^2}{r_1^2} - 1$$

$$x = x^2 - 1$$

$$0 = x^2 - x - 1$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

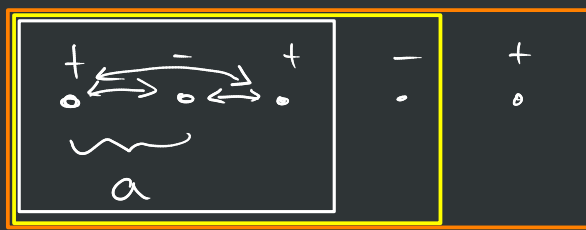
$$= \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2} \rightarrow \begin{matrix} 1.618... \\ -0.618... \end{matrix}$$

$$\frac{r_2}{r_1} = 1.618 \checkmark$$

$$\frac{r_2}{r_1} = -0.618?$$

42



HINT: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

$$-\frac{ke^2}{a} + \frac{ke^2}{2a} - \frac{ke^2}{a}$$

$$-\frac{2ke^2}{a} + \frac{ke^2}{2a} - \frac{ke^2}{3a} + \frac{ke^2}{2a} - \frac{ke^2}{a}$$

$$-\frac{3ke^2}{a} + \frac{2ke^2}{2a} - \frac{ke^2}{3a} + \frac{ke^2}{4a} - \frac{ke^2}{3a} + \frac{ke^2}{2a} - \frac{ke^2}{a}$$

$$-\frac{4ke^2}{a} + \frac{3ke^2}{2a} - \frac{2ke^2}{3a} + \frac{ke^2}{4a}$$

$$\frac{U}{N} = -\frac{ke^2}{Na} \left(\frac{N-1}{1} - \frac{N-2}{2} + \frac{N-3}{3} - \frac{N-4}{4} + \dots \right)$$

$$= -\frac{ke^2}{Na} \left(\frac{N}{1} - \cancel{\frac{1}{1}} - \frac{N}{2} + \cancel{\frac{2}{2}} + \frac{N}{3} - \cancel{\frac{3}{3}} - \frac{N}{4} + \cancel{\frac{4}{4}} + \dots \right)$$

$$= -\frac{ke^2}{a} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \right)$$

$\underbrace{\hspace{10em}}_{\ln(1+1) = \ln(2)}$

HINT: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

$$\frac{U}{N} = -\frac{ke^2}{a} \ln(2)$$

63a) q, m

$$E_{\text{outside}} = \frac{k\sigma A}{r^2}$$

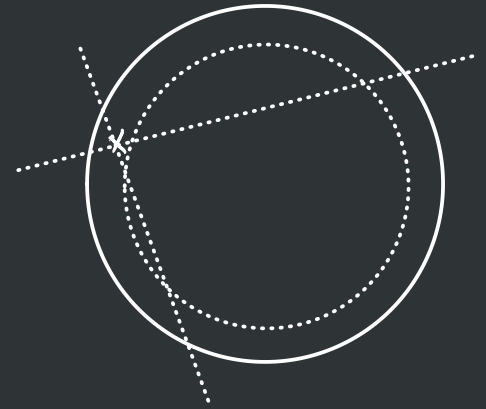
$$E_{\text{inside}} = 0$$

$$\int E \cdot dr = \frac{q_{\text{enc}}}{\epsilon_0}$$

$$F_{\text{net}} = ma$$

$$-\frac{k\sigma A q}{r^2} = ma$$

$$-\frac{k\sigma A q}{mr^2} = a$$



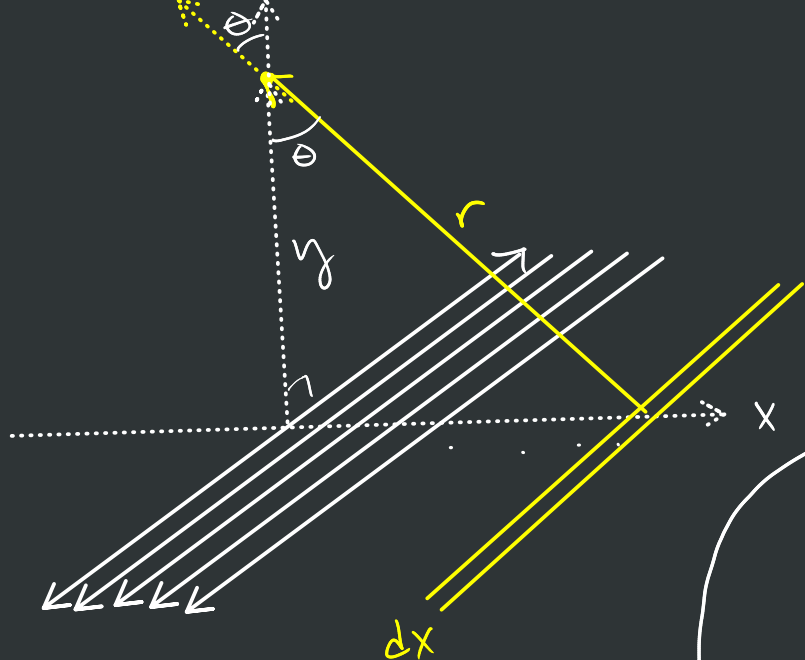
$$a = \frac{dv}{dt}$$

$$a = \frac{dv}{dr} \underbrace{\frac{dr}{dt}}_v$$

$$\frac{a}{v} = \frac{dv}{dr}$$

$$a dr = v dv$$

65



$$\sigma = \frac{Q}{A} = \frac{Q}{l \cdot w} = \frac{\lambda}{w} \rightarrow dx$$

$$\sigma \cdot dx = \lambda$$

$$dE = \frac{\lambda}{2\pi\epsilon_0 r} \cos\theta$$

$$dE = \frac{\sigma dx}{2\pi\epsilon_0 r} \cos\theta$$

$$r = \sqrt{x^2 + y^2}$$

$$\cos\theta = \frac{y}{r}$$

$$\cos\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{1}{r^2} dr &= \int_{-\infty}^{\infty} \frac{1}{r^2} dr \\ &= \left. -\frac{1}{r} \right|_{-\infty}^{\infty} \\ &= \frac{1}{r} \end{aligned}$$

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