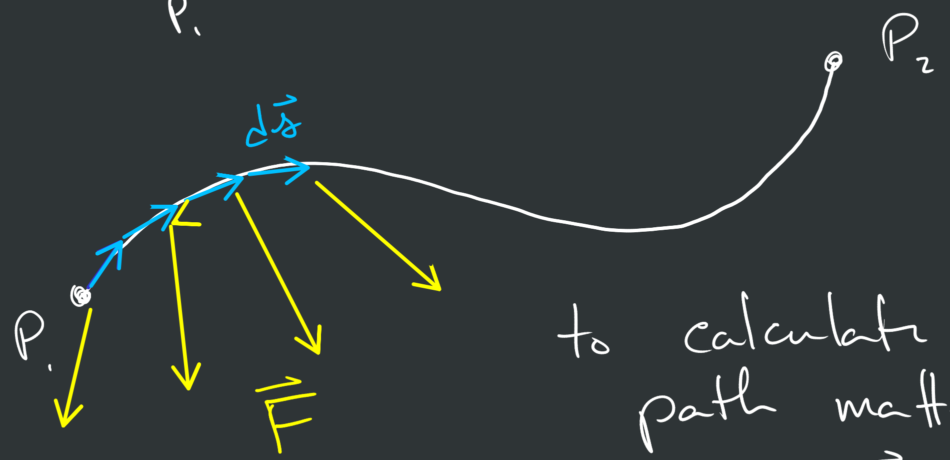
# Chapter 2 - Electric Potential

$$W_{ME} = -\int_{P_1}^{P_2} \vec{F} \cdot d\vec{s} \qquad (d\vec{r}, d\vec{\ell})$$




to calculate the work done
path matters.

**BUT**, if $\vec{F}$ is conservative
then it is path-independent

$$\vec{F} = q \cdot \vec{E}$$

$$W_{ME} = -\int_{P_1}^{P_2} q\vec{E} \cdot d\vec{s}$$

$$\frac{W_{ME}}{q} = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}$$

$$\frac{U(P_2)-U(P_1)}{q} = \frac{\Delta U}{q} \equiv \Delta\phi \equiv \phi_{21} = -\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s}$$

$\Delta K = 0$

electric potential difference

book's notation

$$\frac{[\text{Joules}]}{[\text{Coulomb}]} \quad \frac{U}{q} = \phi \quad [\text{Volts}]$$

sometimes $P_1 \to \infty$, I can set $\phi_1 = 0$.

sometimes $P_1$ is somewhere else, usually $\phi_1 = 0$ at that place

BUT, it is really only $\Delta\phi$ that matters.

Closed loop

$$\oint \vec{E}\cdot d\vec{s} = 0$$

line integral over a closed loop

What about the potential around a point charge?

→ point charge is located at the origin.

→ $\phi(\infty) = 0$

$$\phi(r) = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r}\,'$$

$d r' \hat{r}$

$$= -\int_{\infty}^{r} \frac{q_0}{4\pi\epsilon_0 r'^2} \hat{r} \cdot d\vec{r}\,'$$

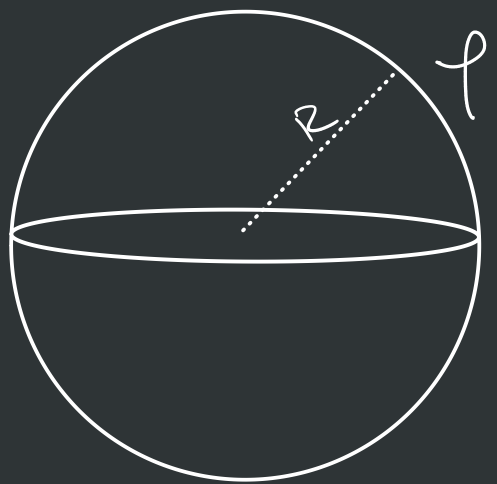
$$= -\frac{q_0}{4\pi\epsilon_0} \int_{\infty}^{r} \frac{1}{r'^2} dr'$$

$$\phi(r) = +\frac{q_0}{4\pi\epsilon_0 r}$$

$$U(r) = \frac{q_0 q_1}{4\pi\epsilon_0 r} = q_1 \phi(r) \checkmark$$

$\boxed{\text{SEE Example on p. 62}}$ – line integral, means the path matters, so you need to break up the path in some way.

Ex: Find $\phi$ inside and outside a uniformly charged sphere.




outside $r > R$

$$\phi_{outside} = -\int_{\infty}^{r} \vec{E} \cdot d\vec{r}$$

$$E = \frac{\rho R^3}{3\epsilon_0 r^2} = \boxed{\frac{Q}{4\pi\epsilon_0 r^2}}$$

$$Q = \rho \cdot V$$

$$Q = \rho \cdot \frac{4}{3}\pi R^3$$

$$\phi_{outside} = -\frac{Q}{4\pi\epsilon_0}\int_{\infty}^{r} \frac{1}{r^2}\hat{r} \cdot dr\,\hat{r}$$

$$= \frac{Q}{4\pi\epsilon_0 r} = \rho\,\frac{4}{3}\pi R^3 \cdot \frac{1}{4\pi\epsilon_0 r}$$

$$\phi_{outside} = \frac{\rho R^3}{3\epsilon_0 r} \quad \checkmark$$

inside $r < R$

$$E = \frac{\rho r}{3\epsilon_0}$$

$$\phi(r)_{inside} = -\int \vec{E} \cdot d\vec{r}$$

$$= -\underbrace{\int_{R}^{r} \frac{\rho r}{3\epsilon_0}\,dr}_{\text{inside}} - \underbrace{\int_{\infty}^{R} \frac{\rho R^3}{3\epsilon_0 r^2}\,dr}_{\text{outside}}$$

$$\frac{\rho R^{\cancel{3}\,2}}{3\epsilon_0 \cancel{R}}$$

$$= \frac{\rho R^2}{3\epsilon_0}$$

$$= -\int_R^r \frac{\rho r}{3\epsilon_0}\, dr$$

$$= -\frac{\rho}{3\epsilon_0}\int_R^r r\, dr$$

$$= -\frac{\rho}{6\epsilon_0} r^2 \Big|_R^r = -\frac{\rho}{6\epsilon_0}\left(r^2 - R^2\right)$$

$$\phi_{inside}(r) = -\frac{\rho}{6\epsilon_0}\left(r^2 - R^2\right) + \frac{\rho R^2}{3\epsilon_0}$$

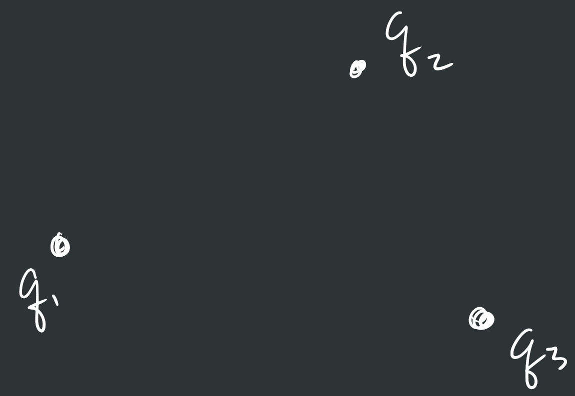
$$= -\frac{\rho}{6\epsilon_0} r^2 + \frac{\rho}{6\epsilon_0} R^2 + \frac{2\rho R^2}{2\cdot 3\epsilon_0}$$

$$\phi_{inside}(r) = -\frac{\rho}{6\epsilon_0} r^2 + \frac{\rho R^2}{2\epsilon_0} \quad ✓$$

Summing up what we have so far:

- potential energy for point charges

$$U = \frac{1}{2} \sum_{j=1}^{N} \sum_{k \neq j} \frac{q_j q_k}{4\pi\epsilon_0 r_{jk}}$$




- electric potential for point charges

$$\phi = \sum_{k}^{N} \frac{q_k}{4\pi\epsilon_0 r_k}$$

- electric potential of a continuous distribution

$$\phi = \int_{\text{all sources}} \frac{\rho \, dV}{4\pi\epsilon_0 r}$$ ← limited to sources of finite extent

Making a comparison now w/ the first two equations:

$$U = \frac{1}{2} \underbrace{\sum_{j}^{N} q_j}_{\text{adding up all the charge}} \underbrace{\sum_{k \neq j}^{N} \frac{q_k}{4\pi\epsilon_0 r_{jk}}}_{\text{like the sum for } \phi}$$

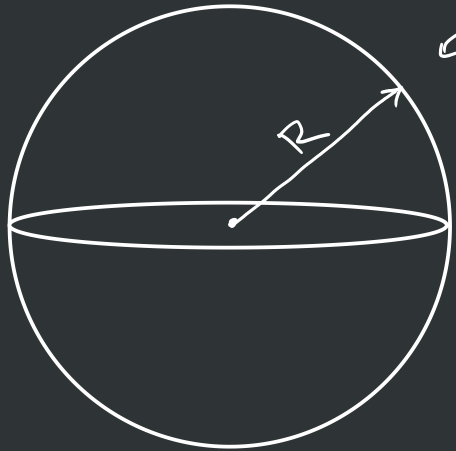
go to continuous charge distribution

$$\boxed{U = \frac{1}{2} \int \rho \phi \, dV}$$

relates the energy to build a charge distribution to the potential of that distribution

another way from Chapter 1 (eq. 1.53)

$$\left| U = \frac{\epsilon_0}{2} \int_{\text{entire field}} E^2 \, dV \right.$$




← a shell of charge, $Q$

$$\vec{E}_{inside} = 0$$

$$\vec{E}_{outside} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

$$U = \frac{\epsilon_0}{2}\int E^2 \, dV$$

$$U = \frac{\epsilon_0}{2}\int_R^\infty\int_0^\pi\int_0^{2\pi}\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)^2 \underbrace{r^2 \sin\theta \, dr \, d\theta \, d\phi}$$

$$= \frac{\cancel{\epsilon_0} Q^2}{2 \, 4^2\pi^2\epsilon_0^{\cancel{2}}}\int_R^\infty\int_0^\pi\int_0^{2\pi}\frac{1}{r^2}\sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{Q^2}{2\cdot 4^{\cancel{2}}\cdot\pi^{\cancel{2}}\epsilon_0}(\cancel{4\pi})\underbrace{\int_R^\infty \frac{1}{r^2}dr}$$

$$-\left(\frac{1}{r}\right)_R^\infty = \frac{1}{R}$$

$$U = \underbrace{\frac{1}{2}\int \rho\,\phi\, dV}$$

$U$ (direct integration)

add $dq$ shells to a radius of $R$ until you have $Q$.

$$dU = \frac{q}{4\pi\epsilon_0 R}\cdot dq$$

work to add $dq$ to $q$ at a radius of $R$

$$U = \int_0^Q \frac{q\,dq}{4\pi\epsilon_0 R}$$

$$U = \frac{Q^2}{8\pi\epsilon_0 R}$$ ✓✓

$$\boxed{U = \frac{Q^2}{8\pi\epsilon_0 R}}$$

$$U = \frac{1}{2}\int \rho \phi \, dV$$

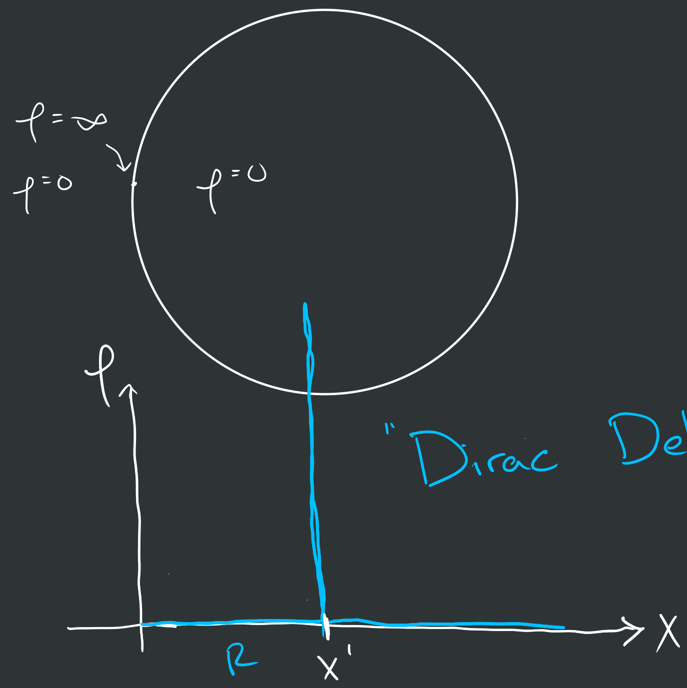
$$\rho = ? \qquad \rho \rightarrow \infty \qquad \rho = \frac{Q}{V}, \quad V \rightarrow 0$$

$$\rho(r) = \rho \cdot \delta(R - r)$$

$$Q = \int \rho(r)\, dV$$

$$Q = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} \rho\, \delta(R-r) \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi$$

$$Q = 4\pi\rho \int_0^{\infty} r^2 \delta(R-r)\, dr$$




"Dirac Delta" function

$$\delta(x' - x) = \begin{cases} 0 & x \neq x' \\ \infty & x = x' \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x' - x)\, dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x' - x)\, dx = f(x')$$

$$Q = 4\pi \sigma R^2$$

$$\sigma = \frac{Q}{4\pi R^2}$$

surface charge density

$$U = \frac{1}{2}\int \sigma \cdot \delta(R-r)\,\Phi(r)\,dV$$

$$U = \frac{1}{2}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \sigma\, \delta(R-r)\,\Phi(r)\, r^2 \underbrace{\sin\theta}\, dr\, \underbrace{d\theta\, d\phi}$$

$$U = \frac{4\pi}{2}\int_0^{\infty} \sigma\, r^2\, \Phi(r)\, \delta(R-r)\, dr$$

$$= \frac{4\pi}{2}\, \sigma\, R^2\, \Phi(R)$$

$$\sigma = \frac{Q}{4\pi R^2} \qquad \Phi(R) = \frac{Q}{4\pi\epsilon_0 R}$$

$$= \frac{\cancel{4\pi}}{2}\cdot\frac{Q}{\cancel{4\pi}\cancel{R^2}}\cdot\cancel{R^2}\cdot\frac{Q}{4\pi\epsilon_0 R} = \boxed{\frac{Q^2}{8\pi\epsilon_0 R}}$$

✓✓

So given $E$ we can find $\phi$.

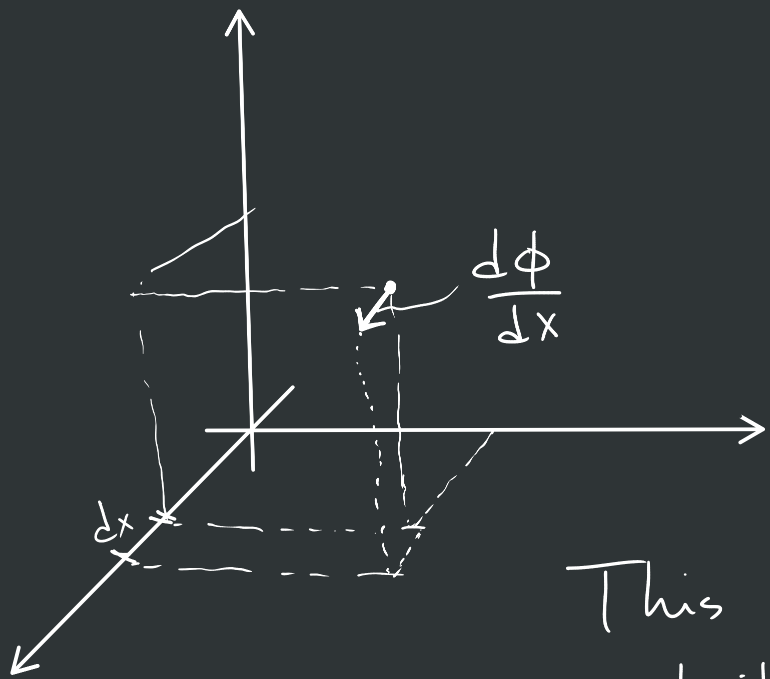
$$\Delta\phi = -\int_{P_1}^{P_2} \vec{E}\cdot d\vec{s}$$

How do we go backwards?

looks like $\rightarrow \dfrac{d\phi}{ds} = -E$, but we are missing vectors!

We will do Cartesian coordinates first.

$$\underbrace{\phi(x,y,z)}_{\text{function of 3 variables}} \longrightarrow \frac{\partial\phi}{\partial x} \longleftarrow$$ partial derivative of $\phi$ w.r.t $x$, holding $y$ and $z$ fixed!




Since we can do the same in the $y$ & $z$ directions, we can construct a vector

$$\frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z}$$

This vector is called the <u>gradient</u> of $\phi$ and it produces a <u>vector field</u>

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z}$$

Now, a small change in any scalar function can be written as (mathematically):

$$\hookrightarrow d\phi = \underbrace{\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz}_{} \leftarrow$$

but also there is physics

$$d\phi = -\vec{E} \cdot d\vec{s}$$

$$\hookrightarrow d\vec{s} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = \vec{\nabla}\phi \cdot d\vec{s} = -\vec{E} \cdot d\vec{s}$$

$$W = \int \vec{F} \cdot d\vec{s}$$

$$\frac{W}{q} = \int \frac{\vec{F}}{q} \cdot d\vec{s}$$

$$\phi = -\int \vec{E} \cdot d\vec{s}$$

$$\boxed{\vec{E} = -\vec{\nabla}\phi}$$

$$\boxed{\phi = -\int_{P_a}^{P_b} \vec{E} \cdot d\vec{s}}$$

## Ex p.62

$$\phi = -Kxy$$

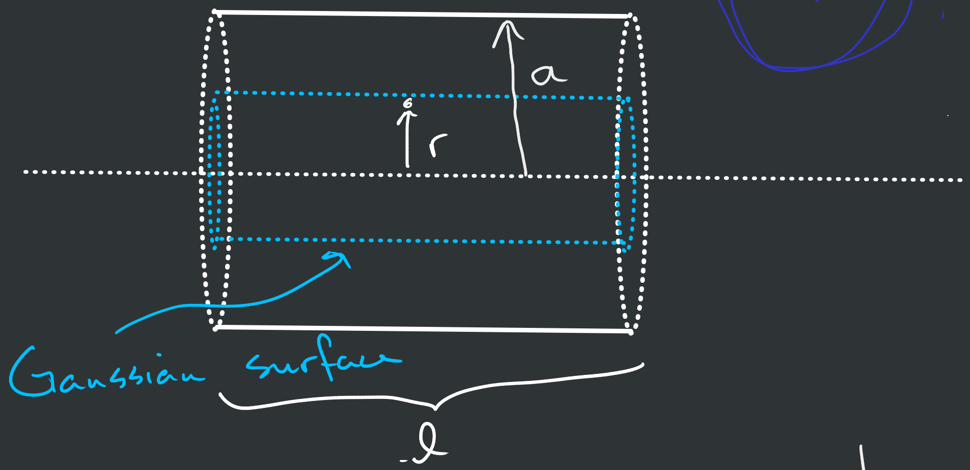
$$\vec{E} = -\vec{\nabla}(-Kxy)$$

$$= K\vec{\nabla}(xy) = K\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y}\right)(xy)$$

$$= K(y\hat{x} + x\hat{y}) \checkmark$$

42) $E_{in} = \frac{\rho r}{2\epsilon_0}$ ⟵ show this w/ Gauss's Law

$E_{out} = \frac{\rho a^2}{2\epsilon_0 r}$




$$\int \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\int \rho \, dV$$

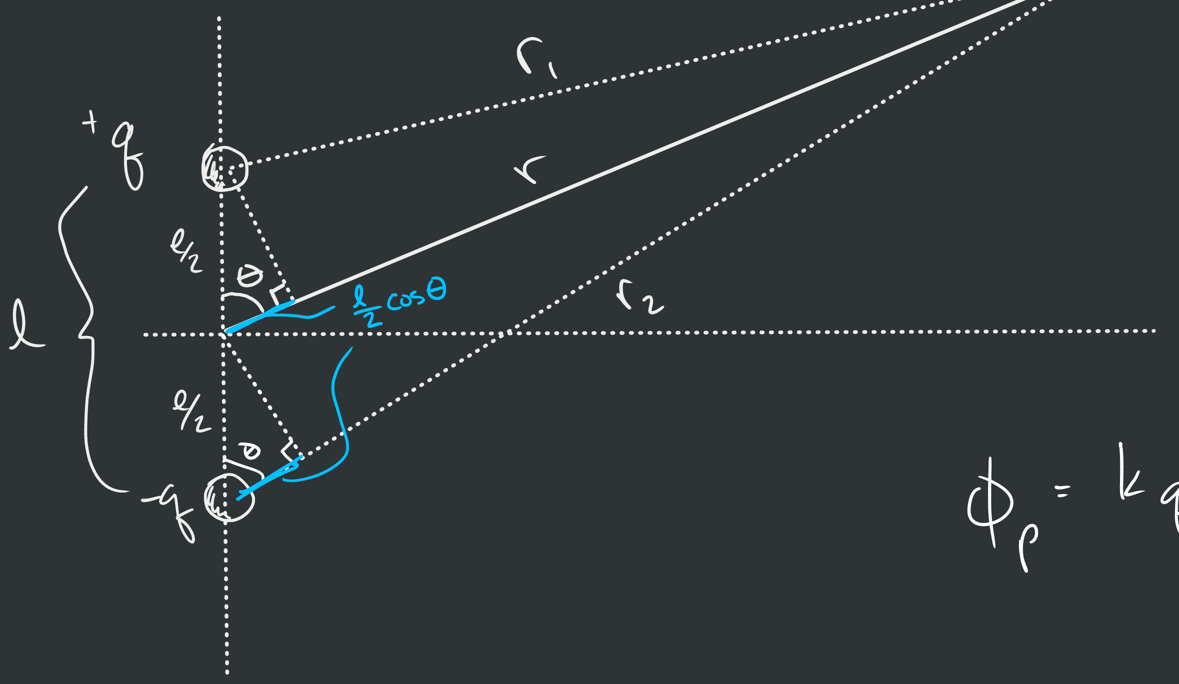
$$E \cdot \cancel{l} \cdot 2\cancel{\pi}\cancel{r} = \frac{1}{\varepsilon_0} \rho \cancel{\pi} r^{\cancel{2}} \cancel{l}$$

$$\underline{\vec{E} = \frac{\rho r}{2\varepsilon_0}\hat{r}}$$

$$\phi = \int_{p_a}^{p_b} \vec{E}\cdot \underbrace{d\vec{s}}_{\underline{\underline{dr\,\hat{r}}}}$$

$$\phi = \frac{\rho a^2}{2\epsilon} \ln(\infty$$

# Dipole




$$\phi_p = \frac{kq}{r_1} + \frac{k(-q)}{r_2}$$

$$r_1 = r - \frac{\ell}{2}\cos\theta$$

$$r_2 = r + \frac{\ell}{2}\cos\theta$$

$$\phi_p = kq\left[\frac{1}{r\left(\frac{r}{r} - \frac{\ell}{2}\frac{\cos\theta}{r}\right)} - \frac{1}{r\left(\frac{r}{r} + \frac{\ell}{2}\frac{\cos\theta}{r}\right)}\right]$$

$$= \frac{kq}{r}\left[\frac{1}{\underbrace{1 - \frac{\ell\cos\theta}{2r}}} - \frac{1}{1 + \frac{\ell\cos\theta}{2r}}\right]$$

$$\left(1 - \frac{\ell\cos\theta}{2r}\right)^{-1}$$

$$(1 \pm x)^{-1} \simeq 1 \mp x$$

Taylor expansion to 1st order

$$\phi_p = \frac{kq}{r}\left[\cancel{1} + \frac{\ell\cos\theta}{2r} - \cancel{1} + \frac{\ell\cos\theta}{2r}\right]$$

$$\phi_p = \frac{kq}{r}\left(\frac{l\cos\theta}{r}\right)$$

$$\phi_p = \frac{kq\,l\cos\theta}{r^2}$$

$\underbrace{ql}_{} \rightsquigarrow$ dipole moment $\equiv p$

plot equipotentials

$$\boxed{\phi_p = \frac{kp\cos\theta}{r^2}}$$

$\vec{E} = ?$

$$\vec{E} = -\vec{\nabla}\Phi = -\left(\frac{\partial\Phi}{\partial r}\hat{r} + \frac{1}{r}\left(\frac{\partial\Phi}{\partial\theta}\right)\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\phi}\right)$$

$$= -\frac{\partial\left(\frac{kp\cos\theta}{r^2}\right)}{\partial r}\hat{r} - \frac{1}{r}\left(\frac{\partial\left(\frac{kp\cos\theta}{r^2}\right)}{\partial\theta}\right)$$

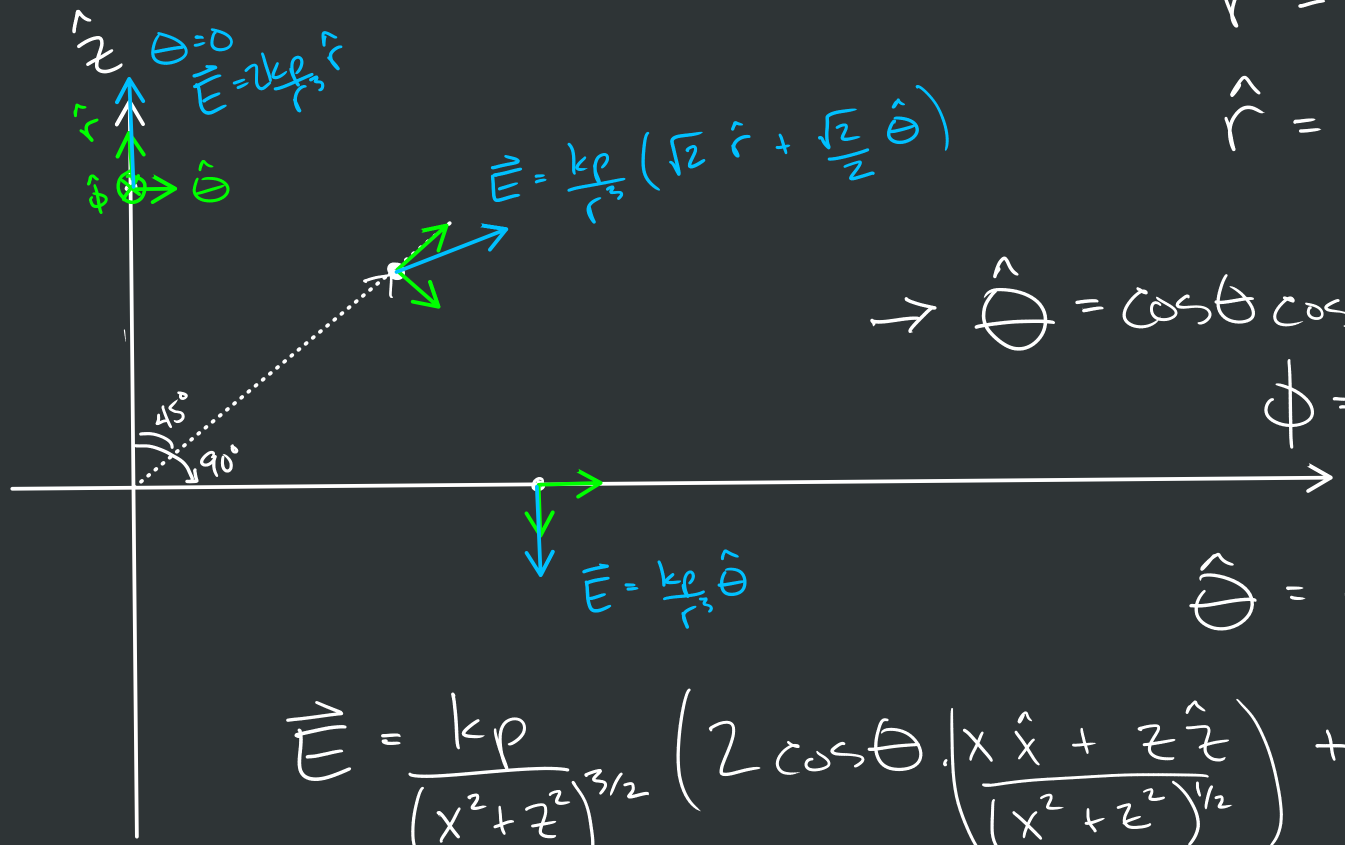
$$= -\left(-\frac{2kp\cos\theta}{r^3}\hat{r}\right) - \frac{1}{r}\left(-\frac{kp\sin\theta}{r^2}\hat{\theta}\right) + 0\hat{\phi}$$

$$= \frac{2kp\cos\theta}{r^3}\hat{r} + \frac{kp\sin\theta}{r^3}\hat{\theta}$$

$$\vec{E} = \frac{kp}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

$$\hat{r} = \frac{x\,\hat{x} + y\,\hat{y} + z\,\hat{z}}{(x^2+y^2+z^2)^{1/2}}$$




$$\rightarrow \hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\phi = 0$$

$$\hat{\theta} = \cos\theta\,\hat{x} - \sin\theta\,\hat{z}$$

$$\vec{E} = \frac{kp}{(x^2+z^2)^{3/2}}\left(2\cos\theta\cdot\left(\frac{x\,\hat{x} + z\,\hat{z}}{(x^2+z^2)^{1/2}}\right) + \sin\theta\cdot\left(\cos\theta\,\hat{x} - \sin\theta\,\hat{z}\right)\right)$$

$$\cos\theta = \frac{z}{r} = \frac{z}{(x^2+z^2)^{1/2}}$$

(after some algebra)

$$\sin\theta = \frac{x}{r} = \frac{x}{(x^2+z^2)^{1/2}}$$

$$\boxed{\vec{E} = k p \left( \frac{3xz}{(x^2+z^2)^{5/2}}\,\hat{x} + \frac{2z^2 - x^2}{(x^2+z^2)^{5/2}}\,\hat{z} \right)}$$

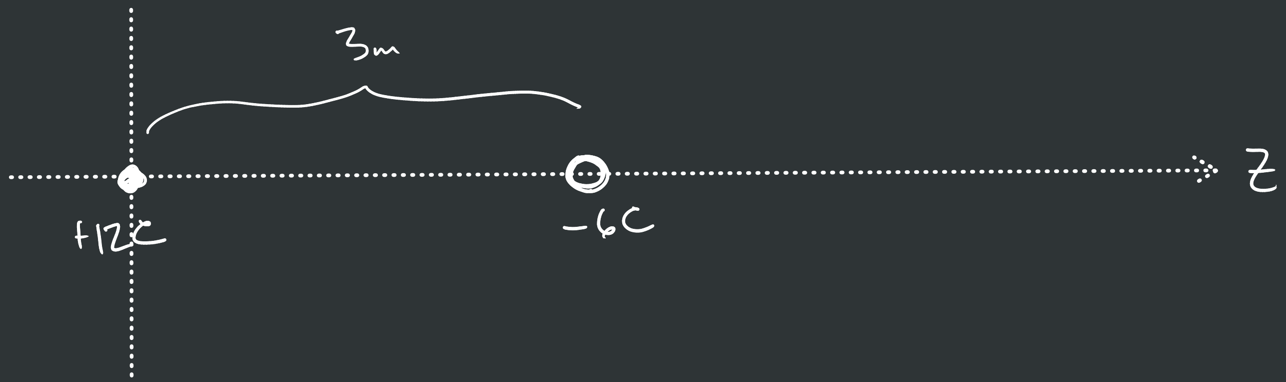
$$\Phi_p = \frac{k p \cos\theta}{r^2}$$

$$r^2 = x^2 + z^2$$

$$\cos\theta = \frac{z}{r} = \frac{z}{(x^2+z^2)^{1/2}}$$

$$\Phi_p = \frac{k p z}{(x^2+z^2)^{3/2}}$$

33)




$$\phi = \frac{q}{4\pi\epsilon_0 r}$$

$$\phi_{12C} = \frac{12C}{4\pi\epsilon_0 |z|}$$

$$\phi_{-6C} = \frac{-6C}{4\pi\epsilon_0 |z-3|}$$

$$\phi(z) = \frac{12C}{4\pi\epsilon_0 |z|} + \frac{-6C}{4\pi\epsilon_0 |z-3|}$$

$$4\pi\epsilon_0 \phi(z) = \frac{12}{|z|} + \frac{-6}{|z-3|}$$

12)