

### Chapter 3 - special Remington section

$$\nabla^2 \phi = 0$$

$$f''(x) = -k \cdot f(x)$$

$$f(x) = A \sin(\omega x)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \cancel{\frac{\partial^2 \phi}{\partial z^2}} = 0$$

Let's stick w/ two dimensions, no dependence in the  $z$ -direction  
→ Assume  $\phi = f(x) \cdot g(y)$

$$\frac{\partial^2 (f \cdot g)}{\partial x^2} + \frac{\partial^2 (f \cdot g)}{\partial y^2} = 0$$

$$g \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2} = 0 \quad \left. \vphantom{\frac{\partial^2 f}{\partial x^2}} \right\} \text{divide this equation by } f \cdot g$$

$$\frac{1}{f} \cdot \frac{\partial^2 f}{\partial x^2} + \frac{1}{g} \frac{\partial^2 g}{\partial y^2} = 0$$

$$\frac{1}{f} \cdot \frac{\partial^2 f}{\partial x^2} = k^2$$

$$\frac{\partial^2 f}{\partial x^2} = k^2 \cdot f$$

$$f(x) = A e^{kx}$$

or  
 $B e^{-kx}$

So

$$f(x) = A e^{kx} + B e^{-kx}$$

$$\frac{1}{g} \frac{\partial^2 g}{\partial y^2} = -k^2$$

$$\frac{\partial^2 g}{\partial y^2} = -k^2 g$$

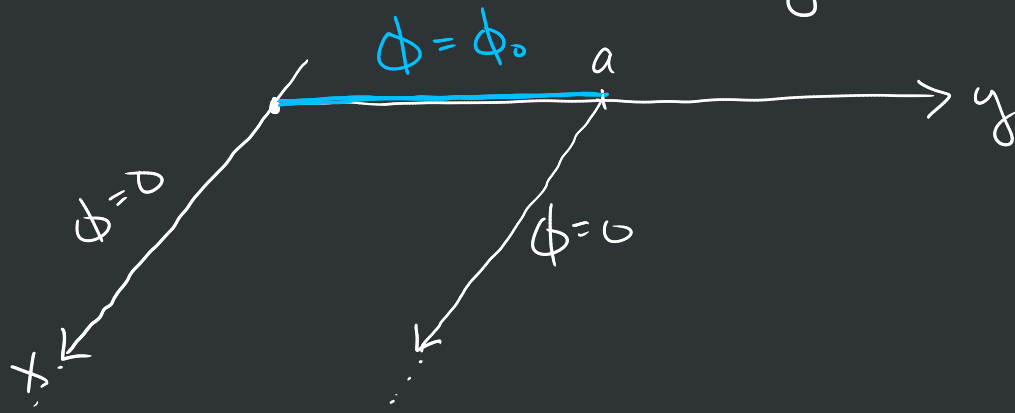
$$g(y) = \begin{cases} C \sin(ky) \\ \text{or} \\ D \cos(ky) \end{cases}$$

linear combination

So

$$g(y) = C \sin(ky) + D \cos(ky)$$

So now we need Boundary Conditions



last B.C. to use  
 $x=0, \phi = \phi_0$

$$\checkmark x \rightarrow \infty, \phi \rightarrow 0$$

$A e^{kx}$  blows up  
 w/  $x \rightarrow \infty$ ,  $\therefore A=0$

$$\begin{aligned} \checkmark y=0, \phi &= 0 \\ \checkmark y=a, \phi &= 0 \end{aligned}$$

$\cos(ky) \neq 0$ , when  
 $y=0$ ,  $\therefore D=0$

$$\text{so } \phi = f(x) \cdot g(y) = B e^{-kx} \cdot C \sin(ky)$$

$\uparrow \quad \quad \uparrow$   
 absorb constants

$$= B e^{-kx} \cdot \sin(ky)$$

now  $y=a, \phi=0 \checkmark$

$$0 = \sin(k \cdot a)$$

$$k \cdot a = 0, \pi, 2\pi, 3\pi, \dots$$

$$= n\pi \quad n = 0, 1, 2, \dots$$

$$k = \frac{n\pi}{a}$$

now  $\phi(x, y) = B e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a}y\right)$

Since this is a valid solution for infinitely many  $n$ ,  
 we need a weighted sum of the individual solutions

↳ linear combination

$$\phi = \alpha_1 \phi_1 + \alpha_2 \phi_2 + \alpha_3 \phi_3 + \dots$$

So that:

$$\nabla^2 \phi = \nabla^2 (\alpha_1 \phi_1 + \alpha_2 \phi_2 + \alpha_3 \phi_3 + \dots) = \alpha_1 \nabla^2 \phi_1 + \alpha_2 \nabla^2 \phi_2 + \dots = 0$$

I'll absorb each  $\alpha$  into  $B \rightarrow B_n$

$$\phi(x, y) = \sum_{n=1}^{\infty} B_n e^{\frac{-n\pi}{a} x} \cdot \sin\left(\frac{n\pi}{a} y\right)$$

So now we apply the last BC. ( $x=0, \phi = \phi_0$ )

$$\phi(0, y) = \phi_0(y) = \sum_{n=1}^{\infty} \underbrace{B_n e^{\frac{-n\pi}{a} \cdot 0}}_1 \cdot \sin\left(\frac{n\pi}{a} y\right)$$

$$\phi_0 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a} y\right)$$

Fourier series!

Multiply both sides of the equation by  $\sin\left(\frac{n'\pi}{a} y\right)$   
where  $n'$  is another constant. And then integrate!

$$\int_0^a \phi_0 \sin\left(\frac{n'\pi}{a} y\right) dy = \sum_{n=1}^{\infty} B_n \underbrace{\int_0^a \sin\left(\frac{n\pi}{a} y\right) \cdot \sin\left(\frac{n'\pi}{a} y\right) dy}_{=\frac{a}{2} \text{ only when } n=n'}$$

$$\int_0^a \phi_0 \sin\left(\frac{n'\pi}{a} y\right) dy = \sum_{n=1}^{\infty} B_n \frac{a}{2} \delta_{nn'} = B_{n'} \frac{a}{2}$$

$$B_{n'} = \frac{2}{a} \int_0^a \phi_0 \sin\left(\frac{n'\pi}{a} y\right) dy$$

$$B_n = \frac{2}{a} \int_0^a \phi_0 \sin\left(\frac{n\pi}{a} y\right) dy$$

$$\begin{aligned} B_n &= \frac{2\phi_0}{a} \int_0^a \sin\left(\frac{n\pi}{a} y\right) dy \\ &= \frac{2\phi_0}{a} \cdot \left(\frac{a}{n\pi}\right) \left(-\cos\left(\frac{n\pi}{a} y\right)\right) \Big|_0^a \end{aligned}$$

$$\hookrightarrow B_n = \frac{2\phi_0}{a} \cdot \left(\frac{a}{n\pi}\right) \cdot 2$$

for odd values of  $n$

$$B_n = \frac{4\phi_0}{n\pi} \text{ where } n \text{ is odd.}$$

$$\phi(x, y) = \sum_{n=1}^{\infty} B_n e^{-\frac{n\pi}{a}x} \cdot \sin\left(\frac{n\pi}{a}y\right)$$

$$\phi(x, y) = \frac{4\phi_0}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} \cdot e^{-\frac{n\pi}{a}x} \cdot \sin\left(\frac{n\pi}{a}y\right)$$

lets plot in mathematica!

$$n \rightarrow 2n+1$$

$$-\left[\cos\left(\frac{n\pi}{a}a\right) - \cos(0)\right]$$

$$-\left[\cos(n\pi) - 1\right]$$

$$-\cos(n\pi) + 1$$

-1 for odd  $n$

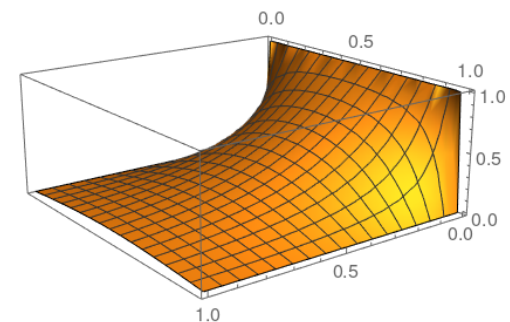
+1 for even  $n$

+2 for odd  $n$

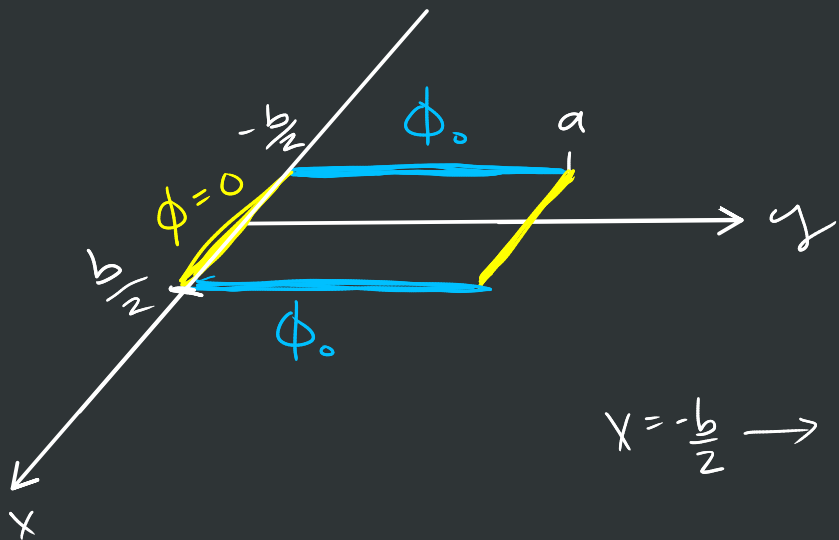
0 for even  $n$

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In[20]:= Plot3D[
  4 * 1 / Pi *
  Sum[1 / (2 n + 1) * Exp[-(2 n + 1) * Pi / 1 * x] *
    Sin[(2 n + 1) * Pi / 1 * y], {n, 0, 1000}],
  {x, 0, 1}, {y, 0, 1}]
```

Out[20]=



Now for another example:



$$\frac{\partial^2 f}{\partial x^2} = k^2 f$$

$$\frac{\partial^2 g}{\partial y^2} = -k^2 g$$

$$f = A e^{kx} + B e^{-kx}$$

$$g = C \sin(ky) + \underbrace{D \cos(ky)}_{D=0}$$

since  $\phi = 0$   
at  $y = 0$

$$x = -\frac{b}{2} \rightarrow \phi_0 = A e^{-kb/2} + B e^{+k \cdot b/2}$$

and

$$x = +\frac{b}{2} \quad \phi_0 = A e^{+kb/2} + B e^{-kb/2}$$

so this can only  
be true if

$$A = B$$

$$f = A (e^{kx} + e^{-kx})$$

$$g = C \sin ky$$

$$\phi = 0, y = a$$

$$\text{so } 0 = C \sin(ka)$$

$$0 = \sin(ka)$$

$$ka = 0\pi, 1\pi, 2\pi, 3\pi$$

$$ka = n\pi \quad n = 1, 2, 3, \dots$$

$$k = \frac{n\pi}{a}$$

$$\text{So ... } \phi(x, y) = \sum_{n=1}^{\infty} A_n \underbrace{(e^{kx} + e^{-kx})}_{\uparrow} \sin(ky)$$

$$2 \cdot \cosh(kx) \quad \rightarrow \quad \cosh(kx) = \frac{e^{kx} + e^{-kx}}{2}$$

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n \cosh(kx) \sin(ky)$$

$$\sinh(kx) = \frac{e^{kx} - e^{-kx}}{2}$$

So now, last BC.

$$\rightarrow \phi = \phi_0 \text{ at } x=b$$

$$\phi_0 = \sum_{n=1}^{\infty} A_n \cosh(kb) \sin(ky)$$

$$\cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2}$$

$$\sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i}$$

now we apply Fourier's trick

$$\int_0^a \phi_0 \sin\left(\frac{n'\pi}{a} y\right) dy = \sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi}{a} b\right) \int_0^a \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{n'\pi}{a} y\right) dy$$

$$\sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi}{a} b\right) \cdot \frac{a}{2} \delta_{nn'}$$

$$A_{n'} \cosh\left(\frac{n'\pi}{a} b\right) \cdot \frac{a}{2}$$

$$\phi_0\left(\frac{a}{n'\pi}\right) \underbrace{\left(-\cos\left(\frac{n'\pi}{a} y\right)\right)_0^a}_{\substack{=2 \text{ odd } n' \\ =0 \text{ even } n'}} =$$



$$\phi_0 \cdot \frac{2a}{n\pi} = A_n \cosh\left(\frac{n\pi}{a} b\right) \cdot \frac{a}{2} \quad \text{drop primes}$$

solve for  $A_n$

$$A_n = \frac{4\phi_0}{n\pi \cosh\left(\frac{n\pi}{a} b\right)} \quad \text{for odd } n$$

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n \cosh(kx) \sin(ky)$$

$$\phi(x, y) = \frac{4\phi_0}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} \frac{\cosh\left(\frac{n\pi}{a} x\right)}{\cosh\left(\frac{n\pi}{a} b\right)} \cdot \sin\left(\frac{n\pi}{a} y\right)$$

Now for 3-d

list the BC's

$$\left. \begin{array}{l} \phi = 0, x=0 \\ x=b \end{array} \right\} \text{sin/cos}$$

$$\left. \begin{array}{l} y=0 \\ y=a \end{array} \right\} \text{sin/cos}$$

$$\left. \begin{array}{l} z \rightarrow \infty \end{array} \right\} \text{exp}$$

$$\phi = \phi_0, z=0$$

separated variable  
for 3D Laplace  
equation

$$\underbrace{\frac{1}{f} \frac{\partial^2 f}{\partial x^2}}_{=-k^2} + \underbrace{\frac{1}{g} \frac{\partial^2 g}{\partial y^2}}_{=-l^2} + \underbrace{\frac{1}{h} \frac{\partial^2 h}{\partial z^2}}_{=k^2+l^2} = 0$$

← b/c of  
BC's

$$\left( \frac{\partial^2 f}{\partial x^2} = -k^2 f \right) \left| \begin{array}{l} x=0 \\ B=0 \end{array} \right| f(x) = A \sin(kx) + B \cos(kx) \left| \begin{array}{l} y=0 \\ D=0 \end{array} \right| g(y) = C \sin(ly) + D \cos(ly) \left| \begin{array}{l} z \rightarrow \infty \\ F=0 \end{array} \right| h(z) = F e^{\sqrt{k^2+l^2} z} + G e^{-\sqrt{k^2+l^2} z}$$

So now  $\rightarrow$   $k b = n\pi$   $\left| \begin{array}{l} k = \frac{n\pi}{b} \\ n = 1, 2, 3, \dots \end{array} \right| \quad \left| \begin{array}{l} l = \frac{m\pi}{a} \\ m = 1, 2, 3, \dots \end{array} \right| \quad \sqrt{k^2 + l^2} = \sqrt{\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2}$

$$\phi(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{b} x\right) \sin\left(\frac{m\pi}{a} y\right) e^{-\pi \sqrt{\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2} z}$$

$$\phi(x, y, 0) = \phi_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{b} x\right) \sin\left(\frac{m\pi}{a} y\right)$$

$$= \sum_n \sum_m A_{nm} \underbrace{\int_0^b \sin\left(\frac{n\pi}{b} x\right) \sin\left(\frac{n'\pi}{b} x\right) dx}_{= \frac{b}{2} \delta_{nn'}} \underbrace{\int_0^a \sin\left(\frac{m\pi}{a} y\right) \sin\left(\frac{m'\pi}{a} y\right) dy}_{= \frac{a}{2} \delta_{mm'}}$$

$$\underbrace{\phi_0 \int_0^b \sin\left(\frac{n'\pi}{b} x\right) dx}_{\quad} \underbrace{\int_0^a \sin\left(\frac{m'\pi}{a} y\right) dy}_{\quad} = A_{n'm'} \left(\frac{b}{2}\right) \left(\frac{a}{2}\right)$$

$$\frac{2b}{n\pi} \quad n=1,3,5,\dots \quad \frac{2a}{m\pi} \quad m=1,3,5,\dots$$

drop primes

$$A_{nm} = \frac{16\phi_0}{nm\pi^2} \quad \underline{n+m \text{ are odd}}$$

$$\phi(x,y,z) = \frac{16\phi_0}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{nm} \sin\left(\frac{n\pi}{b}x\right) \sin\left(\frac{m\pi}{a}y\right) e^{-\pi\sqrt{\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2}z}$$

# Spherical Separation of Variables

- Separation of variables

① completeness  $\rightarrow g(x) = \sum_{n=1}^{\infty} C_n f_n(x)$

↑ any function

↑ "complete set"

② orthogonality

$\hookrightarrow$  "Fourier's trick"

$$\nabla^2 \Phi = 0$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \underbrace{\hspace{2cm}}_{\substack{\uparrow \text{assume azimuthal} \\ \text{symmetry}}} = 0$$

Assume  $\Phi(r, \theta) = R(r) \cdot \Theta(\theta)$   $\leftarrow$  capital theta

plug this in for  $\Phi$  and divide by  $\Phi$   
and multiply by  $r^2$  on both sides.

$$\underbrace{\frac{1}{R} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right)}_{= l(l+1)} + \underbrace{\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right)}_{= -l(l+1)} = 0$$

$$\frac{\partial}{\partial r} \left( r^2 \frac{\partial R}{\partial r} \right) = R \cdot l \cdot (l+1)$$

Guess:  $R(r) = A r^l + B r^{-(l+1)}$

$$\boxed{\phi(r, \theta) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l(\cos \theta)}$$

$$\frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Theta}{\partial \theta} \right) = -l(l+1) \cdot \sin \theta$$

I'll give the solution

$$\Theta(\theta) = P_l(\cos \theta)$$

↑ "Legendre polynomial"

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{3x^2 - 1}{2}$$

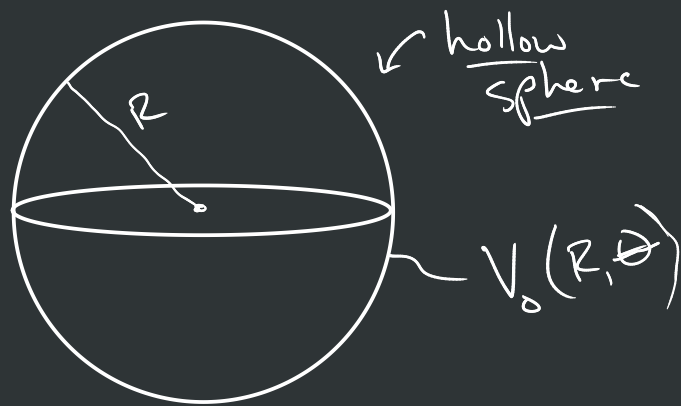
$$P_3(x) = \frac{5x^3 - 3x}{2}$$

$$P_4(x) = \frac{35x^4 - 30x^2 + 3}{8}$$

What is the pattern?

$$P_l(x) \equiv \frac{1}{2^l l!} \left( \frac{d}{dx} \right)^l (x^2 - 1)^l$$

So lets do an example:



What is the potential inside the sphere?

$$\phi(r, \theta) = \sum_{l=0}^{\infty} (A_l r^l + B_l r^{-(l+1)}) P_l(\cos \theta)$$

$B_l = 0$ , otherwise this blows up at origin

$$\phi(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta)$$

$$\phi(R, \theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos \theta)$$

mult. ply on both sides by  $P_{l'}(\cos \theta) \cdot \sin \theta$   
and integrate

$$\int_{-1}^1 P_l(x) P_{l'}(x) dx = \int_{\pi}^0 P_l(\cos\theta) P_{l'}(\cos\theta) (-\sin\theta) d\theta = \underbrace{\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta d\theta}_{= \begin{cases} 0 & l' \neq l \\ \frac{2}{2l+1} & l' = l \end{cases}}$$

$x = \cos\theta \rightarrow \begin{matrix} \theta = \pi & x = -1 \\ \theta = 0 & x = 1 \end{matrix}$

$dx = \frac{dx}{d\theta} d\theta \rightarrow dx = -\sin\theta d\theta$

$$\delta_{l'l} = \begin{cases} 0 & l' \neq l \\ 1 & l' = l \end{cases}$$

$$\int_0^{\pi} \phi(R, \theta) \cdot P_{l'}(\cos\theta) \cdot \sin\theta d\theta = \sum_{l=0}^{\infty} A_l R^l \int_0^{\pi} P_l(\cos\theta) \cdot P_{l'}(\cos\theta) \sin\theta d\theta$$

$$\int_0^{\pi} \phi(R, \theta) \cdot P_{l'}(\cos\theta) \cdot \sin\theta d\theta = A_{l'} R^{l'} \cdot \frac{2}{2l'+1}$$

drop primes

$$A_l = \frac{2l+1}{2R^l} \int_0^{\pi} \phi(R, \theta) \cdot \underline{P_l(\cos\theta)} \cdot \sin\theta d\theta$$

↳ what is the B.C.?



Suppose:  $\phi(R, \theta) = k \sin^2\left(\frac{\theta}{2}\right)$

look at trig identities

$$= k \left( \frac{1 - \cos \theta}{2} \right)$$

$$= \frac{k}{2} (1 - \cos \theta)$$

$$= \frac{k}{2} (P_0(\cos \theta) - P_1(\cos \theta))$$

$$A_l = \frac{2l+1}{2R^2} \cdot \frac{k}{2} \left[ \underbrace{\int_0^\pi P_0(\cos \theta) P_l(\cos \theta) \sin \theta d\theta}_{=2} - \underbrace{\int_0^\pi P_1(\cos \theta) P_l(\cos \theta) \sin \theta d\theta}_{=\frac{2}{3}} \right]$$

$$= \begin{cases} \frac{2}{2l+1} & l=0 \\ 0 & l \neq 0 \end{cases}$$

$$= 2$$

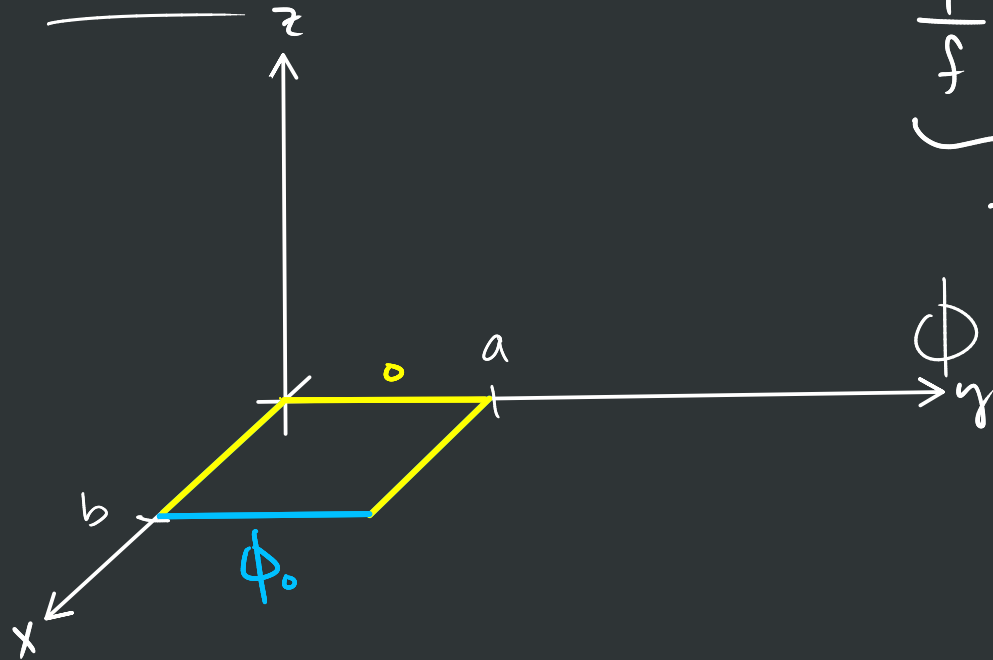
$$= \begin{cases} \frac{2}{2l+1} & l=1 \\ 0 & l \neq 1 \end{cases}$$

$$= \frac{2}{3}$$

$$\phi(r, \theta) = \frac{k}{2} \left( r^0 P_0(\cos \theta) - \frac{r^1}{R} P_1(\cos \theta) \right)$$

$$\phi(r, \theta) = \frac{k}{2} \left( 1 - \frac{r}{R} \cos \theta \right) \leftarrow \text{very simple!}$$

Homework:



$$\underbrace{\frac{1}{f} \frac{\partial^2 f}{\partial x^2}}_{+k^2} + \underbrace{\frac{1}{g} \frac{\partial^2 g}{\partial y^2}}_{-k^2} = 0$$

$$\phi(x, y) = (Ae^{kx} + Be^{-kx}) (\cancel{C \sin ky} + \cancel{D \cos ky})$$

$y=0, \phi=0$

$x=0, \phi=0$

$$0 = Ae^0 + Be^0$$

$$A = -B$$

$$\phi(x, y) = A \underbrace{(e^{kx} - e^{-kx})}_{2 \sinh(kx)} \sin ky$$

$\downarrow y=a, \phi=0$

$$k = \frac{n\pi}{a}$$

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a} x\right) \sin\left(\frac{n\pi}{a} y\right)$$

$x=b, \phi=\phi_0$

$$\phi_0 = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a} b\right) \sin\left(\frac{n\pi}{a} y\right)$$

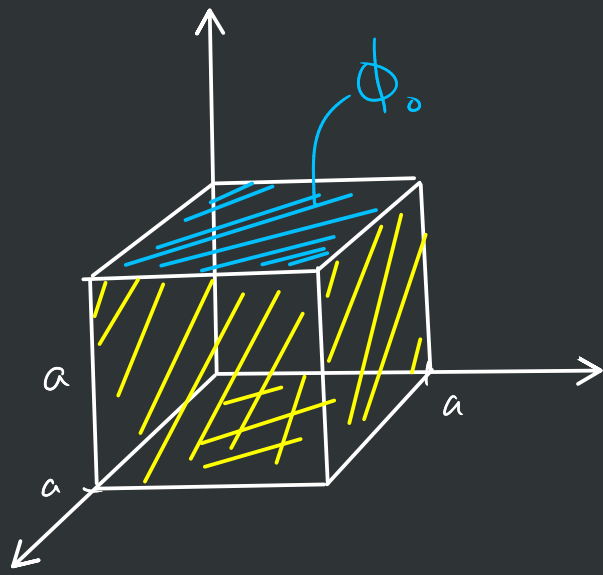
$$\underbrace{\int_0^a \phi_0 \sin\left(\frac{n'\pi}{a} y\right) dy}_{= \frac{a}{2} \delta_{nn'}} = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a} b\right) \underbrace{\int_0^a \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{n'\pi}{a} y\right) dy}_{= \frac{a}{2} \delta_{nn'}}$$

drop primes  $\rightarrow \phi_0 \cdot \underbrace{\left(\frac{\cancel{a}}{n'\pi}\right) \cos\left(\frac{n'\pi}{a} y\right) \Big|_0^a}_{\substack{2 \text{ odd } n' \\ 0 \text{ even } n'}} = A_{n'} \sinh\left(\frac{n'\pi}{a} b\right) \cancel{\frac{a}{2}}$

$$A_n = \frac{4\phi_0}{n\pi} \cdot \frac{1}{\sinh\left(\frac{n\pi}{a} b\right)} \text{ for odd } n$$

$$\begin{aligned} \int_0^a \sin\left(\frac{n\pi}{a} y\right) dy &= -\frac{a}{n\pi} \cos\left(\frac{n'\pi}{a} y\right) \Big|_0^a \\ &= \frac{a}{n\pi} \left[ -\cos(n\pi) + 1 \right] \\ &= \frac{a}{n\pi} \left[ 1 - \cos(n\pi) \right] \\ &= 0 \text{ for even } n \\ &= \underline{\underline{\frac{2a}{n\pi}}} \text{ odd } n \end{aligned}$$

$$\boxed{\phi(x, y) = \frac{4\phi_0}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} \frac{\sinh\left(\frac{n\pi}{a} x\right)}{\sinh\left(\frac{n\pi}{a} b\right)} \sin\left(\frac{n\pi}{a} y\right)}$$



$$\phi(x, y, z) = ?$$

$$\underbrace{\frac{1}{f} \frac{\partial^2 f}{\partial x^2}}_{-k^2} + \underbrace{\frac{1}{g} \frac{\partial^2 g}{\partial y^2}}_{-l^2} + \underbrace{\frac{1}{h} \frac{\partial^2 h}{\partial z^2}}_{l^2 + k^2} = 0$$

$$\phi(x, y, z) = (A \sin kx + B \cos kx) \cdot (C \sin ly + D \cos ly) \cdot \left( F e^{\sqrt{l^2 + k^2} z} + G e^{-\sqrt{l^2 + k^2} z} \right)$$







