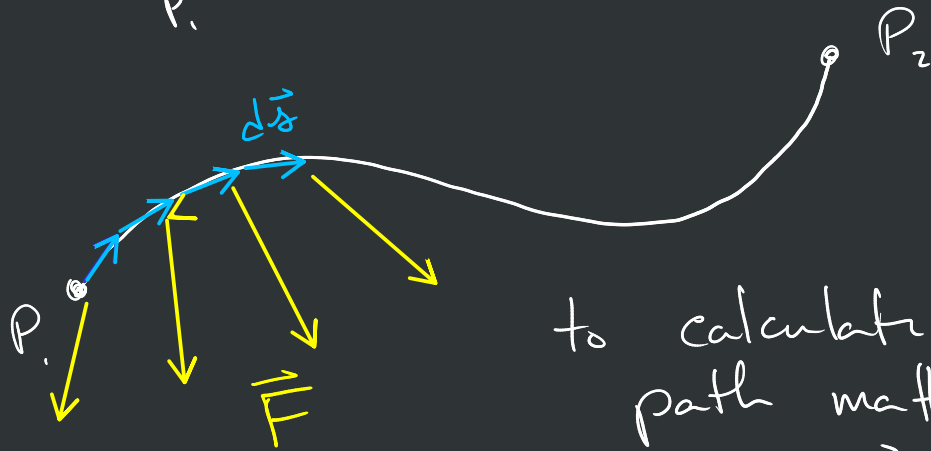


Chapter 2 - Electric Potential

$$W_{ME} = - \int_{P_1}^{P_2} \vec{F} \cdot d\vec{s} \quad (d\vec{r}, d\vec{\ell})$$



to calculate the work done
path matters.

BUT, if \vec{F} is conservative
then it is path-independent

$$\vec{F} = q \cdot \vec{E}$$

$$W_{ME} = - \int_{P_1}^{P_2} q \vec{E} \cdot d\vec{s}$$

$$\frac{W_{ME}}{q} = - \int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}$$

$\downarrow \Delta K = 0$

$$\frac{U(P_2) - U(P_1)}{q} = \frac{\Delta U}{q} \equiv \Delta \phi \equiv \phi_{2,1} = - \int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}$$

\uparrow electric potential difference
 \nwarrow book's notation

$$\frac{[\text{Joules}]}{[\text{Coulomb}]} \frac{U}{q} = \phi \quad [\text{Volts}]$$

sometimes $P_1 \rightarrow \infty$, I can set $\phi_1 = 0$.

sometimes P_1 is somewhere else, usually $\phi_1 = 0$ at that place

BUT, it is really only $\Delta \phi$ that matters.

Closed loop

$$\oint \vec{E} \cdot d\vec{s} = 0$$

\curvearrowright line integral over a closed loop

What about the potential around a point charge?

→ point charge is located at the origin.

$$\rightarrow \phi(\infty) = 0$$

$$\phi(r) = - \int_{\infty}^r \vec{E} \cdot d\vec{r}'$$

$$= - \int_{\infty}^r \frac{q_0}{4\pi\epsilon_0 r'^2} \hat{r}' \cdot d\vec{r}'$$

\swarrow $dr' \hat{r}$

$$= - \frac{q_0}{4\pi\epsilon_0} \int_{\infty}^r \frac{1}{r'^2} dr'$$

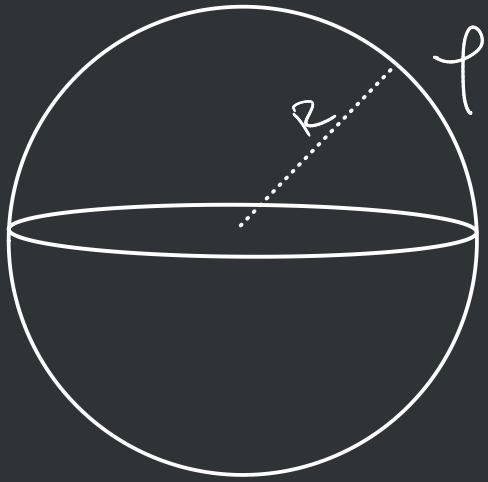
$$\phi(r) = + \frac{q_0}{4\pi\epsilon_0 r}$$

$$U(r) = \frac{q_0 q_1}{4\pi\epsilon_0 r} = q_1 \phi(r) \quad \checkmark$$

SEE Example on p. 62

— line integral, means the path matters, so you need to break up the path in some way.

Ex: Find ϕ inside and outside a uniformly charged sphere.



outside $r > R$

$$\phi_{\text{outside}} = - \int_{\infty}^r \vec{E} \cdot d\vec{r}$$

$$E = \frac{\rho R^3}{3\epsilon_0 r^2} = \left| \frac{Q}{4\pi\epsilon_0 r^2} \right|$$

$$\phi_{\text{outside}} = - \frac{Q}{4\pi\epsilon_0} \int_{\infty}^r \frac{1}{r^2} \hat{r} \cdot d\vec{r}$$

$$Q = \rho \cdot V$$

$$Q = \rho \cdot \frac{4}{3}\pi R^3$$

$$= \frac{Q}{4\pi\epsilon_0 r} = \rho \frac{4}{3}\pi R^3 \cdot \frac{1}{4\pi\epsilon_0 r}$$

$$\phi_{\text{outside}} = \frac{\rho R^3}{3\epsilon_0} \quad \checkmark$$

inside $r < R$

$$E = \frac{\rho r}{3\epsilon_0}$$

$$\phi(r) = - \int \vec{E} \cdot d\vec{r}$$

$$= - \int_{\infty}^r \frac{\rho r}{3\epsilon_0} dr - \underbrace{\int_r^R \frac{\rho r}{3\epsilon_0} dr}_{\text{inside}}$$

$$- \underbrace{\int_{\infty}^R \frac{\rho R^3}{3\epsilon_0 r^2} dr}_{\text{outside}}$$

$$= \frac{\rho R^2}{3\epsilon_0}$$

$$= \frac{\rho R^2}{3\epsilon_0}$$

$$= - \int_R^r \frac{\rho r}{3\epsilon_0} dr$$

$$= - \frac{\rho}{3\epsilon_0} \int_R^r r dr$$

$$= - \frac{\rho}{6\epsilon_0} r^2 \Big|_R^r = - \frac{\rho}{6\epsilon_0} (r^2 - R^2)$$

$$\phi(r)_{\text{inside}} = - \frac{\rho}{6\epsilon_0} (r^2 - R^2) + \frac{\rho R^2}{3\epsilon_0}$$

$$= - \frac{\rho}{6\epsilon_0} r^2 + \frac{\rho}{6\epsilon_0} R^2 + \frac{2\rho R^2}{2 \cdot 3\epsilon_0}$$

$$\phi(r)_{\text{inside}} = - \frac{\rho}{6\epsilon_0} r^2 + \frac{\rho R^2}{2\epsilon_0} \quad \checkmark$$

Summing up what we have so far:

- potential energy for point charges

$$U = \frac{1}{2} \sum_{j=1}^N \sum_{k \neq j} \frac{q_j q_k}{4\pi\epsilon_0 r_{jk}}$$

q_1

q_2

q_3

- electric potential for point charges

$$\phi = \sum_k \frac{q_k}{4\pi\epsilon_0 r_k}$$

- electric potential of a continuous distribution

$$\phi = \int \frac{\rho dV}{4\pi\epsilon_0 r} \quad \leftarrow \text{limited to sources of finite extent}$$

all sources

Making a comparison now w/ the first two equations:

$$U = \frac{1}{2} \sum_j q_j \sum_{k \neq j} \frac{q_k}{4\pi\epsilon_0 r_{jk}}$$

adding like the sum for ϕ
up all
the charge

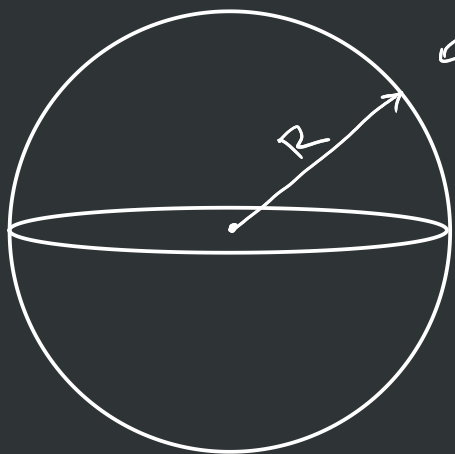
go to continuous
charge distribution

$$U = \frac{1}{2} \int \rho \phi dV$$

relates the energy to
build a charge distribution
to the potential of that
distribution

another way from Chapter 1 (eq. 1.53)

$$U = \frac{\epsilon_0}{2} \int_{\text{entire field}} E^2 dV$$



← a shell of charge, Q

$$\vec{E}_{\text{inside}} = 0$$

$$\vec{E}_{\text{outside}} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}$$

$$U = \frac{\epsilon_0}{2} \int E^2 dV$$

$$U = \frac{1}{2} \int \rho \phi dV$$

$$U = \frac{\epsilon_0}{2} \int_R^\infty \int_0^\pi \int_0^{2\pi} \left(\frac{Q}{4\pi\epsilon_0 r^2} \right)^2 r^2 \sin\theta dr d\theta d\phi$$

$$= \frac{\cancel{\epsilon_0} Q^2}{2 \cancel{4^2} \pi^2 \cancel{\epsilon_0}} \int_R^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{r^2} \sin\theta dr d\theta d\phi$$

$$= \frac{Q^2}{2 \cdot 4 \cdot \pi \epsilon_0} (\cancel{4\pi}) \int_R^\infty \frac{1}{r^2} dr$$

$$-\left(\frac{1}{r}\right)_R^\infty = \frac{1}{R}$$

U (direct integration)
add dq shells to
a radius of R
until you have Q .

$$dU = \frac{q}{4\pi\epsilon_0 R} \cdot dq$$

work to
add dq to
 q at a
radius of R

$$U = \int_0^Q \frac{q dq}{4\pi\epsilon_0 R}$$

$$U = \frac{Q^2}{8\pi\epsilon_0 R} \checkmark \checkmark$$

$$U = \frac{Q^2}{8\pi\epsilon_0 R} \quad \checkmark \checkmark$$

$$U = \frac{1}{2} \int \rho \phi dV$$

$$\rho = ?$$

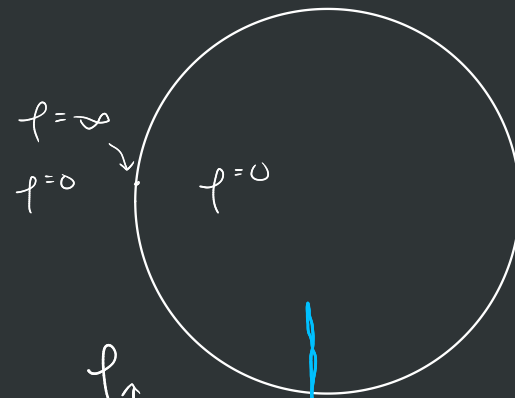
$$\rho = \frac{Q}{V} \rightarrow 0$$

$$\rho(r) = \rho \cdot \delta(R-r)$$

$$Q = \int \rho(r) dV$$

$$Q = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{\infty} \rho \delta(R-r) \cdot r^2 \sin\theta dr d\theta d\phi$$

$$Q = 4\pi \rho \int_0^{\infty} r^2 \delta(R-r) dr$$



"Dirac Delta" function

$$\delta(x'-x) = \begin{cases} 0 & x \neq x' \\ \infty & x = x' \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x'-x) dx = 1$$

$$\int_{-\infty}^{\infty} f(x) \delta(x'-x) dx = f(x')$$

$$Q = 4\pi \rho R^2$$

$$\rho = \frac{Q}{4\pi R^2}$$

surface charge density

$$U = \frac{1}{2} \int \rho \cdot \delta(R-r) \Phi(r) dV$$

$$U = \frac{1}{2} \int_0^{2\pi} \int_0^\pi \int_0^\infty \rho \delta(R-r) \Phi(r) r^2 \underbrace{\sin\theta}_{dr d\theta d\phi}$$

$$U = \frac{4\pi}{2} \int_0^\infty \rho r^2 \Phi(r) \delta(R-r) dr$$

$$= \frac{4\pi}{2} \rho R^2 \Phi(R)$$

$\frac{Q}{4\pi R^2}$
 $\frac{Q}{4\pi\epsilon_0 R}$

$$= \frac{\cancel{4\pi}}{2} \cdot \frac{Q}{\cancel{4\pi R^2}} \cdot \cancel{R^2} \cdot \frac{Q}{4\pi\epsilon_0 R} = \left| \frac{Q^2}{8\pi\epsilon_0 R} \right| \checkmark \checkmark$$

So given E we can find ϕ .

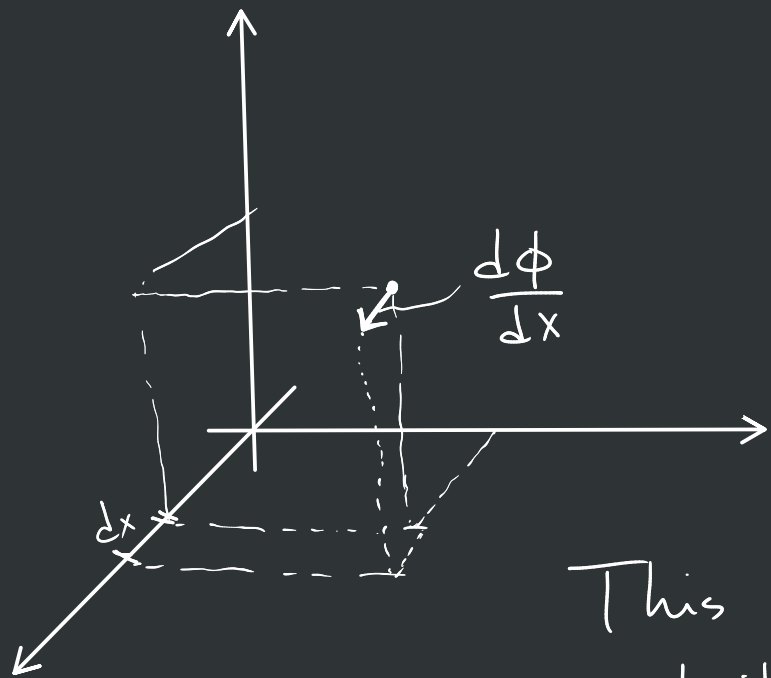
$$\Delta\phi = - \int_{P_1}^{P_2} \vec{E} \cdot d\vec{s}$$

How do we go backwards?

looks like $\rightarrow \frac{d\phi}{ds} = -E$, but we are missing vectors!

We will do Cartesian coordinates first.

$\phi(x, y, z)$ \rightsquigarrow $\frac{\partial\phi}{\partial x}$ \leftarrow partial derivative of ϕ
function of 3 variables w.r.t x , holding y and z fixed!



Since we can do the same in the y & z directions, we can construct a vector

$$\frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y} + \frac{\partial \phi}{\partial z} \hat{z}$$

This vector is called the gradient of ϕ and it produces a vector field

$$\vec{\nabla} \phi = \frac{\partial \phi}{\partial x} \hat{x} + \frac{\partial \phi}{\partial y} \hat{y} + \frac{\partial \phi}{\partial z} \hat{z}$$

Now, a small change in any scalar function can be written as (mathematically):

$$\hookrightarrow d\phi = \underbrace{\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz}_{\text{total differential}} \leftarrow$$

but also there is physics

$$d\phi = -\vec{E} \cdot d\vec{s}$$

$$\hookrightarrow d\vec{s} = dx \hat{x} + dy \hat{y} + dz \hat{z}$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = \vec{\nabla} \phi \cdot d\vec{s} = -\vec{E} \cdot d\vec{s}$$

$$W = \int \vec{F} \cdot d\vec{s}$$

$$\frac{W}{q} = \int \vec{E} \cdot d\vec{s}$$

$$\phi = -\int \vec{E} \cdot d\vec{s}$$

$\vec{E} = -\vec{\nabla} \phi$
$\phi = -\int_{P_a}^{P_b} \vec{E} \cdot d\vec{s}$

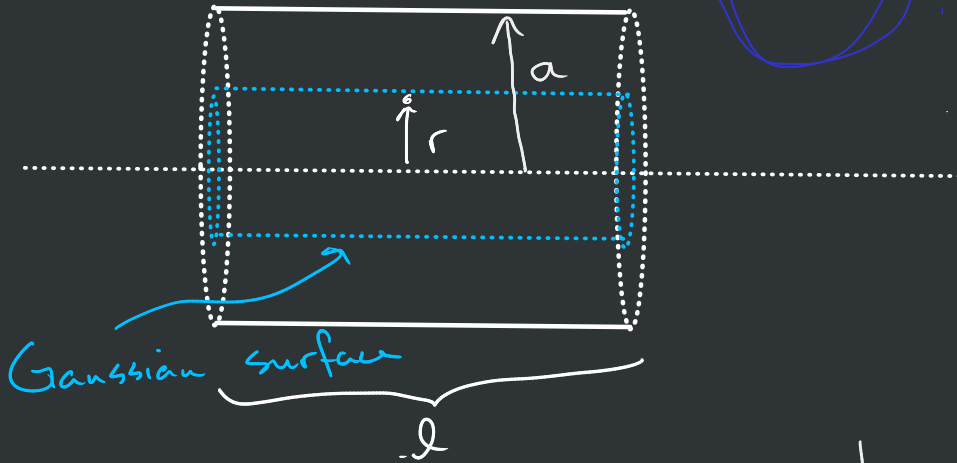
E_x p.62

$$\phi = -Kxy$$

$$\begin{aligned} \vec{E} &= -\vec{\nabla}(-Kxy) \\ &= K \vec{\nabla}(xy) = K \left(\frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y} \right) (xy) \\ &= K(y \hat{x} + x \hat{y}) \quad \checkmark \end{aligned}$$

42 | $E_{in} = \frac{\rho r}{2\epsilon_0}$ \longleftrightarrow show this w/ Gauss's Law

$$E_{out} = \frac{\rho a^2}{2\epsilon_0 r}$$



$$\int \vec{E} \cdot d\vec{a} = \frac{1}{\epsilon_0} \int \rho dV$$

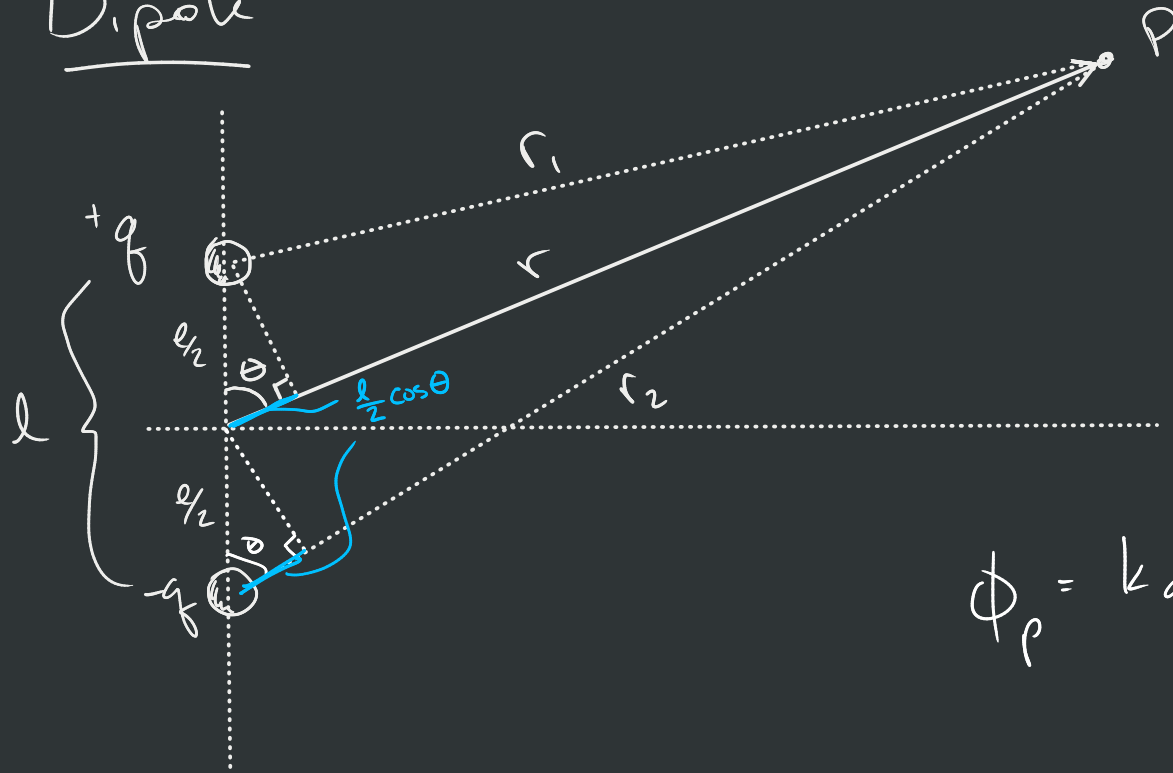
$$E \cdot \cancel{2\pi r} \cdot \cancel{2\pi r} = \frac{1}{\epsilon_0} \rho \cancel{r^2} \cancel{2\pi r}$$

$$\vec{E} = \frac{\rho r}{2\epsilon_0} \hat{r}$$

$$\phi = \int_{P_a}^{P_b} \vec{E} \cdot \frac{d\vec{s}}{\sum dr \hat{r}}$$

$$\phi = \frac{\rho a^2}{2\epsilon} \ln(\infty)$$

Dipole



$$\Phi_P = \frac{kq}{r_1} + \frac{k(-q)}{r_2}$$

$$r_1 = r - \frac{l}{2} \cos \theta$$

$$r_2 = r + \frac{l}{2} \cos \theta$$

$$\Phi_P = kq \left[\frac{1}{r \left(\frac{r - \frac{l}{2} \cos \theta}{r} \right)} - \frac{1}{r \left(\frac{r + \frac{l}{2} \cos \theta}{r} \right)} \right]$$

$$= \frac{kq}{r} \left[\frac{1}{1 - \frac{l \cos \theta}{2r}} - \frac{1}{1 + \frac{l \cos \theta}{2r}} \right]$$

$$\left(1 - \frac{l \cos \theta}{2r} \right)^{-1}$$

$$(1 \pm x)^{-1} \approx 1 \mp x$$

Taylor expansion
to 1st order

$$\Phi_P = \frac{kq}{r} \left[\cancel{1} + \frac{l \cos \theta}{2r} - \cancel{1} + \frac{l \cos \theta}{2r} \right]$$

$$\phi_p = \frac{kq}{r} \left(\frac{l \cos \theta}{r} \right)$$

$$\phi_p = \frac{kq l \cos \theta}{r^2}$$

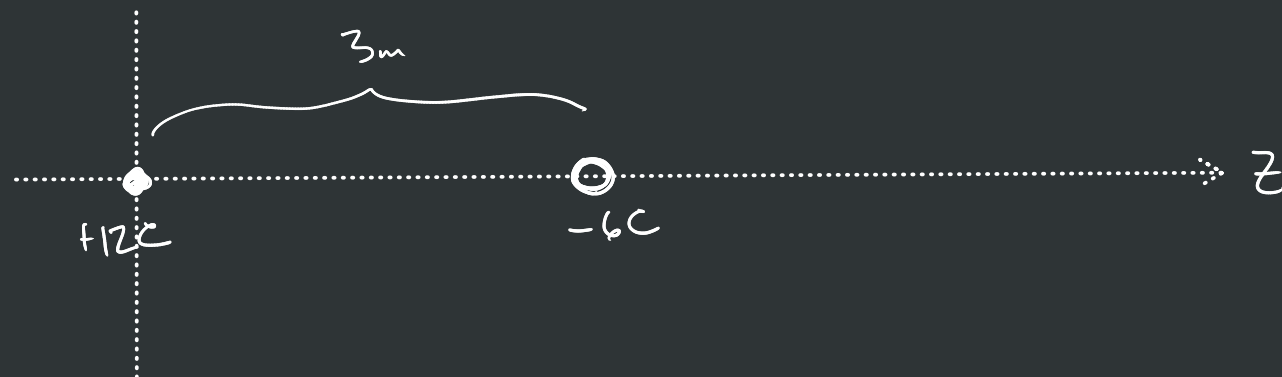
plot equipotentials

$q l \rightsquigarrow$ dipole moment $\equiv p$

$\vec{E} = ?$

$$\boxed{\phi_p = \frac{k p \cos \theta}{r^2}}$$

$$\vec{E} = -\vec{\nabla} \phi$$

33

$$\phi = \frac{q}{4\pi\epsilon_0 r}$$

$$\phi_{12C} = \frac{12C}{4\pi\epsilon_0 |z|}$$

$$\phi_{-6C} = \frac{-6C}{4\pi\epsilon_0 |z-3|}$$

$$\phi(z) = \frac{12C}{4\pi\epsilon_0 |z|} + \frac{-6C}{4\pi\epsilon_0 |z-3|}$$

$$4\pi\epsilon_0 \phi(z) = \frac{12}{|z|} + \frac{-6}{|z-3|}$$

12

