Chapter 3 - special Remington section

$$f''(x) = -k \cdot f(x)$$

$$f(x) = A\sin(\omega x)$$

$$\nabla^2 \phi = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \cancelto{0}{\frac{\partial^2 \phi}{\partial z^2}} = 0$$

Let's stick w/ two dimensions, no dependence in the z-direction

Assume $\phi = f(x) \cdot g(y)$

$$\frac{\partial^2 (f \cdot g)}{\partial x^2} + \frac{\partial^2 (f \cdot g)}{\partial y^2} = 0$$

$$g \frac{\partial^2 f}{\partial x^2} + f \frac{\partial^2 g}{\partial y^2} = 0$$ divide this equation by $f \cdot g$

$$\frac{1}{f} \cdot \frac{\partial^2 f}{\partial x^2} + \frac{1}{g} \frac{\partial^2 g}{\partial y^2} = 0$$

$$\frac{1}{f}\cdot\frac{\partial^2 f}{\partial x^2} = k^2 \qquad\qquad \frac{1}{g}\frac{\partial^2 g}{\partial y^2} = -k^2$$

$$\frac{\partial^2 f}{\partial x^2} = k^2 \cdot f \qquad\qquad \frac{\partial^2 g}{\partial y^2} = -k^2 g$$

$$f(x) = Ae^{kx} \qquad\qquad g(y) = C\sin(ky)$$

or   or

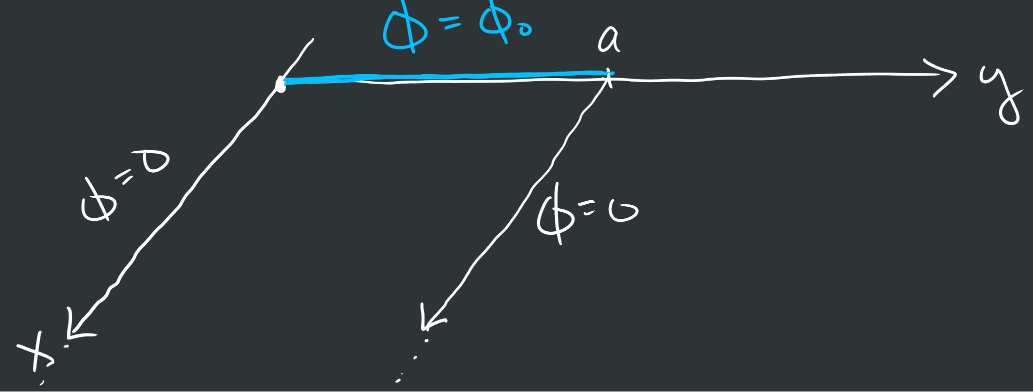
$$Be^{-kx} \qquad\qquad = D\cos(ky)$$

linear combination

So   So

$$f(x) = Ae^{kx} + Be^{-kx} \qquad\qquad g(y) = C\sin(ky) + D\cos(ky)$$

So now we need Boundary Conditions



last B.C. to use

$x=0,\ \phi=\phi_0$ + ✓ $y=0,\ \phi=0$

✓ $\underbrace{x\to\infty,\ \phi\to 0}_{Ae^{kx} \text{ blows up w/ } x\to\infty,\ \therefore A=0}$ ✓ $\underbrace{y=a,\ \phi=0}_{\cos(ky)\neq 0, \text{ when } y=0, \text{ so } D=0}$

so $\phi = f(x)\cdot g(y) = \underbrace{B e^{-kx} \cdot C}_{\text{absorb constants}} \sin(ky)$

$$= B e^{-kx} \cdot \sin(ky)$$

<u>now</u> $y = a,\ \phi = 0$ ✓

$$0 = \sin(k \cdot a)$$

$$k \cdot a = 0, \pi, 2\pi, 3\pi, \ldots$$

$$= n\pi \qquad n = 0, 1, 2 \ldots$$

$$k = \frac{n\pi}{a}$$

now $\phi(x,y) = B e^{-\frac{n\pi}{a}x} \sin\left(\frac{n\pi}{a} y\right)$

Since this is a valid solution for infinitely many n, we need a <u>weighted</u> <u>sum</u> of the individual solutions

↳ <u>linear</u> <u>combination</u>

$$\phi = \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \ldots\ldots$$

So that:

$$\nabla^2\phi = \nabla^2(\alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \ldots\ldots) = \alpha_1\nabla^2\phi_1 + \alpha_2\nabla^2\phi_2 + \ldots = 0$$

I'll absorb each α into $B \rightarrow B_n$

$$\phi(x,y) = \sum_{n=1}^{\infty} B_n e^{-\frac{n\pi}{a}x} \cdot \sin\left(\frac{n\pi}{a}y\right)$$

So now we apply the last B.C. $(x=0,\ \phi=\phi_0)$

$$\phi(0,y) = \phi_0(y) = \sum_{n=1}^{\infty} B_n \underbrace{e^{-\frac{n\pi}{a}\cdot 0}}_{1} \cdot \sin\left(\frac{n\pi}{a}y\right)$$

$$\phi_0 = \underbrace{\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{a}\cdot y\right)}_{\text{Fourier series!}}$$

Multiply both sides of the equation by $\sin\left(\frac{n'\pi}{a}y\right)$ where n' is another constant. And then integrate!

$$\int_0^a \phi_0 \sin\left(\frac{n'\pi}{a} y\right) dy = \sum_{n=1}^{\infty} B_n \underbrace{\int_0^a \sin\left(\frac{n\pi}{a} y\right) \cdot \sin\left(\frac{n'\pi}{a} y\right) dy}_{= \frac{a}{2} \text{ only when } n = n'}$$

$$\int_0^a \phi_0 \sin\left(\frac{n'\pi}{a} y\right) dy = \sum_{n=1}^{\infty} B_n \frac{a}{2} \delta_{nn'} = B_{n'} \frac{a}{2}$$

$$B_{n'} = \frac{2}{a} \int_0^a \phi_0 \sin\left(\frac{n'\pi}{a} y\right) dy$$

$$B_n = \frac{2}{a} \int_0^a \phi_0 \sin\left(\frac{n\pi}{a} y\right) dy$$

$$B_n = \frac{2\phi_0}{a} \int_0^a \sin\left(\frac{n\pi}{a} y\right) dy$$

$$= \frac{2\phi_0}{a} \cdot \left(\frac{a}{n\pi}\right) \left(-\cos\left(\frac{n\pi}{a} y\right)\right)\Big|_0^a$$

So $B_n = \frac{2\phi_0}{\cancel{a}} \cdot \left(\frac{\cancel{a}}{n\pi}\right) \cdot 2$

$$-\left[\cos\left(\frac{n\pi}{\cancel{a}}\cancel{a}\right) - \cos(0)\right]$$

$$-\left[\cos(n\pi) - 1\right]$$

$$\underbrace{\underbrace{-\cos(n\pi)}_{\substack{-1 \text{ for odd } n \\ +1 \text{ for even } n}} + 1}_{\substack{+2 \text{ for odd } n \\ 0 \text{ for even } n}}$$

for odd values of n

$$B_n = \frac{4\phi_0}{n\pi} \quad \text{where } n \text{ is odd.}$$

$$\phi(x,y) = \sum_{n=1}^{\infty} B_n e^{-\frac{n\pi}{a}x} \cdot \sin\left(\frac{n\pi}{a}y\right)$$

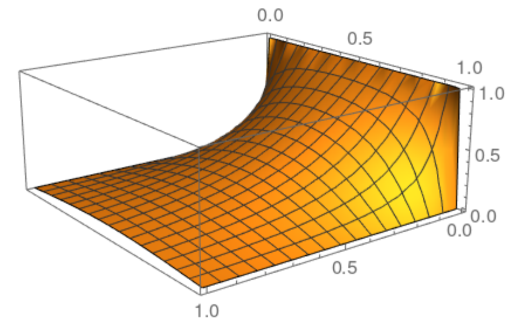
$$\phi(x,y) = \frac{4\phi_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \cdot e^{-\frac{n\pi}{a}x} \cdot \sin\left(\frac{n\pi}{a} \cdot y\right)$$

lets plot in mathematica!

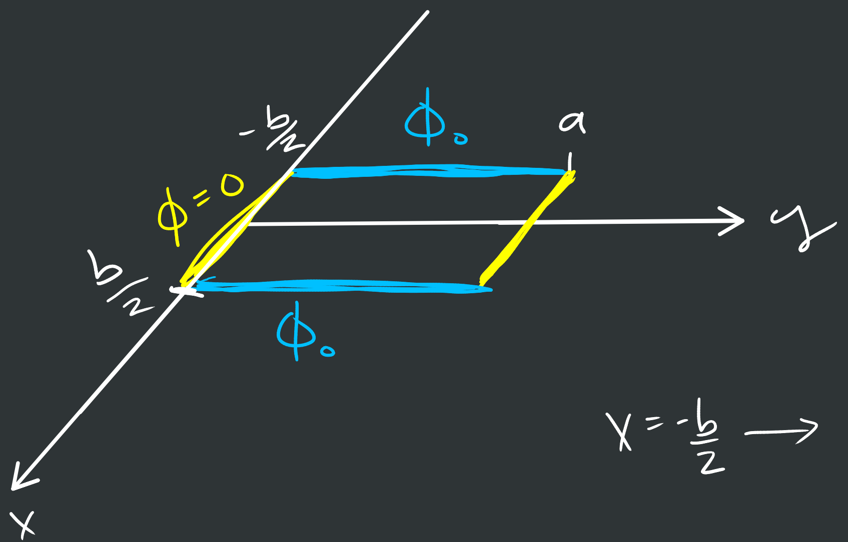
$n \to 2n+1$

In[20]:=
```
Plot3D[
 4 * 1 / Pi *
  Sum[1 / (2 n + 1) * Exp[- (2 n + 1) * Pi / 1 * x] *
    Sin[(2 n + 1) * Pi / 1 * y], {n, 0, 1000}],
 {x, 0, 1}, {y, 0, 1}]
```

Out[20]=



Now for another example:



$$\frac{\partial^2 f}{\partial x^2} = k^2 f \qquad \frac{\partial^2 g}{\partial y^2} = -k^2 g$$

$$f = Ae^{kx} + Be^{-kx} \qquad g = C\sin(ky) + \underbrace{D\cos(ky)}_{D=0}$$

since $\phi = 0$ at $y = 0$

$$x = -\frac{b}{2} \longrightarrow \phi_o = Ae^{-kb/2} + Be^{+k\cdot b/2}$$

and

$$x = +\frac{b}{2} \qquad \underbrace{\phi_o = Ae^{+kb/2} + Be^{-k\frac{b}{2}}}$$

so this can only be true if

$A = B$

$$f = A\left(e^{kx} + e^{-kx}\right)$$

$$g = C\sin ky$$

$\phi = 0,\ y = a$

so $0 = C\sin(ka)$

$0 = \sin(ka)$

$ka = 0\pi, 1\pi, 2\pi, 3\pi$

$ka = n\pi \qquad n = 1, 2, 3 \ldots$

$$k = \frac{n\pi}{a}$$

So...
$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \underbrace{\left(e^{kx} + e^{-kx}\right)}_{A} \sin(ky)$$

$2 \cdot \cosh(kx)$ $\longrightarrow$ $\cosh(kx) = \dfrac{e^{kx} + e^{-kx}}{2}$

$\sinh(kx) = \dfrac{e^{kx} - e^{-kx}}{2}$

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \cosh(kx) \sin(ky)$$

$\cos(kx) = \dfrac{e^{ikx} + e^{-ikx}}{2}$

$\sin(kx) = \dfrac{e^{ikx} - e^{-ikx}}{2i}$

So now, last BC.

$\rightarrow \phi = \phi_0$ at $x = b$

$$\phi_0 = \sum_{n=1}^{\infty} A_n \cosh(kb) \cdot \sin(ky)$$

?

now we apply Fourier's trick

$$\underbrace{\int_0^a \phi_0 \sin\left(\frac{n'\pi}{a} y\right) dy}_{\phi_0 \left(\frac{a}{n'\pi}\right) \underbrace{\left(-\cos\left(\frac{n'\pi}{a} y\right)\right)\Big|_0^a}_{\substack{=2 \text{ odd } n' \\ =0 \text{ even } n'}}} = \underbrace{\sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi}{a} b\right) \int_0^a \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{n'\pi}{a} y\right) dy}_{\sum_{n=1}^{\infty} A_n \cosh\left(\frac{n\pi}{a} b\right) \cdot \frac{a}{2} \delta_{nn'}}$$

$$\phi_0 \left(\frac{a}{n'\pi}\right) \left(-\cos\left(\frac{n'\pi}{a} y\right)\right)\Big|_0^a = A_{n'} \cosh\left(\frac{n'\pi}{a} b\right) \cdot \frac{a}{2}$$

$$\phi_0 \cdot \frac{2\cancel{a}}{n'\pi} = A_{n'} \cosh\left(\frac{n'\pi}{a} b\right) \cdot \frac{\cancel{a}}{2}$$

drop primes

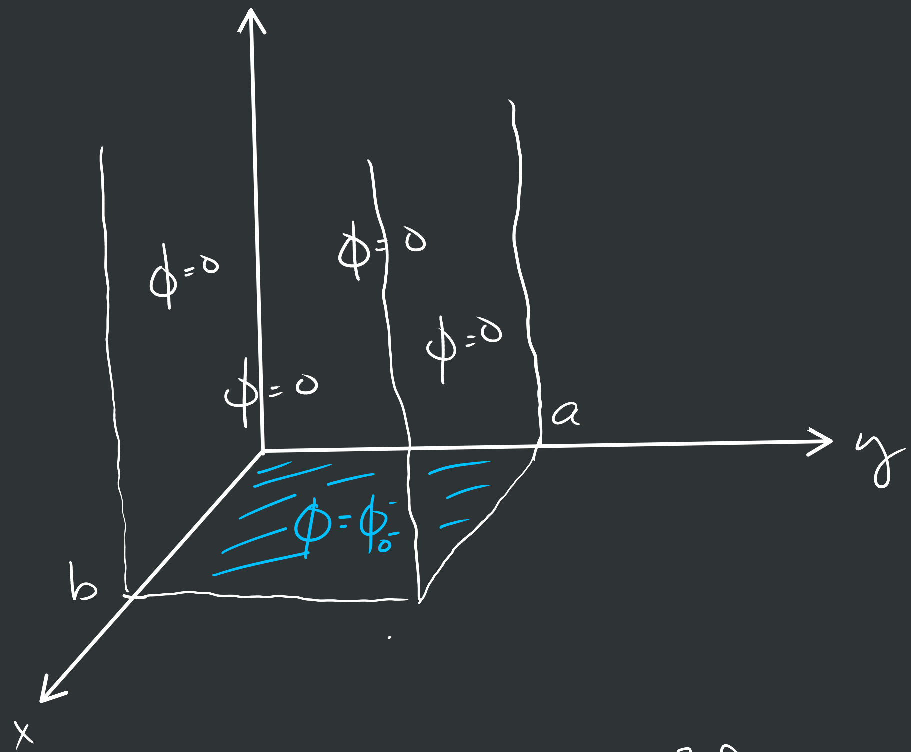
solve for A_n

$$A_n = \underbrace{\frac{4\phi_0}{n\pi \cosh\left(\frac{n\pi}{a} b\right)}}$$ for odd n

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n \cosh(kx) \sin(ky)$$

$$\phi(x,y) = \frac{4\phi_0}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n} \frac{\cosh\left(\frac{n\pi}{a} x\right)}{\cosh\left(\frac{n\pi b}{a}\right)} \cdot \sin\left(\frac{n\pi}{a} y\right)$$

Now for 3-d



list the BCs

$\phi=0,\ x=0,\ x=b$ } sin/cos

$y=0,\ y=a$ } sin/cos

$z\to\infty$, $\phi=\phi_0,\ z=0$ } exp

$$\underbrace{\frac{1}{f}\frac{\partial^2 f}{\partial x^2}}_{=-k^2}+\underbrace{\frac{1}{g}\frac{\partial^2 g}{\partial y^2}}_{=-l^2}+\underbrace{\frac{1}{h}\frac{\partial^2 h}{\partial z^2}}_{=k^2+l^2}=0$$

separated variable for 3D Laplace equation

$$\frac{\partial^2 f}{\partial x^2}=-k^2 f$$

$$f(x)=A\sin(kx)+\cancel{B\cos(kx)} \quad \Big| \quad g(y)=C\sin(ly)+\cancel{D\cos(ly)} \quad \Big| \quad h(z)=\cancel{Fe^{\sqrt{k^2+l^2}\,z}}+Ge^{-\sqrt{k^2+l^2}\,z}$$

$x=0$: $B=0$

$y=0$: $D=0$

$z\to\infty$: $F=0$

b/c of B.C.s

So now $\rightarrow$ $kb = n\pi$ | | $\sqrt{k^2 + l^2} = \sqrt{\left(\frac{n\pi}{b}\right)^2 + \left(\frac{m\pi}{a}\right)^2}$

$k = \frac{n\pi}{b}$ | $l = \frac{m\pi}{a}$ |

$n = 1, 2, 3 \ldots$ $m = 1, 2, 3 \ldots$

$$\phi(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{b} x\right) \sin\left(\frac{m\pi}{a} y\right) e^{-\pi\sqrt{\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2}\, z}$$

$$\phi(x, y, 0) = \phi_0 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin\left(\frac{n\pi}{b} x\right) \sin\left(\frac{m\pi}{a} y\right)$$

$$= \sum_n \sum_m A_{nm} \underbrace{\int_0^b \sin\left(\frac{n\pi}{b} x\right) \sin\left(\frac{n'\pi}{b} x\right) dx}_{= \frac{b}{2} \delta_{nn'}} \underbrace{\int_0^a \sin\left(\frac{m\pi}{a} y\right) \sin\left(\frac{m'\pi}{a} y\right) dy}_{= \frac{a}{2} \delta_{mm'}}$$

$$\phi_0 \underbrace{\int_0^b \sin\left(\frac{n'\pi}{b} x\right) dx}_{1} \underbrace{\int_0^a \sin\left(\frac{m'\pi}{a} y\right) dy}_{1} = A_{n'm'} \left(\frac{b}{2}\right)\left(\frac{a}{2}\right)$$

$\frac{2b}{n'\pi}$ $n = 1, 3, 5 \ldots$ $\frac{2a}{m'\pi}$ $m = 1, 3, 5 \ldots$

drop primes

$$A_{nm} = \frac{16\phi_0}{nm\pi^2}$$ $\underline{\underline{n + m \text{ are odd}}}$

$$\boxed{\phi(x, y, z) = \frac{16\phi_0}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \sum_{m=1,3,5\ldots}^{\infty} \frac{1}{nm} \sin\left(\frac{n\pi}{b} x\right) \sin\left(\frac{m\pi}{a} y\right) e^{-\pi\sqrt{\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2} \cdot z}}$$

Spherical Separation of Variables

- Separation of variables

  ① completeness $\longrightarrow g(x) = \sum_{n=1}^{\infty} C_n f_n(x)$

  any function

  "complete set"

  ② orthogonality

  $\hookrightarrow$ "Fourier's trick"

$$\nabla^2 \Phi = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \underline{\quad\quad} = 0$$

assume azimuthal symmetry

Assume $\Phi(r,\theta) = R(r) \cdot \Theta(\theta)$ capital theta

plug this in for Φ and divide by Φ and multiply by r^2 on both sides.

$$\underbrace{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)}_{=\ell(\ell+1)} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)}_{=-\ell(\ell+1)} = 0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = R\cdot\ell\cdot(\ell+1)$$

Guess: $R(r) = Ar^{\ell} + Br^{-(\ell+1)}$

$$\boxed{\phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^{\ell} + B_\ell r^{-(\ell+1)}\right)P_\ell(\cos\theta)}$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right) = -\ell(\ell+1)\cdot\Theta\sin\theta$$

I'll give the solution

$\Theta(\theta) = P_\ell(\cos\theta)$

"Legendre polynomial"

$P_0(x) = 1$

$P_1(x) = x^1$

$P_2(x) = \dfrac{3x^2-1}{2}$

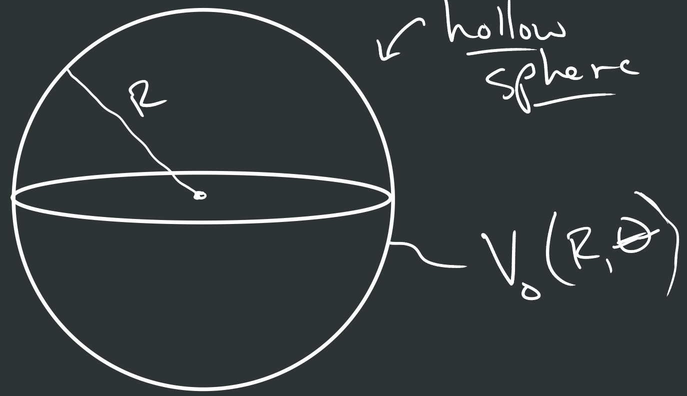
$P_3(x) = \dfrac{5x^3-3x}{2}$

$P_4(x) = \dfrac{35x^4-30x^2+3}{8}$

What is the pattern?

$$P_\ell(x) \equiv \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$

So lets do an example:



What is the potential inside the sphere?

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right)P_\ell(\cos\theta)$$

$B_\ell = 0$, otherwise this blows up at origin

$$\phi(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$\phi(R,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta)$$

multiply on both sides by $P_{\ell'}(\cos\theta)\cdot\sin\theta$ and integrate

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x) dx = \int_{\pi}^{0} P_\ell(\cos\theta) P_{\ell'}(\cos\theta)(-\sin\theta) d\theta = \underbrace{\int_{0}^{\pi} P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta \, d\theta}$$

$$x = \cos\theta \longrightarrow \begin{matrix} \theta = \pi & x = -1 \\ \theta = 0 & x = 1 \end{matrix}$$

$$dx = \frac{dx}{d\theta} d\theta \quad dx = -\sin\theta \, d\theta$$

$$= \begin{cases} 0 & \ell' \neq \ell \\ \dfrac{2}{2\ell+1} & \ell' = \ell \end{cases}$$

$$\delta_{\ell'\ell} = \begin{cases} 0 & \ell' \neq \ell \\ 1 & \ell' = \ell \end{cases}$$

$$\int_0^\pi \phi(R,\theta) \cdot P_{\ell'}(\cos\theta) \cdot \sin\theta \, d\theta = \sum_{\ell=0}^{\infty} A_\ell R^\ell \int_0^\pi P_\ell(\cos\theta) \cdot P_{\ell'}(\cos\theta) \sin\theta \, d\theta$$

$$\int_0^\pi \phi(R,\theta) \cdot P_{\ell'}(\cos\theta) \cdot \sin\theta \, d\theta = A_{\ell'} R^{\ell'} \cdot \frac{2}{2\ell'+1}$$

drop primes

$$A_\ell = \frac{2\ell+1}{2R^\ell} \int_0^\pi \underline{\underline{\phi(R,\theta)}} \cdot P_\ell(\cos\theta) \cdot \sin\theta \, d\theta$$

what is the B.C.?

Suppose: $\phi(R,\theta) = k\sin^2\left(\frac{\theta}{2}\right)$

look at trig identities

$$= k\left(\frac{1-\cos\theta}{2}\right)$$

$$= \frac{k}{2}\left(1-\cos\theta\right)$$

$$= \frac{k}{2}\left(P_0(\cos\theta) - P_1(\cos\theta)\right)$$

$$A_\ell = \frac{2\ell+1}{2R^\ell}\cdot\frac{k}{2}\cdot\left[\underbrace{\int_0^\pi P_0(\cos\theta)\,P_\ell(\cos\theta)\sin\theta\,d\theta}_{=\begin{cases}\frac{2}{2\ell+1} & \ell=0\\ 0 & \ell\neq 0\end{cases}\;=2} - \underbrace{\int_0^\pi P_1(\cos\theta)\cdot P_\ell(\cos\theta)\sin\theta\,d\theta}_{=\begin{cases}\frac{2}{2\ell+1} & \ell=1\\ 0 & \ell\neq 1\end{cases}\;=\frac{2}{3}}\right]$$

$$\phi(r,\theta) = \frac{k}{2}\left(r^0 P_0(\cos\theta) - \frac{r^1}{R}P_1(\cos\theta)\right)$$

$$\phi(r,\theta) = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right) \longleftarrow \text{very simple!}$$