Coherence - Chapter 9

↳ correlation between phases of light phase

coherent light → constant phase relationship

incoherent light → random phase relationship

Fourier Analysis

$$\cos\alpha + \cos\beta \neq \cos\gamma \qquad \gamma(\alpha,\beta)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

periodic, T

$\omega = \frac{2\pi}{T}$

$$\left.f(t) = \sum_{m=0}^{\infty} a_m \cos(m\omega t) + \sum_{m=0}^{\infty} b_m \sin(m\omega t)\right\} \text{Fourier Series}$$

Complete basis set

- sin + cos
- Legendre polynomials
- Hermite
- Laguerre
- Bessel

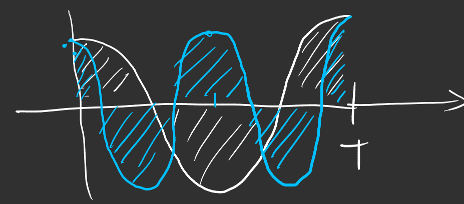
$m=0$, b_m → does not matter

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t)$$

$\frac{1}{2}$ convenience

So how do we find the other coefficients

$$\int_0^T f(t)\,dt = \int_0^T \frac{a_0}{2}\,dt + 0$$



$$\int_0^T f(t)\,dt = \frac{a_0}{2}T$$

$$\omega = \frac{2\pi}{T}$$

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt$$

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t)$$

$$\int_0^T f(t)\cos(n\omega t)\,dt = \underbrace{\int_0^T \frac{a_0}{2}\cos(n\omega t)\,dt}_{=0} + \underbrace{\sum_{m=1}^{\infty}\int_0^T a_m \cos(m\omega t)\cos(n\omega t)\,dt}_{\substack{=0 \quad m\neq n \\ =\frac{T}{2}a_n \quad m=n}} + \underbrace{\sum_{m=1}^{\infty}\int_0^T b_m \sin(m\omega t)\cos(n\omega t)\,dt}_{=0}$$

$$\int_0^T f(t)\cos(n\omega t)\,dt = \frac{T}{2}a_n \implies \boxed{a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt}$$

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(m\omega t) + \sum_{m=1}^{\infty} b_m \sin(m\omega t)$$

$$\int_0^T f(t)\sin(n\omega t)\,dt = \underbrace{\int_0^T \frac{a_0}{2}\sin(n\omega t)\,dt}_{=0} + \underbrace{\sum_{m=1}^{\infty}\int_0^T a_m \cos(m\omega t)\sin(n\omega t)\,dt}_{=0} + \underbrace{\sum_{m=1}^{\infty}\int_0^T b_m \sin(m\omega t)\sin(n\omega t)\,dt}_{\substack{=0 \quad m \neq n \\ =\frac{T}{2}b_n \quad m=n}}$$

$$\int_0^T f(t)\sin(n\omega t)\,dt = \frac{T}{2} b_n \Rightarrow \boxed{b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt}$$

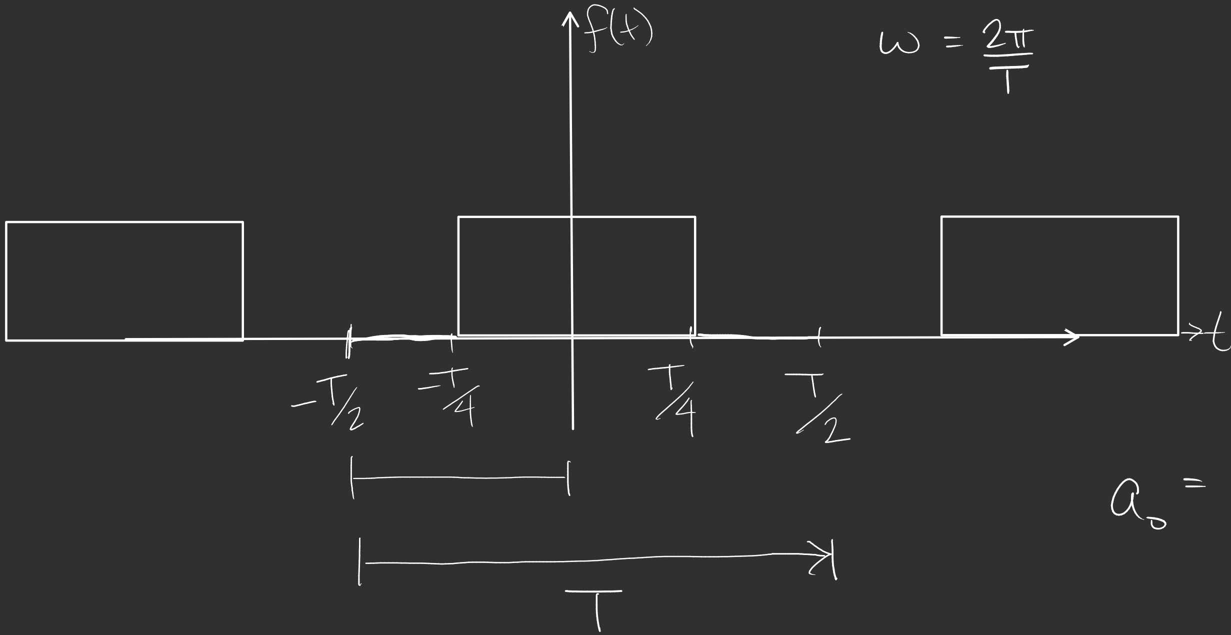
$$\int_0^T \sin(n\omega t)\sin(m\omega t)\,dt = \frac{T}{2}\delta_{nm}$$

Kronecker delta

$$\delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

$$\int_0^T \cos(n\omega t)\cos(m\omega t)\,dt = \frac{T}{2}\delta_{nm}$$

$$\int_0^T \sin(n\omega t)\cos(m\omega t)\,dt = 0$$



$$\omega = \frac{2\pi}{T}$$

$$f(t) = \begin{cases} 0 & -T/2 < t < -T/4 \\ 1 & -T/4 < t < T/4 \\ 0 & T/4 < t < T/2 \end{cases}$$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt = \frac{2}{T}\int_{-T/4}^{T/4} 1\,dt$$

$$= \frac{2}{T}\,T\Big|_{-T/4}^{T/4} = \frac{2}{T}\left[\frac{T}{4} - \left(-\frac{T}{4}\right)\right]$$

$$= \frac{2}{T}\left[\frac{2T}{4}\right]$$

$$= 1$$

$$a_n = \frac{2}{T}\int_{-T/4}^{T/4} 1\cdot\cos(n\omega t)\,dt$$

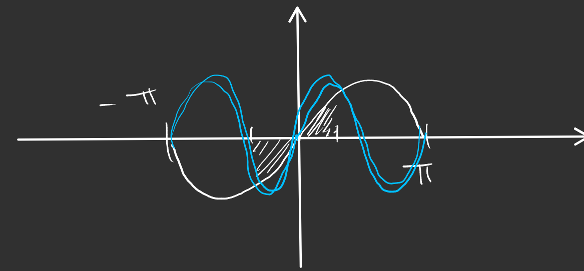
$$= \frac{2}{T}\left(\frac{\sin(n\omega t)}{n\omega}\right)\Bigg|_{-T/4}^{T/4}$$

$$= \frac{2}{n\omega T}\left[\sin\left(\frac{n\omega T}{4}\right) + \sin\left(+\frac{n\omega T}{4}\right)\right]$$

$$= \frac{2}{n\omega T}\cdot 2\sin\left(\frac{n\omega T}{4}\right) \qquad \omega = \frac{2\pi}{T}$$

$$= \frac{2\cancel{T}}{n(\cancel{2\pi})\cancel{T}}\cdot\cancel{2}\sin\left(\frac{n(\cancel{2\pi})T}{\underset{2}{\cancel{4}}\cancel{T}}\right) = \boxed{\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) = a_n}$$

$$b_n = \int_{-T/4}^{T/4} 1 \cdot \sin(n\omega t)\,dt = 0$$



$$f(t) = \frac{1}{2} + \sum_{m=0}^{\infty} \frac{2}{m\pi} \sin\left(\frac{m\pi}{2}\right) \cos(m\omega t)$$

What about complex notation

$$f(t) = \sum_{m=0}^{\infty} a_m \cos(m\omega t) + \sum_{m=0}^{\infty} b_m \sin(m\omega t)$$

$$\cos(m\omega t) = \mathrm{Re}\left[e^{-im\omega t}\right] = \frac{e^{-im\omega t} + e^{+im\omega t}}{2}$$

plane wave

$$Ae^{i(kx-\omega t)}$$

$$x=0$$

$$Ae^{-i\omega t}$$

$$\sin(m\omega t) = \mathrm{Im}\left[e^{-im\omega t}\right] = \frac{e^{-im\omega t} - e^{+im\omega t}}{2i}$$

$$f(t) = \sum_{m=0}^{\infty} a_m e^{-im\omega t} + \sum_{m=0}^{\infty} a_m e^{+im\omega t} + \sum_{m=0}^{\infty} b_m e^{-im\omega t} - \sum_{m=0}^{\infty} b_m e^{+im\omega t}$$

$$= \sum_{m=0}^{\infty} \underbrace{(a_m + b_m)}_{A_m} e^{-im\omega t} + \sum_{m=0}^{\infty} \underbrace{(a_m - b_m)}_{B_m} e^{+im\omega t}$$

$$= \sum_{m=0}^{\infty} A_m e^{-im\omega t} + \sum_{m=0}^{\infty} B_m e^{+im\omega t}$$

$$m = -m'$$

$$\sum_{m'=0}^{-\infty} B_{m'} e^{-im'\omega t}$$

$$m' = m$$

$$\sum_{m=0}^{-\infty} B_m e^{-im\omega t}$$

$$f(t) = \sum_{m=0}^{\infty} A_m e^{-im\omega t} + \sum_{m=0}^{-\infty} B_m e^{-im\omega t}$$

$$f(t) = \sum_{m=-\infty}^{\infty} \underbrace{(A_m + B_m)}\, e^{-im\omega t}$$

$$\boxed{f(t) = \sum_{m=-\infty}^{\infty} c_m e^{-im\omega t}}$$

$$\int^T f(t) e^{+in\omega t} dt = \sum_{m=-\infty}^{\infty} c_m \underbrace{\int^T e^{-im\omega t} \cdot e^{+in\omega t} dt}_{= T\delta_{nm}}$$

$$\int^T f(t) e^{+in\omega t} = c_n T$$

$$\boxed{c_n = \frac{1}{T}\int^T f(t) e^{+in\omega t} dt}$$

Lets add all frequencies (not just the integer multiples)

$$f(t) = \underbrace{\sum_{m=-\infty}^{+\infty} C_m e^{-im\omega t} \underbrace{\Delta\omega}_{1}}_{\text{Riemann Sum}} \longrightarrow f(t) = \int_{-\infty}^{\infty} g(\omega) e^{-i\omega t}\, d\omega$$

$\cancel{C_m}$ $g(\omega)$ is continous

$$\Delta\omega = n\omega - (n-1)\omega$$
$$= \omega = \frac{2\pi}{T}$$

$$\Delta\omega = \frac{2\pi}{T} \qquad T \longrightarrow \infty$$

$\Delta\omega \downarrow d\omega$

not periodic

$f(t)$ is not periodic; any function of t

$$\int_{-\infty}^{\infty} e^{-it(\omega-\omega')}\, dt$$

Employ orthogonality to find $g(\omega)$.

$$\int_{-\infty}^{\infty} f(t) \underline{\ ?\ } dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\omega) e^{-i\omega t} \underline{\ ?\ }\, d\omega\, dt = \int_{-\infty}^{\infty} g(\omega) \underbrace{\left[\int_{-\infty}^{\infty} e^{-i\omega t} e^{+i\omega' t}\, dt\right]}_{\delta(\omega-\omega')} d\omega$$

$e^{+i\omega' t}$

$$= g(\omega')$$

Dirac delta "function")

$$\int_{-\infty}^{\infty} f(t) e^{+i\omega' t}\, dt = 2\pi \cdot g(\omega')$$

$$\boxed{g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\, e^{+i\omega t}\, dt}$$

Fourier Transform of $f(t)$

$$\boxed{f(t) = \int_{-\infty}^{\infty} g(\omega)\, e^{-i\omega t}\, d\omega}$$

Inverse Fourier Transform

$\omega = 2\pi f$

$$f(t) = \mathrm{IFT}[g(\omega)]$$

$$\mathrm{FT}[\mathrm{IFT}[g(\omega)]] = g(\omega)$$

$$f(t) = \mathrm{IFT}[\mathrm{FT}[f(t)]]$$

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\, dx = f(0)$$

$$\delta(x - x') = \begin{cases} \infty & x = x' \\ 0 & \text{elsewhere} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x - x')\, dx = f(x')$$

Spacial Fourier Analysis

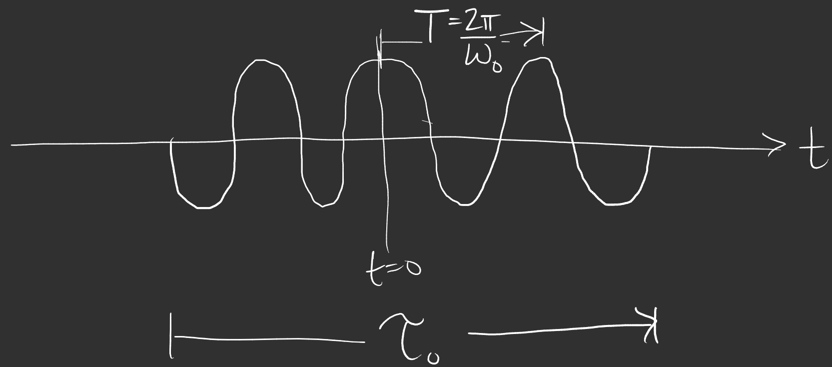
$\omega \longrightarrow k = \frac{2\pi}{\lambda}$

$\omega = \frac{2\pi}{T}$

$T \longrightarrow \lambda$ (book $\to L$)

$$\underbrace{f(x) = \int_{-\infty}^{\infty} g(k)\, e^{-ikx}\, dk \qquad g(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx}_{\text{spacial IFT + FT}}$$

Example



$$f(t) = \begin{cases} e^{-i\omega_0 t} & -\frac{\tau_0}{2} < t < \frac{\tau_0}{2} \\ 0 & \text{at all other times (elsewhen)} \end{cases}$$

frequency spectrum $\rightarrow$ $g(\omega) = \frac{1}{2\pi}\int_{-\tau_0/2}^{\tau_0/2} e^{-i\omega_0 t} \cdot e^{+i\omega t}\, dt$

$$= \frac{1}{2\pi}\int_{-\tau_0/2}^{\tau_0/2} e^{it(\omega-\omega_0)}\, dt$$

$$= \frac{1}{2\pi} \cdot \frac{1}{i(\omega-\omega_0)} \cdot e^{it(\omega-\omega_0)} \Big|_{-\tau_0/2}^{\tau_0/2}$$

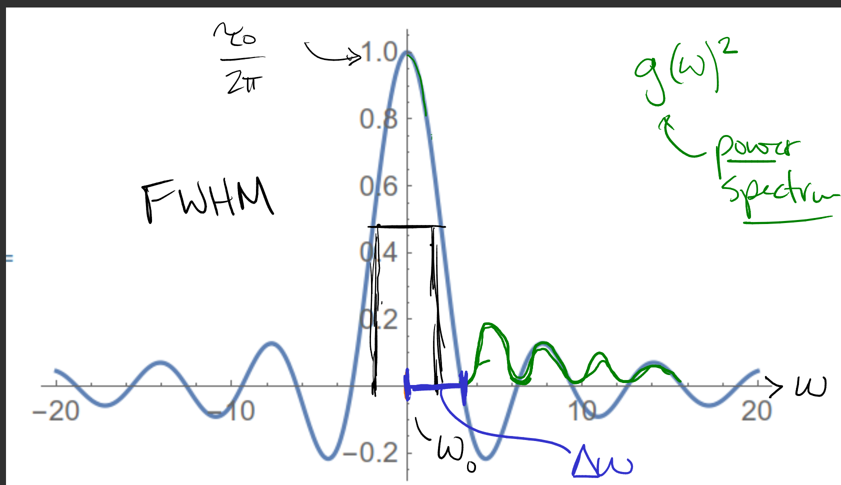
$$= \frac{1}{\pi} \cdot \frac{1}{(\omega-\omega_0)} \cdot \underbrace{\frac{e^{i\frac{\tau_0}{2}(\omega-\omega_0)} - e^{-i\frac{\tau_0}{2}(\omega-\omega_0)}}{2i}}_{\sin\left(\frac{\tau_0}{2}(\omega-\omega_0)\right)}$$

$$g(\omega) = \frac{\sin\left(\frac{\tau_0}{2}(\omega-\omega_0)\right)}{\pi(\omega-\omega_0)} = \frac{\frac{\tau_0}{2}\sin\left(\frac{\tau_0}{2}(\omega-\omega_0)\right)}{\underbrace{\pi\left(\frac{\tau_0}{2}(\omega-\omega_0)\right)}}$$

$$g(\omega_0) = \frac{\tau_0}{2\pi}$$

$$\mathrm{sinc}(u) = \frac{\sin(u)}{u}$$

$$\mathrm{sinc}(0) = 1$$



$$\sin\left(\frac{\tau_0}{2}(\omega-\omega_0)\right) = 0$$

$$\frac{\tau_0}{2}(\omega-\omega_0) = n\pi$$

$$\omega = \omega_0 + \frac{2n\pi}{\tau_0}$$

ω_0

$$\Delta\omega = \frac{2\pi}{\tau_0}$$

$$\Delta\omega = 2\pi\,\Delta\nu$$

$$\boxed{\Delta\nu = \frac{1}{\tau_0}}$$

frequency bandwidth

$\tau_0 \rightsquigarrow$ coherence time

$l_t \rightarrow$ coherence length

$l_t = c\,\tau_0$

$\lambda_{red} = 770\,nm$

$\lambda_{violet} = 380\,nm$

$\} \; 575\,nm \rightsquigarrow \nu = \frac{c}{\lambda} = 5.2 \cdot 10^{14}\,Hz$

$\hookrightarrow \nu_{violet} = 7.9 \cdot 10^{14}\,Hz$

$\Delta\nu = 2.7 \cdot 10^{14}\,Hz$

$\tau_0 = \frac{1}{2.7 \cdot 10^{14}\,Hz} = 1.3 \cdot 10^{-15}\,s$

$l_t = c \cdot \tau_0 = 3.8 \cdot 10^{-7}\,m$

$= 380 \cdot 10^{-9}\,m$

$= \underline{\underline{380\,nm}}$

HW: 1, 2, 3, 4, 6