Chapter 5 - Superposition

superposition – waves at the same place and time, displacements add up.

$$\psi = \psi_1 + \psi_2 + \dots$$

- superposition of harmonic waves

$$E_1(x_1,t) = E_1 \cos(kx_1 - \omega t + \phi_1)$$

$$E_2(x_2,t) = E_2 \cos(\underbrace{kx_2 - \omega t + \phi_2})$$

$$\alpha_1 = ks_1 + \phi_1$$

$$\alpha_2 = ks_2 + \phi_2$$

$$\underbrace{\alpha_2 - \alpha_1}_{\text{phase difference}} = k(s_2 - s_1) + (\phi_2 - \phi_1)$$

what if $\alpha_2 - \alpha_1 = 2\pi m$ } even multiple of π

$$E_R = E_1 + E_2 = E_1 \cos(\alpha_1 - \omega t) + E_2 \cos(\alpha_2 - \omega t)$$

$\alpha_2 = \alpha_1 + 2\pi m$

$\cos(\alpha_1 + 2\pi m - \omega t)$

$\cos\theta = \cos(\theta + 2\pi m)$

constructive interference

$$\hookrightarrow E_R = (E_1 + E_2)\cos(\alpha_1 - \omega t)$$

what if $\alpha_2 - \alpha_1 = (2m-1)\pi$ odd multiple of π

$$E_R = E_1 + E_2 = E_1 \cos(\alpha_1 - \omega t) + E_2 \cos(\alpha_2 - \omega t)$$

$$\Big|\ \alpha_2 = \alpha_1 + (2m-1)\pi$$

$$\downarrow\ -\cos\theta = \cos(\theta + (2m-1)\pi)$$

destructive interference

$$E_R = (E_1 - E_2)\cos(\alpha_1 - \omega t)$$

what about others?

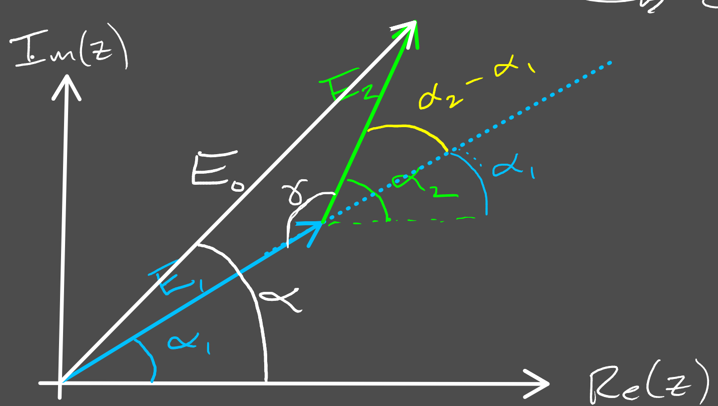
$$E_R = Re\left(E_1 e^{i(\alpha_1 - \omega t)} + E_2 e^{i(\alpha_2 - \omega t)}\right)$$

$$E_1 e^{i\alpha_1} e^{-i\omega t} + E_2 e^{i\alpha_2} e^{-i\omega t}$$

$$E_R = Re\left(e^{-i\omega t}\left(E_1 e^{i\alpha_1} + E_2 e^{i\alpha_2}\right)\right)$$

"phasor diagram"

$\hookrightarrow$ complex as vector



$$E_0^2 = E_1^2 + E_2^2 - 2E_1E_2\cos\gamma$$

$$\Big|\ \gamma = \pi - (\alpha_2 - \alpha_1)$$

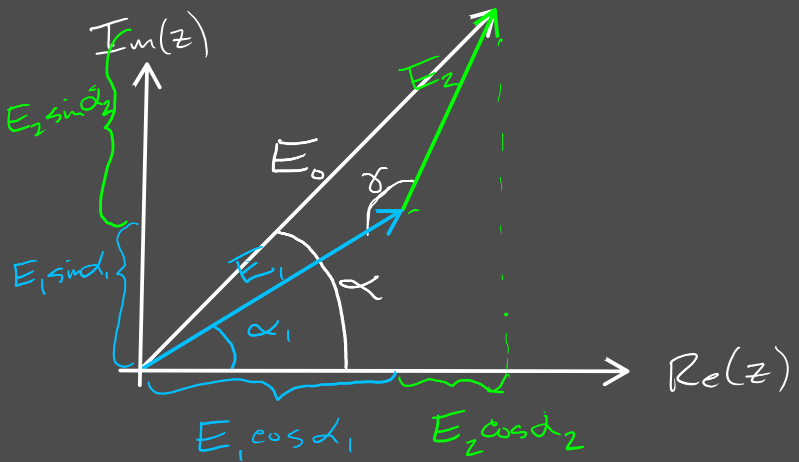
$$= \pi - \alpha_2 + \alpha_1$$

$$= \pi + \alpha_1 - \alpha_2$$

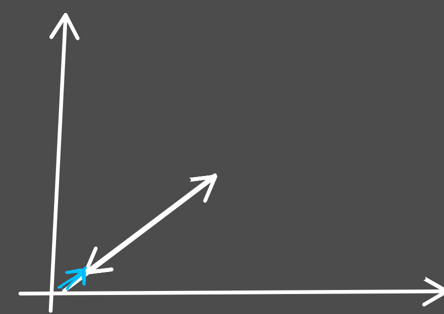
$$\cos(\pi + \alpha_1 - \alpha_2)$$

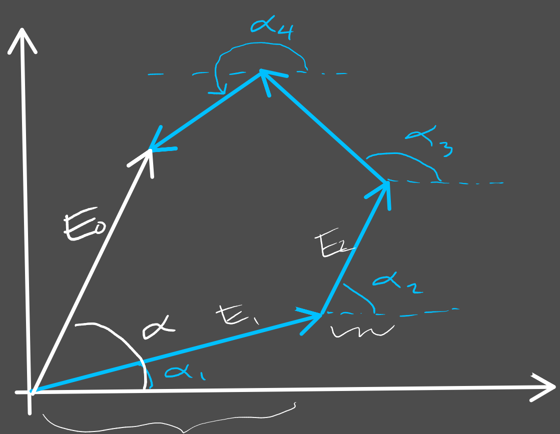
$$= -\cos(\alpha_1 - \alpha_2)$$

$$E_0^2 = E_1^2 + E_2^2 + 2E_1E_2\cos(\alpha_1 - \alpha_2)$$



$$\tan\alpha = \frac{E_1\sin\alpha_1 + E_2\sin\alpha_2}{E_1\cos\alpha_1 + E_2\cos\alpha_2}$$





$$\tan\alpha = \frac{\sum_i^n E_i \sin\alpha_i}{\sum_i^N E_i \cos\alpha_i}$$

$$E_0^2 = \left(\sum_i^N E_i \cos\alpha_i\right)^2 + \left(\sum_i^n E_i \sin\alpha_i\right)^2$$

$$\left(\sum_i^N E_i \cos\alpha_i\right)^2 = \sum_i^N E_i^2 \cos^2\alpha_i + \sum_i^N 2E_i\cos\alpha_i \sum_{j>i}^N E_j\cos\alpha_j$$

$$2\sum_i^N \sum_{j>i}^N E_iE_j\cos\alpha_i\cos\alpha_j$$

$$\left(\sum_i^n E_i \sin\alpha_i\right)^2 = \sum_i^N E_i^2 \sin^2\alpha_i + 2\sum_i^N \sum_{j>i}^N E_iE_j\sin\alpha_i\sin\alpha_j$$

$$E_0^2 = \sum_i^N E_i^2 \cos^2\alpha_i + \sum_i^N E_i^2 \sin^2\alpha_i + \boxed{}$$

$$= \sum_i^N E_i^2 (\underbrace{\cos^2\alpha_i + \sin^2\alpha_i}_{1})$$

$$= \sum_i^N E_i^2 + 2\sum_i^N \sum_{j>i}^N E_i E_j (\underbrace{\cos\alpha_i \cos\alpha_j + \sin\alpha_i \sin\alpha_j}_{\cos(\alpha_j - \alpha_i)})$$

$$\boxed{E_0^2 = \sum_i^N E_i^2 + 2\sum_i^N \sum_{j>i}^N E_i E_j \cos(\alpha_j - \alpha_i)}$$

For random sources $\longrightarrow$ ~~if these α_i are random what does this sum approach?~~ $\rightarrow 0$

$$E_0^2 = \sum_i^N E_i^2$$

if they are equal magnitude sources

$$E_0^2 = N E_1^2 \rightarrow E_0 = E_1 \sqrt{N}$$

irradiance $\rightarrow$ $E_e = \frac{1}{2}\epsilon_0 c^2 E_0 B_0$ or $E_e = \frac{1}{2}\epsilon_0 c E_0^2$

$$E_e = \frac{1}{2}\epsilon_0 c N E_1^2 = N\left(\frac{1}{2}\epsilon_0 c E_1^2\right)$$

$$\boxed{E_e \propto N}$$ $\leftarrow$ Experimental verified

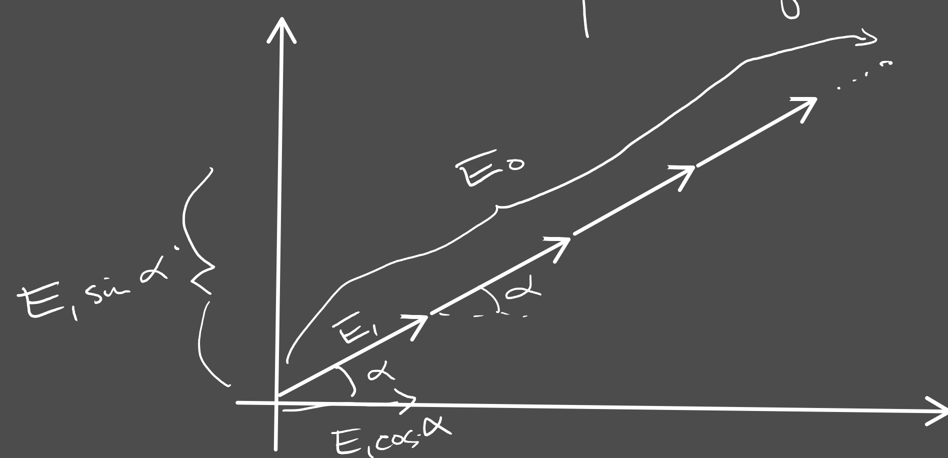
now what about coherent sources

↳ same frequency and waveform and phase

$$\boxed{E_0^2 = \sum_i^N E_i^2 + 2\sum_i^N \sum_{j>i}^N E_i E_j \underbrace{\cos(\alpha_j - \alpha_i)}_{\cos(\alpha_j - \alpha_i) = \cos(0) = 1}}$$

$$E_0^2 = \sum_i^N E_i^2 + 2\sum_i^N \sum_{j>i}^N E_i E_j$$

- if sources of equal magnitude



$$E_0^2 = (NE_1\cos\alpha)^2 + (NE_1\sin\alpha)^2 = N^2E_1^2 \underbrace{(\cos^2\alpha + \sin^2\alpha)}_{1}$$

$$E_0^2 = N^2E_1^2 \rightsquigarrow E_0 = NE_1$$

back to irradiance

$$E_e = \frac{1}{2}\epsilon_0 c N^2 E_1^2$$

$$\boxed{E_e \propto N^2}$$

$$\frac{E_{e,coher}}{E_{e,rand}} = \frac{N^2}{N} = N$$

$\cos(-x) = \cos(x)$

$\sin(-x) = -\sin(x)$

Standing waves

two waves traveling in opposite directions

$E_1 = E_0 \sin(-kx + \omega t + \phi_R)$ ← to right

$E_2 = E_0 \sin(kx + \omega t)$ ← to left

$$E_R = E_0\left(\sin(kx + \omega t) + \sin(-kx + \omega t + \phi_R)\right)$$

$$\sin\beta_1 + \sin\beta_2 = 2\sin\left(\frac{1}{2}(\beta_1 + \beta_2)\right)\cos\left(\frac{1}{2}(\beta_1 - \beta_2)\right)$$

$$E_R = 2E_0 \sin\left(\omega t + \frac{\phi_R}{2}\right)\cos\left(kx + \phi_R/2\right)$$

$$\phi_R = \pi$$

$$E_R = 2E_0 \sin\left(\omega t + \frac{\pi}{2}\right)\cos\left(kx + \pi/2\right)$$

$$E_R = \underbrace{2E_0 \sin(kx)}_{\text{specially varying amplitude}} \underbrace{\cos(\omega t)}$$

→ variation in amplitude in time

→ $= 0$ always in certain places → <u>nodes</u>

$$kx = m\pi \qquad m = 0, \pm 1, \pm 2, \ldots$$

$$k = \frac{2\pi}{\lambda}$$

$$\frac{2\cancel{\pi}}{\lambda} \cdot x = m\cancel{\pi}$$

$$x = m\frac{\lambda}{2} \qquad m = 0, \pm 1, \pm 2, \ldots$$

$$\Delta x = \frac{\lambda}{2}$$ ← distance between adjacent nodes

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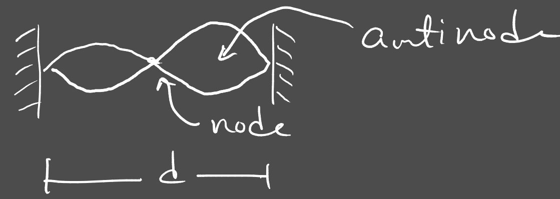
$$\Delta t = \frac{T}{2}$$

$$\omega = 2\pi\nu \qquad \nu = \frac{1}{T}$$

$$\omega = \frac{2\pi}{T}$$

laser cavity



$$d = m\left(\frac{\lambda}{2}\right)$$

$$\lambda = \frac{2d}{m} \qquad m = 1, 2, 3 \dots$$

Frequency Beating

Group velocity + Phase Velocity

dispersion – in materials, waves w/ different frequency travel w/ different speed

$$n = \frac{c}{v} \qquad\qquad n(\lambda) = A + \frac{B}{\lambda^2} + \frac{C}{\lambda^4} + \dots$$

$$\hookrightarrow v = \frac{c}{n}$$

$$\underbrace{\frac{dn}{d\lambda} = -\frac{2B}{\lambda^3}}_{\text{dispersion relationship (normal materials)}} + \underbrace{\dots}_{\text{ignore these}}$$

increase $\lambda \rightarrow$ smaller n

increase $\nu \rightarrow$ higher n
(smaller wavelength)

$\rightarrow$ slower speed

$$V = \lambda \nu = \frac{\omega}{K}$$

higher frequency carrier wave

$$V_p = \frac{\omega_p}{K_p} = \frac{\omega_1 + \omega_2}{K_1 + K_2} \approx \frac{\omega}{K}$$

close in value

$$V_p = \frac{\omega}{K}$$

phase velocity

lower frequency envelope wave

$$V_g = \frac{\omega_g}{K_g} = \frac{\omega_1 - \omega_2}{K_1 - K_2}$$

again very similar

$$V_g \approx \frac{d\omega}{dk}$$

put them together

$$V_p = \frac{\omega}{k} \Rightarrow \omega = v_p k$$

$$V_g = \frac{d(v_p k)}{dk}$$

$$V_g = V_p + k\frac{dv_p}{dk}$$

does the velocity of the waves depend on wavelength

— in a non-dispersive medium

$$\frac{dv}{dk} = 0$$

$$\therefore V_g = V_p$$

— in a dispersive medium

$$V_p = \frac{c}{n}$$

can be a function of λ (so k)

$$\frac{dv_p}{dk} = \frac{d\left(\frac{c}{n}\right)}{dk} = \frac{c\, d(n^{-1})}{dk} = c(-1)(n^{-2})\frac{dn}{dk}$$

$$\frac{dv_p}{dk} = -\frac{c}{n^2}\frac{dn}{dk}$$

$$v_g = v_p + k\frac{dv_p}{dk}$$

$$= \frac{c}{n} + k\left(-\frac{c}{n^2}\frac{dn}{dk}\right)$$

$$= \frac{c}{n}\left(1 - \frac{k}{n}\frac{dn}{dk}\right)$$

$$\frac{dn}{dk} = \frac{dn}{d\lambda}\cdot\frac{d\lambda}{dk}$$

$$k = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{k}$$

$$d\lambda = \underbrace{\frac{d\lambda}{dk}}_{\frac{d\left(\frac{2\pi}{k}\right)}{dk}} dk$$

$$d\lambda = \underbrace{-\frac{2\pi}{k^2}}\, dk$$

$$= \frac{c}{n}\left(1 + \frac{1}{n}\left(\frac{\cancel{2\pi}}{\cancel{\lambda}}\right)\left(+\frac{\cancel{2\pi}}{\cancel{(2\pi)^2}}\cdot \cancel{\lambda^2}\cdot\frac{dn}{d\lambda}\right)\right)$$

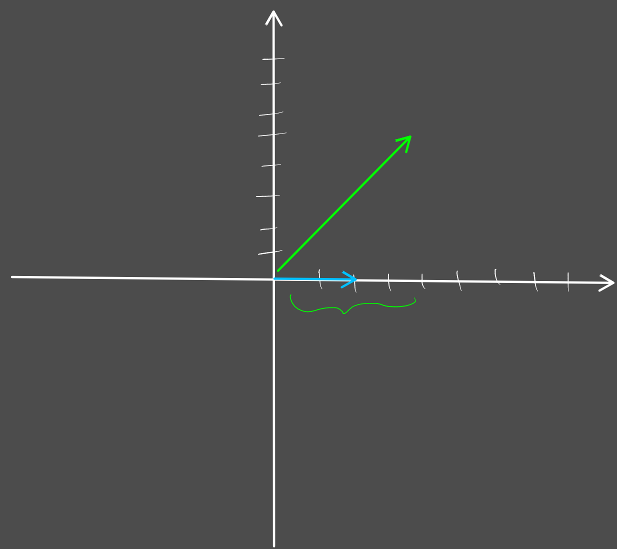
$$\boxed{v_g = \frac{c}{n}\left(1 + \frac{\lambda}{n}\frac{dn}{d\lambda}\right)}$$

normal materials $\quad \frac{dn}{d\lambda} = -\frac{2B}{\lambda^3}$

$$\frac{dn}{d\lambda} < 0$$

$$v_g < v_p$$

#2



$E_1 = 2\cos\omega t$ $\quad E_2 = 7\cos\left(\frac{\pi}{4} - \omega t\right)$

$\downarrow$

$E_1 = 2\cos(-\omega t)$ $\quad \alpha = \frac{\pi}{4}$

$\alpha = 0$

$2e^{-i\alpha}$

$E_R = \vec{E}_1 + \vec{E}_2$

$E_R^2 = \left(2\cos(0) + 7\cos\left(\frac{\pi}{4}\right)\right)^2 + \left(7\sin\left(\frac{\pi}{4}\right)\right)^2$