Chapter 11 - Fraunhofer Diffraction

From ch. 5

for coherent light

$$E_0^2 = N^2 E^2$$

incoherent

$$E_0^2 = N E^2$$

Go to chapter 7

$$E_e \rightarrow I = \frac{1}{2} \epsilon_0 c^2 E_0^2$$

$$I \propto N^2$$

so for double slit 4 times the irradiance
from one source $\rightarrow$ constructive interference

In general for any obstruction.

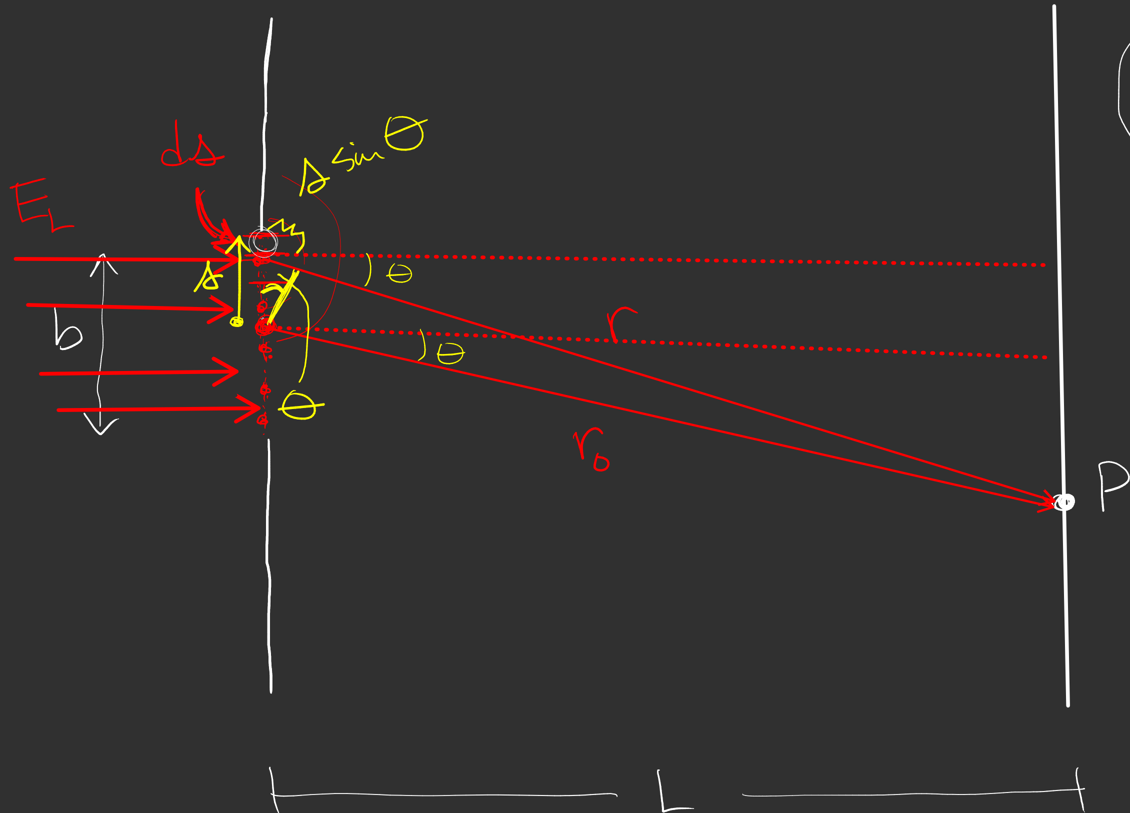
Huygens - Fresnel Principle

any point on a wavefront can be considered as a source of spherical wavelets

actual field beyond the wavefront is the superposition of the wavelets, considering their phase + amplitude diffraction → far field approximation → Fraunhofer Diffraction

diffraction → near field approximation → Fresnel Diffraction

↳ distance from the source to the screen.



$$\boxed{dE_p = \frac{E_L\,ds}{r} e^{i(kr-\omega t)}}$$

$$r = r_0 + s\sin\theta$$

$$dE_p = \frac{E_L\,ds}{r_0 + \underbrace{\cancel{s\sin\theta}}_{\text{tiny}}} \underbrace{e^{i(k(r_0+s\sin\theta)-\omega t)}}$$

$$dE_p = \frac{E_L\,ds}{r_0} e^{i(kr_0-\omega t)}\, e^{iks\sin\theta}$$

$$E_p = \int_{-b/2}^{b/2} \frac{E_L}{r_0} e^{i(kr_0-\omega t)}\, e^{iks\sin\theta}\, ds$$

$$E_p = \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \int_{-b/2}^{b/2} e^{iks\sin\theta}\, ds$$

$$E_p = \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \left[\frac{e^{iks\sin\theta}}{ik\sin\theta} \right]_{-b/2}^{b/2}$$

$$\sin\left(\frac{kb}{2}\sin\theta\right)$$

$$E_p = \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \cdot \cancel{2}\left(\frac{b}{\cancel{2}}\right) \underbrace{\frac{e^{ik\frac{b}{2}\sin\theta} - e^{-ik\frac{b}{2}\sin\theta}}{\left(\frac{b}{2}\right) 2ik\sin\theta}}_{\underbrace{\frac{\sin\left(\frac{kb}{2}\sin\theta\right)}{\frac{kb}{2}\sin\theta}}_{\operatorname{sinc}\left(\frac{kb}{2}\sin\theta\right)}}$$

$$E_p = \underbrace{\frac{E_L b}{r_0} \operatorname{sinc}\left(\frac{kb}{2}\sin\theta\right)}_{\text{amplitude at point } P} e^{i(kr_0 - \omega t)}$$

$$\hookrightarrow E_0 = \frac{E_L b}{r_0} \operatorname{sinc}\left(\frac{kb}{2}\sin\theta\right)$$

Where will this be equal to zero?

$$\rightarrow \underbrace{\frac{kb}{2}}_{\text{what is this}} \sin\theta = m\pi \qquad m = \pm 1, \pm 2, \ldots$$

what is this phase difference of light from the edge and light from the center of the slit

$$k = \frac{2\pi}{\lambda}$$

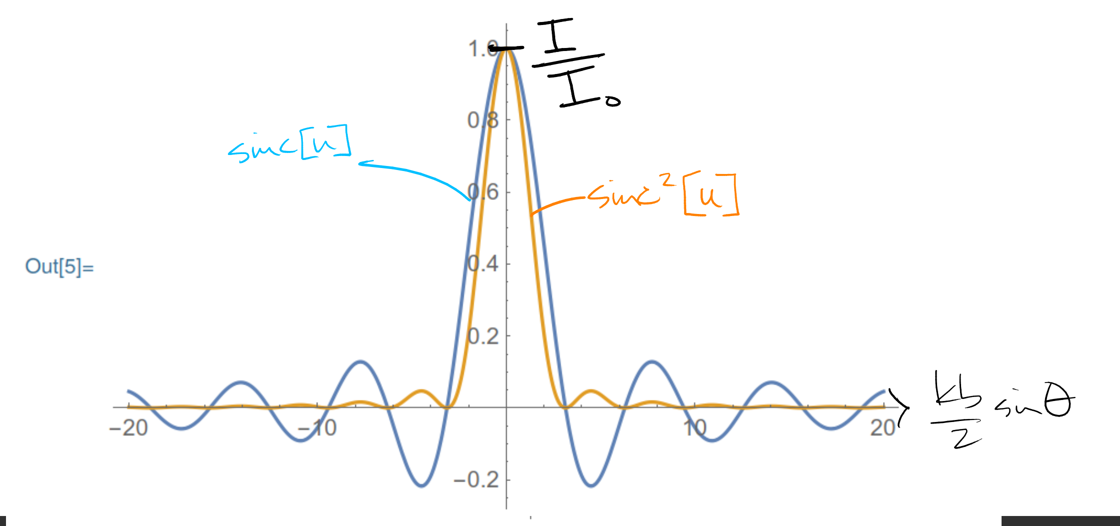
$$\frac{\cancel{2}\cancel{\pi}}{\lambda} \cdot \frac{b}{\cancel{2}} \sin\theta = m\cancel{\pi}$$

$$\boxed{b\sin\theta = m\lambda}$$

Condition for destructive interference

In[5]:= `Plot[{Sinc[u], Sinc[u]^2}, {u, -20, 20}, PlotRange → All]`

Out[5]=



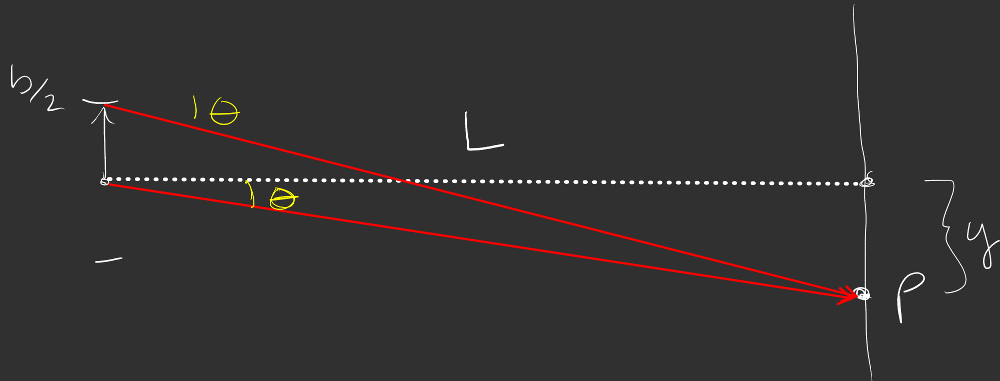
$$\sin(k(x+\Delta) - \omega t)$$

$$\sin(kx - \omega t + \underbrace{k\Delta}_{\text{phase difference}})$$

$$\beta = \frac{kb}{2}\sin\theta$$

$$\rightarrow \boxed{E_0 = \frac{E_L b}{r_0} \operatorname{sinc}\beta}$$

$$I = \frac{1}{2}\epsilon_0 c^2 E_0^2 = \frac{1}{2}\epsilon_0 c^2 \left(\frac{E_L b}{r_0}\right)^2 \operatorname{sinc}^2\beta$$



$$\tan\theta = \frac{y}{L}$$

far field approx.

$$\tan\theta \approx \sin\theta \approx \theta$$

$$b\sin\theta = m\lambda$$

$$b\frac{y}{L} = m\lambda$$

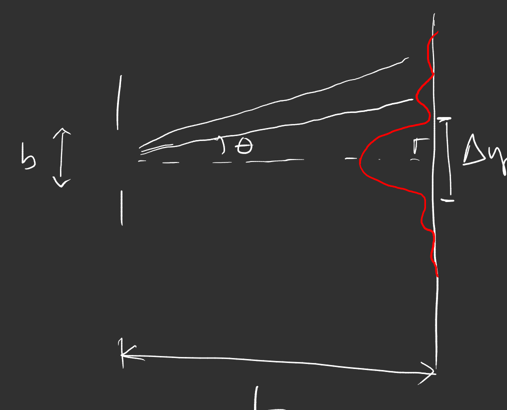
destr. interf. $\longrightarrow$ $\boxed{y = \frac{m\lambda L}{b}}$ $m = \pm 1, \pm 2, \ldots$

What about maxima?

$$E_0 = \frac{E_L b}{r_0}\operatorname{sinc}\beta$$

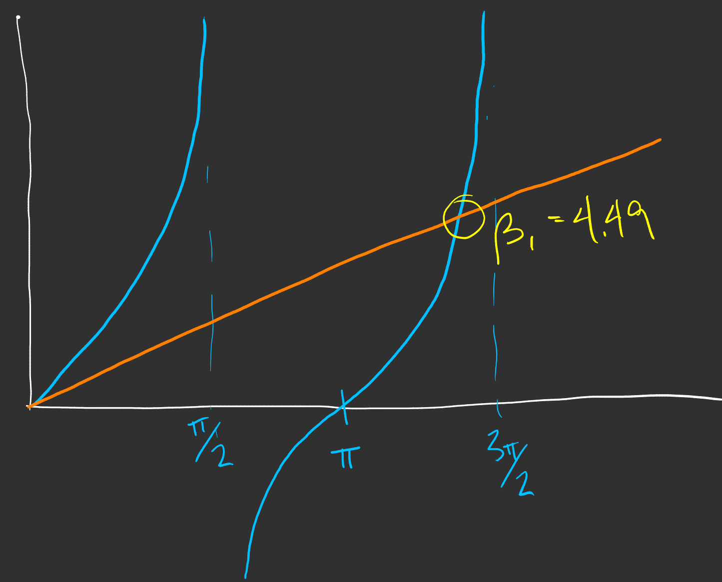
$$\beta = \frac{kb}{2}\sin\theta$$

what values of θ produce maxima



$$\frac{\partial \operatorname{sinc}\beta}{\partial\beta} = \frac{\partial}{\partial\beta}\left(\frac{\sin\beta}{\beta}\right) = \frac{\cos\beta}{\beta} - \frac{\sin\beta}{\beta^2} = 0$$

$$\frac{\cos\beta}{\cancel{\beta}} = \frac{\sin\beta}{\beta^{\cancel{2}}}$$



$$\beta = \frac{\sin\beta}{\cos\beta}$$

$$\rightarrow \quad \beta = \tan\beta$$

$$\beta_1 = 4.49$$
$$\beta_2 = 7.72$$
$$\beta_3 = 10.9$$

$\beta = 2 \quad \tan 2 =$

$\beta = 1 \quad \tan(1) = 1.55$

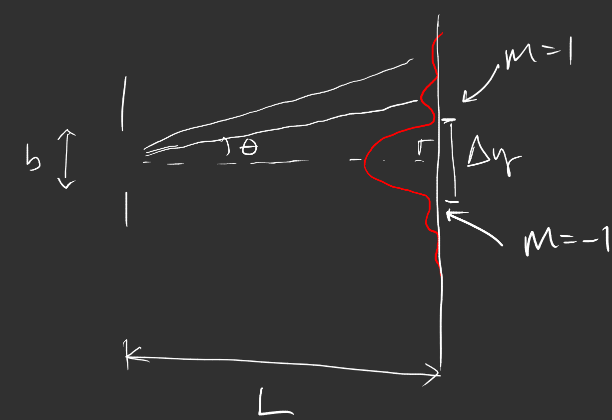
$\beta = 0 \quad \tan(0) = 0$

$$\beta_1 = \frac{kb}{2}\sin\theta$$

$$\sin\theta_1 = \frac{2\beta_1}{kb}$$

What about width of the central peak?

$$\boxed{y = \frac{m\lambda L}{b}} \quad m = \pm 1, \pm 2, \ldots$$

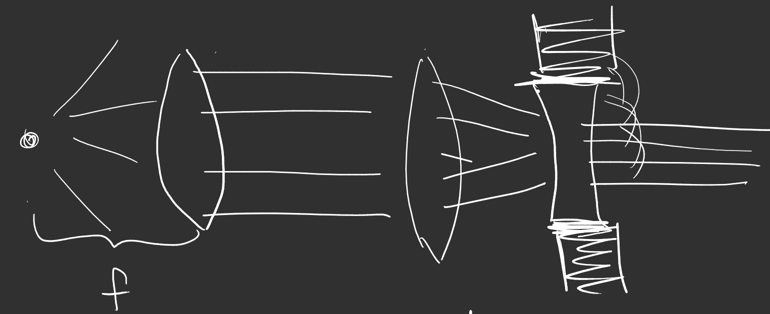


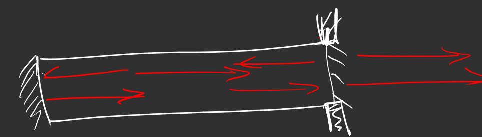
$$\Delta y = y_{m=1} - y_{m=-1}$$

$$W = \Delta y = \frac{1\lambda L}{b} - \frac{(-1)\lambda L}{b}$$

$$W = \frac{2\lambda L}{b}$$

$W \to b$ the smaller L gets





$$W = b = \frac{2\lambda L_{min}}{b}$$

$$L_{min} = \frac{b^2}{2\lambda}$$

$$L \gg \frac{b^2}{\lambda}$$ } criteria for far field approximatin

$$dE = \frac{E_L\, ds}{r} e^{i(kr-\omega t)}$$

$$\psi(r,t) = \frac{A}{r} e^{i(kr-\omega t)}$$

$$E(r,t) = \frac{E_0}{r} e^{i(kr-\omega t)}$$

E_0 → Volts

$$E(r,t) = \frac{E_0 \cdot 1m}{r} e^{i(kr-\omega t)}$$

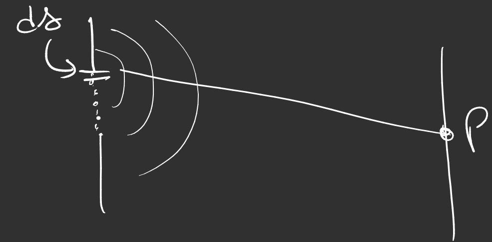
$$E(r) \rightarrow \frac{E_2}{E_1} = \frac{r_1}{r_2}$$

r_1 ← 1m

r_2 ← any r

E_1 ← $E(1m)$

$$E(r) = \frac{E_0 (1m)}{r} e^{i(kr-\omega t)}$$



$$\frac{dp}{dt} = F = qE$$

$$\frac{F}{q} = E = \left[\frac{N}{C}\right]$$

$$W = \int F \cdot dx$$

energy

$$W = \int qE\,dx \Rightarrow \frac{\Delta U}{q} = \int \vec{E} \cdot d\vec{x}$$

Volts

$$[E] = \left[\frac{J}{Cm}\right] = \left[\frac{V}{m}\right]$$

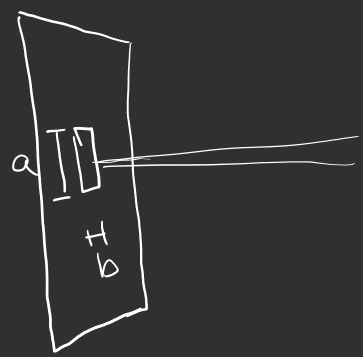
$$E = \frac{\varepsilon_0}{r} e^{i(kr-\omega t)}$$

ε_0 → source strength

$I\,ds$ ← how many oscillators are in ds?

$$b\Big\{ \quad \frac{N}{b}\, ds$$

$$dE = \frac{\varepsilon_0}{r} e^{i(kr-\omega t)} \cdot \frac{N}{b}\, ds$$



$$\boxed{\mathcal{E}_L} = \lim_{N\to\infty} \frac{\mathcal{E}_0 N}{b}$$

$$dE = \frac{\mathcal{E}_L\, ds}{r} e^{i(kr-\omega t)}$$

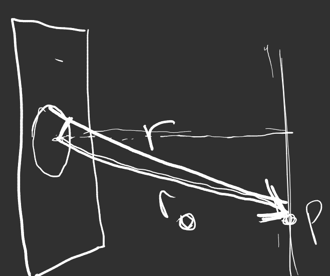
$$\beta = \frac{kb}{2}\sin\theta$$

$$I_b = I_{ob}\,\mathrm{sinc}^2\beta$$

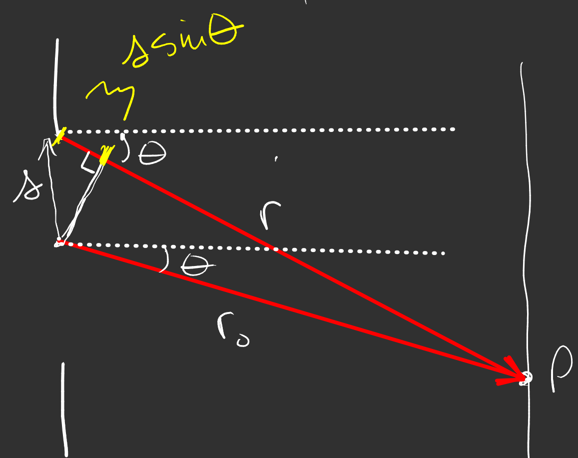
$$I_a = I_{oa}\,\mathrm{sinc}^2\alpha$$

$$\alpha = \frac{ka}{2}\sin\phi$$

$$\rightarrow I = I_0\,\mathrm{sinc}^2\alpha\,\mathrm{sinc}^2\beta$$



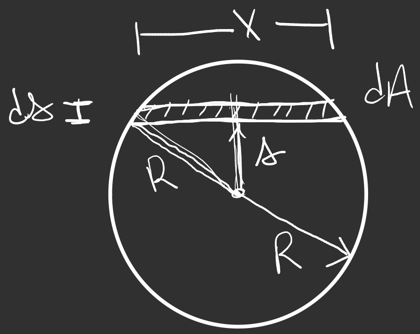
$$dE = \frac{\mathcal{E}_A\, dA}{r} e^{i(kr-\omega t)} = \frac{\mathcal{E}_A\, dA}{r_0 + \underbrace{s\sin\theta}_{\approx 0}} e^{i(k(r_0+s\sin\theta)-\omega t)}$$



$$E_p = \frac{\mathcal{E}_A}{r_0} e^{i(kr_0-\omega t)} \int_A e^{iks\sin\theta}\, dA$$

$$dA = X\, ds$$

$$X = 2\sqrt{R^2 - s^2}$$



$$E_p = \frac{\mathcal{E}_A}{r_0} e^{i(kr_0 - \omega t)} \int_{-R}^{R} e^{iks\sin\theta} \cdot 2 \cdot \sqrt{R^2 - s^2}\, ds$$

$$\frac{s}{R} = v \qquad s = Rv$$

$$\boxed{ds = R\,dv}$$

$$\boxed{\int_{-R}^{R} \to \int_{-1}^{1}}$$

$$kR\sin\theta = \gamma$$

$$k s \sin\theta \cdot \frac{R}{R}$$

$$\boxed{\gamma \cdot v = ks\sin\theta}$$

$$\sqrt{R^2 - s^2} = \sqrt{R^2\left(1 - \frac{s^2}{R^2}\right)}$$

$$\boxed{\sqrt{R^2 - s^2} = R\sqrt{1 - v^2}}$$

$$E_p = \frac{2\mathcal{E}_A R^2 e^{i(kr_0 - \omega t)}}{r_0} \int_{-1}^{1} e^{i\gamma v} \sqrt{1 - v^2}\, dv$$

$$\int_{-1}^{1} e^{i\gamma v} \sqrt{1 - v^2}\, dv = \frac{\pi \boxed{J_1(\gamma)}}{\gamma}$$

Bessel Function of the first kind

$$J_1(\gamma) = \frac{\gamma}{2} - \frac{(\gamma/2)^2}{1^2 \cdot 2} + \frac{(\gamma/2)^5}{1^2 \cdot 2^2 \cdot 3} - \ldots$$

$$\frac{d}{du}\left[u^m J_m(u)\right] = u^m J_{m-1}(u)$$

$m=1$

$$u\,J_1(u) = \int_0^u u'\,J_0(u')\,du'$$