

## Chapter 5 - Superposition

superposition - waves at the same place and time, displacements add up.

$$\psi = \psi_1 + \psi_2 + \dots$$

• Superposition of harmonic waves

$$E_1(x_1, t) = E_1 \cos(kx_1 - \omega t + \phi_1)$$

$$E_2(x_2, t) = E_2 \cos(\underbrace{kx_2 - \omega t + \phi_2}_{\text{phase}})$$

$$\alpha_1 = ks_1 + \phi_1$$

$$\alpha_2 = ks_2 + \phi_2$$

$$\underbrace{\alpha_2 - \alpha_1}_{\text{phase difference}} = k(s_2 - s_1) + (\phi_2 - \phi_1)$$

What if  $\alpha_2 - \alpha_1 = 2\pi m$  } even multiple of  $\pi$

$$E_R = E_1 + E_2 = E_1 \cos(\alpha_1 - \omega t) + E_2 \cos(\alpha_2 - \omega t)$$

$$\downarrow \alpha_2 = \alpha_1 + 2\pi m$$

$$\cos(\alpha_1 + 2\pi m - \omega t)$$

$$\cos \theta = \cos(\theta + 2\pi m)$$

constructive interference

$$\rightarrow E_R = (E_1 + E_2) \cos(\alpha_1 - \omega t)$$

What if  $\alpha_2 - \alpha_1 = (2m-1)\pi$  odd multiple of  $\pi$

$$E_R = E_1 + E_2 = E_1 \cos(\alpha_1 - \omega t) + E_2 \cos(\alpha_2 - \omega t)$$

destructive interference

$$\alpha_2 = \alpha_1 + (2m-1)\pi$$

$$-\cos \Theta = \cos(\Theta + (2m-1)\pi)$$

$$E_R = (E_1 - E_2) \cos(\alpha_1 - \omega t)$$

What about others?

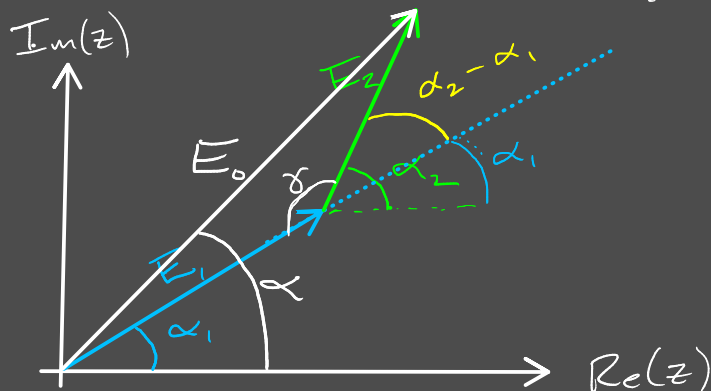
$$E_R = \text{Re}(E_1 e^{i(\alpha_1 - \omega t)} + E_2 e^{i(\alpha_2 - \omega t)})$$

$$E_1 e^{i\alpha_1} e^{-i\omega t} + E_2 e^{i\alpha_2} e^{-i\omega t}$$

$$E_R = \text{Re}(e^{-i\omega t} (E_1 e^{i\alpha_1} + E_2 e^{i\alpha_2}))$$

"phasor diagram"

→ complex as vector



$$E_0^2 = E_1^2 + E_2^2 - 2E_1 E_2 \cos \gamma$$

$$\gamma = \pi - (\alpha_2 - \alpha_1)$$

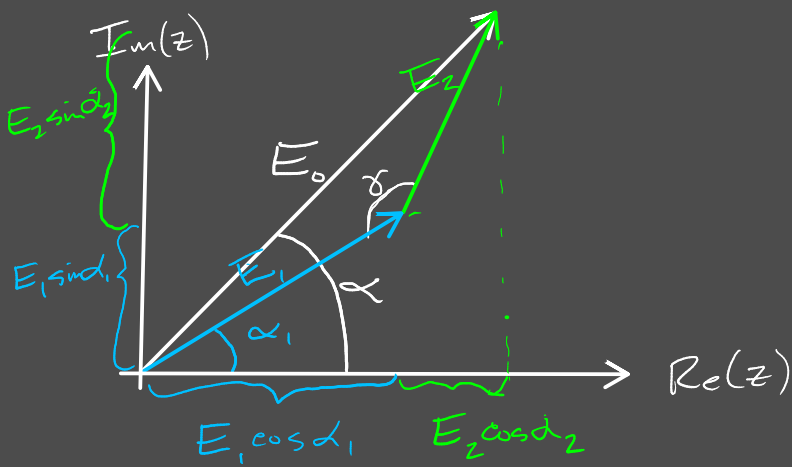
$$= \pi - \alpha_2 + \alpha_1$$

$$= \pi + \alpha_1 - \alpha_2$$

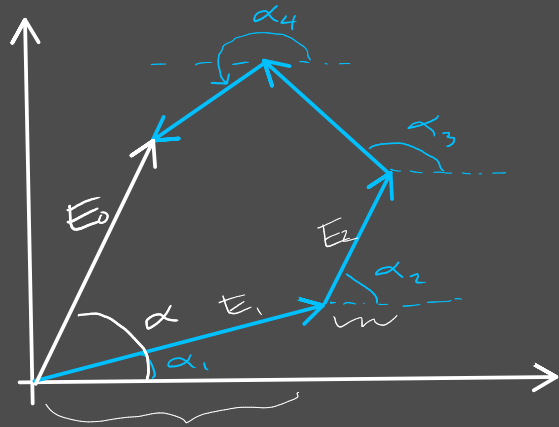
$$\cos(\pi + \alpha_1 - \alpha_2)$$

$$= -\cos(\alpha_1 - \alpha_2)$$

$$E_0^2 = E_1^2 + E_2^2 + 2E_1 E_2 \cos(\alpha_1 - \alpha_2)$$



$$\tan \alpha = \frac{E_1 \sin \alpha_1 + E_2 \sin \alpha_2}{E_1 \cos \alpha_1 + E_2 \cos \alpha_2}$$



$$\tan \alpha = \frac{\sum_{i=1}^n E_i \sin \alpha_i}{\sum_{i=1}^n E_i \cos \alpha_i}$$

$$E_0^2 = \left( \sum_{i=1}^n E_i \cos \alpha_i \right)^2 + \left( \sum_{i=1}^n E_i \sin \alpha_i \right)^2$$

$$\left( \sum_{i=1}^n E_i \cos \alpha_i \right)^2 = \sum_{i=1}^n E_i^2 \cos^2 \alpha_i + \sum_{i=1}^n 2 E_i \cos \alpha_i \sum_{j>i}^n E_j \cos \alpha_j$$

$$2 \sum_{i=1}^n \sum_{j>i}^n E_i E_j \cos \alpha_i \cos \alpha_j$$

$$\left( \sum_{i=1}^n E_i \sin \alpha_i \right)^2 = \sum_{i=1}^n E_i^2 \sin^2 \alpha_i + 2 \sum_{i=1}^n \sum_{j>i}^n E_i E_j \sin \alpha_i \sin \alpha_j$$

$$\begin{aligned}
 E_o^2 &= \sum_i^N E_i^2 \cos^2 \alpha_i + \sum_i^N E_i^2 \sin^2 \alpha_i + \boxed{\phantom{0}} \checkmark \\
 &= \sum_i^N E_i^2 (\underbrace{\cos^2 \alpha_i + \sin^2 \alpha_i}_1) \\
 &= \sum_i^N E_i^2 + 2 \underbrace{\sum_i^N \sum_{j>i}^N E_i E_j (\cos \alpha_i \cos \alpha_j + \sin \alpha_i \sin \alpha_j)}_{\cos(\alpha_j - \alpha_i)}
 \end{aligned}$$

$$E_o^2 = \sum_i^N E_i^2 + 2 \sum_i^N \sum_{j>i}^N E_i E_j \cos(\alpha_j - \alpha_i) \quad \leftarrow$$

For random sources  $\rightarrow$  ~~if these  $\alpha_i$  are random  
what does this sum approach?  $\rightarrow 0$~~

$$E_o^2 = \sum_i^N E_i^2$$

if they are equal magnitude sources

$$E_o^2 = N E_i^2 \rightarrow E_o = E_i \sqrt{N}$$

$$\text{irradiance} \rightarrow E_e = \frac{1}{2} \epsilon_0 c^2 E_o B_o \quad \text{or} \quad E_e = \frac{1}{2} \epsilon_0 c E_o^2$$

$$E_e = \frac{1}{2} \epsilon_0 c N E_i^2 = N \left( \frac{1}{2} \epsilon_0 c E_i^2 \right)$$

$$\boxed{E_e \propto N}$$

$\leftarrow$  Experimental verified

now what about coherent sources

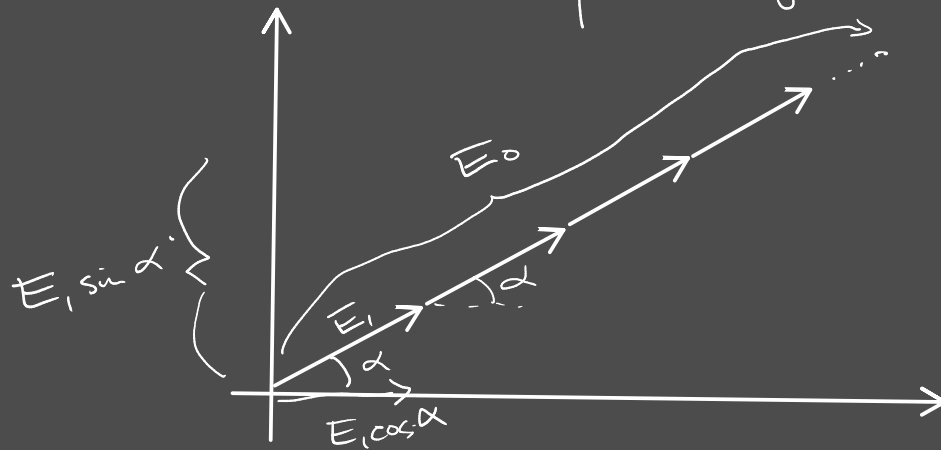
↳ same frequency and waveform  
and phase

$$E_o^2 = \sum_i^N E_i^2 + 2 \sum_i^N \sum_{j>i}^N E_i E_j \cos(\alpha_j - \alpha_i)$$

$$\cos(\alpha_j - \alpha_i) = \cos(0) = 1$$

$$E_o^2 = \sum_i^N E_i^2 + 2 \sum_i^N \sum_{j>i}^N E_i E_j$$

- if sources of equal magnitude



$$E_o^2 = (N E_1 \cos \alpha)^2 + (N E_1 \sin \alpha)^2 = N^2 E_1^2 (\underbrace{\cos^2 \alpha + \sin^2 \alpha}_1)$$
$$E_o^2 = N^2 E_1^2 \Rightarrow E_o = N E_1$$

back to irradiance

$$\hookrightarrow E_e = \frac{1}{2} \epsilon_0 c N^2 E_1^2$$

$$\sqrt{E_e < N^2}$$

$$\frac{E_{e, \text{coher}}}{E_{e, \text{rand}}} = \frac{N^2}{N} = N$$

Standing waves

↳ two waves traveling in opposite directions

$$E_1 = E_0 \sin(-kx + \omega t + \phi_R) \quad \leftarrow \text{to right}$$

$$E_2 = E_0 \sin(kx + \omega t) \quad \leftarrow \text{to left}$$

$$E_R = E_0 \left( \sin(kx + \omega t) + \sin(-kx + \omega t + \phi_R) \right)$$

$$\sin \beta_1 + \sin \beta_2 = 2 \sin\left(\frac{1}{2}(\beta_1 + \beta_2)\right) \cos\left(\frac{1}{2}(\beta_1 - \beta_2)\right)$$

$$E_R = 2E_0 \sin\left(\omega t + \frac{\phi_R}{2}\right) \cos\left(kx + \frac{\phi_R}{2}\right)$$

$$\phi_R = \pi$$

$$E_R = 2E_0 \sin\left(\omega t + \frac{\pi}{2}\right) \cos\left(kx + \frac{\pi}{2}\right)$$

$$\cos(-x) = \cos(x)$$

$$\sin(-x) = -\sin(x)$$

$$E_R = \underbrace{2E_0 \sin(kx)}_{\text{spatially varying amplitude}} \underbrace{\cos(\omega t)}_{\text{variation in amplitude in time}}$$

spatially varying amplitude

variation in amplitude in time

$E = 0$  always in certain planes  $\rightarrow$  nodes

$$kx = m\pi \quad m = 0, \pm 1, \pm 2, \dots$$

$$k = \frac{2\pi}{\lambda}$$

$$\frac{2\pi}{\lambda} \cdot x = m\pi$$

$$x = m \frac{\lambda}{2} \quad m = 0, \pm 1, \pm 2, \dots$$

$$\Delta x = \frac{\lambda}{2} \leftarrow \text{distance between adjacent nodes}$$

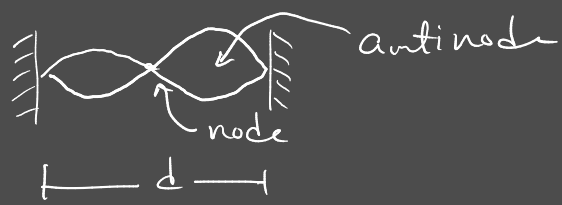
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$$\Delta t = \frac{T}{2}$$

$$\omega = 2\pi\nu \quad \nu = \frac{1}{T}$$

$$\omega = \frac{2\pi}{T}$$

laser cavity



$$d = m \left( \frac{\lambda}{2} \right)$$

$$\lambda = \frac{2d}{m} \quad m = 1, 2, 3, \dots$$

Frequency   Beating

















