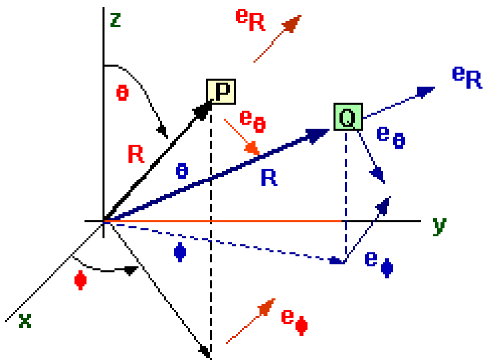
Coordinates: Definition - Spherical

The **spherical coordinate system** is naturally useful for space flights. It is also useful for **three dimensional problems** that have **spherical geometry**. It is the most complex of the three coordinate systems

Coordinates R, θ, ϕ **Unit Vectors** $\hat{e}_R, \hat{e}_\theta, \hat{e}_\phi$

(unit vector R is in the direction of increasing R; unit vector theta is in the direction of increasing theta, unit vector phi is in the direction of increasing phi)

(Note: In some texts the theta and phi definition are reversed to have some connection to the cylindrical system. In that case the following relations will be different)



With the two points **P and Q**, **that are small time unit apart** (***dt***)the changes in the unit vectors can be established as

$$\frac{\partial \hat{e}_R}{\partial R} = 0 \qquad \frac{\partial \hat{e}_\theta}{\partial R} = 0 \qquad \frac{\partial \hat{e}_\phi}{\partial R} = 0$$

$$\frac{\partial \hat{e}_R}{\partial \theta} = \hat{e}_\theta \qquad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_R \qquad \frac{\partial \hat{e}_\phi}{\partial \theta} = 0$$

$$\frac{\partial \hat{e}_R}{\partial \phi} = \sin\theta \hat{e}_\phi \qquad \frac{\partial \hat{e}_\theta}{\partial \phi} = \cos\theta \hat{e}_\phi \qquad \frac{\partial \hat{e}_\phi}{\partial \phi} = -\sin\theta \hat{e}_R - \cos\theta \hat{e}_\theta$$

The **time derivatives** therefore are

$$\frac{\partial \hat{e}_R}{\partial t} = \frac{\partial \hat{e}_R}{\partial R}\frac{\partial R}{\partial t} + \frac{\partial \hat{e}_R}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial \hat{e}_R}{\partial \phi}\frac{\partial \phi}{\partial t} \quad = \dot{\theta}\,\hat{e}_\theta + \dot{\phi}\sin\theta\,\hat{e}_\phi$$

$$\frac{\partial \hat{e}_\phi}{\partial t} = -\sin\theta\dot{\phi}\,\hat{e}_R - \dot{\phi}\cos\theta\,\hat{e}_\theta$$

$$\frac{\partial \hat{e}_\theta}{\partial t} = -\dot{\theta}\hat{e}_R + \dot{\phi}\cos\theta\hat{e}_\phi$$

The **position** vector is $\vec{R} = R\hat{e}_R$

The **velocity** vector is defined as $\vec{V} = V_R\hat{e}_R + V_\theta\hat{e}_\theta + V_\phi\hat{e}_\phi$

$$\vec{V} = \frac{d(R\hat{e}_R)}{dt} = \dot{R}e_R + R\dot{e}_R$$

$$= \dot{R}e_R + R(\dot{\theta}e_\theta + \dot{\phi}\sin\theta e_\phi)$$

$$\bar{a} = a_R \hat{e}_R + a_\theta \hat{e}_\theta + a_\phi \hat{e}_\phi$$

$$a_R = \ddot{R} - R\dot{\theta}^2 - R\dot{\phi}^2 \sin^2\theta$$

$$a_\theta = R\ddot{\theta} + 2\dot{R}\dot{\theta} - R\dot{\phi}^2 \sin\theta\cos\theta$$

$$a_\phi = R\ddot{\phi}\sin\theta + 2\dot{R}\dot{\phi}\sin\theta + 2R\dot{\theta}\dot{\phi}\cos\theta$$

The **gradient** operator is $\vec{\nabla} = \hat{e}_R \frac{\partial}{\partial R} + \hat{e}_\theta \frac{1}{R}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{R\sin\theta}\frac{\partial}{\partial \phi}$

In Fluid Mechanics

The **acceleration** from the use of the substantial derivative is:

$$a_R = \frac{\partial V_R}{\partial t} + V_R \frac{\partial V_R}{\partial R} + \frac{V_\theta}{R}\frac{\partial V_R}{\partial \theta} + \frac{V_\phi}{R\sin\theta}\frac{\partial V_R}{\partial \phi} - \frac{V_\theta^2 + V_\phi^2}{R}$$

$$a_\theta = \frac{\partial V_\theta}{\partial t} + V_R \frac{\partial V_\theta}{\partial R} + \frac{V_\theta}{R}\frac{\partial V_\theta}{\partial \theta} + \frac{V_\phi}{R\sin\theta}\frac{\partial V_\theta}{\partial \phi} + \frac{V_R V_\theta}{R} - \frac{V_\phi^2 \cot\theta}{R}$$

$$a_\phi = \frac{\partial V_\phi}{\partial t} + V_R \frac{\partial V_\phi}{\partial R} + \frac{V_\theta}{R}\frac{\partial V_\phi}{\partial \theta} + \frac{V_\phi}{R\sin\theta}\frac{\partial V_\phi}{\partial \phi} + \frac{V_R V_\phi}{R} + \frac{V_\phi V_\theta \cot\theta}{R}$$