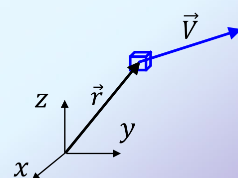
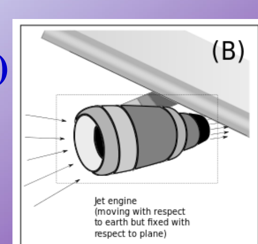
Velocity field



- **Velocity Field $\vec{V}(x, y, z, t)$**
  - **The velocity of a differential control volume of fluid at position (x, y, z) and time t**

$$\vec{V}(\underbrace{x, y, z}_{\vec{r}}, t) = u(x, y, z, t)\hat{\imath} + v(x, y, z, t)\hat{\jmath} + w(x, y, z, t)\hat{k} \qquad (1.4)$$

  - **Mathematics: Functions of 4 independent variables x, y, z, t**
  - **Steady vs. unsteady (or transient) flow: dependence upon t**
    - **Depends on the frame of reference**



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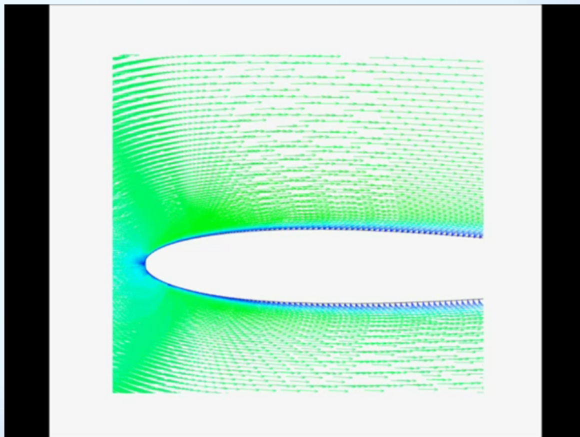
Eulerian view point

- **Velocity Field**



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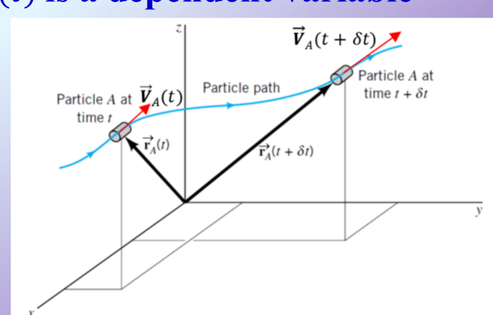
Lagrangian view point

- **Fluid particle: A differential control mass of fluid**
  - **Lagrangian particles A, B, …, instead of a single field**
  - **Particle position $\vec{r}_A(t)$ is a dependent variable**
  - **Particle velocity**

$$\vec{V}_A(t) = \frac{d\vec{r}_A(t)}{dt}$$



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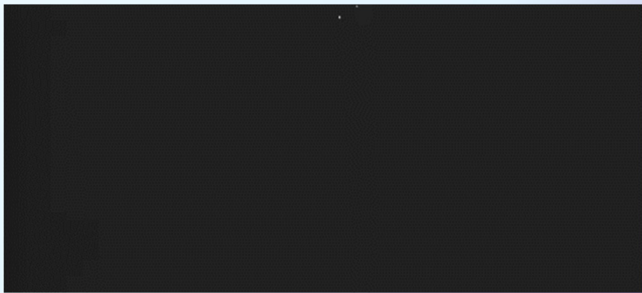
Lagrangian view point

- **Particle velocity**



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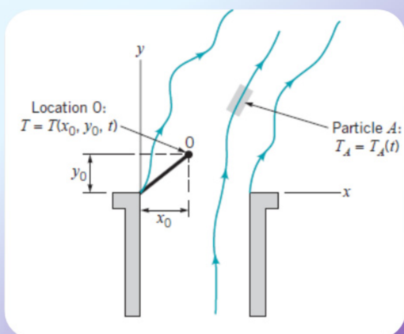
Eulerian vs. Lagrangian view points

- **Fluid mechanics I and II: Usually Eulerian**
- **Connection between Eulerian and Lagrangian points of view**

$$\vec{V}_A(t) = \vec{V}\big(\vec{r} = \vec{r}_A(t), t\big) = \vec{V}(x = x_A, y = y_A, z = z_A, t) \qquad (2.4)$$

$$Q_A(t) = Q\big(\vec{r} = \vec{r}_A(t), t\big)$$



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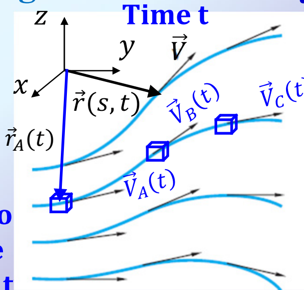
Streamlines

- A streamline is everywhere tangent to the velocity field
- Eulerian view point
- Remember: Parametric representation of a curve



$$\vec{r}_s = \vec{r}_s(s,t) \xrightarrow{\text{steady}} \vec{r}_s = \vec{r}_s(s)$$

$$\partial/\partial s \longrightarrow d/ds$$

Tangent to the curve at a given t

$$\vec{r}_s(s,t) = x_s(s,t)\,\hat{\imath} + y_s(s,t)\hat{\jmath} + z_s(s,t)\hat{k} \longrightarrow \frac{\partial \vec{r}_s}{\partial s} = \frac{\partial x_s}{\partial s}\hat{\imath} + \frac{\partial y_s}{\partial s}\hat{\jmath} + \frac{\partial z_s}{\partial s}\hat{k}$$

At a given t: $\frac{\partial \vec{r}_s}{\partial s} \,||\, \vec{V}(\vec{r} = \vec{r}_s, t) \longrightarrow \frac{\partial x_s/\partial s}{u(\vec{r}_s,t)} = \frac{\partial y_s/\partial s}{v(\vec{r}_s,t)} = \frac{\partial z_s/\partial s}{w(\vec{r}_s,t)}$ (4.4)

Two coupled PDEs

- 2D steady flow

$$\xrightarrow{s = x_s = x} \frac{1}{u} = \frac{\partial y_s/\partial x_s}{v} \longrightarrow \frac{dy}{dx} = \frac{v(x,y(x))}{u(x,y(x))} \quad , y(x = x_0) = y_0 \quad (5.4)$$

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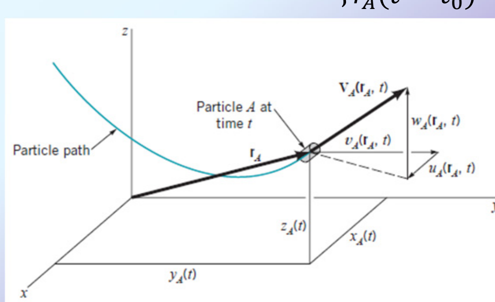
Pathlines

- A line traced out by a given particle
- Lagrangian view point
- How to calculate from Eulerian information $\vec{V}(\vec{r} = \vec{r}_A, t)$

$$\frac{d\vec{r}_A(t)}{dt} = \vec{V}_A(t) = \vec{V}(\vec{r} = \vec{r}_A(t), t) \longrightarrow \frac{d\vec{r}_A(t)}{dt} = \vec{V}(\vec{r}_A(t), t) \quad , \vec{r}_A(t = t_0) = \vec{r}_{A0} \quad (6.4)$$

Three coupled ODEs



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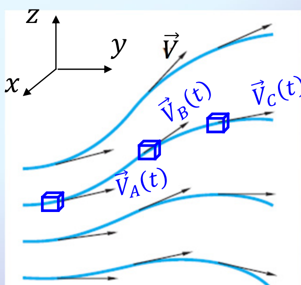
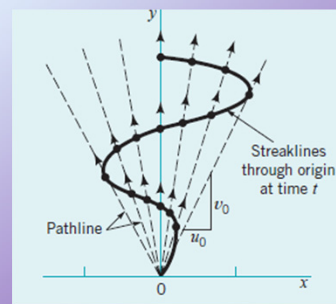
Streaklines

- **Consists of all particles in a flow that have previously passed through a common point**
- **Exercise: What are the differential equations to calculate streaklines?**
- **Steady flow: streamlines, pathlines, and streaklines are the same**





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Stream, path, and streak lines

- **Streamlines =? Streaklines, pathlines**

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Acceleration field

- **Eulerian: At each $(\vec{r}, t)$ gives the acceleration of the particle passing the given location at the given time**

$$\vec{a}\big(\vec{r} = \vec{r}_A(t), t\big) = \vec{a}_A(t)$$

- **How to calculate from Eulerian information**

$$\vec{a}_A(t) = \frac{d\vec{V}_A(t)}{dt}$$

$$\vec{V}_A(t) = \vec{V}(x_A(t), y_A(t), z_A(t), t) = \vec{V}(x = x_A(t), y = y_A(t), z = z_A(t), t) \longrightarrow$$

$$\frac{d\vec{V}_A(t)}{dt} = \left[\frac{\partial \vec{V}}{\partial t} + \underbrace{\frac{\partial \vec{V}}{\partial x_A}}_{\frac{\partial \vec{V}}{\partial x}} \underbrace{\frac{dx_A}{dt}}_{u_A = u(\vec{r}_A, t)} + \frac{\partial \vec{V}}{\partial y_A}\frac{dy_A}{dt} + \frac{\partial \vec{V}}{\partial z_A}\frac{dz_A}{dt}\right]_{\vec{r}=\vec{r}_A} = \underbrace{\left[\frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right]}_{\equiv a(\vec{r}, t)}{}_{\vec{r}=\vec{r}_A}$$

$$\vec{a}_A(t) = a(\vec{r} = \vec{r}_A, t) \longrightarrow a(\vec{r}, t) = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local derivative}} + \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{convective derivative}} \equiv \frac{D\vec{V}}{Dt} \qquad (7.4)$$

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Material derivative

- **Time derivative in a reference frame moving with fluid**

$$\frac{DQ}{Dt} \equiv \frac{\partial Q}{\partial t} + (\vec{V}.\vec{\nabla})Q \qquad (8.4)$$

- **Using Cartesian coordinates:**

$$\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$
$$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \longrightarrow \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

- **See "chap4-cylindrical.pdf" and "chap4-spherical.pdf" for the other coordinate systems.**

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Sample problems

The velocity field for a free-vortex is given in the cylindrical coordinates by $v_r = \frac{A}{r}$, $v_\theta = \frac{B}{r}$, and $v_z = 0$.

a) Determine the pathline of a particle with the initial location of (r_0, θ_0, z_0).

b) Calculate the acceleration field using both Eulerian and Lagrangian view points.

$$\vec{V} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + \dot{z}\,\hat{e}_z$$

The acceleration is

$$\vec{a} = a_r\,\hat{e}_r + a_\theta\,\hat{e}_\theta + a_z\,\hat{e}_z$$

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}, \quad a_z = \ddot{z}$$

⇨ **Lecture Notes**

From “chap4-culindrical.pdf”

$$a_r = V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z\frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r} + \frac{\partial V_r}{\partial t}$$

$$a_\theta = V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z} + \frac{V_\theta V_r}{r} + \frac{\partial V_\theta}{\partial t}$$

$$a_z = V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z} + \frac{\partial V_z}{\partial t}$$

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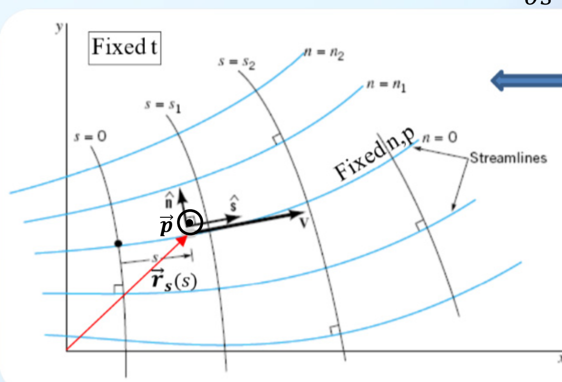
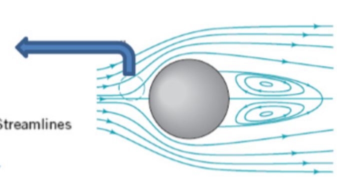
Streamline coordinate system

- **Proof: “chap4-AppendixA.pdf”**

$$\vec{V} = V\hat{s} \quad (12.4) \qquad \vec{a} = \frac{\partial \vec{V}}{\partial t} + V\frac{\partial \vec{V}}{\partial s} \quad (13.4)$$

$$\frac{\partial \vec{V}}{\partial t} = \frac{\partial}{\partial t}(V\hat{s}) = \frac{\partial V}{\partial t}\hat{s} + V(\widehat{\omega}' \times \hat{s}) \quad (15.4) \qquad V\frac{\partial \vec{V}}{\partial s} = V\frac{\partial V}{\partial s}\hat{s} + \frac{V^2}{R}\hat{n} \quad (14.4)$$





■ FIGURE 4.8 Streamline coordinate system for two-dimensional flow.

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The end of chapter 4

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