Fluid mechanics I
by
Dr. Ehsan Amani

Chapter 5

The Bernoulli equation

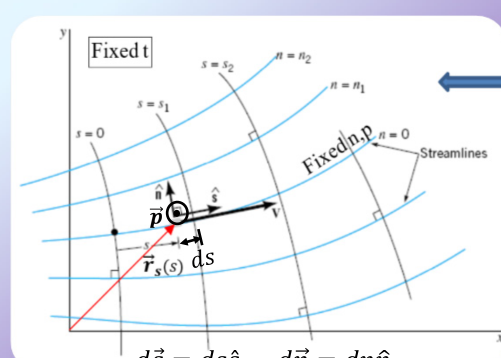
- **Assumptions**
  - **Inviscid flow**
  - **Body force: Gravity only**
  - **Steady flow**

$$-\vec{\nabla}p - \gamma\hat{k} = \rho\vec{a} \qquad (3.3)$$

Remember (math. II):

$$d_{\hat{s}}f = \left(\frac{\partial f}{\partial s}\right)ds = \left(\hat{s}.\vec{\nabla}f\right)ds$$
$$= d\vec{s}.\vec{\nabla}f$$

⇨ **Lecture Notes**



$d\vec{s} = ds\hat{s}$ $\quad d\vec{n} = dn\hat{n}$

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The Bernoulli equation

- **Compressible flow**

Along a streamline: $$\int \frac{dp}{\rho} + gz + \frac{V^2}{2} = cte \qquad (6.5)$$

Across a streamline: $$\int \frac{dp}{\rho} + gz + \int \frac{V^2}{R} dn = cte \qquad (9.5)$$

- **Incompressible flow**

Along a streamline:

$$\frac{p}{\gamma} + z + \frac{V^2}{2g} = cte \qquad (7.5)$$

$$\underbrace{\frac{p_1}{\gamma}}_{\text{Pressure head}} + \underbrace{z_1}_{\text{elevation head}} + \underbrace{\frac{V_1^2}{2g}}_{\text{Velocity head}} = \frac{p_2}{\gamma} + z_2 + \frac{V_2^2}{2g} \qquad (8.5)$$

Across a streamline:

$$\frac{p}{\gamma} + z + \frac{1}{g}\int \frac{V^2}{R} dn = cte \qquad (10.5)$$

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2 + \frac{1}{g}\int_1^2 \frac{V^2}{R} dn \qquad (11.5)$$

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The Bernoulli equation

- **Example: Parallel flow ($R = 0$)**

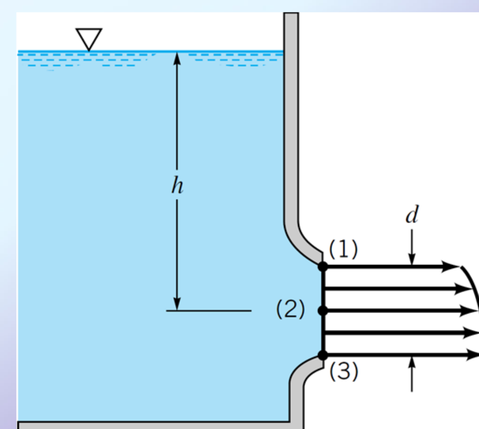
Across the streamline $\rightarrow$

$\perp$: (1), (2)

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2 + \cancelto{0}{\frac{1}{g}\int_1^2 \frac{V^2}{R} dn}$$

$\longrightarrow$ $p_2 - p_1 = \gamma(z_1 - z_2)$

Fluid statics!



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Pressures, heads

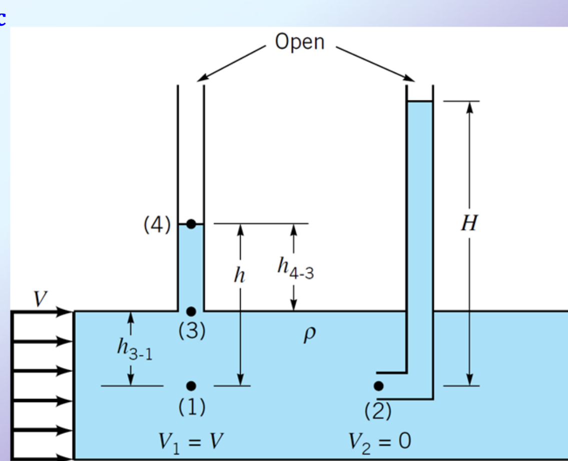
- **Pressures**

$$\underbrace{\underbrace{\underbrace{p}_{\text{Static Or thermodynamic pressure}} + \underbrace{\frac{1}{2}\rho V^2}_{\text{Dynamic pressure}}}_{\text{Stagnation pressure}} + \underbrace{\gamma z}_{\text{Hydrostatic pressure}}}_{\text{Total pressure}} = \text{Constant along a streamline} \qquad (12.5)$$

⊥: (3), (1) → $p_3 - p_1 = \gamma h_{3-1}$

⊥: (4), (3) → $\cancelto{0}{p_4} - p_3 = \gamma h_{4-3}$

→ $p_1 = \gamma h$ **Static pressure measurement**



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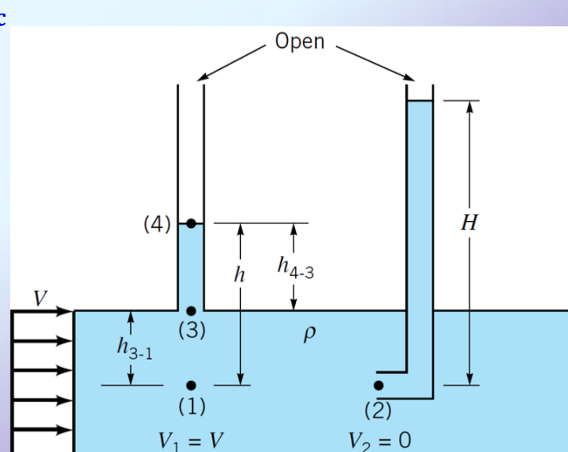
Pressures, heads

- **Pressures**

$$\underbrace{\underbrace{\underbrace{p}_{\text{Static Or thermodynamic pressure}} + \underbrace{\frac{1}{2}\rho V^2}_{\text{Dynamic pressure}}}_{\text{Stagnation pressure}} + \underbrace{\gamma z}_{\text{Hydrostatic pressure}}}_{\text{Total pressure}} = \text{Constant along a streamline} \qquad (12.5)$$

→: (2), (1) → $\overbrace{p_2}^{\gamma H} + 0 + \gamma z_2 = \underbrace{p_1}_{\gamma h} + \frac{1}{2}\rho V_1^2 + \gamma z_1$

→ $\frac{V_1^2}{2g} = H - h$ **Velocity measurement**



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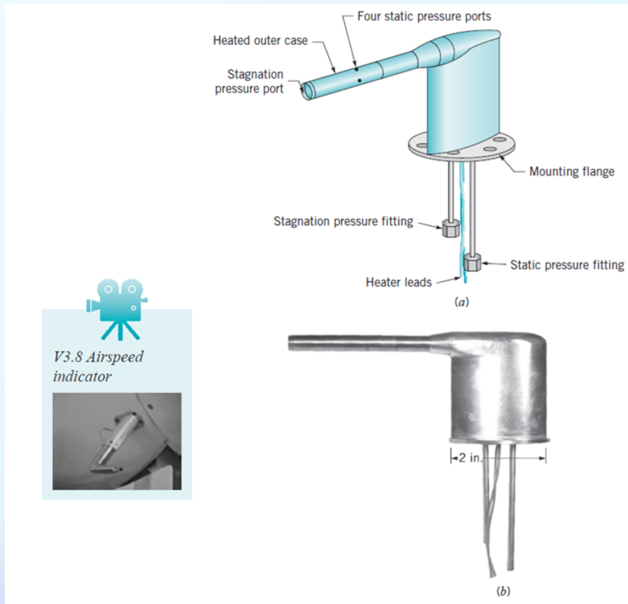
Pressures, heads



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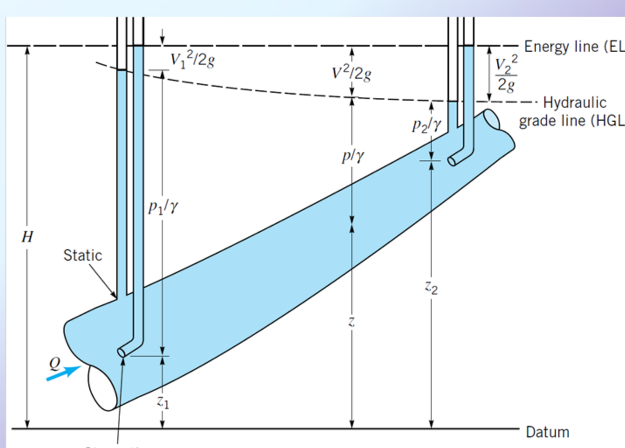
Pressures, heads

- **Heads**

$$\underbrace{\frac{p}{\gamma}}_{\text{Pressure head}} + \underbrace{\frac{V^2}{2g}}_{\text{Velocity head}} + \underbrace{z}_{\text{Elevation head}} = \overbrace{H}^{\text{Total head}} = \text{Constant along a streamline} \qquad (13.5)$$



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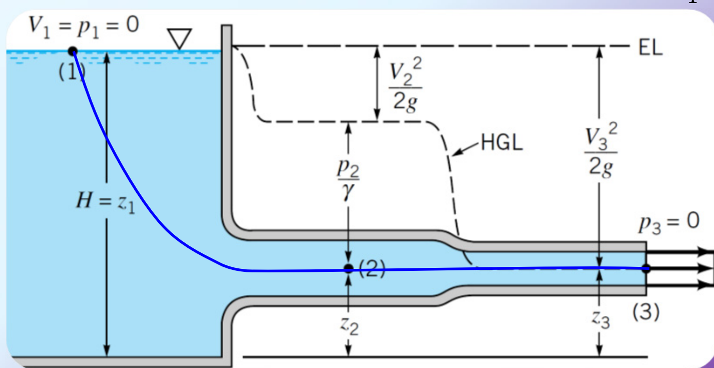
Applications

- **Large containers**
  - **Quasi-steady assumption:** $H = H(t);\ \dfrac{\partial H}{\partial t} \sim 0\ (V_1 \sim 0)$

$$\xrightarrow{\to:\,(1),(2),(3)} \cancelto{0}{\frac{p_1}{\gamma}} + \cancelto{0}{\frac{V_1^2}{2g}} + z_1 = \cancelto{0}{\frac{p_3}{\gamma}} + \frac{V_3^2}{2g} + z_3 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2 \longrightarrow \begin{array}{l} V_3 = \sqrt{2gh} \\ h = z_1 - z_3 \end{array}$$



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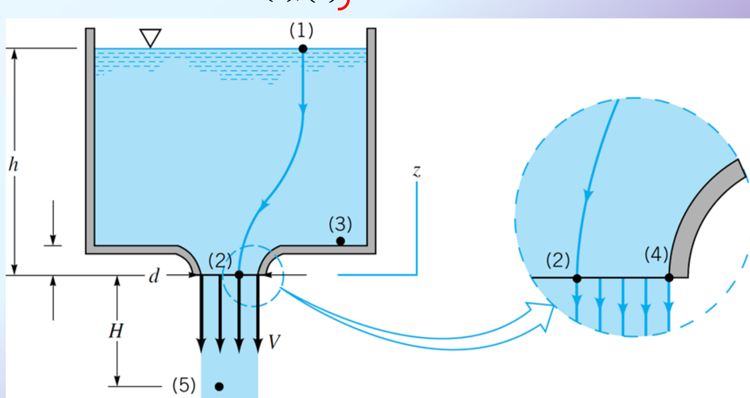
Applications

- **Jets**
  - **Exercise: Assuming no interface pressure jump (?), show that** $\left.\begin{array}{r} \to:(1),(2) \\ \perp:(2),(4) \end{array}\right\} \longrightarrow V = \sqrt{2gh}$



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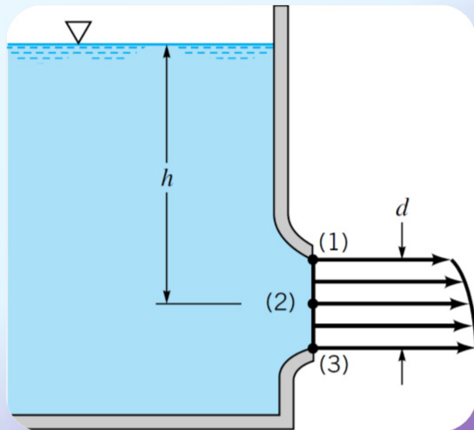
Applications

- **Jets**
  - **Uniform velocity profile?** $d \ll h$



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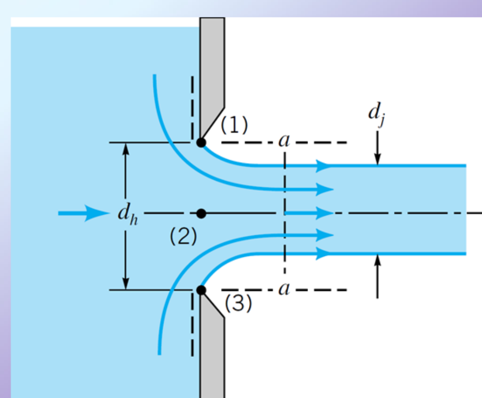
Applications

- **Jets**
  - **Vena contracta**
  - **Exercise: show that** $p_2 > p_1 = p_3 = 0$
  - **An additional (empirical) information is required:**

$$C_c = \frac{A_j}{A_h} = \left(\frac{d_j}{d_h}\right)^2 \qquad (15.5)$$

Contraction coefficient



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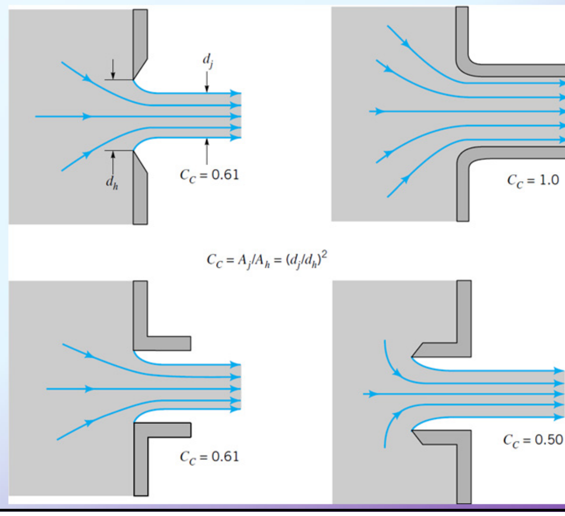
Applications

- Jets
  - Contraction coefficient



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Applications

- Problem solving
  - Bernoulli equations (momentum)
  - Mass conservation
  - Empirical information (if applicable)

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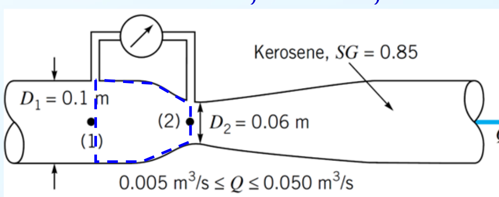
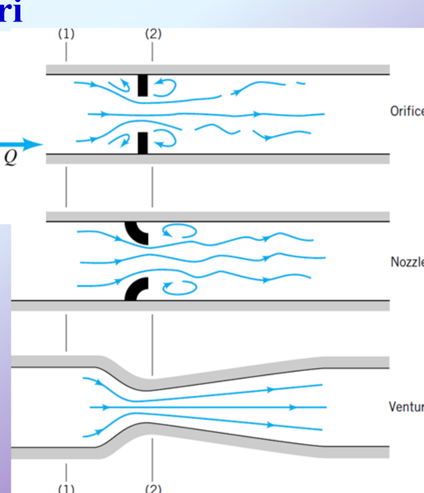
Applications

- **Flow measurement**
  - **Orifice, Nozzle, Venturi**





$\rightarrow$: (1), (2) $\longrightarrow$ $\dfrac{p_1}{\gamma} + \dfrac{V_1^2}{2g} = \dfrac{p_2}{\gamma} + \dfrac{V_2^2}{2g}$

mass $\longrightarrow$ $Q_1 = Q_2 \longrightarrow V_1 A_1 = V_2 A_2$

$$Q = A_2 \sqrt{\frac{2(p_1 - p_2)}{\rho[1 - (A_2/A_1)^2]}} \quad (17.5)$$

$$\longrightarrow Q_{\text{actual}} = cQ \quad (18.5)$$

Experimental correction factor

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Applications

- **Flow measurement**
  - **Exercise: Sluice gate**

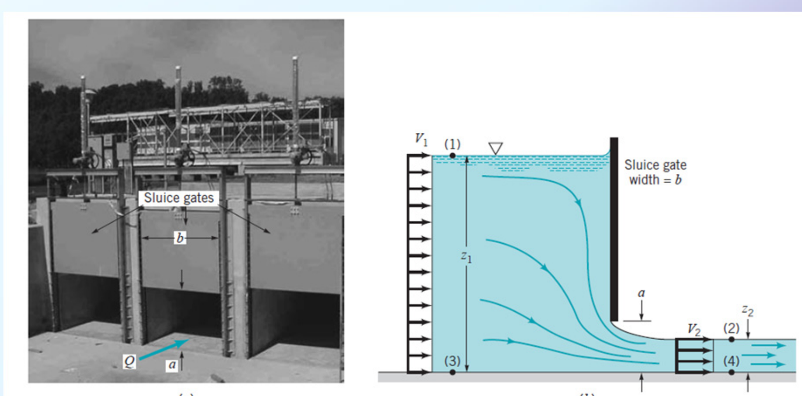

FIGURE 3.19 Sluice gate geometry. (Photograph courtesy of Plasti-Fab, Inc.)

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Applications

- **Flow measurement**
  - **Exercise: Weir**

Triangular sharp-crested weir ▶

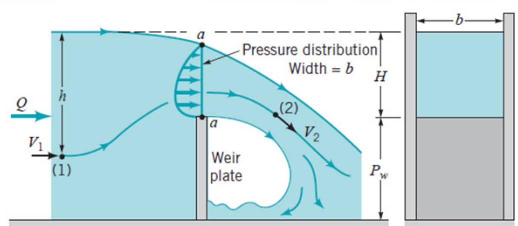

FIGURE 3.20
Rectangular, sharp-crested weir geometry.

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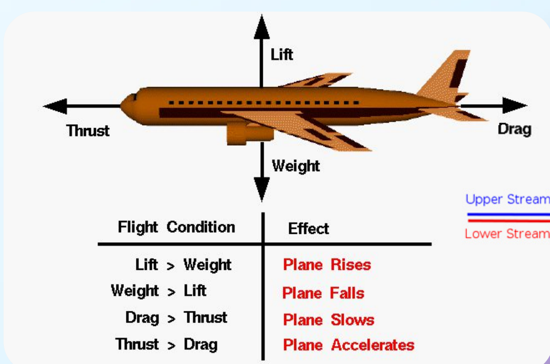
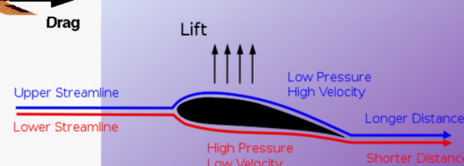
Applications

- **Lift force**
  - **Exercise: Search lift force estimation based on the Bernoulli equation.**





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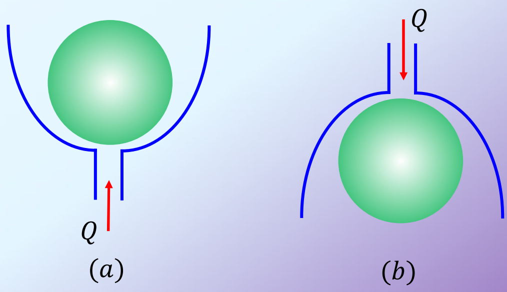
Sample problems

Observations show that it is not possible to throw away a tennis ball from a cone by blowing (figure a). In fact, the ball can be suspended by blowing (figure b). How do you justify this?



Lecture Notes

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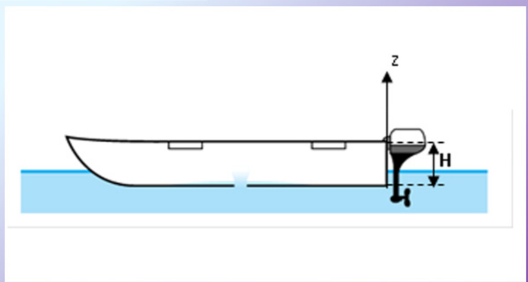
Sample problems

A small hole of area A_h develops in the bottom of the stationary rowboat of weight W. Estimate the amount of time it will take for the boat to sink List all assumptions you make. The area of each section of the boat, parallel to water surface, is given by A=A(z). Assume that A=A(z)=const.



Lecture Notes

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Bernoulli equation extensions

- **Compressibility effect (ideal gas)**
  - **Exercise: For isothermal flow of an ideal gas show that**

Absolute pressure only

Along a streamline: $$\frac{V_1^2}{2g}+z_1+\frac{RT}{g}\ln p_1=\frac{V_2^2}{2g}+z_2+\frac{RT}{g}\ln p_2 \quad (21.5)$$

  - **Exercise: For $\frac{p_1-p_2}{p_1}=\varepsilon\ll 1$, show that Eq. (21.5) is simplified to Eq. (8.5). Hint:** $\lim_{\varepsilon\to 0}\ln(1+\varepsilon)\approx\varepsilon$
  - **Exercise: For isentropic flow of an ideal gas show that**

Along a streamline: $$\left(\frac{k}{k-1}\right)\frac{p_1}{\rho_1}+\frac{V_1^2}{2}+gz_1=\left(\frac{k}{k-1}\right)\frac{p_2}{\rho_2}+\frac{V_2^2}{2}+gz_2 \quad (22.5)$$

  - **Neglecting compressibility effects:** $Ma<0.3$

Mach number, Flow velocity, Sound speed

$$Ma=\frac{V}{c} \quad (23.5)$$

Ideal gas only

$$c=\sqrt{kRT} \quad (24.5)$$

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Bernoulli equation extensions

- **Unsteady effect**

⇨ **Lecture Notes**

Along a streamline: $$p_1+\frac{1}{2}\rho V_1^2+\gamma z_1=p_2+\frac{1}{2}\rho V_2^2+\gamma z_2+\rho\int_1^2\frac{\partial V}{\partial t}ds \quad (25.5)$$

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Bernoulli equation limitations

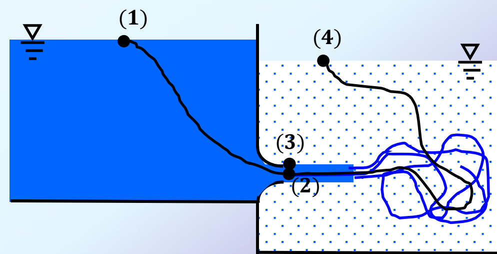
- **Highly viscous regions**
  - **Jet wakes**

$\rightarrow$: (1), (4) ✗

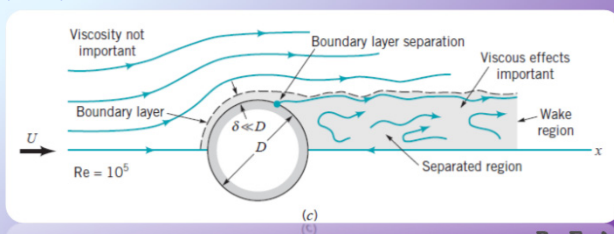
$\rightarrow$: (1), (2) ✓

$\perp$: (2), (3) ✓

γh: (3), (4) ✓



  - **Boundary layers**
  - **Wakes**



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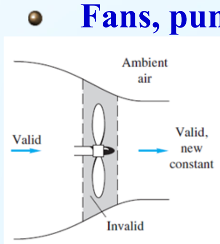
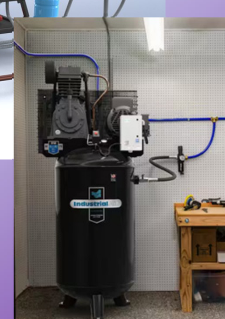
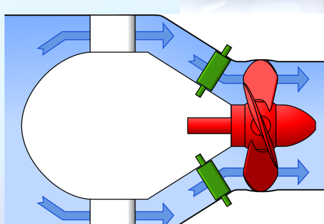
Bernoulli equation limitations

- **Power to/from flow**
  - **Fans, pumps, compressors**





  - **Turbines**



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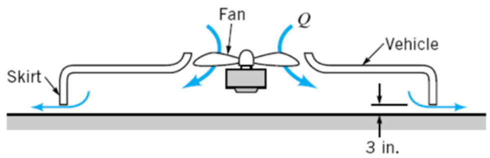
Sample problems

An air cushion vehicle is supported by forcing air into the chamber created by a skirt around the periphery of the vehicle as shown in the figure. The air escapes through the 3-in. clearance between the lower end of the skirt and the ground (or water). Assume the vehicle weighs 10,000 lb and is essentially rectangular in shape, 30 by 50 ft. The volume of the chamber is large enough so that the kinetic energy of the air within the chamber is negligible. Determine the flowrate, Q, needed to support the vehicle (neglect compressibility). If air behaves like a isothermal ideal gas, what flowrate would be needed?



Lecture Notes

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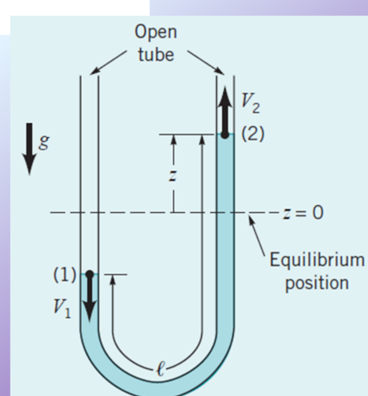
Sample problems

An incompressible, inviscid liquid is placed in a vertical, constant-diameter U-tube as indicated in the figure. When released from the nonequilibrium position shown, the liquid column will oscillate at a specific frequency. Determine this frequency.

Lecture Notes



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The end of chapter 5

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