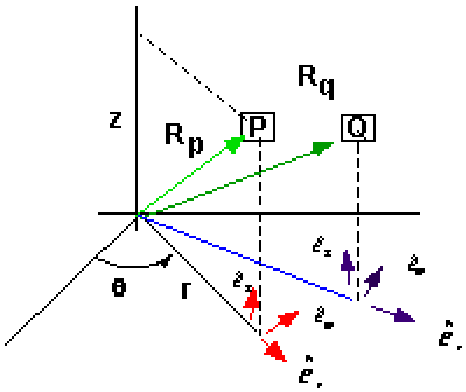
Coordinate System: Cylindrical - Definition

The **cylindrical coordinate system** is another useful way **to track a particle in three dimensional space.** A practical example is the airport control tower (the origin) keeping track of the planes (particles) coming in for landing. In a **cylindrical coordnate system**

Coordinates: r, θ, z **unit vectors:** $\hat{e}_r, \hat{e}_\theta, \hat{e}_z$ **(the unit vector in the r direction is along increasing r; the unit vector in theta direction is at right angles to r and in the direction of increasing theta; the unit vector in the z direction is defined in the direction of the cross product of the r and theta unit vectors)**



Consider the particle at point **P** and point **Q** at successive times (***dt***). The location at point **P,** the particle is determined as the addition of the vectors

$$\bar{R} = r\hat{e}_r + z\hat{e}_z$$

Notice that the **unit vectors** $\hat{e}_r, \hat{e}_\theta$ **change direction** from **P** to **Q especially with** θ**.** It can be established that

$$\frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta, \quad \frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_r$$ **The** rest of the derivatives **are zero**

This implies the **time** derivatives of $\hat{e}_r, \hat{e}_\theta$ are not zero. They can be established as

$$\frac{d\hat{e}_r}{dt} = \dot{\theta}\,\hat{e}_\theta \quad \frac{d\hat{e}_\theta}{dt} = -\dot{\theta}\,\hat{e}_r$$

By definition the **velocity** can be defined as

$$\vec{V} = V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{e}_z$$

$$\vec{V} = \frac{d\vec{R}}{dt} = \frac{d(r\,\hat{e}_r)}{dt} + \frac{d(z\hat{e}_z)}{dt}$$

$$\vec{V} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + \dot{z}\,\hat{e}_z$$

The **acceleration** is

$$\vec{a} = a_r\,\hat{e}_r + a_\theta\,\hat{e}_\theta + a_z\,\hat{e}_z$$

$$a_r = \ddot{r} - r\dot{\theta}^2, \quad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}, \quad a_z = \ddot{z}$$

Note: In Solid Mechanics the acceleration above is $\frac{d\vec{V}}{dt}$

In Fluid Mechanics

the accleration defined above is only $\frac{\partial\vec{V}}{\partial t}$ **Convince yourself of the** difference

The **acceleration** is obtained using the **substantial derivative**: $\vec{a} = \frac{D\vec{V}}{Dt} = (\vec{V} \bullet \nabla + \frac{\partial}{\partial t})\vec{V}$

The **gradient operator** in the cylindrical coordinates: $\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$

The **acceleration** expressions are:

$$a_r = V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^{\ 2}}{r} + \frac{\partial V_r}{\partial t}$$

$$a_\theta = V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z \frac{\partial V_\theta}{\partial z} + \frac{V_\theta V_r}{r} + \frac{\partial V_\theta}{\partial t}$$

$$a_z = V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} + \frac{\partial V_z}{\partial t}$$