**Project: 2D steady conduction – composite fin**

Consider a 2D rectangular fin of Aluminum alloy ($k_a = 100\ W/m.K$) shown in **Figure 1** with $t = 10\ mm$ and $L = 100\ mm$ in a convective environment ($h = 200\ W/m^2.K$ and $T_\infty = 300\ K$) attached to a wall with $T_b = 350\ K$.

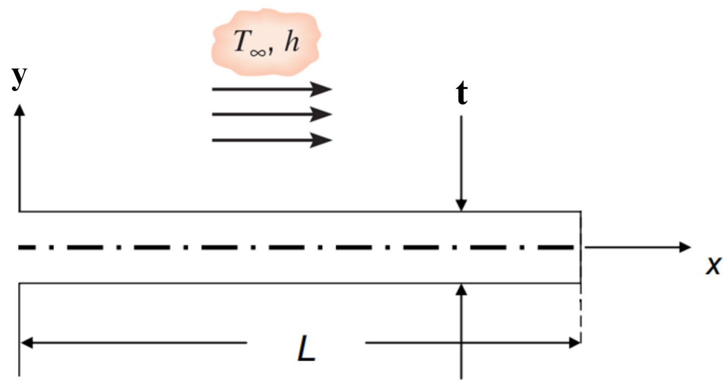


**Figure 1** *The rectangular fin (base design).*

Due to structural design constrains, the geometry of the fin cannot be changed. It is intended to enhance the fin heat transfer rate, $q_f$, using a superconducting material, i.e. Carbon NanoFiber (CNF) ($k_{CNF} = 1000\ W/m.K$). An engineer suggests the designs shown in **Figure 2**. In all designs, the same amount of CNF, indicated with the gray color in the figure, is used, and all designs are symmetric with respect to the $x$ axis, $t = 10\ mm = cte$, and $L = 100\ mm = cte$. You are going to investigate these designs by a detailed design procedure and find the best solution.

You can develop a single code for the detailed design of all above cases. The computational domain is shown in **Figure 3** which is composed of 4 different materials attached to each other. The contact thermal resistance is negligible. Note that all cases of **Figure 2** are symmetric with respect to the $x$ axis and only the upper half of the fin geometries is solved using the symmetry boundary condition at $y = 0$ (see **Figure 3**) which reduces the computational cost. The other boundary conditions are convective boundary condition ($h, T_\infty$) on the top and right boundaries and fixed temperature ($T_b$) at the left boundary ($x = 0$).

The computational grid (nodal network) is illustrated in **Figure 4**. $N_x = N_{x_1} + N_{x_2}$ is the number of segments and $N_x + 1$ is the number of nodes (cells) in the $x$ direction. Similarly, $N_y = N_{y_1} + N_{y_2}$ is the number of segments and $N_y + 1$ is the number of nodes (cells) in the $y$ direction. Therefore, $N = (N_x + 1)(N_y + 1)$ is the total number of nodes of the grid.

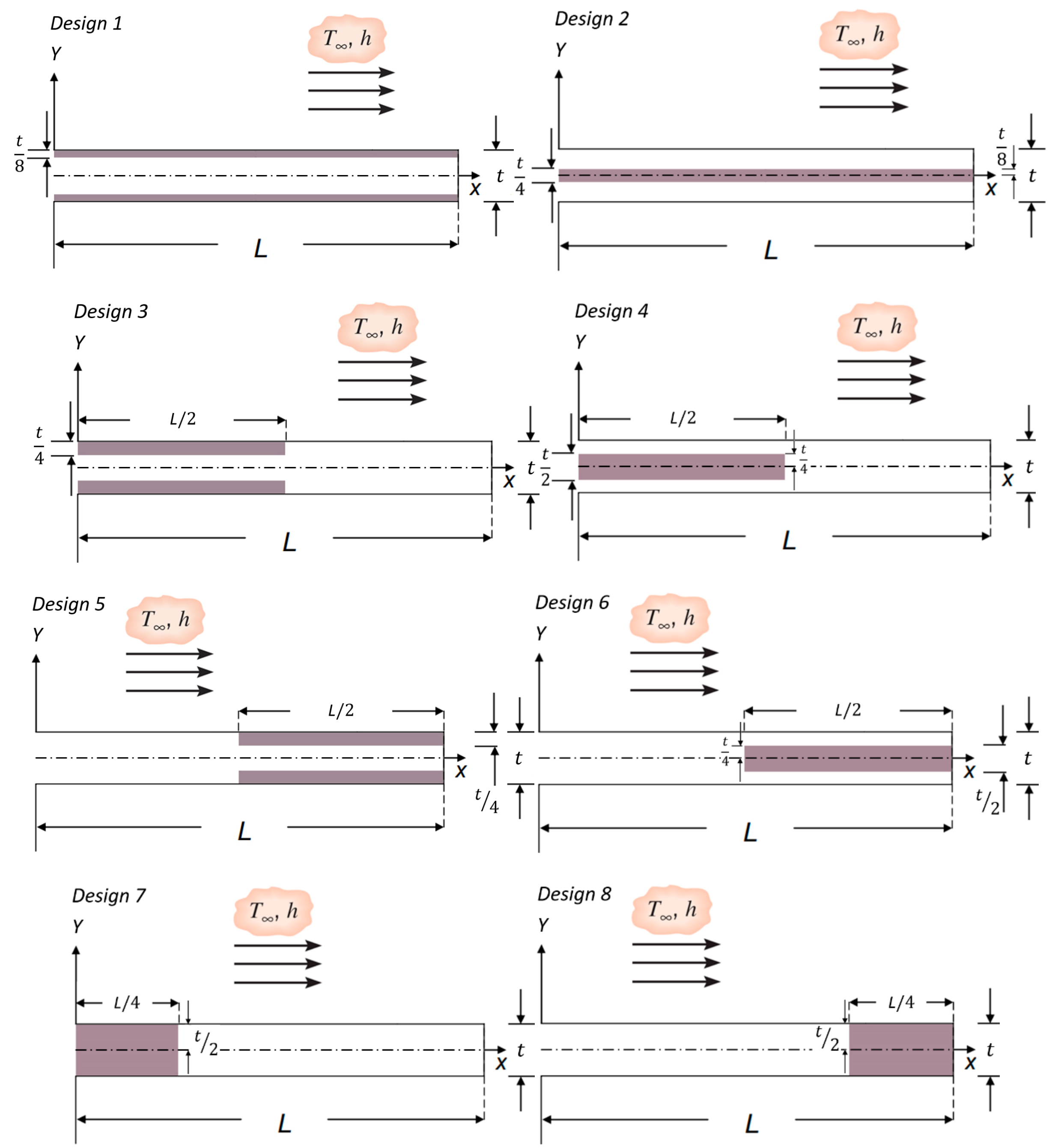


**Figure 2** *The design proposals.*

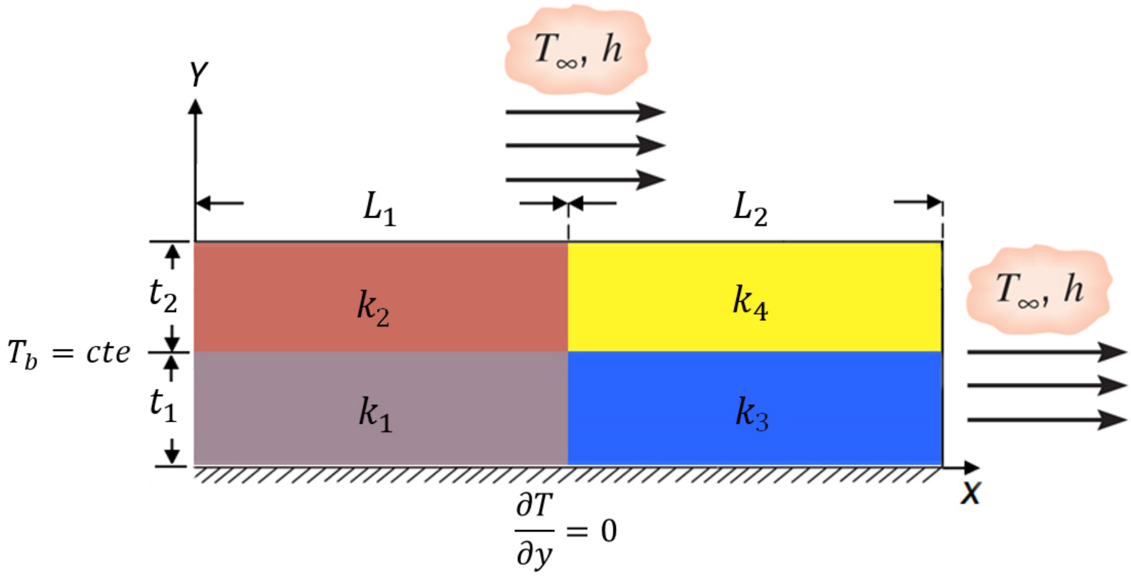


**Figure 3** *The computational domain and boundary conditions.*

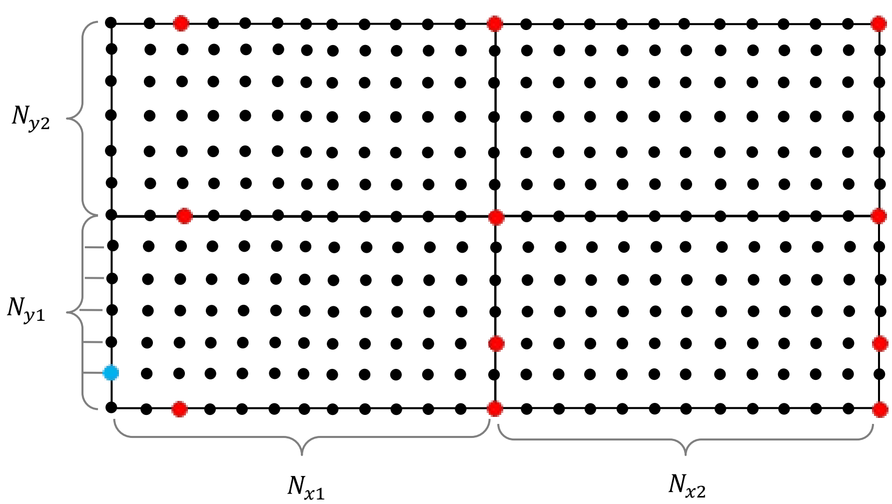


**Figure 4** *The grid configuration.*

Your code should get $(L_1, L_2, t_1, t_2, h, T_\infty, T_b)$ as input physical parameters and $(N_{x_1}, N_{x_2}, N_{y_1}, N_{y_2})$ as input numerical parameters. The numerical parameters will be chosen such that $\Delta x = cte, \Delta y = cte$ (but $\Delta x \neq \Delta y$ in general). Also, note that $t_1 + t_2 = t/2$ and $L_1 + L_2 = L$. Then, you can solve all cases of **Figure 2** by setting the input parameters appropriately. For example, the settings for some designs are given in **Table 1**.

**Table 1** *The code inputs for 4 different designs.*

| Case | Thicknesses | Lengths | Conductivity | $N_{x_1}$ | $N_{x_2}$ | $N_{y_1}$ | $N_{y_2}$ |
|---|---|---|---|---|---|---|---|
| Design 1 | $t_1 = {}^{3t}/_8$<br>$t_2 = {}^{t}/_8$ | $L_1 = {}^{L}/_2$<br>$L_2 = {}^{L}/_2$ | $k_1 = k_3 = k_a$<br>$k_2 = k_4 = k_{CNF}$ | ${}^{N_x}/_2$ | ${}^{N_x}/_2$ | ${}^{3N_y}/_4$ | ${}^{N_y}/_4$ |
| Design 3 | $t_1 = {}^{t}/_4$<br>$t_2 = {}^{t}/_4$ | $L_1 = {}^{L}/_2$<br>$L_2 = {}^{L}/_2$ | $k_1 = k_3 = k_4 = k_a$<br>$k_2 = k_{CNF}$ | ${}^{N_x}/_2$ | ${}^{N_x}/_2$ | ${}^{N_y}/_2$ | ${}^{N_y}/_2$ |
| Design 4 | $t_1 = {}^{t}/_4$<br>$t_2 = {}^{t}/_4$ | $L_1 = {}^{L}/_2$<br>$L_2 = {}^{L}/_2$ | $k_2 = k_3 = k_4 = k_a$<br>$k_1 = k_{CNF}$ | ${}^{N_x}/_2$ | ${}^{N_x}/_2$ | ${}^{N_y}/_2$ | ${}^{N_y}/_2$ |
| Design 8 | $t_1 = {}^{t}/_4$<br>$t_2 = {}^{t}/_4$ | $L_1 = {}^{3L}/_4$<br>$L_2 = {}^{L}/_4$ | $k_1 = k_2 = k_a$<br>$k_3 = k_4 = k_{CNF}$ | ${}^{3N_x}/_4$ | ${}^{N_x}/_4$ | ${}^{N_y}/_2$ | ${}^{N_y}/_2$ |

a) **Discretization**: Derive the discretized equations required for the solution of the present 2D steady conduction problem. Note that besides the internal node type (which is given by Eq. (29.5) of course notes), you have to derive the discretized equation for 11 types of boundary/interface nodes. One node of each boundary/interface node type is shown in **Figure 4** by red color. The node with blue color is located at the constant temperature boundary and the equation is simply $T_{m,n} = T_b$.

b) **Validation**: It is necessary to prove that a numerical code has been implemented correctly. For this purpose, the solution of the code is compared with analytical or experimental results at least for special cases (simplified or "canonical" problems). To do this, note that the problem has a simple analytical solution for the base design (**Figure 1**) given in Table 3.4 of the reference text book [1]. Note that for this case, $Bi_y = \frac{h(t/2)}{k} = 0.01 \ll 0.1$ and the 1D assumption is valid. As a validation of your code, you have to solve this problem on a grid with $(N_x + 1)(N_y + 1) = 401 \times 21 = 8421$ nodes using your general code and by setting all input parameters correctly rather than by writing a simpler code for this case. The proper inputs of the code are given in **Table 2**.

**Table 2** *The code inputs for the validation case.*

| Case | Thicknesses | Lengths | Conductivity | $N_{x_1}$ | $N_{x_2}$ | $N_{y_1}$ | $N_{y_2}$ |
|---|---|---|---|---|---|---|---|
| Validation case | $t_1 = t_2 =$ 2.5 $mm$ | $L_1 = L_2 =$ 50 mm | $k_1 = k_2 = k_3 = k_4 = 100\ W/m.K$ | 200 | 200 | 10 | 10 |

It is recommended that, in debugging stage of your code, use a much coarser grid, e.g. $N_{x_1} = N_{x_2} = 20$ and $N_{y_1} = N_{y_2} = 4$, to find the bugs of your code (if any) easier. When you make sure your code works without error, use the fine grid defined in **Table 2**.

After obtaining the numerical solution, compare the analytical solution of $T(x)$ versus $x$ and numerical temperature profiles versus $x$, 1) at $y = 0\ mm$, 2) at $y = 2.5\ mm$, and 3) at $y = 5\ mm$, all on a single diagram. The diagram may look something like **Figure 5**. Based on your diagram, is the 1D assumption valid? What is the maximum relative error of the numerical solution?

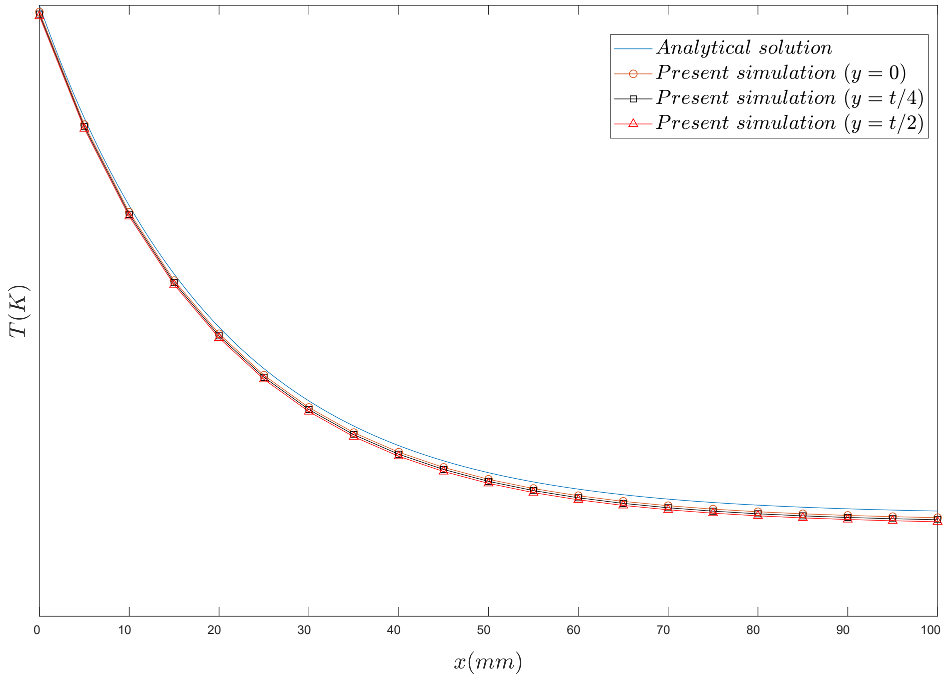


**Figure 5** *Sample validation of fin temperature distribution.*

In addition, compute $q_f$, from your numerical solution, for example by:

$$q_f = 2\left\{h\frac{\Delta x}{2}(T_{1,N_y+1} - T_\infty) + \sum_{n=1}^{N_y+1} q_{m-1,n\to m,n}\right\} \; ; \; m = 2$$

and compare it with the value from the analytical solution in a table like **Table 3**.

**Table 3** *Sample validation of fin heat transfer.*

| | **Analytical solution** | **Present simulation** | **Relative error (%)** |
|---|---|---|---|
| $q_{f,0}$ $(W/m)$ | **...** | **...** | **...** |

c) **Grid independence check**: This step is necessary to make sure that the obtained numerical solution is independent (to a high degree) of the numerical parameters and find the most efficient grid. For design 3, solve the problem using 5 different grids detailed in **Table 4**. For the solution on each grid, calculate the relative error of computing $q_{f,N}$ with respect to the finest grid resolution, i.e. grid 5, as:

$$Er_N = \left|\frac{q_{f,N} - q_{f,28397}}{q_{f,28397}}\right| \times 100\%$$

Note that the ratio of successive $N_x$ (or $N_y$) is roughly $\sqrt{2}$ which results in the ratio of successive $N$ of about 2. Plot the error, $Er_N$, versus $N$ in a log-log diagram. Choose the coarsest grid (the grid with the smallest $N$) that has an error less than 0.2%. This is called the most efficient grid. Use this grid for the solution of all cases of parts (d) and (e).

**Table 4** *Grid resolutions used for the grid independence check study.*

| | Grid 1 | Grid 2 | Grid 3 | Grid 4 | Grid 5 |
|---|---|---|---|---|---|
| $N_y$ | 16 | 24 | 36 | 52 | 72 |
| $N_x$ | 100 | 140 | 196 | 276 | 388 |
| $N$ | 1717 | 3525 | 7289 | 14681 | 28397 |

d) **Sample results:** Use your code to solve design 3 shown in **Figure 2**. Compare the temperature contours of design 3 and the base design (**Figure 1**) solved in part (b). What is the value of $q_f/q_{f_0}$ for design 3? $q_{f_0}$ is the heat transfer rate of the base design obtained in part (b).

e) **Study objective:** Use your code to solve the other designs shown in **Figure 2.** Note that you can use the same grid resolution obtained in part (c) for design 3. Then, calculate $q_f/q_{f_0}$ for all designs and report the results in the form of a bar chart. Your chart may look something like **Figure 6**. Based on your chart, what is the best design? Plot the temperature contour of the best design and compare with the one of the base design.

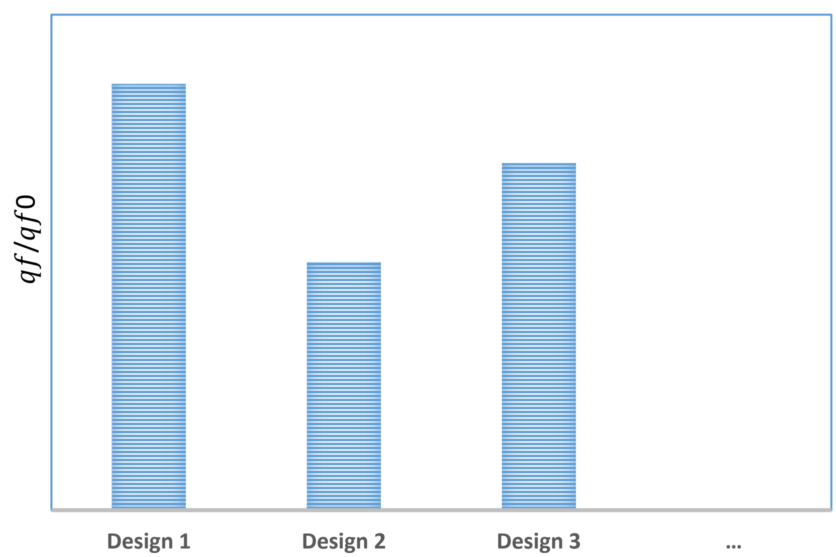


**Figure 6** *The ratio of fin heat transfer to the base fin heat transfer for different designs.*

P.S.

You are strongly advised to search similar 2D steady conduction codes on the Net. It is a good practice to find a similar code to see professional FD or FV programming. Even, it is sufficient to modify such codes for your purpose rather than write a brand new one. Two sample MATLAB and two sample Python codes are included in the compressed file "FDCodeSamples" for your convenience.

Your report should include:

1) A folder named "codes" including the source codes you developed or modified individually. Provide a short "readme" file in the code package on how to run your code. Keep the size of the package to a minimum by only including the main code files. Do not include the results of the code. The results should be generated by running your code according to the "readme" file.
2) The word and pdf of your report. The report should be in conventional scientific article format of maximum 10 pages. Do not include any additional file, image, etc. Include the results in the report file.

## References

[1] T. L. Bergman, F. P. Incropera, A. S. Lavine, and D. P. DeWitt, *Introduction to heat transfer*, 6 ed.: John Wiley & Sons, 2011.