IX.1.1.1 Drag force

An analytical solution for drag force on a particle can be obtained assuming

- Steady-state
- Stokes regime ($Re_p = \frac{\rho |\vec{U}| d_p}{\mu} < 1$) $\longrightarrow$ relaxed by empirical correlations (Re number effect)
- spherical particle $\longrightarrow$ relaxed by equivalent diameter (!)
- no evaporation/condensation $\longrightarrow$ can be relaxed by empirical correlations (blowing effects)
- uniform flow on the particle $\longrightarrow$ can be relaxed (!) e.g. by equivalent $\vec{u}_s$ the effect of wake of other particles
- incompressible low-Mach flow $\longrightarrow$ can be relaxed by empirical correlations turbulence, ...
- no internal flow $\longrightarrow$ relaxed by Hadamard-Rybczynski
- no rotation $\longrightarrow$ (!)

The governing equations of the flow around the particle (in relative coordinates) are

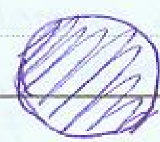
$$\vec{\nabla}\cdot\vec{u} = 0 \qquad (2\text{-}9)$$

$$\rho(\vec{u}\cdot\vec{\nabla})\vec{u} = -\vec{\nabla}p + \mu\nabla^2\vec{u} + \vec{f}$$

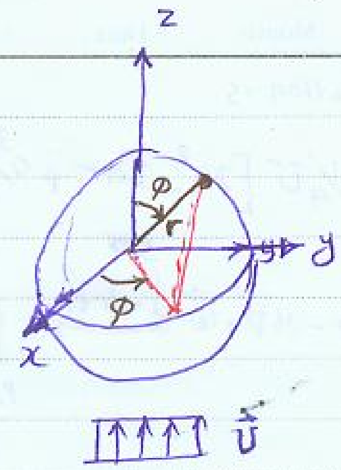
$$(Re<1):\ \vec{\nabla}p = \mu\vec{\nabla}^2\vec{u}$$

vector relations: $\vec{\nabla}\times(\underbrace{\vec{\nabla}\times\vec{u}}_{\vec{\omega}}) = \vec{\nabla}(\underbrace{\vec{\nabla}\cdot\vec{u}}_{0 \text{ continuity}}) - \nabla^2\vec{u} \rightarrow \vec{\nabla}\times\vec{\omega} = -\nabla^2\vec{u}$ $\Big\} \rightarrow \vec{\nabla}p = -\mu\,\vec{\nabla}\times\vec{\omega} \quad \xrightarrow{\vec{\nabla}\times}$

$$\rightarrow \underbrace{\vec{\nabla}\times\vec{\nabla}p}_{0 \text{ vector relations}} = -\mu\,\vec{\nabla}\times(\vec{\nabla}\times\vec{\omega}) \longrightarrow \vec{\nabla}\times(\vec{\nabla}\times\vec{\omega}) = 0$$ ◀ momentum



P4PCO



choosing spherical Coordinates (r, θ, φ)

such tat $\vec{U}_\infty = U\hat{k}$, then the problem is axialy symmetric $\frac{\partial}{\partial\varphi} = 0$

i.e. $\frac{\partial\vec{u}}{\partial\varphi} = 0$, $\vec{u} = u_r\hat{e}_r + u_\theta\hat{e}_\theta$

spherical Coordinates

$$\hat{e}_r = \sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k}$$
$$\hat{e}_\theta = \cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} + \sin\theta\,\hat{k}$$
$$\hat{e}_\varphi = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$$

$$\hat{i} = \sin\theta\cos\varphi\,\hat{e}_r + \cos\theta\cos\varphi\,\hat{e}_\theta - \sin\varphi\,\hat{e}_\varphi$$
$$\hat{j} = \sin\theta\sin\varphi\,\hat{e}_r + \cos\theta\cos\varphi\,\hat{e}_\theta + \cos\varphi\,\hat{e}_\varphi$$
$$\hat{k} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$$

$$\frac{\partial\hat{e}_r}{\partial\theta} = \hat{e}_\theta \qquad \frac{\partial\hat{e}_r}{\partial\varphi} = \sin\theta\,\hat{e}_\varphi$$
$$\frac{\partial\hat{e}_\theta}{\partial\theta} = -\hat{e}_r \qquad \frac{\partial\hat{e}_\theta}{\partial\varphi} = \cos\theta\,\hat{e}_\varphi$$
$$\frac{\partial\hat{e}_\varphi}{\partial\varphi} = -\sin\theta\,\hat{e}_r - \cos\theta\,\hat{e}_\theta$$

$$\vec{\nabla} = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\varphi\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\nabla^2 A,\ \vec{\nabla}A,\ \nabla^2\vec{A},\ \vec{\nabla}\vec{A},\ \vec{\nabla}\times\vec{A},\ \vec{\nabla}\cdot\vec{A} = \ldots$$

(2D flows) $\xrightarrow{(\frac{\partial}{\partial\varphi}=0)}$ streamfunction

$$\vec{\nabla}\cdot\vec{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta) = 0$$

defining $\begin{cases} u_\theta\sin\theta = -\frac{\partial\Psi}{\partial r}\frac{1}{r} \\ r^2 u_r = \frac{\partial\Psi}{\partial\theta}\frac{1}{\sin\theta} \end{cases} \longrightarrow \begin{cases} u_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta} \\ u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} \end{cases}$

the continuity is automatically satisfied

$$\xrightarrow{\text{momentum}} \vec{\nabla}\times(\vec{\nabla}\times(\vec{\nabla}\times\vec{u})) = 0 \longrightarrow \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\Psi = 0 \qquad (3.9)$$

boundary conditions

$$r = a:\ u_r = u_\theta = 0 \longrightarrow \frac{\partial\Psi}{\partial r} = \frac{\partial\Psi}{\partial\theta} = 0 \qquad (4.9)$$

$$r\to\infty:\ \vec{u} = U\hat{k} \longrightarrow u_r\hat{e}_r + u_\theta\hat{e}_\theta = U(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta) \to \begin{cases} u_r = U\cos\theta \\ u_\theta = -U\sin\theta \end{cases} \qquad (5.9)$$

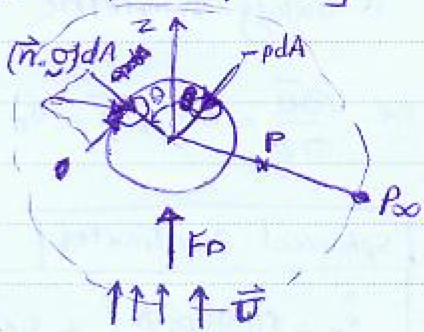
Eq. (3.9) with b.c. (4.9) and (5.9) can be solved using the separation of variables method (see file StokesDrag.pdf)

Subject:
Year. Month. Date. ( )

The solution is

$$\Psi = \tfrac{1}{4} U \left[2r^2 - 3ar + \frac{a^3}{r}\right] \sin^2\theta \quad (6.9) \longrightarrow \begin{cases} u_r = U\cos\theta \left[1 - \frac{3}{2}\frac{a}{r} + \frac{1}{2}\left(\frac{a}{r}\right)^3\right] & (7.9) \\ u_\theta = -U\sin\theta \left[1 - \frac{3}{4}\frac{a}{r} - \frac{1}{4}\left(\frac{a}{r}\right)^3\right] \end{cases}$$

$$\vec{\nabla} p = -\mu \vec{\nabla} \times \vec{\omega} \xrightarrow{\int \cdot\, dr\hat{e}_r} \underbrace{\int_{p_\infty}^{(r,\theta,\varphi)} \frac{\partial p}{\partial r} dr}_{p - p_\infty} = -\mu \int_{r\to\infty}^{(r,\theta,\varphi)} (\vec{\nabla}\times\vec{\omega}) \cdot dr\hat{e}_r$$



$$\longrightarrow p = p_\infty - \frac{3}{2} U a \mu \frac{\cos\theta}{r^2} \quad (8.9)$$

For calculating the drag force

$$\xrightarrow{(28.8)\cdot\hat{k}} F_D = F_{D,p} + F_{D,T} = \int_0^{2\pi}\int_0^{\pi} \left[(\vec{n}\cdot\underset{\sim}{\sigma})\cdot\hat{k}\right]_{r=a} dA \quad (9.9)$$

$\hat{k} \to (\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta)$, $\quad dA \to a^2\sin\theta\, d\theta\, d\varphi$, $\quad \vec{n} = \hat{e}_r$

$$\underset{\sim}{\sigma} = -p\underset{\sim}{I} + \underset{\sim}{T} = (\sigma_{rr}\hat{e}_r\hat{e}_r + \sigma_{r\theta}\hat{e}_r\hat{e}_\theta + \sigma_{r\varphi}\hat{e}_r\hat{e}_\varphi + \sigma_{\theta r}\hat{e}_\theta\hat{e}_r + \dots)$$

$$(\vec{n}\cdot\underset{\sim}{\sigma})\cdot\hat{k} = (\sigma_{rr}\hat{e}_r + \sigma_{r\theta}\hat{e}_\theta + \sigma_{r\varphi}\hat{e}_\varphi)\cdot(\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta)$$

$$= \sigma_{rr}\cos\theta - \sin\theta\,\sigma_{r\theta} \quad (*)$$

Stokes' relation $\quad \underset{\sim}{\sigma} = \mu(\vec{\nabla}\vec{u} + (\vec{\nabla}\vec{u})^T) - p\underset{\sim}{I}$, $\quad \vec{\nabla}\vec{u} = \left(\hat{e}_r\frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta} + \frac{\hat{e}_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\otimes(u_r\hat{e}_r + u_\theta\hat{e}_\theta)$

$$= \left(\hat{e}_r\hat{e}_r\frac{\partial u_r}{\partial r} + e_r u_r \cancel{\frac{\partial \hat{e}}{\partial r}} + \dots\right)$$

$$\xrightarrow[\text{Table}]{\vec{\nabla}\vec{u} \text{ or}} \begin{cases} \sigma_{rr} = \underbrace{-p}_{\text{contribute to } F_{D,p}} + \underbrace{2\mu\frac{\partial u_r}{\partial r}}_{\text{contribute to } F_{D,T}} \\ \sigma_{r\theta} = \underbrace{\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}u_\theta\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]}_{\sim F_{D,T}} \end{cases} \quad (**)$$

Substituting from (7.9) (8.9), u_r and u_θ in (**) and from (**) in (*) and integrating

$$F_D = \underbrace{2\pi\mu U a}_{F_{D,p}} + \underbrace{4\pi\mu U a}_{F_{D,T}} = 6\pi\mu U a \quad (10.9)$$