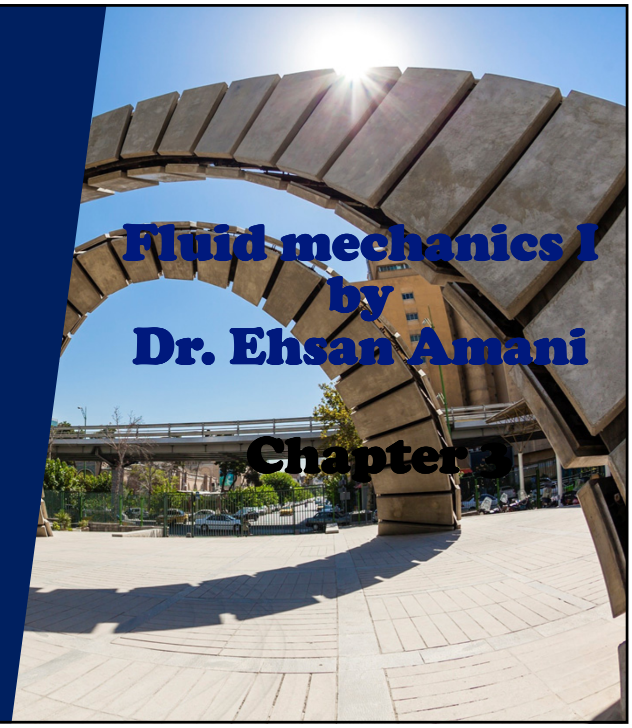
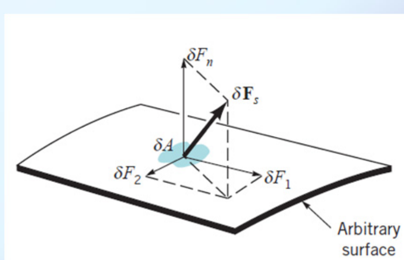


Fluid mechanics I
by
Dr. Ehsan Amani

Chapter 3

Stress field

- **Physical properties types**
  - **Scalar: a magnitude, like pressure**
  - **Vector: a magnitude and a direction (or 3 scalars), like velocity**
  - **Tensor: a magnitude and 2 directions (or 9 scalars), like stress**



$$\sigma_{nn} = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}$$ **Normal stress**

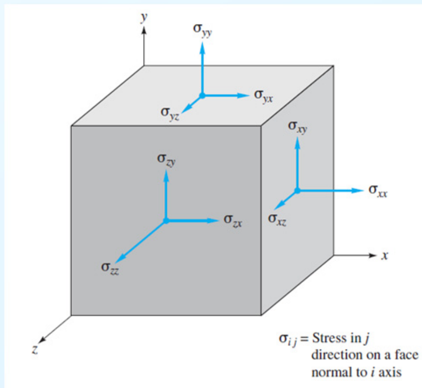
$$\sigma_{n1} = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A}$$ **Shear stress**

$$\sigma_{n2} = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}$$ **Shear stress**

Chapter 3 By E. Amani

Stress field

- **Choosing Cartesian coordinates**
  - **Strength of materials I: Stress**



$$\sigma_{ij} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Stress tensor (symmetric)

Chapter 3 By E. Amani

Stress field in a fluid at rest

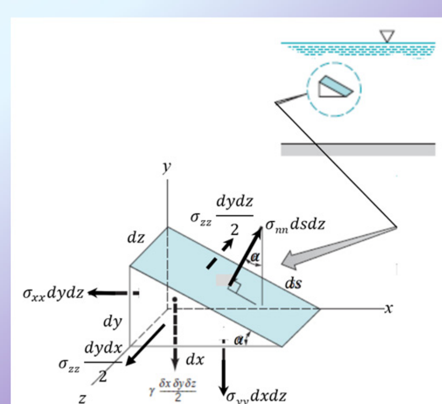
- **Fluid at rest**
  - **Force balance for an arbitrary wedge-shaped element**
  - **No motion (zero shear stresses)**

$$\sum F_x = ma_x \longrightarrow$$

$$-\sigma_{xx} dydz + \sigma_{nn} \overbrace{ds}^{dy/\sin\alpha} dz \sin\alpha = \rho \frac{dxdydz}{2} a_x$$

$$\longrightarrow -\sigma_{xx} + \sigma_{nn} = \rho \frac{dx}{2} a_x \xrightarrow{dx, dy, dz \to 0}$$

$$\sigma_{xx} = \sigma_{nn}$$



  - **Similarly, it can be shown that**

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{nn} \equiv -p \quad (1.3)$$

static pressure

Chapter 3 By E. Amani

Stress field in a fluid at rest

- **Fluid at rest**
  - **Normal stress is independent of direction**
  - **The stress tensor is simplified to**

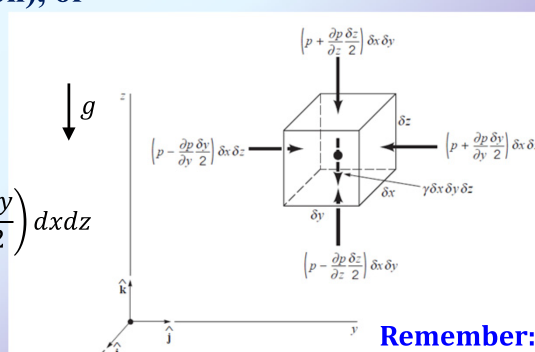
$$\sigma_{ij} \equiv \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

  - **Determining the pressure field, $p(x, y, z, t)$, is the goal**
  - **For compressible flows: The static pressure and thermodynamic pressure are identical**

Chapter 3 **By E. Amani**

Pressure field in a fluid at rest

- **Force balance when viscous forces are negligible**
  - **Fluid with no relative motion between elements (rigid-body motion), or**
  - **Inviscid flow, or**
  - **Fluid at rest**



$$\sum F_y \equiv \delta F_y =$$

$$= \left(p - \frac{\partial p}{\partial y}\frac{dy}{2}\right) dxdz - \left(p + \frac{\partial p}{\partial y}\frac{dy}{2}\right) dxdz$$

$$= -\frac{\partial p}{\partial y} dxdydz$$

  - **Similarly,**

$$\delta F_x = -\frac{\partial p}{\partial x} dxdydz,$$

$$\delta F_z = -\frac{\partial p}{\partial z} dxdydz$$

Remember:

$$f(x + dx, y + dy, z + dz) = f(x, y, z) + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

Chapter 3 **By E. Amani**

Pressure field in a fluid at rest

- **Force balance when viscous forces are negligible**

$$\delta\vec{F}_s = \delta F_x\hat{\imath} + \delta F_y\hat{\jmath} + \delta F_z\hat{k} = -\underbrace{\left(\frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath} + \frac{\partial p}{\partial z}\hat{k}\right)}_{\vec{\nabla}p \text{ or grad}p} dxdydz$$

$$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Net pressure force on an element per unit volume

$$\frac{\delta\vec{F}_s}{dxdydz} = -\vec{\nabla}p = \frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath} + \frac{\partial p}{\partial z}\hat{k} \qquad (2.3)$$

- **Therefore,**

Convention: Gravity in the negative z-direction

$$\sum\vec{F} = m\vec{a} \longrightarrow \delta\vec{F}_s - \gamma dxdydz\hat{k} = \rho dxdydz\vec{a} \longrightarrow -\vec{\nabla}p - \gamma\hat{k} = \rho\vec{a}$$

(3.3)

1 vectorial partial differential equation

- **Solutions to Eq. (3.3)?**

Chapter 3 **By E. Amani**

Pressure field in a fluid at rest

- **Fluid at rest ($\vec{a} = 0$)**

$$\xrightarrow{(3.3)} \vec{\nabla}p = -\gamma\hat{k} \longrightarrow \begin{cases} \dfrac{\partial p}{\partial x} = 0 \longrightarrow p = p(y,z) \\ \dfrac{\partial p}{\partial y} = 0 \longrightarrow p = p(z) \\ \dfrac{\partial p}{\partial z} = -\gamma \longrightarrow \dfrac{dp}{dz} = -\gamma \qquad (4.3) \end{cases}$$

- **Pressure is the same within a connected fluid at rest at all points with the same elevation**
- **For incompressible fluids ($\rho, \gamma = cte$)**

$$\frac{dp}{dz} = -\gamma \longrightarrow dp = -\gamma dz \xrightarrow{\text{integration}} p = -\gamma z + C \xrightarrow[p(z=z_{ref})=p_{ref}]{\text{Boundary condition}}$$

$$p = p_{ref} - \gamma\overbrace{\left(z - z_{ref}\right)}^{h} \quad (5.3) \quad p = p_{ref} - \gamma h$$

Chapter 3 **By E. Amani**

Pressure field in a fluid at rest

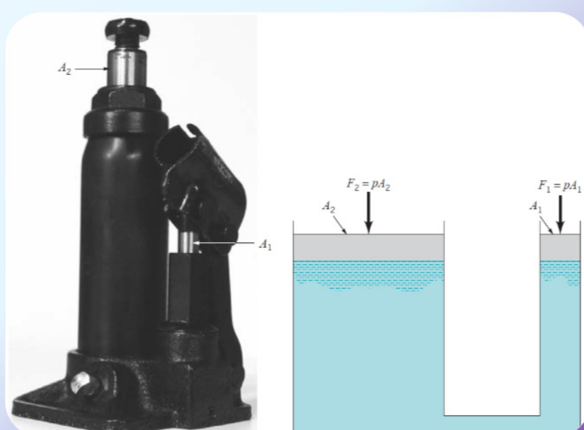
- **Fluid at rest ($\vec{a} = 0$)**
  - **Exercise: For a compressible perfect gas at rest and uniform temperature, show that**

$$\frac{p}{p_{ref}} = \exp\left(-\frac{g}{RT}\left(z - z_{ref}\right)\right) \qquad (6.3)$$

Example

- **Hydraulic jack**

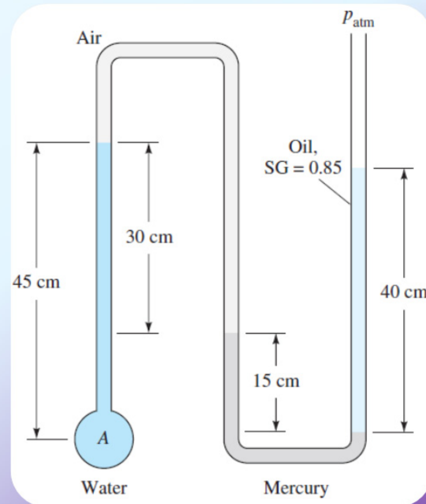
⇨ **Lecture Notes**



Example

- **Manometers**

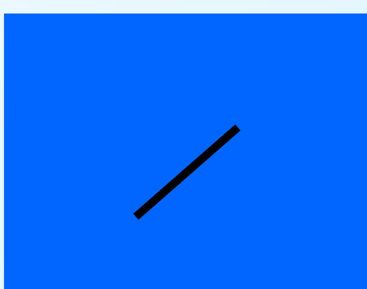
Lecture Notes

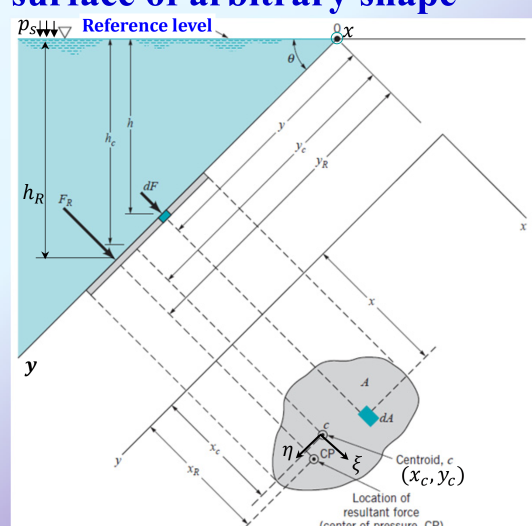


Chapter 3 By E. Amani

Hydrostatic force on a plane surface

- **An inclined plane surface of arbitrary shape**
- **The resultant hydrostatic force and its point (line) of action**



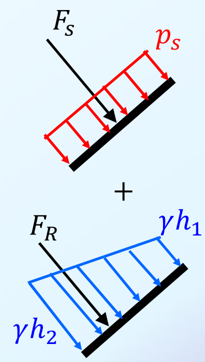


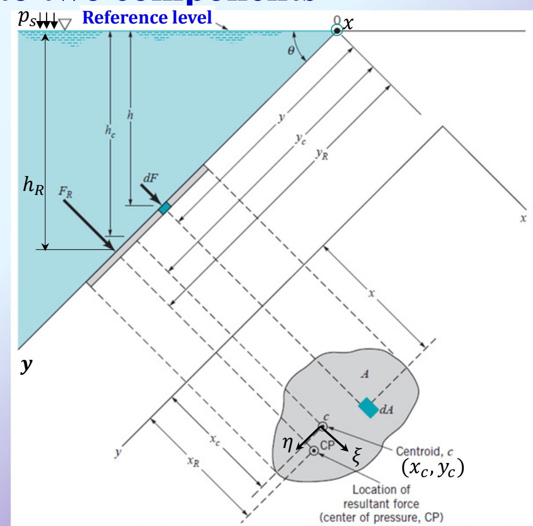
Chapter 3 By E. Amani

Hydrostatic force on a plane surface

- **Decomposition into two components**

$$p = p_s + \gamma h$$





⇨ **Lecture Notes**

Hydrostatic force on a plane surface

- **The resultant hydrostatic force and its point (line) of action**

$$F_R = \gamma \sin\theta \, y_c A = \gamma h_c A \quad (8.3)$$

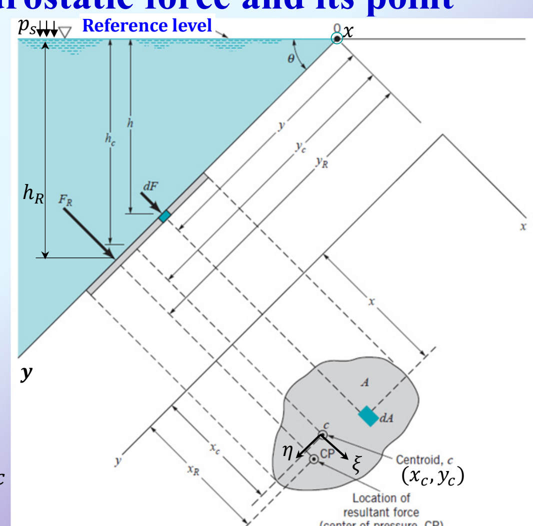
$$y_R = \frac{\int_A y^2 dA}{y_c A} = \frac{I_x}{y_c A} \quad (9.3)$$

$$I_x = I_\xi + A y_c^2 \quad (10.3)$$

$$y_R = \frac{I_\xi}{y_c A} + y_c \quad (11.3)$$

- **Similarly**

$$x_R = \frac{\int_A xy dA}{y_c A} = \frac{I_{xy}}{y_c A} = \frac{I_{\xi\eta}}{y_c A} + x_c \quad (12.3)$$



Hydrostatic force on a plane surface

- **Surface moments:**

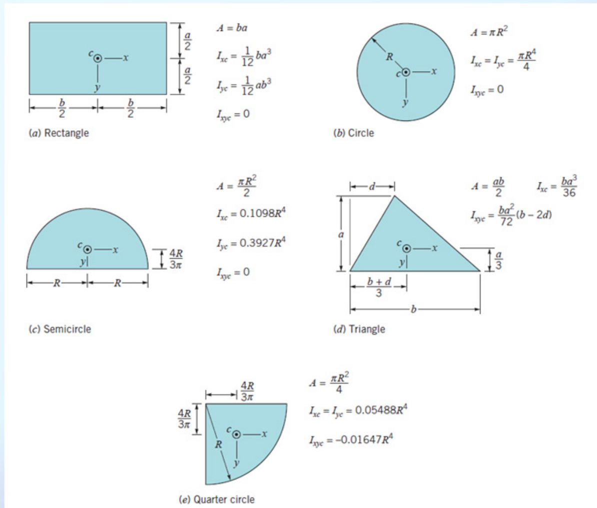


FIGURE 2.18 Geometric properties of some common shapes.

Hydrostatic force on a plane surface

- **The resultant hydrostatic force and its point (line) of action**
  - **Corollary 1: Since $\frac{I_\xi}{y_c A} > 0$, the point of action of the resultant force is always below the centroid**

$$y_R = \frac{I_\xi}{y_c A} + y_c$$

  - **Corollary 2: For a symmetric surface with respect to the ξ or η coordinate, $I_{\xi\eta} = 0$ and $x_R = x_c$**

$$x_R = \frac{I_{\xi\eta}}{y_c A} + x_c$$

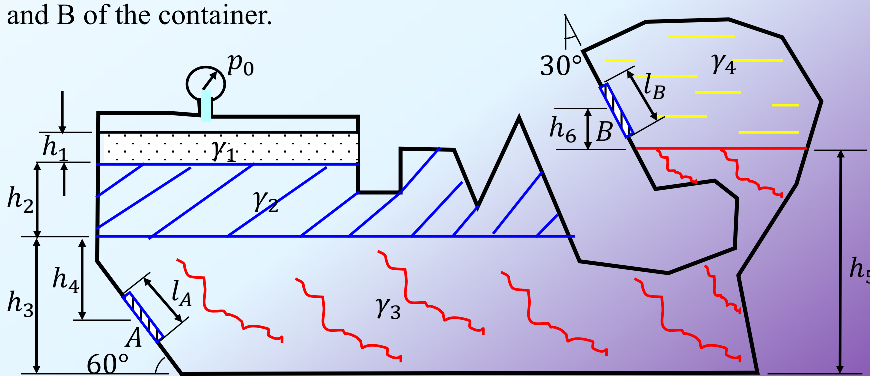
  - **Corollary 3: The total force from the fluid in contact**

$$F = F_R + F_s = \gamma h_c A + p_s A = (\gamma h_c + p_s)A = p_c A$$

Hydrostatic force on a plane surface

Example

Calculate the resultant hydrostatic force and its point of action on the gate A and B of the container.



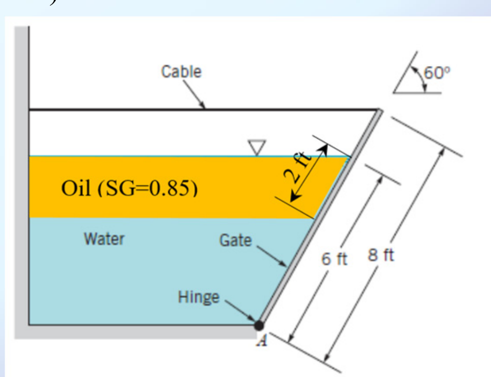
Lecture Notes

Chapter 3 By E. Amani

Hydrostatic force on a plane surface

Problem

Calculate the cable tension (the gate is rectangular with the width of 4 ft and weight of 500 lb).

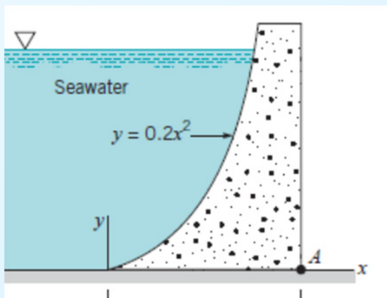


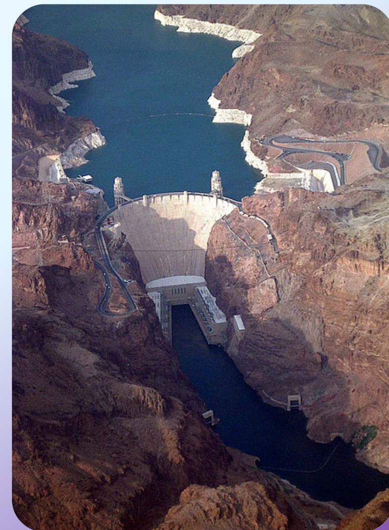
Lecture Notes

Chapter 3 By E. Amani

Hydrostatic force on a curved surface

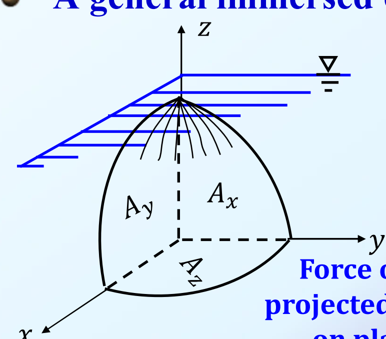
- **Dams**

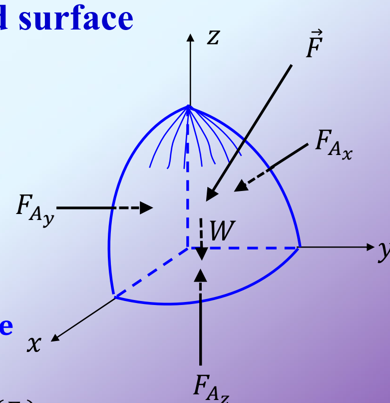




Hydrostatic force on a curved surface

- **A general immersed curved surface**





Force on the projected surface on plane x

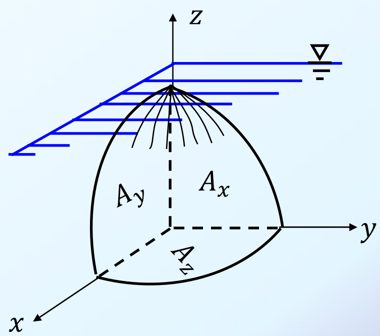
$$\sum F_x = 0 \longrightarrow F_x = F_{A_x} = (F_R)_{A_x} + (F_s)_{A_x}$$

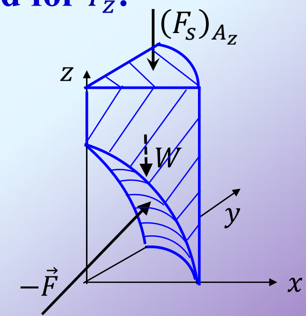
$$\sum F_y = 0 \longrightarrow F_y = F_{A_y} = (F_R)_{A_y} + (F_s)_{A_y} \quad (13.3)$$

Force on the projected surface on plane y

Hydrostatic force on a curved surface

- **The 'Column of fluid' method for F_z:**





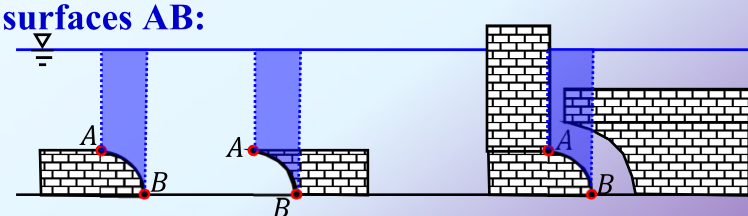
$$\sum F_z = 0 \longrightarrow F_z = (F_R)_{A_z} + (F_s)_{A_z} \quad (14.3)$$

$(F_R)_{A_z} = W$ **(the weight of hypothetical fluid column which can be placed above the surface, acted at the column center of gravity)**

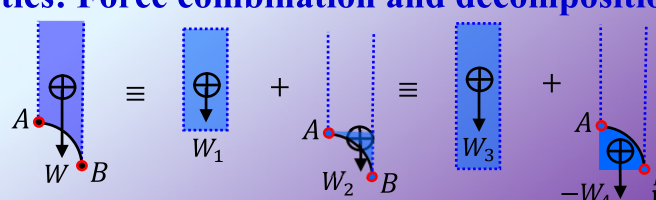
$(F_s)_{A_z} = p_s A_z$ **(acted at the center of A_z surface)**

Hydrostatic force on a curved surface

- **The pressure distribution on a surface only depends on the height of fluid free-surface above the surface**
- **The hydrostatic force magnitude is the same for the surfaces AB:**

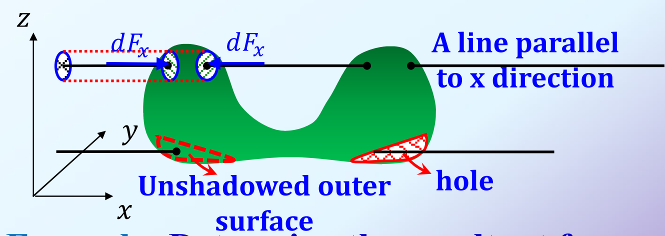


- **Statics: Force combination and decomposition**

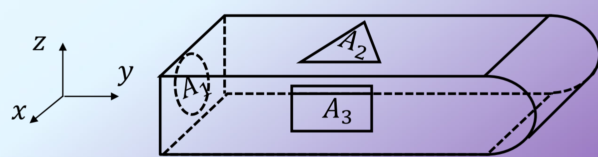


Hydrostatic force on a curved surface

- **The 'unshadowed surface' method: For F_x and F_y**



- **Example: Determine the resultant force on the body and on its outer surfaces.**

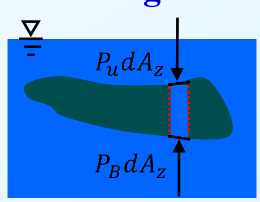


Lecture Notes

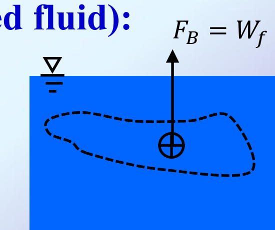
Chapter 3 By E. Amani

Buoyancy and flotation

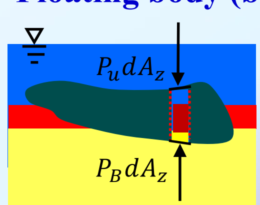
- **Submerged body (in a connected fluid):**



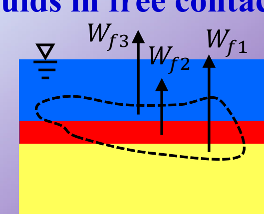
Lecture Notes



- **Floating body (between multiple fluids in free contact):**



Lecture Notes

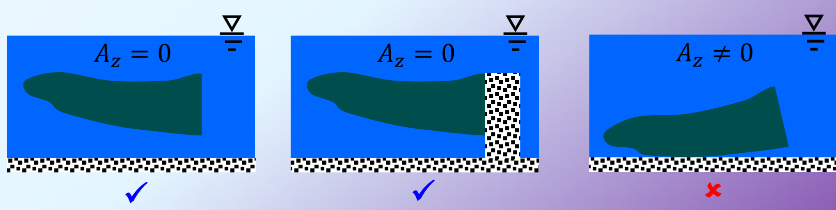


- **Conclusion: The weights of displaced fluids acting upward on their center of gravity**

Chapter 3 By E. Amani

Buoyancy and flotation

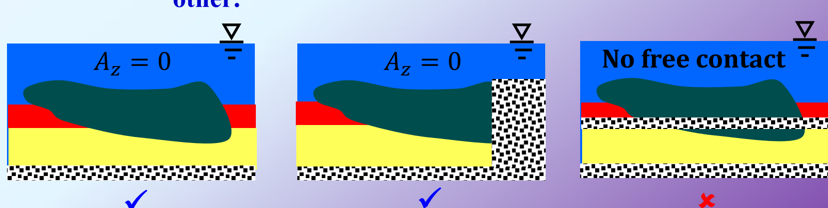
- **Algorithm: Calcuation of the hydrostatic force on a submerged or floating body**
  - **For F_x and F_y: The 'unshadowed surface' method**
  - **For F_z :**
    1. **The Bouyancy force is applicable if $A_z = 0$ and**
       a) **The body is submerged in a single (connected) fluid or,**



Chapter 3 **By E. Amani**

Buoyancy and flotation

- **Algorithm: Calcuation of the hydrostatic force on a submerged or floating body**
  - **For F_x and F_y: The 'unshadowed surface' method**
  - **For F_z :**
    1. **The Bouyancy force is applicable if $A_z = 0$ and**
       a) **The body is submerged in a single (connected) fluid or,**
       b) **is floating between multiple fluids in free contact with each other.**



Chapter 3 **By E. Amani**

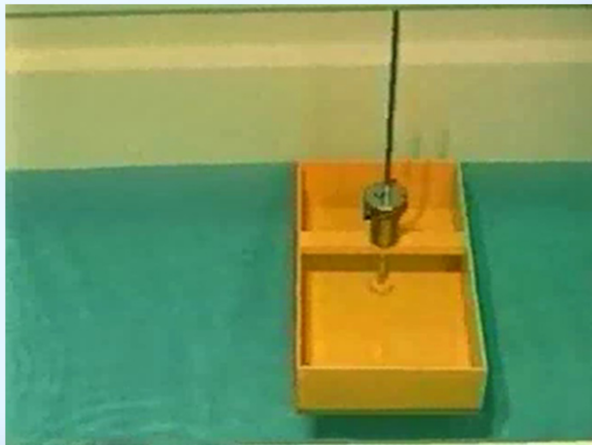
Buoyancy and flotation

- **Algorithm: Calcuation of the hydrostatic force on a submerged or floating body**
  - **For F_x and F_y: The 'unshadowed surface' method**
  - **For F_z :**
    1. **The Bouyancy force is applicable if $A_z = 0$ and**
       a) **The body is submerged in a single (connected) fluid or,**
       b) **is floating between multiple fluids in free contact with each other.**
    2. **Otherwise, the decomposition of the body into surfaces in contact with a single fluid and using the 'Column of fluid' method for each of them**

Importance of stability analysis

- **Costa Concordia**

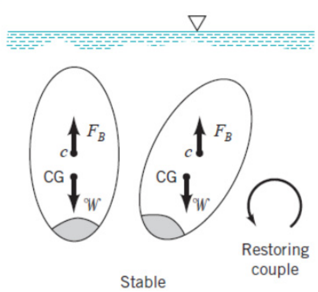
Stability of a floating body



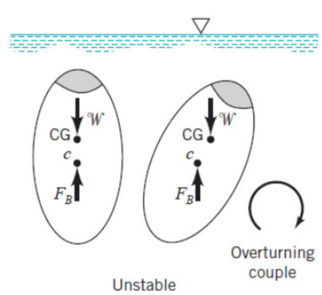
Chapter 3 By E. Amani

Stability analysis

- Submerged body



■ FIGURE 2.25
Stability of a completely immersed body—center of gravity below centroid.



■ FIGURE 2.26
Stability of a completely immersed body—center of gravity above centroid.

Chapter 3 By E. Amani

Stability analysis

- Floating body

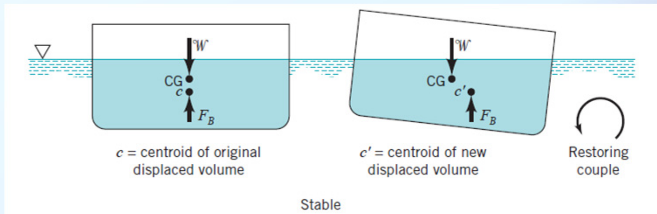


FIGURE 2.27 Stability of a floating body—stable configuration.

- More complicated: The center of buoyancy moves

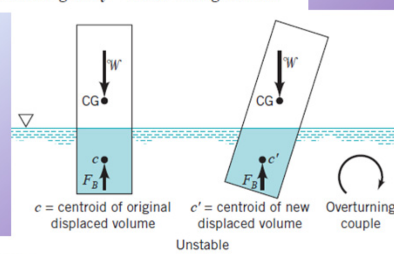
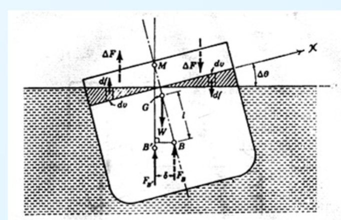


FIGURE 2.28 Stability of a floating body—unstable configuration.

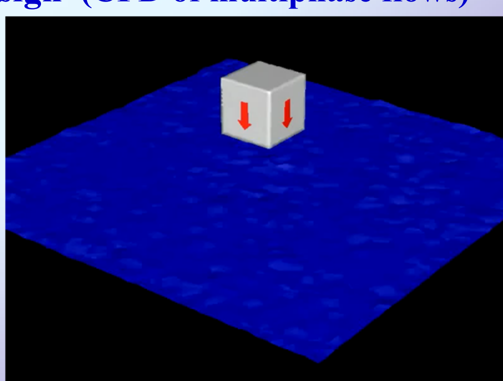
Chapter 3 By E. Amani

Stability analysis

- Preliminary design
  - linear stability
  - smooth shapes



- Detailed design (CFD of multiphase flows)



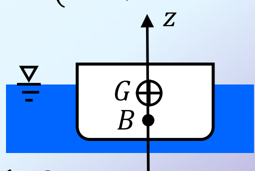
Chapter 3 By E. Amani

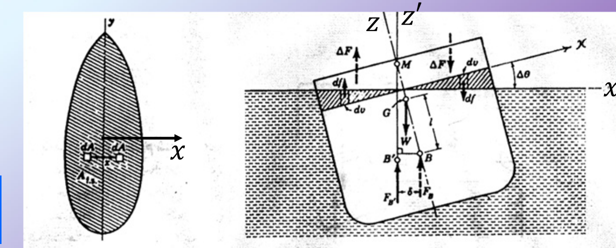
Linear stability analysis

- **Steps**
  1. **Choosing the coordinates system:**
     a) **The z-direction: Passing points B and G at the initial position**
     b) **The y-direction: on the free-surface and in the given direction**
  2. **Using the following condition (see "chap3-appendixA.pdf" for proof):**

$$\overline{MG}=\overline{MB}-\overline{GB}=\frac{\gamma I_{yy}}{W}-\overline{GB} \qquad (18.3)$$

$$\overline{MG}=\begin{cases}>0 & ;\text{stable}\\ =0 & ;\text{neutral}\\ <0 & ;\text{unstable}\end{cases}$$





Chapter 3 By E. Amani

Linear stability analysis

- **Steps**
  1. **Choosing the coordinates system:**
     a) **The z-direction: Passing points B and G at the initial position**
     b) **The y-direction: on the free-surface and in the given direction**
  2. **Using the following condition (see "chap3-appendixA.pdf" for proof):**

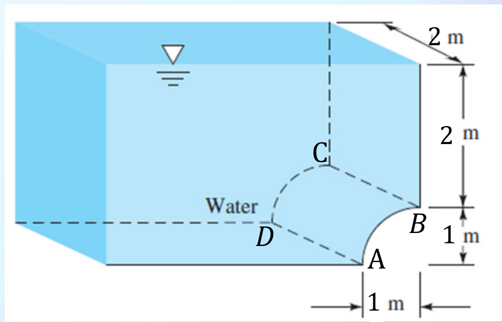
$$\overline{MG}=\overline{MB}-\overline{GB}=\frac{\gamma I_{yy}}{W}-\overline{GB} \qquad (18.3) \qquad \overline{MG}=\begin{cases}>0 & ;\text{stable}\\ =0 & ;\text{neutral}\\ <0 & ;\text{unstable}\end{cases}$$

- **Exercise: Considering Eq. (18.3), what is the critical direction for the linear stability analysis?**

Chapter 3 By E. Amani

Sample problems

Calculate the hydrostatic force and its point of action on the gate ABCD, which is a quarter-cylinder of radius 1m and height 2 m.

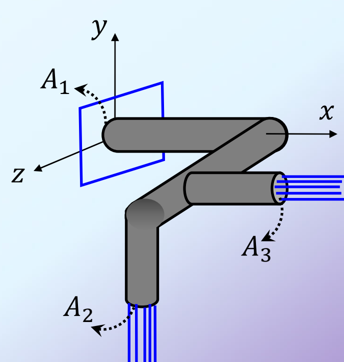


Lecture Notes

Chapter 3 By E. Amani

Sample problems

Calculate the hydrostatic force from air on the outer surface of the piping system.

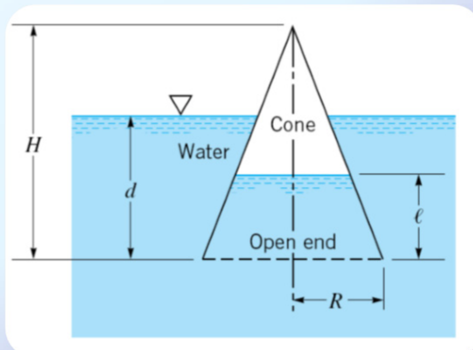


Lecture Notes

Chapter 3 By E. Amani

Sample problems

An inverted hollow cone is pushed into the water as is shown in below figure. Determine equations which are needed for calculating ℓ and d. Assume the temperature of the air within the cone remains constant. Express all assumptions you make.



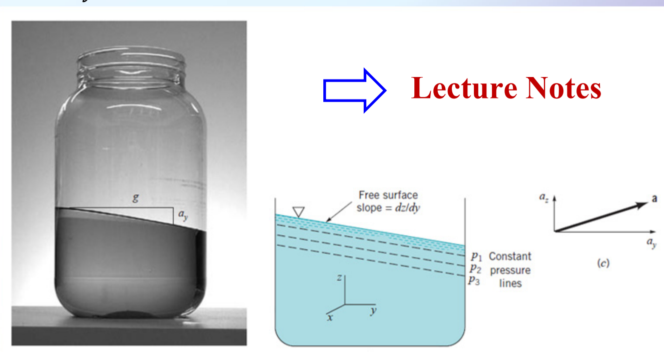
⇨ Lecture Notes

Chapter 3 By E. Amani

Fluid with rigid-body motion

- **Linear motion**
  - **Choosing coordinates system:**
    1. **Attached to the container**
    2. $\vec{g} = -g\hat{k}$
    3. $a_y = 0$ **or** $\vec{a} = a_x\hat{\imath} + a_z\hat{k}$

⇨ Lecture Notes



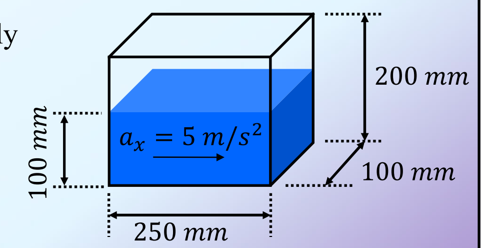
Chapter 3 By E. Amani

Sample problems

A cubic container moves horizontally with an acceleration of 5 m/s^2.

a) Does the water spill out of the container?

b) Determine the force on the left wall of the container from the water.



- **Problem solution steps:**
  1. **Determining the free-surface location using the conservation of mass and free-surface slope**

$$\left(\frac{dz}{dx}\right)_{fs} = -\frac{a_x}{g+a_z} \qquad (21.3)$$

Lecture Notes

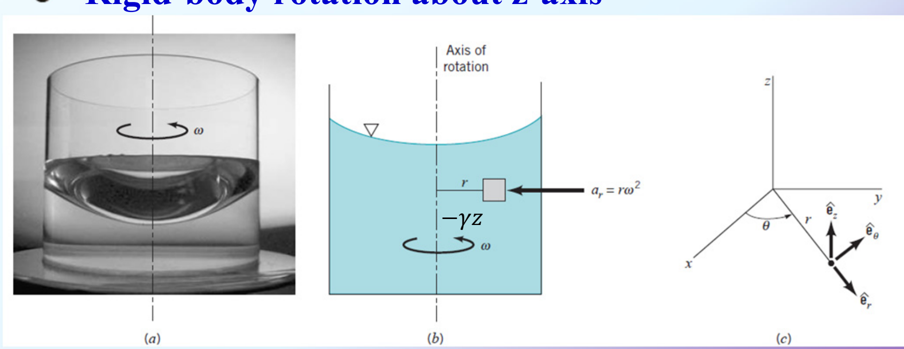
  2. **Calculating the pressure distribution by:**

$$p = p(x,z) = C - \gamma x \frac{a_x}{g} - \gamma z\left(1+\frac{a_z}{g}\right) \qquad (19.3)$$

Chapter 3 **By E. Amani**

Fluid with rigid-body motion

- **Rigid-body rotation about z-axis**



Lecture Notes

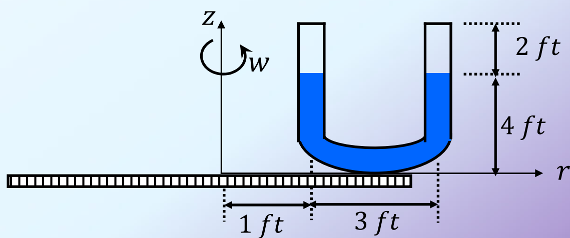
$$p = p(x,r) = C - \gamma z + \frac{\rho\omega^2}{2}r^2 \qquad (23.3)$$

Free-surface equation: $\frac{\rho\omega^2}{2}r^2 - \gamma z = -C$ (24.3)

Chapter 3 **By E. Amani**

Sample problems

The U-tube contains water and is placed on a rotating disk. What is the minimum angular velocity of the disk to spill out water from the U-tube. The tube inner diameter is $0.1\ in$.



Lecture Notes

The end of chapter 3