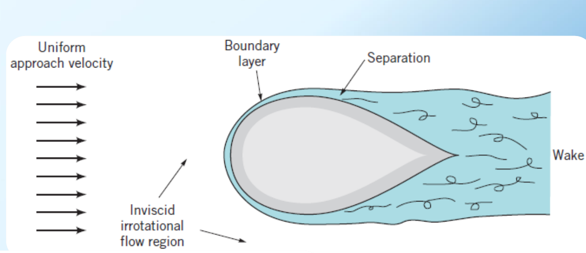
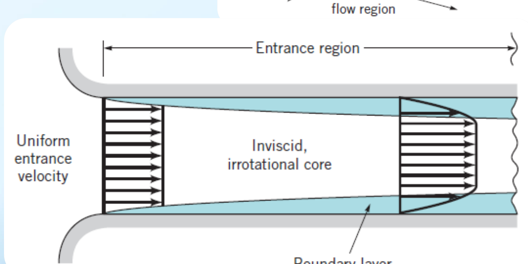
Fluid mechanics II
by
Dr. Ehsan Amani

Chapter 6

Potential flow regions

- Assumptions
  - Incompressible
  - Inviscid
  - Irrotational ($\vec{\nabla} \times \vec{V} = 0$)





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Potential flow governing equations

- **Theorem: For every vector field $\vec{V}$, if $\vec{\nabla} \times \vec{V} = 0$, then $\vec{V}$ can be written as the gradient of a scalar field ϕ:**

$$\vec{V} = \vec{\nabla}\phi \quad (1.6) \qquad u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y}, w = \frac{\partial \phi}{\partial z} \quad (2.6)$$

Cartesian coordinates

(Velocity) potential

  - **Proof: See [3], section 12.4**
  - **See "chap6-AppendixA-Del operat….pdf" for other coordinates systems**
  - **Exercise: Show that the change of variable Eq. (1.6) satisfies the irrotational flow condition ($\vec{\nabla} \times \vec{V} = 0$). You can use Cartesian coordinates.**

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Potential flow governing equations

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Cartesian coordinates

- **Form continuity (incompressible flow):**

$$\vec{\nabla}.\vec{V} = 0 = \vec{\nabla}.(\vec{\nabla}\phi) \longrightarrow$$

$$\underbrace{\nabla^2\phi = 0}_{\text{Laplace equation}}, \quad \nabla^2 = \vec{\nabla}.\vec{\nabla} \quad (3.6) \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (4.6)$$

Laplacian operator

Cartesian coordinates

  - **Exercise: Substituting Eq. (2.6) into Eq. (17.3)', derive Eq. (4.6) directly.**

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Potential flow governing equations

- **Assuming gravity in the negative z-direction, the momentum equation is reduced to Eq. (40.3):**

$$\vec{\nabla}\left(\frac{p}{\rho}+gz+\frac{V^2}{2}\right)+\frac{\partial\vec{V}}{\partial t}=0 \xrightarrow{\frac{\partial\vec{V}}{\partial t}=\frac{\partial\vec{\nabla}\phi}{\partial t}=\vec{\nabla}\frac{\partial\phi}{\partial t}} \vec{\nabla}\left(\frac{p}{\rho}+gz+\frac{(\vec{\nabla}\phi)^2}{2}+\frac{\partial\phi}{\partial t}\right)=0$$

$$\xrightarrow{\text{integration}} \frac{p}{\rho}+gz+\frac{(\vec{\nabla}\phi)^2}{2}+\frac{\partial\phi}{\partial t}=C' \longrightarrow$$

$$p=C-\rho\left(gz+\frac{(\vec{\nabla}\phi)^2}{2}+\frac{\partial\phi}{\partial t}\right) \quad (6.6) \qquad \frac{p}{\rho}+gz+\frac{V^2}{2}=C' \text{ (steady)}$$

  - **Exercise: Assuming $\vec{V}=U\hat{\imath}$, show that Eq. (6.6) is simplified to Eq. (10.5).**

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Potential flow governing equations

- **Solution method**

1. **Solving the Laplace equation for ϕ with appropriate boundary conditions**

$$\nabla^2\phi=0 \quad (3.6) \qquad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \quad (4.6) \quad \text{Cartesian coordinates}$$

2. **Calculating velocity**

$$\vec{V}=\vec{\nabla}\phi \quad (1.6) \qquad u=\frac{\partial\phi}{\partial x}, v=\frac{\partial\phi}{\partial y}, w=\frac{\partial\phi}{\partial z} \quad (2.6) \quad \text{Cartesian coordinates}$$

3. **Calculating pressure**

$$p=C-\rho\left(gz+\frac{(\vec{\nabla}\phi)^2}{2}+\frac{\partial\phi}{\partial t}\right) \quad (6.6) \qquad \frac{p}{\rho}+gz+\frac{V^2}{2}=C' \text{ (steady)}$$

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Stream function in potential flow

- **For 2D plane flow, Eq. (23.3):**

$$u \equiv \frac{\partial \psi}{\partial y}, -v \equiv \frac{\partial \psi}{\partial x} \qquad (23.3)$$

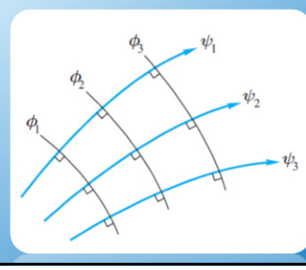
irrotational

$$\rightarrow \vec{\nabla} \times \vec{V} = 0 \rightarrow \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\hat{k} = 0 \xrightarrow{(23.3)} \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) = 0$$

$$\rightarrow \nabla^2 \psi = 0 \quad (12.6)$$

- **Exercise: Using cylindrical coordinates, prove Eq. (12.6).**
- **Are ψ and ϕ the same?**
- **ϕ- and ψ- iso-lines are orthogonal**

$\Rightarrow$ **Lecture Notes**



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Boundary conditions

- **Wall: Using the reference frame attached to the solid body**

$$u_n = \vec{n}.\vec{V} = \vec{n}.\vec{\nabla}\phi = \frac{\partial \phi}{\partial n} \xrightarrow[\text{surface}]{\text{On body}} \left.\frac{\partial \phi}{\partial n}\right|_{body} = 0 \qquad (13.6)$$

$$\psi\big|_{body} = cte \quad \text{or} \quad \left.\frac{\partial \psi}{\partial s}\right|_{body} = 0 \quad (14.6)$$

- **Farfield:**

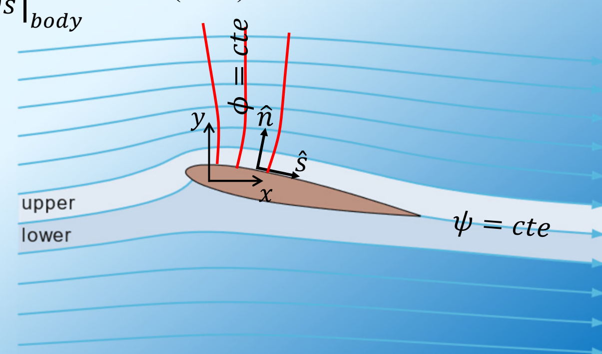
$$u = U_\infty, v = V_\infty \rightarrow$$

$$\frac{\partial \phi}{\partial x} = U_\infty, \frac{\partial \phi}{\partial y} = V_\infty$$

or

$$\frac{\partial \psi}{\partial y} = U_\infty, \frac{\partial \psi}{\partial x} = -V_\infty$$

(15.6)



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Solution of potential flow equations

- Simplifications compared to Navier-Stokes
  - Velocity governed by a linear equation
  - Pressure and velocity is decoupled
  - Momentum is an algebraic equation (Bernoulli)
- Still needs a numerical solution in general case
- Analytical solutions in simple cases
- Superposition if boundary conditions are linear

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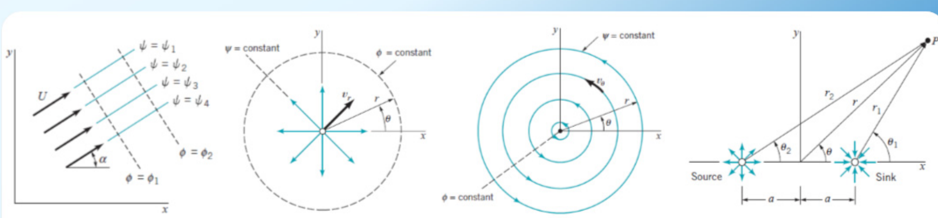
Basic plane potential flows

- Uniform flow
- Source & sink
- Free vortex
- Doublet

⇨ Lecture Notes



uniform flow Source/sink free vortex doublet ($a \longrightarrow 0$)

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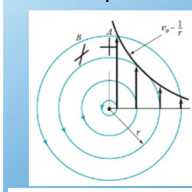
Basic plane potential flows

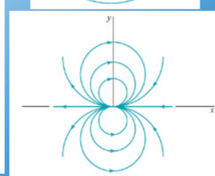
TABLE 6.1
Summary of Basic, Plane Potential Flows

Description of Flow Field	Velocity Potential	Stream Function	Velocity Components[a]
Uniform flow at angle α with the x axis (see Fig. 6.16b)	$\phi = U(x\cos\alpha + y\sin\alpha)$	$\psi = U(y\cos\alpha - x\sin\alpha)$	$u = U\cos\alpha$ $v = U\sin\alpha$
Source or sink (see Fig. 6.17) $m > 0$ source $m < 0$ sink	$\phi = \dfrac{m}{2\pi}\ln r$	$\psi = \dfrac{m}{2\pi}\theta$	$v_r = \dfrac{m}{2\pi r}$ $v_\theta = 0$
Free vortex (see Fig. 6.18) $\Gamma > 0$ counterclockwise motion $\Gamma < 0$ clockwise motion	$\phi = \dfrac{\Gamma}{2\pi}\theta$	$\psi = -\dfrac{\Gamma}{2\pi}\ln r$	$v_r = 0$ $v_\theta = \dfrac{\Gamma}{2\pi r}$
Doublet (see Fig. 6.23)	$\phi = \dfrac{K\cos\theta}{r}$	$\psi = -\dfrac{K\sin\theta}{r}$	$v_r = -\dfrac{K\cos\theta}{r^2}$ $v_\theta = -\dfrac{K\sin\theta}{r^2}$

[a]Velocity components are related to the velocity potential and stream function through the relationships:
$u = \dfrac{\partial\phi}{\partial x} = \dfrac{\partial\psi}{\partial y}$ $\quad v = \dfrac{\partial\phi}{\partial y} = -\dfrac{\partial\psi}{\partial x}$ $\quad v_r = \dfrac{\partial\phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}$ $\quad v_\theta = \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} = -\dfrac{\partial\psi}{\partial r}$.

ψ





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Superposition

- **The superposition of several potential flow is a potential flow**

$$\nabla^2\phi_1 = 0, \nabla^2\phi_2 = 0, \ldots \qquad \phi = \phi_1 + \phi_2 + \cdots \longrightarrow \nabla^2\phi = 0$$

- **Only Boundary Conditions (BC) need to be considered**

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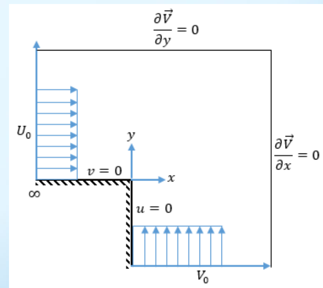
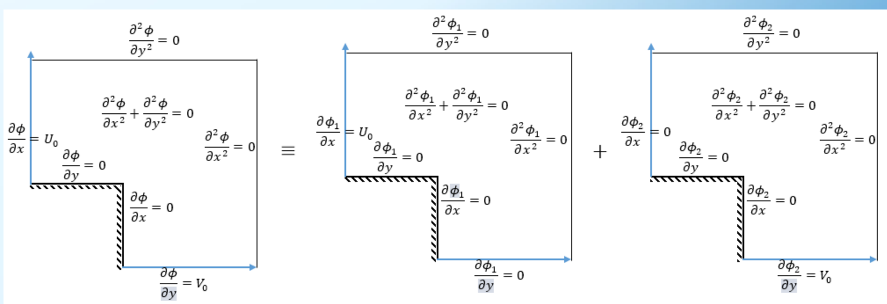
Superposition

- Decomposition of BCs to a linear combination of simpler BCs





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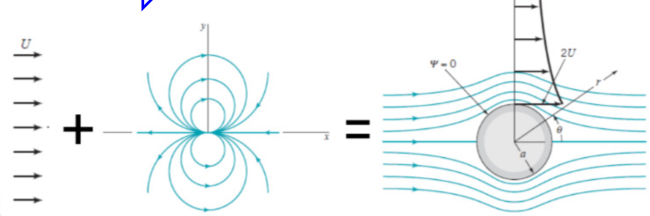
Superposition

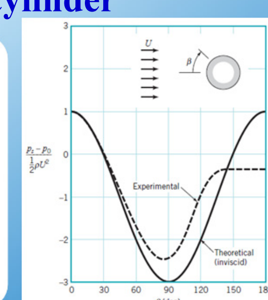
- Linear combination of known potential flows and determining the constant parameter coefficients of each potential flow to satisfy specific BCs

$$BC1: \phi = \phi_1 + \phi_2 + \cdots$$
$$BC2: \phi = \phi_1 + \phi_2 + \cdots$$
$$\cdots$$

- Example: Flow over a circular cylinder

⇨ Lecture Notes





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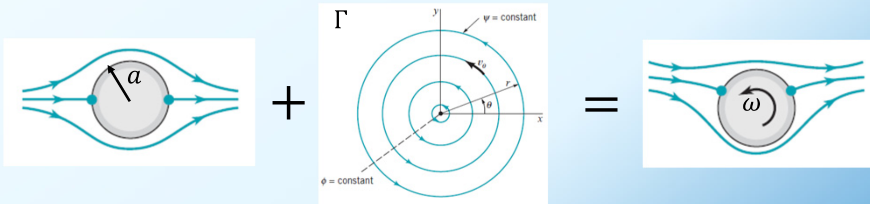
Superposition

- Example: Flow over a rotating circular cylinder



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- Exercise: Determine pressure at the cylinder surface using Bernoulli's equation. Then, show that

$$\mathcal{D} = 0$$

$$\mathcal{L} = -\rho U_0 \Gamma, \Gamma = 2\pi\omega a^2$$

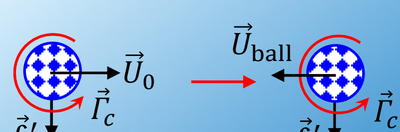
Useful relations in potential flow

- D'Alembert's paradox: For every body of arbitrary shape in a potential flow $\mathcal{D} = 0$
  - Potential flow is not capable of predicting drag
- Kutta-Joukowski lift theorem: For every body of arbitrary shape in a uniform 2D flow

Lift per unit depth ← $\vec{\mathcal{L}}' = \rho \vec{U}_0 \times \vec{\Gamma}_c$ (27.6)

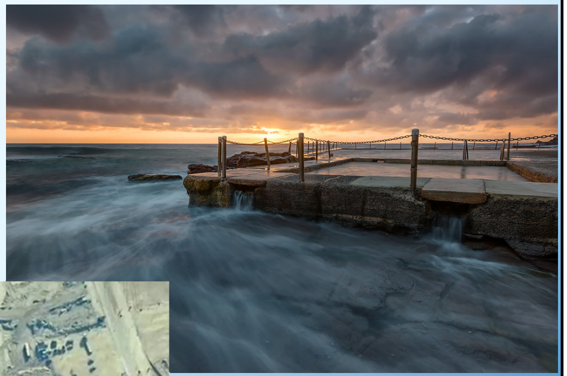
circulation ← $\Gamma_c = \oint_P \vec{V}.d\vec{s}$ (28.6)

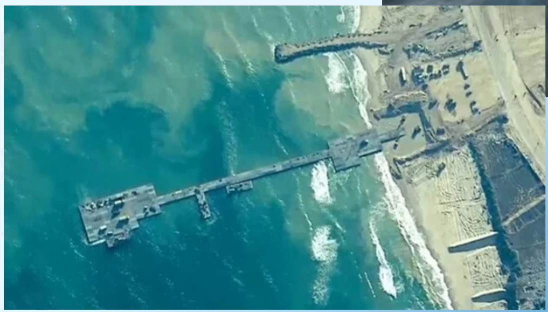
Body perimeter



Sample problem

- Corner flow



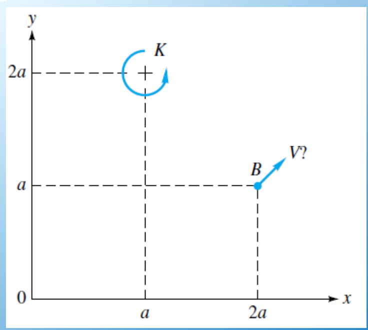


Sample problem

- Corner flow

A positive line vortex K is trapped in a corner, as in Fig. P8.74. Compute the total induced velocity vector at point B, $(x, y) = (2a, a)$, and compare with the induced velocity when no walls are present.

⇨ **Lecture Notes**



The end of chapter 6