Aliasing error: Eq. (7.21)

To prove this behavior, we derive the relation between $\underset{\text{discrete FT}}{\hat{f}_k}$ and $\underset{\text{Continuous FT}}{\tilde{f}_k}$:

$$\text{Eq. (7.15)} \rightarrow \hat{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} \underbrace{f(x_j)}_{\text{Eq. (7.5)}} e^{-ikx_j} = \frac{1}{N}\sum_{j=0}^{N-1}\left[\sum_{k'=-\infty}^{+\infty} \tilde{f}_{k'} e^{ik'x_j}\right] e^{-ikx_j} = \sum_{k'=-\infty}^{+\infty} \tilde{f}_{k'} \underbrace{\frac{1}{N}\sum_{j=0}^{N-1} e^{i(k'-k)x_j}}_{\text{Eq. (7.14)}}$$

$k' = k + mN$

$$\hat{f}_k = \sum_{m=-\infty}^{+\infty} \tilde{f}_{k+mN} = \tilde{f}_k + \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \tilde{f}_{k+mN} \qquad (7.20)$$

Note:

$$I_N f(x) = \sum_{k=-N/2}^{N/2-1} \underset{\text{Eq. (7.20)}}{\hat{f}_k} e^{ikx} = P_N f(x) + \underbrace{\boxed{\sum_{k=-N/2}^{N/2-1}\left(\sum_{\substack{k=-\infty \\ m\neq 0}}^{+\infty} \tilde{f}_{k+mN}\right) e^{ikx}}}_{R_N \text{ (see Eq. (7.18))}} \qquad (7.21)$$

Convolution sum: Eq. (7.22)

We can avoid the aliasing error resulted from the product operation. For this purpose we begin with:

$$\hat{H}_m = (\widehat{fg})_m = \frac{1}{N}\sum_{j=0}^{N-1} \underset{\text{Eq. (7.12)}}{f_j g_j} e^{-imx_j} = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=-N/2}^{N/2-1}\sum_{k'=-N/2}^{N/2-1} \hat{f}_k \hat{g}_{k'} e^{ikx_j} e^{ik'x_j} e^{-imx_j}$$

$$= \sum_{k=-N/2}^{N/2-1}\sum_{k'=-N/2}^{N/2-1} \hat{f}_k \hat{g}_{k'} \underbrace{\frac{1}{N}\sum_{j=0}^{N-1} e^{i\overbrace{(k+k'-m)}^{K} jh}}_{(I)}$$

According to Eq. (7.14), the term (I) is non-zero for $K = k+k'-m = pN$, $p = 0, \pm 1, \pm 2, \ldots$. As k, k', and m are all integers in the range $(-\frac{N}{2}, \frac{N}{2}-1)$ the only possible cases are $K = -N, 0, +N$. As you know the maximum presentable wavenumber in the grid is $|k|_{max} = \frac{N}{2}$, therefore the waves $K = \pm N$ which was generated by product operation aliase the other modes. The remedy is simply to discard these modes, $K = \pm N$, and only keep the mode $K = 0$ (or $k' = m - k$) in the summation. Hence

$$\boxed{\hat{H}_m = \sum_{\substack{k=-N/2 \\ |m-k|\leq N/2}}^{N/2-1} \hat{f}_k \hat{g}_{m-k}} \qquad m = -\frac{N}{2}, \ldots, \frac{N}{2}-1 \qquad (7.22)$$

This is called the convolution sum of the Fourier coefficients of f and g.

Note that only $\hat{g}_k$; $-\frac{N}{2} \leq k \leq \frac{N}{2}-1$ are available and other $\hat{g}_{k'}$ in Eq. (7.22) are considered to be zero.

Energy spectrum function

$$⊛\quad g(\vec{y})\,\delta(\vec{x}-\vec{y}) = g(\vec{x})\,\delta(\vec{x}-\vec{y})$$

proof (7.73')

$$(7.86) \rightarrow \Phi_{ij}(\vec{k}) = \langle \hat{u}_i^*(\vec{k})\,\hat{u}_j(\vec{k})\rangle \xrightarrow{(7.72')} = \sum_{\vec{k}'}\sum_{\vec{k}''} \underbrace{\delta(\vec{k}-\vec{k}')\,\delta(\vec{k}-\vec{k}'')}_{\substack{\vec{k}\to\vec{k}''\\ ⊛\ \delta(\vec{k}''-\vec{k}')\delta(\vec{k}-\vec{k}'')}} \langle \hat{u}_i^*(\vec{k}')\,\hat{u}_j(\vec{k}'')\rangle$$

$$\underbrace{\delta(\vec{k}''-\vec{k}')}_{\vec{k}\to\vec{k}'}\ \hat{u}_j(\vec{k}')$$

$$\vec{k}\to\vec{k}'',\vec{k}' \quad \text{cont} \times \text{disc}$$

$$= \sum_{\vec{k}'}\left[\left(\underbrace{\sum_{\vec{k}''}\delta(\vec{k}''-\vec{k}')\,\delta(\vec{k}-\vec{k}'')}_{?\ \delta(\vec{k}'-\vec{k})}\right)\underbrace{\langle \hat{u}_i^*(\vec{k}')\,\hat{u}_j(\vec{k}')\rangle}_{\hat{R}_{ij}(\vec{k}')}\right]$$

proof (7.82)

$$\int_0^\infty E(k)\,dk \xrightarrow{(7.80')} = \int_0^\infty \sum_{\vec{k}'}\delta(k'-k)\,\hat{E}(\vec{k}')\,dk = \underbrace{\sum_{\vec{k}'}\hat{E}(\vec{k}')}_{(7.64)\ =\ K}\underbrace{\int_0^\infty \delta(k'-k)\,dk}_{1} = K$$

proof (7.83)

$$\int_0^\infty 2\nu k^2 E(k)\,dk = \int_0^\infty \sum_{\vec{k}'}\delta(k'-k)\,\hat{E}(\vec{k}')\,dk\ 2\nu k^2\,dk = \sum_{\vec{k}'}\hat{E}(\vec{k}')\underbrace{\int_0^\infty 2\nu k^2\,\delta(k'-k)\,dk}_{(7.84)\ 2\nu k'^2}$$

$$= \sum_{\vec{k}'} 2\nu k'^2\,\hat{E}(\vec{k}') = \underset{(7.66)}{\varepsilon}$$

proof (7.81)

$$\iiint \frac{1}{2}\Phi_{ii}(\vec{k}')\,\delta(k'-k)\,d\vec{k}' \xrightarrow{(7.73')} = \iiint \frac{1}{2}\sum_{\vec{k}''}\delta(\vec{k}''-\vec{k}')\overbrace{\hat{R}_{ii}(\vec{k}'')}^{2\hat{E}(\vec{k}'')}\,\delta(k'-k)\,d\vec{k}' = \sum_{\vec{k}''}\hat{E}(\vec{k}'')\iiint$$

$$\delta(\vec{k}''-\vec{k}')\,\delta(k'-k)\,d\vec{k}' \xrightarrow[\vec{x}=\vec{k}',\ \vec{y}=\vec{k}'',\ g(\vec{x})=\delta(|\vec{x}|-k)=\delta(k'-k)]{(7.84)} = \sum_{\vec{k}''}\hat{E}(\vec{k}'')\,\delta(k''-k) \xrightarrow{(7.80')} = E(k)$$

Math note: sifting roperty of delta function, see [7] appenidx C

$$\iiint g(\vec{x})\,\delta(\vec{x}-\vec{y})\,d\vec{x} = g(\vec{y}) \quad (7.84)$$