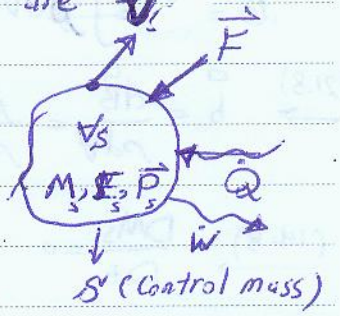
Subject:
Year. Month. Date. ( )

VIII.2 Lagrangian (exact) dispersed-phase equations

VIII-2.1 Background

The ~~basic~~ fundamental laws of mechanics for a closed system are

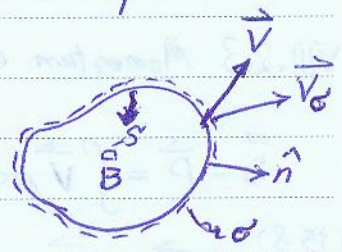


mass conservation: $\dfrac{dM_s}{dt} = 0$ (14.8) (mass)

Newton's second law: $\dfrac{d\vec{P}_s}{dt} = \vec{F}$ (15.8) (momentum)

Energy Conservation: $\dfrac{dE_s}{dt} = \dot{Q} - \dot{W}$ (19.8)

The rate-of-change of an extensive property $\overset{\square}{B}$ in the system S can be related to rate-of-change of this property of a control volume σ coincident with the system at time t using the Reynolds transport theorem as



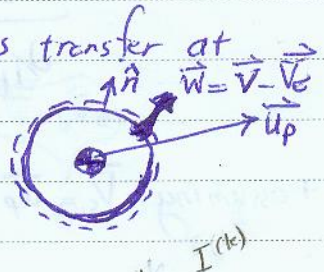
$$\frac{DB_s}{Dt} = \frac{dB_\sigma}{dt} + \int_{A_\sigma} \rho \overset{\square}{b} (\vec{V} - \vec{V}_\sigma) \cdot \hat{n}\, dA \quad (20.8)$$

(intensive) $\overset{\square}{b} = \dfrac{d\overset{\square}{B}}{dm} = \dfrac{d\overset{\square}{B}}{\rho d\forall}$ (21.8)

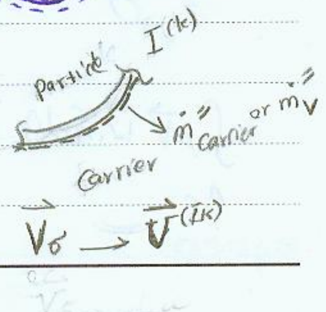
$$d\overset{\square}{B} = \overset{\square}{b}dm \longrightarrow \underbrace{\overset{\square}{B}}_{\binom{S}{\sigma}} = \int_M \overset{\square}{b}\, dm = \int_{\underbrace{\forall}_{\binom{S}{\sigma}}} \rho \overset{\square}{b}\, d\forall \quad (22.8)$$

Now, the general Lagrangian governing equations for a droplet (generalized particle) are derived.

The control volume is the droplet (particle) with mass transfer at its surface:



Other assumptions are mentioned whereever they are applied.



VIII.2.2 Mass equation

choosing

$$\overset{\square}{B} = M = \int \rho d\forall = \int d\overset{\square}{B} \rightarrow d\overset{\square}{B} = \rho d\forall$$

$$\xrightarrow{(21.8)} \overset{\square}{b} = \frac{d\overset{\square}{B}}{\rho d\forall} = \frac{\rho d\forall}{\rho d\forall} = 1$$

$$\xrightarrow{(14.8)} \frac{DM_s}{Dt} = 0 \;, \xrightarrow[\overset{\square}{B}_s = M_s]{(20.8)} \overset{0}{\overset{\parallel}{\frac{DM_s}{Dt}}} = \frac{d\overset{\dot{m}_p \uparrow}{M_p}}{dt} + \int_{A_p} \overbrace{\rho \overset{1}{\overset{\square}{b}} \overbrace{\overbrace{(\vec V - \vec V_\sigma)}^{\vec W} \cdot \hat n}^{W_n} dA}^{\rho_{surf,v} \overline{W_n} A_p} \qquad (23.8)'$$

if.. the average value on the surface $\int_S \rho_{surf,v} \overline{W_n}$

particle mass ← ; average mass flux ; → particle surface area

$$\frac{dm_p}{dt} = -\rho_{surf,v} \overline{W_n} A_p \qquad (23.8) \qquad = \int_{A_p} \rho W_n dA$$

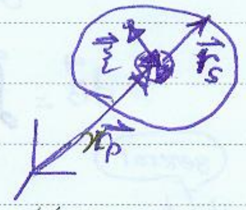
→ values at particle surface $\rho_{surf,v}$ or $\rho_{surf,out}$ (side)

VIII.2.3 Momentum equation

$$\overset{\square}{B} = \vec P = \int \vec V \rho d\forall \longrightarrow d\overset{\square}{B} = \vec V \rho d\forall \longrightarrow \overset{\square}{b} = \frac{\vec V \rho d\forall}{\rho d\forall} = \vec V$$

$$\xrightarrow{(15.8)} \vec F = \frac{d\vec P_s}{dt} \;, \xrightarrow[\overset{\square}{B}_s = \vec P_s]{(20.8)} \frac{d\vec P_s}{dt} = \frac{d\overset{\int_\sigma \rho \vec V d\forall}{\vec P_\sigma}}{dt} + \int_{A_p} \rho \vec V \, \vec W \cdot \hat n \, dA$$

+ assuming $\vec V = \vec u_p + \vec \omega_p \times \vec \xi$ (rigid body like motion neglecting internal flows)



$$\frac{d}{dt}\int_\sigma \rho \vec V d\forall = \underbrace{\frac{d}{dt}\int_\sigma \rho \vec u_p d\forall}_{\frac{d}{dt}(\vec u_p m_p)} + \frac{d}{dt}\Big(\vec\omega_p \times \underbrace{\int_\sigma \vec \xi \rho d\forall}_{0 \text{ definition of center of mass}}\Big) = m_p \frac{d\vec u_p}{dt} + \cancel{\vec u_p \frac{dm_p}{dt}}$$

+ assuming $\vec V_{surf} = \underbrace{\vec u_p + \vec\omega_p \times \vec r_s + \overbrace{\big[(\dot{\vec r}_s)_n \hat n + \underbrace{(\dot{\vec r}_s)_t \hat t}_{\text{neglected}}\big]}^{\dot{\vec r}_s}}_{\vec V_\sigma} + \overbrace{\big[W_n \hat n + \underbrace{W_t \hat t}_{\text{neglected}}\big]}^{\vec W}$ (normal outflow)

$= \vec V_\sigma + \vec W =$

$$\int_{A_p} \rho \vec V \, \vec W \cdot \hat n dA = \vec u_p \underbrace{\int_{A_p} \rho \overset{W_n}{\vec W \cdot \hat n} dA}_{(23.8)^* = -\frac{dm_p}{dt}} + \vec\omega_p \times \int_{A_p} \rho \vec r_s W_n dA + \underbrace{\int_{A_p} \rho (\dot r_s + W_n) \hat n W_n dA}_{-\vec T \text{ thrust}}$$

Therefore,

$$\vec{F} = m_p \frac{d\vec{u_p}}{dt} + \vec{\omega_p} \times \int_{A_p} \rho \vec{r_s} w_n dA - \vec{T} \qquad (24.8)$$

$$\vec{T} = -\int_{A_p} \rho \underbrace{(\dot{r_s} + w_n)}_{w' \rightarrow \text{efflux}} w_n \hat{n}\, dA \qquad (25.8)$$

+ assuming uniform $\dot{r_s}$ and w_n (alternatively for zero mass flux ($w_n = 0$)):

$$\vec{T} = -\rho_{s,v} w' w_n \overbrace{\int_{A_p} \hat{n}\, dA}^{=0 \text{ (for a closed surface)}} = 0$$

Exercise: Using the divergence theorem $\oint_A \vec{A}\cdot\vec{n}\, dA = \int_{\forall} \vec{\nabla}\cdot\vec{A}\, d\forall$, prove that

$$\int_A \hat{n}\, dA = 0 \qquad \text{hint: } \vec{n} = n_x\hat{i} + n_y\hat{j} + n_z\hat{k}$$

assuming
+ spherical particle ($\vec{r_s} = R\hat{n}$)

$$\int_{A_p} \rho \vec{r_s} w_n dA = \underbrace{\rho w_{n\,s,v}}_{\text{uniform } \rho \overline{w}_{n\,s,v}} \int R\hat{n}\, dA = \rho w_{n\,s,v} R \int \hat{n}\, dA = 0$$

Therefore,

$$\vec{F} = m_p \frac{d\vec{u_p}}{dt} \qquad (26.8) \qquad\qquad \frac{d\vec{x_p}}{dt} = \vec{u_p} \qquad (26.8)$$

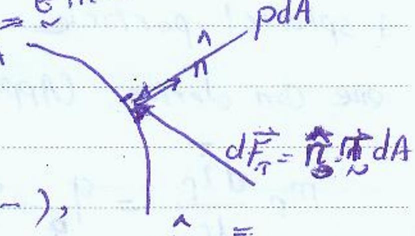
$\vec{x_p} \rightarrow$ position of the center of mass

$$\vec{F} = \vec{F}_{FPI} + \vec{F}_B + \vec{F}_{coll} \qquad (27.8)$$

$\vec{F}_{FPI}$ → fluid-particle interaction; $\vec{F}_B$ → body force; $\vec{F}_{coll}$ → Collision force (Wall, particles, etc) (soft-sphere models)

$$d\vec{F}_{FPI} = -\underbrace{p\hat{n}\, dA}_{\text{pressure}} + \underbrace{\hat{n}\cdot\underset{\sim}{\tau}\, dA}_{\text{viscous stress tensor (exercise)}} = \hat{n}\cdot\underbrace{(-p\underset{\sim}{I} + \underset{\sim}{\tau})}_{\underset{\sim}{\sigma} \rightarrow \text{Stress tensor}}\, dA = \hat{n}\cdot\underset{\sim}{\sigma}\, dA = \underset{\sim}{\sigma}\cdot\hat{n}\, dA \quad (\underset{\sim}{\sigma} \text{ symmetric})$$

$\hat{n}\cdot\underset{\sim}{I} = \underset{\sim}{I}\cdot\hat{n}$



Exercise: Using a Cartesian Coordinate ($\underset{\sim}{\tau} = \tau_{xx}\hat{i}\hat{i} + \tau_{xy}\hat{i}\hat{j} + \cdots$), $\hat{n} = n_x\hat{i} + n_y\hat{j}$

Show that $\hat{n}\cdot\underset{\sim}{\tau}$ is the force on surface $\hat{n}$ (per unit area). (2D element suffices)