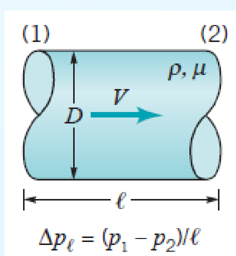
Fluid mechanics I
by
Dr. Ehsan Amani

Chapter 8

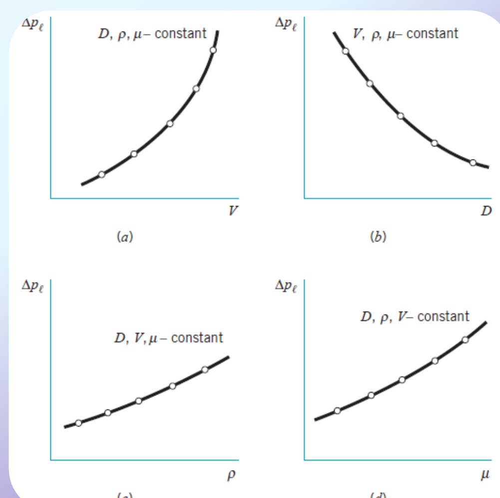
Dimensional analysis

- **Application: experimental correlations**

flow in smooth pipes ▼



$$\Delta p_\ell = f(D, \rho, \mu, V)$$

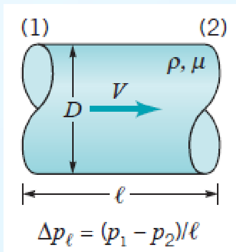


Chapter 8 By E. Amani

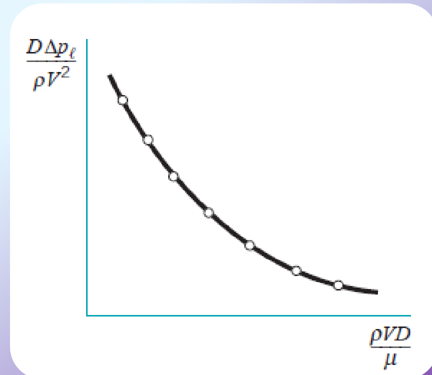
Dimensional analysis

- Application: experimental correlations

flow in smooth pipes ▼



$$\frac{D\,\Delta p_\ell}{\rho V^2} = \phi\left(\frac{\rho V D}{\mu}\right)$$



Chapter 8 By E. Amani

Buckingham Pi theorem

- Dimensionless group or parameter: The product of several dimensional quantities with dimension of 1

height

$$\pi_1 = \frac{h}{D} \qquad \left[\frac{h}{D}\right] = \frac{L}{L} = 1$$

diameter

- Theorem:
  - A phenomenon involving n variables
  - with r is the minimum number of reference dimensions (products of basic dimensions)
  - Can be described by a relation among $n - r$ dimensionless parameters

Chapter 8 By E. Amani

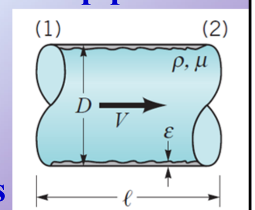
Dimensional analysis

- **Steps:**
  1. **Finding all design variables governing an objective variable.**
  - **Example: Pressure drop in incompressible pipe flow**

Pressure drop **viscosity**

$\Delta p = f(\rho, \mu, V, \ell, D, \varepsilon)$ $\quad n = 7 \quad$ (1.8)

density **Bulk velocity** **roughness**



  2. **Listing the dimensions of all variables (using Table 1.1 FLT and MLT).**

(FLT) **(MLT)**

$[\Delta p] = FL^{-2} = MT^{-2}L^{-1} \qquad [V] = LT^{-1} \qquad [\varepsilon] = L$

$[\rho] = FL^{-4}T^2 = ML^{-3} \qquad [\ell] = L \qquad$ (2.8)

$[\mu] = FL^{-2}T = ML^{-1}T^{-1} \qquad [D] = L$

Dimensional analysis

- **Steps:**
  3. **Determine the minimum reference dimensions.**
     - **The minimum dimensional groups for constructing all dimensions**
     - **Start from simplest dimensions**
     - **Check r using both FLT and MLT**

(FLT) **(MLT)**

$[\Delta p] = FL^{-2} = MT^{-2}L^{-1} \qquad [V] = LT^{-1} \qquad [\varepsilon] = L$

$[\rho] = FL^{-4}T^2 = ML^{-3} \qquad [\ell] = L$

$[\mu] = FL^{-2}T = ML^{-1}T^{-1} \qquad [D] = L \qquad \boxed{r = 3: L, T, M}$

  4. **Choose r design variables to construct the reference dimensions**
     - D, V, ρ **(should involve all reference dimensions)**

$L \longrightarrow D \qquad T \longrightarrow D/V \qquad M \longrightarrow \rho D^3 \qquad$ (3.8)

Dimensional analysis

- **Steps:**

5. **Determine the dimensionless variables:**
   - **Dividing the remaining variables, not used in step 4, to their dimensions** $L \longrightarrow D \quad T \longrightarrow D/V \quad M \longrightarrow \rho D^3$
   - **Replace reference dimensions by equivalent variables from step 4**

$$\pi_1 = \frac{\Delta p}{MT^{-2}L^{-1}} = \frac{\Delta p}{\rho D^3 (D/V)^{-2} D^{-1}} = \frac{\Delta p}{\rho V^2}$$
$$\pi_2 = \frac{\mu}{ML^{-1}T^{-1}} = \frac{\mu}{\rho D^3 D^{-1} (D/V)^{-1}} = \frac{\mu}{\rho V D} \quad (4.8)$$
$$\pi_3 = \frac{\ell}{L} = \frac{\ell}{D}$$
$$\pi_4 = \frac{\varepsilon}{L} = \frac{\varepsilon}{D}$$

$$\pi_1 = f(\pi_2, \pi_3, \pi_4) \quad (5.8) \qquad \frac{\Delta p}{\rho V^2} = f\left(\frac{\mu}{\rho V D}, \frac{\ell}{D}, \frac{\varepsilon}{D}\right)$$

Dimensional analysis

- **Steps:**

6. **Combining dimensionless groups (if desired)**

$$\pi_1 \longrightarrow \pi_1^\alpha$$
$$\pi_1, \pi_2 \longrightarrow \pi_1, \pi_1\pi_2 \text{ or } \pi_2, \pi_1\pi_2$$
$$\pi_1, \pi_2 \longrightarrow \pi_1, \pi_1/\pi_2 \text{ or } \pi_2, \pi_1/\pi_2$$
$$\pi_1, \pi_2 \longrightarrow \pi_1, \pi_1^\alpha\pi_2^\beta \text{ or } \pi_2, \pi_1^\alpha\pi_2^\beta$$

   - **Example:**

$$\frac{\Delta p}{\rho V^2} = f_2\left(\frac{\mu}{\rho V D}, \frac{\ell}{D}, \frac{\varepsilon}{D}\right)$$
$$\pi_2 = \frac{\mu}{\rho V D} \longrightarrow \pi_2^{-1} = \frac{\rho V D}{\mu}$$

Euler number (Eu) **Reynolds number (Re)**

$$\frac{\Delta p}{\rho V^2} = f_3\left(\frac{\rho V D}{\mu}, \frac{\ell}{D}, \frac{\varepsilon}{D}\right) \quad (6.8)$$

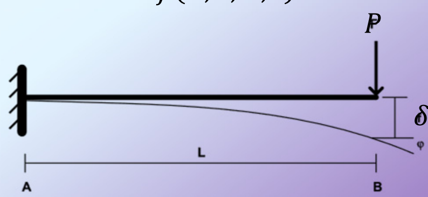
Dimensional analysis

- **Corollary: When there is a single dimensionless parameter**

$$\pi_1 = f(.) = cte$$

- **Example: Perform the dimensional analysis for the deflection of a beam**

$$\delta = f(P, \ell, E, I)$$



Lecture Notes

Dimensional analysis

- **Important Dimensionless Groups (parameters) in Fluid Mechanics**

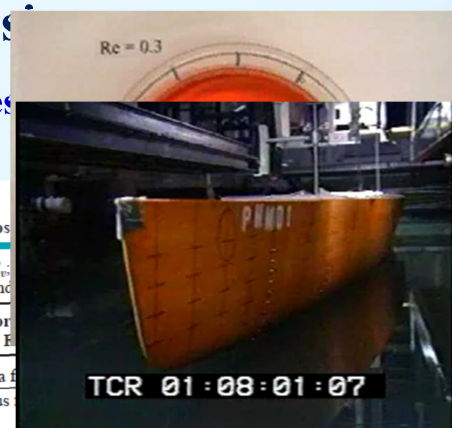
TABLE 7.1
Some Common Variables and Dimensionless Groups in Fluid Mechanics

Variables: Acceleration of gravity, g; Bulk modulus, E_v; Characteristic length, ℓ; Density, ρ; Frequency of oscillating flow, ω; Pressure, p (or Δp); Speed of sound, c; Surface tension, σ; Velocity, V; Viscosity, μ

Dimensionless Groups	Name	Interpretation (Index of Force Ratio Indicated)	Types of Applications
$\frac{\rho V \ell}{\mu}$	Reynolds number, Re	$\frac{\text{inertia force}}{\text{viscous force}}$	Generally of importance in all types of fluid dynamics problems
$\frac{V}{\sqrt{g\ell}}$	Froude number, Fr	$\frac{\text{inertia force}}{\text{gravitational force}}$	Flow with a free surface
$\frac{p}{\rho V^2}$	Euler number, Eu	$\frac{\text{pressure force}}{\text{inertia force}}$	Problems in which pressure, or pressure differences, are of interest
$\frac{\rho V^2}{E_v}$	Cauchy number,[a] Ca	$\frac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important

Dimensional analysi

- **Important Dimensionles in Fluid Mechanics**



■ TABLE 7.1
Some Common Variables and Dimensionless Groups

Variables: Acceleration of gravity, g; Bulk modulus, E_v; oscillating flow, ω; Pressure, p (or Δp); Speed of sounc

Dimensionless Groups	Name	Interpr Force R	
$\frac{\rho V \ell}{\mu}$	Reynolds number, Re	inertia f / viscous	
$\frac{V}{\sqrt{g\ell}}$	Froude number, Fr	inertia force / gravitational force	Flow with a free surface
$\frac{p}{\rho V^2}$	Euler number, Eu	pressure force / inertia force	Problems in which pressure, or pressure differences, are of interest
$\frac{\rho V^2}{E_v}$	Cauchy number,[a] Ca	inertia force / compressibility force	Flows in which the compressibility of the fluid is important

Dimensional analysis

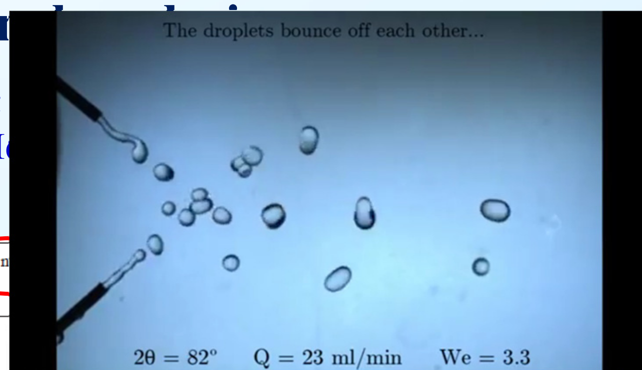
- **Important Dimensionless Groups (parameters) in Fluid Mechanics**

$\frac{V}{c}$	Mach number,[a] Ma	inertia force / compressibility force	Flows in which the compressibility of the fluid is important
$\frac{\omega \ell}{V}$	Strouhal number, St	inertia (local) force / inertia (convective) force	Unsteady flow with a characteristic frequency of oscillation
$\frac{\rho V^2 \ell}{\sigma}$	Weber number, We	inertia force / surface tension force	Problems in which surface tension is important

[a]The Cauchy number and the Mach number are related and either can be used as an index of the relative effects of inertia and compressibility. See accompanying discussion.

Dimension

- **Important**
  in Fluid M



$\frac{V}{c}$ Mach n

$\frac{\omega \ell}{V}$

$\frac{\rho V^2 \ell}{\sigma}$ which surface ... portant

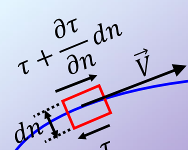
ªThe Cauchy number an... ity. See accompanying d... a and compressibil-

Chapter 8 By E. Amani

Dimensional analysis

- **The Reynolds number: Physical interpretation**
  - **For the sake of simplicity, steady flow**

⇨ **Lecture Notes**



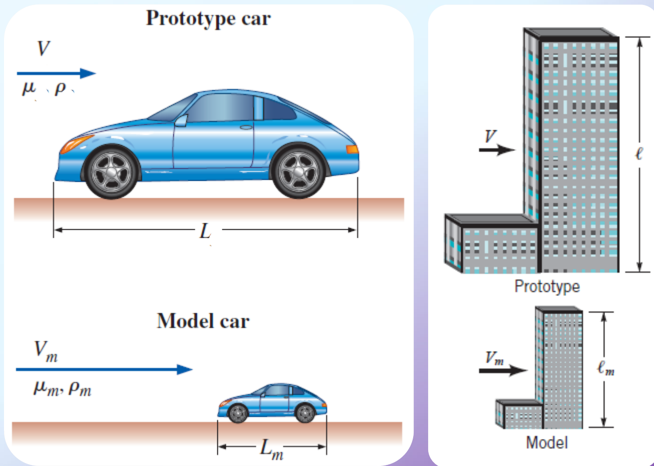
$$\frac{\text{inertial force}}{\text{viscous force}} \propto \frac{\rho V_0 \ell}{\mu} \equiv Re \qquad (8.8)$$

- **Exercise: Similarly prove the physical meanings of the Euler and Froude numbers.**
- **Note: A dimensionless parameter, representing two forces ratio, is important if and only if non of the forces is negligible**

Chapter 8 By E. Amani

Modeling and similitude

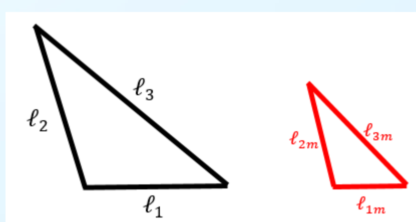
- **Prototype and model**



Chapter 8 By E. Amani

Similitude

- **Geometrical similitude**



$$\frac{\ell_{1m}}{\ell_1} = \frac{\ell_{2m}}{\ell_2} = \frac{\ell_{3m}}{\ell_3}$$

Chapter 8 By E. Amani

Similitude

- **Geometrical similitude**

Length scale — constant for all dimensions

$$\frac{\ell_{im}}{\ell_i} = const = \lambda_\ell; \ \ i = 1, \ldots$$

- **Kinematic similitude**

Velocity scale

$$\vec{V}_m = \lambda_V \vec{V} \ \text{ or } \ \frac{V_m}{V} = \lambda_V, \vec{V}_m || \vec{V}$$

- **Dynamic similitude**

Force scale — constant for all forces

$$\vec{F}_{\alpha,m} = \lambda_F \vec{F}_\alpha \ \text{ or } \ \frac{F_{\alpha,m}}{F_\alpha} = \lambda_F, \vec{F}_{\alpha,m} || \vec{F}_\alpha \ ; \ \alpha = 1, \ldots$$

Chapter 8 By E. Amani

Similitude

- **Design of the model**
  - **The dimensional variables are not the same for the prototype and model**
  - **It is sufficient that all but one of the dimensionless parameters are designed to be the same for the prototype and model**
  - **The last dimensionless parameter would be the same since**

$$\pi_2 = \pi_{2m}, \pi_3 = \pi_{3m}, \ldots \xrightarrow{\pi_1 = f(\pi_2, \pi_3, \ldots)} \pi_1 = \pi_{1m} \qquad (12.8)$$

Chapter 8 By E. Amani

Similitude

- **Example: Pressure drop in pipes** $\frac{\Delta p}{\rho V^2} = f_3\left(Re, \frac{\ell}{D}, \frac{\varepsilon}{D}\right)$

Design: $Re_m = \frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho V D}{\mu} = Re$ (13.8)

$\frac{\ell_m}{D_m} = \frac{\ell}{D}, \frac{\varepsilon_m}{D_m} = \frac{\varepsilon}{D}$ (14.8) $\longrightarrow \frac{\Delta p_m}{\rho_m V_m^2} = \frac{\Delta p}{\rho V^2}$ (15.8)

- **Note 1: Eq. (15.8) can be used to calculate the pressure drop for the prototype using the one measured for model** $\xrightarrow{(15.8)} \Delta p = \frac{\rho V^2}{\rho_m V_m^2} \Delta p_m$
- **Note 2: The model and prototype have geometrical similarity** $\xrightarrow{(14.8)} \frac{\ell_m}{\ell} = \frac{D_m}{D}, \frac{\varepsilon_m}{\varepsilon} = \frac{D_m}{D} \longrightarrow \frac{\ell_m}{\ell} = \frac{\varepsilon_m}{\varepsilon} = \frac{D_m}{D} = cte$
- **Note 3: The model and prototype have dynamic similarity** $\xrightarrow{(13.8)} \frac{F_{Im}}{F_{\tau m}} = \frac{F_I}{F_\tau} \longrightarrow \frac{F_{Im}}{F_I} = \frac{F_{\tau m}}{F_\tau} = cte$

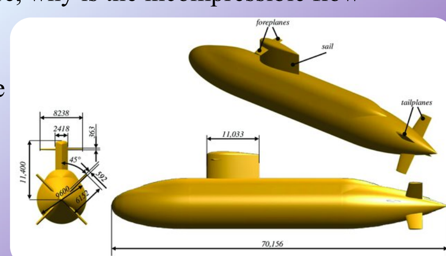
Chapter 8 By E. Amani

Similitude

- **Sample problem**

A 1:20 scale model of a submarine is tested in a wind tunnel under air absolute pressure of 300 lb/in^2 and temperature of 120 °F. The speed of the prototype is 15 kn in the ocean water with a kinematic viscosity of 1.121 ft^3/s.

a) Determine the required air speed in the wind tunnel. The air is assumed as an ideal gas with a gas constant of 1716 $ft.lb/(slug.R)$.
b) What is the ratio of the model to prototype drag forces.
c) In spite of the wind tunnel high pressure, why is the incompressible flow assumption valid?
d) Is the similitude between the model and prototype violated if the submarine moves near the ocean surface?



⇨ **Lecture Notes**

Chapter 7 By E. Amani

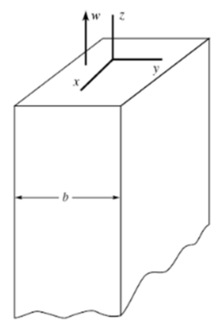
Equation non-dimensionalization

- **Sample problem** ⇨ **Lecture Notes**

A viscous fluid flows through a vertical, square channel as shown in below figure. The velocity w can be expressed as

$$w = f(x, y, b, \mu, \gamma, V, \partial p/\partial z)$$

where μ is the fluid viscosity, γ the fluid specific weight, V the mean velocity, and $\partial p/\partial z$ the pressure gradient in the z direction.



(a) Use dimensional analysis to find a suitable set of dimensionless variables and parameters for this problem.

(b) The differential equation governing the fluid motion for this problem is

$$\frac{\partial p}{\partial z} = -\gamma + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

Write this equation in a suitable dimensionless form, and show that the similarity requirements obtained from this analysis are the same as those resulting from the dimensional analysis of part (a).

- **The governing dimensionless parameters of a phenomenon appear in its governing equations or boundary conditions**

Chapter 8 By E. Amani

The end of chapter 8

Chapter 8 By E. Amani