# Efficient Zeroth-Order Proximal Stochastic Method for Nonconvex Nonsmooth Black-Box Problems

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**Abstract**

Proximal gradient method has a major role in solving nonsmooth composite optimization problems. However in some machine learning problems related to black-box optimization models proximal gradient method could not be leveraged as the derivation of explicit gradients are difficult or entirely infeasible. Several variants of zeroth-order stochastic variance reduced (ZO-SVRG) algorithms have recently been studied for nonconvex optimization problems, however almost all the existing ZO-SVRG type algorithms suffer from a slowdown and increase in function query complexities up to a small-degree polynomial of the problem size. In order to fill this void, we propose a new stochastic gradient algorithm for optimizing nonconvex, nonsmooth finite-sum problems, called ZO-PSVRG+. The main goal of this work is to present an analysis of ZO-PSVRG+, recovering several existing convergence results while improving the complexity of their ZO oracle and proximal oracle calls. We prove that ZO-PSVRG+ under Polyak-Łojasiewicz condition in contrast to the existent ZO-SVRG type methods obtains a global linear convergence for a wide range of minibatch sizes, whereas in the current literature the analysis is limited only to large minibatch sizes, rendering the existing methods unpractical for real-world problems with limited computational capacity. In empirical experiments for black-box models we show that the new analysis provide superior performance and faster convergence to a solution of nonconvex nonsmooth problems compared to the existing ZO-SVRG methods. As a byproduct, the analysis is generic and can be exploited to the other variants of gradient-free variance reduction methods aiming to make them more efficient.

**Keywords:** Nonconvex optimization, Zeroth-order methods, Polyak-Łojasiewicz condition, Nonsmooth optimization, Query efficient methods, Adversarial attacks

## Introduction

In this paper, we consider nonsmooth nonconvex optimization problems of the generic form

$$\min_{x\in\mathbb{R}^d} F(x) = f(x) + h(x), \quad f(x) := \frac{1}{n}\sum_{i=1}^{n} f_i(x) \tag{1}$$

where each $f_i(x)$ is possibly nonconvex and smooth function, and $h(x)$ is a nonsmooth convex function such as $l_1$-norm regularizer. The optimization problem (1) governs numerous machine learning frameworks, ranging from neural networks to generalized linear models and from convex problems like SVM and Lasso to highly nonconvex optimization problems such as loss functions tailored to deep neural models. In this work, we shall explore a set of accelerated variance reduced stochastic zeroth-order (SZO) optimization algorithms for (1) for the purpose of improving their oracle calls. Stochastic variance reduced gradient (SVRG) is a powerful approach to decrease the variance inherited from stochastic sampling Johnson and Zhang (2013); Reddi et al (2016a); Nitanda (2016); Allen-Zhu and Yuan (2016). It is demonstrated in these papers that SVRG enhances the rate of convergence for stochastic gradient descent (SGD) by a factor of $O(1/\epsilon)$ owing to the decrease in the variance of the gradient. The underlying framework for variance reduction methods is to leverage similar ideas from the first-order methods to reduce the variance of first-order optimization methods to improve the convergence rate. The major adversity of first-order methods is their dependency on first-order information from the problem, while there are settings where the first-order gradients are computationally infeasible or costly where zeroth-order information (function information) is accessible. For instance, in online auctions and advertisement selections, only zeroth-order information in the form of responses to the queries is accessible Wibisono et al (2012). This demonstrates the importance of the development of zeroth-order optimization methods for solving many machine learning problems. Currently, there are only a few zeroth-order stochastic methods for solving problem (1), e.g., Ghadimi and Lan (2016) and Huang et al (2019). There are other variants of zeroth-order for nonsmooth problems, e.g., Huang et al (2019) and also momentum accelerated zeroth-order methods Chen et al (2019) to achieve higher rate of convergence. In addition, several zeroth-order projection-free methods were developed in Sahu et al (2019); Huang et al (2020) as well as ADAM-based methods in Gao et al (2018). Recently, Ji et al (2019) refined the ZO estimations to derive an improved ZO complexity with improved convergence rate. However, the analysis in their work is only for smooth functions based on a complicated parameter selection framework which is only valid for large minibatch sizes. Towards improving the convergence analysis in the existing works, in this paper we address this open question: *Is this possible to extend the existing convergence analysis for ZO-SVRG to arbitrary minibatch sizes with improved ZO query complexity and under more general non-smooth, non-convex setting?* Aiming to answer this question, we employ variance reduction methods to design an accelerated ZO proximal method for nonsmooth composite optimization problem (1). Our analysis leads to a lower iteration complexity $O(1/\epsilon)$, which is to our knowledge the best iteration complexity bound obtained thus far for proximal ZO stochastic optimization with nonconvex structure. This demonstrates an improvement to ZO iteration complexity up to a factor of $d$.

In Table 1, we compared the results from our analysis and other state-of-the-art ZO optimization methods for four different items. The table shows that the convergence of ZO-PSVRG+ provides a better dependency on problem dimension $d$ than RSPGF and ZO-ProxSVRG/SAGA for nonconvex nonsmooth optimization. It also shows that RGF has the largest query complexity yet has the worst convergence rate. ZO-SVRG-Coord and ZO-ProxSVRG/SAGA provide an improved rate of convergence $O(d/\epsilon)$ owning to applying variance reduction techniques. Further observation of Table 1 reveals that the existing SVRG type zeroth-order algorithms are highly affected by function query complexities compared to RSPGF, while our algorithm, ZO-PSVRG+ could achieve better trade-offs between the convergence rate and the querying complexity.

## Main Contributions

In this work, we present a novel analysis for ZO-PSVRG+ (Algorithm 1) which is different from the comparable convergence studies. Although recently several accelerated variance reduction techniques for nonconvex problems have been proposed Wang et al (2019); Fang et al (2018), in this work we mainly focus to improve the analysis of the basic ZO-ProxSVRG method and the extension to the novel accelerated frameworks shall be addressed in our future studies. The main contributions of this paper are summarized in the following:

1) Our analysis yields iteration complexity $O(\frac{1}{\epsilon})$ compared to $O(\frac{d}{\epsilon^2})$ of RSPGF Ghadimi and Lan (2016) and $O(\frac{d}{\epsilon})$ of ZO-ProxSVRG/SAGA Huang et al (2019) (the existing variance-reduce SZO proximal algorithm for solving nonconvex nonsmooth problems). It shows that our results have improved complexity for the terms dependent on $d$ in contrast to the existing proximal variance-reduced SZO methods. ZO-PSVRG+ also matches the best result achieved by ZO-SVRG-Coord-Rand

with minibatch size $b = dn^{2/3}$ and epoch size $m = n^{1/3}$ in Ji et al (2019), while being valid for *any minibatch sizes* as detailed in the following sections.

2) The convergence analysis for ZO-PSVRG+ in contrast to ZO-SVRG-Coord Liu et al (2018b); Ji et al (2019) is straightforward and yields simpler proofs. Our analysis achieves new iteration complexity bounds and improves the effectiveness of all the existing ZO-SVRG-based algorithms along with RSPGF for nonconvex nonsmooth composite optimization. Note that the convergence studies for RSPGF and ZO-ProxSVRG/SAGA rely on bounded gradient assumption, which is not our working assumption in this paper.

3) For the nonconvex functions under the Polyak-Łojasiewicz condition Polyak (1963), we show that ZO-PSVRG+ obtains a global linear convergence rate equivalent to the first-order Prox-SVRG. Thus, ZO-PSVRG+ can certainly achieve linear convergence in some zones without restarting. To the best of our knowledge, this is the first paper that leverages the PL condition for improving the convergence of ZO-ProxSVRG for problem (1) with arbitrary minibatch sizes. This analysis generalizes the results Duchi et al (2015) while showing linear convergence in contrast to the sublinear convergence rate in their paper. In Ji et al (2019), the authors show that ZO-SPIDER-Coord achieves linear convergence under PL condition but the analysis is limited to the minibatch of size $b = O(n^{1/2})$. Note that due to both computational and statistical efficiency, convergence analysis for minibatch of moderate sizes is essential (also see the remarks after Theorem 7 for more details).

## Implications of the Proposed Analysis

In application, the number of training data is at least in the order of $n \sim 10^7 - 10^9$ and so $n^{2/3} \sim 10^5 - 10^6$, which is quite large, given the capacity of modern computational infrastructures. On the other hand, if $b$ is too small, the benefits from parallelism of the algorithm would be forfeited. Thus, there is a question if a similar convergence rate could be achieved by applying moderate sizes of $b$, which do not depend on $n$ and $d$. Our results, in this case, matches the best result achieved in Huang et al (2019) for $b = n^{2/3}$, despite the fact we did not apply the bounded gradient assumption similar to their work. Further, we avoided to use all the samples to compute the full gradient and the gradients are computed on the subset of size $\mathcal{B}$. Note that avoiding calculating of full gradient besides computational efficiency can improve the generalization performance of the model. Although this idea has been explored before in Ji et al (2019), but the results therein require the linear dependency of $\mathcal{B}$ to the dimension $d$ (see Corollary 1 in Ji et al (2019)). Our studies provide explicit and direct expressions for the algorithm parameters based on notably new analysis, while the parameters of the analysis presented in Ji et al (2019) depend on implicit, unpractical and complicated equations, being only limited to smooth functions. Furthermore, the analysis in Ji et al (2019) for the single batch case requires adopting a very large number of iterations in the inner loop, $m \sim O(nd)$ to improve the SZO complexity (see Corollary 2 in Ji et al (2019)). This will significantly increase the variance between the full gradient and the gradient over minibatch, as the algorithm applies outdated full gradient for many inner loop iterations.

# Preliminaries

In the following, we illustrate preliminary details of zeroth-order gradient approximations. Considering the loss function $f_i$, a two-point random stochastic gradient estimator (RandSGE) $\hat{\nabla}_r f_i(x)$ is defined as Nesterov and Spokoiny (2017); Gao et al (2018)

$$\hat{\nabla}_r f_i(x, u_i) = \frac{d(f_i(x + \mu u_i) - f_i(x))}{\mu} u_i, \qquad i \in [n] \tag{2}$$

where $d$ is the number of optimization variables, $\{u_i\}$ are i.i.d. random directions drawn from a uniform distribution over a unit sphere and $\mu > 0$ is the smoothing parameter Flaxman et al (2005); Shamir (2017); Gao et al (2018). RandSGE is a biased approximation to the true gradient $\nabla f_i(x)$, and its bias decreases as $\mu$ approaches zero. Nevertheless, in practice, if $\mu$ is too small, the function variation could be magnified by the noise in the function evaluations when the rate of noise to signal is high Lian et al (2016). To obtain a more accurate approximation for ZO gradient, we could apply coordinate gradient estimation (CoordSGE) Gu et al (2018b,a); Liu et al (2018b) to approximate

**Table 1**: Summary of convergence rate and function query complexity of various SZO algorithms. S: Smooth, NS: Nonsmooth, NC: Nonconvex, C: Convex, SC: Strong Convexity, and PL: Polyak-Łojasiewicz Condition. $b$ denotes the minibatch size, $m$ denotes the epoch size, $\lambda$ is the constant in PL condition (15) and $s_n = \min\{n, \frac{1}{\epsilon}\}$. *: The single-minibatch version.

| Method | Problem | Stepsize | Convergence rate | SZO complexity |
|---|---|---|---|---|
| RGF (Nesterov and Spokoiny (2017)) | NS(C) | $O\left(\frac{1}{\sqrt{dT}}\right)$ | $O\left(\frac{d^2}{\epsilon^2}\right)$ | $O\left(\frac{nd^2}{\epsilon^2}b\right)$ |
| RSPGF (Ghadimi and Lan (2016)) | S(NC)+NS(C) | $O(1)$ | $O\left(\frac{d}{\epsilon^2}\right)$ | $O\left(\frac{nd}{\epsilon^2}\right)$ |
| ZO-SVRG-Coord (Liu et al (2018b)) | S(NC) | $O\left(\frac{1}{d}\right)$ | $O\left(\frac{d}{\epsilon}\right)$ | $O(\frac{nd^2}{\epsilon} + \frac{d^2 b}{\epsilon})$ |
| ZO-SVRG-Coord-Rand (Ji et al (2019)) | S(NC) | $O\left(\frac{1}{dn^{2/3}}\right)$ | $O\left(\frac{dn^{2/3}}{\epsilon}\right)$ | $O(\min\{\frac{dn^{2/3}}{\epsilon}, \frac{d}{\epsilon^{5/3}}\})^*$ |
| ZO-ProxSVRG-Coord (Huang et al (2019)) | S(NC)+NS(C) | $O\left(\frac{1}{d}\right)$ | $O\left(\frac{d}{\epsilon}\right)$ | $O(\frac{nd^2}{\epsilon\sqrt{b}} + \frac{md^2\sqrt{b}}{\epsilon})$ |
| ZO-ProxSAGA-Coord (Huang et al (2019)) | S(NC)+NS(C) | $O\left(\frac{1}{d}\right)$ | $O\left(\frac{d}{\epsilon}\right)$ | $O(\frac{nd^2}{\epsilon\sqrt{b}})$ |
| ZO-PSVRG+ (Ours) | S(NC)+NS(C) | $O(1)$ | $O\left(\frac{1}{\epsilon}\right)$ | $O\left(s_n\frac{d}{\epsilon\sqrt{b}} + \frac{bd}{\epsilon}\right)$ |
| ZO-PSVRG+ (RandSGE) (Ours) | S(NC)+NS(C) | $O\left(\frac{1}{\sqrt{d}}\right)$ | $O\left(\frac{\sqrt{d}}{\epsilon}\right)$ | $O\left(s_n\frac{d\sqrt{d}}{\epsilon\sqrt{b}} + \frac{b\sqrt{d}}{\epsilon}\right)$ |
| ZO-PSVRG+ (Ours) | S(PL)+NS(C) | $O(1)$ | $O\left(\log(\frac{1}{\epsilon})\right)$ | $O(s_n\frac{d}{\lambda}\log\frac{1}{\epsilon} + \frac{bd}{\lambda}\log\frac{1}{\epsilon})$ |
| ZO-PSVRG+ (RandSGE) (Ours) | S(PL)+NS(C) | $O\left(\frac{1}{\sqrt{d}}\right)$ | $O\left(\sqrt{d}\log(\frac{1}{\epsilon})\right)$ | $O(s_n\frac{d\sqrt{d}}{\lambda}\log\frac{1}{\epsilon} + \frac{b\sqrt{d}}{\lambda}\log\frac{1}{\epsilon})$ |

the gradients as:

$$\hat{\nabla} f_i(x) = \sum_{j=1}^{d} \frac{f_i(x+\mu e_j) - f_i(x-\mu e_j)}{2\mu} e_j, \quad i \in [n] \tag{3}$$

In this expression, $e_j$ is a standard basis vector with 1 at its $j$-th coordinate and 0 otherwise, and $\mu$ is the smoothing parameter. In contrast to RandSGE, CoordSGE is deterministic and needs $d$ times more ZO function calls. We will later show that ZO variance-reduced method using CoordSGE results in a larger stepsize and a speedier convergence, although the coordinate-wise gradient estimator requires more ZO calls compared to the two-point random gradient approximation.

Finally, we formulate the zeroth-order proximal gradient descent method using ZO gradient estimation (3):

$$x_t^s = \mathrm{Prox}_{\eta h}(x_{t-1}^s - \eta \hat{\nabla} f(x_{t-1}^s)), \qquad t = 1, 2, \dots \tag{4}$$

where $s$ is epoch number, $\hat{\nabla} f = \frac{1}{n}\sum_{i=1}^{n} \hat{\nabla} f_i(x)$ and

$$\mathrm{Prox}_{\eta h}(x) := \arg\min_{y \in \mathbb{R}^d} \left( h(y) + \frac{1}{2\eta}\|y - x\|^2 \right) \tag{5}$$

In the rest of paper we assume that the nonsmooth convex function $h(x)$ in (1) is well-defined, i.e., the proximal operator (5) can be computed effectively.

In general for the convex problems the convergence is measured using the optimality gap $F(x) - F(x^*)$, where we let $x^*$ denote the optimal solution of Problem (1). However, for general nonconvex problems, the gradient norm is commonly used as the convergence metric. For example, for smooth nonconvex optimization (i.e., $h(x) = 0$), in Ghadimi and Lan (2013); Reddi et al (2016a); Lei et al (2017); Liu et al (2018b) the norm of the function gradient $\|\nabla F(x)\|^2$ is applied as the convergence criterion. Aiming to investigate the convergence behavior for nonsmooth nonconvex problems, we define the gradient mapping metric as $g_\eta(x) = \frac{1}{\eta}(x - \mathrm{Prox}_{\eta,h}(x - \eta\nabla f(x)))$. If $h(x)$ is a constant function, the gradient mapping reduces to the ordinary gradient: $g_\eta(x) = \nabla F(x) = \nabla f(x)$. In this work we use the gradient mapping $g_\eta(x)$ as the convergence metric similar to Ghadimi and Lan (2016); Reddi et al (2016b); Parikh et al (2014). For problems with nonconvex structure, the point $x$ is called a stationary point if $g_\eta(x) = 0$, Parikh et al (2014). Therefore, we end up with the following definition for the convergence metric.

**Definition 1** We call point $x \in \mathbb{R}^d$ as an $\epsilon$-accurate point, if $\mathbb{E}\|g_\eta(x)\|^2 \le \epsilon$, for some $\eta > 0$.

# ZO Proximal Stochastic Method (ZO-PSVRG+)

The core idea in variance-reduced methods is to generate an additional sequence $\tilde{x}^{s-1}$ at which the full gradient is computed to obtain a more accurate

**Algorithm 1** Zeroth-Order Proximal Stochastic Method

1: **Input:** initial point $x_0$, batch size $\mathcal{B}$, minibatch size $b$, epoch length $m$, stepsize $\eta$
2: **Initialize:** $\tilde{x}^0 = x_0$
3: **for** $s = 1, 2, \ldots, S$ **do**
4: $\quad x_0^s = \tilde{x}^{s-1}$
5: $\quad \hat{g}^s = \frac{1}{\mathcal{B}} \sum_{j \in I_{\mathcal{B}}} \hat{\nabla} f_j(\tilde{x}^{s-1})$
6: $\quad$ **for** $t = 1, 2, \ldots, m$ **do**
7: $\quad\quad$ Compute $\hat{v}_{t-1}^s$ according to (7) or (8)
8: $\quad\quad x_t^s = \mathrm{Prox}_{\eta h}(x_{t-1}^s - \eta \hat{v}_{t-1}^s)$
9: $\quad$ **end for**
10: $\quad \tilde{x}^s = x_m^s$
11: **end for**
12: **Output:** $\hat{x}$ chosen uniformly from $\{x_t^s\}_{t \in [m], s \in [S]}$

stochastic gradient estimate

$$v_{t-1}^s = \frac{1}{b} \sum_{i \in I_b} \left( \nabla f_i(x_{t-1}^s) - \nabla f_i(\tilde{x}^{s-1}) \right) + g^s \quad (6)$$

where $v_{t-1}^s$ denotes the gradient estimate at $x_{t-1}^s$ and $g^s = \frac{1}{\mathcal{B}} \sum_{i \in I_{\mathcal{B}}} \nabla f_i(\tilde{x}^{s-1})$. We study a proximal stochastic gradient algorithm based on variance reduced approach of ProxSVRG in Xiao and Zhang (2014); Reddi et al (2016b); Li and Li (2018). The description of ZO-PSVRG+ is presented in Algorithm 1. Our method has two types of random sampling. In the outer iteration, we calculate the gradient consisting of $\mathcal{B}$ samples. In the inner iteration, we randomly choose a minibatch of samples of size $b$ to approximate the gradient over the minibatch. We call $\mathcal{B}$ and $b$, batch, and minibatch size, respectively. In our ZO framework, the mix gradient (6) is estimated by applying only function evaluations, given by

$$\hat{v}_{t-1}^s = \frac{1}{b} \sum_{i \in I_b} \left( \hat{\nabla} f_i(x_{t-1}^s) - \hat{\nabla} f_i(\tilde{x}^{s-1}) \right) + \hat{g}^s \quad (7)$$

or

$$\hat{v}_{t-1}^s = \frac{1}{b} \sum_{i \in I_b} \left( \hat{\nabla}_r f_i(x_{t-1}^s, u_i) - \hat{\nabla}_r f_i(\tilde{x}^{s-1}, u_i) \right) + \hat{g}^s \quad (8)$$

where $\hat{g}^s = \frac{1}{\mathcal{B}} \sum_{i \in I_{\mathcal{B}}} \hat{\nabla} f_i(\tilde{x}^{s-1})$, $\hat{\nabla} f_i$ is a ZO gradient approximation using CoordSGE and $\hat{\nabla}_r f_i$ is a ZO gradient estimate using RandSGE. We let ZO-PSVRG+ and ZO-PSVRG+ (RandSGE) denote Algorithm 1 with gradient estimation (7) and (8), respectively. Note that, $\mathbb{E}_{I_b}[\hat{v}_{t-1}^s] = \hat{\nabla} f(x_{t-1}^s) \neq \nabla f(x_{t-1}^s)$, i.e., this stochastic gradient is a biased approximation of the true gradient. In other words, the unbiased assumption on gradient approximates utilized in ProxSVRG Reddi et al (2016b); Li and Li (2018) is no longer valid. Note that the biased ZO gradient estimation yields a fundamental challenge in analyzing ZO-PSVRG+. It means that adjusting the similar concepts from ProxSVRG to zeroth-order algorithm 1 is not effortless and requires an elaborated analysis of ZO-PSVRG+. To tackle this issue, we derive an upper bound for the variance of the gradient approximation $\hat{v}_t^s$ by selecting an appropriate stepsize $\eta$ and smoothing parameter $\mu$ to control the variance of gradient estimation which will be discussed later.

The other major difference of ZO-PSVRG+ and ZO-ProxSVRG is that we avoid the evaluation of the total gradient for each epoch, i.e., the number of samples $\mathcal{B}$ is not necessarily equal to $n$ (see Line 5 of Algorithm 1). If $\mathcal{B} = n$, ZO-PSVRG+ is equivalent to ZO-ProxSVRG, which indicates that our convergence studies yield a novel analysis for ZO-ProxSVRG-Coord (i.e, $\mathcal{B} = n$). In the next section, we will carefully study the convergence of ZO-PSVRG+ under different settings. The proofs of main theorems is deferred to the extended paper.

# Convergence Analysis

Here we provide some minimal assumptions for problem (1):

**Assumption 1** For $\forall i \in [n]$, gradient of the function $f_i$ is Lipschitz continuous with a Lipschitz constant $L > 0$, such that

$$\|\nabla f_i(x) - \nabla f_i(y)\| \le L \|x - y\|, \ \forall x, y \in \mathbb{R}^d$$

**Assumption 2** For $\forall x \in \mathbb{R}^d$, $\mathbb{E}\left[\left\|\hat{\nabla} f_i(x) - \hat{\nabla} f(x)\right\|^2\right] \le \sigma^2$, where $\sigma > 0$ is a constant and $\hat{\nabla} f_i(x)$ is a CoordSGE gradient approximation of $\nabla f_i(x)$.

Assumptions 1 and 2 are standard assumptions applied in SZO optimization. The first assumption is for the convergence studies of the zeroth-order algorithms Ghadimi and Lan (2016); Nesterov and Spokoiny (2017); Liu et al (2018b). The second assumption specifies bounded variance for zeroth-order gradient approximations Lian et al (2016); Liu et al (2018a,b). Assumption 2 is essential in order to obtain a convergence result independent of $n$. Due to the error estimation for CoordSGE, this assumption is equivalent to the bounded variance of true gradients.

Assumption 2 is weaker than the assumption of bounded gradients in Liu et al (2017); Hajinezhad et al (2019), while, we are able to analyze the more complicated problem (1) involving a nonsmooth part and obtain faster convergence rates. Note that according to the error estimation for CoordSGE, this assumption is equivalent to the bounded variance of true gradients.

**Theorem 1** *Suppose Assumptions 1 and 2 hold, and the ZO gradient estimator (7) for mix gradient $\hat{v}_k$ is used. The output $\hat{x}$ of Algorithm 1 satisfies*

$$\mathbb{E}[\|g_\eta(\hat{x})\|^2] \leq \frac{6\left(F(x_0) - F(x^*)\right)}{\eta S m} + \frac{I(\mathcal{B} < n) 12\sigma^2}{\mathcal{B}} + 21 L^2 d^2 \mu^2$$

*where $\eta = \min\{\frac{1}{8L}, \frac{\sqrt{b}}{12mL}\}$ denotes the stepsize.*

In contrast to the convergence rate of SVRG in Reddi et al (2016b), Theorem 1 presents two extra error terms $\frac{I(\mathcal{B}<n)\sigma^2}{\mathcal{B}}$ and $O(L^2 d^2 \mu^2)$, related to the batch gradient estimation $\mathcal{B} < n$ and the use of SZO gradient approximations, respectively. The error related to $\mathcal{B} < n$ is removed only when $\mathcal{B} = n$. Note that the stepsize $\eta$ depends on the epoch length $m$, and the minibatch size $b$. The proof for Theorem 1 is significantly different from the proofs in the existing literature. For instance, the convergence analysis for ZO-SVRG-Coord and ZO-ProxSVRG/SAGA uses the notion of Lyapunov function to show that the accumulated gradient mapping decreases with epoch $s$. However, in our analysis, we explicitly prove that $F(x^s)$ is descending. On the other hand, our convergence result is valid for a wide range of minibatch sizes and any epoch size $m$, whereas the analysis for ZO-SVRG-Coord is valid only for specific values of $m$ with a complicated setting for parameter selection.

The next corollary demonstrates the convergence rate of ZO-PSVRG+ in terms of error of the solution $\hat{x}$, providing explicit descriptions for the parameters in Theorem 1.

**Corollary 2** Let the batch size $\mathcal{B} = \min\{12\sigma^2/\epsilon, n\}$ and $\mu \leq \frac{\sqrt{\epsilon}}{5dL}$ denote the smoothing parameter. Suppose $\hat{x}$ in Algorithm 1 is an $\epsilon$-accurate solution for problem (1). Recalling that CoordSGE require $O(d)$ function queries, the number of SZO calls is at most

$$\begin{aligned} d(S\mathcal{B} + Smb) &= 6d\left(F(x_0) - F(x^*)\right)\left(\frac{\mathcal{B}}{\epsilon\eta m} + \frac{b}{\epsilon\eta}\right) \\ &= O\left(\frac{\mathcal{B}d}{\epsilon\eta m} + \frac{bd}{\epsilon\eta}\right) \end{aligned} \tag{9}$$

and the number of PO calls is equal to $T = Sm = \frac{6(F(x_0)-F(x^*))}{\epsilon\eta} = O\left(\frac{1}{\epsilon\eta}\right)$. In particular, by setting $m = \sqrt{b}$ and $\eta = \frac{1}{12L}$, the number of SZO calls is at most

$$\begin{aligned} & 72dL(F(x_0) - F(x^*))\left(\frac{\mathcal{B}}{\epsilon\sqrt{b}} + \frac{b}{\epsilon}\right) \\ &= O\left(s_n \frac{d}{\epsilon\sqrt{b}} + \frac{bd}{\epsilon}\right) \end{aligned} \tag{10}$$

where $s_n = \min\{n, \frac{1}{\epsilon}\}$ and the number of PO calls equals to $T = Sm = S\sqrt{b} = \frac{72L(F(x_0)-F(x^*))}{\epsilon} = O\left(\frac{1}{\epsilon}\right)$.

Corollary 2 indicates that if the smoothing parameter $\mu$ is sufficiently small and the batch size $\mathcal{B}$ is large enough, then the errors induced from zeroth-order estimation and batch gradient approximation will decrease, leading to a non-dominant term in the convergence rate of ZO-PSVRG+. Indeed, the error term induced by batch size is eliminated only when $\mathcal{B} = n$ (i.e., $I(\mathcal{B} < n) = 0$). In this case, Step 5 of Algorithm 1 converts to $\hat{g}^s = \hat{\nabla} f(\tilde{x}^{s-1})$ and consequently ZO-PSVRG+ changes to ZO-ProxSVRG. In fact equation (10) indicates that a large batch $\mathcal{B}$ for $\mathcal{B} \neq n$ reduces the error inherited by the variance of batch gradient and improves the convergence of ZO-PSVRG+. In summary, with the smoothing parameter and batch size chosen properly, we derive the error term $O(1/\epsilon)$, which is better than the convergence rate of the state-of-the-art SZO algorithms by the factor $\frac{1}{d}$. Moreover, ZO-PSVRG+ uses much fewer SZO oracle calls compared to the methods listed in Table 1. Further, it is worth mentioning that the stepsize $\eta$ in Theorem 1 has a lower dimension-dependency than the existing SZO algorithms in Table 1.

## Analysis for ZO-PSVRG+ (RandSGE)

In this section, we will study the convergence of ZO-PSVRG+ (RandSGE) under different settings. In particular, in the following theorem, we prove that ZO-PSVRG+ (RandSGE) improves the convergence rate and the function query complexity of the existing SZO methods.

**Theorem 3** *Suppose Assumptions 1 and 2 hold, and the coordinate gradient estimator (8) is used to compute the mix gradient $\hat{v}_k$. The output $\hat{x}$ of Algorithm 1 satisfies*

$$\mathbb{E}[\|g_\eta(\hat{x})\|^2] \leq \frac{6\,(F(x_0) - F(x^*))}{\eta S m} + \frac{I(\mathcal{B} < n)12\sigma^2}{\mathcal{B}} + 21L^2d^2\mu^2 \quad (11)$$

*where $\eta = \min\{\frac{1}{8L}, \frac{\sqrt{b}}{12mL\sqrt{d}}\}$ denotes the stepsize.*

**Corollary 4** We Let the batch size $\mathcal{B} = \min\{12\sigma^2/\epsilon, n\}$ and $\mu \leq \frac{\sqrt{\epsilon}}{5dL}$ denote the smoothing parameter. Suppose $\hat{x}$ returned by Algorithm 1 is an $\epsilon$-accurate solution for problem (1). Recalling that CoordSGE and RandSGE require $O(d)$ and $O(1)$ function queries respectively, the number of SZO calls is at most

$$(dS\mathcal{B} + Smb) = 6\left(F(x_0) - F(x^*)\right)\left(\frac{\mathcal{B}d}{\epsilon\eta m} + \frac{b}{\epsilon\eta}\right) = O\left(\frac{\mathcal{B}d}{\epsilon\eta m} + \frac{b}{\epsilon\eta}\right) \quad (12)$$

and the number of PO calls is equal to $T = Sm = \frac{6(F(x_0)-F(x^*))}{\epsilon\eta} = O\left(\frac{1}{\epsilon\eta}\right)$. In particular, by setting $m = \sqrt{b}$ and $\eta = \frac{1}{12L\sqrt{d}}$, the number of SZO calls is at most

$$72L(F(x_0) - F(x^*))\left(\frac{\mathcal{B}d\sqrt{d}}{\epsilon\sqrt{b}} + \frac{b\sqrt{d}}{\epsilon}\right) = O\left(s_n\frac{d\sqrt{d}}{\epsilon\sqrt{b}} + \frac{b\sqrt{d}}{\epsilon}\right) \quad (13)$$

where $s_n = \min\{n, \frac{1}{\epsilon}\}$ and the number of PO calls equals to $T = Sm = S\sqrt{b} = \frac{72L\sqrt{d}(F(x_0)-F(x^*))}{\epsilon} = O\left(\frac{\sqrt{d}}{\epsilon}\right)$.

*Remark 1* The results from Theorem 3 improves the convergence rate $O(\frac{dn^{2/3}}{T})$ for ZO-SVRG-Coord-Rand Ji et al (2019) in single-minibatch setting and with the stepsize $O(\frac{1}{dn^{2/3}})$ to the convergence rate of $O(\frac{\sqrt{d}}{T})$ with the stepsize $O(\frac{1}{\sqrt{d}})$. In addition, it is noted that ZO-SVRG-Coord-Rand in single-minibatch setting requires $m = d$ for the number of inner iterations. If we choose $b = dm^2$ for ProxSVRG+, then $\eta$ reduces to $O(1)$ with the convergence rate $O(\frac{1}{\epsilon})$ which generalizes the best result for ZO-SVRG-Coord-Rand that is only achieved by selecting $m = s_n^{1/3}$.

# Convergence Under PL Condition

In this section, we show the linear convergence of ProxSVRG+ under Polyak-Łojasiewicz (PL) assumption Polyak (1963). The classic structure of PL condition is, for all $x \in \mathbb{R}^d$,

$$\|\nabla f(x)\|^2 \geq 2\lambda(f(x) - f^*) \quad (14)$$

where $\lambda > 0$ and $f^*$ denotes the optimal function value. This condition specifies the rate of increase of the loss function in the vicinity of optimal solutions. It is important to note that if $f$ is $\lambda$-strongly convex then $f$ fulfills the PL condition. We will show that under our analysis the complexity of ZO-PSVRG+ (Algorithm 1) under PL condition is improved. Due to the presence of the nonsmooth term $h(x)$ in problem (1), we utilize the gradient projection to characterize a more generic form of PL condition as follows,

$$\|g_\eta(x)\|^2 \geq 2\lambda(F(x) - F^*) \quad (15)$$

for some $\lambda > 0$ and for all $x \in \mathbb{R}^d$. In particular if $h(x)$ is a constant function, the gradient projection changes to $g_\eta(x) = \nabla f(x)$. The PL condition has been thoroughly investigated in Karimi et al (2016) where the authors proved that PL condition is milder condition than a large family of conditions for functions such as convexity. The revised PL condition (15) is feasibly natural and has been studied in several papers for problems with nonconvex nonsmooth setting, e.g., Li and Li (2018). Similarly, a zeroth-order algorithm under PL condition for smooth functions has been analyzed in Ji et al (2019).

## ZO-PSVRG+ Under PL Condition

In this section, we demonstrate the convergence analysis of ZO-PSVRG+ (Algorithm 1) under PL-condition. More specifically, we provide a generic analysis for enhancing the convergence rate of the existing SZO algorithms for functions satisfying PL condition using variance reduced techniques. It is worth noting that for functions satisfying PL condition (i.e. (15) holds), ZO-PSVRG+ can immediately use the final iteration $\tilde{x}^S$ as the output point rather than using a randomly chosen $\hat{x}$. The following theorem provides the convergence guarantee for ZO-PSVRG+ under PL condition.

**Theorem 5** *Given the Assumptions 1 and 2, suppose that in Algorithm 1 the ZO gradient estimator (7) is applied for mix gradient $\hat{v}_k$ with stepsize $\eta \leq \min\{\frac{1}{8L}, \frac{\sqrt{\gamma b}}{12mL}\}$ where $\gamma = 1 - \frac{2\lambda\eta}{3}m - \frac{\lambda\eta}{3} > 0$. Then*

$$\mathbb{E}[F(\tilde{x}^S) - F^*] \leq \left(1 - \frac{\lambda\eta}{3}\right)^{Sm} \mathbb{E}[F(\tilde{x}^0) - F^*] + \frac{6I(\mathcal{B} < n)\sigma^2}{\lambda\mathcal{B}} + \frac{21L^2d^2\mu^2}{2\lambda} \quad (16)$$

Theorem 5 shows that if the batch size and smoothing parameter are appropriately chosen, ZO-PSVRG+ has a dominant linear convergence rate without restart. Further, compared to Theorem 1, it is evident from (16) that the error term $\frac{6I(\mathcal{B}<n)\sigma^2}{\mathcal{B}} + \frac{7L^2d^2\mu^2}{2}$ is amplified by the factor $1/\lambda$. Thus, the error induced by these terms will be improved if $\lambda >> 1$.

We next explore the number of ZO queries in ZO-PSVRG+ under PL condition to obtain an $\epsilon$-accurate solution, as formalized in Corollary 6.

**Corollary 6** Suppose the final iteration point $\tilde{x}^S$ in Algorithm 1 satisfies $\mathbb{E}[F(\tilde{x}^S) - F^*] \leq \epsilon$ under PL condition. Under Assumptions 1 and 2, we let batch size $\mathcal{B} = \min\{\frac{6\sigma^2}{\lambda\epsilon}, n\}$ and $\mu \leq \frac{\sqrt{\lambda\epsilon}}{4Ld}$ denote the smoothing parameter. Then, the number of SZO calls is bounded by

$$d(S\mathcal{B} + Smb) = O(\frac{s_n d}{\lambda\eta m}\log\frac{1}{\epsilon} + \frac{bd}{\lambda\eta}\log\frac{1}{\epsilon})$$

where $s_n = \min\{n, \frac{1}{\lambda\epsilon}\}$. The number of PO calls equals to the total number of iterations $T$ which is bounded by $T = Sm = O(\frac{1}{\lambda\eta}\log\frac{1}{\epsilon})$. In particular, under the setting $m = \sqrt{b}$ and $\eta = \frac{\sqrt{\gamma}}{12L}$, the number of SZO calls simplifies to $d(S\mathcal{B} + Smb) = O(\frac{\mathcal{B}d}{\lambda\sqrt{\gamma}m}\log\frac{1}{\epsilon} + \frac{bd}{\lambda\sqrt{\gamma}}\log\frac{1}{\epsilon})$.

Here we provide an intuition regarding Corollary 6: this result actually indicates that leveraging the PL condition improves the dominant convergence rate when the error of order $O(1/\epsilon)$ in Corollary 2 is improved to $O(\log(1/\epsilon))$, resulting in a significant speedup. Compared to the sublinear convergence rate for ZO algorithms in Duchi et al (2015); Nesterov and Spokoiny (2017); Liu et al (2018b), the convergence performance of ZO-PSVRG+ under PL condition has a global linear convergence rate and therefore requires a lower number of ZO oracle calls. This also indicates that if ZO-PSVRG+ is initialized in a generic nonconvex domain, the rate of convergence can be automatically accelerated due to entering the PL area. It is an improved result compared with Reddi et al (2016a) where therein PL-SVRG/SAGA is used to restart ProxSVRG/SAGA to obtain a linear convergence rate under PL condition. On the other hand, note that the convergence analysis under PL condition in Ji et al (2019) has complex coupling structures which makes it difficult for practitioners to apply, while our proof is simple and the parameters are properly specified to be suitable for hyperparameter selections.

*Remark 2* Compared to Theorem 1, the convergence rate of ZO-PSVRG+ in Theorem 5 admits additional parameter $\gamma$ for parameter selection due to the PL condition. By assuming the condition number $\lambda/L \leq \frac{1}{n^{1/3}}$ and through choosing $m = n^{1/3}$ and $\eta = \frac{\rho}{L}$ with $\rho \leq \frac{1}{2}$, the definition of $\gamma$ yields

$$\gamma = 1 - \frac{2\lambda\eta}{3}m - \frac{\lambda\eta}{3} \geq 1 - \rho \geq \frac{1}{2} \quad (17)$$

According to Theorem 5, equation (17) implies $\eta \leq \min\{\frac{1}{8L}, \frac{\sqrt{b}}{12\sqrt{2}mL}\}$. Hence, choosing $b = m^2$ implies the constant stepsize $\eta \leq \frac{1}{12\sqrt{2}L}$. Note that the assumption $\lambda/L \leq \frac{1}{n^{1/3}}$ on condition number is milder than the assumption $\lambda/L < \frac{1}{\sqrt{n}}$ in Reddi et al (2016b).

## ZO-PSVRG+ (RandSGE) Under PL Condition

In the following theorem, we explore if ZO-PSVRG+ (RandSGE) achieves a linear convergence rate when it enters a local landscape where the loss function satisfying the PL condition.

**Theorem 7** *Let Assumptions 1 and 2 hold, and ZO gradient estimator (8) for mix gradient $\hat{v}_k$ is used in*

*Algorithm 1 with stepsize* $\eta \leq \min\{\frac{1}{8L}, \frac{\sqrt{\gamma b}}{12mL\sqrt{d}}\}$ *where* $\gamma = 1 - \frac{2\lambda\eta}{3}m - \frac{\lambda\eta}{3} > 0$*. Then*

$$\mathbb{E}[F(\tilde{x}^S) - F^*] \leq \left(1 - \frac{\lambda\eta}{3}\right)^{Sm} \mathbb{E}[F(\tilde{x}^0) - F^*] + \frac{6I(\mathcal{B} < n)\sigma^2}{\lambda\mathcal{B}} + \frac{21L^2d^2\mu^2}{2\lambda} \quad (18)$$

**Corollary 8** Suppose the final iteration point $\tilde{x}^S$ in Algorithm 1 satisfies $\mathbb{E}[F(\tilde{x}^S) - F^*] \leq \epsilon$ under PL condition. Under Assumptions 1 and 2, we let batch size $\mathcal{B} = \min\{\frac{6\sigma^2}{\lambda\epsilon}, n\}$ and the smoothing parameter $\mu \leq \frac{\sqrt{\lambda\epsilon}}{4Ld}$. The number of SZO calls is bounded by

$$(S\mathcal{B}d + Smb) = O(\frac{s_n d}{\lambda\eta m}\log\frac{1}{\epsilon} + \frac{b}{\lambda\eta}\log\frac{1}{\epsilon})$$

where $s_n = \min\{n, \frac{1}{\lambda\epsilon}\}$. The number of PO calls equals the total number of iterations $T$ which is bounded by

$$T = Sm = O(\frac{1}{\lambda\eta}\log\frac{1}{\epsilon})$$

In particular, given the setting $m = \sqrt{b}$ and $\eta = \frac{\sqrt{\gamma}}{12L\sqrt{d}}$, the number of SZO calls simplifies to $(S\mathcal{B}d + Smb) = O(\frac{\mathcal{B}d\sqrt{d}}{\lambda\sqrt{\gamma}m}\log\frac{1}{\epsilon} + \frac{b\sqrt{d}}{\lambda\sqrt{\gamma}}\log\frac{1}{\epsilon})$.

*Remark 3* It should be noted that by selecting $b = O(d)$ in Theorem 7, the stepsize $\eta$ reduces to $O(1)$ with $O(s_n d \log\frac{1}{\epsilon})$ SZO queries. Note that the analysis for ZO-SPIDER-Coord in Ji et al (2019) has no single-sample version for functions satisfying PL condition and the authors only provided a rate of convergence for large minibatch sizes with involved parameter information.

# Experimental Results

In this section we provide our experimental results[1]. We compare the performance of our ZO-PSVRG+ with 1) ZO-ProxSVRG (based on our improved analysis), 2) ZO-ProxSAGA-Coord Gu et al (2018a) and 3) ZO-ProxSGD Ghadimi and Lan (2016) over two empirical experiments: black-box binary classification and adversarial attacks for black-box deep neural networks (DNNs). We let ZO-ProxSGD denote RSPGF based on CoordSGE (3) for gradient estimation. We also let ZO-ProxSVRG and ZO-ProxSVRG (RandSGE) denote ZO-PSVRG+ and ZO-PSVRG+ (RandSGE) with $\mathcal{B} = n$, respectively. The learning rates are tuned in the experiments for competitive algorithms according to their convergence guarantees in Table 1, and the results shown in this section are based on the best learning rate we obtained for each algorithm. More specifically, Table 1 shows that the base stepsize demands to be dependent on the dimension $d$. We set stepsize $\eta$ and smoothing parameter $\mu$ for ZO-PSVRG+ and ZO-PSVRG+ (RandSGE) according to the convergence guarantee we obtained in the studied lemmas and theorems.

## Binary Classification

In the first set of our experiments, we investigate the logistic regression loss function with $L_1$ and $L_2$ regularization for training the black-box binary classification problem. The problem can be described as the optimization problem (1) with $f_i(x) = \frac{1}{1+e^{y_i z_i^T x}}$, $h(x) = \lambda_1\|x\|_1 + \frac{\lambda_2}{2}\|x\|^2$, where $z_i \in \mathbb{R}^d$ and $y_i$ is the corresponding label for each $i$. The $L_1$ and $L_2$ regularization weights $\lambda_1$ and $\lambda_2$ in all the experiments are set respectively to $10^{-4}$ and $10^{-6}$. We also set $\mathcal{B} = \lfloor\frac{n}{5}\rfloor$ for ZO-PSVRG+. We run our experiments on datasets from LIBSVM website[2], as listed in Table 2. The epoch size is chosen $m = 30$ over all the experiments and the minibatch size $b$ is set to 50.

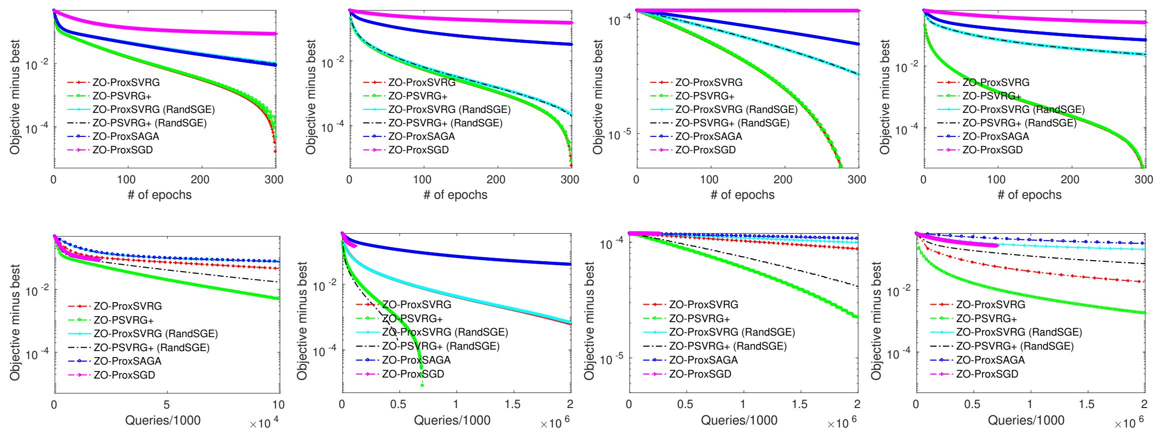
**Table 2**: Summary of training datasets.

| Datasets | Data | Features |
|---|---|---|
| ijcnn | 49990 | 22 |
| a9a | 32561 | 123 |
| w8a | 64,700 | 300 |
| mnist | 60000 | 784 |

In Figure 1 (top), we show the training loss versus the number of epochs (i.e., iterations divided by the epoch length $m = 30$). Note that ZO-PSVRG+ is evaluated using mix gradient CoordSGE (3) and mix gradient RandSGE (2). Results in Figure 1 (bottom) compare the performance of ZO-PSVRG+ with the variants of ZO variance reduced stochastic gradient descent described earlier in Table 1 against the number of function queries. In these figures, ZO-PSVRG+ shows a faster convergence rate compared to ZO-PSVRG+ (RandSGE). Note

[1] The code is available in the anonymous repository: https://anonymous.4open.science/r/ZO-Data-Mining-AA1F

[2] https://www.csie.ntu.edu.tw/ cjlin/libsvmtools/datasets/binary.html




(a) ijcnn (b) comparison on covtype (c) w8a (d) mnist

**Fig. 1**: Comparison of different zeroth-order algorithms for logistic regression loss residual $f(x) - f(x^*)$ versus the number of epochs (top) and ZO queries (bottom)

that ZO-ProxSVRG based on our improved analysis presents a better convergence than both of ZO-ProxSAGA and ZO-ProxSGD. On the other hand, the application of $\mathcal{B} = \lfloor \frac{n}{5} \rfloor$ in ZO-PSVRG+ significantly improves ZO-ProxSVRG with respect to the number of ZO-queries (see Table 1), leading to a non-dominant factor $O(I_{\{\mathcal{B}<n\}}/\mathcal{B})$ in the convergence rate of ZO-PSVRG+. In particular, ZO-PSVRG+ exhibits better performance in terms of the number of function queries than ZO-ProxSAGA using CoordSGE. The degradation in the convergence of ZO-ProxSAGA is due to the requirement for small stepsizes $O(\frac{1}{d})$. Similarly, the large number of function queries to construct coordinate-wise gradient estimates significantly increases the number of SZO queries for ZO-ProxSVRG. On the other hand, ZO-ProxSGD consumes an extremely large number of iterations while exhibiting marginal convergence compared with variance reduced algorithms. Thus, ZO-PSVRG+ obtains the best tradeoffs between the iteration and the function query complexity.

## Adversarial Attacks on Black-Box DNNs

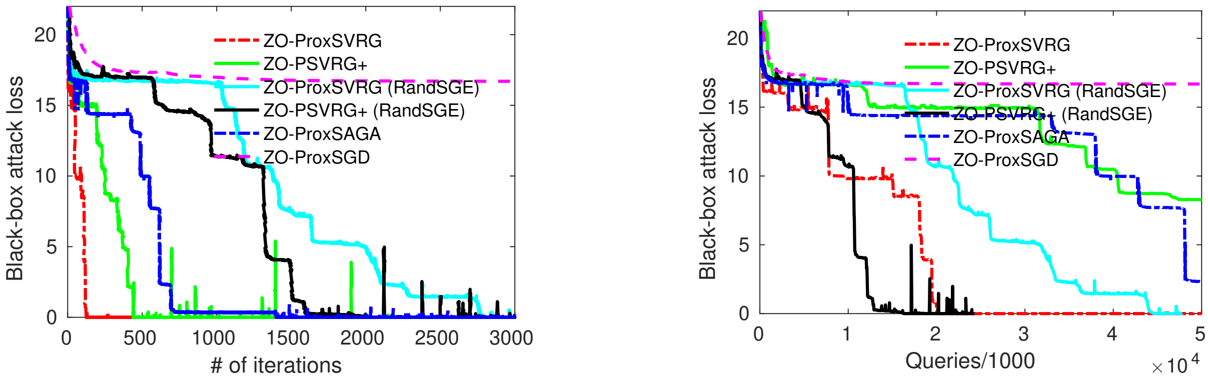
Adversarial examples in image classification are related to designing unperceptive perturbations such that they lead to misclassifying the target model by adding to the natural images. In the framework of zeroth-order attacks Chen et al (2017); Liu et al (2018b), the model parameters are hidden and obtaining its gradient is not feasible, while only the model evaluations are available. We can then consider the task of producing a universal adversarial example with respect to $n$ natural images as an ZO optimization problem of the form (1). More precisely, we apply the zeroth-order algorithms to obtain a global adversarial perturbation $x \in \mathbb{R}^d$ that could mislead the classifier on samples $\{a_i \in \mathbb{R}^d, y_i \in \mathbb{N}\}_{i=1}^n$. This problem can be specified as the following elastic-net attacks to black-box DNNs problem:

$$\min_{x\in\mathbb{R}^d} \frac{1}{n}\sum_{i=1}^{n} \max\{F_{y_i}(a_i^{adv}) - \max_{j\neq y_i} F_j(a_i^{adv}), 0\} + c\left\|a_i^{adv} - a_i\right\|^2 + \lambda_1 \left\|x\right\|_1 + \lambda_2 \left\|x\right\|^2 \tag{19}$$

where $a_i^{adv} = 0.5\tanh(\tanh^{-1}(2a_i)+x)$ and $\lambda_1$ and $\lambda_2$ are nonnegative parameters to obtain consistency between attack success rate, distortion and sparsity. Here $F(a) = [F_1(a), \dots, F_K(a)] \in [0,1]^K$ describes a trained DNN[3] for the MNIST handwritten digit classification, where $F_i(a)$ returns the prediction score of $i$-th class. The parameter $c$ in (19) compensate the rate of adversarial success and the distortion of adversarial examples. In our experiment, we set the regularization parameter $c = 0.2$ and $\lambda_1 = \lambda_2 = 10^{-5}$.

We perform two experiments by choosing $n = 10$ and $n = 100$ images from the same class, and set the minibatch sizes, respectively $b = 5$ and

[3] https://github.com/carlini/nn_robust_attacks




(a) Loss vs iterations: $n = 10$

(b) Loss vs queries: $n = 10$

**Fig. 2**: Comparison of different zeroth-order algorithms for generating black-box adversarial examples from a black-box DNN

$b = 30$. We select the batch size $\mathcal{B} = \lfloor \frac{n}{2} \rfloor$ for ZO-PSVRG+. Figure 2 shows the performance of different ZO algorithms considered in this paper. Our two algorithms ZO-PSVRG+ (RandSGE) and ZO-ProxSVRG (under our improved analysis) show better performance both in convergence rate (iteration complexity) and function query complexity than ZO-ProxSGD and ZO-ProxSAGA. The performance of ZO-PSVRG+ (CoordSGE) algorithm degrades due to large number of function queries for CoordSGE and the variance inherited by $\mathcal{B} \neq n$. ZO-PSVRG+ (RandSGE) shows faster convergence in the initial optimization stage, and more importantly, has much lower function query complexity, which is largely due to efficient ZO queries for computing mix gradient (8) and the $O(\frac{1}{\sqrt{d}})$-level stepsize required by ZO-PSVRG+ (RandSGE). ZO-ProxSAGA and ZO-PSVRG+ (CoordSGE) exhibit relatively similar convergence behaviors. Furthermore, the convergence performance of ZO-ProxSGD is poor compared to other algorithms due to not using variance reduced techniques.

# Conclusion

In this paper, we developed a novel analysis for two zeroth-order variance-reduced proximal algorithms, ZO-PSVRG+ and ZO-PSVRG+ (RangSGE). Our analysis for ZO-PSVRG+ generalizes and improves the analysis of several well-known convergence results, e.g., ZO-ProxSVRG. Compared with ZO-SVRG-Coord-Rand Ji et al (2019), our analysis allows single minibatch size and larger stepsizes while improving the function query complexity. Moreover, for nonconvex functions under Polyak-Łojasiewicz condition, we prove that ZO-PSVRG+ obtains global linear convergence rate for a wide range of minibatch sizes without restart. The empirical results demonstrate the effectiveness of our novel approaches. As a byproduct, our analysis provides the first step towards improving the query complexity of ZO methods for nonconvex optimization where it can be applied to the other variance reduced gradient-free methods.

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# Supplemental Materials

In this section, we present the complete proofs of the above lemmas and theorems.

**Lemma 9** For a given $x \in \mathbb{R}^d$, let $\overline{x} = \text{Prox}_{\eta h}(x - \eta v)$, then we have for all $w \in \mathbb{R}^d$

$$\begin{aligned} F(\overline{x}) \le F(w) + \langle \nabla f(x) - v, \overline{x} - w \rangle - \frac{1}{\eta} \langle \overline{x} - x, \overline{x} - w \rangle \\ + \frac{L}{2} \|\overline{x} - x\|^2 + \frac{L}{2} \|w - x\|^2 \end{aligned} \tag{S1}$$

*Proof* First, we recall the proximal operator

$$\text{Prox}_{\eta h}(x - \eta v) := \arg \min_{y \in \mathbb{R}^d} \left( h(y) + \frac{1}{2\eta} \|y - x\|^2 + \langle v, y \rangle \right) \tag{S2}$$

For the nonsmooth function $h(x)$, for all $w \in \mathbb{R}^d$ we have

$$\begin{aligned} h(\overline{x}) &\le h(w) + \langle p, \overline{x} - w \rangle \\ &= h(w) - \left\langle v + \frac{1}{\eta}(\overline{x} - x), \overline{x} - w \right\rangle \end{aligned} \tag{S3}$$

where $p \in \partial h(\overline{x})$ such that $p + \frac{1}{\eta}(\overline{x} - x) + v = 0$ according to the optimality condition of (S2), and (S3) due to the convexity of $h$. In addition, since since $f(x)$ is $L$-Lipschitz continuous, we have

$$f(\overline{x}) \le f(x) + \langle \nabla f(x), \overline{x} - x \rangle + \frac{L}{2} \|\overline{x} - x\|^2 \tag{S4}$$

and

$$f(x) \le f(w) + \langle \nabla f(x), x - w \rangle + \frac{L}{2} \|w - x\|^2 \tag{S5}$$

This lemma is obtained by adding (S3), (S4), (S5), and using $F(x) = f(x) + h(x)$. □

**Lemma 10** Let $x_t^s = \text{Prox}_{\eta h}(x_{t-1}^s - \eta \hat{v}_{t-1}^s)$ and $\overline{x}_t^s$ be the proximal projection using full true gradient, i.e., $\overline{x}_t^s = \text{Prox}_{\eta h}(x_{t-1}^s - \eta \nabla f(x_{t-1}^s))$. Then the following inequality holds

$$\left\langle \nabla f(x_{t-1}^s) - \hat{v}_{t-1}^s, x_t^s - \overline{x}_t^s \right\rangle \le \eta \left\| \nabla f(x_{t-1}^s) - \hat{v}_{t-1}^s \right\|^2$$

*Proof* Based on inequality (S3) we obtain

$$h(x_t^s) \le h(\overline{x}_t^s) - \left\langle \hat{v}_{t-1}^s + \frac{1}{\eta}(x_t^s - x_{t-1}^s), x_t^s - \overline{x}_t^s \right\rangle \tag{S6}$$

$$h(\overline{x}_t^s) \le h(x_t^s) - \left\langle \nabla f(x_{t-1}^s) + \frac{1}{\eta}(\overline{x}_t^s - x_{t-1}^s), \overline{x}_t^s - x_t^s \right\rangle \tag{S7}$$

By summing (S6) and (S7), we have

$$\begin{aligned} \frac{1}{\eta} \left\langle x_t^s - \overline{x}_t^s, x_t^s - \overline{x}_t^s \right\rangle &\le \left\langle \nabla f(x_{t-1}^s) - \hat{v}_{t-1}^s, x_t^s - \overline{x}_t^s \right\rangle \\ \frac{1}{\eta} \left\| x_t^s - \overline{x}_t^s \right\|^2 &\le \left\| \nabla f(x_{t-1}^s) - \hat{v}_{t-1}^s \right\| \left\| x_t^s - \overline{x}_t^s \right\| \end{aligned} \tag{S8}$$

where (S8) holds by Cauchy-Schwarz inequality. Thus, we obtain

$$\left\| x_t^s - \overline{x}_t^s \right\| \le \eta \left\| \nabla f(x_{t-1}^s) - \hat{v}_{t-1}^s \right\| \tag{S9}$$

Now the proof is complete using Cauchy-Schwarz inequality and (S9). □

Below, we start by deriving an upper bound for the variance of estimated gradient $\hat{v}_{t-1}^s$ based on CoordSGE.

**Lemma 11** Given the mix gradient estimation $\hat{v}_{t-1}^s = \frac{1}{b}\sum_{i\in I_b}\left(\hat\nabla f_i(x_{t-1}^s) - \hat\nabla f_i(\tilde{x}^{s-1})\right) + \hat{g}^s$ with $\hat{g}^s = \frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\hat\nabla f_j(\tilde{x}^{s-1})$, the following inequality holds.

$$\begin{aligned}\mathbb{E}\left[\eta\left\|\nabla f(x_{t-1}^s) - \hat{v}_{t-1}^s\right\|^2\right] &\le \frac{6\eta L^2}{b}\mathbb{E}\left[\left\|x_{t-1}^s - \tilde{x}^{s-1}\right\|^2\right] \\ &+ 2\frac{I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}} + \eta\frac{7L^2d^2\mu^2}{2} \end{aligned} \tag{S10}$$

where $I(A)=1$ if the scenario $A$ occurs and 0 otherwise.

Lemma 11 provides an upper bound for the variance of $\hat{v}_{t-1}^s$. We will show later that the points $x_{t-1}^s$ and $\tilde{x}^{s-1}$ both will converge to the same stationary point. This results in reducing the variance of stochastic gradient, however the variance is not totally diminished due to the zeroth-order gradient estimation and the variance of the gradient over batch.

*Proof* We have

$$\begin{aligned}
&\mathbb{E}\left[\eta\left\|\nabla f(x_{t-1}^s) - \hat{v}_{t-1}^s\right\|^2\right] \\
=&\mathbb{E}\left[\eta\left\|\frac{1}{b}\sum_{i\in I_b}\left(\hat\nabla f_i(x_{t-1}^s) - \hat\nabla f_i(\tilde{x}^{s-1})\right) - \left(\nabla f(x_{t-1}^s) - \hat{g}^s\right)\right\|^2\right] \\
=&\mathbb{E}\left[\eta\left\|\frac{1}{b}\sum_{i\in I_b}\left(\hat\nabla f_i(x_{t-1}^s) - \hat\nabla f_i(\tilde{x}^{s-1})\right) - \left(\nabla f(x_{t-1}^s) - \frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\hat\nabla f_j(\tilde{x}^{s-1})\right)\right\|^2\right] \\
=&\mathbb{E}\left[\eta\left\|\frac{1}{b}\sum_{i\in I_b}\left(\hat\nabla f_i(x_{t-1}^s) - \hat\nabla f_i(\tilde{x}^{s-1})\right) - \left(\nabla f(x_{t-1}^s) - \hat\nabla f(\tilde{x}^{s-1})\right) + \left(\frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\hat\nabla f_j(\tilde{x}^{s-1}) - \hat\nabla f(\tilde{x}^{s-1})\right)\right\|^2\right] \\
=&\eta\mathbb{E}\left[\left\|\frac{1}{b}\sum_{i\in I_b}\left(\left(\hat\nabla f_i(x_{t-1}^s) - \hat\nabla f_i(\tilde{x}^{s-1})\right) - \left(\nabla f(x_{t-1}^s) - \hat\nabla f(\tilde{x}^{s-1})\right)\right) + \frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\left(\hat\nabla f_j(\tilde{x}^{s-1}) - \hat\nabla f(\tilde{x}^{s-1})\right)\right\|^2\right] \\
\le&2\eta\mathbb{E}\left[\left\|\frac{1}{b}\sum_{i\in I_b}\left(\left(\hat\nabla f_i(x_{t-1}^s) - \hat\nabla f_i(\tilde{x}^{s-1})\right) - \left(\hat\nabla f(x_{t-1}^s) - \hat\nabla f(\tilde{x}^{s-1})\right)\right) + \frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\left(\hat\nabla f_j(\tilde{x}^{s-1}) - \hat\nabla f(\tilde{x}^{s-1})\right)\right\|^2\right] \\
&+ 2\eta\mathbb{E}\left\|\hat\nabla f(x_{t-1}^s) - \nabla f(x_{t-1}^s)\right\|^2 \qquad \text{(S11)} \\
=&2\eta\mathbb{E}\left[\left\|\frac{1}{b}\sum_{i\in I_b}\left(\left(\hat\nabla f_i(x_{t-1}^s) - \hat\nabla f_i(\tilde{x}^{s-1})\right) - \left(\hat\nabla f(x_{t-1}^s) - \hat\nabla f(\tilde{x}^{s-1})\right)\right)\right\|^2\right] \\
&+ 2\eta\mathbb{E}\left[\left\|\frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\left(\hat\nabla f_j(\tilde{x}^{s-1}) - \hat\nabla f(\tilde{x}^{s-1})\right)\right\|^2\right] + 2\eta\mathbb{E}\left\|\hat\nabla f(x_{t-1}^s) - \nabla f(x_{t-1}^s)\right\|^2 \qquad \text{(S12)} \\
=&\frac{2\eta}{b^2}\mathbb{E}\left[\sum_{i\in I_b}\left\|\left(\left(\hat\nabla f_i(x_{t-1}^s) - \hat\nabla f_i(\tilde{x}^{s-1})\right) - \left(\hat\nabla f(x_{t-1}^s) - \hat\nabla f(\tilde{x}^{s-1})\right)\right)\right\|^2\right] \\
&+ 2\eta\mathbb{E}\left[\left\|\frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\left(\hat\nabla f_j(\tilde{x}^{s-1}) - \hat\nabla f(\tilde{x}^{s-1})\right)\right\|^2\right] + 2\eta\mathbb{E}\left\|\hat\nabla f(x_{t-1}^s) - \nabla f(x_{t-1}^s)\right\|^2 \qquad \text{(S13)}
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{2\eta}{b^2}\mathbb{E}\left[\sum_{i\in I_b}\left\|\hat{\nabla} f_i(x_{t-1}^s)-\hat{\nabla} f_i(\tilde{x}^{s-1})\right\|^2\right] \\
&\quad + 2\eta\mathbb{E}\left[\left\|\frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\left(\hat{\nabla} f_j(\tilde{x}^{s-1})-\hat{\nabla} f(\tilde{x}^{s-1})\right)\right\|^2\right] + 2\eta\mathbb{E}\left\|\hat{\nabla} f(x_{t-1}^s)-\nabla f(x_{t-1}^s)\right\|^2 \qquad \text{(S14)} \\
&\le \frac{3\eta L^2 d^2\mu^2}{b} + \frac{6\eta L^2}{b}\mathbb{E}\left[\left\|x_{t-1}^s-\tilde{x}^{s-1}\right\|^2\right] + 2\eta\mathbb{E}\left[\left\|\frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\left(\hat{\nabla} f_j(\tilde{x}^{s-1})-\hat{\nabla} f(\tilde{x}^{s-1})\right)\right\|^2\right] \\
&\quad + 2\eta\mathbb{E}\left\|\hat{\nabla} f(x_{t-1}^s)-\nabla f(x_{t-1}^s)\right\|^2 \qquad \text{(S15)} \\
&\le \frac{6\eta L^2}{b}\mathbb{E}\left[\left\|x_{t-1}^s-\tilde{x}^{s-1}\right\|^2\right] + 2\frac{I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}} + 2\eta\mathbb{E}\left\|\hat{\nabla} f(x_{t-1}^s)-\nabla f(x_{t-1}^s)\right\|^2 \\
&\quad + \frac{3\eta L^2 d^2\mu^2}{b} \qquad \text{(S16)} \\
&\le \frac{6\eta L^2}{b}\mathbb{E}\left[\left\|x_{t-1}^s-\tilde{x}^{s-1}\right\|^2\right] + 2\frac{I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}} \\
&\quad + \eta\frac{L^2 d^2\mu^2}{2} + \frac{3\eta L^2 d^2\mu^2}{b} \qquad \text{(S17)} \\
&\le \frac{6\eta L^2}{b}\mathbb{E}\left[\left\|x_{t-1}^s-\tilde{x}^{s-1}\right\|^2\right] + 2\frac{I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}} + \eta\frac{7L^2 d^2\mu^2}{2} \qquad \text{(S18)}
\end{aligned}
$$

where, recalling that a deterministic gradient estimator is employed and the expectations are taking with respect to $I_b$ and $I_{\mathcal{B}}$. The inequality (S11) holds by the Jensen's inequality. (S12) and (S13) are due to $\mathbb{E}[\|x_1+x_2+\ldots+x_k\|^2] = \sum_{i=1}^k \mathbb{E}[\|x_i\|^2]$ if $x_1, x_2, \ldots, x_k$ are independent and of mean zero. Recall that $I_b$ and $I_{\mathcal{B}}$ are also independent. (S14) applies the fact that for any random variable $z$, $\mathbb{E}[\|z-\mathbb{E}[z]\|^2] \le \mathbb{E}[\|z\|^2]$. (S15) employs following inequality

$$
\begin{aligned}
\mathbb{E}\left\|\hat{\nabla} f_i(x_t^s)-\hat{\nabla} f_i(\tilde{x}^s)\right\|^2 &= \mathbb{E}\left\|\hat{\nabla} f_i(x_t^s)-\nabla f_i(x_t^s)+\nabla f_i(x_t^s)-\hat{\nabla} f_i(\tilde{x}^s)+\nabla f_i(\tilde{x}^s)-\nabla f_i(\tilde{x}^s)\right\|^2 \\
&\le 3\mathbb{E}\left\|\hat{\nabla} f_i(x_t^s)-\nabla f_i(x_t^s)\right\|^2 + 3\left\|\hat{\nabla} f_i(\tilde{x}^s)-\nabla f_i(\tilde{x}^s)\right\|^2 \\
&+ 3\left\|\nabla f_i(x_t^s)-\nabla f_i(\tilde{x}^s)\right\|^2 \\
&\le \frac{3L^2 d^2\mu^2}{2} + 3L^2\left\|x_t^s-\tilde{x}^s\right\|^2
\end{aligned}
\qquad \text{(S19)}
$$

where the last inequality used the fact that $f_{i,\mu_j}$ is $L$-smooth. (S16) is by Assumption 2 and (S17) uses Lemma 15. The proof is now complete. □

Blow we present the counterpart of Lemma 11 for the mix gradient estimation in (8).

**Lemma 12** Given the mix gradient estimation $\tilde{v}_{t-1}^s = \frac{1}{b}\sum_{i\in I_b}\left(\hat{\nabla}_r f_i(x_{t-1}^s)-\hat{\nabla}_r f_i(\tilde{x}^{s-1})\right)+\hat{g}^s$ with $\hat{g}^s = \frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\hat{\nabla} f_j(\tilde{x}^{s-1})$, the following inequality holds.

$$
\begin{aligned}
\mathbb{E}\left[\eta\left\|\nabla f(x_{t-1}^s)-\tilde{v}_{t-1}^s\right\|^2\right] &\le \frac{6\eta L^2 d}{b}\mathbb{E}\left[\left\|x_{t-1}^s-\tilde{x}^{s-1}\right\|^2\right] \\
&+ 2\frac{I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}} + \eta\frac{7L^2 d^2\mu^2}{2}
\end{aligned}
\qquad \text{(S20)}
$$

*Proof* We have

$$
\mathbb{E}\left[\eta\left\|\nabla f(x_{t-1}^s)-\tilde{v}_{t-1}^s\right\|^2\right]
$$

$$
\begin{aligned}
=&\mathbb{E}\left[\eta\left\|\frac{1}{b}\sum_{i\in I_b}\left(\hat{\nabla}_r f_i(x_{t-1}^s)-\hat{\nabla}_r f_i(\tilde{x}^{s-1})\right)-\left(\nabla f(x_{t-1}^s)-\hat{g}^s\right)\right\|^2\right]\\
=&\mathbb{E}\left[\eta\left\|\frac{1}{b}\sum_{i\in I_b}\left(\hat{\nabla}_r f_i(x_{t-1}^s)-\hat{\nabla}_r f_i(\tilde{x}^{s-1})\right)-\left(\nabla f(x_{t-1}^s)-\frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\hat{\nabla} f_j(\tilde{x}^{s-1})\right)\right\|^2\right]\\
=&\mathbb{E}\left[\eta\left\|\frac{1}{b}\sum_{i\in I_b}\left(\hat{\nabla}_r f_i(x_{t-1}^s)-\hat{\nabla}_r f_i(\tilde{x}^{s-1})\right)-\left(\nabla f(x_{t-1}^s)-\hat{\nabla} f(\tilde{x}^{s-1})\right)+\left(\frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\hat{\nabla} f_j(\tilde{x}^{s-1})-\hat{\nabla} f(\tilde{x}^{s-1})\right)\right\|^2\right]\\
=&\eta\mathbb{E}\left[\left\|\frac{1}{b}\sum_{i\in I_b}\left(\left(\hat{\nabla}_r f_i(x_{t-1}^s)-\hat{\nabla}_r f_i(\tilde{x}^{s-1})\right)-\left(\nabla f(x_{t-1}^s)-\hat{\nabla} f(\tilde{x}^{s-1})\right)\right)+\frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\left(\hat{\nabla} f_j(\tilde{x}^{s-1})-\hat{\nabla} f(\tilde{x}^{s-1})\right)\right\|^2\right]\\
\le&2\eta\mathbb{E}\left[\left\|\frac{1}{b}\sum_{i\in I_b}\left(\left(\hat{\nabla}_r f_i(x_{t-1}^s)-\hat{\nabla}_r f_i(\tilde{x}^{s-1})\right)-\left(\hat{\nabla} f(x_{t-1}^s)-\hat{\nabla} f(\tilde{x}^{s-1})\right)\right)+\frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\left(\hat{\nabla} f_j(\tilde{x}^{s-1})-\hat{\nabla} f(\tilde{x}^{s-1})\right)\right\|^2\right]\\
&+2\eta\mathbb{E}\left\|\hat{\nabla} f(x_{t-1}^s)-\nabla f(x_{t-1}^s)\right\|^2 \qquad\text{(S21)}\\
=&2\eta\mathbb{E}\left[\left\|\frac{1}{b}\sum_{i\in I_b}\left(\left(\hat{\nabla}_r f_i(x_{t-1}^s)-\hat{\nabla}_r f_i(\tilde{x}^{s-1})\right)-\left(\hat{\nabla} f(x_{t-1}^s)-\hat{\nabla} f(\tilde{x}^{s-1})\right)\right)\right\|^2\right]\\
&+2\eta\mathbb{E}\left[\left\|\frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\left(\hat{\nabla} f_j(\tilde{x}^{s-1})-\hat{\nabla} f(\tilde{x}^{s-1})\right)\right\|^2\right]+2\eta\mathbb{E}\left\|\hat{\nabla} f(x_{t-1}^s)-\nabla f(x_{t-1}^s)\right\|^2 \qquad\text{(S22)}\\
=&\frac{2\eta}{b^2}\mathbb{E}\left[\sum_{i\in I_b}\left\|\left(\left(\hat{\nabla}_r f_i(x_{t-1}^s)-\hat{\nabla}_r f_i(\tilde{x}^{s-1})\right)-\left(\hat{\nabla} f(x_{t-1}^s)-\hat{\nabla} f(\tilde{x}^{s-1})\right)\right)\right\|^2\right]\\
&+2\eta\mathbb{E}\left[\left\|\frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\left(\hat{\nabla} f_j(\tilde{x}^{s-1})-\hat{\nabla} f(\tilde{x}^{s-1})\right)\right\|^2\right]+2\eta\mathbb{E}\left\|\hat{\nabla} f(x_{t-1}^s)-\nabla f(x_{t-1}^s)\right\|^2 \qquad\text{(S23)}\\
\le&\frac{2\eta}{b^2}\mathbb{E}\left[\sum_{i\in I_b}\left\|\hat{\nabla}_r f_i(x_{t-1}^s)-\hat{\nabla}_r f_i(\tilde{x}^{s-1})\right\|^2\right]+2\eta\mathbb{E}\left[\left\|\frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\left(\hat{\nabla} f_j(\tilde{x}^{s-1})-\hat{\nabla} f(\tilde{x}^{s-1})\right)\right\|^2\right]\\
&+2\eta\mathbb{E}\left\|\hat{\nabla} f(x_{t-1}^s)-\nabla f(x_{t-1}^s)\right\|^2 \qquad\text{(S24)}\\
\le&\frac{3\eta L^2d^2\mu^2}{b}+\frac{6\eta L^2 d}{b}\mathbb{E}\left[\left\|x_{t-1}^s-\tilde{x}^{s-1}\right\|^2\right]+2\eta\mathbb{E}\left[\left\|\frac{1}{\mathcal{B}}\sum_{j\in I_{\mathcal{B}}}\left(\hat{\nabla} f_j(\tilde{x}^{s-1})-\hat{\nabla} f(\tilde{x}^{s-1})\right)\right\|^2\right]\\
&+2\eta\mathbb{E}\left\|\hat{\nabla} f(x_{t-1}^s)-\nabla f(x_{t-1}^s)\right\|^2 \qquad\text{(S25)}\\
\le&\frac{6\eta L^2 d}{b}\mathbb{E}\left[\left\|x_{t-1}^s-\tilde{x}^{s-1}\right\|^2\right]+2\frac{I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}}+2\eta\mathbb{E}\left\|\hat{\nabla} f(x_{t-1}^s)-\nabla f(x_{t-1}^s)\right\|^2\\
&+\frac{3\eta L^2d^2\mu^2}{b} \qquad\text{(S26)}\\
\le&\frac{6\eta L^2 d}{b}\mathbb{E}\left[\left\|x_{t-1}^s-\tilde{x}^{s-1}\right\|^2\right]+2\frac{I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}}
\end{aligned}
$$

$$+\eta\frac{L^2d^2\mu^2}{2}+\frac{3\eta L^2d^2\mu^2}{b} \tag{S27}$$

$$\leq\frac{6\eta L^2 d}{b}\mathbb{E}\left[\left\|x_{t-1}^s-\tilde{x}^{s-1}\right\|^2\right]+2\frac{I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}}+\eta\frac{7L^2d^2\mu^2}{2} \tag{S28}$$

where, the expectations are taking with respect to $I_b$ and $I_{\mathcal{B}}$ and random directions $\{u_i\}$ in (2). The inequality (S21) holds by the Jensen's inequality. (S22) and (S23) are based on $\mathbb{E}[\|x_1+x_2+\ldots+x_k\|^2]=\sum_{i=1}^k\mathbb{E}[\|x_i\|^2]$ if $x_1,x_2,\ldots,x_k$ are independent and of mean zero. Note that $I_b$ and $I_{\mathcal{B}}$ are also independent. (S24) uses the fact that $\mathbb{E}[\|x-\mathbb{E}[x]\|^2]\leq\mathbb{E}[\|x\|^2]$, for any random variable $x$. (S25) holds due to Lemma 16. (S26) is by Assumption 2 and (S27) is by Lemma 15. (S28) uses $b\geq 1$. The proof is now complete. □

In the sequel we frequently use the following inequality

$$\|x-z\|^2\leq(1+\frac{1}{\alpha})\|x-y\|^2+(1+\alpha)\|y-z\|^2,\forall\alpha>0 \tag{S29}$$

## Proof of Theorem 1

*Proof* Now, we apply Lemma 9 to prove Theorem 1. Let $x_t^s=\mathrm{Prox}_{\eta h}(x_{t-1}^s-\eta\hat{v}_{t-1}^s)$ and $\overline{x}_t^s=\mathrm{Prox}_{\eta h}(x_{t-1}^s-\eta\nabla f(x_{t-1}^s))$. By letting $x^+=x_t^s$, $x=x_{t-1}^s$, $v=\hat{v}_{t-1}^s$ and $z=\overline{x}_t^s$ in (S1), we have

$$\begin{aligned}F(x_t^s)\leq&F(\overline{x}_t^s)+\langle\nabla f(x_{t-1}^s)-\hat{v}_{t-1}^s,x_t^s-\overline{x}_t^s\rangle-\frac{1}{\eta}\langle x_t^s-x_{t-1}^s,x_t^s-\overline{x}_t^s\rangle\\&+\frac{L}{2}\|x_t^s-x_{t-1}^s\|^2+\frac{L}{2}\|\overline{x}_t^s-x_{t-1}^s\|^2.\end{aligned} \tag{S30}$$

Besides, by letting $x^+=\overline{x}_t^s$, $x=x_{t-1}^s$, $v=\nabla f(x_{t-1}^s)$ and $z=x=x_{t-1}^s$ in (S1), we have

$$\begin{aligned}F(\overline{x}_t^s)\leq&F(x_{t-1}^s)-\frac{1}{\eta}\langle\overline{x}_t^s-x_{t-1}^s,\overline{x}_t^s-x_{t-1}^s\rangle+\frac{L}{2}\|\overline{x}_t^s-x_{t-1}^s\|^2\\=&F(x_{t-1}^s)-(\frac{1}{\eta}-\frac{L}{2})\|\overline{x}_t^s-x_{t-1}^s\|^2.\end{aligned} \tag{S31}$$

Combining (S30) and (S31) we have

$$\begin{aligned}F(x_t^s)\leq&F(x_{t-1}^s)+\frac{L}{2}\|x_t^s-x_{t-1}^s\|^2-\left(\frac{1}{\eta}-L\right)\|\overline{x}_t^s-x_{t-1}^s\|^2+\langle\nabla f(x_{t-1}^s)-\hat{v}_{t-1}^s,x_t^s-\overline{x}_t^s\rangle\\&-\frac{1}{\eta}\langle x_t^s-x_{t-1}^s,x_t^s-\overline{x}_t^s\rangle\\=&F(x_{t-1}^s)+\frac{L}{2}\|x_t^s-x_{t-1}^s\|^2-\left(\frac{1}{\eta}-L\right)\|\overline{x}_t^s-x_{t-1}^s\|^2+\langle\nabla f(x_{t-1}^s)-\hat{v}_{t-1}^s,x_t^s-\overline{x}_t^s\rangle\\&-\frac{1}{2\eta}\left(\|x_t^s-x_{t-1}^s\|^2+\|x_t^s-\overline{x}_t^s\|^2-\|\overline{x}_t^s-x_{t-1}^s\|^2\right)\\=&F(x_{t-1}^s)-(\frac{1}{2\eta}-\frac{L}{2})\|x_t^s-x_{t-1}^s\|^2-\left(\frac{1}{2\eta}-L\right)\|\overline{x}_t^s-x_{t-1}^s\|^2+\langle\nabla f(x_{t-1}^s)-\hat{v}_{t-1}^s,x_t^s-\overline{x}_t^s\rangle\\&-\frac{1}{2\eta}\|x_t^s-\overline{x}_t^s\|^2\\\leq&F(x_{t-1}^s)-(\frac{1}{2\eta}-\frac{L}{2})\|x_t^s-x_{t-1}^s\|^2-\left(\frac{1}{2\eta}-L\right)\|\overline{x}_t^s-x_{t-1}^s\|^2+\langle\nabla f(x_{t-1}^s)-\hat{v}_{t-1}^s,x_t^s-\overline{x}_t^s\rangle\\&-\frac{1}{8\eta}\|x_t^s-x_{t-1}^s\|^2+\frac{1}{6\eta}\|\overline{x}_t^s-x_{t-1}^s\|^2\end{aligned} \tag{S32}$$

$$\begin{aligned}=&F(x_{t-1}^s)-(\frac{5}{8\eta}-\frac{L}{2})\|x_t^s-x_{t-1}^s\|^2-\left(\frac{1}{3\eta}-L\right)\|\overline{x}_t^s-x_{t-1}^s\|^2+\langle\nabla f(x_{t-1}^s)-\hat{v}_{t-1}^s,x_t^s-\overline{x}_t^s\rangle\\\leq&F(x_{t-1}^s)-(\frac{5}{8\eta}-\frac{L}{2})\|x_t^s-x_{t-1}^s\|^2-\left(\frac{1}{3\eta}-L\right)\|\overline{x}_t^s-x_{t-1}^s\|^2+\eta\|\nabla f(x_{t-1}^s)-\hat{v}_{t-1}^s\|^2\end{aligned} \tag{S33}$$

where the second inequality uses (S29) with $\alpha=3$ and the last inequality holds due to the Lemma 10.

Note that $x_t^s = \mathrm{Prox}_{\eta h}(x_{t-1}^s - \eta \hat{v}_{t-1}^s)$ is the iterated form in our algorithm. By taking the expectation with respect to all random variables in (S33) we obtain

$$\mathbb{E}[F(x_t^s)] \leq \mathbb{E}\left[F(x_{t-1}^s) - (\frac{5}{8\eta} - \frac{L}{2})\left\|x_t^s - x_{t-1}^s\right\|^2 - \left(\frac{1}{3\eta} - L\right)\left\|\overline{x}_t^s - x_{t-1}^s\right\|^2 + \eta\left\|\nabla f(x_{t-1}^s) - \hat{v}_{t-1}^s\right\|^2\right] \tag{S34}$$

In (S34), we further bound $\eta\left\|\nabla f(x_{t-1}^s) - \hat{v}_{t-1}^s\right\|^2$ using Lemma 11 to obtain

$$\begin{aligned}
&\mathbb{E}[F(x_t^s)] \\
\leq&\mathbb{E}\left[F(x_{t-1}^s) - (\frac{5}{8\eta} - \frac{L}{2})\left\|x_t^s - x_{t-1}^s\right\|^2 - \left(\frac{1}{3\eta} - L\right)\left\|\overline{x}_t^s - x_{t-1}^s\right\|^2\right] \\
&+ \frac{6\eta L^2}{b}\mathbb{E}\left\|x_{t-1}^s - \tilde{x}^{s-1}\right\|^2 + \frac{2I(\mathcal{B} < n)\eta\sigma^2}{\mathcal{B}} + \eta\frac{7L^2d^2\mu^2}{2} \\
=&\mathbb{E}\left[F(x_{t-1}^s) - (\frac{5}{8\eta} - \frac{L}{2})\left\|x_t^s - x_{t-1}^s\right\|^2 - \left(\frac{\eta}{3} - L\eta^2\right)\left\|g_\eta(x_{t-1}^s)\right\|^2\right] \\
&+ \frac{6\eta L^2}{b}\mathbb{E}\left\|x_{t-1}^s - \tilde{x}^{s-1}\right\|^2 + \frac{2I(\mathcal{B} < n)\eta\sigma^2}{\mathcal{B}} + \eta\frac{7L^2d^2\mu^2}{2} \qquad \text{(S35)}\\
\leq&\mathbb{E}\left[F(x_{t-1}^s) - \frac{1}{2t}(\frac{5}{8\eta} - \frac{L}{2})\left\|x_t^s - \tilde{x}^{s-1}\right\|^2 - \left(\frac{\eta}{3} - L\eta^2\right)\left\|g_\eta(x_{t-1}^s)\right\|^2\right] \\
&+ (\frac{6\eta L^2}{b} + \frac{1}{2t-1}(\frac{5}{8\eta} - \frac{L}{2}))\mathbb{E}\left\|x_{t-1}^s - \tilde{x}^{s-1}\right\|^2 + \frac{2I(\mathcal{B} < n)\eta\sigma^2}{\mathcal{B}} + \eta\frac{7L^2d^2\mu^2}{2} \qquad \text{(S36)}
\end{aligned}$$

where recalling $\overline{x}_t^s := \mathrm{Prox}_{\eta h}(x_{t-1}^s - \eta\nabla f(x_{t-1}^s))$, (S35) is based on the definition of gradient mapping $g_\eta(x_{t-1}^s)$. (S36) uses (S29) by choosing $\alpha = 2t - 1$.

Taking a telescopic sum for $t = 1, 2, \ldots, m$ in epoch $s$ from (S36) and recalling that $x_m^s = \tilde{x}^s$ and $x_0^s = \tilde{x}^{s-1}$, we obtain

$$\begin{aligned}
&\mathbb{E}[F(\tilde{x}^s)] \\
\leq&\mathbb{E}\left[F(\tilde{x}^{s-1}) - \sum_{t=1}^{m}\frac{1}{2t}(\frac{5}{8\eta} - \frac{L}{2})\left\|x_t^s - \tilde{x}^{s-1}\right\|^2 - \left(\frac{\eta}{3} - L\eta^2\right)\sum_{t=1}^{m}\left\|g_\eta(x_{t-1}^s)\right\|^2\right] \\
&+ \sum_{t=1}^{m}(\frac{6\eta L^2}{b} + \frac{1}{2t-1}(\frac{5}{8\eta} - \frac{L}{2}))\mathbb{E}\left\|x_{t-1}^s - \tilde{x}^{s-1}\right\|^2 \\
&+ \sum_{t=1}^{m}\frac{2I(\mathcal{B} < n)\eta\sigma^2}{\mathcal{B}} + \sum_{t=1}^{m}\eta\frac{7L^2d^2\mu^2}{2} \\
\leq&\mathbb{E}\left[F(\tilde{x}^{s-1}) - \sum_{t=1}^{m-1}\frac{1}{2t}(\frac{5}{8\eta} - \frac{L}{2})\left\|x_t^s - \tilde{x}^{s-1}\right\|^2 - \left(\frac{\eta}{3} - L\eta^2\right)\sum_{t=1}^{m}\left\|g_\eta(x_{t-1}^s)\right\|^2\right] \\
&+ \sum_{t=2}^{m}(\frac{6\eta L^2}{b} + \frac{1}{2t-1}(\frac{5}{8\eta} - \frac{L}{2}))\mathbb{E}\left\|x_{t-1}^s - \tilde{x}^{s-1}\right\|^2 \\
&+ \sum_{t=1}^{m}\frac{2I(\mathcal{B} < n)\eta\sigma^2}{\mathcal{B}} + \sum_{t=1}^{m}\eta\frac{7L^2d^2\mu^2}{2} \qquad \text{(S37)}\\
=&\mathbb{E}\left[F(\tilde{x}^{s-1}) - \left(\frac{\eta}{3} - L\eta^2\right)\sum_{t=1}^{m}\left\|g_\eta(x_{t-1}^s)\right\|^2\right] \\
&- \sum_{t=1}^{m-1}\left((\frac{1}{2t} - \frac{1}{2t+1})(\frac{5}{8\eta} - \frac{L}{2}) - \frac{6\eta L^2}{b})\right)\mathbb{E}\left\|x_t^s - \tilde{x}^{s-1}\right\|^2 \\
&+ \sum_{t=1}^{m}\frac{2I(\mathcal{B} < n)\eta\sigma^2}{\mathcal{B}} + \sum_{t=1}^{m}\eta\frac{7L^2d^2\mu^2}{2} \\
\leq&\mathbb{E}\left[F(\tilde{x}^{s-1}) - \left(\frac{\eta}{3} - L\eta^2\right)\sum_{t=1}^{m}\left\|g_\eta(x_{t-1}^s)\right\|^2\right]
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{t=1}^{m-1} \left( \frac{1}{6t^2} (\frac{5}{8\eta} - \frac{L}{2}) - \frac{6\eta L^2}{b} ) \right) \mathbb{E} \left\| x_t^s - \tilde{x}^{s-1} \right\|^2 \\
&+ \sum_{t=1}^{m} \frac{2I(\mathcal{B} < n)\eta\sigma^2}{\mathcal{B}} + \sum_{t=1}^{m} \eta \frac{7L^2d^2\mu^2}{2} \\
\leq & \mathbb{E} \left[ F(\tilde{x}^{s-1}) - \frac{\eta}{6} \sum_{t=1}^{m} \left\| g_\eta(x_{t-1}^s) \right\|^2 \right] + \sum_{t=1}^{m} \frac{2I(\mathcal{B} < n)\eta\sigma^2}{\mathcal{B}} + \sum_{t=1}^{m} \eta \frac{7L^2d^2\mu^2}{2}
\end{aligned} \tag{S38}
$$

where (S37) holds since norm is always non-negative and $x_0^s = \tilde{x}^{s-1}$. In (S38) we have used the fact that $(\frac{1}{6t^2}(\frac{5}{8\eta} - \frac{L}{2}) - \frac{6\eta L^2}{b}) \geq 0$ for all $1 \leq t \leq m$ and $\frac{\eta}{6} \leq \frac{\eta}{3} - L\eta^2$ since $\eta = \min\{\frac{1}{8L}, \frac{\sqrt{b}}{12mL}\}$. Telescoping the sum for $s = 1, 2, \ldots, S$ in (S38), we obtain

$$
\begin{aligned}
0 &\leq \mathbb{E}[F(\tilde{x}^S) - F(x^*)] \\
&\leq \mathbb{E} \left[ F(\tilde{x}^0) - F(x^*) - \sum_{s=1}^{S} \sum_{t=1}^{m} \frac{\eta}{6} \left\| g_\eta(x_{t-1}^s) \right\|^2 + \sum_{s=1}^{S} \sum_{t=1}^{m} (\frac{2I(\mathcal{B} < n)\eta\sigma^2}{\mathcal{B}} + \eta \frac{7L^2d^2\mu^2}{2}) \right]
\end{aligned}
$$

Thus, we have

$$
\mathbb{E}[\|g_\eta(\hat{x})\|^2] \leq \frac{6\,(F(x_0) - F(x^*))}{\eta S m} + \frac{I(\mathcal{B} < n)12\sigma^2}{\mathcal{B}} + 21L^2d^2\mu^2 \tag{S39}
$$

where (S39) holds since we choose $\hat{x}$ uniformly randomly from $\{x_{t-1}^s\}_{t\in[m],s\in[S]}$. □

## Proof of Corollary 2

*Proof* Using Theorem 1, we have $\eta = \min\{\frac{1}{8L}, \frac{\sqrt{b}}{12mL}\}$

$$
\begin{aligned}
\mathbb{E}[\|g_\eta(\hat{x})\|^2] \leq & \frac{6\,(F(x_0) - F(x^*))}{\eta S m} \\
& + \frac{I(\mathcal{B} < n)12\sigma^2}{\mathcal{B}} + 21L^2d^2\mu^2 = 3\epsilon
\end{aligned} \tag{S40}
$$

Now we derive the total number of iterations $T = Sm = \frac{6(F(x_0)-F(x^*))}{\epsilon\eta}$. Since $\mu \leq \frac{\sqrt{\epsilon}}{5dL}$, and for $\mathcal{B} = n$, the second term in the bound (S40) is 0, the proof is completed as the number of SZO call equals to $Sn + Smb = 6\,(F(x_0) - F(x^*))\,(\frac{n}{\epsilon\eta m} + \frac{b}{\epsilon\eta})$. If $\mathcal{B} < n$ the number of SZO calls equal to $d(S\mathcal{B}+Smb) = 6d\,(F(x_0) - F(x^*))\,(\frac{\mathcal{B}}{\epsilon\eta m} + \frac{b}{\epsilon\eta})$ by noting that $\frac{I(\mathcal{B}<n)12\sigma^2}{\mathcal{B}} \leq \epsilon$ due to $\mathcal{B} \geq 12\sigma^2/\epsilon$. The second part of corollary is obtained by setting $m = \sqrt{b}$ in the first part. □

**Corollary 13** We set the batch size $\mathcal{B} = \min\{12\sigma^2/\epsilon, n\}$ and the smoothing parameter $\mu \leq \frac{\sqrt{\epsilon}}{5dL}$. Suppose $\hat{x}$ returned by Algorithm 1 is an $\epsilon$-accurate solution for problem (1). Recalling that CoordSGE and RandSGE require $O(d)$ and $O(1)$ function queries respectively, the number of SZO calls is at most

$$
\begin{aligned}
(dS\mathcal{B} + Smb) &= 6\left(F(x_0) - F(x^*)\right)(\frac{\mathcal{B}d}{\epsilon\eta m} + \frac{b}{\epsilon\eta}) \\
&= O\left(\frac{\mathcal{B}d}{\epsilon\eta m} + \frac{b}{\epsilon\eta}\right)
\end{aligned} \tag{S41}
$$

and the number of PO calls is equal to $T = Sm = \frac{6(F(x_0)-F(x^*))}{\epsilon\eta} = O\left(\frac{1}{\epsilon\eta}\right)$. In particular, by setting $m = \sqrt{b}$ and $\eta = \frac{1}{12L\sqrt{d}}$, the number of ZO calls is at most

$$
\begin{aligned}
& 72L(F(x_0) - F(x^*)) \left( \frac{\mathcal{B}d\sqrt{d}}{\epsilon\sqrt{b}} + \frac{b\sqrt{d}}{\epsilon} \right) \\
&= O\left( s_n \frac{d\sqrt{d}}{\epsilon\sqrt{b}} + \frac{b\sqrt{d}}{\epsilon} \right)
\end{aligned} \tag{S42}
$$

where $s_n = \min\{n, \frac{1}{\epsilon}\}$ and the number of PO queries is equal to $T = Sm = S\sqrt{b} = \frac{72L\sqrt{d}(F(x_0)-F(x^*))}{\epsilon} = O\left(\frac{\sqrt{d}}{\epsilon}\right)$.

## Proof of Theorem 5

*Proof* We start by recalling inequality (S35) from the proof of Theorem 1, i.e.,

$$
\begin{aligned}
&\mathbb{E}[F(x_t^s)]\\
\leq&\mathbb{E}\left[F(x_{t-1}^s) - \frac{1}{2t}(\frac{5}{8\eta} - \frac{L}{2})\left\|x_t^s - \tilde{x}^{s-1}\right\|^2 - \left(\frac{\eta}{3} - L\eta^2\right)\left\|g_\eta(x_{t-1}^s)\right\|^2\right]\\
&+(\frac{2\eta L^2 d}{b} + \frac{1}{2t-1}(\frac{5}{8\eta} - \frac{L}{2}))\mathbb{E}\left\|x_{t-1}^s - \tilde{x}^{s-1}\right\|^2 + \frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}} + \eta\frac{L^2d^2\mu^2}{2}\\
\leq&\mathbb{E}\left[F(x_{t-1}^s) - \frac{1}{2t}(\frac{5}{8\eta} - \frac{L}{2})\left\|x_t^s - \tilde{x}^{s-1}\right\|^2 - \frac{\eta}{6}\left\|g_\eta(x_{t-1}^s)\right\|^2\right]\\
&+(\frac{6\eta L^2 d}{b} + \frac{1}{2t-1}(\frac{5}{8\eta} - \frac{L}{2}))\mathbb{E}\left\|x_{t-1}^s - \tilde{x}^{s-1}\right\|^2 + \frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}} + \eta\frac{7L^2d^2\mu^2}{2}
\end{aligned}
\tag{S43}
$$

where in (S43) inequality we applied $\eta L \leq \frac{1}{6}$. Moreover, substituting PL inequality, i.e.,

$$
\left\|g_\eta(x)\right\|^2 \geq 2\lambda(F(x) - F^*) \tag{S44}
$$

into (S43), we obtain

$$
\begin{aligned}
&\mathbb{E}[F(x_t^s)]\\
\leq&\mathbb{E}\left[F(x_{t-1}^s) - \frac{1}{2t}(\frac{5}{8\eta} - \frac{L}{2})\left\|x_t^s - \tilde{x}^{s-1}\right\|^2 - \lambda\frac{\eta}{3}(F(x_{t-1}^s) - F^*)\right]\\
&+(\frac{6\eta L^2 d}{b} + \frac{1}{2t-1}(\frac{5}{8\eta} - \frac{L}{2}))\mathbb{E}\left\|x_{t-1}^s - \tilde{x}^{s-1}\right\|^2 + \frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}} + \eta\frac{7L^2d^2\mu^2}{2}
\end{aligned}
\tag{S45}
$$

Thus, we have

$$
\begin{aligned}
&\mathbb{E}[F(x_t^s)]\\
\leq&\mathbb{E}\left[(1-\lambda\frac{\eta}{3})(F(x_{t-1}^s) - F^*) - \frac{1}{2t}(\frac{5}{8\eta} - \frac{L}{2})\left\|x_t^s - \tilde{x}^{s-1}\right\|^2\right]\\
&+(\frac{6\eta L^2 d}{b} + \frac{1}{2t-1}(\frac{5}{8\eta} - \frac{L}{2}))\mathbb{E}\left\|x_{t-1}^s - \tilde{x}^{s-1}\right\|^2 + \frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}} + \eta\frac{7L^2d^2\mu^2}{2}
\end{aligned}
\tag{S46}
$$

Let $\beta := 1 - \lambda\frac{\eta}{3}$ and $\Psi_t^s := \frac{\mathbb{E}[F(x_t^s) - F^*]}{\beta^t}$. Combining these definitions with (S46), we have

$$
\begin{aligned}
&\Psi_t^s\\
\leq&\Psi_{t-1}^s - \frac{1}{\beta^t}\mathbb{E}\left[\frac{1}{2t}(\frac{5}{8\eta} - \frac{L}{2})\left\|x_t^s - \tilde{x}^{s-1}\right\|^2 - (\frac{6\eta L^2 d}{b} + \frac{1}{2t-1}(\frac{5}{8\eta} - \frac{L}{2}))\left\|x_{t-1}^s - \tilde{x}^{s-1}\right\|^2\right]\\
&+\frac{1}{\beta^t}\frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}} + \frac{1}{\beta^t}\eta\frac{7L^2d^2\mu^2}{2}
\end{aligned}
\tag{S47}
$$

Similar to the proof of Theorem 1, summing (S47) for $t = 1, 2, \ldots, m$ in epoch $s$ and remembering that $x_m^s = \tilde{x}^s$ and $x_0^s = \tilde{x}^{s-1}$, we have

$$
\begin{aligned}
&\mathbb{E}[F(\tilde{x}^s) - F^*]\\
\leq&\ \beta^m\mathbb{E}\left[(F(\tilde{x}^{s-1}) - F^*)\right] + \beta^m\sum_{t=1}^m\frac{1}{\beta^t}\frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}} + \beta^m\sum_{t=1}^m\frac{1}{\beta^t}\eta\frac{7L^2d^2\mu^2}{2}\\
&-\beta^m\mathbb{E}\left[\sum_{t=1}^m\frac{1}{2t\beta^t}(\frac{5}{8\eta} - \frac{L}{2})\left\|x_t^s - \tilde{x}^{s-1}\right\|^2 - \sum_{t=1}^m\frac{1}{\beta^t}(\frac{6\eta L^2}{b} + \frac{1}{2t-1}(\frac{5}{8\eta} - \frac{L}{2}))\left\|x_{t-1}^s - \tilde{x}^{s-1}\right\|^2\right]\\
\leq&\ \beta^m\mathbb{E}\left[(F(\tilde{x}^{s-1}) - F^*)\right] + \frac{1-\beta^m}{1-\beta}\frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}} + \frac{1-\beta^m}{1-\beta}\frac{7\eta L^2d^2\mu^2}{2}
\end{aligned}
$$

$$-\beta^m\mathbb{E}\left[\sum_{t=1}^{m}\frac{1}{2t\beta^t}(\frac{5}{8\eta}-\frac{L}{2})\left\|x_t^s-\tilde{x}^{s-1}\right\|^2-\sum_{t=1}^{m}\frac{1}{\beta^t}(\frac{6\eta L^2}{b}+\frac{1}{2t-1}(\frac{5}{8\eta}-\frac{L}{2}))\left\|x_{t-1}^s-\tilde{x}^{s-1}\right\|^2\right]$$

$$\begin{aligned}
&\leq \beta^m\mathbb{E}\left[(F(\tilde{x}^{s-1})-F^*)\right]+\frac{1-\beta^m}{1-\beta}\frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}}+\frac{1-\beta^m}{1-\beta}\frac{7\eta L^2d^2\mu^2}{2}\\
&-\beta^m\mathbb{E}\left[\sum_{t=1}^{m-1}\frac{1}{2t\beta^t}(\frac{5}{8\eta}-\frac{L}{2})\left\|x_t^s-\tilde{x}^{s-1}\right\|^2\right]\\
&+\beta^m\mathbb{E}\left[\sum_{t=2}^{m}\frac{1}{\beta^t}(\frac{6\eta L^2}{b}+\frac{1}{2t-1}(\frac{5}{8\eta}-\frac{L}{2}))\left\|x_{t-1}^s-\tilde{x}^{s-1}\right\|^2\right] \qquad \text{(S48)}\\
&\leq \beta^m\mathbb{E}\left[(F(\tilde{x}^{s-1})-F^*)\right]+\frac{1-\beta^m}{1-\beta}\frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}}+\frac{1-\beta^m}{1-\beta}\frac{7\eta L^2d^2\mu^2}{2}\\
&-\beta^m\mathbb{E}\left[\sum_{t=1}^{m-1}\frac{1}{\beta^{t+1}}\left((\frac{\beta}{2t}-\frac{1}{2t+1})(\frac{5}{8\eta}-\frac{L}{2})-\frac{6\eta L^2}{b})\right)\left\|x_t^s-\tilde{x}^{s-1}\right\|^2\right]\\
&\leq \beta^m\mathbb{E}\left[(F(\tilde{x}^{s-1})-F^*)\right]+\frac{1-\beta^m}{1-\beta}\frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}}+\frac{1-\beta^m}{1-\beta}\frac{7\eta L^2d^2\mu^2}{2} \qquad \text{(S49)}
\end{aligned}$$

where (S48) since $\|.\|^2$ always is non-negative and $x_0^s=\tilde{x}^{s-1}$. (S49) holds since it is sufficient to show $(\frac{\beta}{2t}-\frac{1}{2t+1})(\frac{5}{8\eta}-\frac{L}{2})-\frac{6\eta L^2}{b}\geq 0$, for all $t=1,2,\ldots,m$. It is easy to see that this inequality is valid due to $\eta\leq\min\{\frac{1}{8L},\frac{\sqrt{\gamma b}}{12mL}\}$, where $\gamma=1-\frac{2\lambda\eta}{3}m-\frac{\lambda\eta}{3}>0$. Similarly, let $\tilde{\beta}=\beta^m$ and $\tilde{\Psi}^s=\frac{\mathbb{E}[F(\tilde{x}^s)-F^*]}{\tilde{\beta}^s}$. Substituting these definitions into (S49), we have

$$\tilde{\Psi}^s\leq\tilde{\Psi}^{s-1}+\frac{1}{\tilde{\beta}^s}\frac{1-\tilde{\beta}}{1-\beta}\frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}}+\frac{1}{\tilde{\beta}^s}\frac{1-\tilde{\beta}}{1-\beta}\frac{7\eta L^2d^2\mu^2}{2} \qquad \text{(S50)}$$

Taking a telescopic sum from (S50) for all epochs $1\leq s\leq S$, we obtain

$$\begin{aligned}
\mathbb{E}[F(\tilde{x}^S)-F^*]&\leq\tilde{\beta}^S\mathbb{E}[F(\tilde{x}^0)-F^*]+\tilde{\beta}^S\sum_{s=1}^{S}\frac{1}{\tilde{\beta}^s}\frac{1-\tilde{\beta}}{1-\beta}\frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}}+\tilde{\beta}^S\sum_{s=1}^{S}\frac{1}{\tilde{\beta}^s}\frac{1-\tilde{\beta}}{1-\beta}\frac{7\eta L^2d^2\mu^2}{2}\\
&=\beta^{Sm}\mathbb{E}[F(\tilde{x}^0)-F^*]+\frac{1-\tilde{\beta}^S}{1-\tilde{\beta}}\frac{1-\tilde{\beta}}{1-\beta}\frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}}+\frac{1-\tilde{\beta}^S}{1-\tilde{\beta}}\frac{1-\tilde{\beta}}{1-\beta}\frac{7\eta L^2d^2\mu^2}{2}\\
&\leq\beta^{Sm}\mathbb{E}[F(\tilde{x}^0)-F^*]+\frac{1}{1-\beta}\frac{2I(\mathcal{B}<n)\eta\sigma^2}{\mathcal{B}}+\frac{1}{1-\beta}\frac{7\eta L^2d^2\mu^2}{2}\\
&=\left(1-\frac{\lambda\eta}{3}\right)^{Sm}\mathbb{E}[F(\tilde{x}^0)-F^*]+\frac{6I(\mathcal{B}<n)\sigma^2}{\lambda\mathcal{B}}+\frac{21L^2d^2\mu^2}{2\lambda} \qquad \text{(S51)}
\end{aligned}$$

where in (S51) we recall that $\beta=1-\frac{\lambda\eta}{3}$. □

## Proof of Corollary 6

*Proof* From Theorem 5, we have

$$\begin{aligned}
\mathbb{E}[F(\tilde{x}^S)-F^*]\leq&\left(1-\frac{\lambda\eta}{3}\right)^{Sm}\mathbb{E}[F(\tilde{x}^0)-F^*]\\
&+\frac{6I(\mathcal{B}<n)\sigma^2}{\lambda\mathcal{B}}+\frac{21L^2d^2\mu^2}{2\lambda}=3\epsilon
\end{aligned}$$

which gives the total number of iterations $T=Sm=O(\frac{1}{\lambda\eta}\log\frac{1}{\epsilon})$ and is equal to the number of PO calls. Since $\mu\leq\frac{\sqrt{\lambda\epsilon}}{4Ld}$, we have $\frac{21L^2d^2\mu^2}{2\lambda}\leq\epsilon$. The number of SZO calls is equal to $d(S\mathcal{B}+Smb)=O(\frac{\mathcal{B}d}{\lambda\eta m}\log\frac{1}{\epsilon}+\frac{bd}{\lambda\eta}\log\frac{1}{\epsilon})$. Note that if $\mathcal{B}<n$ then $\frac{6I(\mathcal{B}<n)\sigma^2}{\lambda\mathcal{B}}\leq\epsilon$ since $\mathcal{B}\geq 6\sigma^2/\lambda\epsilon$. With $m=\sqrt{b}$ and $\eta=\frac{\sqrt{\gamma}}{12L}$, the number of PO queries to $T=Sm=O(\frac{1}{\lambda\sqrt{\gamma}}\log\frac{1}{\epsilon})$ and the number of SZO calls is equal to $d(S\mathcal{B}+Smb)=O(\frac{\mathcal{B}d}{\lambda\sqrt{\gamma}m}\log\frac{1}{\epsilon}+\frac{bd}{\lambda\sqrt{\gamma}}\log\frac{1}{\epsilon})$. □

## Corollary 14

**Corollary 14** Suppose the final iteration point $\tilde{x}^S$ in Algorithm 1 satisfies $\mathbb{E}[F(\tilde{x}^S) - F^*] \le \epsilon$ under PL condition. Under Assumptions 1 and 2, we set batch size $\mathcal{B} = \min\{\frac{6\sigma^2}{\lambda\epsilon}, n\}$ and the smoothing parameter $\mu \le \frac{\sqrt{\lambda\epsilon}}{4Ld}$. The number of SZO calls is bounded by

$$(S\mathcal{B}d + Smb) = O(\frac{s_n d}{\lambda\eta m}\log\frac{1}{\epsilon} + \frac{b}{\lambda\eta}\log\frac{1}{\epsilon})$$

where $s_n = \min\{n, \frac{1}{\lambda\epsilon}\}$. The number of PO calls is equal to the total number of iterations $T$ which is given by

$$T = Sm = O(\frac{1}{\lambda\eta}\log\frac{1}{\epsilon})$$

In particular, given the setting $m = \sqrt{b}$ and $\eta = \frac{\sqrt{\gamma}}{12L\sqrt{d}}$, the number of SZO calls simplifies to $(S\mathcal{B}d + Smb) = O(\frac{\mathcal{B}d\sqrt{d}}{\lambda\sqrt{\gamma}m}\log\frac{1}{\epsilon} + \frac{b\sqrt{d}}{\lambda\sqrt{\gamma}}\log\frac{1}{\epsilon})$.

*Proof* From Theorem 5, we have

$$\begin{aligned}\mathbb{E}[F(\tilde{x}^S) - F^*] \le& \left(1 - \frac{\lambda\eta}{3}\right)^{Sm}\mathbb{E}[F(\tilde{x}^0) - F^*] \\ &+ \frac{6I(\mathcal{B} < n)\sigma^2}{\lambda\mathcal{B}} + \frac{21L^2d^2\mu^2}{2\lambda} = 3\epsilon\end{aligned} \tag{S52}$$

which gives the total number of iterations $T = Sm = O(\frac{1}{\lambda\eta}\log\frac{1}{\epsilon})$ and equals to the number of PO calls. Since $\mu \le \frac{\sqrt{\lambda\epsilon}}{4Ld}$, we have $\frac{21L^2d^2\mu^2}{2\lambda} \le \epsilon$. The number of SZO calls equals to $(S\mathcal{B}d + Smb) = O(\frac{\mathcal{B}d}{\lambda\eta m}\log\frac{1}{\epsilon} + \frac{b}{\lambda\eta}\log\frac{1}{\epsilon})$. Note that if $\mathcal{B} < n$ then $\frac{6I(\mathcal{B}<n)\sigma^2}{\lambda\mathcal{B}} \le \epsilon$ since $\mathcal{B} \ge 6\sigma^2/\lambda\epsilon$. With $m = \sqrt{b}$ and $\eta = \frac{\sqrt{\gamma}}{12L\sqrt{d}}$, the number of PO calls equals to $T = Sm = O(\frac{\sqrt{d}}{\lambda\sqrt{\gamma}}\log\frac{1}{\epsilon})$ and the number of SZO calls equals to $(S\mathcal{B}d + Smb) = O(\frac{\mathcal{B}d\sqrt{d}}{\lambda\sqrt{\gamma}m}\log\frac{1}{\epsilon} + \frac{b\sqrt{d}}{\lambda\sqrt{\gamma}}\log\frac{1}{\epsilon})$. □

Now we give some useful properties of CoordSGE and RandSGE, respectively.

**Lemma 15** (Liu et al (2018b)) Suppose that the function $f(x)$ is $L$-smooth. Let $\hat{\nabla} f(x)$ denote the estimated gradient defined by CoordSGE. Define $f_\mu = \mathbb{E}_{u\sim U[-\mu,\mu]} f(x + ue_j)$, where $U[-\mu, \mu]$ denotes the uniform distribution on the interval $[-\mu, \mu]$. Then for any $x \in \mathbb{R}^d$ we have

1. $f_\mu$ is $L$-smooth, and $\hat{\nabla} f(x) = \sum_{j=1}^d \frac{\partial f_\mu(x)}{\partial x_j} e_j$.
2. $|f_\mu(x) - f(x)| \le \frac{L\mu^2}{2}$ and $|\frac{\partial f_\mu(x)}{\partial x_j}| \le \frac{L\mu^2}{2}$.
3. $\left\|\hat{\nabla} f(x) - \nabla f(x)\right\|^2 \le \frac{L^2d^2\mu^2}{4}$.

**Lemma 16** Assume that the function $f(x)$ is $L$-smooth. Let $\hat{\nabla}_r f(x)$ denote the estimated gradient defined by RandSGE. Define $f_\mu = \mathbb{E}_{u\sim U_S}[f(x + \mu u)]$, where $U$ is uniform distribution over a $d$-dimensional unit ball $S$. Then, we have

1. For any $x \in \mathbb{R}^d$, $\nabla f_\mu(x) = \mathbb{E}_u[\hat{\nabla}_r f(x, u)]$.
2. $|f_\mu(x) - f(x)| \le \frac{\mu^2 L}{2}$ and $\|f_\mu(x) - f(x)\| \le \frac{\mu L d}{2}$ for any $x \in \mathbb{R}^d$.
3. $\mathbb{E}_u \left\|\hat{\nabla}_r f(x, u) - \hat{\nabla}_r f(y, u)\right\|^2 \le 3dL^2\|x - y\|^2 + \frac{3L^2d^2\mu^2}{2}$.

*Proof* The proof of items 1 and 2 can be found in Gao et al (2018). Item 3 is due to Lemma 5 in Ji et al (2019). □