The puzzle of eggs and floors

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1 Introduction

This document is prepared to answer a puzzle posed by R. Nandakumar in his blog `http://nandakumarr.blogspot.com/2005/04/glass-bulbs-puzzle.html`. I had tackled this problem a while back, when my friend Sharad Kedia asked this question in Mentor Graphics India puzzle group. My solution is given below.

Sharad used *eggs* while Nandakumar used *bulbs*. That is the only difference.

2 Question

You have two eggs. you need to figure out how high an egg can fall from a 100 story building before it breaks. The eggs might break from the first floor, or might even survive a drop from the 100th floor – you have no a priori information. What is the largest number of egg drops you would ever have to do to find the right floor? (i.e. what's the most efficient way to drop the eggs and determine an answer?) You are allowed to break both eggs, as long as you identify the correct floor afterwards.

After you've solved the above problem, generalize. Define the *break floor* as the lowest floor in a building from which an egg would break if dropped. Given an n story building and a supply of d eggs, find the strategy which minimizes (in the worst case) the number of experimental drops required to determine the break floor.

3 Notations

For discussing the solutions, the following notations are used in this chapter.

$\binom{n}{r}$ = Binomial coefficient ${}_nC_r = \frac{n!}{r!(n-r)!}$

$F_{[d,e]}$	=	The highest floor that can be tested with d drops and e eggs.
$D_{[f,e]}$	=	The minimum number of drops needed to test f floors with e eggs.
$E_{[f,d]}$	=	The minimum number of eggs needed to test $f$floors in d drops.
d	=	Drop counter
e	=	Egg counter.
N_f	=	A given number of floors (= 100 in the question)
N_e	=	A given number of eggs (= 2 in the question)
N_d	=	A given number of drops

4 Particular solution for 100 floors and two eggs

4.1 Initial solution

If there is only one egg, the solution is quite obvious. There is nothing better than trying at every floor 1, 2, 3, ..., N_f, requiring N_f drops. When there are two eggs, and we finally find the minimum number of drops N_d needed for N_f floors, we should span them in such a way that every sub-case should take N_d drops in the worst case.

So, the initial drop should be from floor N_d, so that if it breaks, the right floor can be found by trying floors 1~~xxx~~ N_d.-1

Now we have $N_d - 1$ drops left, so the next drop should be at $N_d + N_d - 1 = 2.N_d - 1$. This can be continued like this:

Let $f_{[d,e,N_d]}$ denote the floor that need to be tried at the d^{th} drop when there are e eggs and we know the correct number of drops N_d.

$$
\begin{aligned}
f_{[1,2,N_d]} &= N_d \\
f_{[2,2,N_d]} &= F_{1,2} + N_d - 1 = N_d + N_d - 1 = 2.N_d - 1 \\
f_{[3,2,N_d]} &= F_{2,2} + N_d - 2 = 3.N_d - 3 \\
f_{[4,2,N_d]} &= F_{3,2} + N_d - 3 = 4.N_d - 6 \\
&\vdots \\
F_{[d,2,N_d]} &= d.N_d - \frac{d(d-1)}{2} \qquad (1)
\end{aligned}
$$

Now, to solve the particular problem, we need to find d such that $F_{[d,2,d]} \geq 100$. So,

$$d^2 - \frac{d(d-1)}{2} \geq 100$$

$$\frac{d^2}{2} + \frac{d}{2} \geq 100$$
$$d^2 + d - 200 \geq 0$$

Solving for equality, we get

$$d = \frac{-1 \pm \sqrt{1^2 + 4.200}}{2} = 13.65 \text{ or } -14.65$$

So the least value of d is 14[1]. Now the values are

$$\begin{aligned}
f_{[1,2,14]} &= 14 \\
f_{[2,2,14]} &= 27 \\
f_{[3,2,14]} &= 39 \\
f_{[4,2,14]} &= 50 \\
f_{[5,2,14]} &= 60 \\
f_{[6,2,14]} &= 69 \\
f_{[7,2,14]} &= 77 \\
f_{[8,2,14]} &= 84 \\
f_{[9,2,14]} &= 90 \\
f_{[10,2,14]} &= 95 \\
f_{[11,2,14]} &= 99
\end{aligned}$$

That is only 11 drops, and the 12th drop can be at 100.

So, do we need only 12 drops? Not really. If the break floor is one 13, 26, 38 etc., still we need 14 drops. An attempt to reduce the number of drops to 13 fails, because we will be stuck at 91 after 13, 25, 36, 46, 55, 63, 70, 76, 81, 85, 88, 90, 91, and we cannot determine where it breaks between 92 and 100 if it hasn't broken yet[2].

4.2 A simpler solution

Somewhere in the middle of the solution given above, I realized we need to start from the top for an efficient solution. A second thought revealed that we can arrive at a solution by doing only that and without doing any other complicated calculations. Here is that approach:

[1] It is unfortunate this doesn't solve to an integer. Because of this, we have many solutions. If the above quadratic equation has only one positive root, we would have a unique solution. Perhaps it is intentional, to tempt the solver to make a wrong conclusion that only 12 drops are necessary. Please read further.

[2] May be this is another trap, as I mentioned. With 14 drops and two eggs, we can test upto 105 floors. See **section 5.2**.

Let us abbreviate $f_{[d,2,N_d]}$ as $\mathcal{F}_d$. The second egg is reserved for the 100^{th} floor, or $\mathcal{F}_{N_d} = 100$. In order to get this efficiently, we need $\mathcal{F}_{N_d-1} = 99$, for the previous drop, $\mathcal{F}_{N_d-2} = 97$, (so that the remaining 2 drops could be used at 99 and 98/100), $\mathcal{F}_{N_d-3} = 94$ (with 97, 95, 96 *or* 97, 99, 98 *or* 97, 99, 100), $\mathcal{F}_{N_d-4} = 90$ etc. Counting like this we get the drop floors from the top are

$$100, 99, 97, 94, 90, 85, 79, 72, 64, 55, 45, 34, 22, 9$$

and this is *a* solution, and it takes 14 drops.

5 General Solution

5.1 General solution with 2 eggs (expression for $F_{[d,2]}$)

The first attempt is to fix $N_e = 2$, and then computing the maximum $F_{[d,2]}$ for any given d.

The first drop should be at d, the second at $2.d - 1$, the third at $3.d - 3$ etc., as explained in section 4.1, so

$$F_{[d,2]} = d.d - \frac{d.(d-1)}{2} \tag{2}$$

or

$$F_{[d,2]} = d^2 - \frac{d.(d-1)}{2} = \frac{d.(d+1)}{2} \tag{3}$$

so that

$$\begin{aligned} F_{[1,2]} &= 1 \\ F_{[2,2]} &= 3 \\ F_{[3,2]} &= 6 \\ F_{[4,2]} &= 10 \end{aligned}$$

etc.

5.2 A recurrance relation for $F_{[d,e]}$

The next attempt is to generalize for any N_e.

Now, let us say we know $F_{[d,e]}$ for some d and e. Now, we might have broken all the e eggs by now. What if we have got another egg and another drop? We can try that egg first, so that after it is broken, the resulting problem is finding $F_{[d,e]}$.

This means, initially we have $(d+1)$ drops and $(e+1)$ eggs. We will go to the $(F_{[d,e]}+1)^{th}$ floor first and try that egg, so that

1. if it breaks, we have e eggs and d drops to tackle $F_{[d,e]}$ floors, which we know we can.
2. If it doesn't break, we have $(e+1)$ eggs and d drops left, with which we can cover another $F_{[d,e+1]}$ floors.

So, we get the recurrance relation

$$F_{[d+1,e+1]} = F_{[d,e]} + F_{[d,e+1]} + 1 \tag{4}$$

or

$$F_{[d,e]} = F_{[d-1,e-1]} + F_{[d-1,e]} + 1 \tag{5}$$

with the obvious specializations

$$F_{[1,1]} = 1 \tag{6}$$
$$F_{[d,1]} = d \tag{7}$$

Also, $F_{[d,e]} = f[d,d]$ if $d < e$.

Here are the values till d=16 and e=5.

	e=1	e=2	e=3	e=4	e=5
d = 1	1				
d = 2	2	3			
d = 3	3	6	7		
d = 4	4	10	14	15	
d = 5	5	15	25	30	31
d = 6	6	21	41	56	62
d = 7	7	28	63	98	119
d = 8	8	36	92	162	218
d = 9	9	45	129	255	381
d = 10	10	55	175	385	637
d = 11	11	66	231	561	1023
d = 12	12	78	298	793	1585
d = 13	13	91	377	1092	2379
d = 14	14	105	469	1470	3472
d = 15	15	120	575	1940	4943
d = 16	16	136	696	2516	6884

6 Finding general expressions

In this section, we are trying to arrive at formulae for $F_{[d,e]}$, $D_{[f,e]}$, $E_{[f,d]}$ etc.

6.1 A particular case with 56 floors and 4 eggs

Let us take a particular case big enough to analyze. We have 56 floors and 4 eggs. How will we solve it?

We know we need 6 drops. So we proceed as given in figure 1 on the following page

6.2 Finding general expression for $F_{[d,e]}$

By observing the case with 6 drops and 4 eggs (see figure 1 on the next page), we can infer that, in the optimal case, each break floor corresponds to a definite unique path. For example, if the break floor is, say, 37, the first egg will survive 26 and break at 41, the second egg will survive 33 and break at 37, the third egg will survive 35 and 36, giving the break floor 37. It is already clear that a particular path and result set correspond to a particular floor. So, there is a one-to-one correspondence between the break floor and the result set in a path.

So, the problem is reduced to find how many unique paths are there. It is less than 2^{N_e}, because some cases do not require N_e drops. For example, if the break floor is 1, we need only 4 drops in the example, at 26, 11, 4 and 1.

The maximum number of floors that can be tested using d drops and e eggs is given by

$$F_{[d,e]} = \sum_{i=1}^{e} \binom{d}{i} \tag{8}$$

Proof by Induction: This proof uses the well-known identity[3]

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \tag{9}$$

Let 8 is true for some d and e. Now, let us calculate $F_{[d+1,e]}$.

[3]This can be proved easily by observing the Pascal's triangle. See [1].

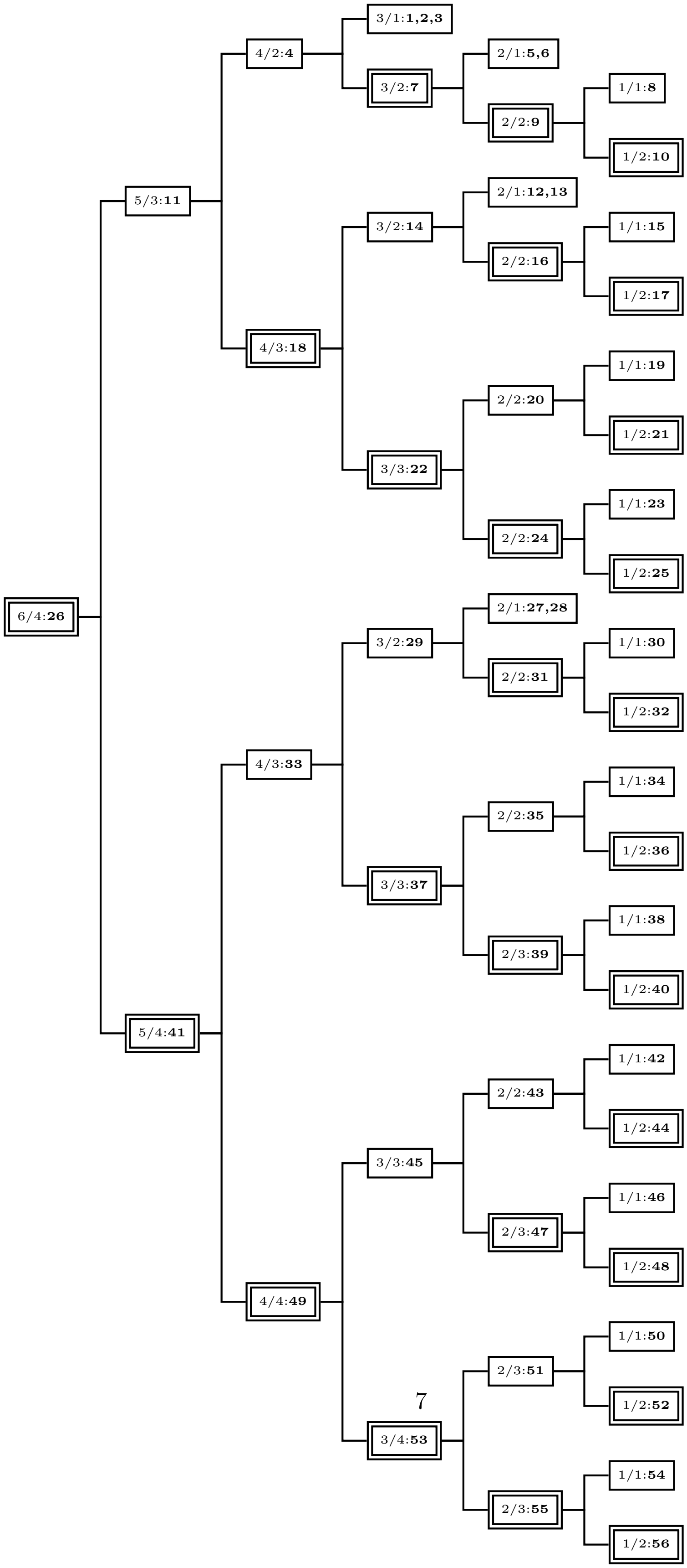

Figure 1: Particular case with 6 drops and 4 eggs

$$
\begin{aligned}
F_{[d+1,e]} &= F_{[d,e-1]} + F_{[d,e]} + 1 \\
&= \sum_{i=1}^{e-1} \binom{d}{i} + \sum_{i=1}^{e} \binom{d}{i} + 1 \\
&= \binom{d}{1} + \binom{d}{2} + \ldots + \binom{d}{e-1} + \binom{d}{1} + \binom{d}{2} + \ldots + \binom{d}{e} + 1 \\
&= \binom{d}{1} + \binom{d+1}{2} + \binom{d+1}{3} + \ldots + \binom{d+1}{e} + 1 \\
&= \sum_{i=1}^{e} \binom{d+1}{i}
\end{aligned} \tag{10}
$$

because

$$\binom{d}{1} + 1 = d + 1 = \binom{d+1}{1}$$

Similarly, we can find $F_{[d,e+1]}$ also.

$$
\begin{aligned}
F_{[d,e+1]} &= F_{[d-1,e]} + F_{[d-1,e+1]} + 1 \\
&= \sum_{i=1}^{e} \binom{d-1}{i} + \sum_{i=1}^{e+1} \binom{d-1}{i} + 1 \\
&= \binom{d-1}{1} + \binom{d-1}{2} + \ldots + \binom{d-1}{e} \\
&+ \binom{d-1}{1} + \binom{d-1}{2} + \ldots + \binom{d-1}{e+1} + 1 \\
&= \binom{d-1}{1} + \binom{d}{2} + \binom{d}{3} + \cdots + \binom{d}{e+1} + 1 \\
&= \sum_{i=1}^{e+1} \binom{d}{i}
\end{aligned} \tag{11}
$$

Since eq. 8 on page 6 is true for d=1, e=1, the validity of the expression is proved by induction.

Proof by derivation: Consider the example worked out, with 56 floors, 4 eggs and 6 drops. Let us consider in which drops the eggs break in each case. Table 1 on the next page lists the number of eggs broken (0 or 1) at each drop corresponding to each floor.

Floor	**Drop #**					
	1	2	3	4	5	6
1	1	1	1	1		
2	1	1	1	0	1	
3	1	1	1	0	0	1
4	1	1	1	0	0	0
5	1	1	0	1	1	
..	..	..	..	..	..	..
..	..	..	..	..	..	..
54	0	0	0	0	1	1
55	0	0	0	0	1	0
56	0	0	0	0	0	1
> 56	0	0	0	0	0	0

Table 1: Number of eggs broken in each drop

We can observe the following from this table:

1. No eggs will be broken in only one case - when the break floor is > 56. We need not consider this case.
2. Exactly one egg will be broken in six cases - (1,0,0,0,0,0) $\Rightarrow$ 25, (0,1,0,0,0,0) $\Rightarrow$ 40, (0,0,1,0,0,0) $\Rightarrow$ 48, (0,0,0,1,0,0) $\Rightarrow$ 52, (0,0,0,0,1,0) $\Rightarrow$ 55 and (0,0,0,0,0,1) $\Rightarrow$ 56. That is in $\binom{6}{1}$ cases.
3. Exactly two eggs will be broken in $\binom{6}{2} = 15$ cases.
4. Exactly three eggs will be broken in $\binom{6}{3} = 20$ cases.
5. Exactly four eggs will be broken in $\binom{6}{4} = 15$ cases.

So, the total number of cases is $6 + 15 + 20 + 15 = 56$.

This can be generalized with d drops and e eggs. In d drops, 1 egg will be broken in $\binom{d}{1}$ floors, two eggs will be broken in $\binom{d}{2}$ floors $\ldots e$ eggs will be broken in $\binom{d}{e}$, which will cover all the floors considered. So,

$$F_{[d,e]} = \sum_{i=1}^{e} \binom{d}{i}$$

which is (8).

6.3 Finding general expression for $D_{[f,e]}$

The task here is, given the number of floors f and the number of eggs e, find the minimum number of drops needed to find the break floor. The original puzzle is in this form where $f = 100$ and $e = 2$.

The best way to tackle this problem is to use eq. 8 on page 6 for each $d (d \geq e)$, and find where $F_{[d,e]}$ equals or exceeds f.

I could not find a straight formula for $D_{[f,e]}$ yet.

6.4 Finding general expression for $E_{[f,d]}$

Not even tried it so far.

References

[1] R.L. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics, 2nd Ed.* Addison-Wesley, 1994.