

## Lambda Calculus Homework 1

Due one week from time assigned

\*\*Deadline is FIRM as sample solutions will be available afterwards\*\*

Write your solutions clearly by hand and submit scans (in pdf) or clear photos of your solutions, or submit hard copy at end of class.

0. Identify the free and bound variables in the following term:

$(\lambda u. (\lambda x. ([y (\lambda y. x y) u] y)))$

↑ bound      bound      free

Remember: whether a term is free or bound is always relative to a given term. What are the free and bound variables for just the term inside the []'s?

$[y (\lambda y. x y) u y]$

↑ free      bound      free

1. Determine if the following pairs of terms are alpha-equivalent.

a.  $\lambda x. \lambda y. (x y x)$  and  $\lambda x. \lambda y. (y x y)$

NO bc they are structurally different. [i.e. "ab" ≠ "ba"]

b.  $\lambda x. \lambda y. x (\lambda y. x) y$  and  $\lambda a. \lambda b. a (\lambda u. a) b$

$\lambda x. M \Rightarrow \lambda a. M[a/x]$

$\lambda y. x (\lambda y. x) y$   
 $\lambda b. a (\lambda u. a) b$

$\lambda y. z \Rightarrow \lambda b. z[b/y]$

Yes, alpha-equivalent.

For the rest of the problems, find the beta-normal forms of the following terms. Remember: you can rename bound variables to avoid possible clash, but never free variables. Also, application associates to the left:  $a b c$  is the same as  $(a b) c$ , but not the same as  $a (b c)$ . Also, do not confuse  $\lambda x. x y$  with  $(\lambda x. x) y$ : application has precedence over abstraction.

2.  $(\lambda x. (\lambda y. (y x)) u (\lambda x. x))$  let  $M[u/x]$

$\Rightarrow_{\beta} \lambda y. (y x) (\lambda x. x)$  let  $z [\lambda x. x / y]$

$\Rightarrow_{\beta} \lambda x. x x$

3. given S, K, I as they are in the handout, find the normal forms of KII, SIK, SK(KI), and S(KI). Application associates to the left, and the order is important. Also remember that application binds tighter than lambda-abstraction, so  $(\lambda x.x y)$  is NOT the same as  $(\lambda x.x) y$ .

$$\begin{aligned} (S I) K &\Rightarrow_{\beta} (\lambda x \lambda y \lambda z. x z (y z)) I K \\ &\quad (\lambda x \lambda z. x z (z)) I K \\ &\quad (\lambda x. x z) I K \\ &\quad (I z) K \\ &\quad (\lambda u. u z) K \\ &\quad (\lambda u. u z) (\lambda a \lambda b. a) \end{aligned}$$

4. given  $A = \lambda m. \lambda n. \lambda f. \lambda x. m f (n f x)$ ,  
 $T = \lambda m. \lambda n. \lambda f. \lambda x. m (n f) x$   
 $Zero = \lambda f. \lambda x. x$   
 $One = \lambda f. \lambda x. (f x)$   
 $Two = \lambda f. \lambda x. f (f x)$

Ⓐ Find the normal form of  $(A \text{ Two } One)$ .

Ⓑ Then find the normal form of  $(T \text{ Zero } Two)$

Ⓒ What do A and T stand for?

$$\begin{aligned} \textcircled{A} (\lambda m. \lambda n. \lambda f. \lambda x. m f (n f x)) \text{ Two } One \\ ((\lambda n. \lambda x. \text{Two}) (n f x)) One &\Rightarrow_{\beta} \text{Two } (One) \end{aligned}$$

$$\begin{aligned} \textcircled{B} (\lambda m. \lambda n. \lambda f. \lambda x. m (n f) x) \text{ Zero } Two \\ (\lambda x. \lambda n. \lambda f. \text{Zero}) (n f) x \\ \text{Zero } (x) \text{ Zero } Two \end{aligned}$$

Ⓓ  $A = \text{Arithmetic Plus}$   
 $T = \text{Arithmetic Times}$

$$\begin{aligned} (K I) I &\Rightarrow_{\beta} ((\lambda x. \lambda y. x) I) I \\ &\Rightarrow_{\beta} \lambda y. I \Rightarrow_{\beta} \lambda u. u \\ SK(KI) &\Rightarrow (\lambda x \lambda y \lambda z. x z (y z)) K \\ &\Rightarrow_{\beta} (\lambda y \lambda z. K z (y z)) (KI) \\ &\quad \downarrow_{\beta} \\ &\quad ((\lambda a \lambda b. a) I) \\ &\quad \lambda b. I \Rightarrow_{\beta} \lambda b \lambda u. u \\ &(\lambda y \lambda z. K z (y z)) (\lambda b \lambda u. u) \\ &\Rightarrow_{\beta} \lambda y. K (\lambda b \lambda u. u) (y z) \\ &\Rightarrow_{\beta} K (y z) \Rightarrow_{\beta} \lambda e \lambda f. e (y z) \\ &\quad \Rightarrow_{\beta} \lambda f. y z \end{aligned}$$

$$\begin{aligned} S(KI) &\Rightarrow_{\beta} (\lambda x \lambda y \lambda z. x z (y z)) (KI) \\ &\Rightarrow_{\beta} \lambda y \lambda z. (KI) z (y z) \\ &\Rightarrow_{\beta} \lambda y. (KI) (y z) \Rightarrow_{\beta} KI \\ &\Rightarrow_{\beta} (\lambda a \lambda b. a) I \Rightarrow_{\beta} \lambda b. I \\ &\quad \Rightarrow_{\beta} \lambda b. \lambda u. u \end{aligned}$$

6. Let T and F be as they were in the above problem, and let

$N = \text{lambda } x. (x \text{ F T})$  ~~NOT~~

what are the normal forms of

a. (N F)

a) NOT FALSE  $\rightarrow$  TRUE  
B

b. (N T)

b) NOT TRUE  $\rightarrow$  FALSE  
B

c. (N (N T))

c) NOT (NOT TRUE)

NOT FALSE  $\Rightarrow$  TRUE

(he he he ...)

CONS (cab)

7. Let T and F be as above and let IF =  $\text{lambda } c. \text{lambda } a. \text{lambda } b. (c \text{ a } b)$ .

Find the normal forms of

(IF T A B)

CONS (IFELSE TRUE A B)

(IF F A B)

CONS (IFELSE FALSE A B)

For arbitrary terms A, B (what A and B are shouldn't be important).



4b. (NEW - OPTIONAL CHALLENGE)

Church exponentiation,  $m$  to the  $n$ th power, is  $\text{lambda } m.\text{lambda } n.(n\ m)$ .

Show that 3 to 2nd power = 9 using the Church representation.

5. Given  $T = \text{lambda } x.\text{lambda } y.x$  (same as K)

TRUE

$F = \text{lambda } x.\text{lambda } y.y$  (same as KI)

FALSE

$D = \text{lambda } p.\text{lambda } q.(p\ T\ q)$  (where  $F$  is as above)

Find the normal forms of:

a. (D F F)

a) IF ELSE FALSE FALSE  $\Rightarrow$  FALSE

b. (D F T)

b) IF ELSE FALSE TRUE

c. (D T F)

c) IF ELSE TRUE FALSE

d. (D T T)

d) IF ELSE TRUE TRUE  $\Rightarrow$  TRUE

Does anything about the behavior of these lambda terms look familiar?

(pretend you're all excited now and want to know more!)

Boolean expression +  
Control statements.

8. let PAIR = lambda a. lambda b. lambda c. (c a b) *CONS*

let FST = lambda p. p T (where T is as above) *CAR*

let SND = lambda p. p F *CDR*

let M = (PAIR a (PAIR b c))

Find the normal forms of (FST M), (SND M) and (SND (SND M))

①  $FST\ M$   
 $\Rightarrow_{\beta}$   
 $\lambda p. (p\ T)\ M \Rightarrow_{\beta} TRUE\ (M) \Rightarrow_{\beta} TRUE\ (PAIR\ a\ (PAIR\ bc)) \Rightarrow_{\beta} Pair\ a$

②  $SND\ M$   
 $\Rightarrow_{\beta}$   
 $\lambda p. (p\ F)\ M \Rightarrow_{\beta} False\ (Pair\ a\ (Pair\ bc)) \Rightarrow_{\beta} Pair\ bc$

③  $FALSE\ (FALSE\ (Pair\ a\ (Pair\ bc)))$   
 $\Rightarrow_{\beta} c$

9. (challenge). Design a pure lambda term ISZERO that determines if a church numeral is zero, that is:

ISZERO ZERO should beta-reduce to TRUE (K), but for non-zero values:

ISZERO (lambda f. lambda x. f (f x)) should beta-reduce to FALSE.

Show that these reductions indeed hold for your ISZERO.

10. (not recommended unless you have a lot of time..) find the normal form of

(lambda x. x x) (lambda x. x x) *NOT reducible, no beta-redexes*

$\Rightarrow_{\beta}$

$(\lambda x. x x) (\lambda x. x x)$

