Robotic Pulling

***Abstract*—**

I. Introduction

II. Theory

A. Planar Pulling Subject to Friction

We present a preliminary quasi-static analysis of planar pulling subject to friction. Planar pulling is the manipulation skill where the robot contacts a rigid body at a point and pulls that contact point along a trajectory.

Let the *generalized velocity* of a planar rigid body be $\mathbf{v}^+ = [v_x, v_y, \omega]^T$, where v_x and v_y are the linear velocities of a reference point and ω is the angular velocity about that point. Taking the origin as the reference point, the velocity of a point $\mathbf{x}$ on the body is then given by $\mathbf{v}(\mathbf{x}) = [v_x, v_y]^T + \omega\hat{\mathbf{k}} \times \mathbf{x}$ and can be written in matrix notation as

$$\mathbf{v}(\mathbf{x}) = A(\mathbf{x})\mathbf{v}^+, \tag{1}$$

where

$$A(\mathbf{x}) = \begin{bmatrix} 1 & 0 & -x_2 \\ 0 & 1 & x_1 \end{bmatrix}. \tag{2}$$

In this analysis, we choose a coordinate frame such that the origin is the contact point and the y-axis aligns with the contact point velocity (see Figure 1 for an example). Then the total frictional force and moment of the body at the origin are

$$\mathbf{f}_f = -\mu \int_R \frac{A(\mathbf{r})\mathbf{v}^+}{\|A(\mathbf{r})\mathbf{v}^+\|} p(\mathbf{r}) dA \tag{3}$$

$$\mathbf{m}_f = -\mu \int_R \mathbf{r} \times \frac{A(\mathbf{r})\mathbf{v}^+}{\|A(\mathbf{r})\mathbf{v}^+\|} p(\mathbf{r}) dA, \tag{4}$$

where μ is the coefficient of friction, R is the region of the rigid body in contact with the plane, $\mathbf{v}(\mathbf{r})$ is the body point velocity given by (1) and $p(\mathbf{r})$ is a pressure distribution over R.

The principle of minimal dissipation states that the motion of the pulled body minimizes the instantaneous work dissipated by friction [1]. That is, the motion minimizes the following:

$$\begin{aligned} \underset{\mathbf{v}^+}{\text{minimize}} \quad & \mu \int_R \|A(\mathbf{r})\mathbf{v}^+\| p(\mathbf{r}) dA \\ \text{subject to} \quad & \mathbf{v}^+ \in \mathcal{C}. \end{aligned} \tag{5}$$

Without loss of generality, we take $\mathcal{C} = \{[0, 1, \omega]^T, \omega \in \mathbb{R}\}$ so that the contact point motion is aligned with the coordinate frame. We call the objective in (5) the frictional dissipation equation $\mathrm{P}(\mathbf{v}^+)$.

The principle of minimal dissipation is equivalent to the quasi-static model of planar sliding with friction [1]. In fact, the quasi-static model is identical to the first order optimality conditions of (5). To see this, take the Lagrangian of (5)

$$\mathrm{L}(\mathbf{v}^+, \lambda) = \mathrm{P}(\mathbf{v}^+) + \lambda_1 v_1^+ + \lambda_2 (v_2^+ - 1) \tag{6}$$

and set gradient of L with respect to $\mathbf{v}^+$,

$$\nabla \mathrm{L}(\mathbf{v}^+, \lambda) = \mu \int_R \frac{A(\mathbf{r})^T A(\mathbf{r})\mathbf{v}^+}{\|A(\mathbf{r})\mathbf{v}^+\|} p(\mathbf{r}) dA + \left[\lambda_1, \lambda_2, 0\right]^T, \tag{7}$$

to zero. After some work, we end up with the following first order conditions on the force and the moment

$$\mathbf{f}_f = \left[\lambda_1, \lambda_2\right]^T \tag{8}$$

$$\mathbf{m}_f = 0, \tag{9}$$

that is, the quasi-static motion model.

We prefer the principle of minimal dissipation in our theoretical analysis because the frictional dissipation equation is continuous and convex in μ, $p(\mathbf{x})$ and $\mathbf{v}^+$.

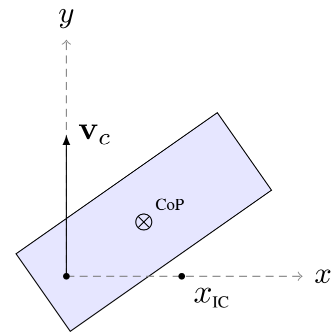


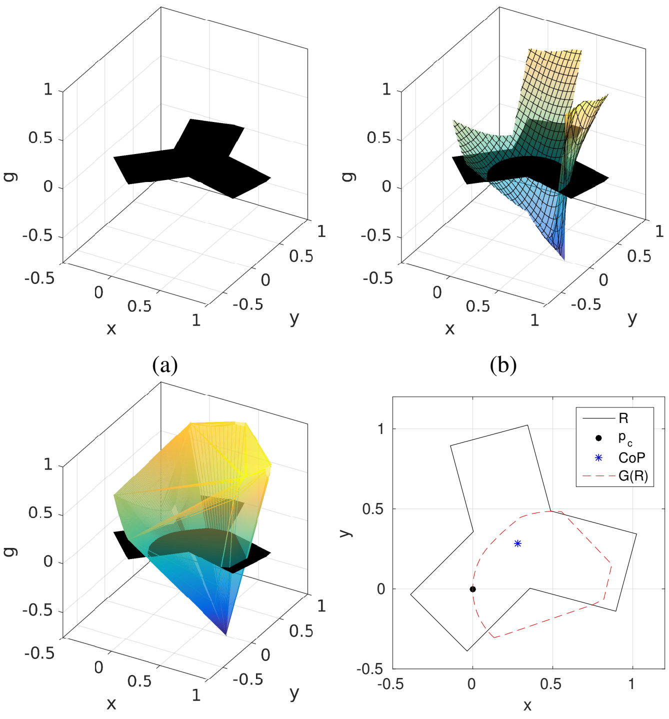
Fig. 1: Motion of a press-pulled slider.

B. Quasi-static Stability

Theorem 1. *For pulling of a rigid body in the plane, the line from the pulling contact point to the body's center of pressure converges to the line of motion.*

Proof: Let the y-axis of the coordinate frame be aligned with the line of motion of the press-pull contact point (see Figure 1 for an example). Then the instantaneous rotation center of the body must fall on the x-axis, and we can write $\mathbf{r}_{\mathrm{IC}} = (x_{\mathrm{IC}}, 0)^T$.

Suppose the center of pressure is strictly to the right of the line of motion (as in Figure 1). Then by Theorem 7.4 in [4], the rotation center lies on the positive x-axis and has negative rotation. The velocity of a point on the rigid body is given by $\mathbf{v}(\mathbf{r}) = \dot{\theta}\,\hat{\mathbf{k}} \times (\mathbf{r} - \mathbf{r}_{\mathrm{IC}})$. We can compute the motion of the



(a) (b) (c) (d)

Fig. 2: Example moment envelope for a 2D tetrapod with rotation center $x_{\text{IC}} = 0.75$. (a) Support region R. (b) Normalized-moment surface $G(R)$. (c) Convex moment envelope of $G(R)$. (d) Intersection of the moment envelope and the xy-plane. The intersection bounds the set of feasible centers of pressure with zero moment.

body relative to the contact point by

$$\begin{aligned}\mathbf{v}(\mathbf{r}) - \mathbf{v}_c &= \dot{\theta}\,\hat{\mathbf{k}} \times (\mathbf{r} - \mathbf{r}_{\text{IC}}) - \dot{\theta}\,\hat{\mathbf{k}} \times (\mathbf{0} - \mathbf{r}_{\text{IC}}) \\ &= \dot{\theta}\,\hat{\mathbf{k}} \times (\mathbf{r} - \mathbf{0}),\end{aligned} \tag{10}$$

where $\mathbf{v}_c$ is the velocity of the contact point. In other words, the body is rotating clockwise relative to the contact point with angular velocity

$$\dot{\theta} = -\frac{\|\mathbf{v}_c\|}{x_{\text{IC}}}. \tag{11}$$

Let $\theta \in (\pi/2, -\pi/2)$ be the orientation of the center of pressure in the frame of the contact point. When the line of motion passes through the center of pressure, a pure translation takes place (Theorem 7.4 in [4]). This implies that $\dot{\theta} = 0$ if and only if $\theta = \pi/2, -\pi/2$. Since θ is monotonically decreasing, we see that it must converge to $-\pi/2$ in the limit as $t \to \infty$.

The case when the center of pressure is strictly to the left of the line of motion follows from symmetry about the y-axis. ∎

Note that the above proof holds for *all pressure distributions* of the body on the support surface.

C. Frictional Force & Moment Envelopes

The frictional force and moment envelopes provide a nice geometric model of the constraints between the center of pressure, force and moment [4]. A frictional envelope has as its parameters the support region R and the velocity $\mathbf{v}^+$. Let f_0 be the total pressure. Then the functions

$$\mathbf{f}(\mathbf{x}) = -\mu f_0 \frac{A(\mathbf{x})\mathbf{v}^+}{\|A(\mathbf{x})\mathbf{v}^+\|}, \tag{12}$$

$$g(\mathbf{x}) = -\mu f_0\, \mathbf{x} \times \frac{A(\mathbf{x})\mathbf{v}^+}{\|A(\mathbf{x})\mathbf{v}^+\|} \tag{13}$$

evaluate the frictional force and frictional torque that would result from a unit normalized pressure at $\mathbf{x}$. For this exposition, we will focus on how to generate the frictional moment envelope illustrated in Figure 2.

Let G map R into a surface in $\mathbb{R}^3$ by associating each point $\mathbf{x} \in R$ with its maximum potential moment $g(\mathbf{x})$, i.e.

$$G(\mathbf{x}) = \begin{bmatrix} x \\ y \\ g(\mathbf{x}) \end{bmatrix}, \tag{14}$$

and let $\hat{p} = p/f_0$ be the normalized pressure. Then the set $\{\int_R G(\mathbf{r})\hat{p}(\mathbf{r})dA \,|\, \int_R \hat{p}(\mathbf{r})dA = 1\}$ is the convex hull of the surface $G(R)$. Moreover, any point in the convex hull of $G(R)$ satisfies

$$\begin{aligned}\int_R G(\mathbf{r})\hat{p}(\mathbf{r})dA &= \int_R \begin{bmatrix} x \\ y \\ g(\mathbf{x}) \end{bmatrix} \hat{p}(\mathbf{r})dA \\ &= \begin{bmatrix} x_0 \\ y_0 \\ \int_R g(\mathbf{x})\hat{p}(\mathbf{r})dA \end{bmatrix}.\end{aligned} \tag{15}$$

Thus, the convex hull of $G(R)$ is the set of all feasible centers of pressure and moments for a given support region R and velocity $\mathbf{v}^+$. An identical procedure generates the frictional force envelope, i.e. the convex hull of $F(R)$.

The frictional force and moment envelopes are employed in Sections III-A and III-C to compute bounds on feasible angular velocities and centers of pressure.

III. Methods

A. Exact Angular Velocities Bounds

Proposition 1. *The constrained frictional dissipation equation* $\mathrm{D}_{\mathcal{C}}(\mathbf{v}^+)$ *has a unique minimizer for pressure distributions with some support force off of the x-axis. When otherwise, the equation admits an interval of minimizers.*

Proof: The work of [3] was the first to prove the above observation for finite pressure distributions. For the case of infinite (discrete) pressure distributions, we can appeal to the convexity properties of $\|A(\mathbf{x})\mathbf{v}^+\|$.

Minkowski's inequality gives $\|x + y\| \leq \|x\| + \|y\|$ with equality if and only if x and y are positively linearly dependent. For two generalized velocities $\mathbf{v}_1^+, \mathbf{v}_2^+ \in \mathcal{C}$, the resulting body point velocities are linearly dependent if and only if the determinant

$$\det A(\mathbf{x}) \begin{bmatrix} \mathbf{v}_1^+ & \mathbf{v}_2^+ \end{bmatrix} = \det \begin{bmatrix} -x_2\omega_1 & -x_2\omega_2 \\ 1 + x_1\omega_1 & 1 + x_1\omega_2 \end{bmatrix} \tag{16}$$

$$= -x_2(\omega_1 - \omega_2) \tag{17}$$

is zero. We immediately see that $\|A(\mathbf{x})\mathbf{v}^+\|$ is strictly convex relative to $\mathcal{C}$ if and only if the body point $\mathbf{x}$ lies off of the x-axis. The proposition follows from the fact that the positive sum of convex functions and a strictly convex function is strictly convex. ■

An object whose contact region is two discrete points, e.g. a spoon, can have multiple minimizing angular velocities when aligned with the x-axis.

Proposition 2. *Suppose $P(u) := \arg\min_x f(x,u)$ with $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ continuous and level-bounded in x locally uniformly in u. Then the set-valued mapping $P(u)$ is outer-semicontinuous and locally bounded.*

Proof: Proposition adapted from Corollary 7.42 and Theorem 1.17 in [5]. ■

Theorem 2. *For pulling of a planar rigid body with known center of pressure, the set of all feasible angular velocities is connected.*

Proof: Let Ω be the set of feasible angular velocities, w.r.t. (5), for a given support region R with known center of pressure. Ω is nonempty since a single support point located at the center of pressure has an analytic solution. Let $\omega_1, \omega_2 \in \Omega$ and let p_1, p_2 be their corresponding pressure distributions. Then the new distribution $p_t = tp_1 + (1-t)p_2$, with $t \in [0,1]$, shares the same center of pressure as p_1, p_2. Define the function

$$f(\omega, t) = \mu \int_R \|A(\mathbf{r})\mathbf{v}^+(\omega)\| p_t(\mathbf{r}) dA, \tag{18}$$

where $\mathbf{v}^+ : \omega \to [0, 1, \omega]^T$ and $\operatorname{dom} f := \mathbb{R} \times [0,1]$. By inspection, f is continuous. For all $t \in [0,1]$ and $\alpha \in \mathbb{R}$, the set $\{(\omega, t) \mid f(\omega, t) \leq \alpha\}$ is bounded because $f \to \infty$ as $|x| \to \infty$. Hence, f is level-bounded in ω locally uniformly in t, and f satisfies the conditions of Proposition 2.

The image of a connected set by an outer-semicontinuous set-valued mapping whose values are nonempty and connected is connected [2]. Let $P(t) := \arg\min_\omega f(\omega, t)$. For a given t, the set $P(t)$ is convex-valued and therefore connected. Since f is continuous and level-bounded, the set is also nonempty (Theorem 1.9 [5]). Therefore, the image $P([0,1])$ contains the interval connecting ω_1 and ω_2. Because the choice of ω_1, ω_2 was arbitrary, Ω is connected. ■

In general, Theorem 2 holds for any convex set of pressure distributions.

Corollary 1. *For pulling of a planar rigid body with known center of pressure, the set of all feasible angular velocities is a bounded interval.*

Proof: By Theorem 7.4 of [4], the angular velocity of the body is strictly negative (positive) when the center of pressure falls strictly in the right (left) half-plane. This shows that Ω is bounded from below (above). The Bisector Bound [4] states that the set of feasible rotation centers falls behind the line bisecting the contact point and the center of pressure. With respect to our coordinate frame (Section II-A), we see that Ω is bounded above (below) by

$$\omega^* = -\frac{\|\mathbf{v}_c\|}{x^*}, \tag{19}$$

where x^*, strictly positive (negative), is the intersection of the bisector line and the x-axis. Therefore, Ω is a bounded interval.

When the center of pressure falls on the line of motion, the rigid body translates and $\Omega = \{0\}$ [4]. ■

Algorithm 1 finds exact angular velocity bounds for a given support region R and center of pressure $[x_0, y_0]^T$. It uses bisection search to estimate the end-points of Ω. The bisection search tests the feasibility of an angular velocity ω by checking whether the point $[x_0, y_0, 0]^T$ is contained in the associated frictional moment envelope (see Section II-C). In quasi-static terminology, this is equivalent to checking whether there exists a pressure distribution with center of pressure $[x_0, y_0]^T$ such that the velocity $\mathbf{v}^+ = [\mathbf{v}_c^T, \omega]^T$ generates zero moment about the contact point (Equation 4).

The run-time of the algorithm is $\mathcal{O}(d\, n \log n)$, where d is the number of significant digits returned and n is the number of points in the discretization of R.

Algorithm 1 Exact Angular Velocity Bounds

function FIND EXTREMA(R, x_0, y_0)
  if x_0 is 0 **then return** $[0, 0]$
  $l \leftarrow 0$
  $u \leftarrow -\|\mathbf{v}_c\|/(x_0 + y_0^2/x_0)$
  $\omega_1 \leftarrow$ BISECTION SEARCH(R, x_0, y_0, l, u)
  $l \leftarrow u \leftarrow -\|\mathbf{v}_c\|/(x_0 + y_0^2/x_0)$
  do
    $u \leftarrow 2u$
    $\mathbf{v}^+ \leftarrow [\mathbf{v}_c^T, u]^T$
    $G \leftarrow \{-\mathbf{x} \times A(\mathbf{x})\mathbf{v}^+ / \|A(\mathbf{x})\mathbf{v}^+\| \mid \mathbf{x} \in R\}$
  while $[x_0, y_0, 0]^T \in$ CONVHULL(G)
  $\omega_2 \leftarrow$ BISECTION SEARCH(R, x_0, y_0, l, u)
  $l \leftarrow$ MIN(ω_1, ω_2)
  $u \leftarrow$ MAX(ω_1, ω_2)
  return $[l, u]$
function BISECTION SEARCH(R, x_0, y_0, l, u)
  while $\varepsilon < |u - l|$ **do**
    $\omega \leftarrow (u + l)/2$
    $\mathbf{v}^+ \leftarrow [\mathbf{v}_c^T, \omega]^T$
    $G \leftarrow \{-\mathbf{x} \times A(\mathbf{x})\mathbf{v}^+ / \|A(\mathbf{x})\mathbf{v}^+\| \mid \mathbf{x} \in R\}$
    if $[x_0, y_0, 0]^T \in$ CONVHULL(G) **then**
      $u \leftarrow \omega$
    else
      $l \leftarrow \omega$
  return $(u + l)/2$

B. Exact Position Bounds

C. Center of Pressure Bounds

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