H

1 Question 1 (Linear Diffusion)

Let $u(x,t)$ be a real-valued function. Consider the 1D, unbounded domain, linear diffusion, defined by

$$u_t = u_{xx}, \quad u(x,0) = f(x), \quad t \in [0, \infty) \tag{1}$$

where t is the time variable, x is the spatial variable, u_t and u_{xx} are respectively the time derivative and the second spatial derivative of u, and the function $f(x)$ is the initial condition.

(a) For a real-valued function $u \in \mathbb{R}$, show that the solution of the above PDE is a Gaussian convolution with the initial condition, namely

$$u(x,t) = f(x) * g_\sigma(t)(x) \tag{2}$$

where $g_\sigma(t)$ is the Gaussian kernel.

Solution:

Then, find the relation between the time variable t and the standard deviation σ.

(b) Assume the initial condition $f(x)$ is

$$f(x) = \sin(\omega_1 x) + \sin(\omega_2 x) \tag{3}$$

where ω_1, ω_2 are positive constants. Use $u(x,t) = f(x) * g_\sigma(t)(x)$ to show that $u(t,x)$ is

$$u(x,t) = \sin(\omega_1 x)e^{-\omega_1^2 t} + \sin(\omega_2 x)e^{-\omega_2^2 t}. \tag{4}$$

(c) For the solution:

$$u(x,t) = \sin(\omega_1 x)e^{-\omega_1^2 t} + \sin(\omega_2 x)e^{-\omega_2^2 t}.$$

1. Write the explicit backward-difference in time expression of $u(x, \Delta t)$. Keep the spatial coordinates continuous.

2. Write the analytic solution of the approximation of $u(x, \Delta t)$ given above in (i).

3. Show that the extremum principle is kept under some condition. What is the condition? Is it similar to a stability condition learned in class? Explain.