Variational Methods 049064
Exercise 1

Part 1

Analytic Exercises

Question 1 (Linear Diffusion)

Let $u(x,t)$ be a real-valued function. Consider the 1D, unbounded domain, linear diffusion, defined by

$$u_t = u_{xx}, \quad u(x,0) = f(x), \quad t \in [0, \infty) \tag{1}$$

where t is the time variable, x is the spatial variable, u_t and u_{xx} are respectively the time derivative and the second spatial derivative of u, and the function $f(x)$ is the initial condition.

(a) For a real-valued function $u \in \mathbb{R}$, show that the solution of the above PDE is a Gaussian convolution with the initial condition, namely

$$u(x,t) = f(x) * g_{\sigma(t)}(x) \tag{2}$$

where $g_{\sigma(t)} = a$ is the Gaussian kernel.

Solution:

Denote:

$$(\mathcal{F}\{u\})(s) = \widehat{u}(s,t) = \int_{-\infty}^{\infty} e^{-i2\pi sx} u(x,t)\, dx$$

We apply the spatial FT to both sides of Eq. (1), and get:

$$\int_{-\infty}^{\infty} e^{-i2\pi sx} u_t\, dx = \int_{-\infty}^{\infty} e^{-i2\pi sx} u_{xx}\, dx$$

Using the Fourier Transform of the n-th derivative:

$$\mathcal{F}\left\{\frac{d^{(n)}g(x)}{dx^n}\right\} = (i2\pi s)^n \mathcal{F}\{g\}$$

and the fact that the spatial FT of a time derivative is the time derivative of the spatial FT, we get:

$$\frac{\partial}{\partial t}\widehat{u}(s,t) = -(2\pi s)^2 \cdot \widehat{u}(s,t)$$

This is a simple ODE wrt to t. We solve this and get:

$$\widehat{u}(s,t) = c(s) e^{-(2\pi s)^2 t}$$

we plug $t = 0$ into the above equation, and get:

$$\widehat{u}(s,0) = c(s)$$

and since

$$\widehat{u}(s,0) = \int_{-\infty}^{\infty} e^{-i2\pi sx} u(x,0)\, dx = \int_{-\infty}^{\infty} e^{-i2\pi sx} f(x)\, dx = \widehat{f}(s)$$

we get:

$$\widehat{u}(s,t) = \widehat{f}(s) e^{-(2\pi s)^2 t}$$

i.e. we notice that the spatial FT of u is a multiplication of two functions. Applying the inverse spatial FT, we get that $u(x,t)$ is a convolution of the inverse FT of those functions, i.e.:

$$u(x,t) = f(x) * \left(\mathcal{F}^{-1}\left\{e^{-4\pi^2 s^2 t}\right\}(x)\right)$$

Recalling that:

$$\mathcal{F}^{-1}\left\{e^{-\pi s^2}\right\}(x) = e^{-\pi x^2}$$

and the Fourier stretch theorem:

$$\big(\mathcal{F}\left\{f(ax)\right\}\big)(s) = \frac{1}{|a|}\,\widehat{f}(\frac{s}{a})$$

we get that:

$$\big(\mathcal{F}^{-1}\left\{e^{-4\pi^2 s^2 t}\right\}\big)(x) = \frac{1}{\sqrt{4\pi t}} e^{-\pi\frac{x^2}{4\pi t}} = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

If we denote:

$$g_{\sigma(t)}(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

we get:

$$\boxed{u(x,t) = f(x) * g_{\sigma(t)}(x)}$$

Then, find the relation between the time variable t and the standard deviation σ.

Solution:

By the formula:

$$g_{\sigma(t)}(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

We notice that ———————————— (b) Assume the initial condition $f(x)$ is

$$f(x) = \sin(\omega_1 x) + \sin(\omega_2 x) \tag{3}$$

where ω_1, ω_2 are positive constants. Use $u(x,t) = f(x) * g_\sigma(t)(x)$ to show that $u(t,x)$ is

$$u(x,t) = \sin(\omega_1 x)e^{-\omega_1^2 t} + \sin(\omega_2 x)e^{-\omega_2^2 t}. \tag{4}$$

(c) For the solution:

$$u(x,t) = \sin(\omega_1 x)e^{-\omega_1^2 t} + \sin(\omega_2 x)e^{-\omega_2^2 t}.$$

1. Write the explicit backward-difference in time expression of $u(x,\Delta t)$. Keep the spatial coordinates continuous.

2. Write the analytic solution of the approximation of $u(x,\Delta t)$ given above in (i).

3. Show that the extremum principle is kept under some condition. What is the condition? Is it similar to a stability condition learned in class? Explain.

Let $u(x,t) : \mathbb{R} \times [0,\infty] \to \mathbb{R}$ denote the temperature at position x at time t. Solve the IVP:

$$u_t = u_{xx} \quad , u(x,t=0) = f(x) \tag{5}$$

where f is the initial condition.

a) Claim: The solution of Eq. (5) is a Gaussian convolution with the initial condition, namely:

$$u(x,t) = f(x) * g_{\sigma(t)}(x)$$

where $g_{\sigma(t)}$ is the Gaussian Kernel.

Proof. a □

b)