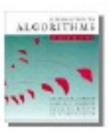
Analysis of Algorithms (Fall 2013) Istanbul Technical University Computer Eng. Dept.



Chapter 18

B-Trees

Last updated: December 16, 2009

Purpose

- Understand B-tree properties and why B-trees are important
- Understand search, insert, and delete operations on B-trees
- Learn B+ and B* tree definitions

Outline

- B-Tree
  - B-Tree Properties
  - B-Tree Search, Insert, Delete
- B* Tree
- B+ Tree

B-Trees

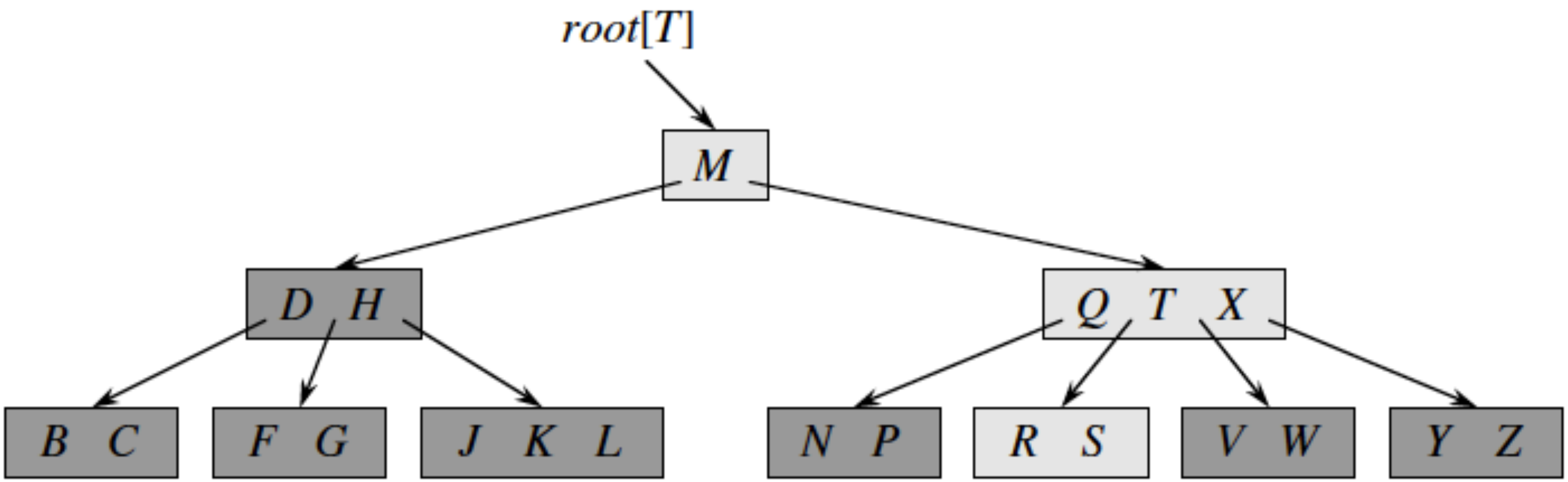
- Relatively new research area
- 1972: R. Bayer and E. McCreight (Boeing Corp.), "Organization and Maintenance of Large Ordered Indexes" first introduced B-trees
- 1979: B-trees had become "the standard organization for indexes in a database system" (D. Comer, "The Ubiquitous B-Tree")
- Also new publications, e.g.:
  – "A practical scalable distributed B-tree," M.K. Aguilera, W. Golab, and M.A. Shah, Proc. of VLDB Endowment, 2008

Statement of the Problem

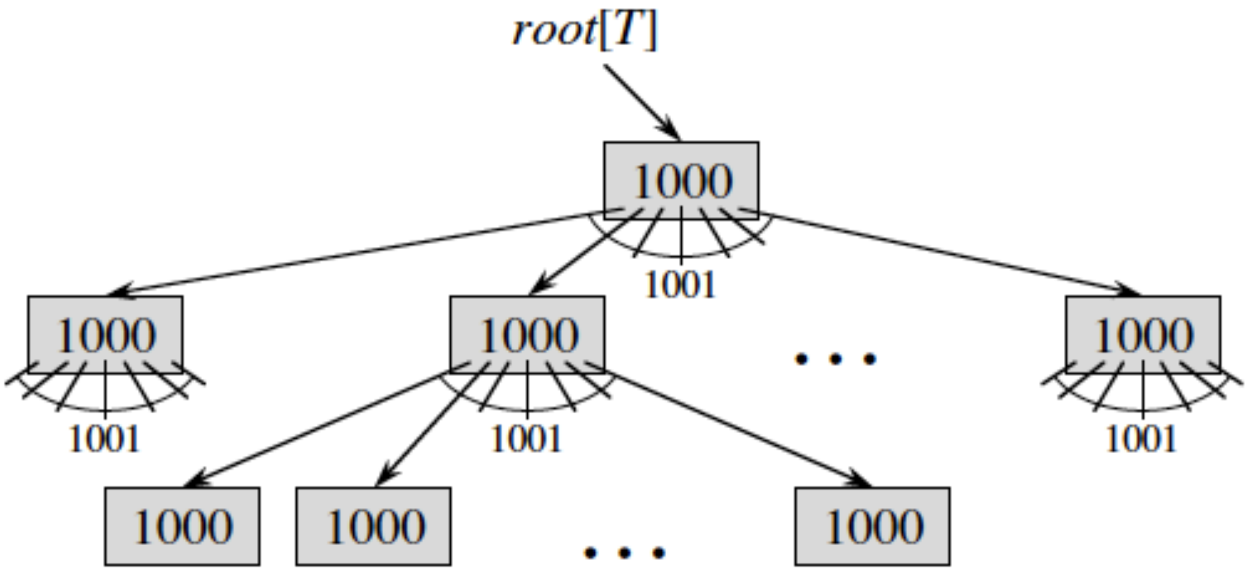
- Fundamental Problem
  - Secondary storage is slow
  - Keeping an index on secondary storage is slow, too!

- Approaches
  - Faster index searching (than binary search)
  - Fast insertion and deletion

B-Trees



B-Trees



1 node,
1000 keys

1001 nodes,
1,001,000 keys

1,002,001 nodes,
1,002,001,000 keys

B-Trees

- Are balanced search trees designed to work well on magnetic disks or other direct-access
- Are similar to red-black trees but they are better at minimizing disk I/O operations
- Have height O(log n)
- Can also be used to implement many dynamic-set operations in time O(logn)

B-Trees

- B-Tree algorithms copy selected pages from disk into main memory as needed and write back onto disk the pages that have changed
- Example: B-Tree with a branching factor of 1001 and height 2
  - can store over one billion keys
  - only two disk accesses at most are required to find any key

B-Tree

B-tree, T, is a rooted tree (whose root is root[T]) having the following properties:

1. Every node x has the following properties
   - n[x], number of keys currently stored in node x
   - n[x] keys themselves in non decreasing order, so that $key_1[x] \le key_2[x] \le \ldots \le key_{n[x]}[x]$
   - leaf[x], a boolean value that is TRUE if x is a leaf and FALSE if x is an internal node
2. Each internal node x also contains n[x]+1 pointers $c_1[x], c_2[x], \ldots, c_{n[x]+1}[x]$ to its children
   - Leaves have no children, hence their c_i are undefined

B-Tree (continued)

B-tree, T, is a rooted tree (whose root is root[T]) having the following properties:

3. Keys $key_i[x]$ separate ranges of keys stored in each subtree:
   – if k_i is any key stored in subtree with root $c_i[x]$, then

$$k_1 \le key_1[x] \le k_2 \le key_2[x] \le \ldots \le key_{n[x]}[x] \le k_{n[x]+1}$$

4. All leaves have same depth, which is tree's height h

B-Tree (continued)

B-tree, T, is a rooted tree (whose root is root[T]) having the following properties:

5. There are lower and upper bounds on the number of keys a node can contain. These bounds can be expressed in terms of a fixed integer $t \geq 2$ called the minimum degree of the B-tree
   - Every node other than the root must have at least $t - 1$ keys
   - Every internal node other than the root thus has at least t children
   - If the tree is nonempty, the root must have at least one key
   - Every node can contain at most $2t - 1$ keys
   - Therefore an internal node can have at most $2t$ children
   - We say that a node is full if it contains exactly $2t - 1$ keys

B-Tree

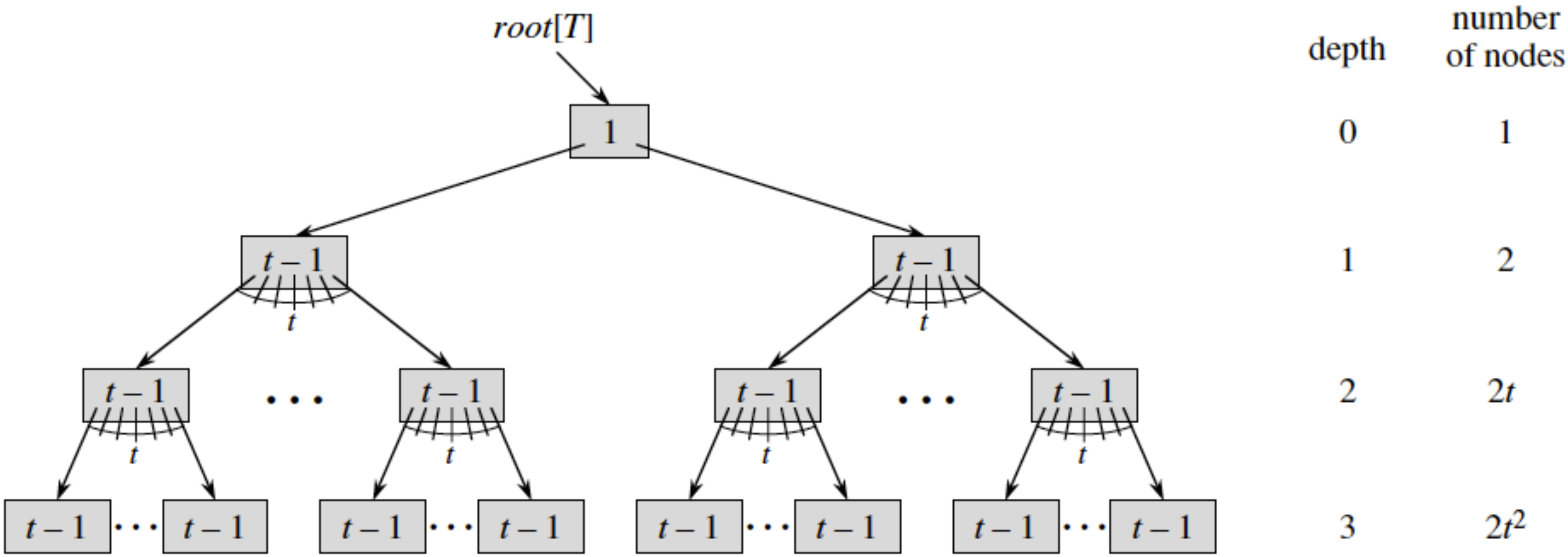
- 2-3-4 tree is the simplest B-tree with t = 2
- Typically much larger values of t are used

Theorem:

If n ≥ 1, then for any n-key B-tree of height h and minimum degree t ≥ 2,

$$h \leq \log_t \frac{n+1}{2}$$

B-Tree



$$
\begin{aligned}
n &\geq 1 + (t-1)\sum_{i=1}^{h} 2t^{i-1} \\
&= 1 + 2(t-1)\left(\frac{t^h - 1}{t - 1}\right) \\
&= 2t^h - 1\,.
\end{aligned}
$$

$$h \leq \log_t \frac{n+1}{2}$$

High Capacity of B-Trees

- B-tree of height 2 can contain over 1 billion keys when each internal node and leaf contains 1000 keys!

- depth 0: 1 node 1,000 keys
- depth 1: 1001 nodes, 1,001,000 keys
- depth 2: 1,002,001 nodes, 1,002,001,000 keys

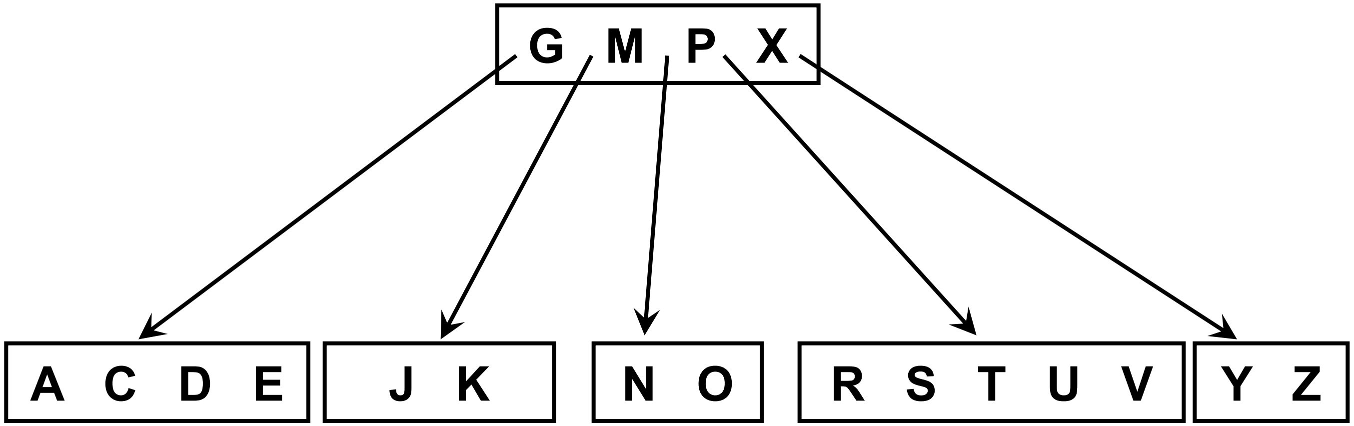
Why t-1 to 2t-1 Keys?

As opposed to t keys at every node for example?

Answer: Because we can keep at least t-1 and at most 2t-1 keys in a node, we

- do not have to increase the tree height right after every insertion and
- need to increase height only after inserting t more keys

Example of a B-Tree



- Minimum degree t = 3
- Check:
  - – Max: 2t – 1 = 5
  - – Min: t - 1 = 2

Searching

- Search for key k at node x

```
B-TREE-SEARCH(x,k)
i ← 1
while i ≤ n[x] and k > key_i[x]
   do i ← i+1
if i ≤ n[x] and k = key_i[x]
   then return (x,i)
if leaf[x]
   then return NIL
   else DISK-READ(c_i[x])
      return B-TREE-SEARCH(c_i[x],k)
```

Searching

- Number of disk accesses is:
  $\Theta(h) = \Theta(\log_t n)$
  where h: tree height
  n: number of keys in B tree

- Total CPU time:
  $O(th) = O(t\log_t n)$

- Could do better with binary search:
  $O(\log_2 t \log_t n)$

Inserting

- More complicated than inserting a key into a binary search tree
- Need to insert at a leaf node
- Can not insert into a full leaf node, hence need to split around the median if the leaf node is full

Inserting (2)

- y is the ith child of x and is the node being split

B-TREE-SPLIT-CHILD(x,i,y)

z ← Allocate_Node
leaf[z] ← leaf[y]
n[z] ← t - 1
for j ← 1 **to** t - 1
    do $key_j[z] \leftarrow key_{j+t}[y]$
if not leaf[y]
    then for j ← 1 **to** t
        do $c_j[z] \leftarrow c_{j+t}[y]$
n[y] ← t-1
for j← n[x]+1 **downto** i+1
    do $c_{j+1}[x] \leftarrow c_j[x]$
$c_{i+1}[x] \leftarrow z$
for j ← n[x] **downto** i
    do $key_{j+1}[x] \leftarrow key_j[x]$
$key_i[x] \leftarrow key_t[y]$
n[x] ← n[x]+1
DISK-WRITE(y) ; DISK-WRITE(z) ; DISK-WRITE(x)

Inserting (3)

```
B-TREE-INSERT(T,k)
r ← root[T]
if n[r] = 2t - 1 ▹ if root is full, add a new layer
    then s ← ALLOCATE-NODE()
         root[T] ← s
         leaf[s] ← FALSE
         n[s] ← 0
         c1[s] ← r
         B-TREE-SPLIT-CHILD(s,1,r)
         B-TREE-INSERT-NONFULL(s,k)
    else B-Tree-INSERT-NONFULL(r,k)
```

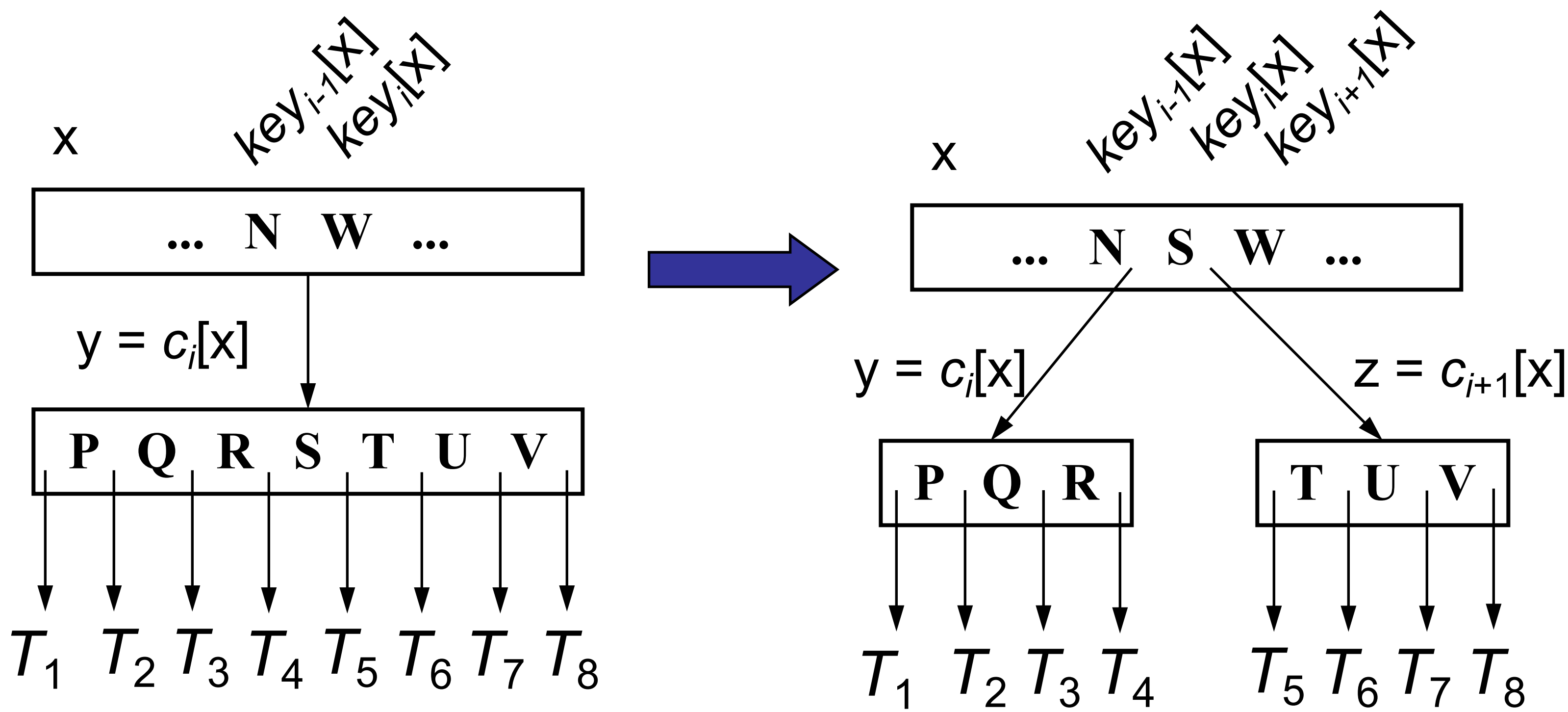
Inserting (4)

```
B-TREE-INSERT-NONFULL(x,k)
i ← n[x]
if leaf[x]
   then while i ≥ 1 and k < key_i[x]
              do key_i+1[x] ← key_i[x]
                 i ← i -1
        key_i+1[x] ← k
        n[x] ← n[x]+1
        DISK-WRITE(x)
   else while i ≥ 1 and k < key_i[x]
              do i ← i - 1
        i ← i + 1
        DISK-READ(c_i[x])
        if n[c_i[x]] = 2t-1
           then B-TREE-SPLIT-CHILD(x,i,c_i[x])
                if k > key_i[x]
                   then i ← i + 1
        B-TREE-INSERT-NONFULL(c_i[x],k)
```

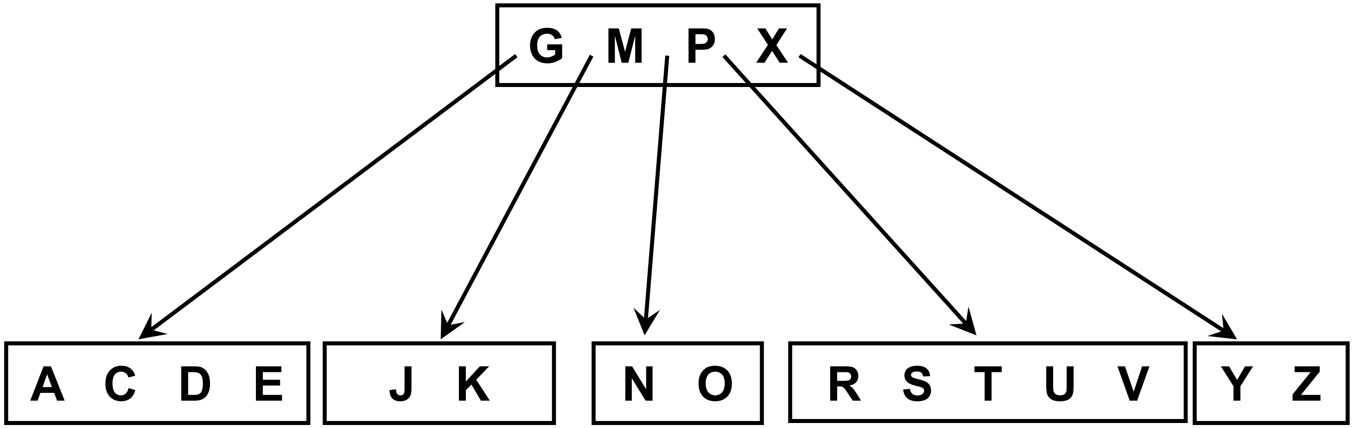
Inserting and Splitting

- Insert into a node to its limit, i.e., $2t - 1$ keys
- A node with $2t - 1$ keys, i.e. $2t$ children has to split
- After a split
  - One key (median of 2t - 1 keys) moves up
  - Two new nodes with t -1 keys
  - New item inserted into the appropriate node

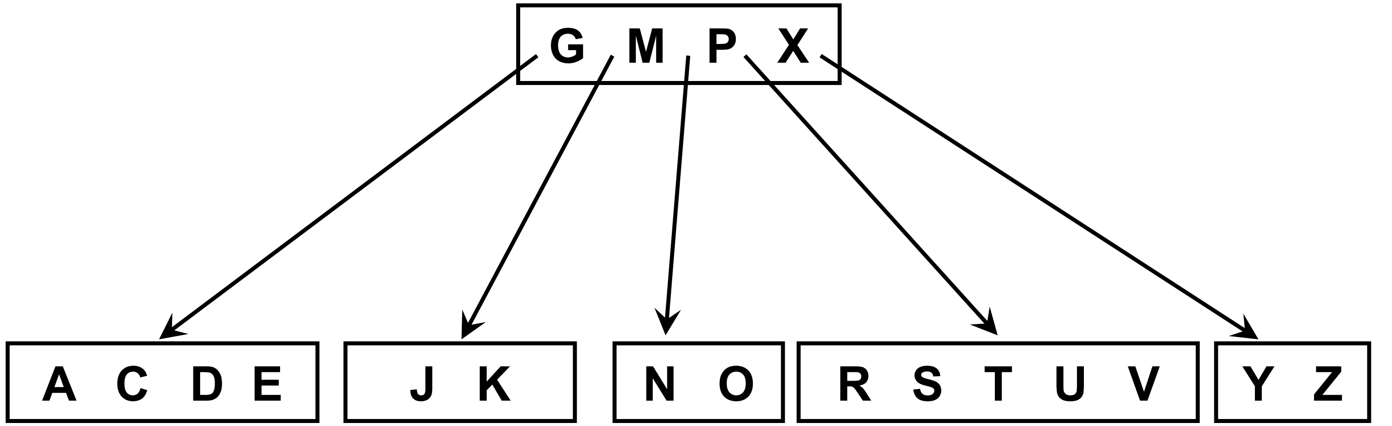
Split


x
$key_{i-1}[x]$
$key_i[x]$
… N W …
$y = c_i[x]$
P Q R S T U V
T_1 T_2 T_3 T_4 T_5 T_6 T_7 T_8
x
$key_{i-1}[x]$
$key_i[x]$
$key_{i+1}[x]$
… N S W …
$y = c_i[x]$
$z = c_{i+1}[x]$
P Q R
T U V
T_1 T_2 T_3 T_4
T_5 T_6 T_7 T_8

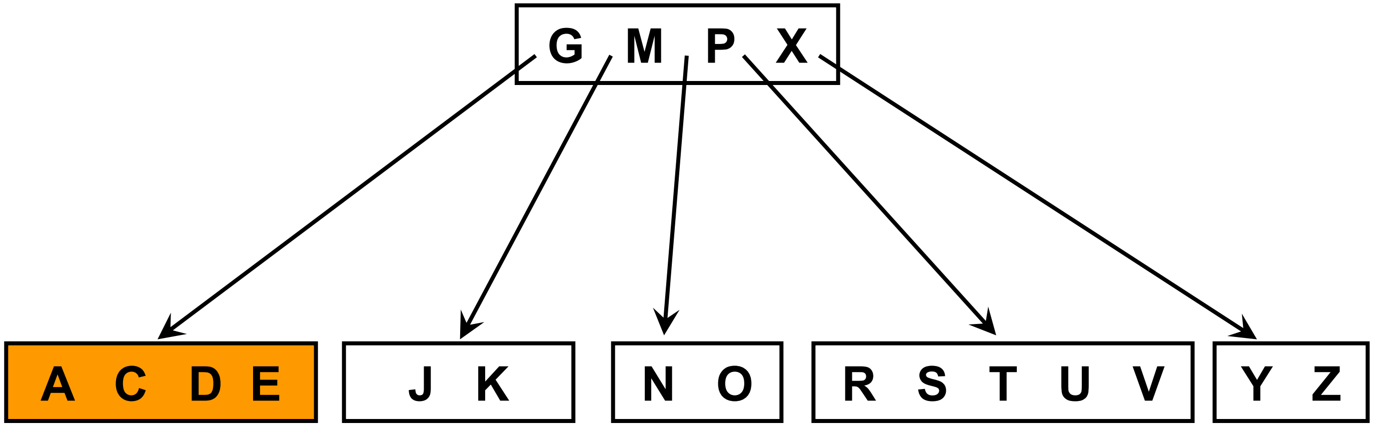
Initial Tree



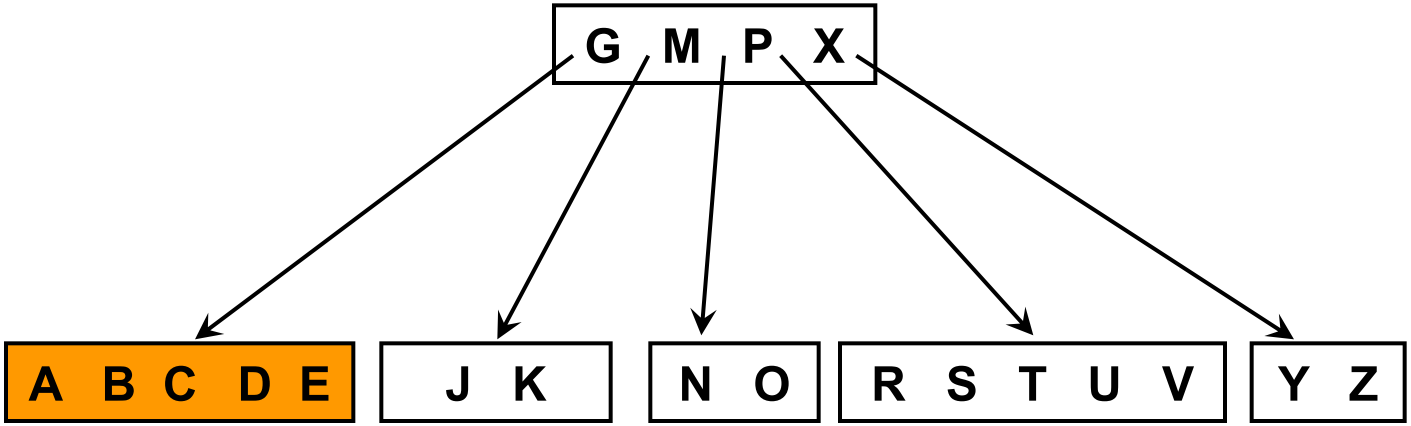
Inserting B



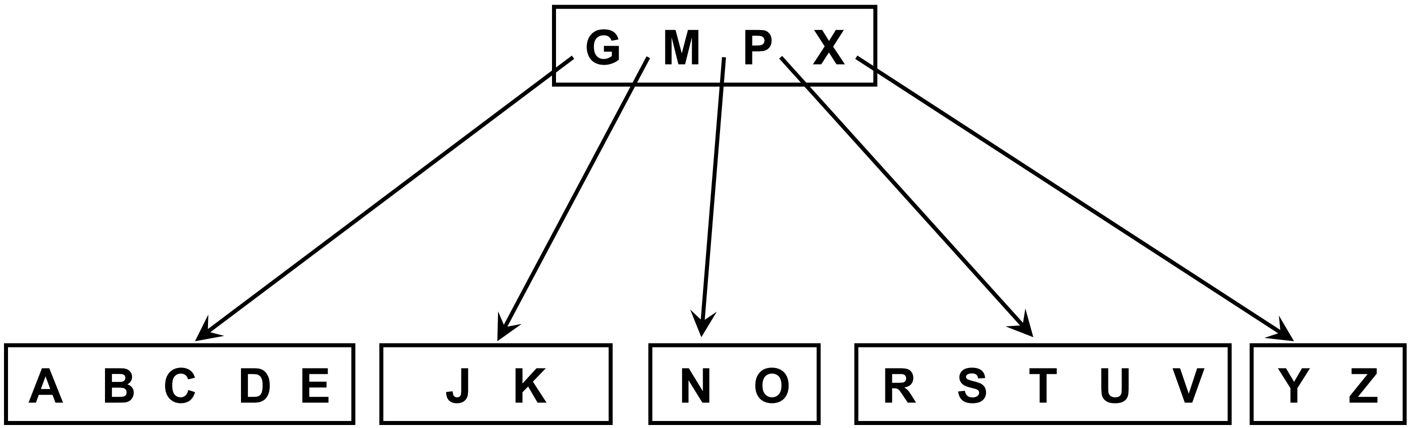
Inserting B



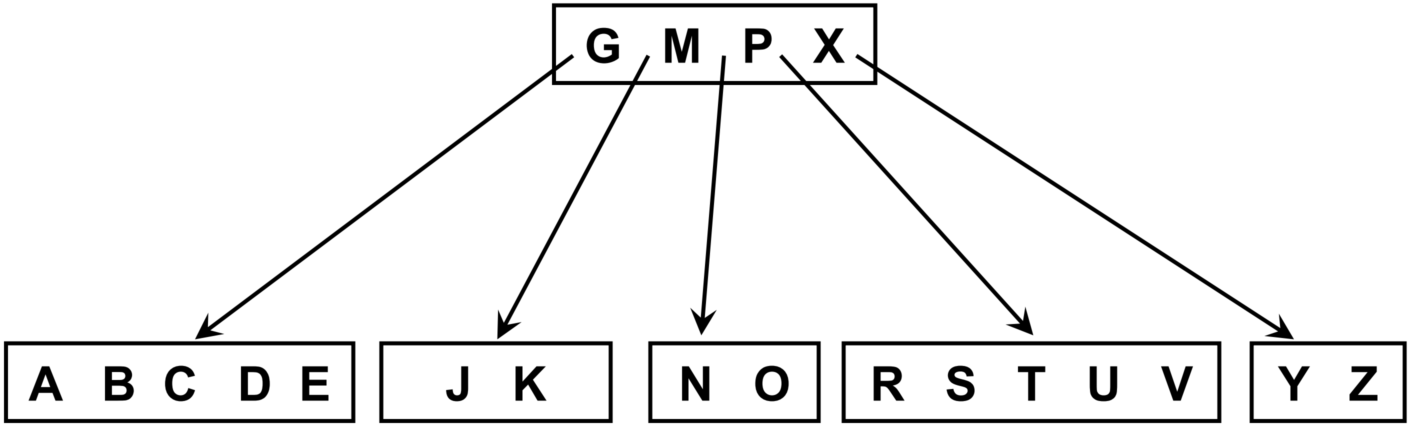
Inserting B



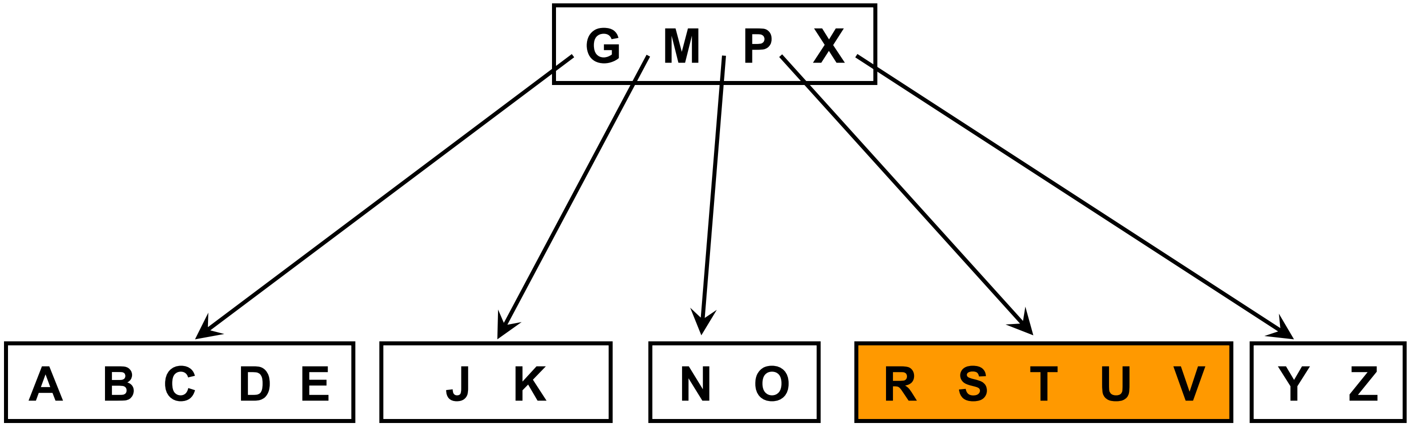
Inserting B



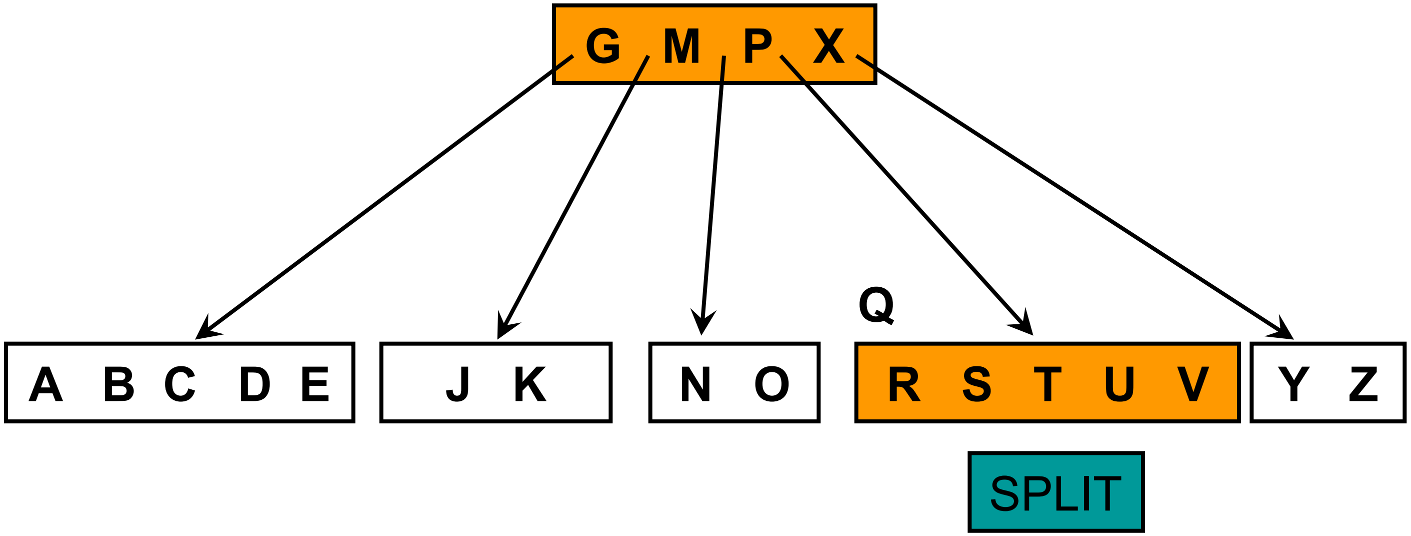
Inserting Q



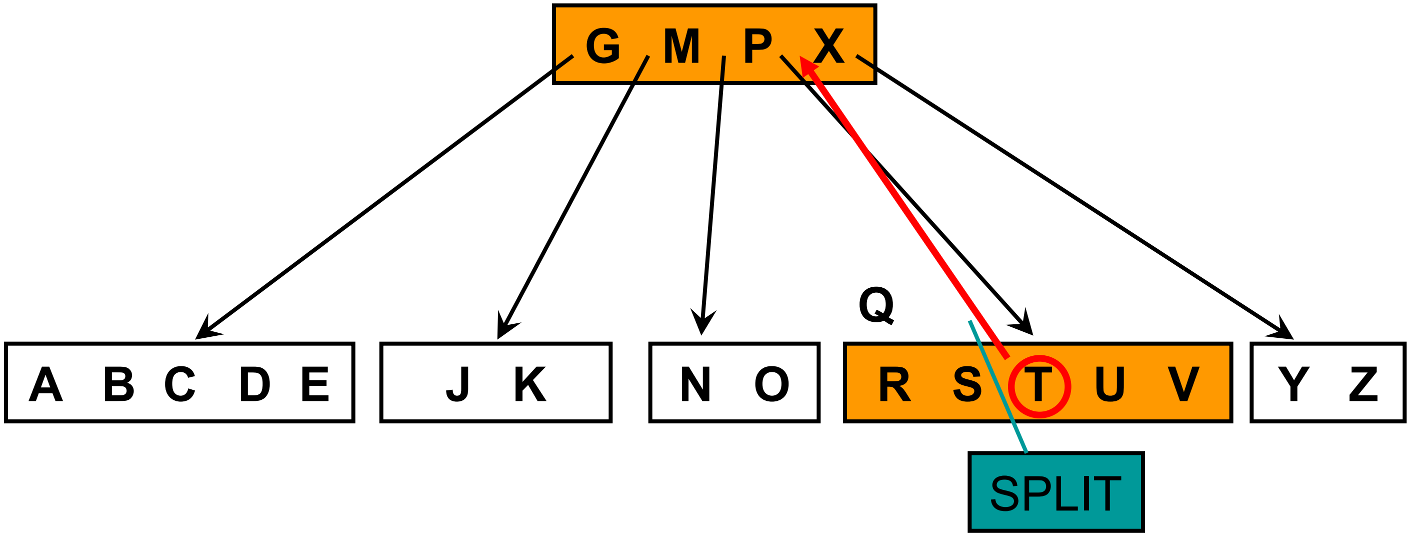
Inserting Q



Inserting Q



Inserting Q



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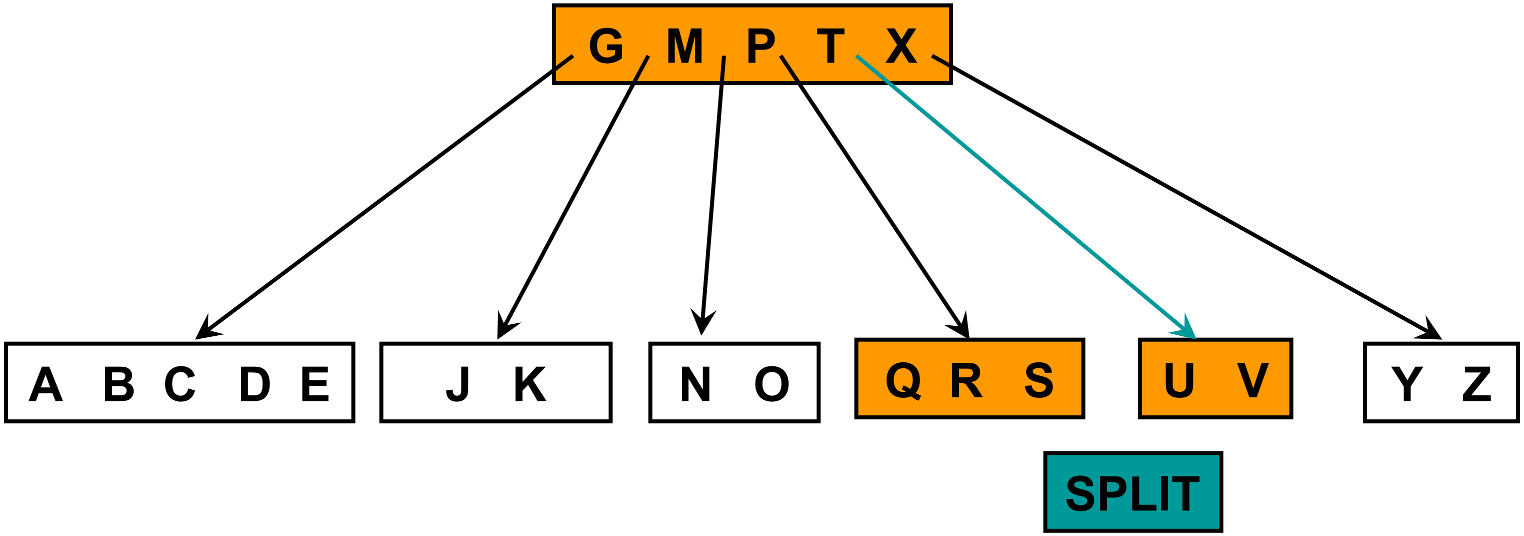
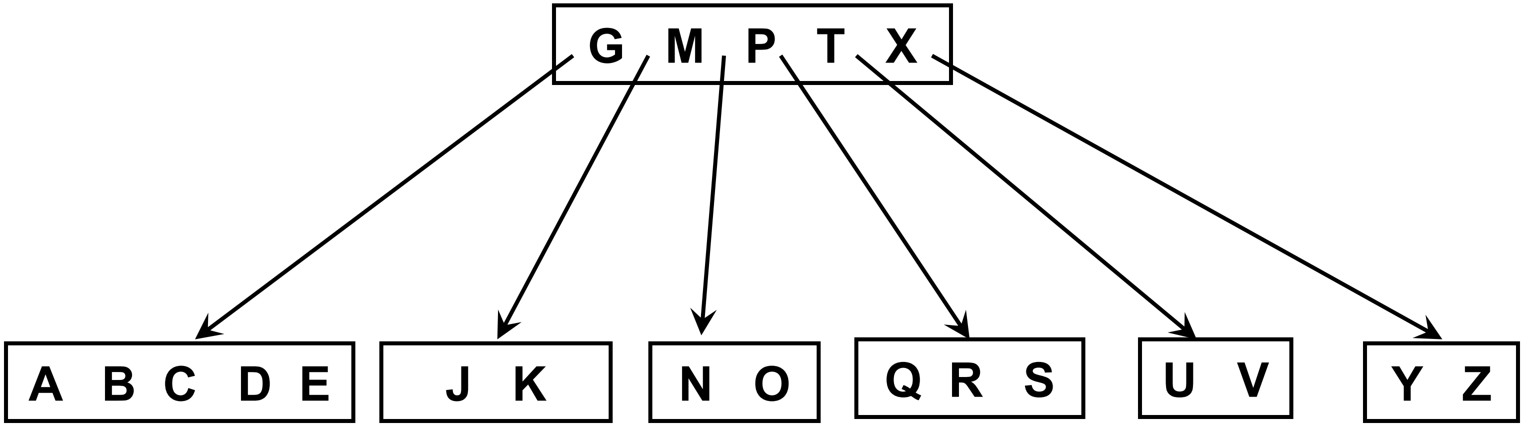
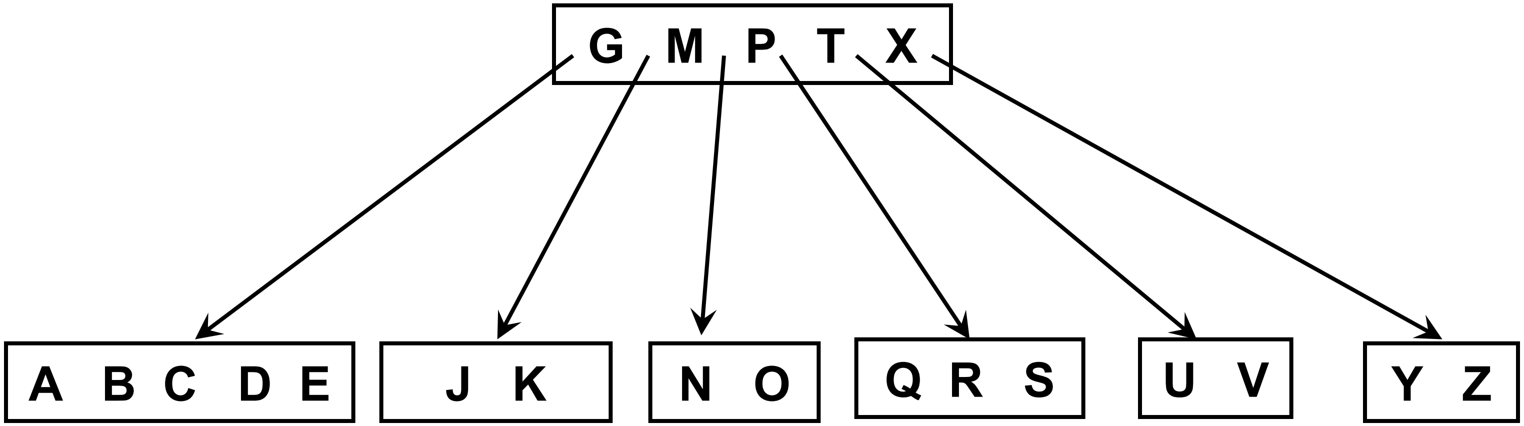


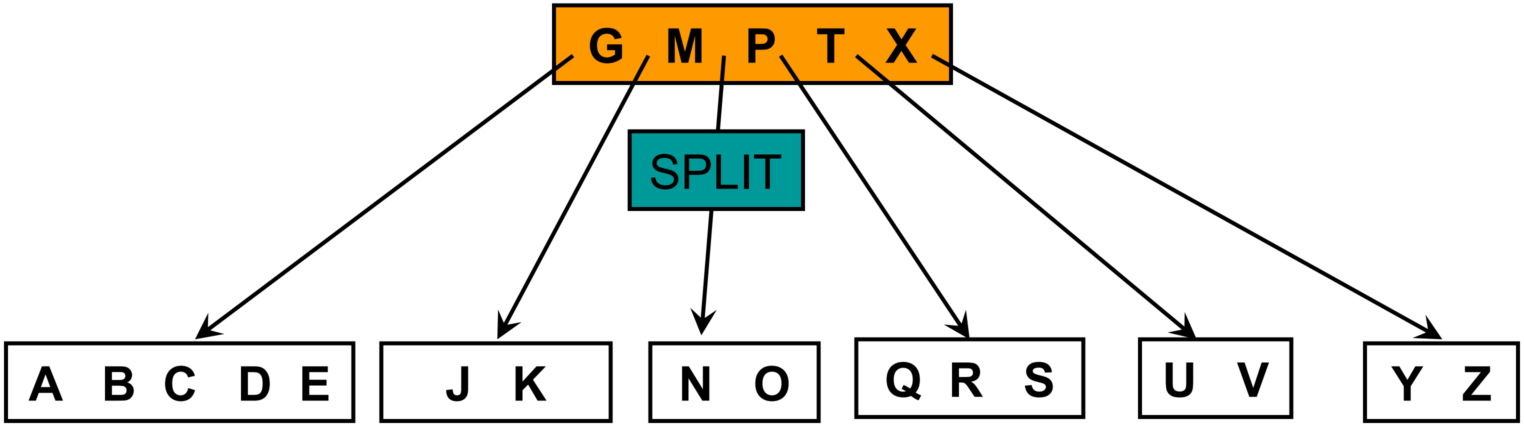
Inserting Q



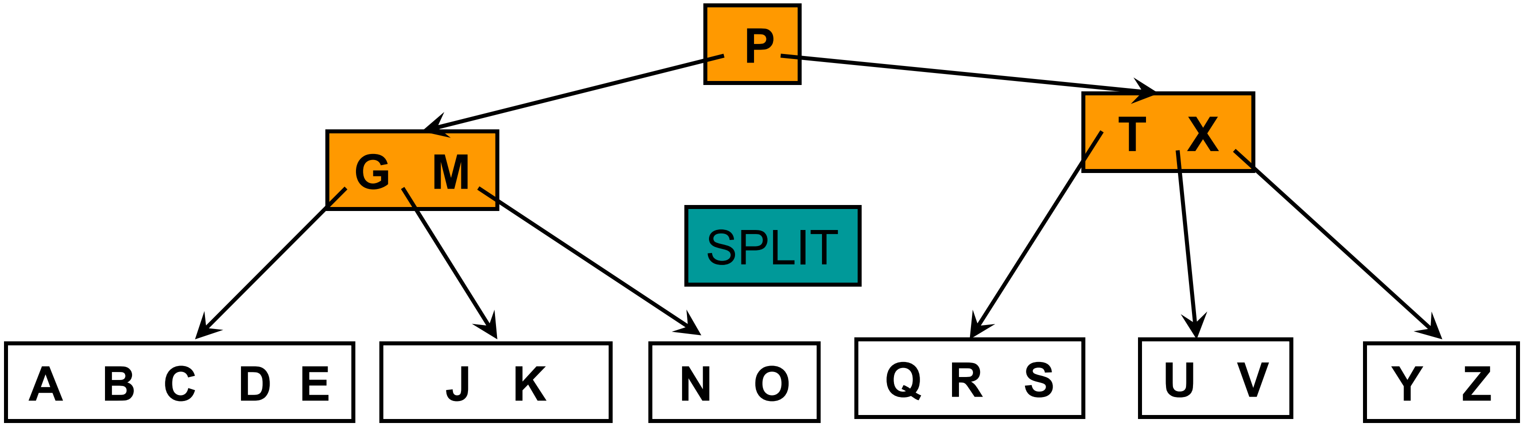
Inserting L



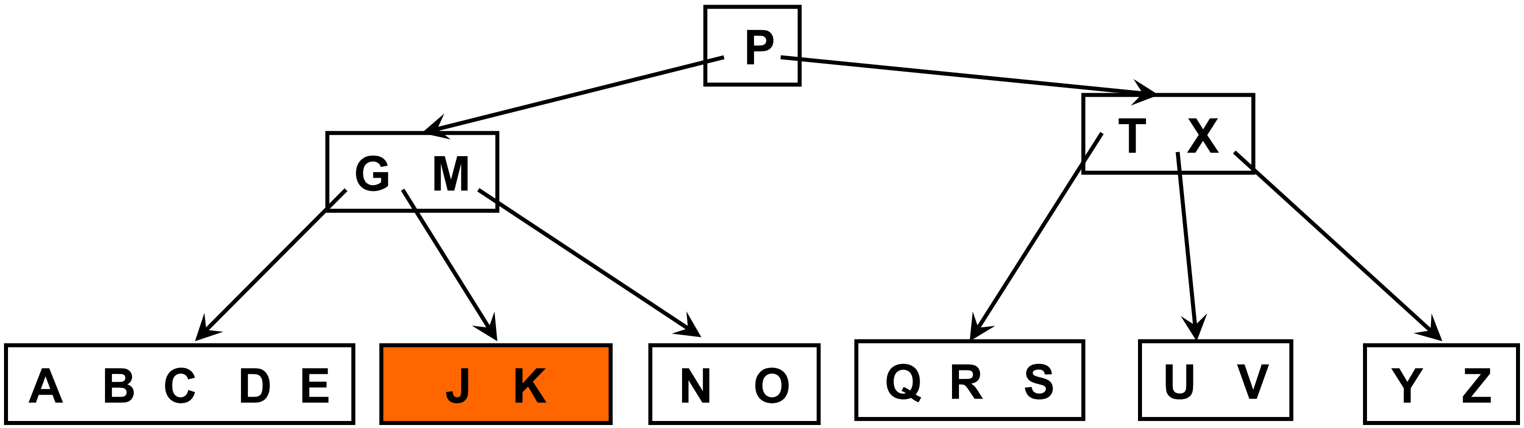
Inserting L



Inserting L



Inserting L



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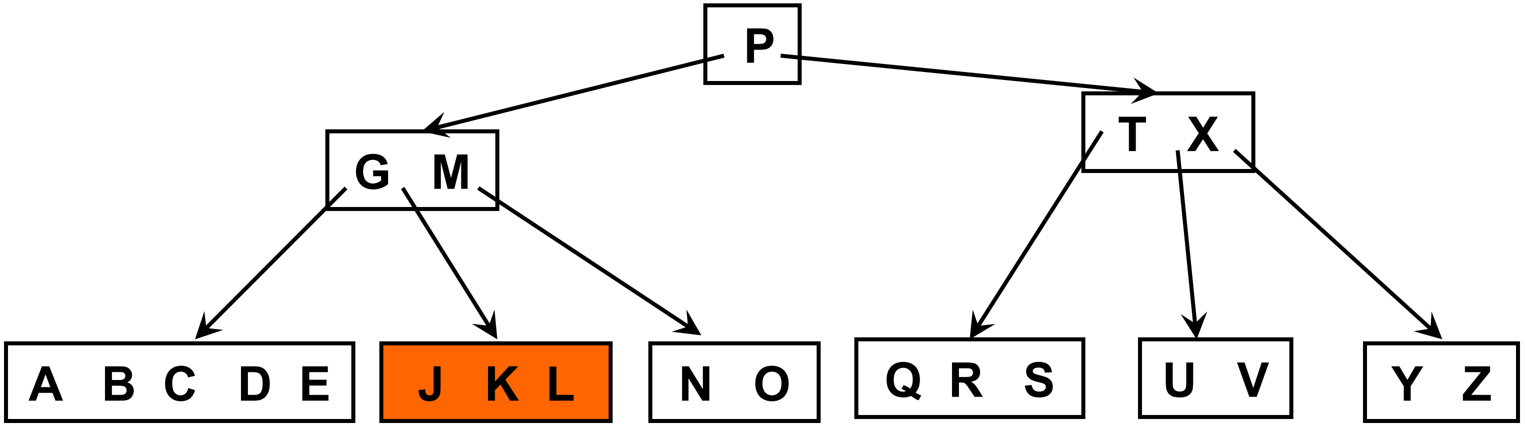

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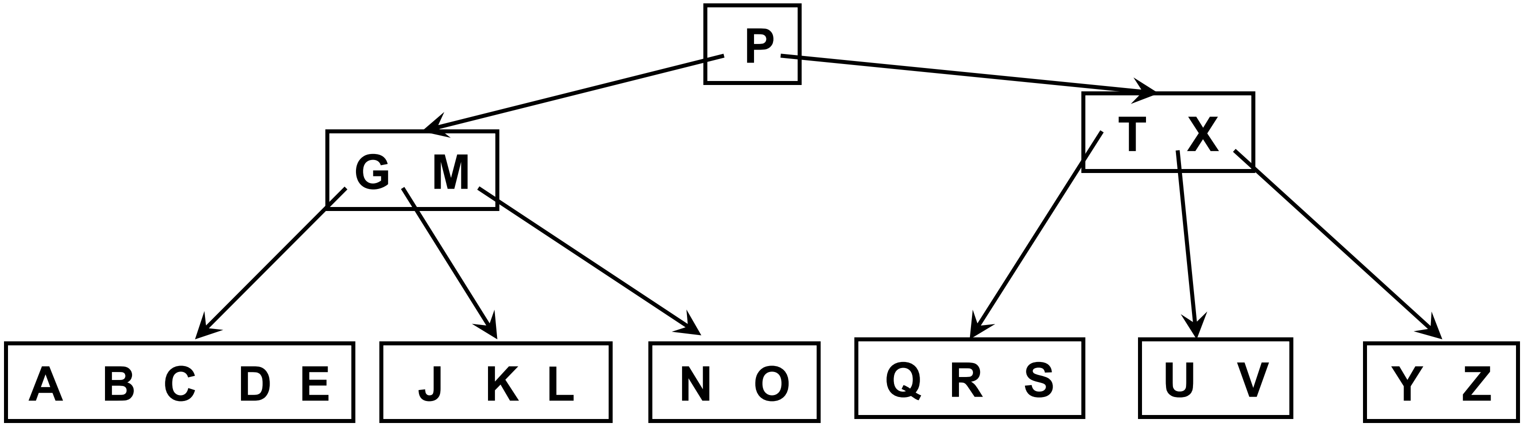

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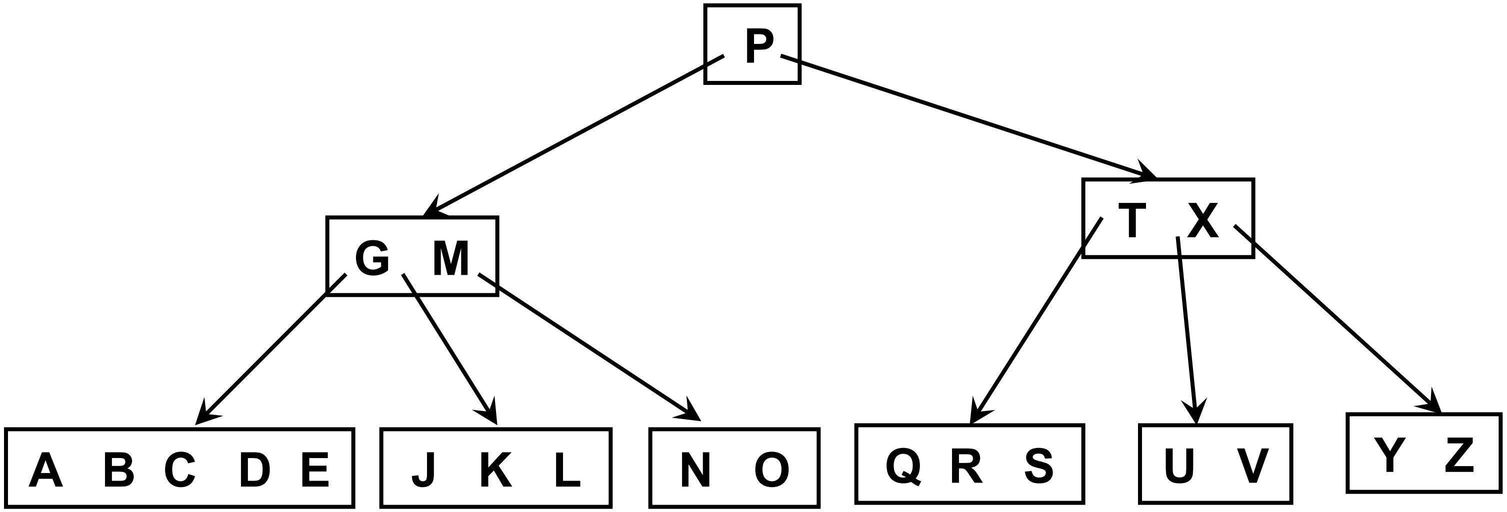

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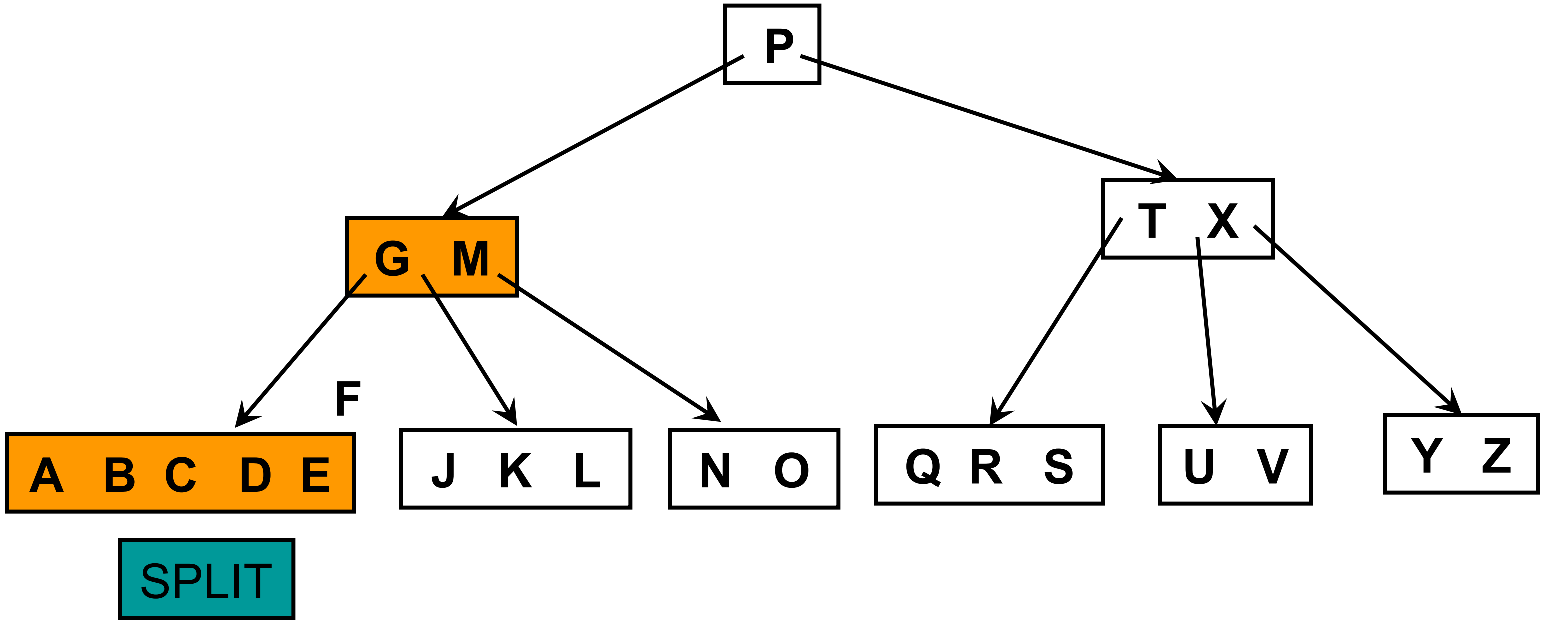

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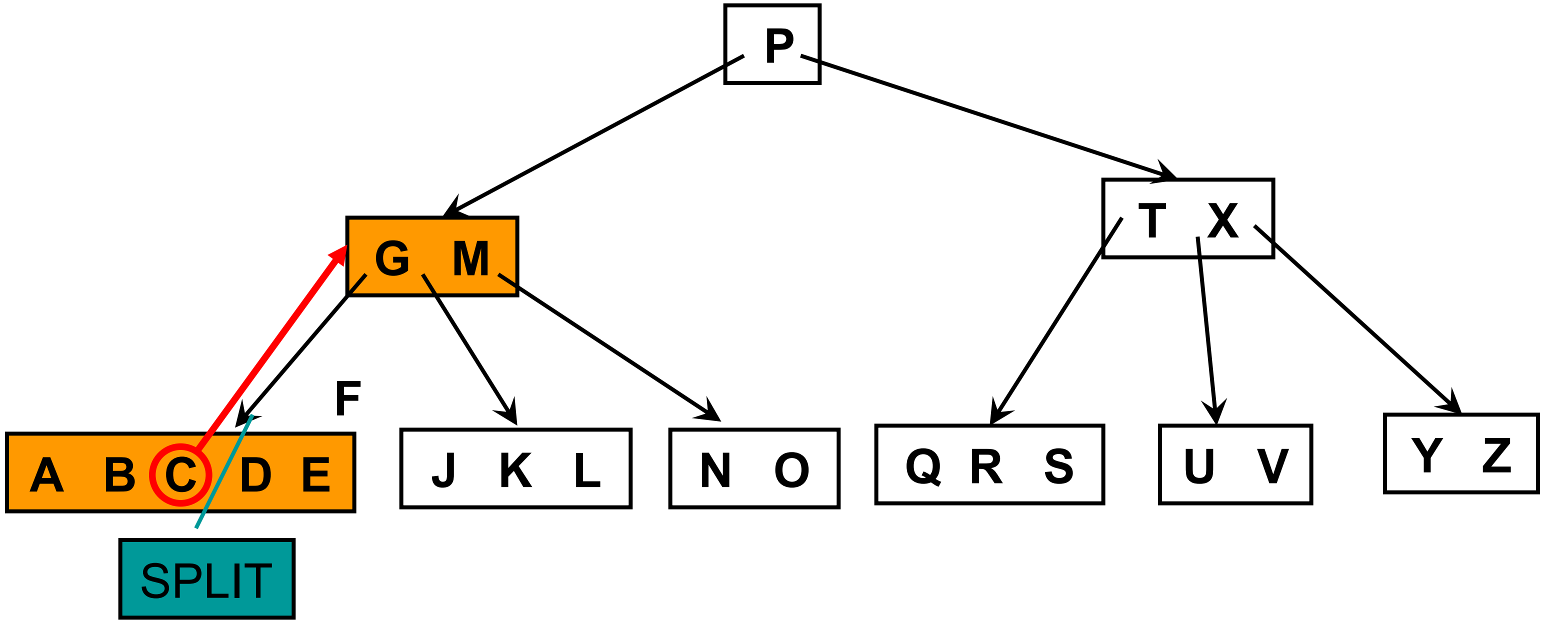

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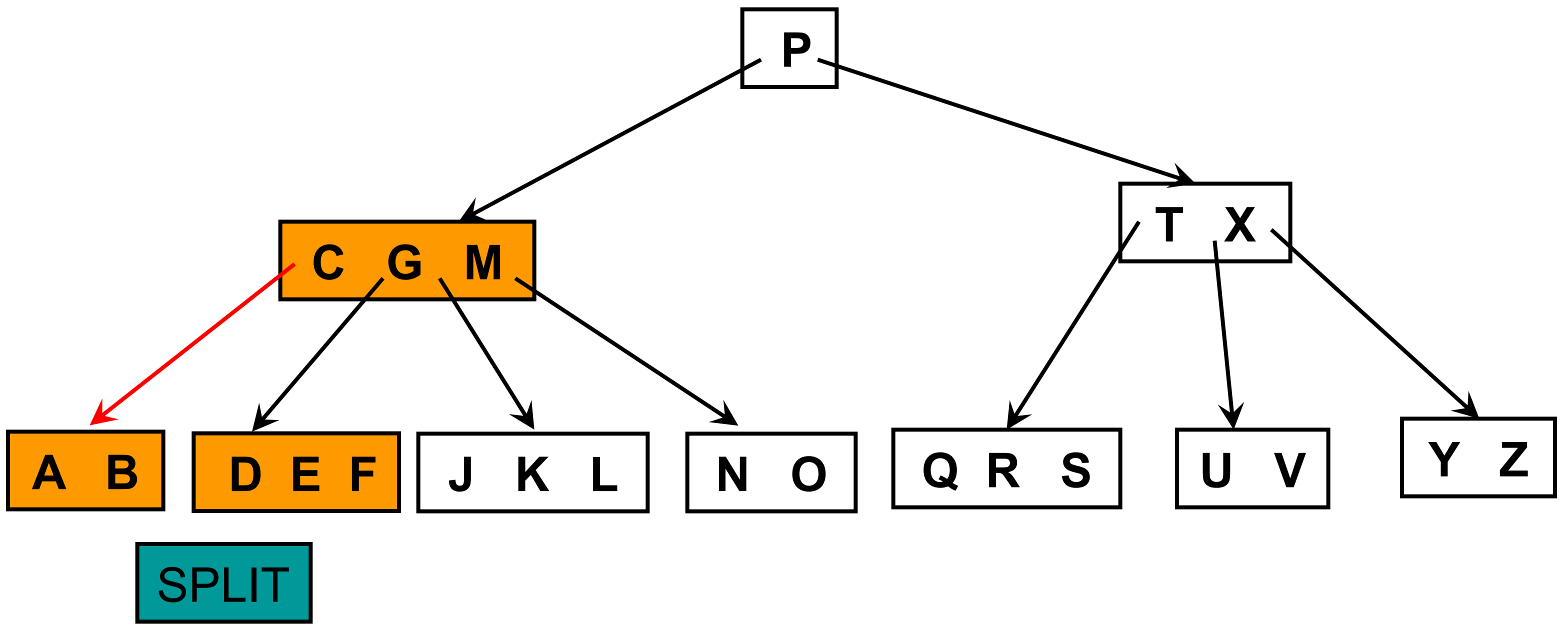

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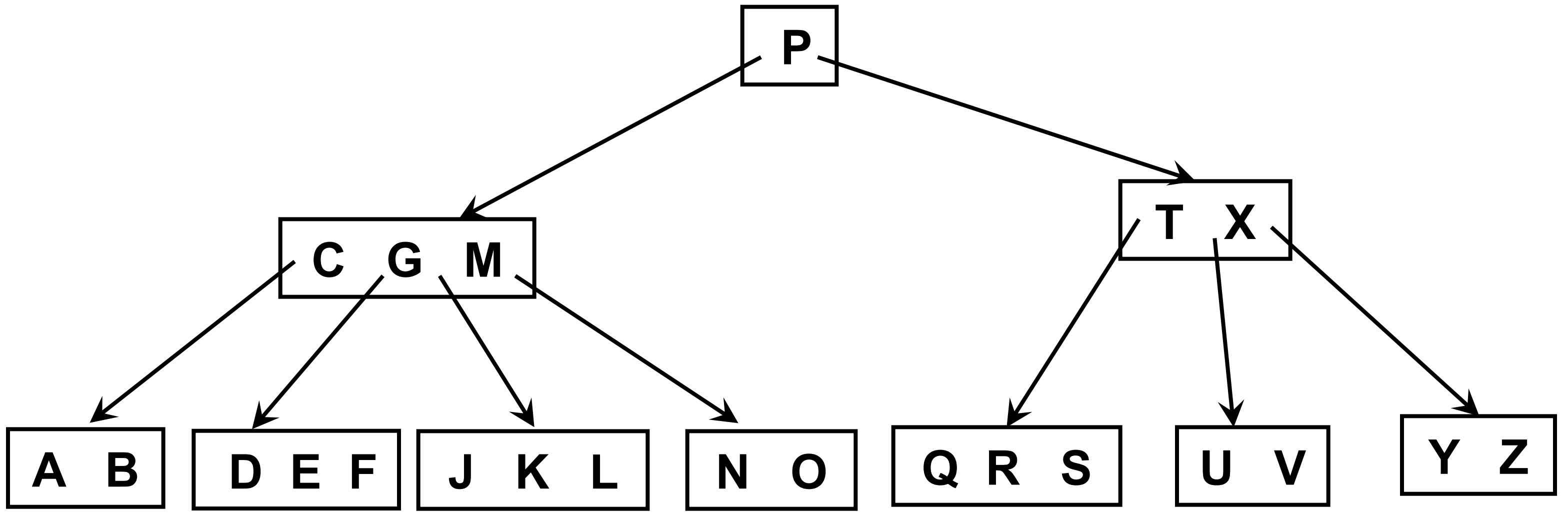
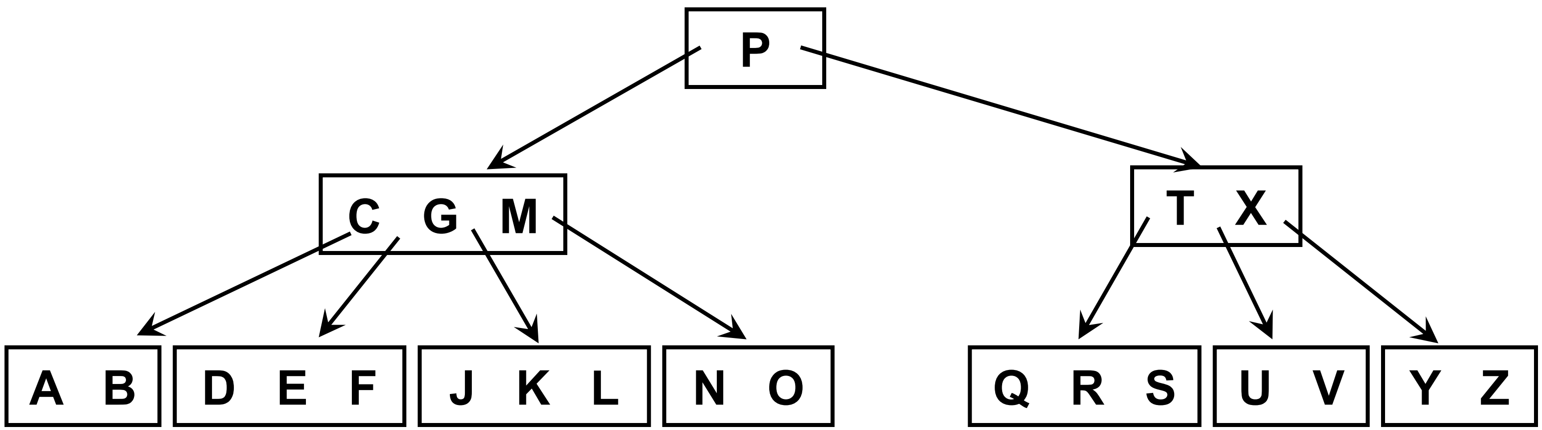


Deleting

- Case 1: key *k* is in a leaf node
- Case 2: key *k* is in an internal node
  - Subcase a: having a child with at least *t* keys preceding *k*
  - Subcase b: having a child with at least *t* keys following k
  - Subcase c: both have *t-1* keys
- Case 3: key *k* is not in an internal node and root of an appropriate subtree has only t-1 keys
  - Subcase a: subtree has only *t-1* keys having a sibling with at least *t* keys
  - Subcase b: both subtree and immediate siblings have *t-1* keys

Deleting F (Case 1)



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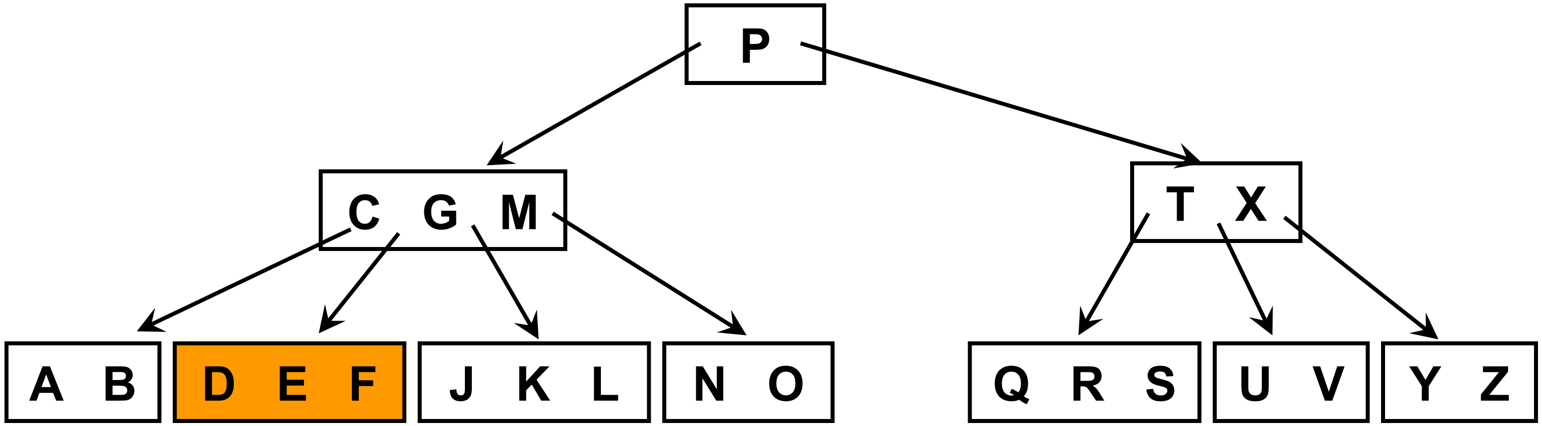

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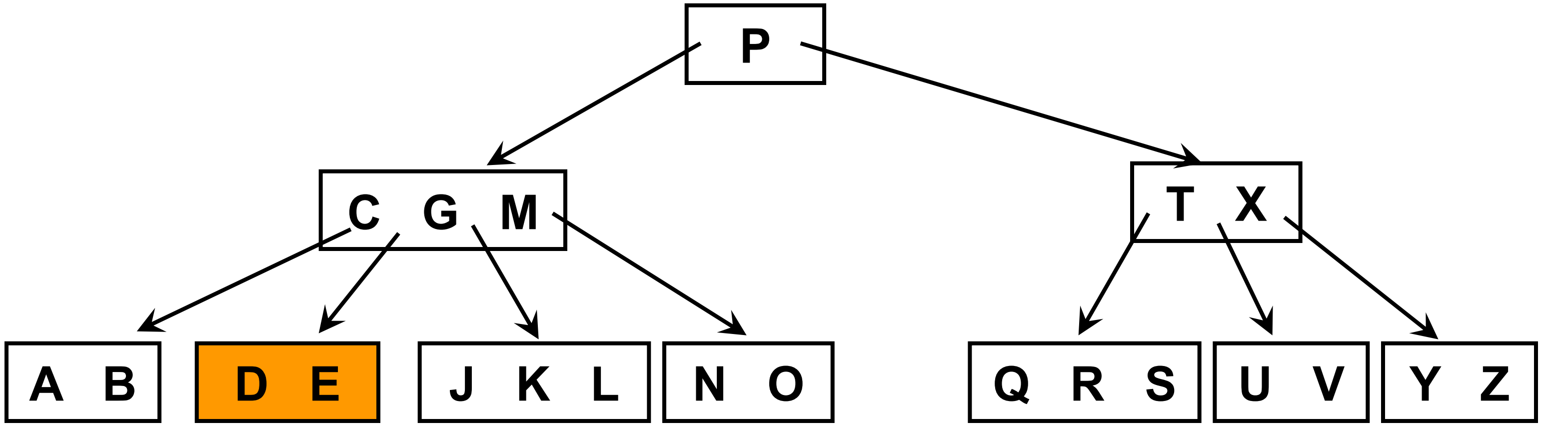
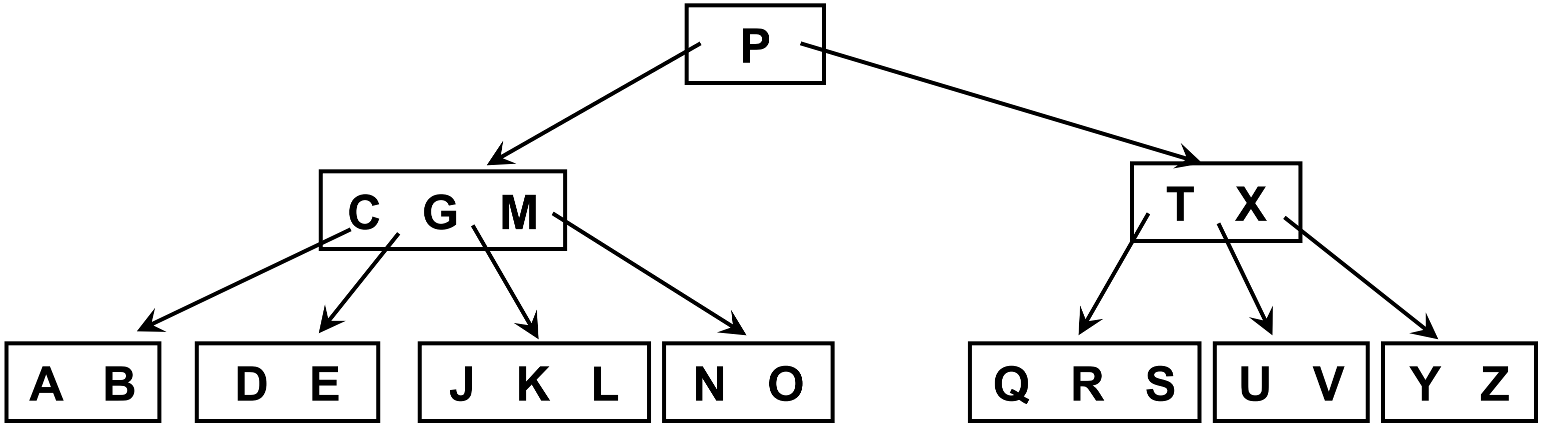
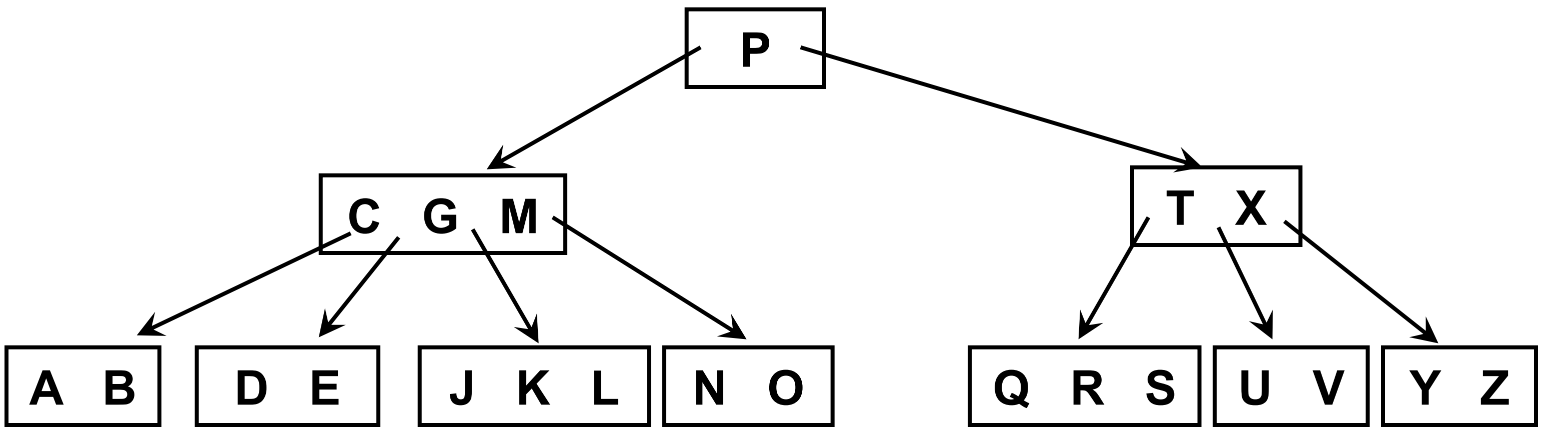


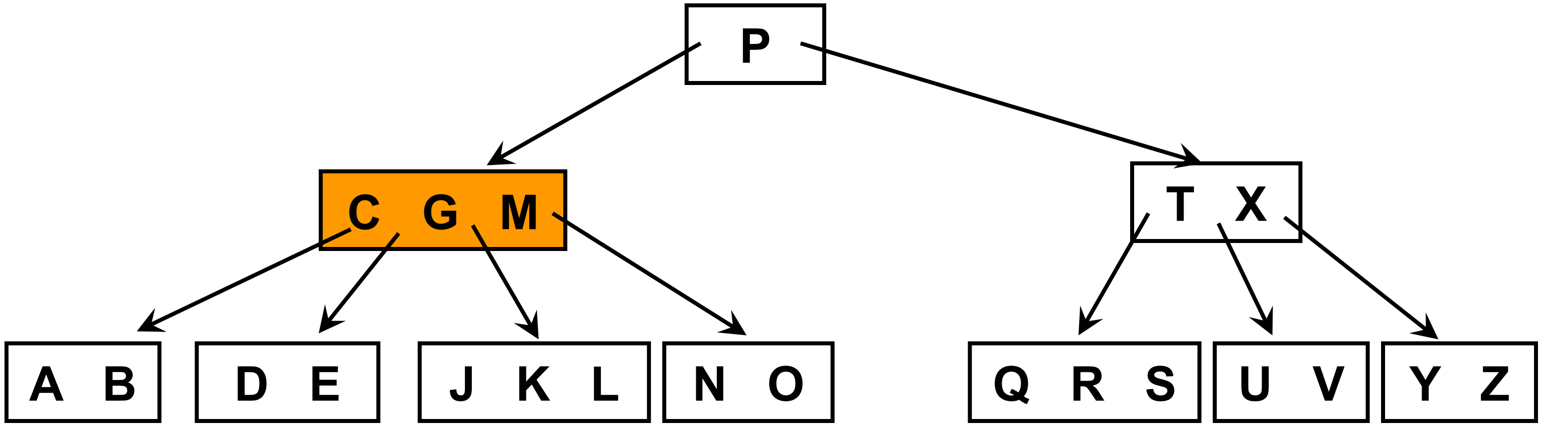
Deleting F



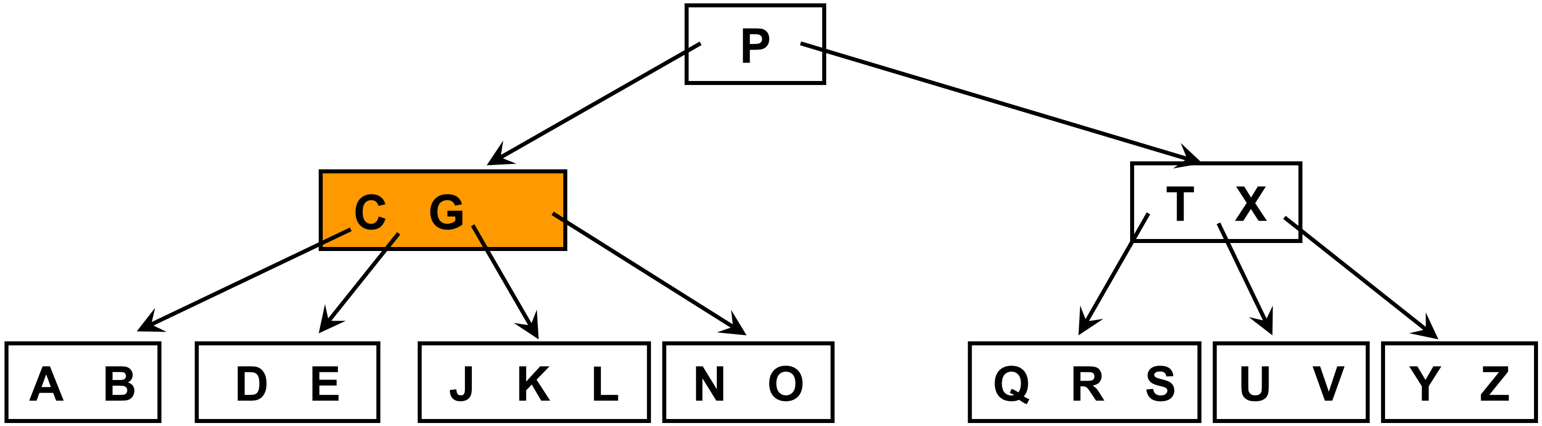
Deleting M (Case 2a)



Deleting M



Deleting M



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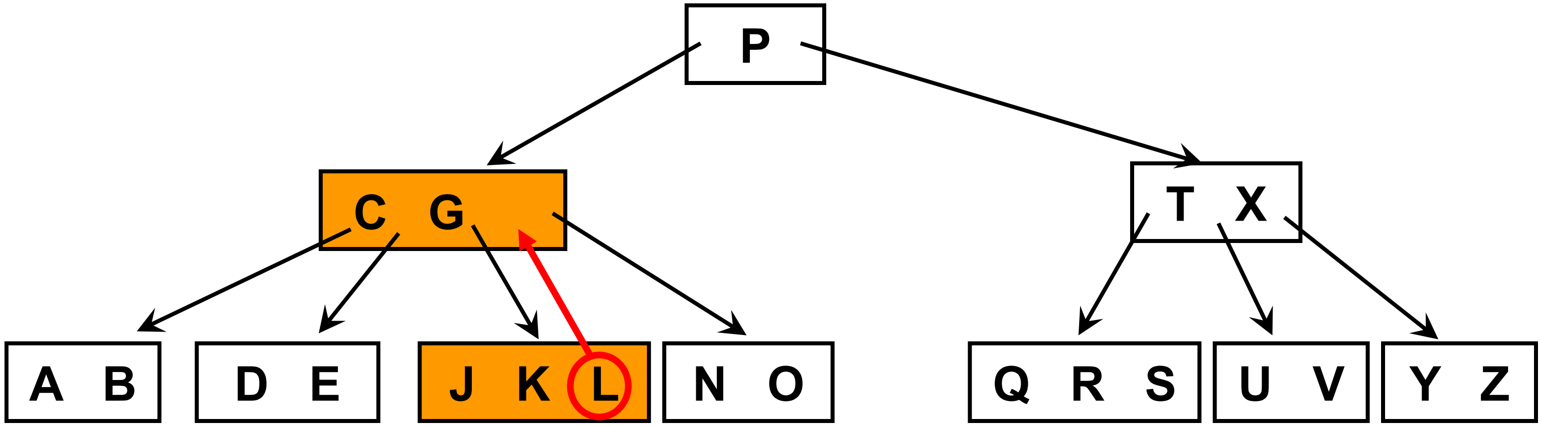
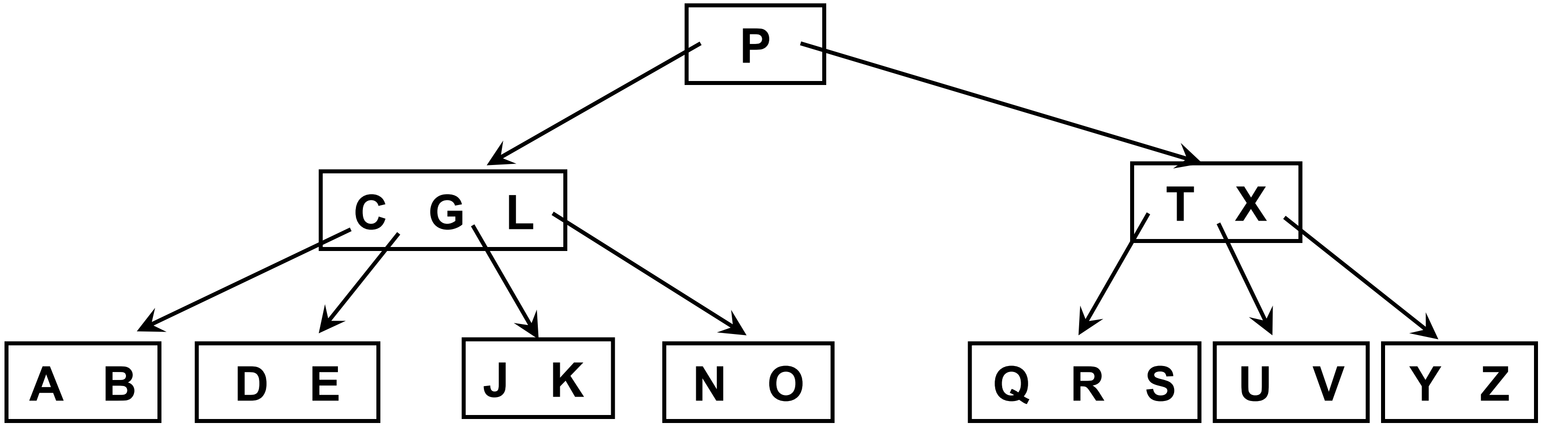
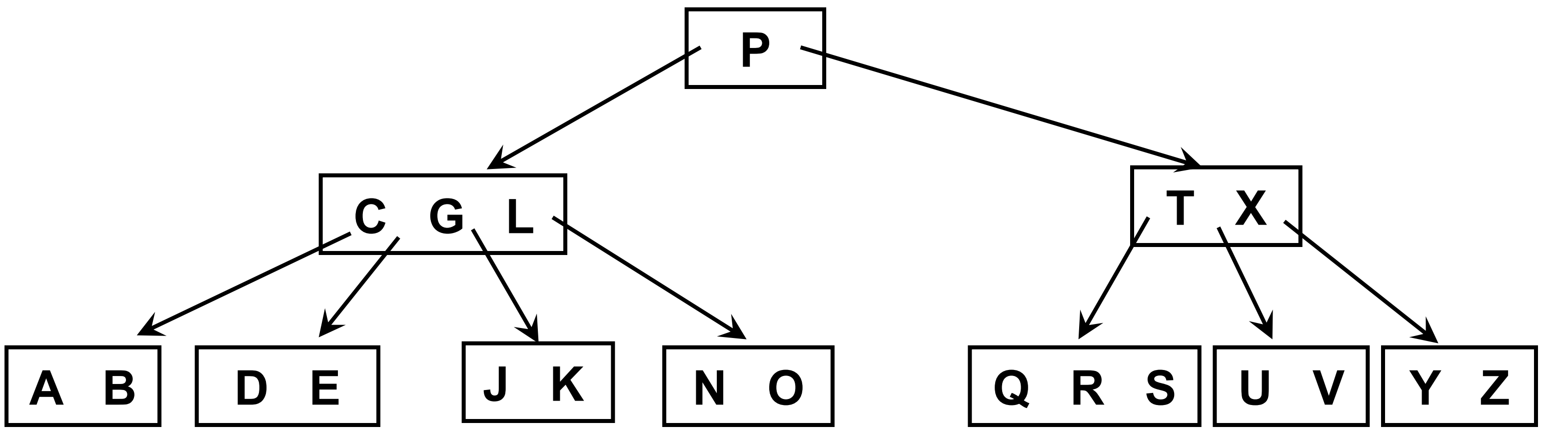


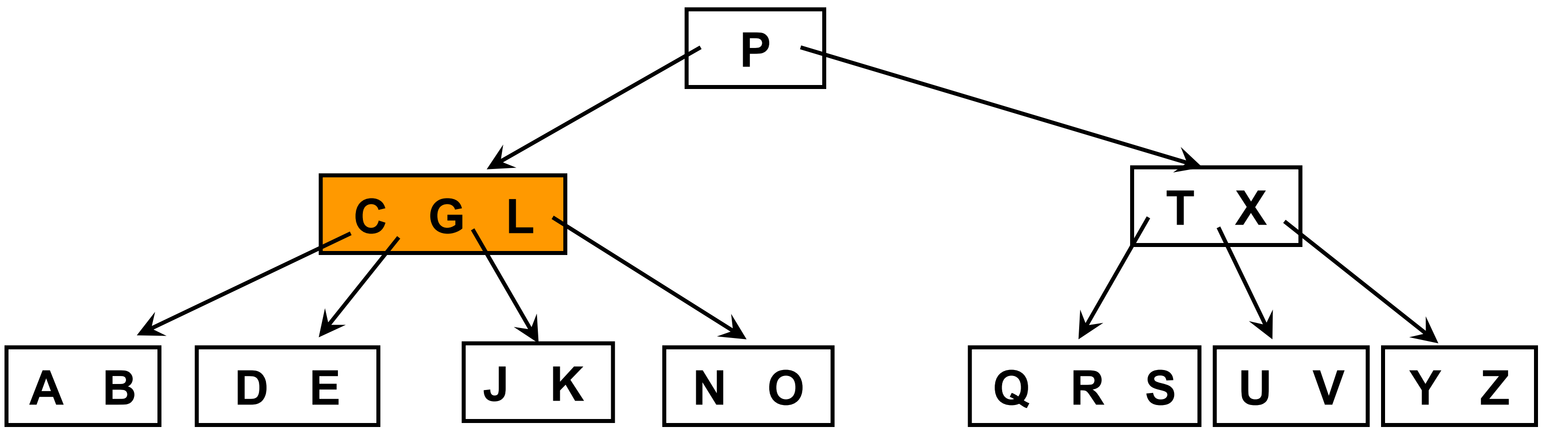
Deleting M



Deleting G (Case 2c)



Deleting G



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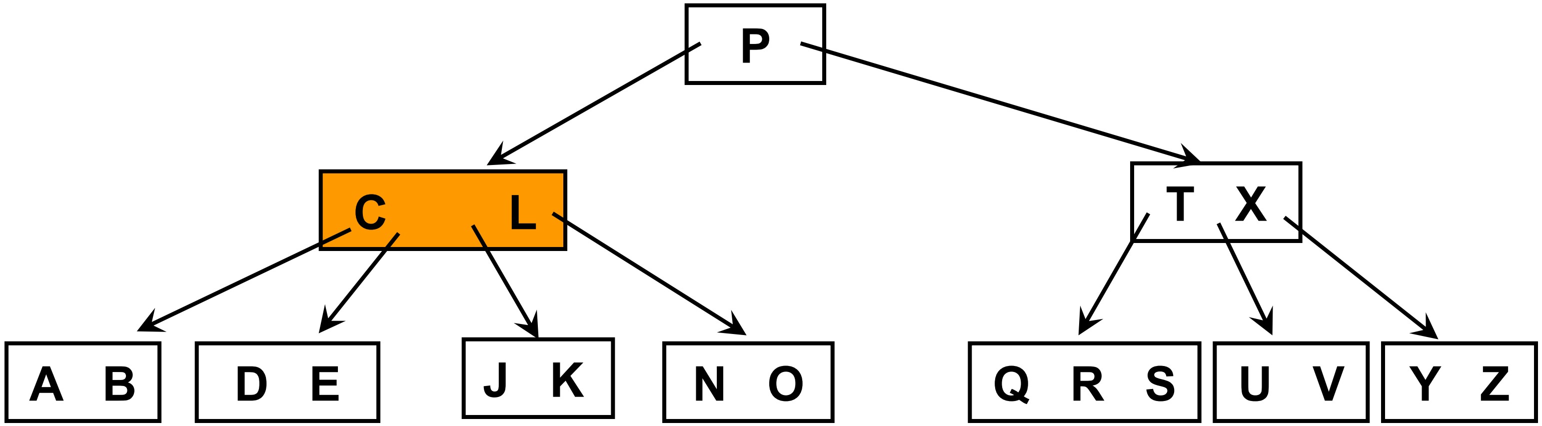

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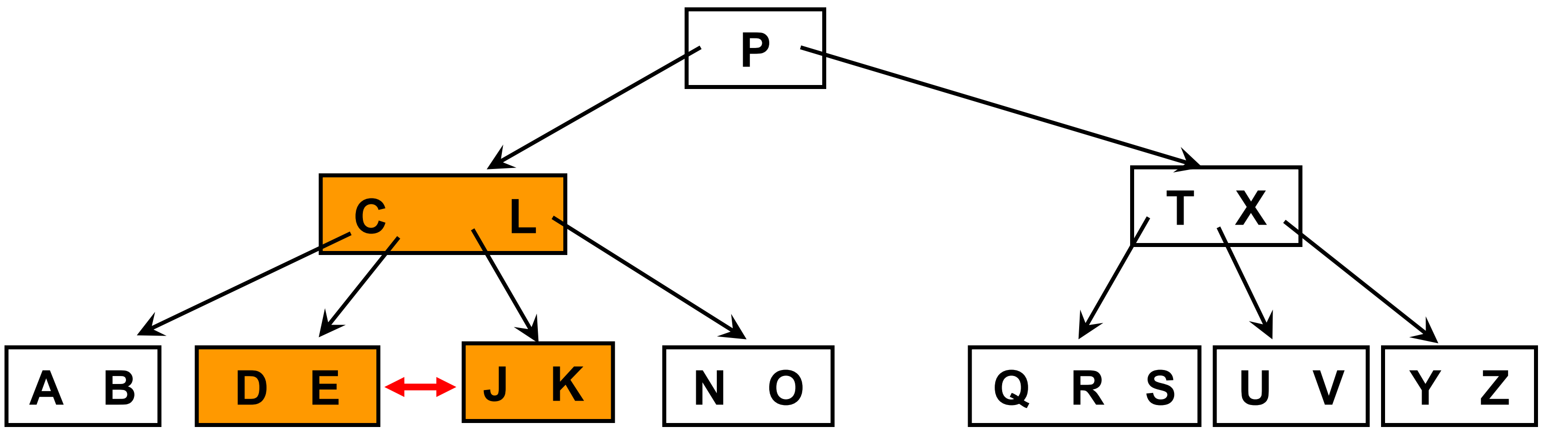

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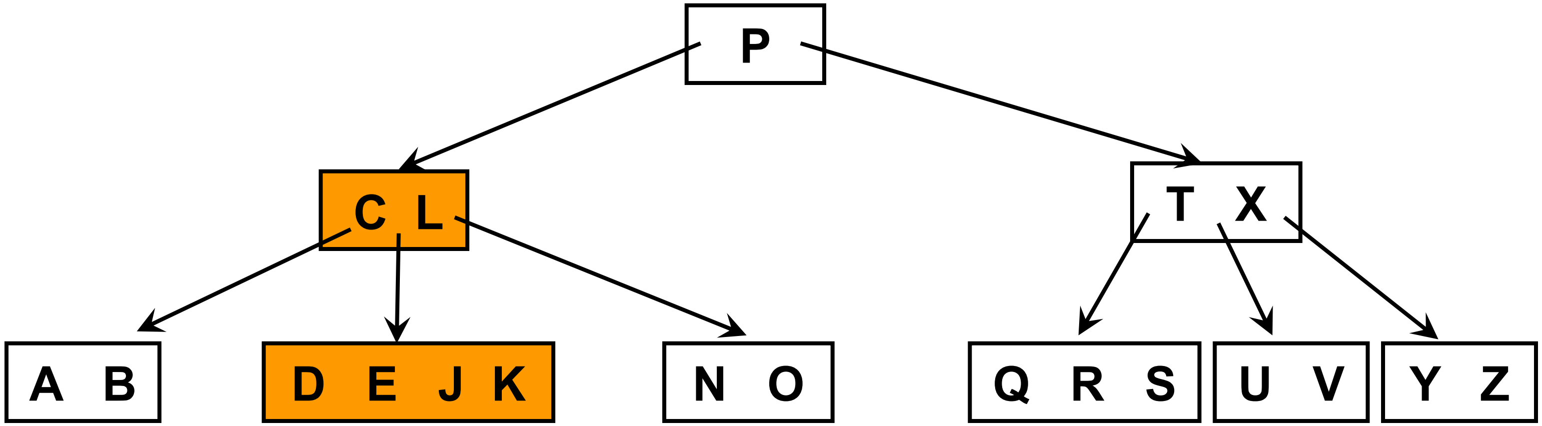
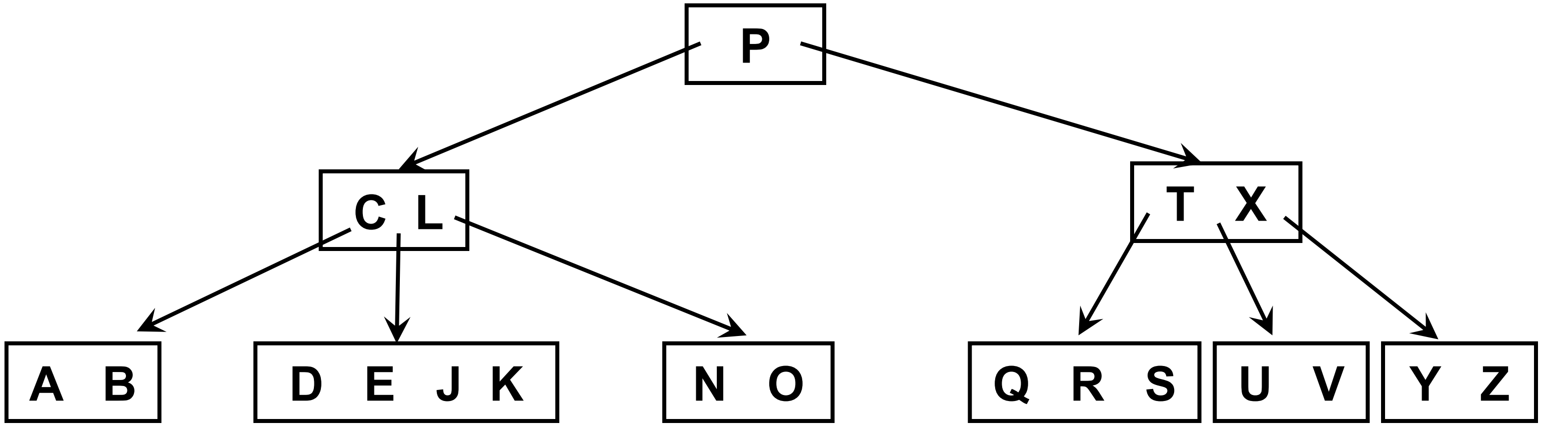
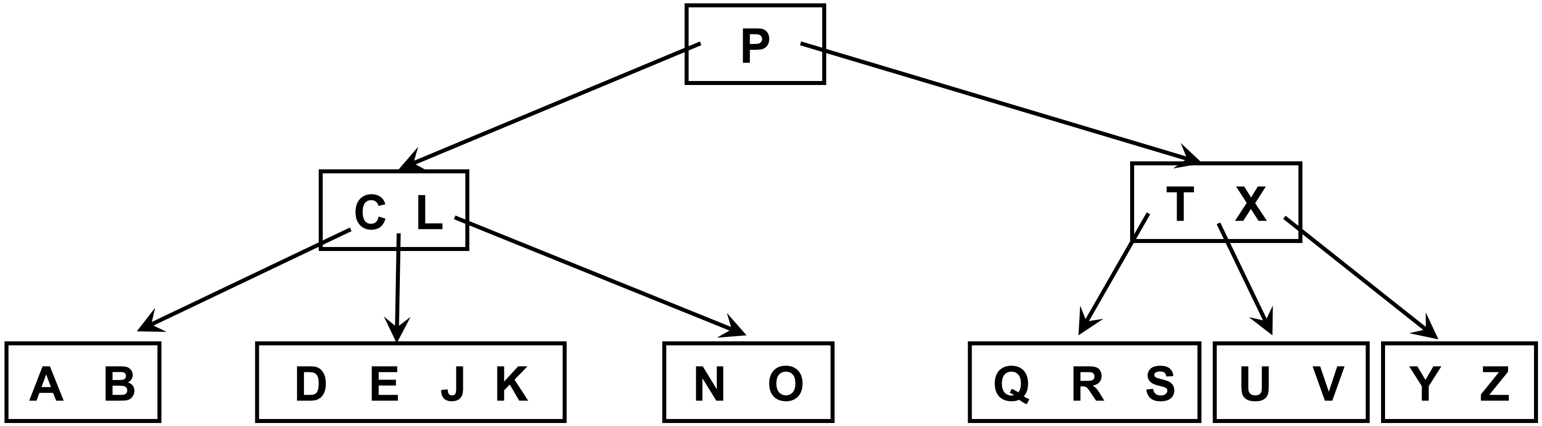


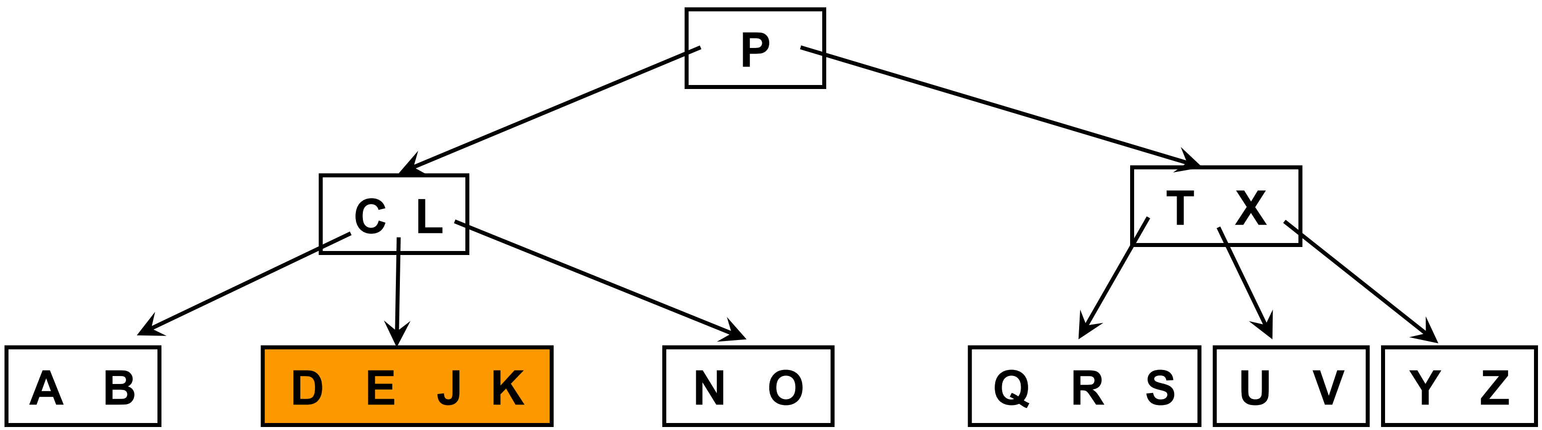
Deleting G



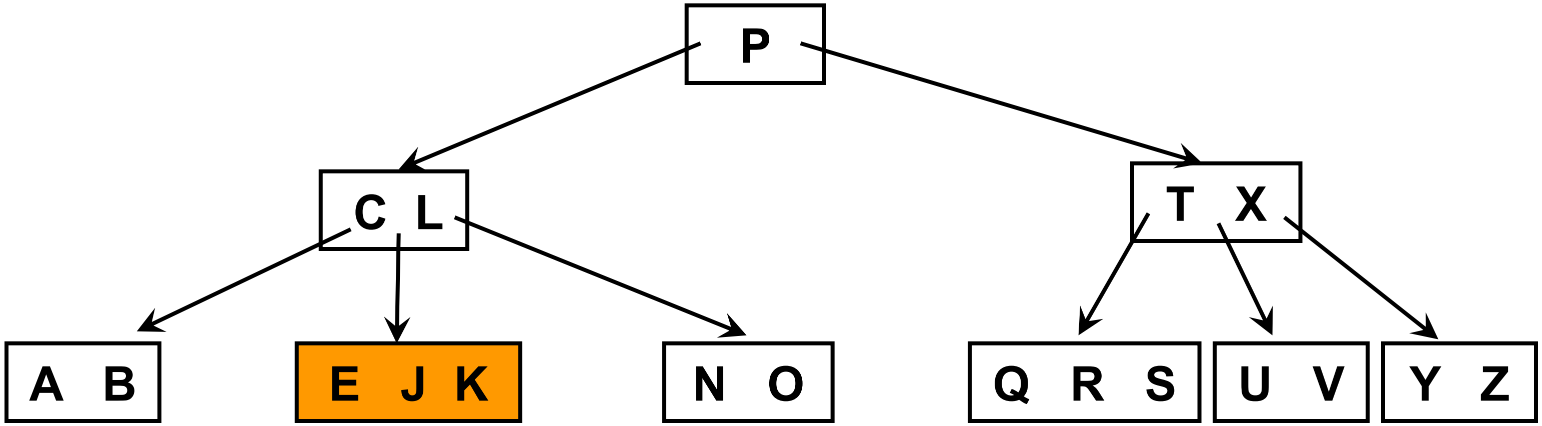
Deleting D (Case 3b)



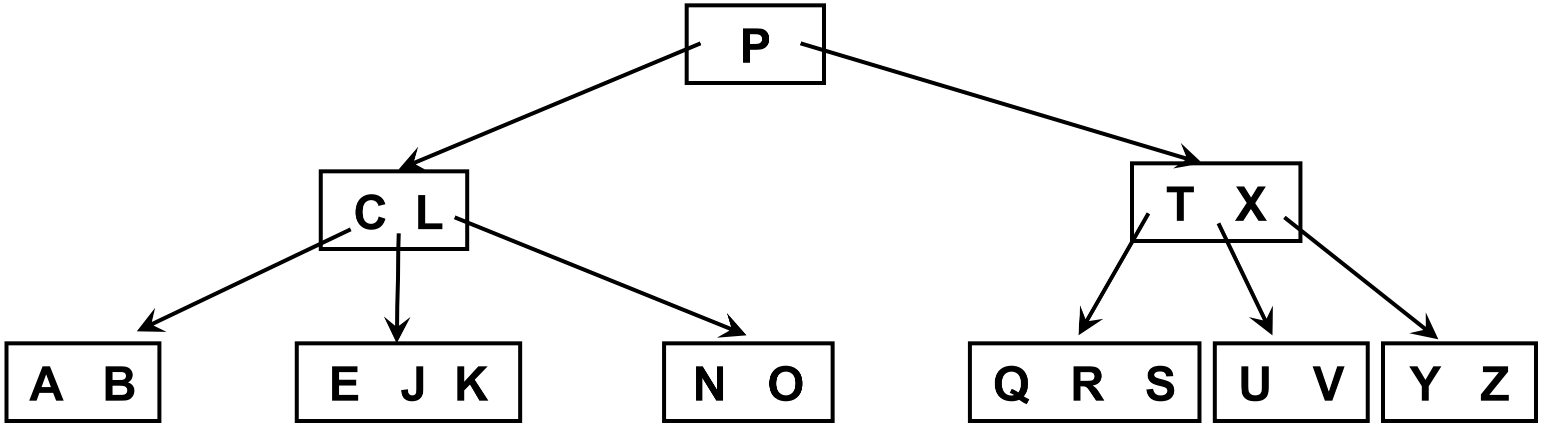
Deleting D



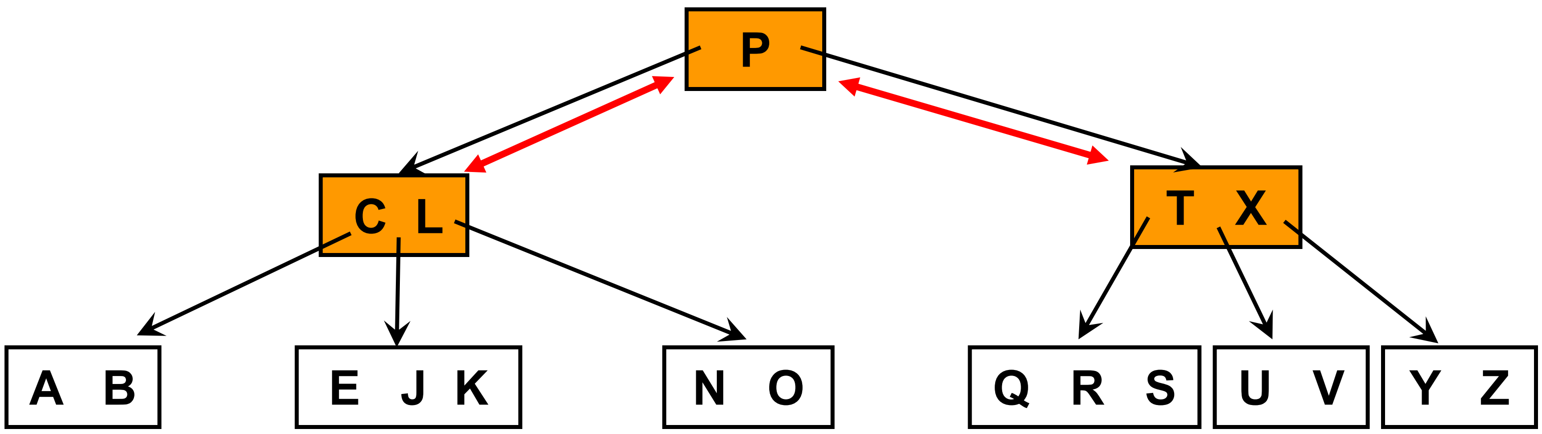
Deleting D



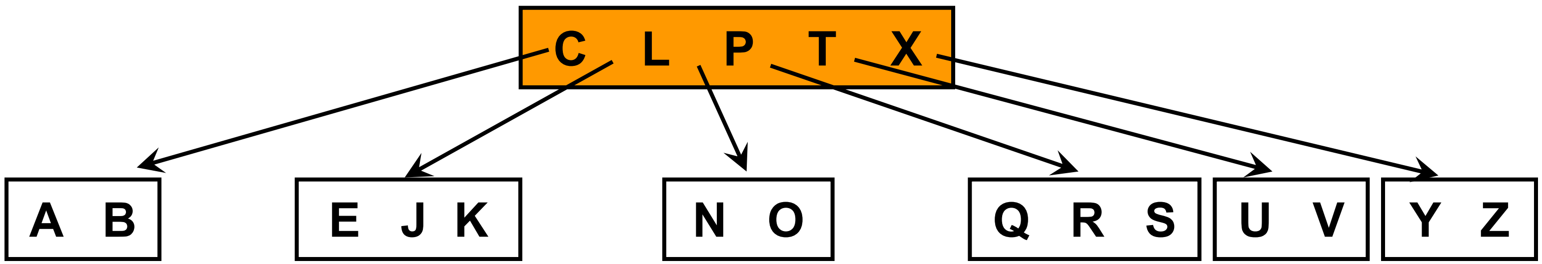
Deleting D



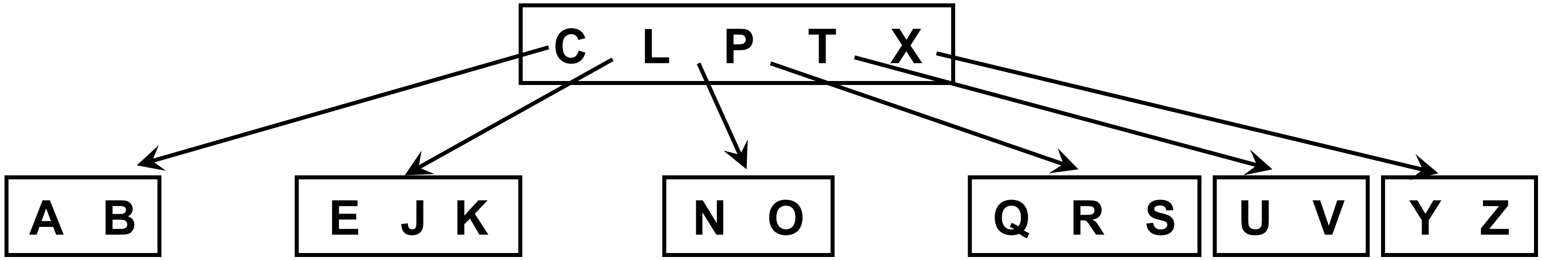
Deleting D



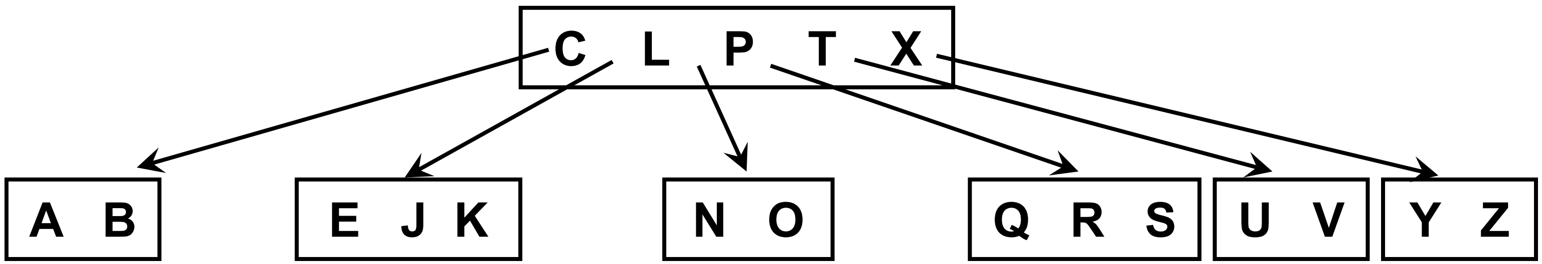
Deleting D



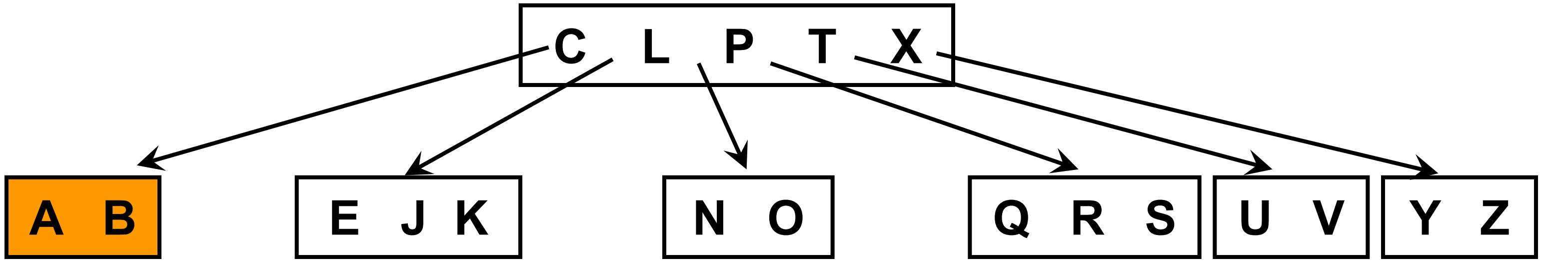
Deleting D



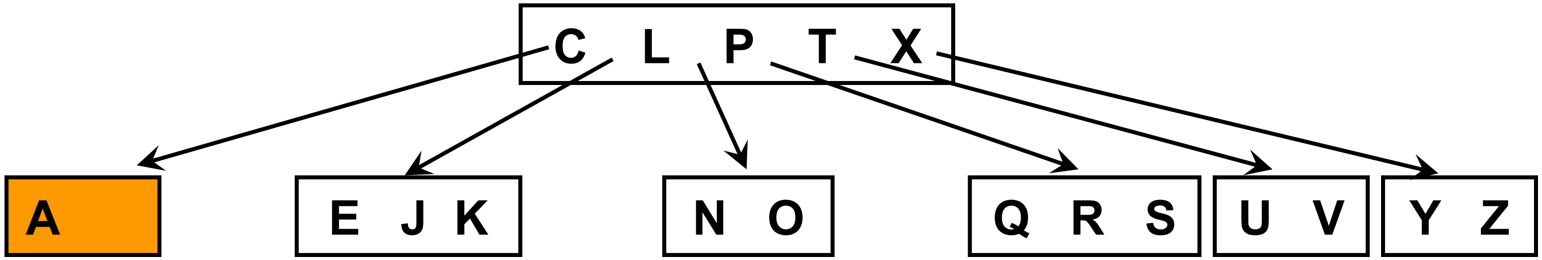
Deleting B (Case 3a)



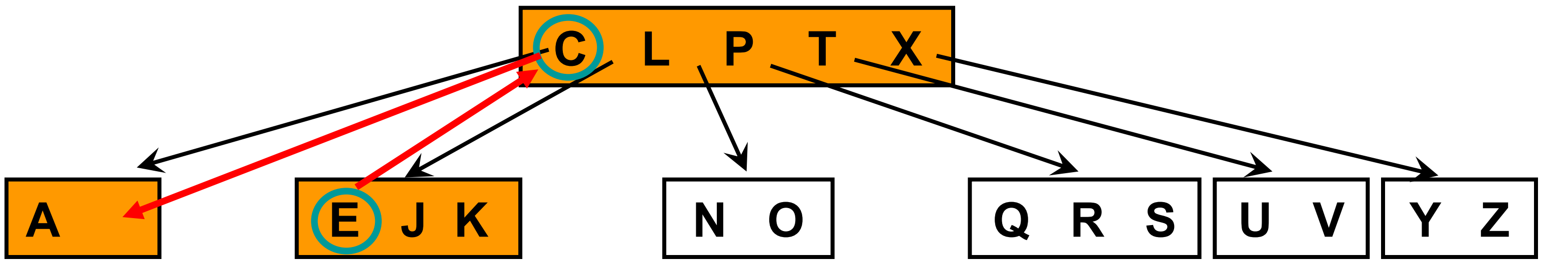
Deleting B



Deleting B



Deleting B



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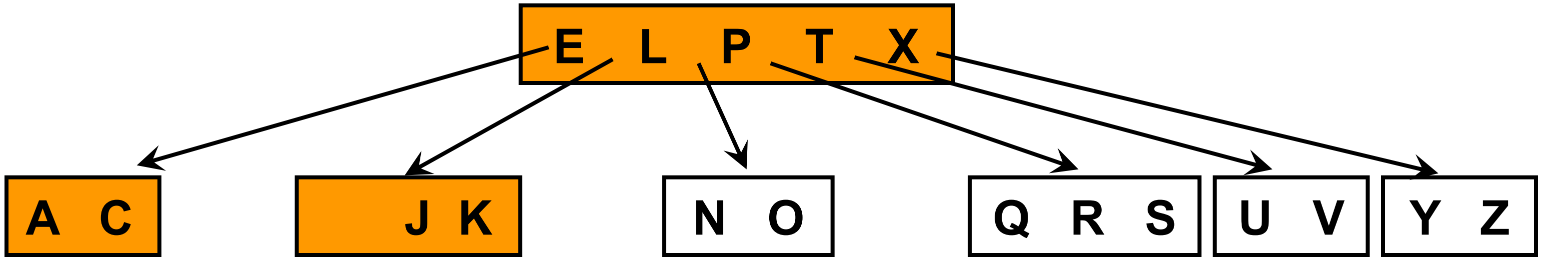
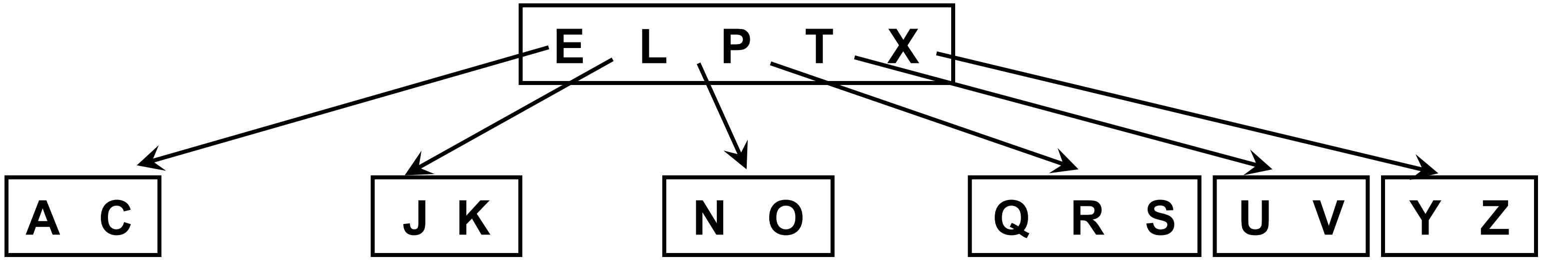


Deleting B



Another type of B-Tree: m-way

- m-way: m pointers and m keys
- For most of the database applications satellite data is kept only in the leaves
- Requires changes in search, insert delete algorithms!
- Input order

  C S D T A M P I B W N G U

  R K E H O L J Y Q Z F X V

- Insert into a 4-way B-Tree

Insert C, S, D, T

•	**C**	•	**S**	•		•	

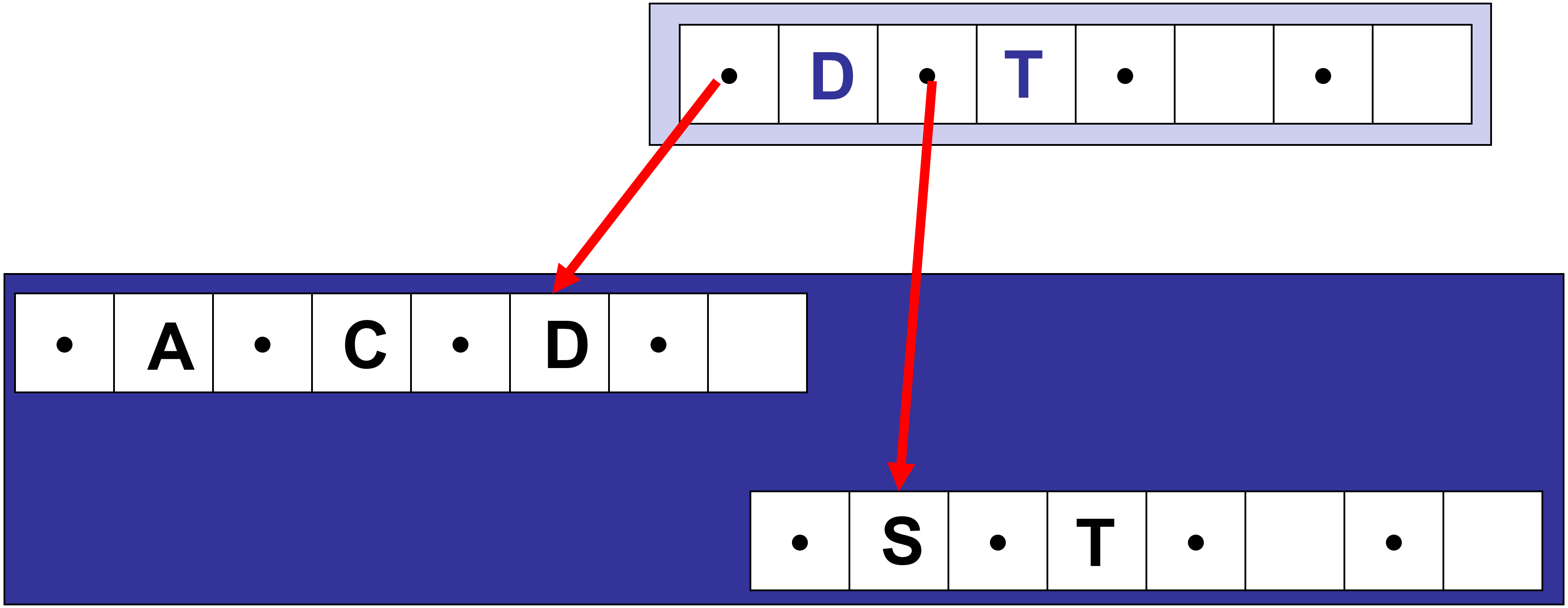
Insert C, S, D, T

•	**C**	•	**D**	•	**S**	•	

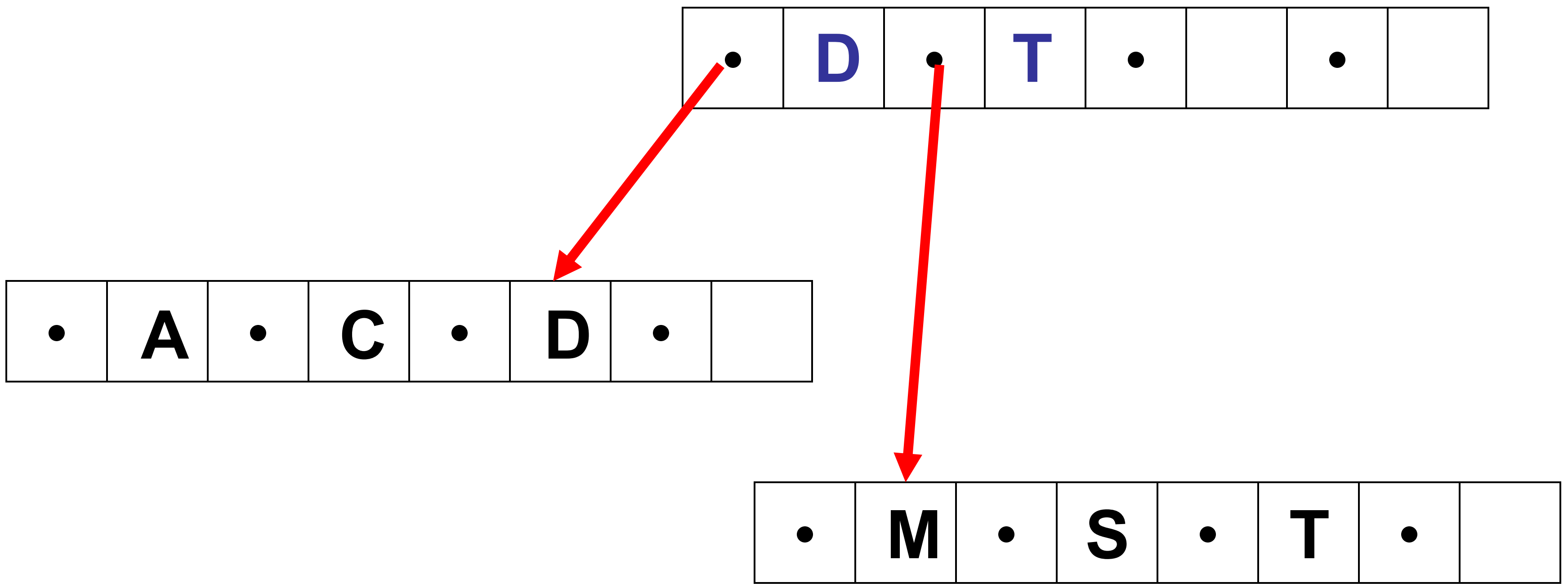
Insert C, S, D, T, A

•	**C**	•	**D**	•	**S**	•	**T**

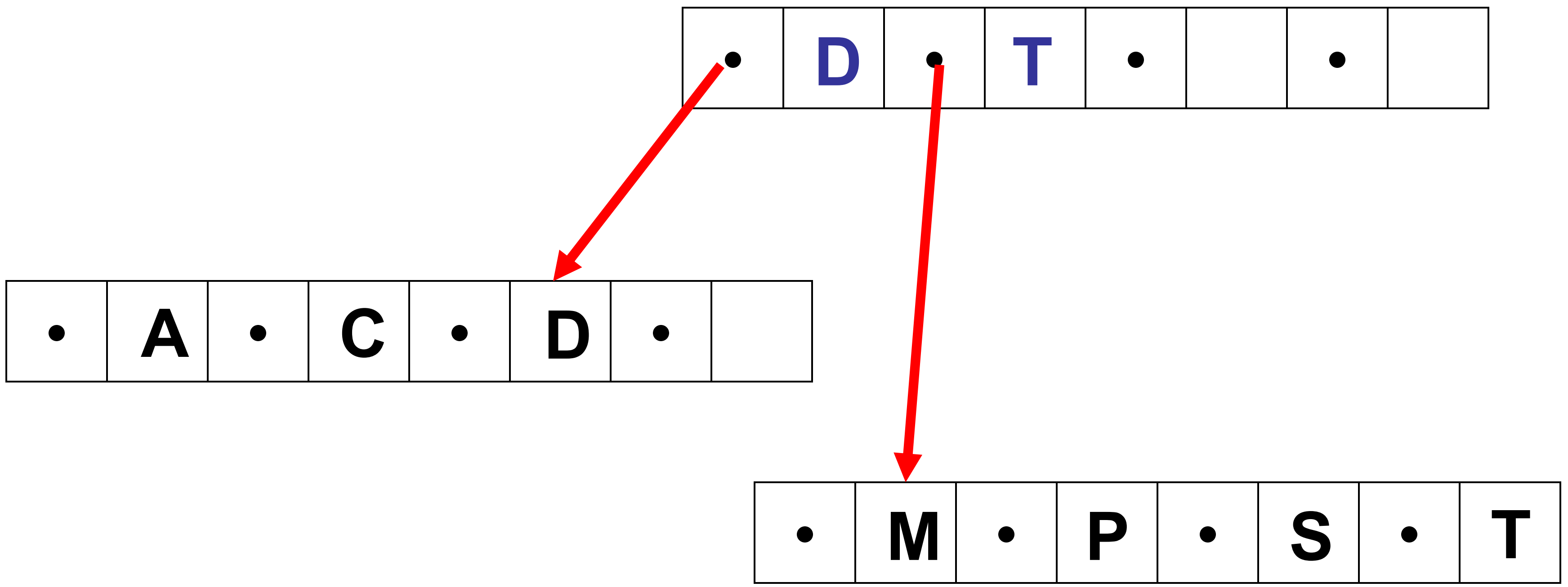
Insert A, M



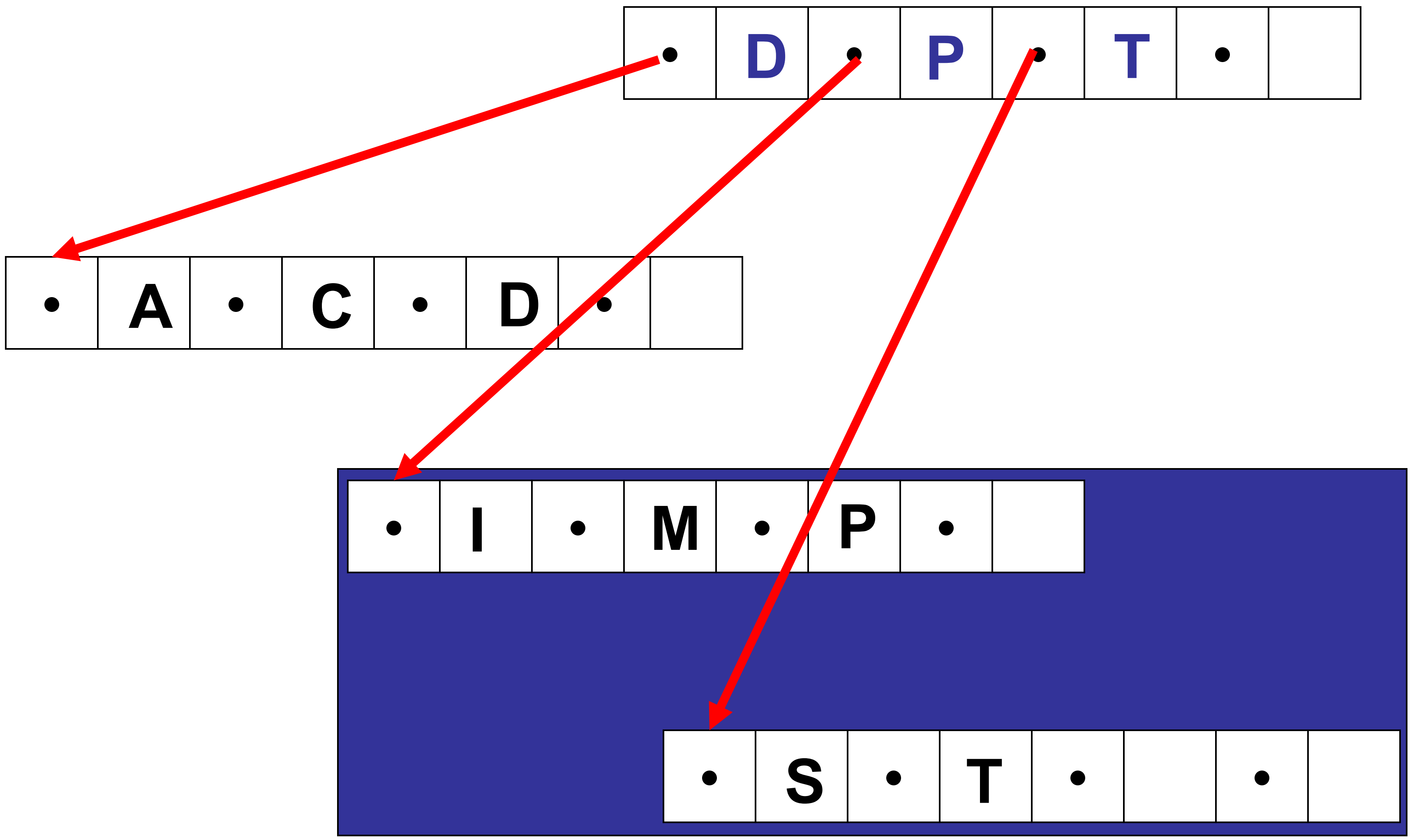
Insert M, P



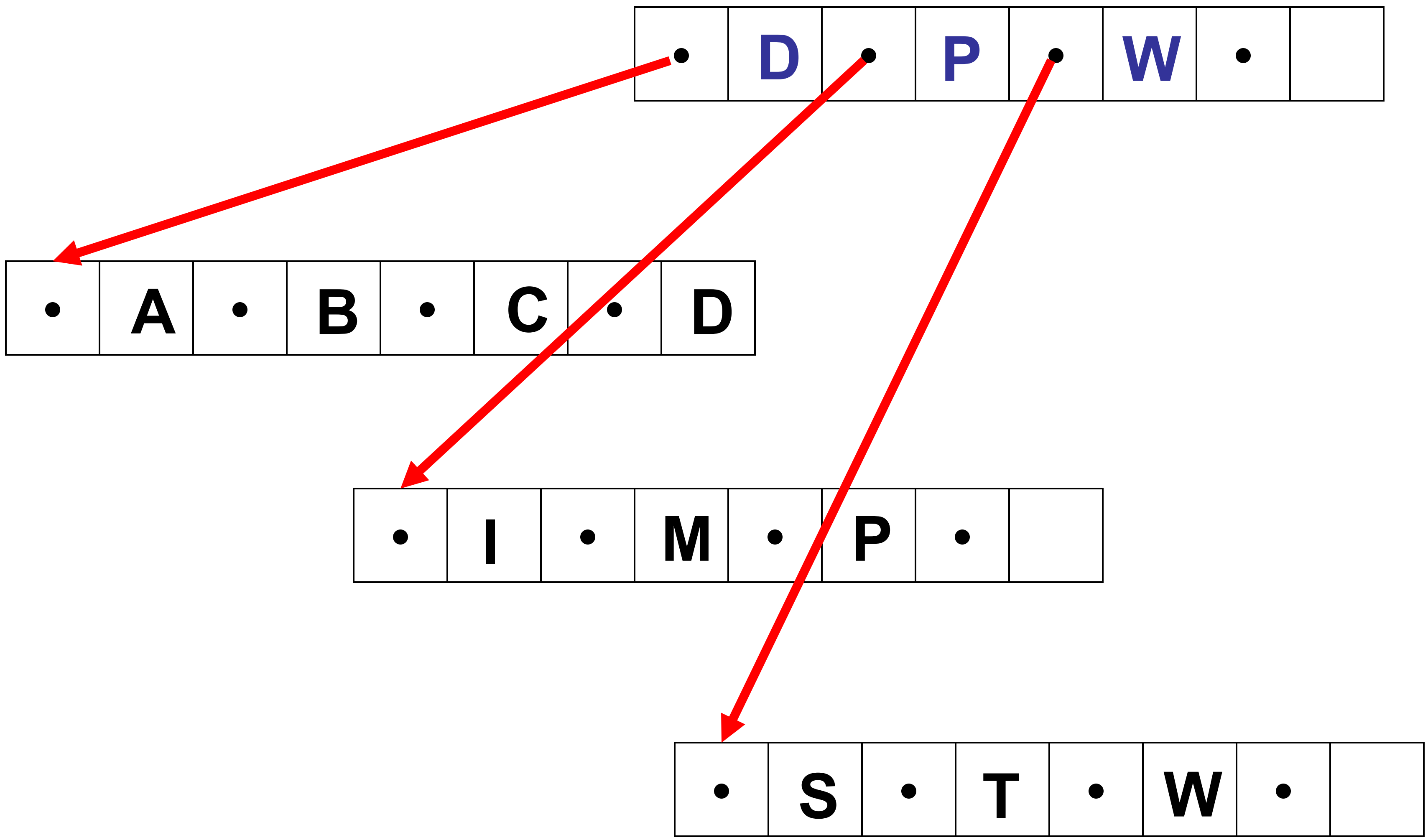
Insert P, I



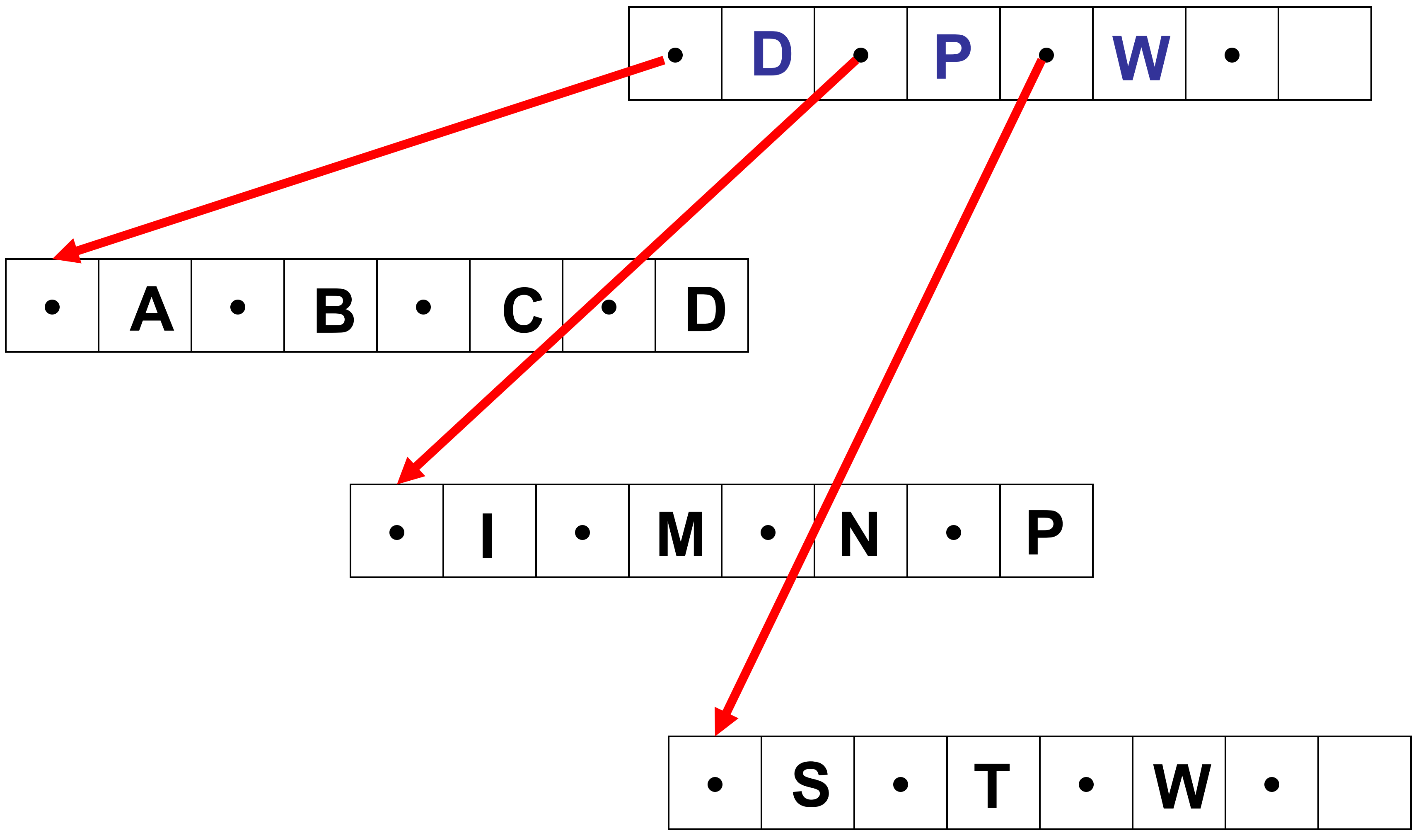
Insert I, B



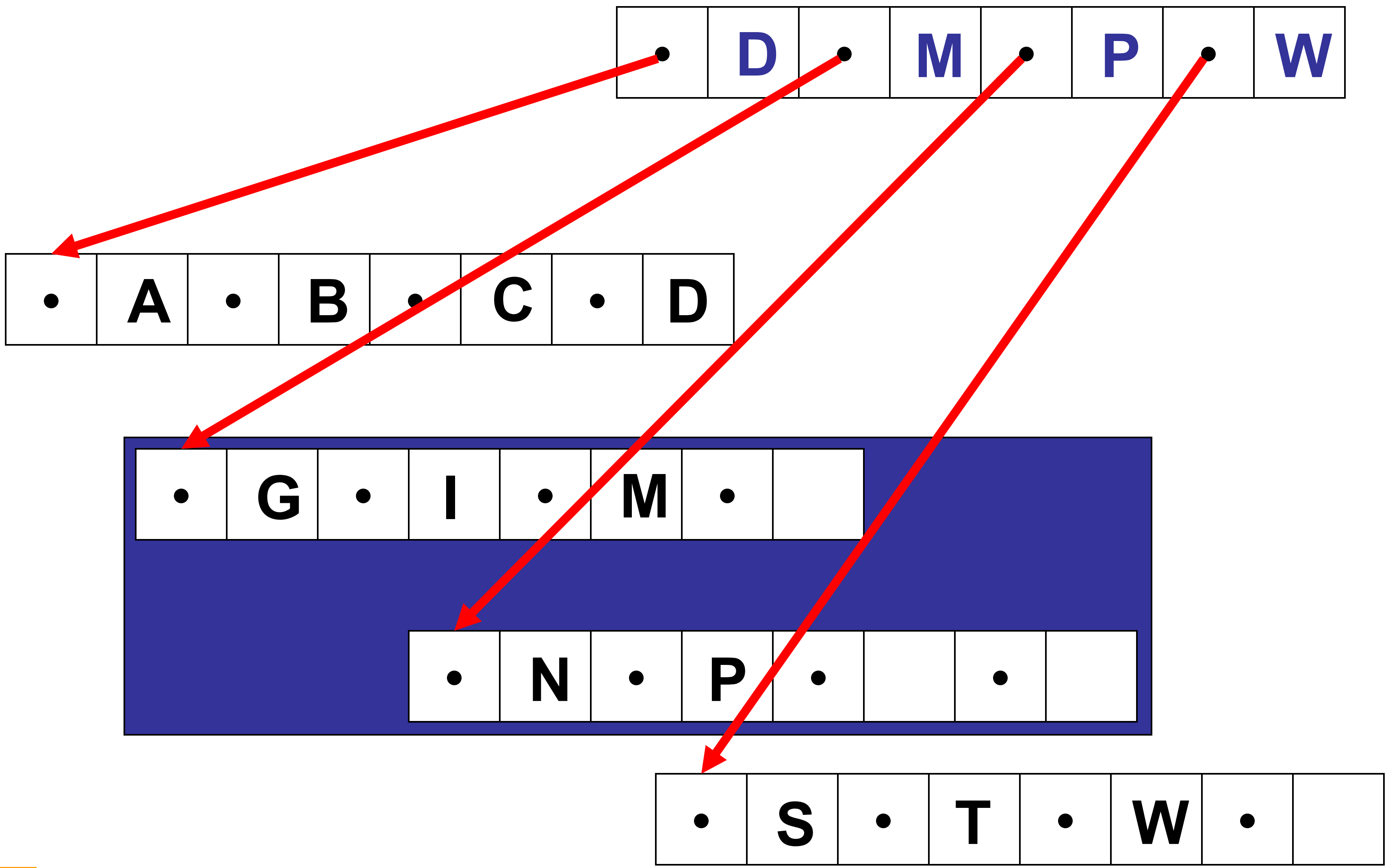
Insert B, W, N



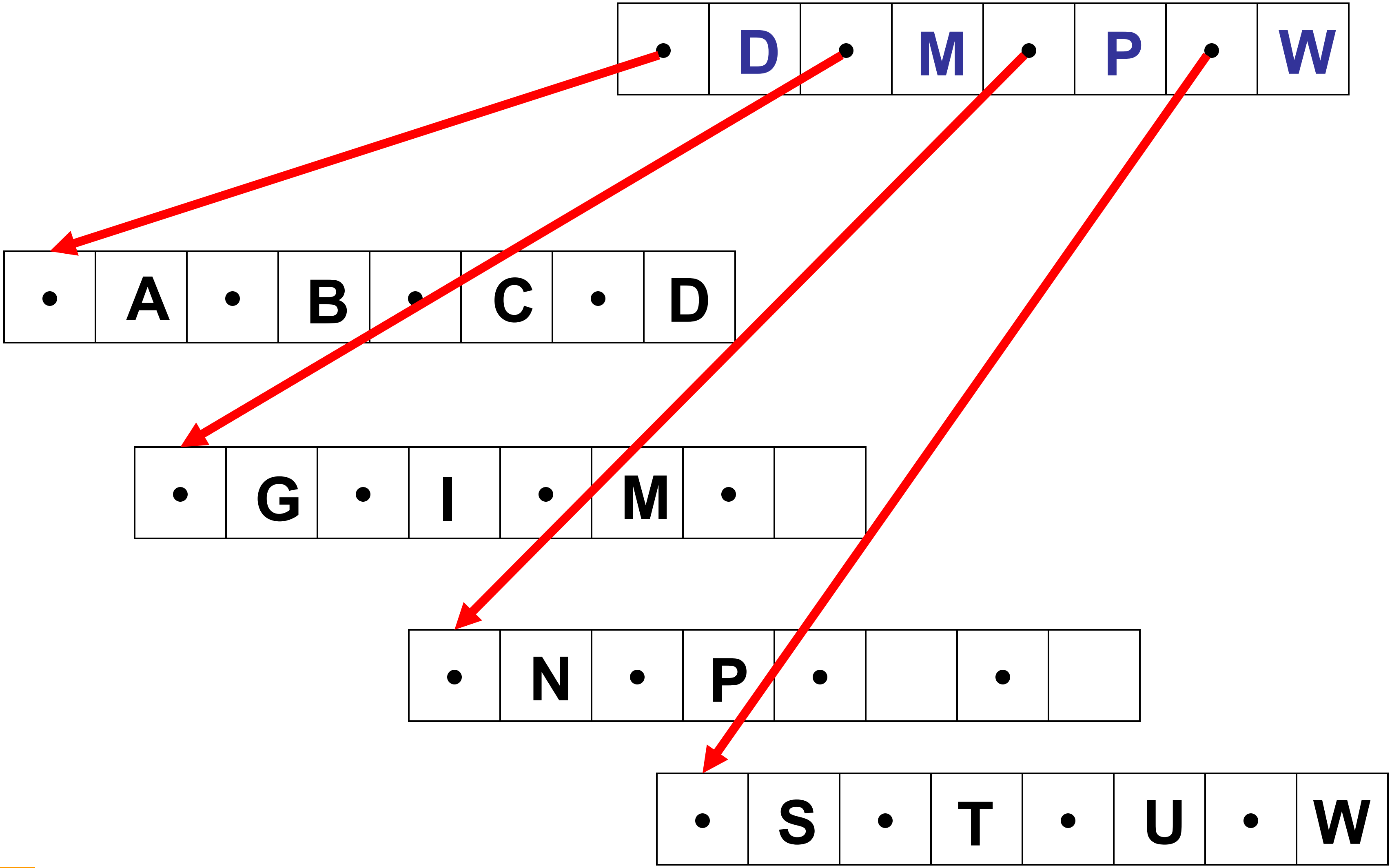
Insert N, G



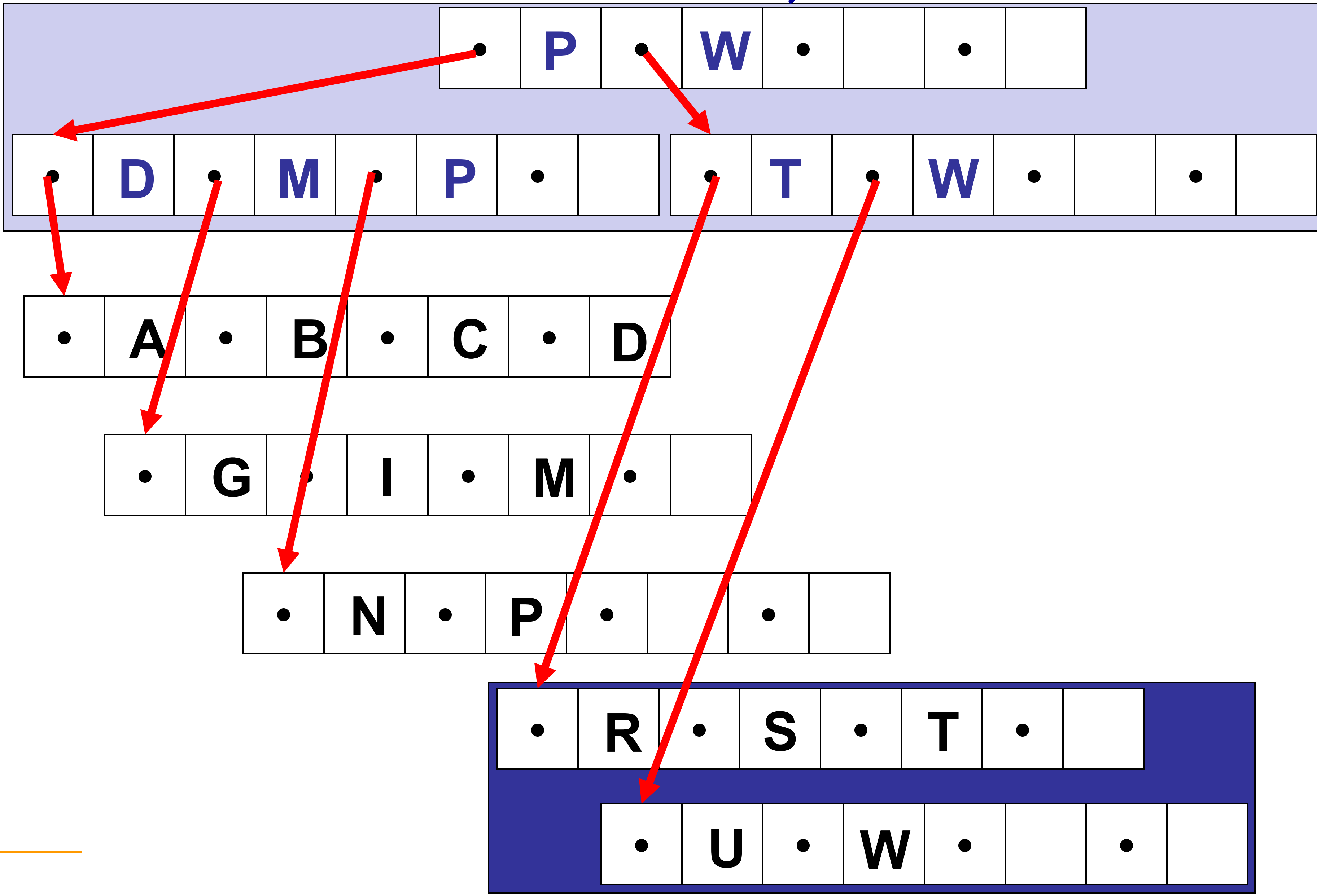
Insert G, U



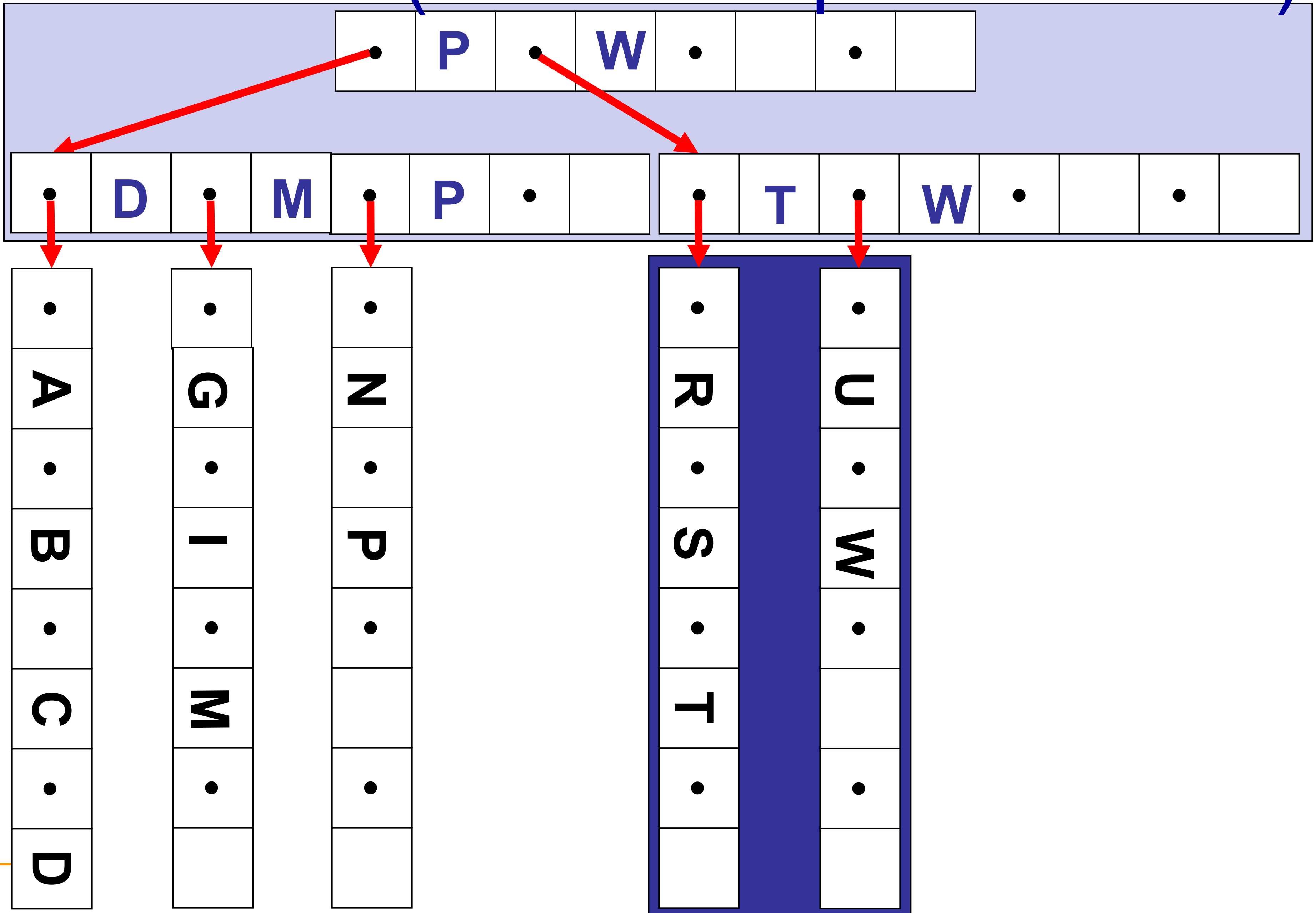
Insert U, R



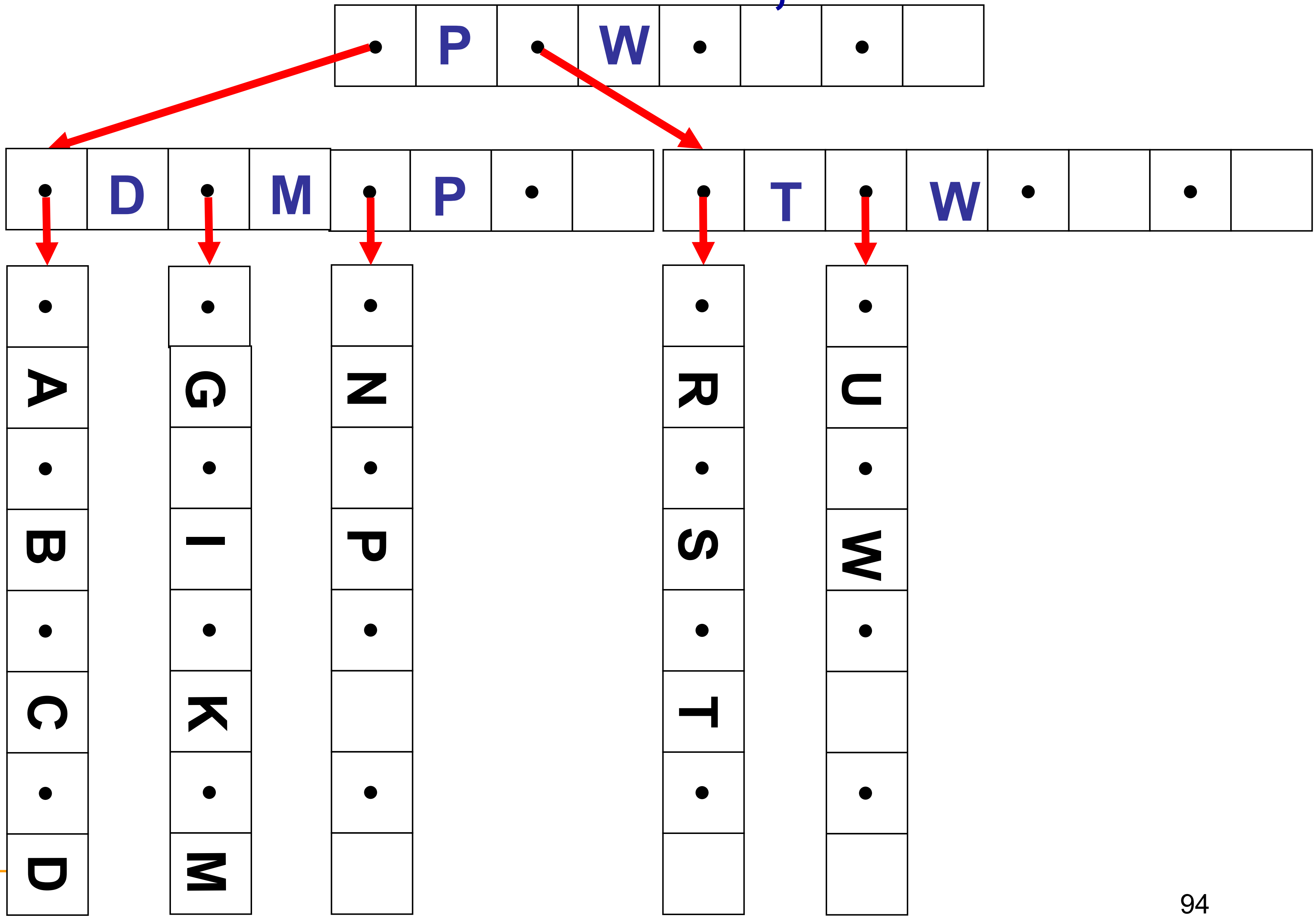
Insert R, K



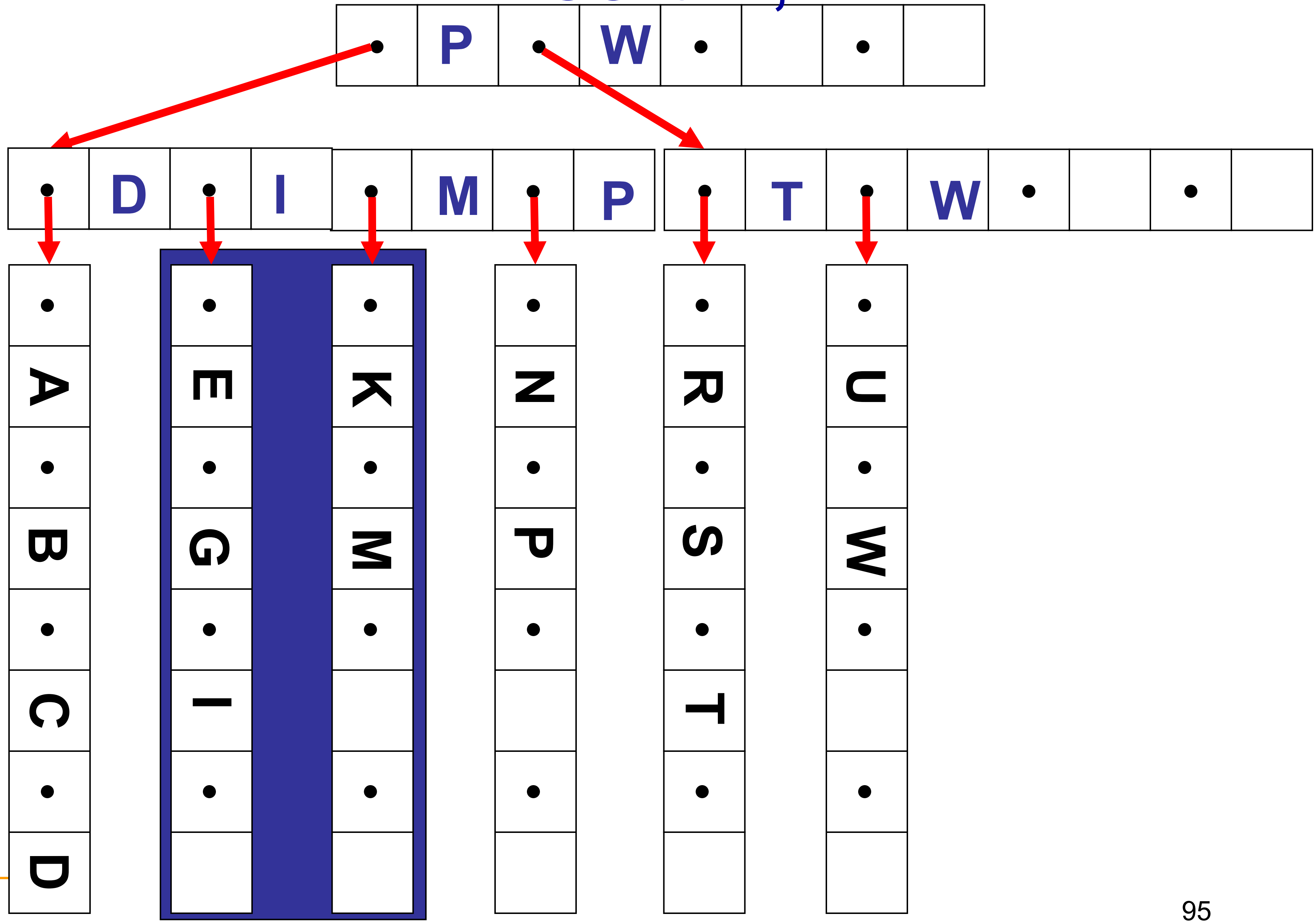
Insert R (another representation)



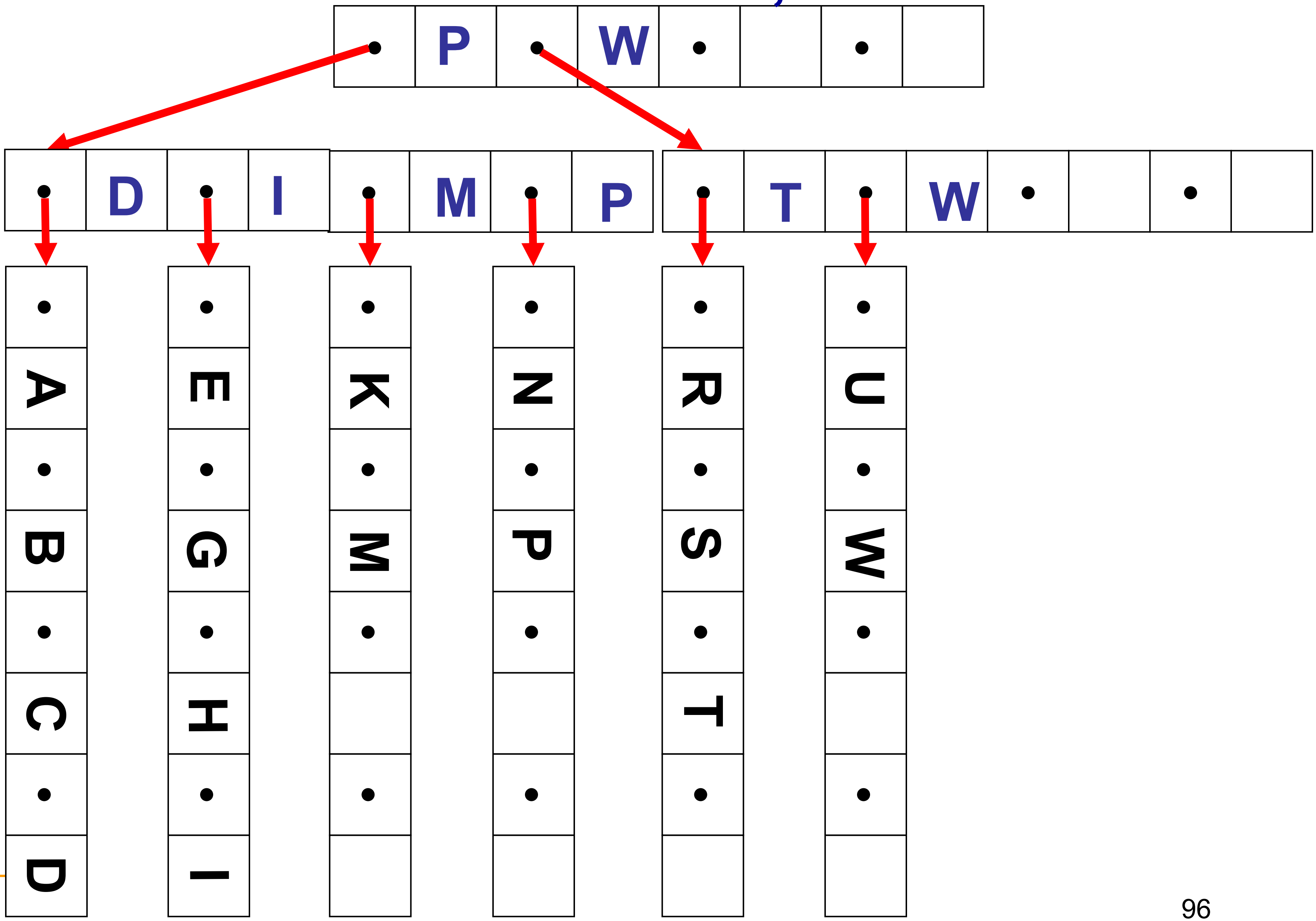
Insert K, E



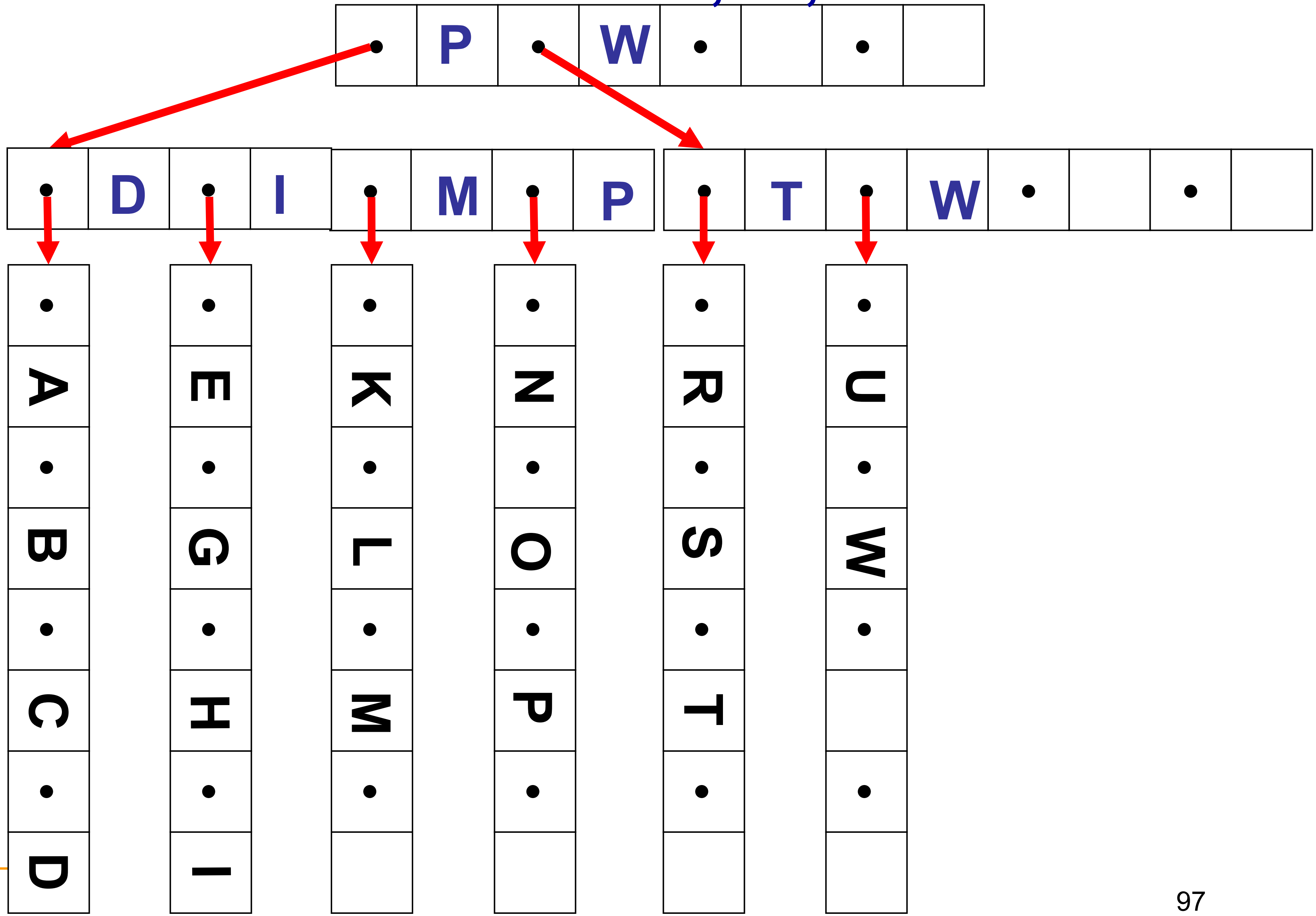
Insert E, H



Insert H, O

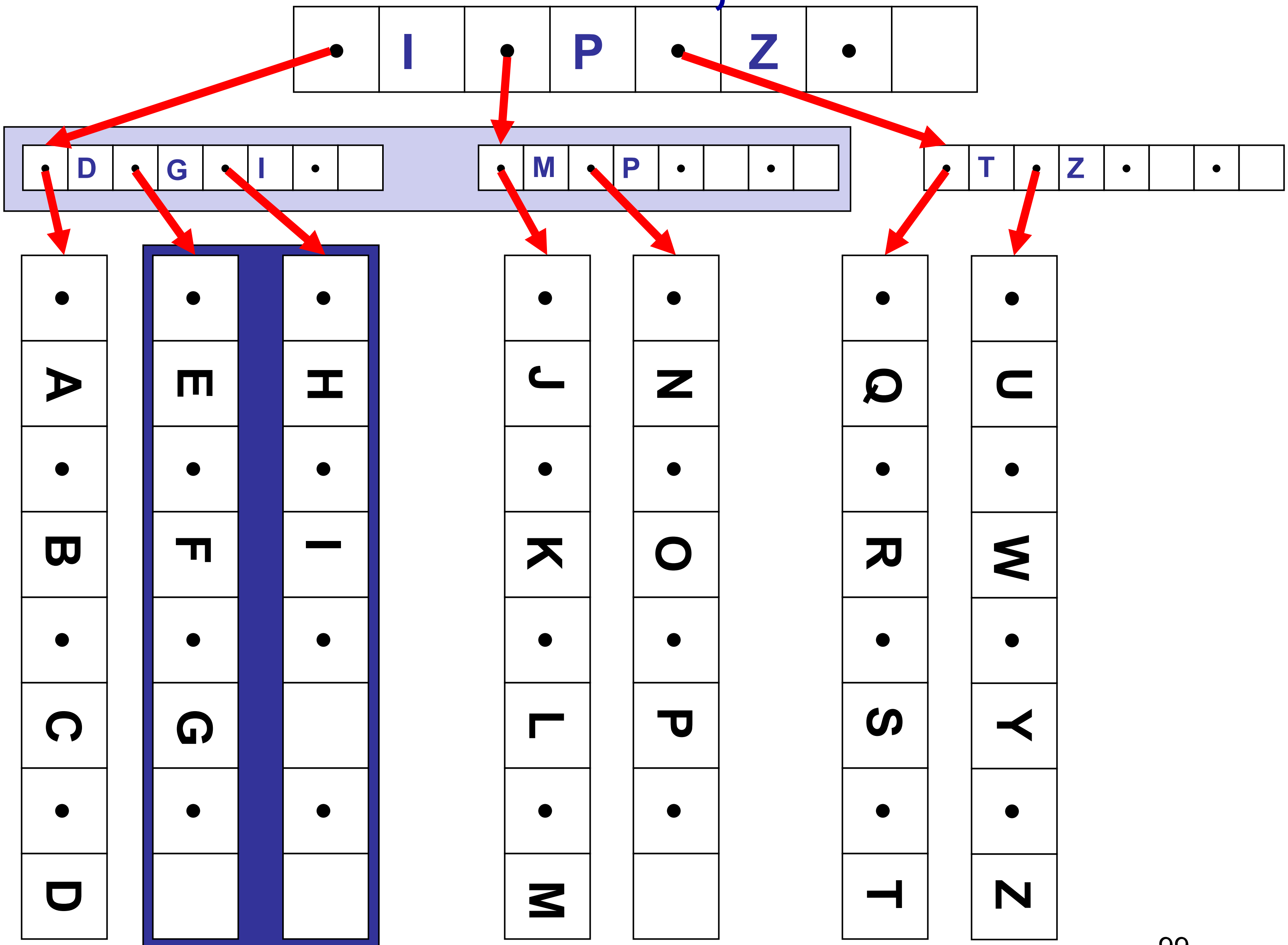


Insert O, L, J

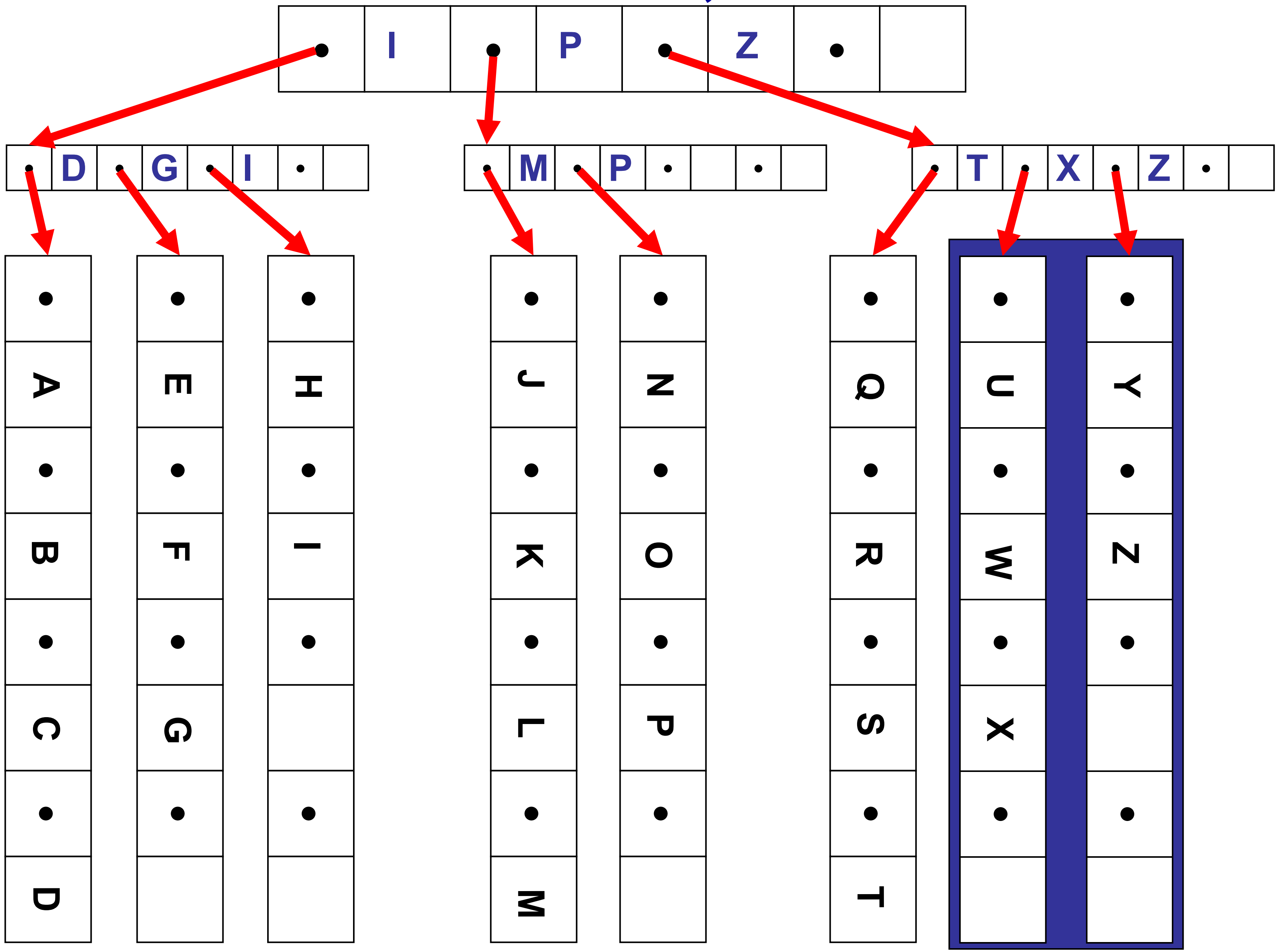


Insert J, Y, Q, Z, F

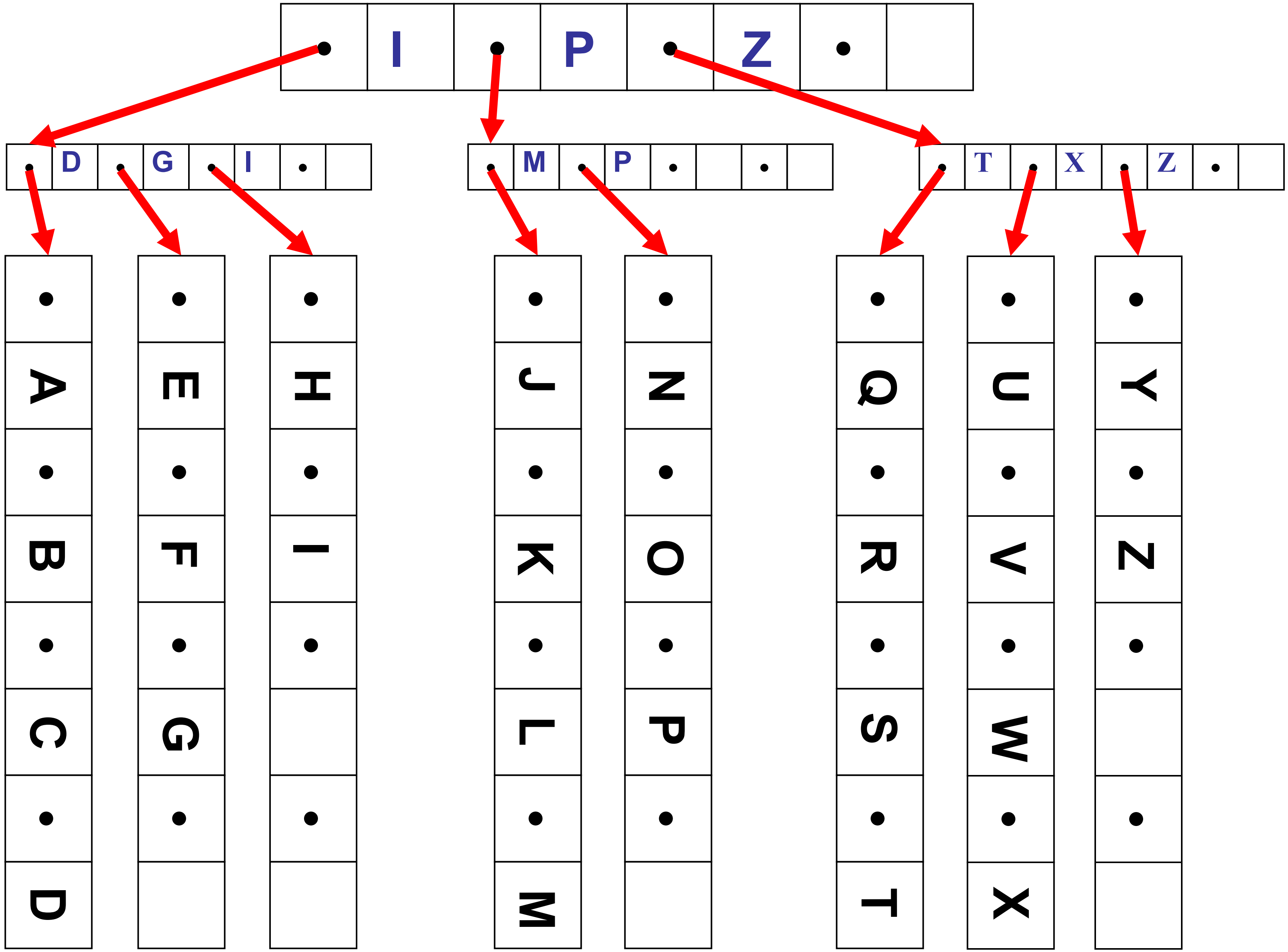
Insert F, X



Insert X, V



Insert V



B-Tree Revisited

- Linear cost of insertion and deletion
- Index records not to be fully occupied
- Does not shift record to another node but splits
- Some variations on B Trees:
  - B* Trees
  - B+ Trees

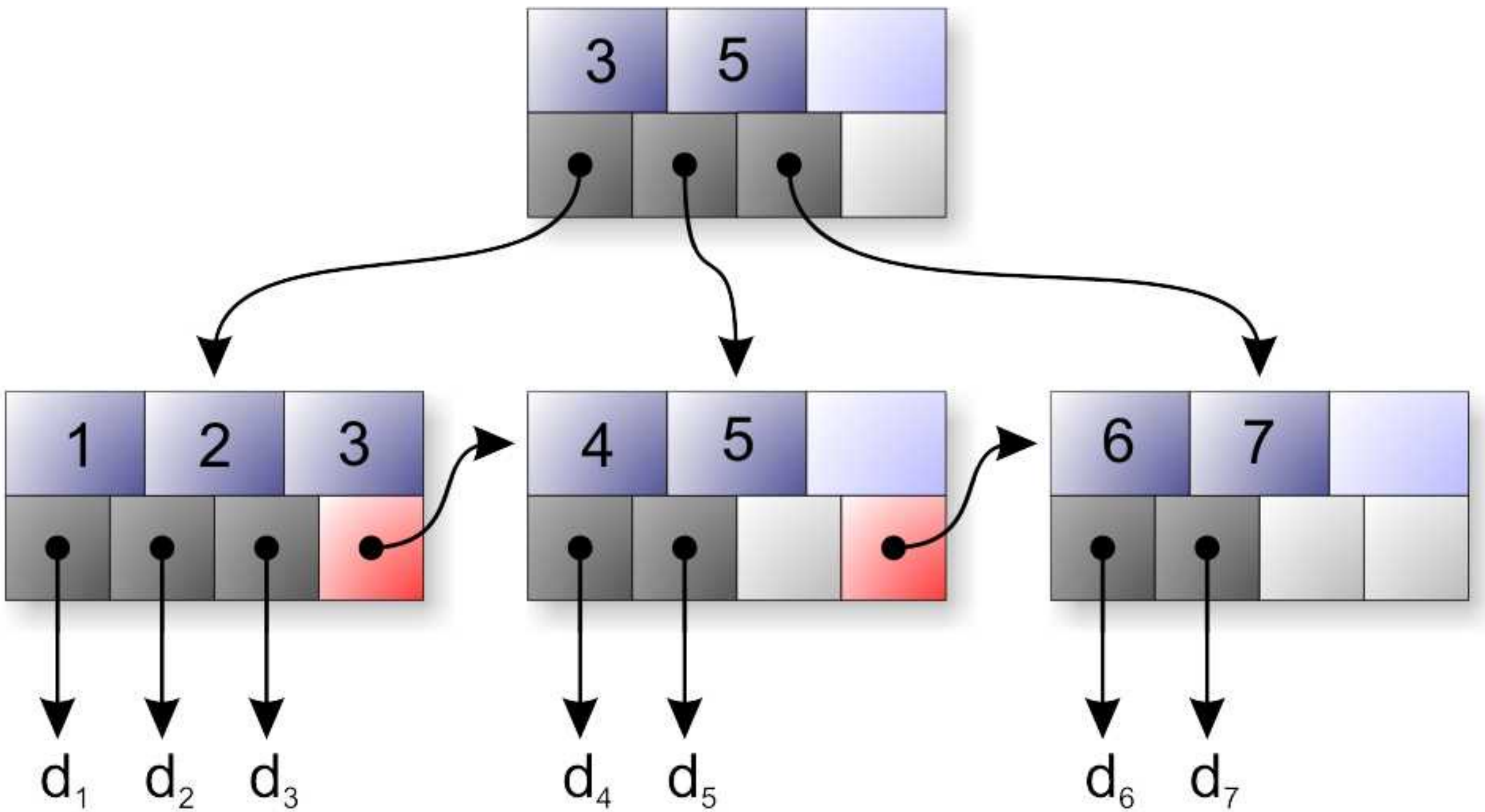
B*-Trees

- A variant of B-Tree
- Essential approach is to delay a split as long as possible
- Two thirds of a node (as opposed to 1/2) has to be full to split (except the root)
  - If sibling is not full then redistribute
  - If sibling is full then split
  - Split into three, not two

B^+-Trees

- B+ tree of order m consists of a root, internal nodes and leaves
- The root may be either leaf or node with two or more children
- Internal nodes contain between m and 2m keys, and a node with k keys has k + 1 children
- Leaves are always on the same level
- If a leaf is a primary index, it consists of a bucket of records, sorted by search key
- If it is a secondary index, it will have many short records consisting of a key and a pointer to the actual record

B+ Tree Example



Linked list (red) allows rapid inorder traversal

Source: encyclopedia.thefreedictionary.com

Theoretical Results

- Robert Tarjan proved
  - amortized number of splits/merges for a B Tree is 2

Summary

- B-Tree
- B-Tree Search, Insert, Delete
- B* Tree, B+ Tree

References on B-Trees

- Cormen, Leiserson, Rivest, Stein, Ch. 18
- Folk, Zoellick, Riccardi, Ch.9