BLG 453E

Week 14

Basic Geometric 2D shape analysis

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Parametrized Curves

- Our goal is to characterize certain subsets of R^3 (to be called curves) that are:
  - in a certain sense, one-dimensional
  - the methods of differential calculus can be applied.

- A natural way of defining such subsets is through differentiable functions.
  - It has, at all points, derivatives of all orders
  - Autonomically continuous
- **Definition:** A parametrized differentiable curve is a differentiable map

of an open interval $I = (a, b)$ of the real line R into R^3.

$$\alpha = I \rightarrow R^3$$

- α is a correspondence which maps each $t \in I$ into a point $\alpha(t) = \big(x(t), y(t), z(t)\big) \in R^3$ in such a way that the functions $x(t), y(t), z(t)$ are differentiable.
  - t: Curve parameter

From De Carmo's book

Tangent Vector

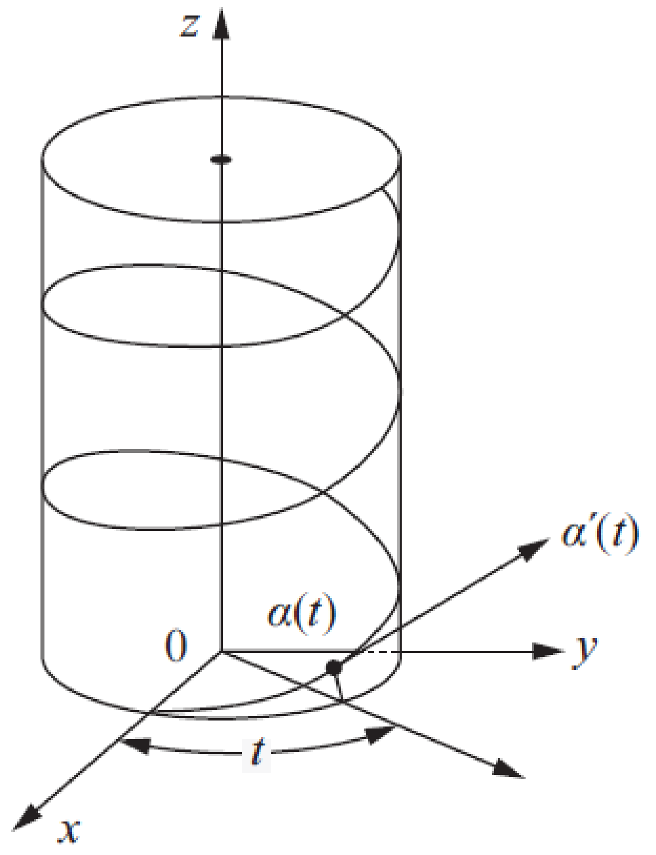
- Considering the first derivatives of $x(t)$, $y(t)$, $z(t)$ the tangent vector is defined as

$$\alpha'(t) = \big(x'(t), y'(t), z'(t)\big)$$

- Also called velocity vector.
- The set $\alpha(t) \subset \mathrm{R}^3$ is called the trace of α.

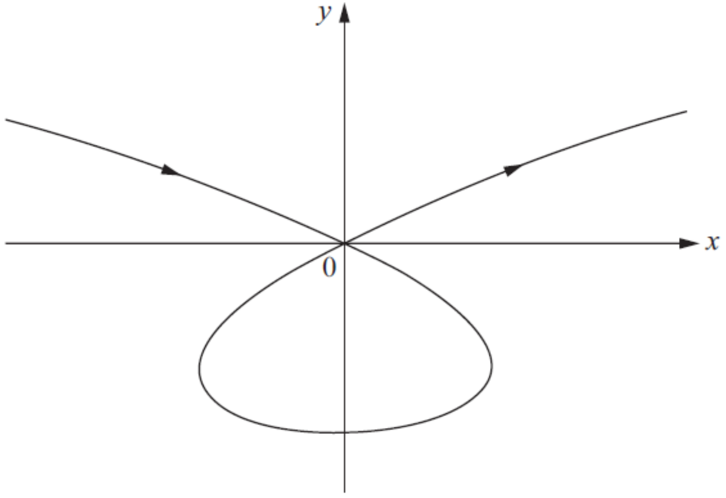
$$\alpha(t) = (a\cos t, a\sin t, bt), t \in R$$

Trace: A helix of pitch $2\pi b$ on the cylinder $x^2 + y^2 = a^2$.

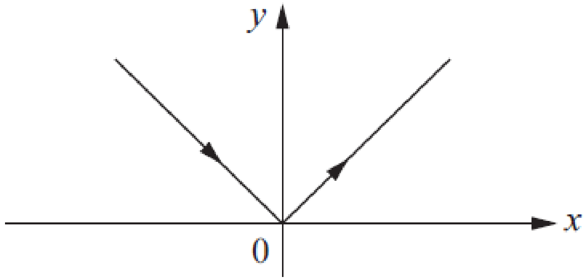


Parametrized Curves

- **Ex. 1:** The map $\alpha: R \to R^2$ given by $\alpha(t) = (t^3 - 4t, t^2 - 4), t \in R$ is a parametrized differentiable curve.
  - Could not be one-to-one.



- **Ex. 2:** The map $\alpha: R \to R^2$ given by $\alpha(t) = (t, |t|), t \in R$ is is not a parametrized differentiable curve, since $|t|$ is not differentiable at t = 0

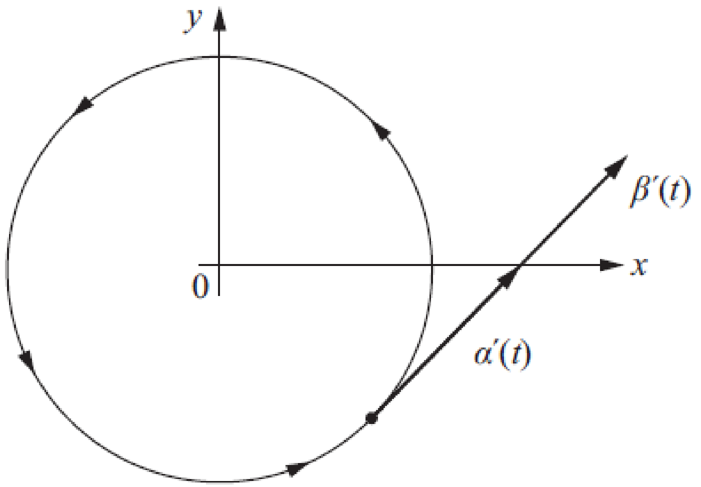


Trace & Velocity

- The two distinct parametrized curves

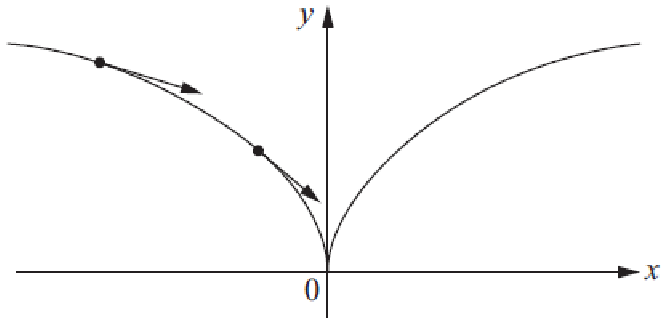
$$\alpha(t) = (\cos t\,, \sin t)$$
$$b(t) = (\cos 2t\,, \sin 2t)$$

where $t \in (0 - \epsilon, 2\pi + \epsilon), \epsilon > 0$ have the same trace: $x^2 + y^2 = 1$. However, the velocity vector of the second curve is the double of the first one.



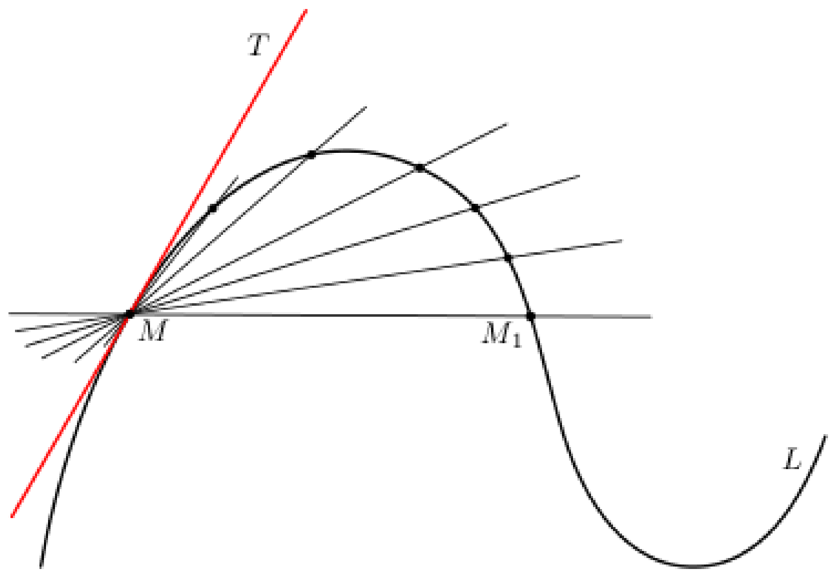
Regular Curves

- Tangent Line:
  - For each $t \in I$ where $\alpha' \neq 0$, there is a well-defined straight line, which contains the point $\alpha(t)$ and the vector $\alpha'(t)$.
- We call any point t where $\alpha'(t) = 0$ is a singular point.



$\alpha(t) = (t^3, t^2), t \in R$

t=0 is a singular point.



https://encyclopediaofmath.org/wiki/Tangent_line

- A parametrized differentiable curve $\alpha: I \to R^3$ is said to be regular if $\alpha'(t) \neq 0$ for all $t \in I$.

Arc Lengths

- Given $t_0 \in I$, the *arc length* of a regular parametrized curve $\alpha: I \rightarrow R^3$ from the point t_0 is

$$s(t) = \int_{t_0}^{t} |\alpha'(t)|\, dt$$

where

$$|\alpha'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

is the length of vector $\alpha'(t)$.

- Since $\alpha'(t) \neq 0$, the arc length s is a differentiable function of t and $\frac{ds}{dt} = |\alpha'(t)|$.

Parametrization by arc length

- To work with velocity vectors with constant magnitude, the parameter t could be chosen to represent the arc length s.

- In this case, $\frac{ds}{dt} = |\alpha'(t)| = 1$.

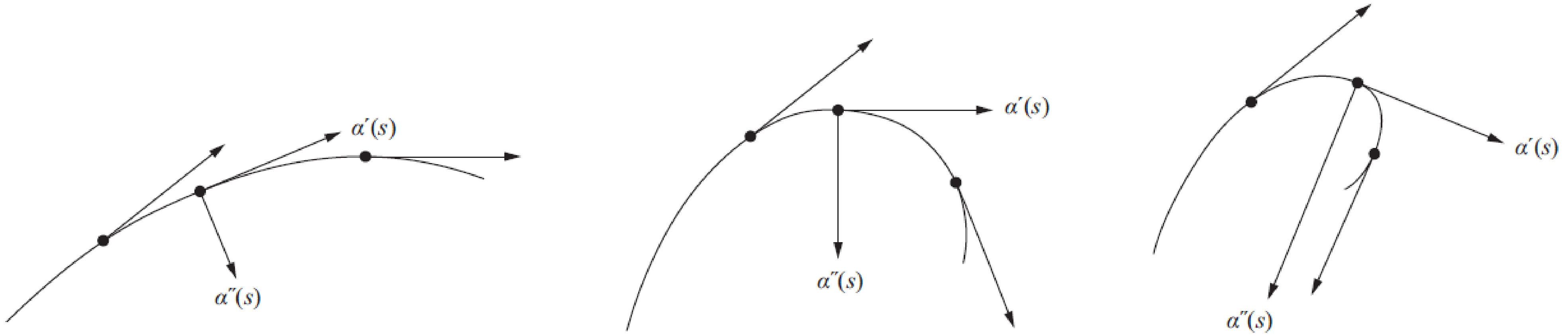
- Then

$$s = \int_{t_0}^{t} dt = t - t_0$$

- t is the arc length of α measured from some point.
- It is not necessary to mention the origin of the arc length s, since most concepts are defined only in terms of the derivatives of α(s).

Curvature

- The curve $\alpha: I = (a, b) \rightarrow R^3$ could also be parametrized by arc length s.
- Using tangent vectors of unit length $\alpha'(s)$, the norm $|\alpha''(s)|$ of the second derivative measures the rate of change of the angle which neighboring tangents make with the tangent at s.
  - how rapidly the curve pulls away from the tangent line at s, in a neighborhood of s
- **Curvature:** For a curve parametrized by arc length $s \in I$, the number $|\alpha''(s)| = k(s)$ is called the curvature of α at s.



Curvature

- If α is a straight line, $\alpha(s) = us + v$, where u and v are constant vectors ($|u| = 1$), then k = 0.

- By a change of orientation, the tangent vector changes its direction; that is, if $\alpha(s) = \beta(-s)$, then

$$\frac{d\beta}{d(-s)}(-s) = -\frac{d\alpha}{ds}(s)$$

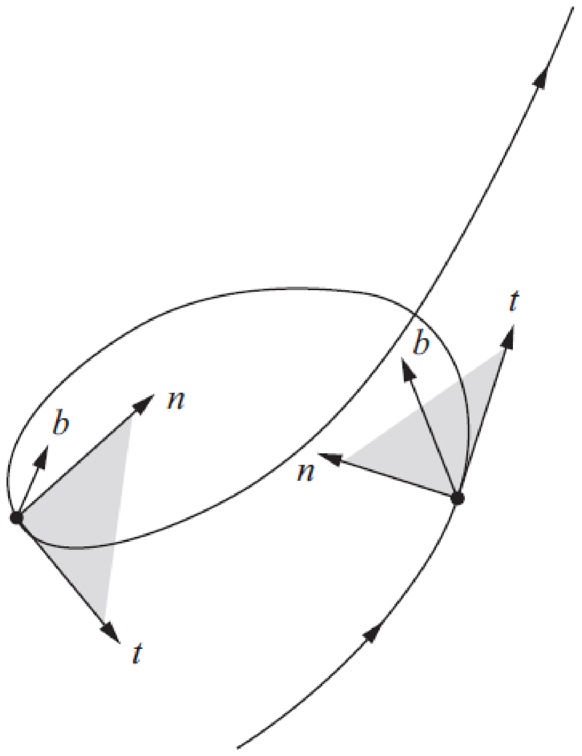
$\alpha''(s)$ and the curvature remain invariant under a change of orientation.

- At points where $k(s) \neq 0$, we can define:

$$\alpha''(s) = k(s)n(s)=t'(s)$$

$$b(s) = t(s) \times n(s)$$

$$b'(s) = t'(s) \times n(s) + t(s) \times n'(s)= t(s) \times n'(s)$$



t(s): unit tangent vector
n(s): unit normal vector
b(s): binormal vector, normal of the osculating plane

Torsion

$$b'(s) = t'(s) \times n(s) + t(s) \times n'(s) = t(s) \times n'(s)$$

- Since $b'(s)$ is parallel to $n(s)$, we may write

$$b'(s) = \tau(s)n(s)$$

for some function $\tau(s)$.

- **Definition:** Let $\alpha: I \to R^3$ be a curve parametrized by arc length s such that $\alpha''(s) \neq 0, s \in I$, $\tau(s)$ defined by $b'(s) = \tau(s)n(s)$ is called the torsion of α at point s.

- If α is contained in a plane then the plane of the curve agrees with the osculating plane; hence $\tau(s)$=0.

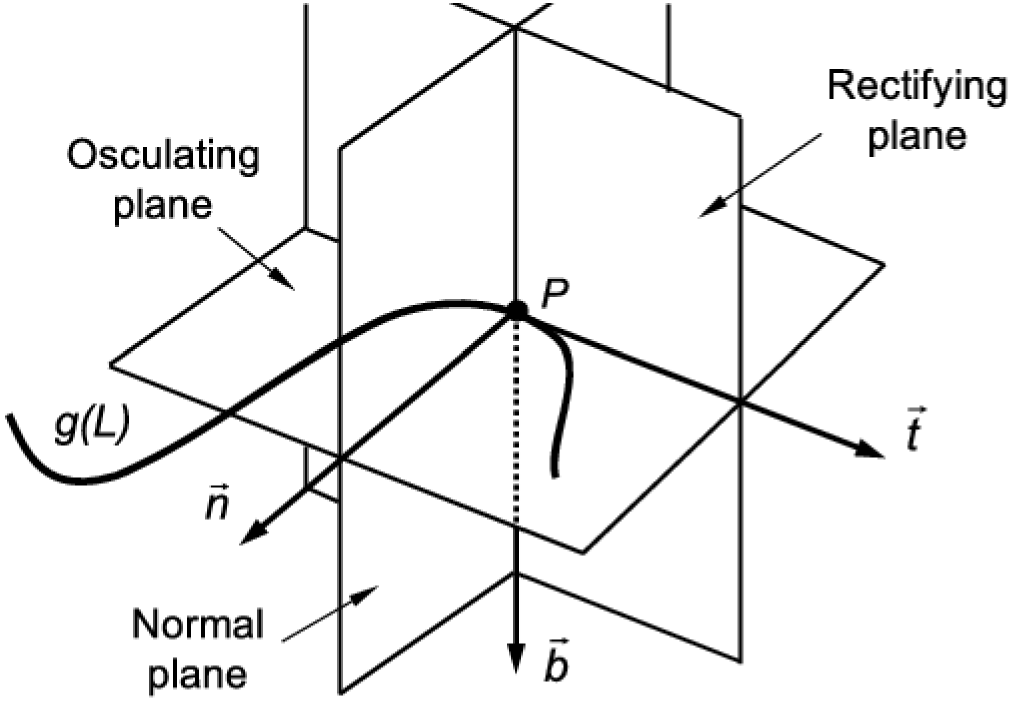
Frenet Equations

- To each value of the parameter s, we have associated three orthogonal unit vectors t(s), n(s), b(s).

- The trihedron thus formed is referred to as the Frenet trihedron at s.

- Derivaties of these vectors will give us geometrical information around the neighborhood.

$$t'(s) = k(s)n(s)$$
$$b'(s) = \tau(s)n(s)$$
$$n'(s) = b'(s) \times t(s) + t(s) \times t'(s) = -\tau(s)b(s) - k(s)t(s)$$

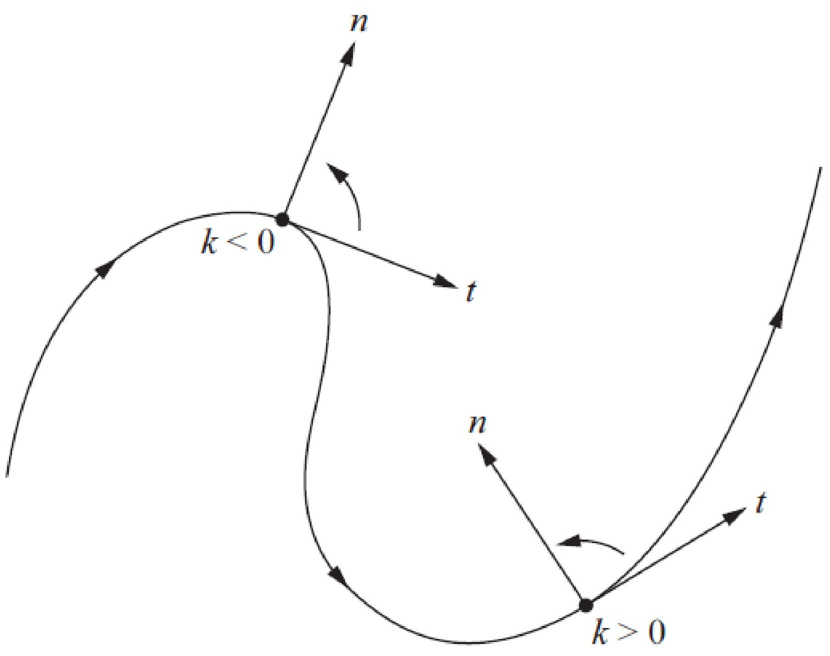


Pombo, J., & Ambrósio, J. (2012). An alternative method to include track irregularities in railway vehicle dynamic analyses. *Nonlinear Dynamics*, *68*, 161-176.

Curvature Orientation

$$t'(s) = k(s)n(s)$$

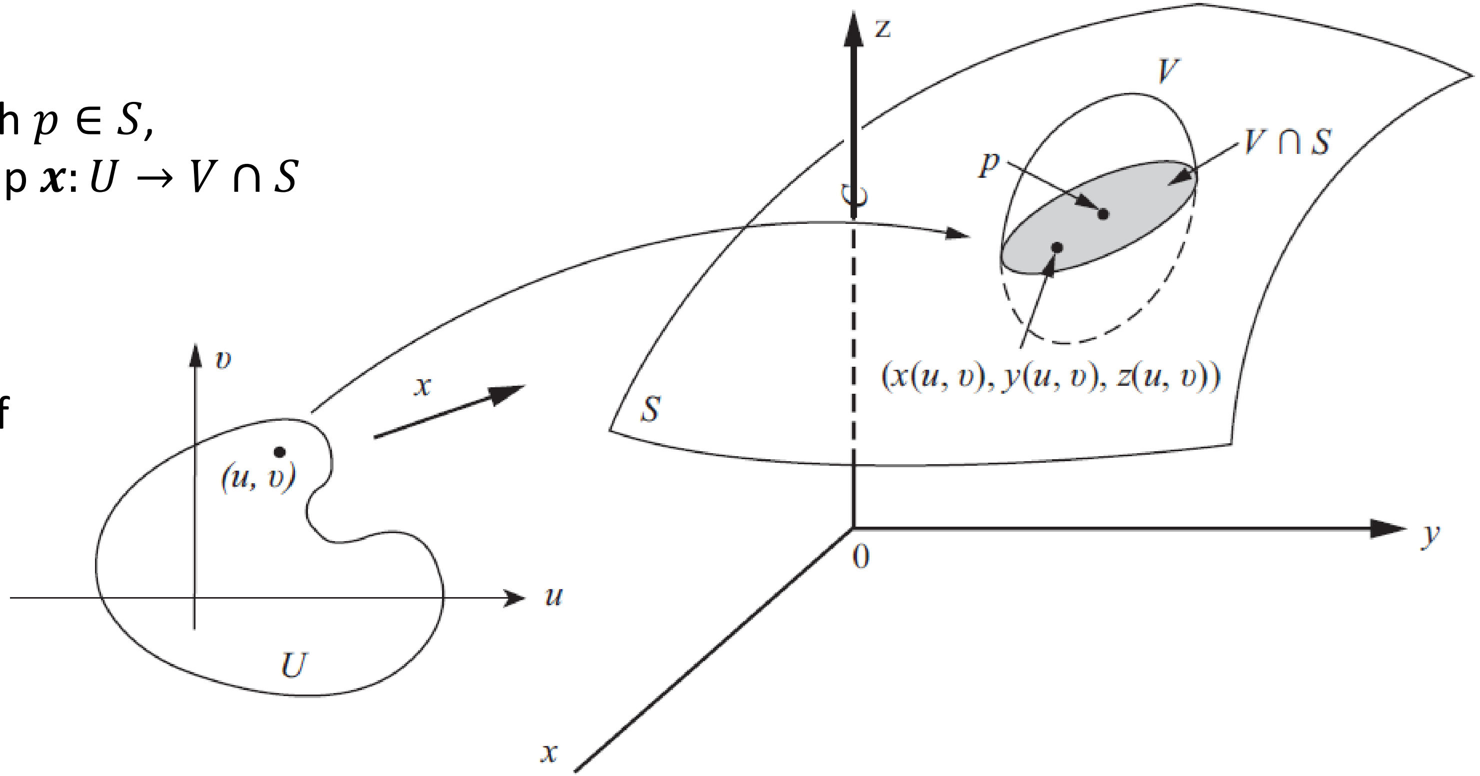
- **k(s)>0:** The curve bends towards the side of n, and the curvature vector aligns with n
- $\boldsymbol{k(s)}$**<0:** The curve bends away from the side of n, and the curvature vector points opposite to n.
- The sign of $k(s)$ determines whether n points "inside" or "outside" the curve's bend.



Regular Surfaces

- A regular surface in R^3 is obtained by taking pieces of a plane, deforming them, and arranging them in such a way that the resulting figure has no sharp points, edges, or self-intersections.

A subset $S \subset \mathrm{R}^3$ is a regular surface if, for each $p \in S$, there exists a neighborhood V in R^3 and a map $\boldsymbol{x}: U \to V \cap S$ of an open set $U \subset \mathrm{R}^2$ onto $V \cap S \subset \mathrm{R}^3$.



- The mapping **x** is called a parametrization of local coordinates in the neighborhood of p.

- The neighborhood $V \cap S$ is called the coordinate neighborhood.

Properties of Regular Surfaces

1. x is differentiable. This means that if we write

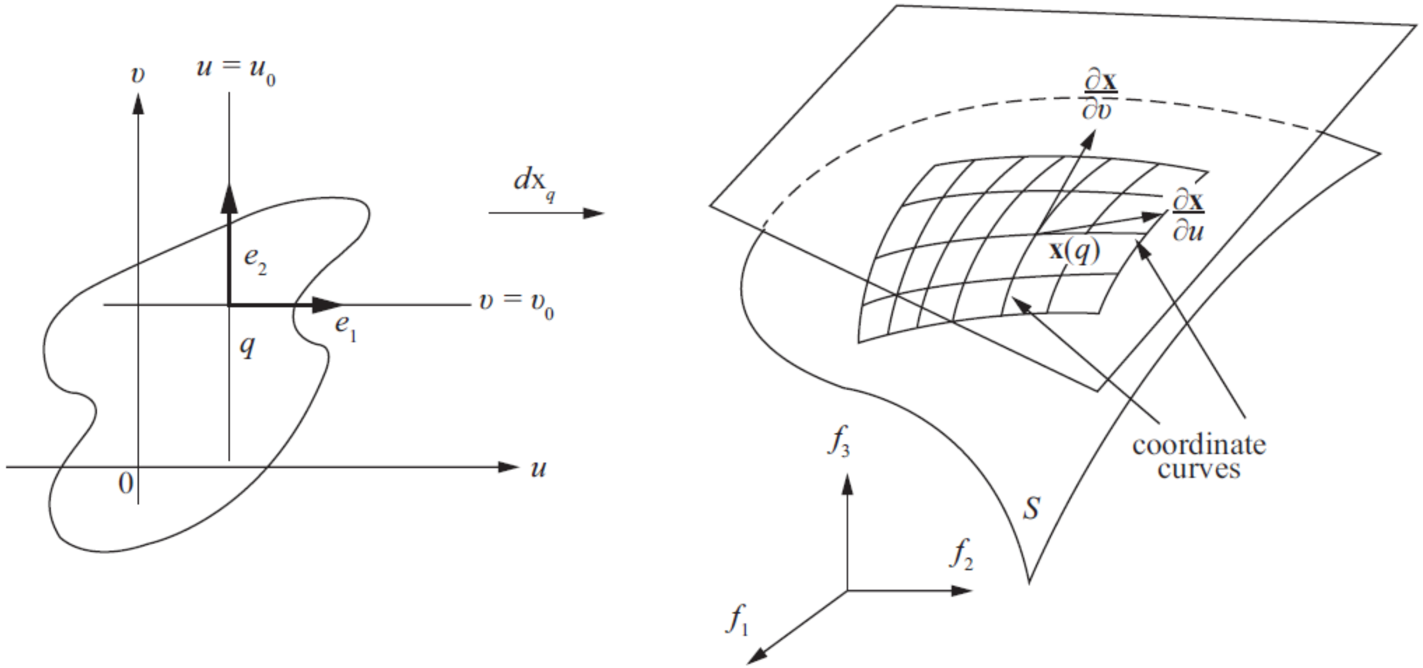
$$\boldsymbol{x}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big), (u,v) \in U,$$

the functions $x(u,v), y(u,v), z(u,v)$ have continuous partial derivatives of all orders in U.

2. $\boldsymbol{x}$ is a homeomorphism. Both $\boldsymbol{x}$ and its inverse $\boldsymbol{x}^{-1}: V \cap S \to U$ are continuous.

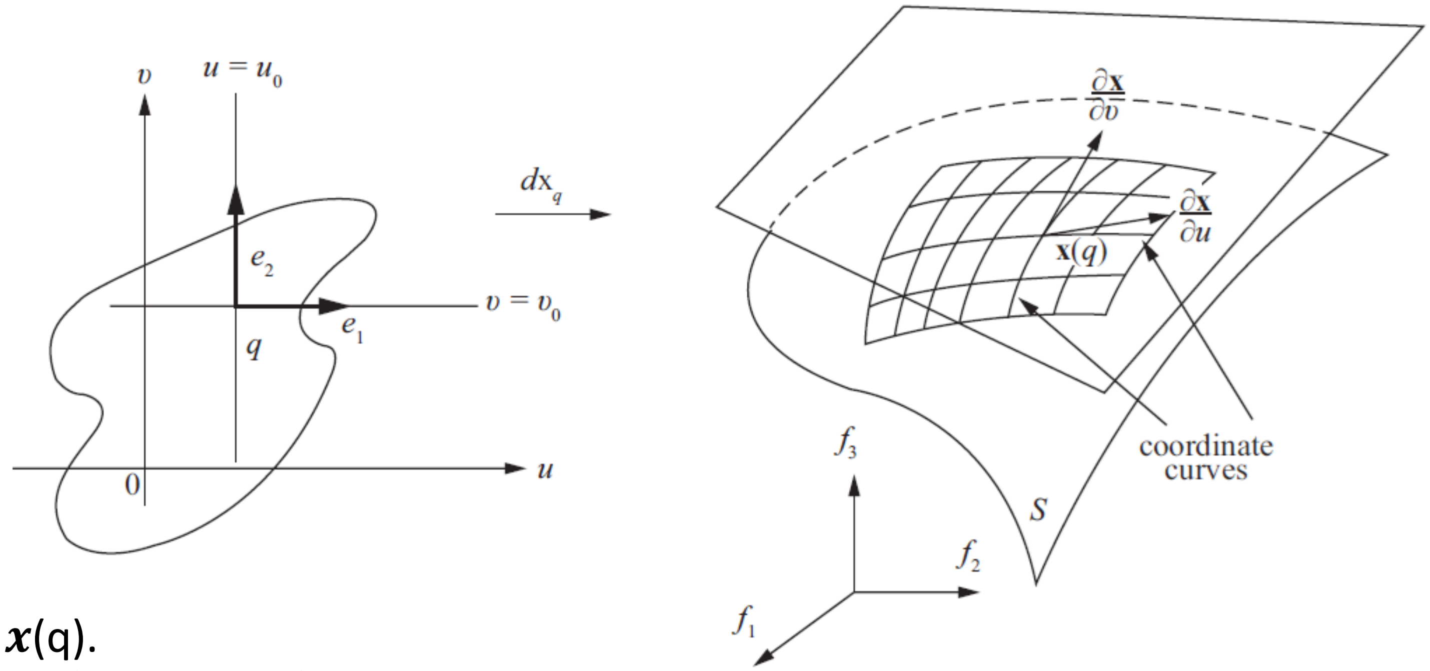
3. For each $q \in U$, the differential $d\boldsymbol{x}_q: R^2 \to R^3$ is one-to-one.

$$e_1 = (1,0)$$
$$e_2 = (0,1)$$



$$q_1 = (1,0,0)$$
$$q_2 = (0,1,0)$$
$$q_3 = (0,0,1)$$

$d\boldsymbol{x}_q$



- Let $q = (u_0, v_0)$.
- For a constant v_0, by aligning the values of u, we can create the coordinate curve (e_1 in R^2)

$$u \to \big(x(u, v_0), y(u, v_0), z(u, v_0)\big)$$

which lies on S and has a tangent vector at $\boldsymbol{x}$(q).

$$d\boldsymbol{x}_q(e_1) = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = \frac{\partial \boldsymbol{x}}{\partial u}$$

- Similarly, for e_2:

$$v \to \big(x(u_0, v), y(u_0, v), z(u_0, v)\big)$$

$$d\boldsymbol{x}_q(e_2) = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = \frac{\partial \boldsymbol{x}}{\partial v}$$

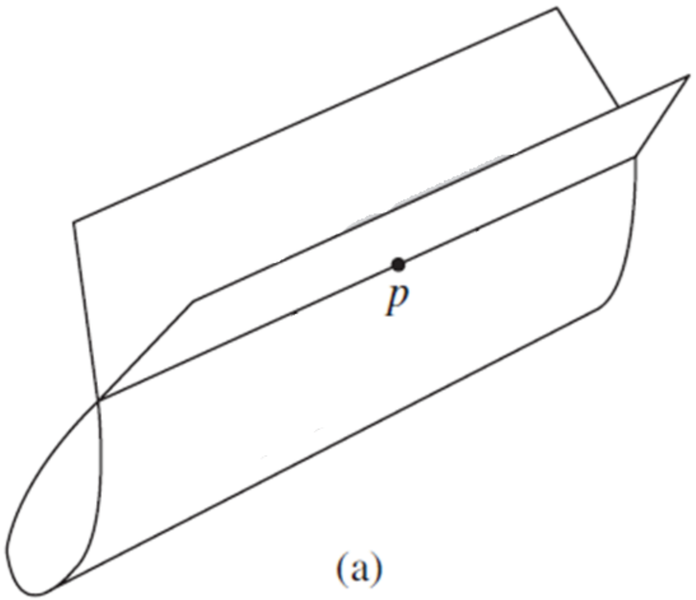
- Thus, the matrix of the linear map $d\boldsymbol{x}_q$ is:

$$d\boldsymbol{x}_q = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix}$$

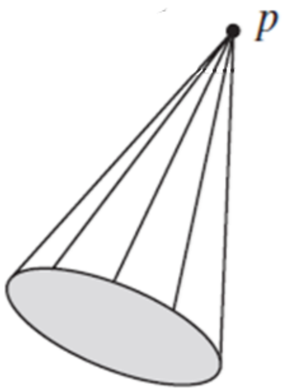
Cond 3.
The two column vectors of this matrix to be linearly independent.

Q:

- Could we define regular surfaces for the given points?



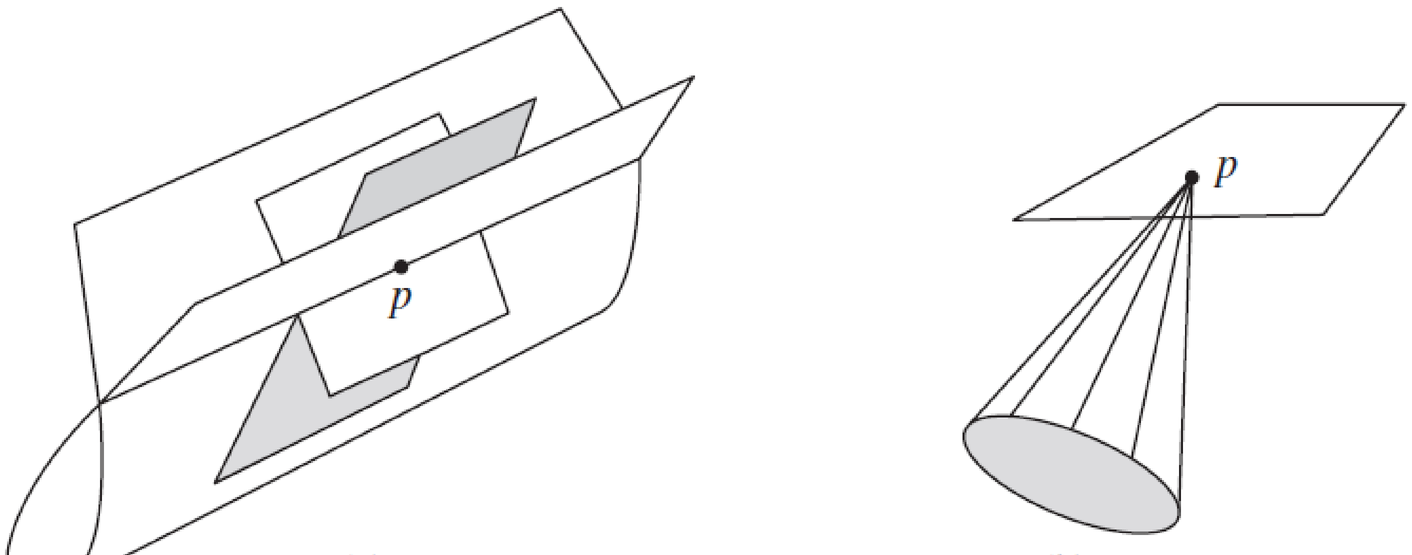
(a)



(b)

Q:

- Could we define regular surfaces for the given points?



(a) (b)

Unit Sphere

- Let us show that the unit sphere

$$S^2 = \{(x, y, z) \in R^3; x^2 + y^2 + z^2 = 1\}$$

is a regular surface.

A subset $S \subset \mathrm{R}^3$ is a regular surface if, for each $p \in S$, there exists a neighborhood V in R^3 and a map $\boldsymbol{x}: U \to V \cap S$ of an open set $U \subset \mathrm{R}^2$ onto $V \cap S \subset \mathrm{R}^3$.

First, check for $\boldsymbol{x}$:

$$\boldsymbol{x}_1(x, y) = \left(x, y, \sqrt{1 - (x^2 + y^2)}\right), \qquad (x, y) \in U$$
$$U = \{(x, y) \in R^2; x^2 + y^2 < 1\}$$

Since $x^2 + y^2 < 1$, the function $\sqrt{1 - (x^2 + y^2)}$ has continuous partial derivatives of all orders. Thus Condition 1 holds.

Condition 2 holds, since $\boldsymbol{x}_1$ is one-to-one and $\boldsymbol{x}_1^{-1}$ could be reobtained by projection.

Condition 3 holds, since $\frac{\partial x_1}{\partial x} = \left(1,0, \frac{-x}{\sqrt{1-(x^2+y^2)}}\right)$ and $\frac{\partial x_1}{\partial y} = \left(0,1, \frac{-y}{\sqrt{1-(x^2+y^2)}}\right)$ are linearly independent.

Unit Sphere

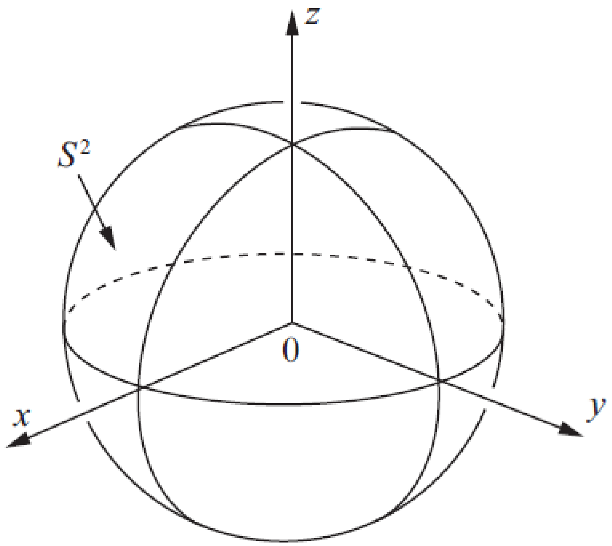
- We shall now cover the whole sphere with similar parametrizations as:

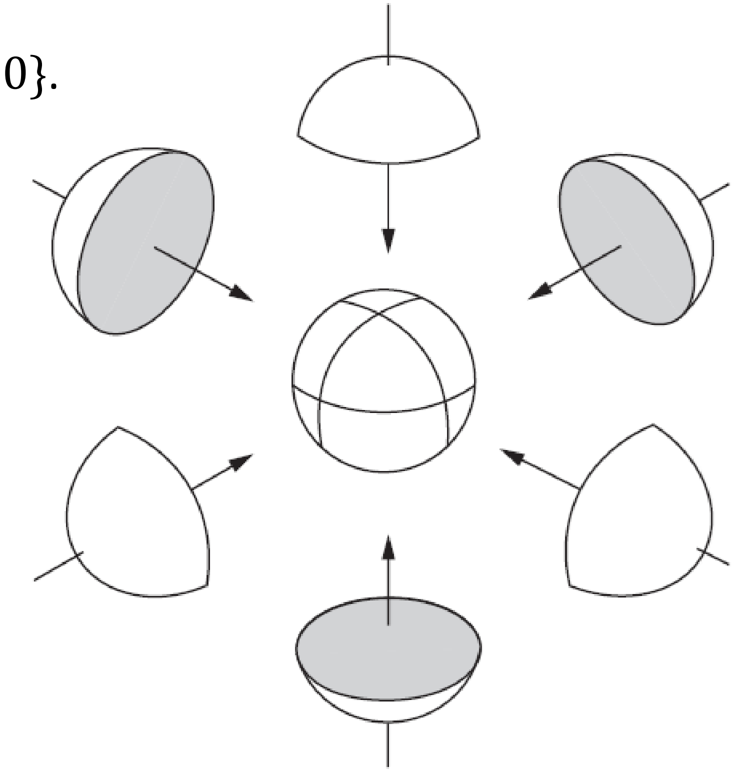
$$\boldsymbol{x}_2(x,y) = \left(x, y, -\sqrt{1-(x^2+y^2)}\right)$$

- $x_1(U) \cup x_2(U)$ covers S^2 minus the equator $\{(x,y,z) \in R^3; x^2+y^2=1, z=0\}$.

- Using the xz and zy planes, we can define other parametrizations:

$$\boldsymbol{x}_3(x,z) = \left(x, \sqrt{1-(x^2+z^2)}, z\right)$$
$$\boldsymbol{x}_4(x,z) = \left(x, -\sqrt{1-(x^2+z^2)}, z\right)$$
$$\boldsymbol{x}_5(y,z) = \left(\sqrt{1-(y^2+z^2)}, y, z\right)$$
$$\boldsymbol{x}_6(y,z) = \left(-\sqrt{1-(y^2+z^2)}, y, z\right)$$





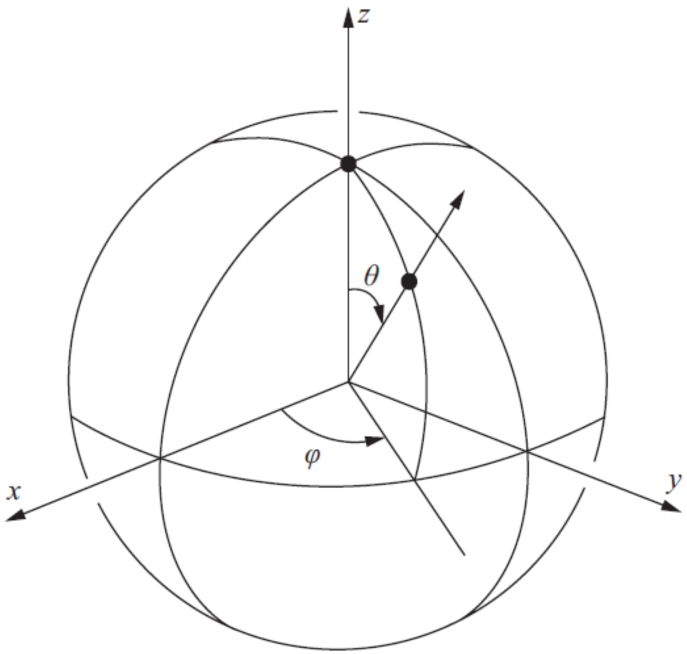
Geographical Coordinates

- It is convenient to relate parametrizations to the geographical coordinates on S^2.

$$V = \{(\theta, \phi); 0 < \theta < \pi, 0 < \phi < 2\pi\}$$
$$\boldsymbol{x}: V \rightarrow R^3$$

$$x(\theta, \phi) = (sin\theta cos\phi, sin\theta sin\phi, cos\theta)$$

colatitude longitude



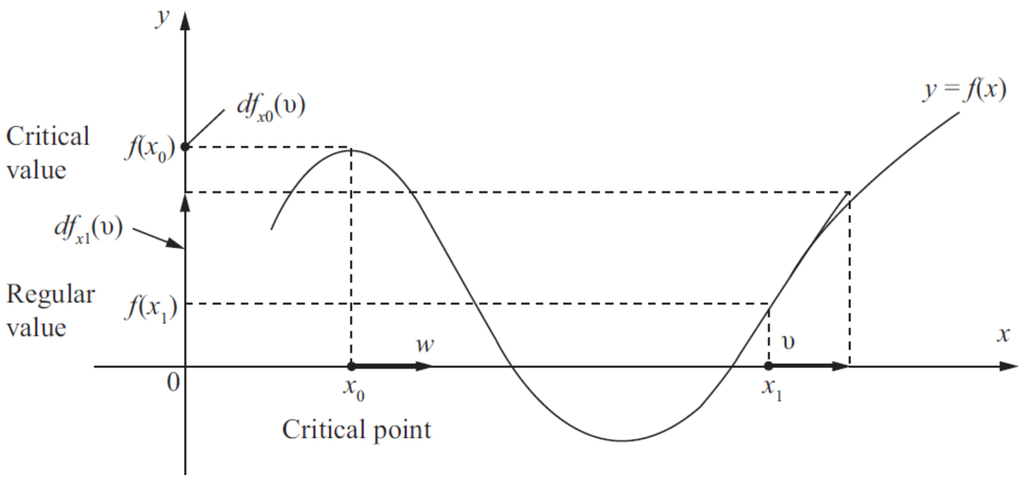
Differential Functions in R^n

- If $f: U \to R$ is a differentiable function in an open set U of R^2, then the graph of f:

$$\big(x, y, f(x, y)\big) \text{ for } (x, y) \in U$$

is a regular surface.

- Given a differentiable map $F: U \subset \mathrm{R^n} \to \mathrm{R^m}$, we say that $\mathrm{p} \in U$ is a critical point of F if the differential $dF_p: R^n \to R^m$ is not a surjective mapping.



- If $f: U \subset \mathrm{R}^3 \to R$ is a differentiable function and $a \in f(U)$ is a regular (non-critical) value of f, then $f^{-1}(a)$ is a regular surface in R^3.

Ex:

- The ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is a regular surface.

$$f(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$$

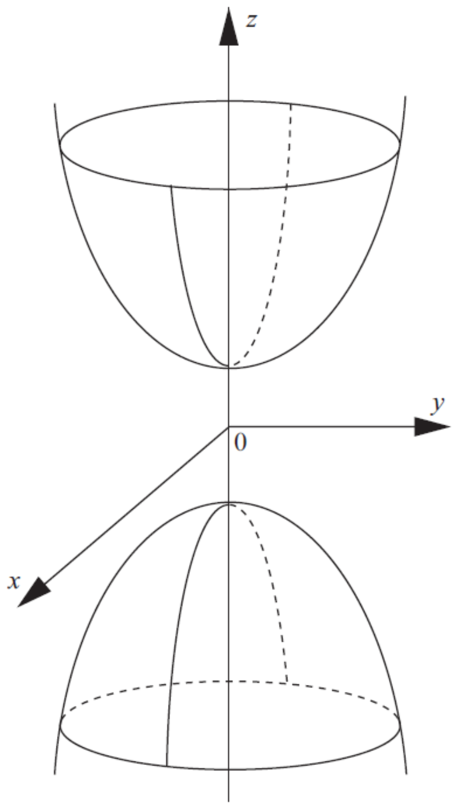
is a differentiable function and 0 is a regular value of f.

$$f_x=\frac{2x}{a^2},\ f_y=\frac{2y}{b^2},\ f_z=\frac{2z}{c^2}$$

- The hyperboloid of two sheets $-x^2-y^2+z^2=1$ is a regular surface.

$$f(x,y,z)=-x^2-y^2+z^2-1$$

Note that the surface S is not connected.



Ex:

- The torus T is a regular surface generated by rotating a circle S^1 of radius r about a straight line belonging to the plane of the circle and at a distance a > r away from the center of the circle.

$$f(x, y, z) = z^2 + \left(\sqrt{x^2 + y^2} - a\right)^2$$