Approximating the Complete Euclidean Graph

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1. Introduction

Let S be a set of N points in the plane. Then a Euclidean graph G(S) on S has the property that each vertex corresponds to a point in S and the weight of an edge is equal to the Euclidean distance between the points it connects [3]. Let d(p,q) be the Euclidean distance between points p and q and let G(p,q) be the length of the shortest path in G(S) between p and q. Only if G is the complete Euclidean graph can we guarantee that G(p,q) = d(p,q).

We say a Euclidean graph G(S) approximates the complete Euclidean graph if the maximum value of the ratio $\frac{G(p,q)}{d(p,q)}$ is bounded by a constant for any pair of points p and q in S. The problem is to identify classes of graphs which closely approximate the complete Euclidean graph yet contain only a linear number of edges.

The first result of this type concerns the graph of the Delaunay triangulation in the L_1 norm, DT_1(S). Chew [1] has shown that the ratio of shortest distances in DT_1(S) to the Euclidean distances is bounded above by $\sqrt{10}$ independent of S and N. Along similar lines it has recently been shown [2] that if DT(S) is the graph of the Delaunay triangulation with the Euclidean norm then the ratio of shortest distances in DT(S) to the Euclidean distance is bounded above by $\frac{1+\sqrt{5}}{2}\pi$ independent of S and N. Both of these types of graphs contain only a linear number of edges and can be computed from the point set in O(NlogN) time.

Approximating the complete Euclidean graph has potential application in the development of approximation algorithms or heuristics for problems that involve finding shortest distances in the plane [1,2,3]. In some applications it may be that the Delaunay approximations are not sufficiently close. In this paper we describe a class of Euclidean graphs each type of which very closely approximates the complete Euclidean graph, yet contains only a linear number of edges and can be efficiently computed.

2. Fixed Angle Theta Graph

Consider the shortest path from p to q in a Euclidean graph G. Let r be the intermediate vertex adjacent to p on this path. If the angle qpr is small we might suspect that G(p,q) is not much longer than d(p,q), whereas if angle qpr is large we might suspect a greater difference between G(p,q) and d(p,q). This intuition suggests that by defining a type of graph in which each point will be connected to a near neighbour in each of a variety of directions we may approximate the complete Euclidean graph.

Given a set S of points in the plane we define the θ-graph, for $\theta = \frac{2\pi}{k}$, k an integer constant such that k > 8, to be the Euclidean graph, θ(S), whose edges are defined as follows. Each point will be the source of at most k edges. A point p is the source of a type 1 edge if there exists a point q such that if the origin is located at p then the angle ϕ between the x-axis and the ray $\vec{pq}$ is such that $0 \le \phi < \frac{2\pi}{k}$. If there are several such points, the type 1 edge from p will go to the point of minimum x-coordinate. In general, a point p will have a type i edge, $1 \le i \le k$, if there exists a point q such that, if the origin is located at p, the angle ϕ between the x-axis and the ray $\vec{pq}$ is such that $\frac{2\pi(i-1)}{k} \le \phi < \frac{2\pi i}{k}$. If there are several such points the type i edge from p will go to the point whose projection on the ray from p at angle $\frac{2\pi(i-1)}{k}$ with the x-axis is closest to p.

To construct the θ-graph for a given point set we present a simple plane sweep algorithm. The algorithm makes use of a separate plane sweep for each of the k different types of edges.

Let us consider the plane sweep in which the type i edges are identified. During the performance of this plane sweep, three different orderings of the points are employed. In each of the orderings the points are ordered by the ordering of their projections onto the oriented line through the origin which forms an angle of ϕ with the x-axis and which is oriented by the ray based at the origin which forms an angle of ϕ with the x-axis. In the α ordering $\phi = \frac{2\pi(i-1)}{k}$, while in the β ordering $\phi = \frac{2\pi(i-1)}{k} + \frac{\pi}{2}$, while in the γ ordering $\phi = \frac{2\pi i}{k} + \frac{\pi}{2}$. Let α(p) (β(p),γ(p)) be the rank of point p in the α (β,γ) ordering. Note that if i = 1 the α ordering is the ordering of the points by x-coordinate and the β ordering is the ordering of the points by y-coordinate.

The point set is swept in nonincreasing order of β rank. As the sweep progresses a table T of active points to which type i edges may be destined is maintained in γ order. When a point p is encountered on the sweep the following operations are performed.

1) Insert point p into table T.

2) If p has a predecessor q in T then report that pq is a type i edge.

3) Repeat Forever

    If p has a successor r in T then

        if $\alpha(r) > \alpha(p)$ then delete r from T else exit loop

    else exit loop

If the table of active points T is maintained as a balanced binary tree (e.g. 2-3-tree) then the operations insert, delete, successor and predecessor can be performed in O(logN) time. Thus each of the k plane sweeps can be performed in O(NlogN) time. Since k is a constant the total time required to form the θ-graph is O(NlogN).

3. The Bound

Given a set S of points in the plane and the θ-graph $\theta(S)$, for $\theta = \frac{2\pi}{k}$, k an integer constant such that $k > 8$, the purpose of this section is to show that the graph $\theta(S)$ closely approximates the complete Euclidean graph. We must thus show that the ratio $\frac{\theta(p,q)}{d(p,q)}$ is bounded by a small constant for any pair of points p and q in S. The shortest path from p to q in the graph $\theta(S)$ will pass through m, $0 \le m \le N - 2$ intermediate points. The following lemma bounds the ratio $\frac{\theta(p,q)}{d(p,q)}$ with a function of m.

Lemma

If the shortest path from p to q in $\theta(S)$, for $\theta = \frac{2\pi}{k}$, k an integer constant such that $k > 8$, passes through m intermediate points then

$$\frac{\theta(p,q)}{d(p,q)} \le \frac{1}{\cos\theta}\left[\frac{\tan^m\theta - 1}{\tan\theta - 1}\right] + \tan^m\theta$$

Proof

Let s_i be the ith intermediate point on the path from p to q, $0 \le i \le m$, $s_0 = p$. Let $\theta'(p,q)$ be the length of the shortest path from p to q under the following restriction. If the origin is located at s_i and q is located so that the ray $\overrightarrow{s_i q}$ forms an angle ϕ with the x-axis where $\frac{2\pi(i-1)}{k} \le \phi < \frac{2\pi i}{k}$, then the edge from s_i to s_{i+1} in the path from p to q is of type i. Clearly $\theta(p,q) \le \theta'(p,q)$ thus the lemma will follow if we can prove the bound for $\theta'(p,q)$.

We proceed by induction on m. In the base case, m = 0, there are no intermediate points on the path from p to q and $\theta'(p,q) = d(p,q)$.

As an inductive assumption we assume that if there are m-1 intermediate points on the shortest restricted path between two points p and q that

$$\frac{\theta'(p,q)}{d(p,q)} \leq \frac{1}{\cos\theta}\left(\frac{\tan^{(m-1)}\theta - 1}{\tan\theta - 1}\right) + \tan^{(m-1)}\theta$$

As an inductive step we consider the case where there are m intermediate points on the shortest restricted path between p and q. Let the origin be located at p. Without loss of generality let q be located so that the angle ϕ between the x-axis and the ray $\overrightarrow{pq}$ is such that $0 \leq \phi < \frac{2\pi}{k}$. We also have then that s_1 is located so that the angle α between the x-axis and the ray $\overrightarrow{ps_1}$ is such that $0 \leq \alpha < \frac{2\pi}{k}$. To proceed further we need more information as to where q and s_1 are located. The following claim allows us to restrict their locations.

Claim:

The ratio $\frac{\phi'(p,q)}{d(p,q)}$ can attain its maximum when q is located on the x-axis and s_1 is located to maximize the angle α such that $0 \leq \alpha < \frac{2\pi}{k}$.

Proof of Claim:

Since there are m - 1 intermediate points on the shortest restricted path from s_1 to q in $\theta(S)$ the inductive assumption implies that

$$\theta'(s_1,q) \leq d(s_1,q)\left[\frac{1}{\cos\theta}\left(\frac{\tan^{(m-1)}\theta - 1}{\tan\theta - 1}\right) + \tan^{(m-1)}\theta\right]$$

We thus have

$$\theta'(p,q) \leq d(p,s_1) + d(s_1,q)\left[\frac{1}{\cos\theta}\left(\frac{\tan^{(m-1)}\theta - 1}{\tan\theta - 1}\right) + \tan^{(m-1)}\theta\right] \qquad (*)$$

We will show how q and s_1 can be moved to their desired locations without decreasing the value of the ratio $\frac{\theta'(p,q)}{d(p,q)}$. If we move q and s_1 so that the values of $d(p,s_1)$ and $d(s_1,q)$ do not decrease and the value of $d(p,q)$ does not increase then the value of the ratio $\frac{\theta'(p,q)}{d(p,q)}$ will not decrease.

If s_1 and q are located such that angle $\alpha <$ angle ϕ, then consider the polar coordinates of s_1 and q. s_1 has coordinate (α, r_{s_1}) and q has coordinate (ϕ, r_q). If we move q to location (α, r_q)

and s_1 to the location (ϕ, r_{s_1}), we preserve the values $d(p,s_1) = r_{s_1}$, $d(s_1, q)$ and $d(p,q) = r_q$. We may thus henceforth assume that if ϕ is the angle between the x-axis and the ray $\overrightarrow{pq}$ and α is the angle between the x-axis and the ray $\overrightarrow{ps_1}$ that $\alpha > \phi$.

We now show how q can be moved to the x-axis. To do this we rotate the segment s_1q about s_1 until q lies on the x-axis. This transformation clearly preserves the values $d(p,s_1)$ and $d(s_1,q)$. That $\alpha > \phi$ ensures that the transformation does not increase the value of d(p,q).

To complete the proof of the claim we need to show that we can move s_1 so that the angle α approaches $\frac{2\pi}{k}$, without decreasing the ratio $\frac{\theta'(p,q)}{d(p,q)}$. To do this we move s_1 by increasing its y coordinate as much as possible without violating the condition that $\alpha < \frac{2\pi}{k}$. Since both p and q lie on the x-axis, this transformation will increase the values of $d(p,s_1)$ and $d(s_1,q)$, but will not affect the value of d(p,q). □

The claim implies that in (*) $0 \le d(p,s_1) < \frac{d(p,q)}{\cos\theta}$. Using elementary calculus it can be shown that the right hand side of (*) will attain its maximum when s_1 is located so that $d(p,s_1)$ is maximized. In this situation we have that $d(s_1,q) = d(p,q)\tan\theta$ thus (*) becomes

$$\theta'(p,q) \le \frac{d(p,q)}{\cos\theta} + d(p,q) tan\,\theta\left[\frac{1}{\cos\theta}\left[\sum_{i=0}^{m-2}\tan^i\theta\right] + \tan^{(m-1)}\theta\right].$$

Therefore the ratio

$$\frac{\theta'(p,q)}{d(p,q)} \le \frac{1}{\cos\theta}\left[\sum_{i=0}^{m-1}\tan^i\theta\right] + \tan^m\theta \;=\; \frac{1}{\cos\theta}\left[\frac{\tan^m\theta - 1}{\tan\theta - 1}\right] + \tan^m\theta$$

as required. □

The bound given by the lemma is strictly increasing with m. By taking the limit as m approaches infinity we have the following theorem.

Theorem

Given a set S of N points in the plane and the θ-graph $\theta(S)$, for $\theta = \frac{2\pi}{k}$, k an integer constant such that $k > 8$, then for any two points p and q in S

$$\frac{\theta(p,q)}{d(p,q)} \le \frac{1}{\cos\theta}\left[\frac{1}{1-\tan\theta}\right]$$

independent of S and N.

Given that $\theta = \frac{2\pi}{k}$, the following table illustrates the nature of the bound $B = \frac{1}{\cos\theta}\left[\frac{1}{1-\tan\theta}\right]$.

k	B
10	4.52
15	1.97
20	1.56
25	1.39
30	1.30
35	1.24
40	1.20

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References

[1] Chew, P., *There is a planar graph almost as good as the complete graph*, Proceedings of the 2nd Symposium on Computational Geometry, Yorktown Heights NY, 1986, 169-177.

[2] Dobkin, D., S. Friedman and K. Supowit, *Delaunay Graphs are Almost as Good as Complete Graphs*, Proceedings of the 28th Annual Symposium on Foundations of Computing, Los Angeles Ca., 1987, 20-26.

[3] Sedgewick, R. and J. Vitter, *Shortest paths in Euclidean graphs*, Algorithmica, 1,1(1986), 31-48.