Ten Binomial Identities

Emrul Hasan Zawad

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1 Factorial Expansion

$$\binom{n}{k} = \frac{n}{k!(n-k)!}$$

Proof: The symmetric group S_k on k symbols consists of all permutations of these k symbols. For instance, S_3 consists of all permutations of $1, 2, 3$ which means that the elements of S_3 are the triples $(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)$.

First choose k elements among the n elements in some order, which can be done in $n.(n-1)\cdots(n-k+1)$ways.

In this count, any group of k elements have been counted $k!$ times, which you have to compensate for, giving

$$\frac{n.(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

2 Binomial Theorem

Proof:
Expanding $(x+y)^n$ yields the sum of the 2^n products of the form $e_1e_2\ldots e_n$ where each e_i is x or y. Rearranging factors shows that each product equals $x^{(n-k)}y^k$ for some k between 0 and n. For a given k, the following are proved equal in succession:

• the number of copies of $x^{(n-k)}y^k$ in the expansion
• the number of n-$k$character x,y strings having y in exactly k positions
• the number of k-element subsets of $1, 2, \ldots, n \ldots$ So $\binom{n}{k}$ by a short combinatorial argument as $\frac{n!}{k!(n-k)!}$

This proves the Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

3 Pascal Identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

Proof:

$$\begin{aligned}
\binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} \\
&= \frac{n!}{k(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)(n-k)!} \\
&= \frac{n!(n-k+1)}{k(k-1)!(n-k)!(n-k+1)} + \frac{n!(k)}{(k-1)!(n-k+1)(n-k)!(k)} \\
&= \frac{n!(n-k+1)}{k!(n-k+1)!} + \frac{n!(k)}{k!(n-k+1)!} \\
&= \frac{n!(n-k+1) + n!(k)}{k!(n-k+1)!} \\
&= \frac{n!(n-k+1+k)}{k!(n-k+1)!} \\
&= \frac{n!(n+1)}{k!(n-k+1)!} \\
&= \frac{(n+1)!}{k!((n+1)-k)!} \\
&= \binom{n+1}{k}
\end{aligned}$$

4 Symmetry Identity

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof:

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&= \frac{n!}{(n-k)!k!} \\
&= \frac{n!}{(n-k)!(n-(n-k))!} \\
&= \binom{n}{n-k}
\end{aligned}$$

5 Absorption/Extraction Identity

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

Proof:

$$\begin{aligned}
\binom{n}{k} &= \frac{n!}{k!(n-k)!} \\
&= \frac{n(n-1)!}{k(k-1)!(n-k)!} \\
&= \frac{n}{k}\frac{(n-1)!}{(k-1)!(n-1-(k-1))!} \\
&= \frac{n}{k}\binom{n-1}{k-1}
\end{aligned}$$

6 trinomial revision

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$$

Proof:

$$\begin{aligned}
\binom{r}{m}\binom{m}{k} &= \frac{r!}{m!(r-m)!}\frac{m!}{k!(m-k)!} \\
&= \frac{r!(r-k)!}{(r-m)!k!(m-k)!(r-k)!} \\
&= \frac{r!}{k!(r-k)!}\frac{(r-k)!}{(m-k)!(r-k-m+k)!} \\
&= \binom{r}{k}\binom{r-k}{m-k}
\end{aligned}$$

7 Vandermonde's Identity

$$S(a,b,n) = \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}$$

Proof: Since $\binom{a}{k}$ is the coefficient of x^k in the polynomial $(x+1)^a$ and $\binom{b}{n-k}$ is the coefficient of $x^{(n-k)}$ in the polynomial $(1+x)^b$, the sum $S(a,b,n)$ of their products collects all the contributions to the coefficient of x^n in the polynomial $(1+x)^a(1+x)^b = (1+x)^{a+b}$.

This proves that $S(a,b,n) = \binom{a+b}{n}$

A Counting argument: If there are a items of type A and b items of type B, then

$$\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k}$$

is the number of ways to choosing n items from them: choose k of type A and $n-k$ of type B, vary from o to n and add up.

Thus,

$$\binom{a+b}{n}$$

8 Hockey-Stick Identity

$$\sum_{i=0}^{n} \binom{i+k-1}{k-1} = \binom{n+k}{k}$$

Proof by argument: One way to interpret this identity is to consider the number of ways to choose k integers from the set $1,2,3,\ldots,n+k$.

There are $\binom{n+k}{k}$ ways to do this, and we can also count the number of possibilities by considering the largest integer chosen. This can vary from k up to $n+k$, and if the largest integer chosen is l, then there are $\binom{l-1}{k-1}$ ways to choose the remaining $k-1$ integers

Therefore

$$\sum_{l=k}^{n+k} \binom{l-1}{k-1} = \binom{n+k}{k}$$

, and letting $i = l-k$ gives

$$\sum_{i=0}^{n} \binom{i+k-1}{k-1} = \binom{n+k}{k}$$

9 Sum of Binomial Coefficients over Upper Index

$$\sum_{j=0}^{n} \binom{j}{m} = \binom{n+1}{m+1}$$

Proof:

$$\begin{aligned}
\sum_{0 \le j \le n} \binom{j}{m} &= \sum_{0 \le m+j \le n} \binom{m+j}{m} \\
&= \sum_{-m \le j < 0} \binom{m+j}{m} + \sum_{0 \le j \le n-m} \binom{m+j}{m} \\
&= 0 + \sum_{0 \le j \le n-m} \binom{m+j}{m} \\
&= \binom{m+(n-m)+1}{m+1} \\
&= \binom{n+1}{m+1}
\end{aligned}$$

10 Sum of Binomial Coefficients over Lower Index

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Proof:
From the Binomial Theorem, we have that:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{i} x^{(n-i)} y^i$$

Putting x = y = 1 we get:

$$\begin{aligned}
2^n &= (1+1)^n \\
&= \sum_{i=0}^{n} \binom{n}{i} 1^{(n-1)} 1^i \\
&= \sum_{i=0}^{n} \binom{n}{i}
\end{aligned}$$