Rechnersehen Theorieaufgaben

3. Übungsserie

Aufgabe 1:

$$
\begin{aligned}
g(x,y) &= f(x,y)\cdot(-1)^{x+y} \\
\mathcal{F}(g)(u,v) &= \sum_{x=1}^{M}\sum_{y=1}^{N} f(x,y)\cdot(-1)^{x+y}e^{-2\pi i\left(\frac{ux}{M}+\frac{vy}{N}\right)} \\
&= \sum_{x=1}^{M}\sum_{y=1}^{N} f(x,y)\cdot e^{i\pi(x+y)}\cdot e^{-2\pi i\left(\frac{ux}{M}+\frac{vy}{N}\right)} \\
&= \sum_{x=1}^{M}\sum_{y=1}^{N} f(x,y)e^{-2\pi i\left(\frac{ux}{M}+\frac{vy}{N}-\frac{x+y}{2}\right)} \\
&= \sum_{x=1}^{M}\sum_{y=1}^{N} f(x,y)e^{-2\pi i\left(\frac{(u-\frac{M}{2})x}{M}+\frac{(v-\frac{N}{2})y}{N}\right)} \\
&= \mathcal{F}(f)(u-\frac{M}{2}, v-\frac{N}{2}) \quad \square
\end{aligned}
$$

Aufgabe 2:

$$
\begin{aligned}
(f*g)(x) &= \int_{-\infty}^{\infty} g(x')f(x-x')\,dx' \\
&= \int_{-\infty}^{\infty} g(x')\left(\int_{-\infty}^{\infty}\mathcal{F}(f)(\omega)e^{2\pi i\omega(x-x')}\,d\omega\right)dx' \\
&= \int_{-\infty}^{\infty} g(x')\left(\int_{-\infty}^{\infty}\mathcal{F}(f)(\omega)e^{2\pi i\omega x}\cdot e^{-2\pi i\omega x}\,d\omega\right)dx' \\
&= \int_{-\infty}^{\infty}\mathcal{F}(f)(\omega)e^{2\pi i\omega x}\left(\int_{-\infty}^{\infty} g(x')e^{-2\pi i\omega x}\,dx'\right)d\omega \\
&= \int_{-\infty}^{\infty}\mathcal{F}(f)(\omega)\mathcal{F}(g)(\omega)e^{2\pi i\omega x}\,d\omega \\
&= \mathcal{F}^{-1}(\mathcal{F}(F)\mathcal{F}(g))(x) \quad \square
\end{aligned}
$$

Aufgabe 3:

$$
\begin{aligned}
\mathcal{F}(\lambda f+\mu g)(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\lambda f(x,y)+\mu g(x,y))e^{-2\pi i(ux+vy)}\,dxdy \\
&= \lambda\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-2\pi i(ux+vy)}\,dxdy \\
&\quad +\mu\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-2\pi i(ux+vy)}\,dxdy \\
&= \lambda\mathcal{F}(f)(u,v)+\mu\mathcal{F}(g)(u,v) \quad \square
\end{aligned}
$$

$$\begin{aligned}
\mathcal{F}^{-1}(\lambda f + \mu g)(u, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\lambda f(x, y) + \mu g(x, y)) e^{2\pi i(ux+vy)} \, dxdy \\
&= \lambda \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{2\pi i(ux+vy)} \, dxdy \\
&\quad + \mu \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{2\pi i(ux+vy)} \, dxdy \\
&= \lambda \mathcal{F}^{-1}(f)(u, v) + \mu \mathcal{F}^{-1}(g)(u, v) \quad \square
\end{aligned}$$

Aufgabe 4:

$$\begin{aligned}
A_{\text{avg}}(x) &= \frac{1}{n} \text{box}\left(\frac{x}{n}\right) \\
\mathcal{F}(A_{\text{avg}})(\omega) &= \frac{1}{n} \mathcal{F}\left(\text{box}\left(\frac{x}{n}\right)\right) = \text{sinc}(n\omega)
\end{aligned}$$

Die sinc-Funktion wird mit steigender Frequenz immer geringer. Hohe Frequenzen (Kanten) werden also gestaucht, wodurch es zu einer Glättung des Bildes kommt.