

Rechnersehen Theorieaufgaben

3. Übungsserie

Aufgabe 1:

$$\begin{aligned}
 g(x, y) &= f(x, y) \cdot (-1)^{x+y} \\
 \mathcal{F}(g)(u, v) &= \sum_{x=1}^M \sum_{y=1}^N f(x, y) \cdot (-1)^{x+y} e^{-i2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} \\
 &= \sum_{x=1}^M \sum_{y=1}^N f(x, y) \cdot e^{i\pi(x+y)} \cdot e^{-i2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} \\
 &= \sum_{x=1}^M \sum_{y=1}^N f(x, y) e^{-i2\pi\left(\frac{ux}{M} + \frac{vy}{N} - \frac{x+y}{2}\right)} \\
 &= \sum_{x=1}^M \sum_{y=1}^N f(x, y) e^{-i2\pi\left(\frac{(u-\frac{M}{2})x}{M} + \frac{(v-\frac{N}{2})y}{N}\right)} \\
 &= \mathcal{F}(f)\left(u - \frac{M}{2}, v - \frac{N}{2}\right) \quad \square
 \end{aligned}$$

Aufgabe 2:

$$\begin{aligned}
 (f * g)(x) &= \int_{-\infty}^{\infty} g(x') f(x - x') dx' \\
 &= \int_{-\infty}^{\infty} g(x') \left(\int_{-\infty}^{\infty} \mathcal{F}(f)(\omega) e^{i2\pi\omega(x-x')} d\omega \right) dx' \\
 &= \int_{-\infty}^{\infty} g(x') \left(\int_{-\infty}^{\infty} \mathcal{F}(f)(\omega) e^{i2\pi\omega x} \cdot e^{-i2\pi\omega x'} d\omega \right) dx' \\
 &= \int_{-\infty}^{\infty} \mathcal{F}(f)(\omega) e^{i2\pi\omega x} \left(\int_{-\infty}^{\infty} g(x') e^{-i2\pi\omega x'} dx' \right) d\omega \\
 &= \int_{-\infty}^{\infty} \mathcal{F}(f)(\omega) \mathcal{F}(g)(\omega) e^{i2\pi\omega x} d\omega \\
 &= \mathcal{F}^{-1}(\mathcal{F}(f) \mathcal{F}(g))(x) \quad \square
 \end{aligned}$$

Aufgabe 3:

$$\begin{aligned}
 \mathcal{F}(\lambda f + \mu g)(u, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\lambda f(x, y) + \mu g(x, y)) e^{-i2\pi(ux+vy)} dx dy \\
 &= \lambda \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i2\pi(ux+vy)} dx dy \\
 &\quad + \mu \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-i2\pi(ux+vy)} dx dy \\
 &= \lambda \mathcal{F}(f)(u, v) + \mu \mathcal{F}(g)(u, v) \quad \square
 \end{aligned}$$

$$\begin{aligned}\mathcal{F}^{-1}(\lambda f + \mu g)(u, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\lambda f(x, y) + \mu g(x, y)) e^{i2\pi(ux+vy)} dx dy \\ &= \lambda \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{i2\pi(ux+vy)} dx dy \\ &\quad + \mu \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{i2\pi(ux+vy)} dx dy \\ &= \lambda \mathcal{F}^{-1}(f)(u, v) + \mu \mathcal{F}^{-1}(g)(u, v) \quad \square\end{aligned}$$

Aufgabe 4:

$$\begin{aligned}A_{\text{avg}}(x) &= \frac{1}{n} \text{box}\left(\frac{x}{n}\right) \\ \mathcal{F}(A_{\text{avg}})(\omega) &= \frac{1}{n} \mathcal{F}\left(\text{box}\left(\frac{x}{n}\right)\right) = \text{sinc}(n\omega)\end{aligned}$$

Die sinc-Funktion wird mit steigender Frequenz immer geringer. Hohe Frequenzen (Kanten) werden also gestaucht, wodurch es zu einer Glättung des Bildes kommt.