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Exploring Einsum as a Universal Inference Language

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von Maurice Wenig **geboren am** 11.08.1999 **in** Jena

Betreuer: Prof. Dr. Joachim Giesen, Dr. Julien Klaus

Abstract

Frameworks like SAT and ILP are essential tools for coming up with quick solutions to natural problems. For inference problems, a fitting framework, which is powerful enough to solve modern inference problems and specific enough to find good general optimization techniques, is not yet known. Einsum seems like a promising candidate for that role because of its ability to naturally express tensor expressions. Its freedom in the choice of contraction path also allows for good general optimization techniques. To see if Einsum is a good fit for a universal inference language, we investigated how a small set of modern inference techniques can be translated to Einsum. When translating chained operations, nested Einsum expressions naturally occur, which forbid fully applying optimization techniques. To avoid this, we provide a procedure which compresses nested Einsum expressions into flat Einsum expressions, provided the nested expressions use the same semiring. We were able to translate inference techniques including the DFT, the Hadamard transform, convolution, and max-pooling. Sadly, not all inference techniques could be translated to Einsum. Some of the standard architectures in deep learning could only be translated to mixtures of Einsum expressions over different semirings and element-wise functions. We conclude that Einsum is probably not powerful enough for standard architectures in deep learning, but such architectures could still benefit from the translation to said mixtures.

Contents

1 Introduction

In the process of building software, it can often happen that one encounters optimization problems. When such a problem occurs in this natural way, it can lead to vast amounts of time spent on implementing a solution to the specific problem, which can severely slow down production time. To help with this, we can use frameworks like SAT and ILP. The benefit of such framworks is that we do not need to spend much time building an efficient solver for each individual problem. Instead we can translate the individual problems to such a framework, which is often much easier. Then a general solver can be applied to the now embedded problem, which often yields efficient solutions. This can make the process of coming up with good solutions to naturally occuring problems much faster.

Inference problems are a subset of naturally occuring problems for which no such framework is known. Such a framework would have to be powerful enough to solve modern inference problems and specific enough to find good general optimization techniques. Einsum seems like a promising candidate for that role. It is powerful enough to support SAT [1] and ILP (see Chapter 6), and is a natural framework for operations on tensors, which occur frequently in inference problems. It is also specific enough so that efficient solvers can be built. Einsum can include multiple reductions, which allows us to optimize the order in which these reductions are applied. An example of this is matrix-matrix-vector multiplication $w = ABv$ for matrices A and B and a vector v. Here it is more efficient to first compute the right matrix-vector product $u = Bv$, and then compute the product of the remaining matrix with the resulting vector $w = Au$, instead of first computing the left matrix-matrix product $C = AB$, and then computing the the product of the resulting matrix with the remaining vector $w = Cv$. Additionally Einsum's restriction on tensor operations allows the use of specific hardware which is optimized for exactly those operations. To explore if Einsum is a good fit for a universal inference language, we try to translate a small set of modern inference techniques to Einsum.

Another problem we investigate is the nesting of Einsum expressions, which naturally occurs when trying to translate chained operations to Einsum. Nested expressions partially force the order in which reductions are applied, meaning the order cannot be optimized freely. To avoid this constraint, we explore a way of compressing nested Einsum expressions into flat Einsum expressions.

2 Einsum

Given two third-order tensors $A \in \mathbb{R}^{3\times4\times5}$ and $B \in \mathbb{R}^{3\times3\times5}$, and a vector $v \in \mathbb{R}^4$. Consider the following computation resulting in a matrix $C \in \mathbb{R}^{3\times3}$:

$$\forall i \in [3] : \forall j \in [4] : C_{ij} = \sum_{k=1}^{5} A_{ijk} B_{iik} v_j$$

The original Einstein-notation for summation removes redundant formalism ("boilerplate") from this expression:

$$C_{ij} = A_{ijk} B_{iik} v_j$$

where it is assumed that C is defined for all possible i, j. We sum over all indices that are not used to index the output. In this example, we therefore have to sum over all possible values of k, because it is not used to index C_{ij}. Note how it is clear what the shape of C is, because i and j were used to index the tensors A, B, and v, for which we defined the dimensions on every axis.

This notation is essentially the inspiration for Einsum, which might be apparent given the name Einsum. Einsum is just an adaptation of this style, which makes it easier to use in programming. With it, we can write the above expression like this:

$$C = (ijk, iik, j \to ij, A, B, v)$$

Through the following definition, we hope to clear up why this Einsum expression results in the computation above, and what computation is the result of a general Einsum expression.

Definition 1. Einsum expressions specify how several input tensors are combined into a single output tensor. Let $T^{(1)}, \ldots, T^{(n)}$ be our input tensors, where $T^{(i)}$ is an n_i-th order tensor for $i \in [n]$. The core of the Einsum expression are index strings. For this, we first need a collection of symbols S. The respective index string for a tensor $T^{(i)}$ is then just a tuple $\boldsymbol{s_i} \in S^{n_i}$, composed of symbols $s_{ij} \in S$ for $j \in [n_i]$. The index string that is right of the arrow ($\to$) belongs to the output tensor T and is referred to as output string $\boldsymbol{s_t}$.

In our example this could be $S = \{i, j, k\}$. The tensor $T^{(1)} = A$ has the index string $\boldsymbol{s_1} = ijk$, $T^{(2)} = B$ has $\boldsymbol{s_2} = iik$, $T^{(3)} = v$ has $\boldsymbol{s_3} = j$, and the output string is $\boldsymbol{s_t} = ij$. The individual symbols are $s_{11} = i$, $s_{12} = j$, $s_{13} = k$, $s_{12} = i$, $s_{22} = i$, $s_{23} = k$, $s_{31} = j$, $s_{t1} = i$, $s_{t2} = j$.

The next step in the definition is to speak about axis sizes. If we want to iterate over shared indices, it is necessary that the axes, that these indices are used for, share the same size. In our example, A_{ijk} and v_j share the symbol $s_{12} = s_{31} = j$. This means that the second axis of A and the first axis of v have to have the same size, which happens to be four. Let us express this formally.

Let $d_{ij} \in \mathbb{N}$ denote the size of the j-th axis of $T^{(i)}$ for $i \in [n], j \in [n_i]$. Then it must hold that $s_{ij} = s_{i'j'} \implies d_{ij} = d_{i'j'}$ for all $i, i' \in [n], j \in [n_i], j' \in [n_{i'}]$.

Therefore we can also denote the size of all axes, that a symbol $s \in S$ corresponds to, as $d_s := d_{ij}$ for all $i \in [n], j \in [n_i]$ with $s = s_{ij}$. Note that not all same size axes have to be assigned the same symbol. For instance a square matrix could have index strings $\boldsymbol{s} = (i, i)$ or $\boldsymbol{s} = (i, j)$.

The next step of the definition is figuring out which symbols are used for summation and which symbols are used for saving the result of the computation. In order to do this, it is useful to know which symbols are in an index string, because symbols can occur more than once in just one index string (as seen in B_{iik} in our example). Therefore, let $\sigma(\boldsymbol{s})$ denote the set with all symbols used in an index string $\boldsymbol{s}$. That is, in out example $\sigma(\boldsymbol{s_2}) = \sigma(iik) = \{i, k\}$.

All symbols to the right of the arrow ($\rightarrow$) are used as an index for the result of the computation. These symbols are called *free* symbols $F = \sigma(\boldsymbol{s_t})$. All other symbols used in the expression are called *bound* symbols $B = \bigcup_{i \in [n]} \sigma(\boldsymbol{s_i}) \setminus \sigma(\boldsymbol{s_t})$. The reasoning behind this name is, that these symbols are bound by the summation symbol in the original computation. In Einsum, we sum over all axes that belong to bound symbols. It follows that the multi-index space that we iterate over is $\mathcal{F} = \prod_{s \in F} [d_s]$ and the multi-index space we sum over is $\mathcal{B} = \prod_{s \in B} [d_s]$. In our example, the free symbols are $F = \{ij\}$ and the bound symbols are $B = \{k\}$. The multi-index space we iterate over is $d_i \times d_j = [3] \times [4]$. The multi-index space we sum over is $d_k = [5]$.

From the definition of $\mathcal{F}$, it follows that d_s has to be defined for all symbols $s \in F$. This means we have to add the constraint $\sigma(\boldsymbol{s_t}) \subseteq \bigcup_{i \in [n]} \sigma(\boldsymbol{s_i})$.

However, we do not use every symbol in the multi-index spaces to index every input tensor. Instead, we use the index strings $\boldsymbol{s}$ to index the tensor. To formally express this, we need a projection from a multi-index $(\boldsymbol{f}, \boldsymbol{b}) \in \mathcal{F} \times \mathcal{B}$ [1] to another multi-index, which includes only the indices, that are represented by the symbols used in $\boldsymbol{s}$, in the same order as present in $\boldsymbol{s}$. We denote this as $(\boldsymbol{f}, \boldsymbol{b}) : \boldsymbol{s}$. Notice how this still allows duplication of indices given in $(\boldsymbol{f}, \boldsymbol{b})$. This is needed, as can be seen in our example for B_{iik}, where a multi-index, e.g. $(i = 1, j = 4, k = 2) \in \mathcal{F} \times \mathcal{B}$, is projected onto a different multi-index, by the index string iik. With this index string, the index that is represented by the symbol i is projected onto the first and second position, and the index that is represented by the symbol k is projected onto the third position. Therefore, the resulting multi-index is $(i = 1, j = 4, k = 2) : iik = (1, 1, 2)$.

[1] Here, we use $(\boldsymbol{f}, \boldsymbol{b})$ as the notation for concatenating the tuples $\boldsymbol{f}$ and $\boldsymbol{b}$. This means, $(\boldsymbol{f}, \boldsymbol{b})$ is not a tuple of multi-indices, but another multi-index.

In our example, we used the standard sum and multiplication as operators for computing our result. But with Einsum, we allow the more general use of any semiring $R = (M, \oplus, \odot)$. With this, we can finally define a general Einsum expression

$$T = (\boldsymbol{s_1}, \ldots, \boldsymbol{s_n} \to \boldsymbol{s_t}, T^{(1)}, \ldots, T^{(n)})_R$$

in terms of semiring operations. Namely, T is the $|\boldsymbol{s_t}|$-th order tensor

$$\forall \boldsymbol{f} \in \mathcal{F} : T_{\boldsymbol{f}:\boldsymbol{s_t}} = \bigoplus_{\boldsymbol{b} \in \mathcal{B}} \bigodot_{i=1}^{n} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}}.$$

Because we also project the indices $\boldsymbol{f}$ with the output string $\boldsymbol{s_t}$, we allow to iterate over duplicate indices, e.g. $\mathrm{diag}(v) = (j \to jj, v)$. This leaves some entries of the result undefined. We define these entries to be the additive neutral element in the given semiring R. This may sound arbitrary at first, but will be useful later.

In case the semiring can be derived from the context, or if it is irrelevant, it can be left out from the expression.

The careful reader might have noticed two potential problems that could arise in the above definition. The first potential problem could arise when one of the input tensors is a scalar, which is a 0-th order tensor. This would mean that the index string $\boldsymbol{s}$ for that input tensor has to be the empty string ϵ. Now when the multi-index $(\boldsymbol{f}, \boldsymbol{b})$ is projected by this empty index string, then the resulting multi-index can only be the empty multi-index $\lambda := ()$. One might expect that this leads to a problem, because we can not access any entries of a tensor with an empty multi-index. But for scalars, it makes sense to define the empty multi-index in such a way, that it accesses precisely the only entry that is stored in the scalar, i.e. $T_\lambda := T$ for a scalar T. This way, we can easily support scalars with empty index strings in Einsum.

The second potential problem could arise when either the free symbols F or the bound symbols B are empty, because the universal quantor over an empty multi-index space $\mathcal{F}$ is always trivially true, and the sum over an empty multi-index space $\mathcal{B}$ is always trivially zero. But in fact, this leads to no problem, because the induced multi-index spaces of empty F or B are not empty themselves. They contain one element, namely the set including only the empty multi-index $\{\lambda\}$. In the following, we will explain why this is the case, and how this solves any problems with empty F or B.

Notice the definition of the product we use for to sets M, N:

$$M \times N = \{(m, n) \mid m \in M, n \in N\}.$$

This looks like an ordinary cartesian product, but the hidden difference lies in the meaning of (m, n). Namely, if m and n are multi-indices $\boldsymbol{m} = (m_1, \ldots, m_{k_1})$ and

$\boldsymbol{n} = (n_1, \ldots, n_{k_2})$ for $k_1, k_2 \in \mathbb{N}$, then we defined $(\boldsymbol{m}, \boldsymbol{n})$ to be the concatenation of the multi-indices:

$$(\boldsymbol{m}, \boldsymbol{n}) = (m_1, \ldots, m_{k_1}, n_1, \ldots, n_{k_2}),$$

which is one tuple with the entries of $\boldsymbol{m}$ and $\boldsymbol{n}$, instead of the tuple of tuples

$$((m_1, \ldots, m_{k_1}), (n_1, \ldots, n_{k_2})).$$

Therefore we can name a neutral element for concatenation, which is the empty multi-index λ with $(\boldsymbol{i}, \lambda) = \boldsymbol{i}$ for any multi-index $\boldsymbol{i}$. From this, we can derive a neutral element for our product of multi-index spaces, which is the set including only the empty multi-index $\{\lambda\}$ with $\mathcal{I} \times \{\lambda\} = \mathcal{I}$ for any multi-index space $\mathcal{I}$.

Now, because it makes sense to define an operation over an empty set of operands as the neutral element of said operation, we can safely define

$$\prod_{s \in \emptyset} [d_s] := \{\lambda\}\,.$$

Therefore, if $F = \emptyset$, then

$$T = (\boldsymbol{s_1}, \ldots, \boldsymbol{s_n} \to \boldsymbol{s_t}, T^{(1)}, \ldots, T^{(n)})_R$$

results in the computation of a $|\boldsymbol{s_t}|$-th order tensor T with

$$\begin{aligned} \forall \boldsymbol{f} \in \mathcal{F} : T_{\boldsymbol{f}:\boldsymbol{s_t}} &= \sum_{\boldsymbol{b} \in \{\lambda\}} \bigodot_{i=1}^{n} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \\ \iff \forall \boldsymbol{f} \in \mathcal{F} : T_{\boldsymbol{f}:\boldsymbol{s_t}} &= \bigodot_{i=1}^{n} T^{(i)}_{\boldsymbol{f}:\boldsymbol{s_i}}. \end{aligned}$$

And if $B = \emptyset$, then

$$T = (\boldsymbol{s_1}, \ldots, \boldsymbol{s_n} \to, T^{(1)}, \ldots, T^{(n)})_R$$

results in the computation of a scalar T with

$$\begin{aligned} \forall \boldsymbol{f} \in \{\lambda\} : T_{\boldsymbol{f}:\epsilon} &= \bigoplus_{\boldsymbol{b} \in \mathcal{B}} \bigodot_{i=1}^{n} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \\ \iff T &= \bigoplus_{\boldsymbol{b} \in \mathcal{B}} \bigodot_{i=1}^{n} T^{(i)}_{\boldsymbol{b}:\boldsymbol{s_i}}. \end{aligned}$$

All following examples use the standard semiring $R = (\mathbb{R}, +, \cdot)$.

- matrix-vector multiplication: Let $A \in \mathbb{R}^{m \times n}, v \in \mathbb{R}^n$. Then

$$A \cdot v = (ij, j \to i, A, v)$$

- matrix-matrix multiplication: Let $A \in \mathbb{R}^{m \times r}, B \in \mathbb{R}^{r \times n}$. Then

$$A \cdot B = (ik, kj \to ij, A, B)$$

- trace: Let $A \in \mathbb{R}^{n \times n}$. Then

$$\text{trace}(A) = (ii \to, A)$$

- squared Frobenius norm: Let $A \in \mathbb{R}^{n \times n}$. Then

$$|A|_2^2 = (ij, ij \to, A, A)$$

- diagonal matrix: Let $v \in \mathbb{R}^n$. Then

$$\text{diag}(v) = (i \to ii, v)$$

3 Nested Expressions

In practice, concatenations of operations arise naturally, e.g. computing the squared norm of a matrix-vector product $|A \cdot v|_2^2$ for $A \in \mathbb{R}^{m \times n}, v \in \mathbb{R}^n$. This would lead to a nested Einsum expression $|A \cdot v|_2^2 = (i, i \to, (ij, j \to i, A, v), (ij, j \to i, A, v))$. This expression dictates the order of evaluating the expression. In the example of the norm, the expression $(ij, j \to i, A, v)$ has to be evaluated before squaring and summing over the results of this computation.

This is limiting, because the order of evaluation might not yield optimal runtime. This can be seen with a simple matrix-matrix-vector multiplication, which can be written as follows:

$$(A \cdot B) \cdot v = (ij, j \to i, (ik, kj \to ij, A, B), v)$$

which is clearly worse than the optimal contraction order

$$A \cdot (B \cdot v) = (ij, j \to i, A, (ij, j \to i, B, v))$$

for $A \in \mathbb{R}^{m \times r}, B \in \mathbb{R}^{r \times n}, v \in \mathbb{R}^n$. Another limitation of nested Einsum expressions is that we can not fully benefit from the computational advantages that come with the optimization of single Einsum expressions.

But fortunately, all nested Einsum expressions can be compressed into a single Einsum expression, if they are computed over the same semiring. For instance,

$$\begin{aligned} |A \cdot v|_2^2 &= (i, i \to, (ij, j \to i, A, v), (ij, j \to i, A, v)) \\ &= (ij, j, ij, j \to, A, v, A, v) \end{aligned}$$

and

$$\begin{aligned} (A \cdot B) \cdot v &= (ij, j \to i, (ik, kj \to ij, A, B), v) \\ &= (ik, kj, j \to i, A, B, v). \end{aligned}$$

This leaves the path of contraction up to the implementation, and lets us benefit from all the computational advantages mentioned in Chapter 2. For the following theorems, we assume that the computations are all over the same semiring $R = (M, \oplus, \odot)$.

3.1 Simple Nested Expressions

In the following, we will explore how to compress such expressions:

$$\underbrace{(ij, j \to i, \overbrace{(ik, kj \to ij, A, B)}^{\text{inner expression}}, v)}_{\text{outer expression}}$$

for $A \in \mathbb{R}^{m \times r}, B \in \mathbb{R}^{r \times n}, v \in \mathbb{R}^n$.

Theorem 1: For $i \in [m+n]$, let $T^{(i)}$ be an n_i-th order tensor with index string $\boldsymbol{s_i} \in S^{n_i}$. Let $\boldsymbol{s_u}, \boldsymbol{s_v}$ be index strings. Let

$$U = (\boldsymbol{s_{m+1}}, \ldots, \boldsymbol{s_{m+n}} \to \boldsymbol{s_u}, T^{(m+1)}, \ldots, T^{(m+n)})$$

and

$$V = (\boldsymbol{s_1}, \ldots, \boldsymbol{s_m}, \boldsymbol{s_u} \to \boldsymbol{s_v}, T^{(1)}, \ldots, T^{(m)}, U)$$

where the bound symbols of the second Einsum expression share no symbols with the first Einsum expression, then

$$V = (\boldsymbol{s_1}, \ldots, \boldsymbol{s_{m+n}} \to \boldsymbol{s_v}, T^{(1)}, \ldots, T^{(m+n)})$$

is the compressed Einsum expression for V that includes the computation of U.

Proof. Let F, F', B, B' be the free and bound symbols of the outer and inner Einsum expression respectively. W.l.o.g. they are all non-empty. From them we can derive the multi-index spaces $\mathcal{F}, \mathcal{F}', \mathcal{B}, \mathcal{B}'$ as in the definition. Then

$$\begin{aligned}
V & = (\boldsymbol{s_1}, \ldots, \boldsymbol{s_m}, \boldsymbol{s_u} \to \boldsymbol{s_v}, T^{(1)}, \ldots, T^{(m)}, U) \\
\iff \forall \boldsymbol{f} \in \mathcal{F} : V_{\boldsymbol{f}:\boldsymbol{s_v}} & = \bigoplus_{\boldsymbol{b} \in \mathcal{B}} \bigodot_{i=1}^{m} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \odot U_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_u}} \\
& = \bigoplus_{\boldsymbol{b} \in \mathcal{B}} \bigodot_{i=1}^{m} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \odot \left[\bigoplus_{\boldsymbol{b'} \in \mathcal{B}'} \bigodot_{i'=m+1}^{m+n} T^{(i')}_{(\boldsymbol{f},\boldsymbol{b},\boldsymbol{b'}):\boldsymbol{s_{i'}}} \right] \\
& = \bigoplus_{\boldsymbol{b} \in \mathcal{B}} \bigoplus_{\boldsymbol{b'} \in \mathcal{B}'} \bigodot_{i=1}^{m} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \odot \bigodot_{i=m+1}^{m+n} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b},\boldsymbol{b'}):\boldsymbol{s_i}} \\
& = \bigoplus_{\boldsymbol{b} \in \mathcal{B} \times \mathcal{B}'} \bigodot_{i=1}^{m+n} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \\
\iff V & = (\boldsymbol{s_1}, \ldots, \boldsymbol{s_{m+n}} \to \boldsymbol{s_v}, T^{(1)}, \ldots, T^{(m+n)})
\end{aligned}$$

where the third equality follows from the definition of U:

$$\forall \boldsymbol{f'} \in \mathcal{F}' : U_{\boldsymbol{f'}:\boldsymbol{s_u}} = \bigoplus_{\boldsymbol{b'} \in \mathcal{B}'} \bigodot_{i'=m+1}^{m+n} T^{(i')}_{(\boldsymbol{f'},\boldsymbol{b'}):\boldsymbol{s_{i'}}}$$

and from the fact, that the symbols in $\boldsymbol{s_u}$ are used in the outer expression as an input string, and in the inner expression as the output string, and therefore $F' \subseteq B \cup F$. Additionally, because of the stated requirement $(B \cup F) \cap B' = \emptyset$, the symbols representing $\boldsymbol{b'}$ do not clash with the symbols representing $(\boldsymbol{f}, \boldsymbol{b})$, and therefore $(\boldsymbol{f}, \boldsymbol{b}, \boldsymbol{b'}) : \boldsymbol{s_{i'}}$ is well-defined and projects on the same indices as $(\boldsymbol{f'}, \boldsymbol{b'}) : \boldsymbol{s_{i'}}$. The fourth equality follows from the distributivity of a semiring. □

This means that we can compress all nested Einsum expressions, where the output string of the inner expression, which is used to compute U, is exactly the same as the respective input string in the outer expression, where U is used as an input tensor. This is already helpful for some naturally occuring expressions in linear algebra, e.g.

$$\begin{aligned} |A \cdot v|_2^2 &= (i, i \to, (ij, j \to i, A, v), (ij, j \to i, A, v)) \\ &= (ij, j, ij, j \to, A, v, A, v) \end{aligned}$$

for $A \in \mathbb{R}^{m \times n}, v \in \mathbb{R}^n$, or

$$\begin{aligned} A \cdot B \cdot v &= (ij, j \to i, (ik, kj \to ij, A, B), v) \\ &= (ik, kj, j \to i, A, B, v) \end{aligned}$$

for $A \in \mathbb{R}^{m \times r}, B \in \mathbb{R}^{r \times n}, v \in \mathbb{R}^n$. However, sometimes we need to access a different multi-index set than the one we computed, e.g.

$$\text{trace}(A \cdot B) = (ii \to, (ik, kj \to ij, A, B))$$

or

$$A \cdot \text{diag}(v) = (ik, kj \to ij, A, (i \to ii, v))$$

for $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times m}, v \in \mathbb{R}^n$. For this, we need more general ways of compressing nested Einsum expressions.

3.2 General Nested Expressions

The following is an example of an expression, which we cannot compress with the previous theorem:

$$(a, b, c, d, e, abbcde \to bc, v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(5)}, (i, j, k, l \to iijkkl, v^{(6)}, v^{(7)}, v^{(8)}, v^{(9)}))$$

for $v^{(i)} \in \mathbb{R}^{d_{vi}}$ with $i \in [9]$, where $d_{vi} \in \mathbb{N}$ are appropriate dimensions. This is because the output string $\boldsymbol{s_u} = iijkkl$ and the input string $\hat{\boldsymbol{s}}_{\boldsymbol{u}} = abbde$ are not the same. In the following, we will explore how to compress such expressions. Note that, for the theorem, we use disjoint sets of symbols for the inner and outer expression. This helps in the proof, and is not a real constraint in practice, because we can just

rename the symbols in different scopes. For example, we could write the matrix-matrix-vector multiplication of $A \in \mathbb{R}^{m \times r}, B \in \mathbb{R}^{r \times n}, v \in \mathbb{R}^n$ as

$$A \cdot (B \cdot v) = (ij, j \to i, A, (ab, b \to a, B, v))$$

or as

$$A \cdot (B \cdot v) = (ij, j \to i, A, (ij, j \to i, B, v)),$$

because the scope of each symbol does not reach into nested expressions, and therefore the i and j used in the outer expression are treated as different symbols than the i and j used in the inner expression.

Theorem 2: For $i \in [m+n]$, let $T^{(i)}$ be an n_i-th order tensor with index string $\boldsymbol{s_i} \in S^{n_i}$. Let $\boldsymbol{s_u}$ be an index string for the n_u-th order tensor U, which is defined as follows:

$$U = (\boldsymbol{s_{m+1}}, \dots, \boldsymbol{s_{m+n}} \to \boldsymbol{s_u}, T^{(m+1)}, \dots, T^{(m+n)})$$

Also let $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ be alternative index strings for U.

Let s_v be an index string and

$$V = (\boldsymbol{s_1}, \dots, \boldsymbol{s_m}, \hat{\boldsymbol{s}}_{\boldsymbol{u}} \to \boldsymbol{s_v}, T^{(1)}, \dots, T^{(m)}, U)$$

where the first and second Einsum expression share no symbols. Then these nested Einsum expressions can also be compressed into a single Einsum expression.

Let us clarify that the index string $\boldsymbol{s_u}$ corresponds to the output string of the inner expression, and the index string $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ corresponds to the input string that is used for the input tensor U in the outer expression. In our example, these are $\boldsymbol{s_u}$ and $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$:

$$(a, b, c, d, e, \overbrace{abbcde}^{\hat{\boldsymbol{s}}_{\boldsymbol{u}}} \to bc, v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(5)}, (i, j, k, l \to \overbrace{iijkkl}^{\boldsymbol{s_u}}, v^{(6)}, v^{(7)}, v^{(8)}, v^{(9)})).$$

In contrast to Theorem 1, we cannot just replace the input index string $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ by all the input index strings in the inner Einsum expression $\boldsymbol{s_{m+1}}, \dots, \boldsymbol{s_{m+n}}$. Instead, we first need to apply a symbol map $\nu : S \to S$ to each of the index strings. This symbol map holds information about which symbols are effectively the used for the same index.

For the definition of the map ν, we first construct an undirected graph $G = (V, E)$ that we call *symbol graph*. In the symbol graph, the nodes consist of all symbols from $\boldsymbol{s_u}$ and $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$. The edges are $E = \{\{s_{uj}, \hat{s}_{uj}\} \mid j \in [n_u]\}$, which connects all symbols from $\boldsymbol{s_u}$ and $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ that share an index. The symbol graph for our example is illustrated in Figure 3.1.

In the symbol graph, if two symbols are connected, then they will both be mapped to the same symbol. Therefore, it makes sense assigning a symbol $s_C \in S \setminus V$ to

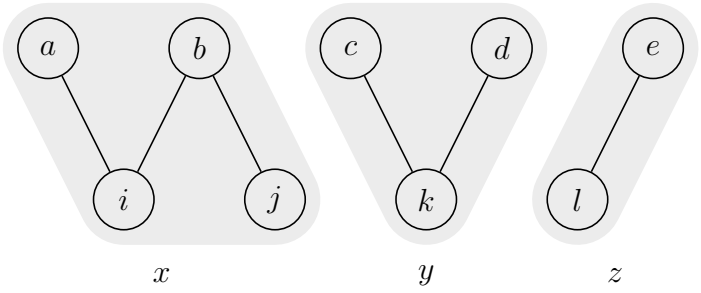


Figure 3.1: Symbol graph for the example

each of the graphs components C. Then we can define ν as follows:

$$\nu(s) := \begin{cases} s_C & \text{if } \exists C : C \text{ is a component of } G \text{ and } s \in C, \\ s & \text{else.} \end{cases}$$

In our example, the components are $\{a, b, i, j\}$, $\{c, d, k\}$, and $\{e, l\}$. Therefore we could use

$$\nu(s) := \begin{cases} x & \text{if } s \in \{a, b, i, j\}, \\ y & \text{if } s \in \{c, d, k\}, \\ z & \text{if } s \in \{e, l\}, \\ s & \text{else.} \end{cases}$$

The symbol map ν can be extended, such that it maps entire index strings instead of just symbols, by setting $\nu(\boldsymbol{s_i}) \in S^{n_i}, \nu(\boldsymbol{s_i})_j := \nu(s_{ij})$. Then we can write the substituted index strings by setting $\hat{\boldsymbol{s}}_{\boldsymbol{i}} := \nu(\boldsymbol{s_i})$ for $i \in [m+n]$ and $\hat{\boldsymbol{s}}_{\boldsymbol{t}} = \nu(\boldsymbol{s_t})$. With these index strings, the compressed Einsum expression is the following:

$$V = (\hat{\boldsymbol{s}}_{\boldsymbol{1}}, \ldots, \hat{\boldsymbol{s}}_{\boldsymbol{m+n}} \to \hat{\boldsymbol{s}}_{\boldsymbol{v}}, T^{(1)}, \ldots, T^{(m+n)})$$

which helps us to compress the example:

$$\begin{aligned}(a, b, c, d, e, abbde \to bc, v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(5)}, (i, j, k, l \to iijkkl, v^{(6)}, v^{(7)}, v^{(8)}, v^{(9)})) \\ = (x, x, y, y, z, x, x, y, z \to xy, v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(5)}, v^{(6)}, v^{(7)}, v^{(8)}, v^{(9)}).\end{aligned}$$

For the proof of this theorem, we first need three lemmata, which essentially boil down to one intuitive thought: The effective equality of two symbols can be expressed by multiplication with the unity matrix $\mathbb{1}_d$:

$$(\mathbb{1}_d)_{ij} := \begin{cases} \mathbb{1} & \text{if } i = j, \\ \mathbb{0} & \text{else} \end{cases}$$

for $i, j \in [d]$, where $\mathbb{0}$ and $\mathbb{1}$ indicate the neutral element of addition and multiplication in the given semiring respectively.

Lemma 3: For $i \in [n]$, let $T^{(i)}$ be an n_i-th order tensor with index string $\boldsymbol{s_i} \in S^{n_i}$. Let $\boldsymbol{s_t}$ be the index string for T with

$$T = (\boldsymbol{s_1}, \dots, \boldsymbol{s_n} \to \boldsymbol{s_t}, T^{(1)}, \dots, T^{(n)}).$$

Let F and B be the free and bound symbols of this expression. Let $k \in [n]$ and $j \in [n_k]$, then we can replace the j-th symbol of the k-th index string with a new symbol $s_{\text{new}} \in S \setminus (F \cup B)$ by adding the unity matrix $\mathbb{1}_{d_{kj}}$ as an input tensor in the following way:

Let $\boldsymbol{s'_k}$ be a new index string such that

$$s'_{ki} := \begin{cases} s_{\text{new}} & \text{if } i = j, \\ s_{ki} & \text{else} \end{cases}$$

for $i \in [n_k]$. Let $\boldsymbol{s}_{\mathbb{1}} = (s_{kj}, s_{\text{new}})$. Then

$$T = (\boldsymbol{s_1}, \dots, \boldsymbol{s'_k}, \dots, \boldsymbol{s_n}, \boldsymbol{s}_{\mathbb{1}} \to \boldsymbol{s_t}, T^{(1)}, \dots, T^{(n)}, \mathbb{1}_{d_{kj}}).$$

Proof. Let $\mathcal{F}$ and $\mathcal{B}$ be the induces multi-index spaces for the free and bound symbols of the Einsum expression. Then

$$\begin{aligned}
& T = (\boldsymbol{s_1}, \dots, \boldsymbol{s_n} \to \boldsymbol{s_t}, T^{(1)}, \dots, T^{(n)}) \\
\iff \forall \boldsymbol{f} \in \mathcal{F} : T_{\boldsymbol{f}:\boldsymbol{s_t}} &= \bigoplus_{\boldsymbol{b} \in \mathcal{B}} \bigodot_{i \in [n]} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \\
&= \bigoplus_{\boldsymbol{b} \in \mathcal{B} \times [d_{kj}]} \bigodot_{1 \le i < k} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \odot T^{(k)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s'_k}} \odot \bigodot_{k < i \le n} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \\
&\qquad \odot \begin{cases} \mathbb{1} & \text{if } (\boldsymbol{f},\boldsymbol{b}) : s_{kj} = (\boldsymbol{f},\boldsymbol{b}) : s_{\text{new}}, \\ \mathbb{0} & \text{else} \end{cases} \\
&= \bigoplus_{\boldsymbol{b} \in \mathcal{B} \times [d_{kj}]} \bigodot_{1 \le i < k} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \odot T^{(k)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s'_k}} \odot \bigodot_{k < i \le n} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \\
&\qquad \odot \left(\mathbb{1}_{d_{kj}}\right)_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s}_{\mathbb{1}}} \\
\iff T &= (\boldsymbol{s_1}, \dots, \boldsymbol{s'_k}, \dots, \boldsymbol{s_n}, \boldsymbol{s}_{\mathbb{1}} \to \boldsymbol{s_t}, T^{(1)}, \dots, T^{(n)}, \mathbb{1}_{d_{kj}})
\end{aligned}$$

where the third equality holds because in the summation over $\mathcal{B} \times [d_{kj}]$, exactly those summands get selected by the condition, which are also valid summands in the previous summation over $\mathcal{B}$. All other summands are disregarded because they are multiplied by $\mathbb{0}$, which is the additive neutral element in the semiring and *annihilates* every element, which means $a \odot \mathbb{0} = \mathbb{0}$ for every $a \in M$. □

This lemma intuitively means that we can replace any symbol in an index string of an input tensor of our choice with a new symbol by introducing the unity matrix with an appropriate index string as a factor. Now the same holds for the index string of the output tensor $\boldsymbol{s_t}$, which will be the content of the next lemma.

Lemma 4: For $i \in [n]$, let $T^{(i)}$ be an n_i-th order tensor with index string $\boldsymbol{s_i} \in S^{n_i}$. Let $\boldsymbol{s_t}$ be the index string for T with

$$T = (\boldsymbol{s_1}, \dots, \boldsymbol{s_n} \to \boldsymbol{s_t}, T^{(1)}, \dots, T^{(n)}).$$

Let F and B be the free and bound symbols of this expression. Let $n_t := |\boldsymbol{s_t}|$, $j \in [n_t]$, and $d_{tj} := d_{s_{tj}}$, then we can replace the j-th symbol of the output string with a new symbol $s_{\text{new}} \in S \setminus (F \cup B)$ by adding the unity matrix $\mathbb{1}_{d_{tj}}$ as an input tensor in the following way:

Let $\boldsymbol{s'_t}$ be a new index string such that

$$s'_{ti} := \begin{cases} s_{\text{new}} & \text{if } i = j, \\ s_{ti} & \text{else} \end{cases}$$

for $i \in [n_t]$. Let $\boldsymbol{s}_{\mathbb{1}} = (s_{tj}, s_{\text{new}})$. Then

$$T = (\boldsymbol{s_1}, \dots, \boldsymbol{s_n}, \boldsymbol{s}_{\mathbb{1}} \to \boldsymbol{s'_t}, T^{(1)}, \dots, T^{(n)}, \mathbb{1}_{d_{kj}}).$$

Proof. Let $\mathcal{F}$ and $\mathcal{B}$ be the induces multi-index spaces for the free and bound symbols of the Einsum expression. If s_{tj} occurs in $\boldsymbol{s_t}$ even after replacing it with s_{new}, then

$$\begin{aligned}
& T = (\boldsymbol{s_1}, \dots, \boldsymbol{s_n} \to \boldsymbol{s_t}, T^{(1)}, \dots, T^{(n)}) \\
\iff \forall \boldsymbol{f} \in \mathcal{F} : T_{\boldsymbol{f}:\boldsymbol{s_t}} &= \bigoplus_{\boldsymbol{b} \in \mathcal{B}} \bigodot_{i \in [n]} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \\
\iff \forall \boldsymbol{f} \in \mathcal{F} \times [d_{tj}] : T_{\boldsymbol{f}:\boldsymbol{s'_t}} &= \bigoplus_{\boldsymbol{b} \in \mathcal{B}} \bigodot_{i \in [n]} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \odot \begin{cases} \mathbb{1} & \text{if } \boldsymbol{f} : s_{tj} = \boldsymbol{f} : s_{\text{new}}, \\ \mathbb{0} & \text{else} \end{cases} \\
&= \bigoplus_{\boldsymbol{b} \in \mathcal{B}} \bigodot_{i \in [n]} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \odot \left(\mathbb{1}_{d_{tj}}\right)_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s}_{\mathbb{1}}} \\
\iff & T = (\boldsymbol{s_1}, \dots, \boldsymbol{s_n}, \boldsymbol{s}_{\mathbb{1}} \to \boldsymbol{s'_t}, T^{(1)}, \dots, T^{(n)}, \mathbb{1}_{d_{tj}})
\end{aligned}$$

where the third equality holds because exactly those indices get selected by the condition, where T was originally defined. If s_{tj} no longer occurs in $\boldsymbol{s_t}$ after replacing it with s_{new}, then s_{tj} turns into a bound symbol. Therefore we have to define $\mathcal{F}' = \prod_{s \in \boldsymbol{s'_t}} [d_s]$. Then

$$\begin{aligned}
& T = (\boldsymbol{s_1}, \dots, \boldsymbol{s_n} \to \boldsymbol{s_t}, T^{(1)}, \dots, T^{(n)}) \\
\iff \forall \boldsymbol{f} \in \mathcal{F} : T_{\boldsymbol{f}:\boldsymbol{s_t}} &= \bigoplus_{\boldsymbol{b} \in \mathcal{B}} \bigodot_{i \in [n]} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \\
\iff \forall \boldsymbol{f} \in \mathcal{F}' : T_{\boldsymbol{f}:\boldsymbol{s'_t}} &= \bigoplus_{\boldsymbol{b} \in \mathcal{B} \times [d_{tj}]} \bigodot_{i \in [n]} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \odot \begin{cases} \mathbb{1} & \text{if } (\boldsymbol{f},\boldsymbol{b}) : s_{tj} = (\boldsymbol{f},\boldsymbol{b}) : s_{\text{new}}, \\ \mathbb{0} & \text{else} \end{cases} \\
&= \bigoplus_{\boldsymbol{b} \in \mathcal{B} \times [d_{tj}]} \bigodot_{i \in [n]} T^{(i)}_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s_i}} \odot \left(\mathbb{1}_{d_{tj}}\right)_{(\boldsymbol{f},\boldsymbol{b}):\boldsymbol{s}_{\mathbb{1}}} \\
\iff & T = (\boldsymbol{s_1}, \dots, \boldsymbol{s_n}, \boldsymbol{s}_{\mathbb{1}} \to \boldsymbol{s'_t}, T^{(1)}, \dots, T^{(n)}, \mathbb{1}_{d_{tj}})
\end{aligned}$$

where the third equality holds because exactly those summands get selected by the

condition, where $(\boldsymbol{f}, \boldsymbol{b}) : \boldsymbol{s_t}$ could also get used as an index for T. All others are annihilated. □

Now with these two lemmata, we can replace any symbol in any index string, regardless whether it is an input string or the output string, by introducing the unity matrix with an appropriate index string as a factor. In the following lemma, we will show that any unity matrix factors can be removed again, by renaming certain symbols in all other index strings in the Einsum expression.

Lemma 5: For $i \in [n]$, let $T^{(i)}$ be an n_i-th order tensor with index string $\boldsymbol{s_i} \in S^{n_i}$, where $T^{(n)} = \mathbb{1}_m$ for some $m \in \mathbb{N}$. Let $\boldsymbol{s_t}$ be the index string for T with

$$T = (\boldsymbol{s_1}, \ldots, \boldsymbol{s_n} \to \boldsymbol{s_t}, T^{(1)}, \ldots, T^{(n)}).$$

Let F and B be the free and bound symbols of this expression. Then we can introduce a symbol map $\mu : S \to S$, which maps both symbols in $\boldsymbol{s_n} = s_{n1}s_{n2}$ to the same symbol $s_{\text{new}} \in S \setminus (F \cup B)$:

$$\mu(s) := \begin{cases} s_{\text{new}} & \text{if } s \in \{s_{n1}, s_{n2}\}, \\ s & \text{else.} \end{cases}$$

The symbol map μ can be extended, such that it maps entire index strings instead of just symbols, by setting $\mu(\boldsymbol{s_i}) \in S^{n_i}, \mu(\boldsymbol{s_i})_j := \mu(s_{ij})$. Then we can write the substituted index strings by setting $\boldsymbol{s'_i} := \mu(\boldsymbol{s_i})$ for $i \in [n]$ and $\boldsymbol{s'_t} = \mu(\boldsymbol{s_t})$. With these index strings, the following holds:

$$T = (\boldsymbol{s'_1}, \ldots, \boldsymbol{s'_{n-1}} \to \boldsymbol{s'_t}, T^{(1)}, \ldots, T^{(n-1)}).$$

Proof. For this proof, we provide the following example of an Einsum expression on which we demonstrate the given arguments for better understanding:

$$(ij, kl, mn, ij, kl, mn \to imn, A, B, C, \mathbb{1}_a, \mathbb{1}_b, \mathbb{1}_c)$$

for $A \in \mathbb{R}^{a \times a}, B \in \mathbb{R}^{b \times b}, C \in \mathbb{R}^{c \times c}$ and some $a, b, c \in \mathbb{N}$.

We need to consider three cases for the symbols used in the index string $\boldsymbol{s_n} = (s_{n1}, s_{n2})$:

- s_{n1} and s_{n2} are both free symbols,
- s_{n1} and s_{n2} are both bound symbols,
- one symbol of s_{n1} and s_{n2} is a free symbol, the other is a bound symbol.

Every one of these cases leads to the same result, but in a slightly different way.

First let us consider the case where both symbols are free. In this case, both symbols can be replaced by a single symbol, because T is $\mathbb{0}$ for all entries with a multi-index, where the indices projected by the symbols are not equal.

In our example, this is equivalent to the following:

$$\begin{aligned}\forall i,m,n : T_{imn} &= \bigoplus_{j,k,l} A_{ij}B_{kl}C_{mn}\,(\mathbb{1}_a)_{ij}\,(\mathbb{1}_b)_{kl}\,(\mathbb{1}_c)_{mn} \\ &= \begin{cases} \bigoplus_{j,k,l} A_{ij}B_{kl}C_{mn}\,(\mathbb{1}_a)_{ij}\,(\mathbb{1}_b)_{kl} & \text{if } m = n, \\ \mathbb{0} & \text{else} \end{cases} \\ \iff \forall i,z : T_{izz} &= \bigoplus_{j,k,l} A_{ij}B_{kl}C_{zz}\,(\mathbb{1}_a)_{ij}\,(\mathbb{1}_b)_{kl}\,.\end{aligned}$$

Next let us consider the case where both symbols are bound. In this case, those summands are multiplied with $\mathbb{0}$, which have a multi-index where the projected indices are not equal. Therefore, those summands are annihilated and left out from the summation. This means that both symbols can be replaced by a single symbol.

In our example, this is equivalent to the following:

$$\begin{aligned}\forall i,z : T_{izz} &= \bigoplus_{j,k,l} A_{ij}B_{kl}C_{zz}\,(\mathbb{1}_a)_{ij}\,(\mathbb{1}_b)_{kl} \\ &= \bigoplus_{j,k,l} A_{ij}B_{kl}C_{zz}\,(\mathbb{1}_a)_{ij} \odot \begin{cases} \mathbb{1} & \text{if } k = l, \\ \mathbb{0} & \text{else} \end{cases} \\ &= \bigoplus_{j,y} A_{ij}B_{yy}C_{zz}\,(\mathbb{1}_a)_{ij}\,.\end{aligned}$$

Next let us consider the case where one symbol is free and one symbol is bound. W.l.o.g. we consider the case where s_{n1} is free and s_{n2} is bound. In this case, those summands are multiplied with $\mathbb{0}$, which have a multi-index where the index projected by the bound symbol s_{n2} is not the same as the index projected by the free symbol s_{n1}. Therefore those summands are annihilated and left out from the summation, and the symbol s_{n2} can be replaced by the symbol s_{n1}. Additionally, we can rename the s_{n1} to some new symbol.

In our example, this is equivalent to the following:

$$\begin{aligned}\forall i,z : T_{izz} &= \bigoplus_{j,y} A_{ij}B_{yy}C_{zz}\,(\mathbb{1}_a)_{ij} \\ &= \bigoplus_{j,y} A_{ij}B_{yy}C_{zz} \odot \begin{cases} \mathbb{1} & \text{if } i = j, \\ \mathbb{0} & \text{else} \end{cases} \\ &= \bigoplus_{y} A_{ii}B_{yy}C_{zz} \\ \iff \forall x,z : T_{xzz} &= \bigoplus_{y} A_{xx}B_{yy}C_{zz}.\end{aligned}$$

Therefore, in all three cases, the symbols, that are used in an index string for a unity matrix, can simply be replaced by a single symbol. □

From these three lemmata, Theorem 2 follows with the following procedure:

Step 1: Apply Lemma 3 to all of the symbols in the input string $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ and therefore replace it with a new index string $\boldsymbol{s}'_{\boldsymbol{u}} \in S^{n_u}$ where $\boldsymbol{s}'_{\boldsymbol{u}}$ contains no duplicate symbols.

Step 2: Apply Lemma 4 to all of the symbols in the output string $\boldsymbol{s}_{\boldsymbol{u}}$ and therefore replace it with the same new index string $\boldsymbol{s}'_{\boldsymbol{u}}$.

Step 3: Apply Theorem 1 to the nested expression and therefore compress the nested expression into a single expression with lots of unity matrices. This is possible because the input string and output string for U are both $\boldsymbol{s}'_{\boldsymbol{u}}$ now.

Step 4: Apply Lemma 5 to remove unity matrices from the compressed expression until there are no more of those unity matrices left, which were introduced in Step 1 and Step 2.

Proof. For the proof, we again demonstrate the arguments on the example used in Theorem 2 for better understanding:

$$(a, b, c, d, e, abbcde \to bc, v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(5)}, (i, j, k, l \to iijkkl, v^{(6)}, v^{(7)}, v^{(8)}, v^{(9)}))$$

Applying Step 1 and Step 2 to this example results in the following Einsum expression if $\boldsymbol{s}'_{\boldsymbol{u}} = s_1 s_2 s_3 s_4 s_5 s_6$:

$$\begin{gathered}(a, b, c, d, e, s_1 s_2 s_3 s_4 s_5 s_6, as_1, bs_2, bs_3, cs_4, ds_5, es_6 \to bc,\\ v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(5)}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1},\\ (i, j, k, l, is_1, is_2, js_3, ks_4, ks_5, ls_6 \to s_1 s_2 s_3 s_4 s_5 s_6,\\ v^{(6)}, v^{(7)}, v^{(8)}, v^{(9)}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1})),\end{gathered}$$

where we used $\mathbb{1}$ to indicate unity matrices of different sizes, because the sizes can be derived from the context and are not important for better understanding. Applying Step 3 results in the following compressed expression:

$$\begin{gathered}(a, b, c, d, e, i, j, k, l, is_1, is_2, js_3, ks_4, ks_5, ls_6, as_1, bs_2, bs_3, cs_4, ds_5, es_6 \to bc,\\ v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}, v^{(5)}, v^{(6)}, v^{(7)}, v^{(8)}, v^{(9)}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1}, \mathbb{1})\end{gathered}$$

The only fact that remains to be understood is why the removal of the unity matrices in Step 4 leads to the transformation described in Theorem 2. For this, we construct another undirected graph $G' = (V', E')$ that we call *extended symbol graph*. In the extended symbol graph, the nodes consist of all symbols from $\boldsymbol{s}_{\boldsymbol{u}}$, $\boldsymbol{s}'_{\boldsymbol{u}}$, and $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$. An edge $\{u, v\}$ exists precisely when the compressed expression contains a unity matrix with index string uv, that was introduced in Step 1 or Step 2. The extended symbol graph for our example is illustrated in Figure 3.2.

Now, every newly introduced unity matrix is represented by an edge in the extended symbol graph. If a unity matrix is removed with Lemma 5, the symbols connected by the representing edge collapse into one symbol, and the rest of the extended symbol graph stays the same. An example of this collapse is illustrated in Figure 3.3.

Therefore, after repeatedly applying Lemma 5, all nodes that are part of the same component in the extended symbol graph will collapse into one symbol. Now the only

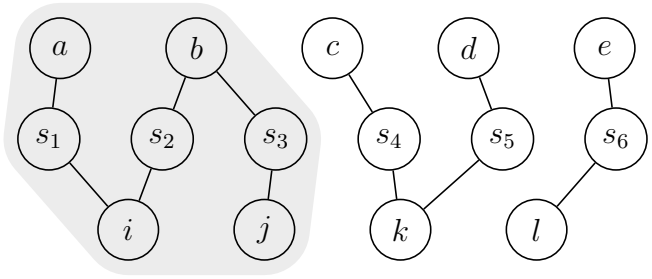


Figure 3.2: Extended symbol graph for the example with the first component highlighted

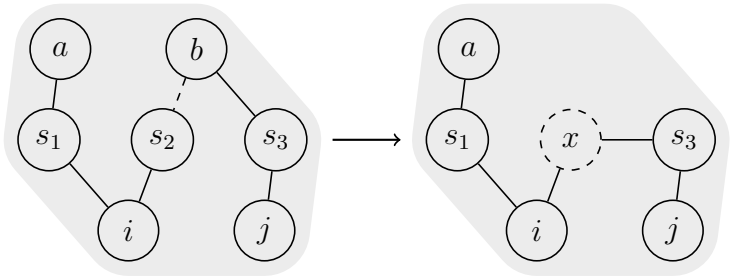


Figure 3.3: First component of the extended symbol graph after removing the unity matrix represented by the edge $\{b, s_2\}$

thing left to show is that two nodes are connected in G exactly if they are also connected in G'. This can be seen by understanding the edges of G and G'. In G, two symbols s_{ui} and $\hat{s}_{uj}$ share an edge if they share a position $i = j$. In G', two symbols s_{ui} and $\hat{s}_{uj}$ share a neighbour s'_{uk} if both were replaced by s'_{uk} in Step one and Step two, which happens exactly if they share a position $i = j = k$. Therefore every edge in G is represented by a shared neighbour in G'. Now, because every s'_{uk} has exactly two neighbours, and because there are no direct edges between any symbols of $\boldsymbol{s_u}$ and $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$, there are no more edges in G' other than the ones that contribute to a shared neighbour s'_{uk}. Then, because sharing an edge in G is the same as sharing a neighbour in G', two symbols are in the same component in G precisely when they are also in the same component in G'.

Therefore collapsing every edge in the extended symbol graph leads to the symbol map defined in Theorem 2. □

From this general theorem we can derive two more specific theorems, which make the process of compressing simpler for a subset of nested expressions.

3.3 Duplications

The following two theorems revolve around duplicated symbols in index strings and the way these duplications are *broken*. We speak of a broken duplication in $\boldsymbol{s_u}$, if the symbols s_{ui} and s_{uj} at two positions $i, j \in [n_u]$ are the same, meaning $s_{ui} = s_{uj}$, but symbols at the same positions in $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$, $\hat{s}_{ui}$ and $\hat{s}_{uj}$ are different, meaning $\hat{s}_{ui} \neq \hat{s}_{uj}$. In the same way, duplications in $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ can be broken by $\boldsymbol{s_u}$.

Because $\boldsymbol{s_u}$ and $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ are easily confused, we can also use a different terminology that does not include these similar variable names. If $\boldsymbol{s_u}$ breaks duplications in $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$, we say that the outer expression *introduces* duplications to the inner expression, because the outer expression uses duplications in the input string for the inner expression, that were not used in the output string of the inner expression. If $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ breaks duplications in $\boldsymbol{s_u}$, we say that the outer expression *removes* duplications from the inner expression, because the outer expression uses different symbols in the input string for the inner expression, where there was orignally a duplication in the output string of the inner expression.

3.3.1 Introducing Duplications

The first duplication theorem handles all nested expressions, where the outer expression can only introduce duplications and cannot remove any. This means that if two positions hold the same symbol in $\boldsymbol{s_u}$, these positions also have to hold the same symbol in $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$. This brings the advantage, that the symbol map is much simpler and only has to be applied to the inner expression.

The following example is an expression, which respects this condition:

$$(ij, jjj \to i, A, (kl, lo \to kko, B, C))$$

for $A \in \mathbb{R}^{a \times b}, B \in \mathbb{R}^{b \times c}, C \in \mathbb{R}^{c \times b}$. Again, we use disjoint sets of symbols for the inner and outer expression to help us in the formulation and the proof.

Theorem 6: For $i \in [m+n]$, let $T^{(i)}$ be an n_i-th order tensor with index strings $\boldsymbol{s_i} \in S^{n_i}$. Let $\boldsymbol{s_u}$ be an index string for the n_u-th order tensor U, which is defined as follows:

$$U = (\boldsymbol{s_{m+1}}, \ldots, \boldsymbol{s_{m+n}} \to \boldsymbol{s_u}, T^{(m+1)}, \ldots, T^{(m+n)})$$

Also let $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ be alternative index strings for U with $s_{uj} = s_{uj'} \implies \hat{s}_{uj} = \hat{s}_{uj'}$ for all $j, j' \in [n_u]$, which means that the outer expression can only introduce new symbol duplications, and cannot remove any.

Let s_v be an index string and

$$V = (\boldsymbol{s_1}, \ldots, \boldsymbol{s_m}, \hat{\boldsymbol{s}}_{\boldsymbol{u}} \to \boldsymbol{s_v}, T^{(1)}, \ldots, T^{(m)}, U)$$

such that the first and second Einsum expression share no symbols. Then these nested Einsum expressions can also be compressed into a single Einsum expression.

As in Theorem 2, we need to apply a symbol map in order to compress the nested expression. Let $\nu : S \to S$ such that

$$\nu(s) := \begin{cases} \hat{s}_{uj} & \text{if } \exists j \in [n_u] : s_{uj} = s, \\ s & \text{else,} \end{cases}$$

which maps symbols in $\boldsymbol{s_u}$ to the symbol at the same index in $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ and all other symbols to themselves.

In our example, we have the following symbols on the same positions:

- $s_{u1} = k$ and $\hat{s}_{u1} = j$,
- $s_{u2} = k$ and $\hat{s}_{u2} = j$,
- $s_{u3} = o$ and $\hat{s}_{u3} = j$.

Therefore these are the important mappings:

$$k \to j,$$
$$o \to j.$$

The symbol map ν can be extended, such that it maps entire index strings instead of just symbols as in Theorem 2. Then we can write the substituted index strings by setting $\hat{\boldsymbol{s}}_{\boldsymbol{i}} := \nu(\boldsymbol{s_i})$ for $i \in [m+1, m+n]$. With these index strings, the compressed Einsum expression is the following:

$$V = (\boldsymbol{s_1}, \ldots, \boldsymbol{s_m}, \hat{\boldsymbol{s}}_{\boldsymbol{m+1}}, \ldots, \hat{\boldsymbol{s}}_{\boldsymbol{m+n}} \to \boldsymbol{s_v}, T^{(1)}, \ldots, T^{(m+n)})$$

which helps us to compress the example:

$$(ij, jjj \to i, A, (kl, lo \to kko, B, C)) = (ij, jl, lj \to i, A, B, C).$$

Proof. Because $s_{uj} = s_{uj'} \implies \hat{s}_{uj} = \hat{s}_{uj'}$ for all $j, j' \in [n_u]$, each symbol in $\boldsymbol{s_u}$ only has one neighbour in the symbol graph. This is illustrated for our example in Figure 3.4 Therefore each symbol in $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ defines its own component, and can be used as the symbol in the symbol map, that replaces all symbols in their component. Therefore the symbols in $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ are not changed by applying the symbol map, which are the only symbols in the outer expression that could have been changed by the map. Therefore the symbol map only has to be applied to the inner expression.

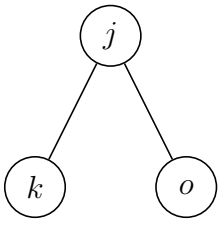


Figure 3.4: Symbol graph for the example

□

This theorem suffices to prove a property of the trace in a relatively simple manner, namely that for $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times m}$, it holds that

$$\operatorname{trace}(A \cdot B) = \operatorname{trace}(B \cdot A).$$

Proof.

$$\begin{aligned}\operatorname{trace}(A \cdot B) &= (ll \to, (ik, kj \to ij, A, B)) \\ &= (lk, kl \to, A, B) \\ &= (kl, lk \to, B, A) \\ &= (kk \to, (il, lj \to ij, B, A)) \\ &= \operatorname{trace}(B \cdot A)\end{aligned}$$

where the second and fourth equality hold because of Theorem 6, and the third equality holds because of the commutativity of multiplication in the standard semiring. □

3.4 Removing Duplications

The second duplication theorem handles all nested expressions, where the outer expression can only remove duplications and cannot remove any. This means that if two positions hold the same symbol in $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$, these positions also have to hold the same symbol in $\boldsymbol{s}_{\boldsymbol{u}}$. This brings the advantage, that the symbol map is much simpler and only has to be applied to the outer expression.

The following example is an expression, which respects this condition:

$$(ij, kl, mn, ijklmn \to ijk, A, B, C, (abc \to aabbcc, D))$$

for $A \in \mathbb{R}^{x \times x}, B \in \mathbb{R}^{y \times y}, C \in \mathbb{R}^{z \times z}, D \in \mathbb{R}^{x \times y \times z}$. This is because the duplications aa, bb, and cc in the output string $\boldsymbol{s}_{\boldsymbol{u}}$ are broken by ij, kl, and mn respectively. Again, we use disjoint sets of symbols for the inner and outer expression to help us in the formulation and the proof.

Theorem 7: For $i \in [m+n]$, let $T^{(i)}$ be an n_i-th order tensor with index strings $\boldsymbol{s_i} \in S^{n_i}$. Let $\boldsymbol{s_u}$ be an index string for the n_u-th order tensor U, which is defined as follows:

$$U = (\boldsymbol{s_{m+1}}, \ldots, \boldsymbol{s_{m+n}} \to \boldsymbol{s_u}, T^{(m+1)}, \ldots, T^{(m+n)}).$$

Also let $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ be alternative index strings for U with $s_{uj} \neq s_{uj'} \implies \hat{s}_{uj} \neq \hat{s}_{uj'}$ for all $j, j' \in [n_u]$, which means that $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ can only remove symbol duplications, and cannot introduce any. Note that this is the converse of the constraint in Theorem 6.

Let s_v be an index string and

$$V = (\boldsymbol{s_1}, \ldots, \boldsymbol{s_m}, \hat{\boldsymbol{s}}_{\boldsymbol{u}} \to \boldsymbol{s_v}, T^{(1)}, \ldots, T^{(m)}, U)$$

where the first and second Einsum expression share no symbols. Then these nested Einsum expressions can also be compressed into a single Einsum expression.

As in Theorem 2, we need to apply a symbol map in order to compress the nested expression. Let $\nu : S \to S$ such that

$$\nu(s) := \begin{cases} s_{uj} & \text{if } \exists j \in [n_u] : \hat{s}_{uj} = s, \\ s & \text{else,} \end{cases}$$

which can be extended to map entire index strings as in Theorem 2. In our example, these are the important mappings:

$$\begin{aligned} i &\to a, & k &\to b, & m &\to c, \\ j &\to a, & l &\to b, & n &\to c. \end{aligned}$$

This means that i and j will be iterated over at the same time, k and l will be iterated over at the same time, and m and l will be iterated over at the same time.

We can write the substituted index strings by setting $\hat{\boldsymbol{s}}_{\boldsymbol{i}} := \nu(\boldsymbol{s}_{\boldsymbol{i}})$ for $i \in [m]$, $\hat{\boldsymbol{s}}_{\boldsymbol{v}} := \nu(\boldsymbol{s}_{\boldsymbol{v}})$ With these index strings, the compressed Einsum expression is the following:

$$V = (\hat{\boldsymbol{s}}_{\boldsymbol{1}}, \ldots, \hat{\boldsymbol{s}}_{\boldsymbol{m}}, \boldsymbol{s}_{\boldsymbol{m+1}}, \ldots, \boldsymbol{s}_{\boldsymbol{m+n}} \to \hat{\boldsymbol{s}}_{\boldsymbol{v}}, T^{(1)}, \ldots, T^{(m+n)})$$

which helps us to compress the example:

$$\begin{aligned} &(ij, kl, mn, ijklmn \to ijk, A, B, C, (abc \to aabbcc, D)) \\ &\quad = (aa, bb, cc, abc \to aab, A, B, C, D). \end{aligned}$$

Proof. Because $\hat{s}_{uj} = \hat{s}_{uj'} \implies s_{uj} = s_{uj'}$ for all $j, j' \in [n_u]$, each symbol in $\hat{\boldsymbol{s}}_{\boldsymbol{u}}$ only has one neighbour in the symbol graph. This is illustrated for our example in Figure 3.5 Therefore each symbol in $\boldsymbol{s}_{\boldsymbol{u}}$ defines its own component, and can be used as the symbol in the symbol map, that replaces all symbols in their component. Therefore the symbols in $\boldsymbol{s}_{\boldsymbol{u}}$ are not changed by applying the symbol map, which are the only symbols in the inner expression that could have been changed by the map. Therefore the symbol map only has to be applied to the outer expression.

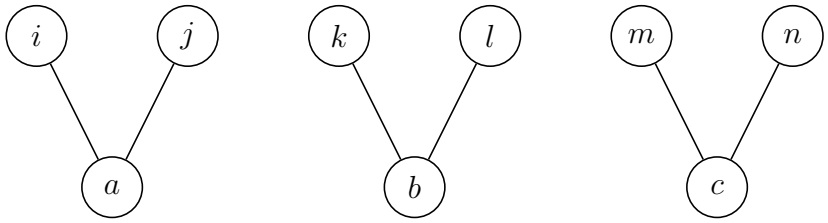


Figure 3.5: Symbol graph for the example

□

With these two more specific theorems, we can write every naturally occuring complex expression from linear algebra as a single Einsum expression. The reason for this is, that in linear algebra, only up to two indices are used for a single tensor,

which means that with two index strings, it cannot happen that a duplication is removed and anonther duplication is introduced simultaniously. Here are some more complex examples of expressions, which we can compress with the two more specific theorems:

- Squared norm of matrix-vector multiplication: Let $A \in \mathbb{R}^{m \times n}, v \in \mathbb{R}^n$. Then

$$\begin{aligned} |A \cdot v|_2^2 &= (i, i \to, (ij, j \to i, A, v), (ij, j \to i, A, v)) \\ &= (ij, j, ij, j \to, A, v, A, v). \end{aligned}$$

- Trace of matrix-matrix multiplication: Let $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times m}$. Then

$$\begin{aligned} \text{trace}(A \cdot B) &= (ii \to, (ik, kj \to ij, A, B)) \\ &= (ik, ki \to, A, B). \end{aligned}$$

- Matrix multiplication with a diagonal matrix: Let $A \in \mathbb{R}^{m \times n}, v \in \mathbb{R}^n$. Then

$$\begin{aligned} A \cdot \text{diag}(v) &= (ik, kj \to ij, A, (i \to ii, v)) \\ &= (ij, j \to ij, A, v). \end{aligned}$$

But writing every expression from linear algebra as a single Einsum expression respectively was already possible before [2]. With these theorems, we just derived a different way of achieving that. For this, we can state a simple procedure. First, every function is translated to their respective Einsum expression, which results in a nested Einsum expression. Then, the nested expressions are compressed from the bottom up.

But there are naturally occuring examples of nested Einsum expressions, which need more than the two duplication theorems to be compressed. For instance, when symbolically deriving Einsum expressions to find out if they compute a convex function, nested statements can arise, that break duplications in both directions. An example of a function, where this happens in the second derivative, is the trace of a matrix where all entries have been squared:

$$(ii, ii \to, A, A)$$

for some $A \in \mathbb{R}^{n \times n}$. Now, with Theorem 2, this procedure allows us to compress even these expressions. Generally, it allows us to write every computation, where the single steps can be translated to Einsum over a single semiring, as one big Einsum expression that does not blow up in size with levels of nesting.

4 Naturally Occuring Einsum Expressions

4.1 Discrete Fourier Transform

Let T be an n-th order tensor where all axes have size m. Then the Discrete Fourier Transform $\mathrm{DFT}(T)$ is the n-th order tensor where all axes have size m with:

$$\mathrm{DFT}(T)_{x_1,\ldots,x_n} = \sum_{y_1,\ldots,y_n \in [m]} T_{y_1,\ldots,y_n} \cdot \prod_{j+k \leq n+1} \exp\left(-i2\pi \frac{(x_j - 1)(y_k - 1)}{m^{n-j-k+2}}\right)$$

for $x_1, \ldots, x_n \in [m]$. This formulation of the DFT is slightly rewritten form of the formulation by Aji [3].

To write the DFT as an Einsum expression, we need to define $\binom{n+1}{2}$ matrices $T^{(j,k)} \in \mathbb{C}^{m \times m}$ for $j + k \leq n + 1$ in the following way:

$$T^{(j,k)}_{x_j y_k} := \exp\left(-i2\pi \frac{(x_j - 1)(y_k - 1)}{m^{n-j-k+2}}\right)$$

for $x_j, y_k \in [m]$. Then

$$\mathrm{DFT}(T)_{x_1,\ldots,x_n} = \sum_{y_1,\ldots,y_n \in [m]} T_{y_1,\ldots,y_n} \cdot \prod_{j+k \leq n+1} T^{(j,k)}_{x_j y_k}$$

for $x_1, \ldots, x_n \in [m]$. Therefore the DFT can be written as an Einsum expression with index strings consisting of symbols s_{xi} and s_{yj} for $j, k \in [n]$:

- $\boldsymbol{s_x} = s_{x1} \ldots s_{xn}$
- $\boldsymbol{s_y} = s_{y1} \ldots s_{yn}$
- $\boldsymbol{s_{j,k}} = s_{xj} s_{yk}$ for $j, k \in [n]$

With these index strings, the Einsum expression for a general DFT over any number of dimensions is the following:

$$\begin{aligned}\mathrm{DFT}(T) = (&\boldsymbol{s_y}, \boldsymbol{s_{1,1}}, \ldots, \boldsymbol{s_{1,n}}, \boldsymbol{s_{2,1}}, \ldots, \boldsymbol{s_{2,n-1}}, \ldots, \boldsymbol{s_{n,1}} \to \boldsymbol{s_x}, \\ &T, T^{(1,1)}, \ldots, T^{(1,n)}, T^{(2,1)}, \ldots, T^{(2,n-1)}, \ldots, T^{(n,1)})\end{aligned}$$

Because this notation is hard to read, we will explore an example of the DFT of a third-order tensor with axes of size 32. In this example, we have to define 6 matrices $T^{(1,1)}$, $T^{(1,2)}$, $T^{(1,3)}$, $T^{(2,1)}$, $T^{(2,2)}$, and $T^{(3,1)}$. We will use a, b, c as symbols for the indices of T, and x, y, z as the symbols for the indices of $\mathrm{DFT}(T)$. Then the matrices are defined in the following way:

$$T^{(1,1)}_{xa} = \exp\left(-i2\pi\frac{(x-1)(a-1)}{32^3}\right) \qquad T^{(1,3)}_{xc} = \exp\left(-i2\pi\frac{(x-1)(c-1)}{32}\right)$$
$$T^{(1,2)}_{xb} = \exp\left(-i2\pi\frac{(x-1)(b-1)}{32^2}\right) \qquad T^{(2,2)}_{yb} = \exp\left(-i2\pi\frac{(y-1)(b-1)}{32}\right)$$
$$T^{(2,1)}_{ya} = \exp\left(-i2\pi\frac{(y-1)(a-1)}{32^2}\right) \qquad T^{(3,1)}_{za} = \exp\left(-i2\pi\frac{(z-1)(a-1)}{32}\right)$$

for $x, y, z, a, b, c \in [32]$. Then the Einsum expression is the following:

$$\begin{aligned}\mathrm{DFT}(T) = (&abc, xa, xb, xc, ya, yb, za \to xyz,\\ &T, T^{(1,1)}, T^{(1,2)}, T^{(1,2)}, T^{(2,1)}, T^{(2,2)}, T^{(3,1)})\end{aligned}$$

The inverse transform is anologue with

$$\hat{T}^{(j,k)}_{x_j y_k} := \exp\left(i2\pi\frac{(x_j-1)(y_k-1)}{m^{n-j-k+2}}\right).$$

4.2 Hadamard Transform

Let T be an n-th order tensor where all axes have size m. Then the Hadamard Transform $H(T)$ is the n-th order tensor where all axes have size m with:

$$H(T)_{x_1\dots x_n} = \sum_{y_1,\dots,y_n\in[m]} T_{y_1\dots y_n}(-1)^{x_1y_1+\dots+x_ny_n}$$

for $x_1, \dots, x_n \in [m]$.

Then we can define n matrices $T^{(i)} \in \mathbb{R}^{m\times m}$ for $i \in n$:

$$T^{(i)}_{x_iy_i} = (-1)^{x_iy_i}$$

for $x_i, y_i \in [m]$. Then

$$H(T)_{x_1\dots x_n} = \sum_{y_1\dots y_n\in[m]} T_{y_1\dots y_n} \cdot \prod_{i\in[n]} T^{(i)}_{x_iy_i}$$

for $x_1, \dots, x_n \in [m]$. Therefore the Hadamard Transform can be written as an Einsum expression with index strings consisting of symbols s_{xi} and s_{yj} for $j, k \in [n]$:

- $\boldsymbol{s_x} = s_{x1} \dots s_{xn}$
- $\boldsymbol{s_y} = s_{y1} \dots s_{yn}$
- $\boldsymbol{s_i} = s_{xi} s_{yi}$ for $i \in [n]$

Then

$$H(T) = (\boldsymbol{s_y}, \boldsymbol{s_1}, \dots, \boldsymbol{s_n} \to \boldsymbol{s_x}, T, T^{(1)}, \dots, T^{(n)}).$$

5 Deep Learning

5.1 Fully Connected Feed-Forward Net

A single layer of a fully connected Feed-Forward Neural Net with ReLU activations can be expressed as a nested Einsum expression, with the use of multiple semirings. For this, let

- $R_{(+,\cdot)}$ be the standard semiring $(\mathbb{R}, +, \cdot)$,
- $R_{(\max,+)}$ be the tropical semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$,
- $R_{(\min,\max)}$ be the minimax semiring $(\mathbb{R} \cup \{-\infty, +\infty\}, \min, \max)$.

Let $\nu : \mathbb{R}^n \to \mathbb{R}^m$ be the function that computes the output of the layer for a given input vector $x \in \mathbb{R}^n$, with weights $A \in \mathbb{R}^{m \times n}$ and biases $b \in \mathbb{R}^m$. Then

$$\begin{aligned}\nu(x) &= \max(Ax + b, 0) \\ &= (i, i \to i, 0, (i, i \to i, b, (ij, j \to i, A, x)_{R_{(+,\cdot)}})_{R_{(\max,+)}})_{R_{(\min,\max)}}.\end{aligned}$$

A multi-layer network can the be achieved by nesting the respective Einsum expressions of each layer.

Because each level of nesting needs a different semiring, we can not use the theorems from Chapter 3 to compress the expression. But if we could find a way of compressing the massively nested expressions of deeper neural networks despite of that, then we could benefit from the advantages mentioned in Chapter 2 such as the optimisation of contraction paths. Unfortunately, it is unlikely that this is possible, because we found that expanding the matrix multiplication, that transforms the outputs of another layer, results in an exponentially big term.

To see this, we use the tropical semiring $(\mathbb{R}, \oplus, \odot)$, where $a \oplus b = \max(a, b)$ and $a \odot b = a + b$. We do this because tropical semiring can naturally express all the operations used in a fully connected neural network. For this we need to define the tropical power:

$$a^{\odot n} = \underbrace{a \odot a \odot \ldots \odot a}_{n \text{ times}}$$

for $n \in \mathbb{N}$. The following property of the tropical power is also needed:

$$\left(a^{\odot b}\right)^{\odot c} = a^{\odot(b+c)}.$$

The tropical semiring also allows us to use the distributive law of maximization and addition

$$a \odot (b \oplus c) = a \odot b \oplus a \odot c,$$

as well as the distributive law of addition and multiplication

$$(a \odot b)^{\odot n} = a^{\odot n} \odot b^{\odot n},$$

and the distributive law of maximization and multiplication, which is restricted on natural numbers

$$(a \oplus b)^{\odot n} = a^{\odot n} \oplus b^{\odot n}.$$

Let $\nu : \mathbb{R}^n \to \mathbb{R}^l$ be the function that computes the output of a two-layer fully connected neural network ($n \to m \to l$ neurons) with ReLU activations, which maps inputs $x \in \mathbb{R}^n$ to outputs $\nu(y) \in R^l$, with parameters $A^{(0)} \in \mathbb{R}^{m \times n}, A^{(1)} \in \mathbb{R}^{l \times m}, b^{(0)} \in \mathbb{R}^m, b^{(1)} \in \mathbb{R}^l$. Then the computation of the neural network is:

$$\nu(x) = \max(A^{(1)} \max(A^{(0)} x + b^{(0)}, 0) + b^{(1)}, 0)$$

In order to reasonably work with matrix multiplication in the tropcial semiring, we can only view matrices with positive integer entries. This is not a limitation, because making the entries integers does not impact the strength of the neural network [see 4, sec. 4].

In order to only use positive valued matrices, we can rewrite the expression of computing the next layer from a previous layer:

$$\begin{aligned} \max(Ax + b, 0) &= \max(A_+ x - A_- x + b, A_- x - A_- x) \\ &= \max(A_+ x + b, A_- x) - A_- x \end{aligned}$$

where $A_+ = \max(A, 0), A_- = \max(-A, 0)$ and therefore $A = A_+ - A_-$. This turns the network output into a tropical rational function [see 4, sec. 5]:

$$\begin{aligned} \nu(x) = \max(&\overbrace{A_+^{(1)} \max(A_+^{(0)} x + b^{(0)}, A_-^{(0)} x)}^{z} + A_-^{(1)} A_+^{(0)} x + b^{(1)}, \\ &A_-^{(1)} \max(A_+^{(0)} x + b^{(0)}, A_-^{(0)} x) + A_+^{(1)} A_+^{(0)} x) \\ &- \left[A_-^{(1)} \max(A_+^{(0)} x + b^{(0)}, A_-^{(0)} x) + A_+^{(1)} A_+^{(0)} x \right] \end{aligned}$$

We focus on the subexpression z, which makes the calculation a bit simpler, but keeps the point.

Now, if we want to avoid switching semirings, we need to apply the distributive law

multiple times.

$$
\begin{aligned}
z &= A_+^{(1)} \max(A_+^{(0)} x + b^{(0)}, A_-^{(0)} x) \\
z_i &= \bigodot_{j=1}^{m} \left(b_j^{(0)} \odot \bigodot_{k=1}^{n} x_k^{\odot A_{jk+}^{(0)}} \oplus \bigodot_{k=1}^{n} x_k^{\odot A_{jk-}^{(0)}} \right)^{\odot A_{ij+}^{(1)}} \\
&= \bigodot_{j=1}^{m} \left(\left(b_j^{(0)} \right)^{\odot A_{ij+}^{(1)}} \odot \bigodot_{k=1}^{n} x_k^{\odot \left(A_{ij+}^{(1)} + A_{jk+}^{(0)} \right)} \oplus \bigodot_{k=1}^{n} x_k^{\odot \left(A_{ij+}^{(1)} + A_{jk-}^{(0)} \right)} \right) \\
&= \bigoplus_{J \in 2^{[m]}} \bigodot_{j \in J} \left[\left(b_j^{(0)} \right)^{\odot A_{ij+}^{(1)}} \odot \bigodot_{k=1}^{n} x_k^{\odot \left(A_{ij+}^{(1)} + A_{jk+}^{(0)} \right)} \right] \odot \bigodot_{j \in [n] \setminus J} \left[\bigodot_{k=1}^{n} x_k^{\odot \left(A_{ij+}^{(1)} + A_{jk-}^{(0)} \right)} \right]
\end{aligned}
$$

Where the second equality is just the first equality written with the operations of the tropical semiring, the third equality follows from the distributive law of standard addition and multiplication and the distributive law of maximization and multiplication, and the last equality follows from the distributive law of maximization and addition.

This expression maximizes over a number of subexpressions that grows exponentially in the width of the inner layer. Which subexpressions can be removed before the evaluation remains an open question. Note that it depends on the non-linearities of the neural network, which might make it hard to find a general answer to this question.

5.2 Attention

For $Q \in \mathbb{R}^{d_v \times d_k}, K \in \mathbb{R}^{d_v \times d_k}, V \in \mathbb{R}^{d_v \times d_v}$, the attention mechanism [5] is the following:

$$\text{Attention}(Q, K, V) = \text{softmax}\left(\frac{QK^\top}{\sqrt{d_K}} \right) V.$$

It is comprised of multiple steps, which can all be expressed with Einsum expressions and element-wise operations:

- matrix multiplication QK^T:

$$
\begin{aligned}
(QK^\top)_{ij} &= \sum_{k \in [d_k]} Q_{ik} K_{jk} \\
QK^\top &= (ik, jk \to ij, Q, K)
\end{aligned}
$$

- scaling by $\frac{1}{\sqrt{d_k}}$. Let $X \in \mathbb{R}^{d_v \times d_v}$:

$$\frac{1}{\sqrt{d_k}} X = (, ij \to ij, \frac{1}{\sqrt{d_k}}, X)$$

- normalizing with softmax: Let $X \in \mathbb{R}^{m \times n}$, then

$$\text{softmax}(X)_{ij} := \frac{\exp(X_{ij})}{\omega_i}$$

where

$$\omega_i := \sum_{j \in [n]} \exp(X_{ij}).$$

Therefore

$$\text{softmax}(X) = (ij, i \to ij, \exp(X), 1/(ij \to i, \exp(X)))$$

- another matrix multiplication with V. Let $X \in \mathbb{R}^{d_v \times d_v}$:

$$XV = (ik, kj \to ij, X, V)$$

Then the whole attention mechanism can be expressed with Einsum expressions and the use of element-wise operations:

$$\begin{aligned}
\text{Attention}(Q, K, V) &= \text{softmax}\left(\frac{QK^\top}{\sqrt{d_K}}\right) V \\
&= (ik, kj \to ij, (ij, i \to ij, \\
&\quad \exp((, ij \to ij, \frac{1}{\sqrt{d_k}}, (ik, jk \to ij, Q, K))), \\
&\quad 1/(ij \to i, \exp((, ij \to ij, \frac{1}{\sqrt{d_k}}, (ik, jk \to ij, Q, K))))), V) \\
&= (ik, kj, k \to ij, \exp((, ik, jk \to ij, \frac{1}{\sqrt{d_k}}, Q, K)) \\
&\quad 1/(ij \to i, \exp((, ik, jk \to ij, \frac{1}{\sqrt{d_k}}, Q, K))), V)
\end{aligned}$$

5.3 Batch Norm

Let $x^{(1)}, \ldots, x^{(m)}$ be m 4th-order tensors. Let X be the 5th-order tensor that consists of all those tensors combined along a new axis. Then the Batch Norm [6] with parameters $\gamma, \beta \in \mathbb{R}$ over this *mini-batch* of tensors is defined in the following way:

$$\text{BN}_{\gamma,\beta}(X)_i = \frac{x_i - \text{E}_j[x_j]}{\sqrt{\text{Var}_j[x_j] + \epsilon}} \cdot \gamma + \beta$$

for $i \in [m]$, where ϵ is some constant added for numerical stability. This computation is comprised of multiple steps, which can all be expressed with Einsum expressions and element-wise operations:

- mini-batch mean:

$$\mu_{abcd} = \frac{1}{m}\sum_{i=1}^{m} x_{abcd}^{(i)}$$
$$\mu = \frac{1}{m}(abcdi \to abcd, X)$$

- centralize:

$$\overline{x}_{abcd}^{(i)} = x_{abcd}^{(i)} - \mu_{abcd}$$
$$\overline{X} = (abcdi, abcd \to abcd, X, -\mu)_{R_{(\max,+)}}$$

- mini-batch variance:

$$\sigma_{abcd}^2 = \frac{1}{m}\sum_{i=1}^{m}\left(\overline{x}_{abcd}^{(i)}\right)^2$$
$$\sigma^2 = \frac{1}{m}(abcdi, abcdi \to abcd, \overline{X})$$

- normalize:

$$\hat{x}_{abcd}^{(i)} = \frac{\overline{x}_{abcd}^{(i)}}{\sqrt{\sigma_{abcd}^2 + \epsilon}}$$
$$\hat{X} = (abcdi, abcd \to abcd, \overline{X}, (\sigma^2 + \epsilon)^{-\frac{1}{2}})$$

- scale and shift:

$$y_{abcd}^{(i)} = \gamma \hat{x}_{abcd}^{(i)} + \beta$$
$$Y = \gamma \hat{X} + \beta$$

Therefore the whole attention mechanism can be expressed with Einsum expressions and the use of element-wise operations:

$$\begin{aligned}
\mathrm{BN}_{\gamma,\beta}(X) = (&abcdi, abcd \to abcd,\\
&(abcdi, abcd \to abcd, X, -\frac{1}{m}(abcdi \to abcd, X))_{R_{(\max,+)}},\\
&(\frac{1}{m}(abcdi, abcdi \to abcd,\\
&\quad (abcdi, abcd \to abcd, X, -\frac{1}{m}(abcdi \to abcd, X))_{R_{(\max,+)}}\\
&) + \epsilon)^{-\frac{1}{2}}\\
) \cdot \gamma + \beta&
\end{aligned}$$

5.4 Convolution

Let F and G be two n-th order tensors where all axes of F have size d_F and all axes of G have size d_G with $d_F < d_G$. Let $d_O = d_G - d_F + 1$. Then the convolution $F * G$ is defined in the following way:

$$(F * G)_x := \sum_{y \in [d_F]^n} F_y \cdot G_{x+d_F-y}$$

for all $x \in [d_O]^n$, where $x+d_F-y$ indicates the element-wise addition $(x+d_F-y)_i = x_i + d_F - y_i$ for $i \in [n]$. Let

$$G'_{(x,y)} := G_{x+d_F-y}$$

for $x \in [d_O]^n, y \in [d_F]^n$. Then

$$(F * G)_x = \sum_{y \in [d_G]^n} F_y G'_{(x,y)}$$

for $x \in [d_O]^n$. Let

$$P_{(x,y,z)} := \begin{cases} 1 & \text{if } z = x + d_F - y, \\ 0 & \text{else} \end{cases}$$

for $x \in [d_O]^n, y \in [d_F]^n, z \in [d_G]^n$. Then

$$G'_{(x,y)} = \sum_{z \in [d_G]^n} P_{(x,y,z)} G_z$$

for $x \in [d_O]^n, y \in [d_F]^n$. Therefore, convolution can be expressed as an Einsum expression:

$$(F * G) = (\boldsymbol{s_x s_y s_z}^{1}, \boldsymbol{s_y}, \boldsymbol{s_z} \to \boldsymbol{s_x}, P, F, G)$$

where $\boldsymbol{s_x}, \boldsymbol{s_y}, \boldsymbol{s_z} \in S^n$ use distinct symbols.

The manual computation of the design tensor[2] P is quite expensive, and therefore this expression could be inefficient. It could lead to a more efficient computation, if this design tensor could be expressed as an *outer product* of 2 or more smaller tensors. Sadly, this is not possible.

Proof. To prove this, we will first show that, if P can be expressed as an outer product, then the factors also have to be scaled design tensors. Then we will give an example of a convolution, where P can not be expressed as an outer product of design tensors.

Let U be an m-th order tensor, and V an $(3n-m)$-th order tensor for $1 \leq m < 3n$. Let $\boldsymbol{s_u} \in S^m$ be the index string for U, and let $\boldsymbol{s_v} \in S^{3n-m}$ be the index string for V such

[1] Here, we use $\boldsymbol{s_x s_y s_z}$ as the notation for concatenating the index strings $\boldsymbol{s_x}$, $\boldsymbol{s_y}$, and $\boldsymbol{s_z}$.

[2] A design tensor is a tensor that is filled only with the additive and multiplicative neutral element of the used semiring.

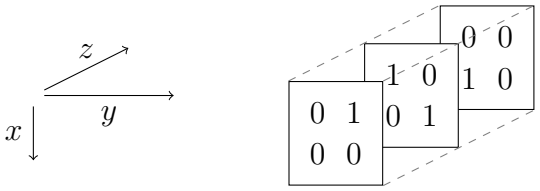


Figure 5.1: P for $F \in \mathbb{R}^2$ and $G \in \mathbb{R}^3$

that $\boldsymbol{s_u}$ and $\boldsymbol{s_v}$ use distinct symbols. Then P being the outer product of U and V means

$$P = (\boldsymbol{s_u}, \boldsymbol{s_v} \to \boldsymbol{s_u s_v}, U, V)$$

up to reordering of axes.

Let u and v be entries of U and V respectively, and let $\boldsymbol{i_u}$ and $\boldsymbol{i_v}$ be a multi-index of U and V respectively, where this value occurs. Then the value $u \cdot v$ occurs in P at the multi-index $(\boldsymbol{i_u}, \boldsymbol{i_v})$. Therefore, if the sets of values contained in U and V are not of the form $\{0, c\}$ and $\{0, \frac{1}{c}\}$ for some $c \in \mathbb{R}$, then we can produce more values than 0 and 1 in P. Therefore U and V must be design tensors that were scaled by c and $\frac{1}{c}$ respectively for some $c \in \mathbb{R}$.

W.l.o.g. we now assume that U and V are both unscaled design tensors, because if

$$P = (\boldsymbol{s_u}, \boldsymbol{s_v} \to \boldsymbol{s_u s_v}, cU, \frac{1}{c}V)$$

then

$$\begin{aligned} P &= c \cdot \frac{1}{c} \cdot (\boldsymbol{s_u}, \boldsymbol{s_v} \to \boldsymbol{s_u s_v}, U, V) \\ &= (\boldsymbol{s_u}, \boldsymbol{s_v} \to \boldsymbol{s_u s_v}, U, V). \end{aligned}$$

Consider a convolution $F * G$ of vectors $F \in \mathbb{R}^2$ and $G \in \mathbb{R}^3$. Then

$$(F * G)_x = \sum_{y \in [2]} F_y G_{x+2-y}$$

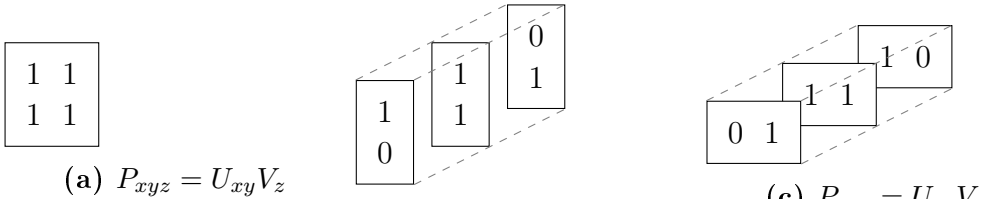
for $x \in [2]$, and

$$P_{(x,y,z)} = \begin{cases} 1 & \text{if } z = x + 2 - y, \\ 0 & \text{else} \end{cases}$$

for $x \in [2], y \in [2], z \in [3]$. This is illustrated in Figure 5.1.

Now, the only two possibilities for m are $m = 1$ and $m = 2$, because $n = 1$ and therefore $1 \leq m < 3$. This means, U and V are a matrix and a vector. W.l.o.g. we assume that U is a matrix and V is a vector. Then for U to be a matrix and a factor of P, there are only three possibilities:

$$\begin{aligned} P_{xyz} &= U_{xy} V_z, \\ P_{xyz} &= U_{xz} V_y, \\ P_{xyz} &= U_{yz} V_x. \end{aligned}$$



(a) $P_{xyz} = U_{xy}V_z$ **(b)** $P_{xyz} = U_{xz}V_y$ **(c)** $P_{xyz} = U_{yz}V_x$

Figure 5.2: Possiblities for U in the factorization of P

In the first case, it has to hold that $U_{xy} = 1$ where $P_{xyz} = 1$ for any $z \in [3]$, because otherwise, no design tensor V could produce the entry $P_{xyz} = 1$. For the other two cases, the analogous fact holds: all the entries, where missing index exists such that $P_{xyz} = 1$, have to hold one as well. And unless V is full of zeros (which it cannot be), the converse is true as well, meaning if all entries of the missing index hold $P_{xyz} = 0$, then the entry in U also has to hold zero. Therefore there is only possibility for U for each of the cases, which are illustrated in Figure 5.2:

$$P_{xyz} = U_{xy}V_z \qquad U = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$P_{xyz} = U_{xz}V_y \qquad U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$P_{xyz} = U_{yz}V_x \qquad U = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Therefore, the multiplication with V has to spread out the ones across different values of the missing index, which is not possible, because depending on the entries of V on the missing index:

- if V is one at a value of the missing index, then P is a copy of U at this value of the missing index, and
- if V is zero at a value of the missing index, then P is zero at this value of the missing index.

Therefore, there is no factorization of P when $F \in \mathbb{R}^2$ and $G \in \mathbb{R}^3$, which means P can not generally be expressed as an outer product of two lower-order tensors. □

5.5 Max-Pooling

Let T be an n-th order tensor where all axes have size m. Let $p \in \mathbb{N}$ be the *pool size* such that $m = k \cdot p$ for $k \in \mathbb{N}$. Then the max-pooling of T is defined in the following way:

$$M(T)_x := \max_{y \in [p]^n} T_{p(x-1)+y}$$

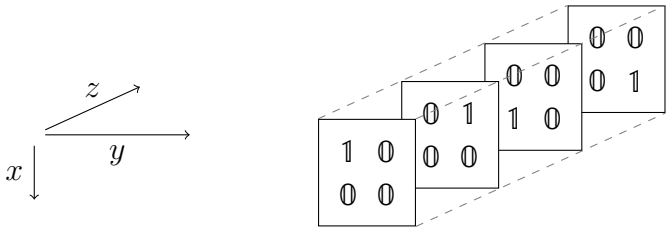


Figure 5.3: P for $F \in \mathbb{R}^2$ and $G \in \mathbb{R}^3$

for $x \in [k]^n$, where $x - 1$ indicates the element-wise substraction $(x-1)_i = x_i - 1$ for $i \in [n]$. Let

$$T'_{(x,y)} = T_{p(x-1)+y}$$

for $x \in [k]^n, y \in [p]^n$. Then

$$M(T)_x = \max_{y \in [p]^n} T'_{(x,y)}$$

for $x \in [k]^n$. Let

$$P_{(x,y,z)} := \begin{cases} 0 & \text{if } z = p(x-1)+y, \\ -\infty & \text{else} \end{cases}$$

for $x \in [k]^n, y \in [p]^n, z \in [m]^n$. Then

$$T'_{(x,y)} = \max_{z \in [m]^n} P_{(x,y,z)} + T_z$$

for $x \in [k]^n, y \in [p]^n$. Therefore, Max-Pooling can be expressed as an Einsum expression:

$$M(T) = (\boldsymbol{s_x s_y s_z}, \boldsymbol{s_z} \to \boldsymbol{s_x}, P, T)_{R_{(\max,+)}}$$

where $\boldsymbol{s_x}, \boldsymbol{s_y}, \boldsymbol{s_z} \in S^n$ use distinct symbols and $R_{(\max,+)}$ denotes the tropical semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$.

As in Section 5.4, the efficiency of the computation could benefit from decomposition of the design tensor P into an outer product of lower-order design tensors. But yet again, this is not possible.

Proof. Because $-\infty$ and 0 are the additive neutral and multiplicative neutral element of the used semiring respectively, the argument in Section 5.4, that the factors have to be scaled design tensors, holds here as well.

Therefore, w.l.o.g. we assume that U and V are design tensors of order one or higher. Consider max-pooling where $n = 1$, $m = 4$, $p = 2$, and $k = 2$. Then

$$P_{(x,y,z)} := \begin{cases} 0 & \text{if } z = 2(x-1)+y, \\ -\infty & \text{else} \end{cases}$$

for $x \in [2], y \in [2], z \in [4]$. This is illustrated in Figure 5.3, where $\mathbb{0}$ and $\mathbb{1}$ were used to indicate the additive and multiplicative neutral element of the tropical semiring, $-\infty$ and 0.

The rest of the proof is analogous to the proof in Section 5.4, where the key argument is, that the multiplicative neutral element cannot be distributed with an outer product in such a way, that there are no copied slices in P. □

5.6 Discussion

The techniques that we were able to describe with a single Einsum expression respectively are convolution and max-pooling. But even here, the mapping is not optimal because we still need design tensors to compute the permutations required for this techniques. Because these design tensors have a potentially high order, they can be computationally inefficient. We also showed that they can not be split up into a product of lower order tensors, which might have been a way to reduce the computational work load with the optimization of contraction paths. However, this problem might be smaller in practice. In the case of convolution in inference, the same constant masks are repeatedly used on different input data. This allows us to precompute the contraction of the mask and reuse the result on different inputs.

We did not manage to describe the other presented examples with a single Einsum expression respectively. Instead, we had to use mixtures of multiple Einsum expressions over different semirings and element-wise functions. Such mixtures are technically not in the framework of Einsum, but they might still be helpful, because they can be computed in a sequential manner. Because the steps in this sequential computation are all Einsum expressions or element-wise functions, they could all benefit from specific hardware and efficient Einsum engines individually, which could result in a solution that is still efficient overall.

Because these mixtures are not supported by Einsum, this result means that Einsum is probably not powerful enough to describe the standard architectures in deep learning. We do not know this for sure, as there might be semirings we did not explore, that allow to translate even these architectures naturally, but we think it is unlikely, because of the high number of operations used both in the combination of tensor entries and in the aggregation over axes. Another possibility which we did not explore is, that there are other architectures which approximate standard architectures, and therefore produce similar outputs, that can naturally be translated to Einsum with the semirings we used. Architectures like this could be explored in future work.

6 Conclusion

We have shown that nested Einsum expressions over the same semiring can always be compressed into flat Einsum expressions, which makes the translation of chained operations much easier.

We managed to naturally translate the DFT and the Hadamard Transform to Einsum. For deep learning, we found acceptable translations for convolution and max-pooling, but could not find flat Einsum expressions for fully connected feed-forward nets, attention, and batch norm with the vanilla semirings we used. Instead, we managed to translate these to mixtures of multiple Einsum expressions over different semirings and element-wise functions. Such mixtures, although not optimal, might still be helpful because they can be computed in a sequential manner which might still lead to efficient solutions. Because we could not find natural translations, Einsum is probably not powerful enough to describe the standard architectures in deep learning. There might also be different architectures that are similar to the ones we explored, which can naturally be translated. Such architectures and more exotic semirings could be explored in future work.

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Mapping Integer Linear Programs to Einsum

First translate the ILP

$$\min_{x \in \mathbb{Z}_+^n} c^\top x$$
$$\text{s.t.} Ax = b$$

to a QUBO

$$\min_{\overline{x} \in \{0,1\}^m} \overline{x}^\top Q \overline{x}$$

by

1. binarizing the input $x \in \mathbb{Z}_+^n$ to $\overline{x} \in \{0,1\}^m$ with an upper bound on x,
2. turning the constraints
$$Ax = b$$
to penalties in the objective function
$$\rho(Ax - b)^\top (Ax - b)$$
for a big enough $\rho \in \mathbb{R}$, and
3. embedding linear terms on the diagonal of Q.

Then consider a symbol $s_i \in S$ for each row $i \in [m]$ of $\overline{x}$. We introduce a factor $T^{(k,k)}$ with index string $\boldsymbol{s_{kk}} = s_k$ for every diagonal entry, and a factor $T^{(i,j)}$ with index string $\boldsymbol{s_{ij}} = s_i s_j$ for every non-diagonal entry:

$$T^{(k,k)} := \begin{pmatrix} 0 \\ q_{kk} \end{pmatrix}$$
$$T^{(i,j)} := \begin{pmatrix} 0 & 0 \\ 0 & q_{ij} \end{pmatrix}$$

We can then solve the QUBO by computing the Einsum expression

$$(\boldsymbol{s_{11}}, \ldots, \boldsymbol{s_{1m}}, \ldots, \boldsymbol{s_{m1}}, \ldots, \boldsymbol{s_{mm}} \to, T^{(1,1)}, \ldots, T^{(1,m)}, \ldots, T^{(m,1)}, \ldots, T^{(m,m)})$$

over the semiring $R = (\mathbb{R}, \min, +)$.

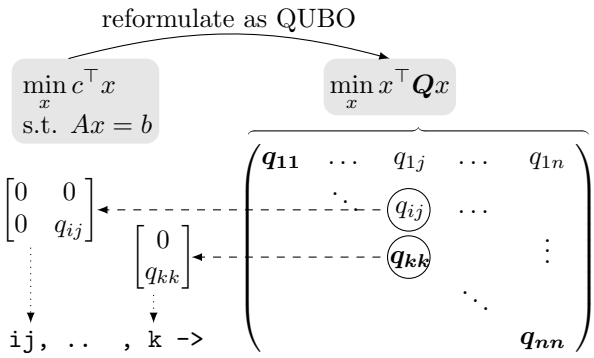

Figure 1: Translation of ILP to Einsum ©Julien Klaus

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Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe. Seitens des Verfassers bestehen keine Einwände die vorliegende Masterarbeit für die öffentliche Benutzung im Universitätsarchiv zur Verfügung zu stellen.

Jena, 07.08.2023