Tutorial 1

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Contents

1 Convex sets

(a) Closed sets and convex sets.

i. A polyhedron $\{x \in \mathbb{R}^n : Ax \leq b\}$, for some $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, is both convex and closed.

ii. Show that if $S_i \subseteq \mathbb{R}^n$, $i \in I$ is a collection of convex sets, then their intersection $\cap_{i \in I} S_i$ is also convex. Show that the same statement holds if we replace "convex" with "closed".

iii. Given an example of a closed set in $\mathbb{R}^2$ whose convex hull is not closed.

iv. Let $A \in \mathbb{R}^{m \times n}$. Show that if $S \subseteq \mathbb{R}^m$ is convex then so is $A^{-1}(S) = \{x \in \mathbb{R}^n : Ax \in S\}$, which is called the preimage of S under the map $A : \mathbb{R}^n \to \mathbb{R}^m$. Show that the same statement holds if we replace "convex" with "closed".

v. Let $A \in \mathbb{R}^{m \times n}$. Show that if $S \subseteq \mathbb{R}^n$ is convex then so is $A(S) = \{Ax : x \in S\}$, called the image of S under A.

vi. Give an example of a matrix $A \in \mathbb{R}^{m \times n}$ and a set $S \subseteq \mathbb{R}^n$ that is closed and convex but such that $A(S)$ is not closed.

Proofs:

i. Not difficult.

ii. By noting that $\{x \in \mathbb{R}^n : Ax \leq b\} = \cap_{i=1}^n \{x \in \mathbb{R}^n : a_i^\top x \leq b_i\}$, it is left to prove that each halfspace associated with (a_i, b_i) is closed (resp. convex), then we can use the result established in the past item. As the convexity can be proved in a straightforward manner, we will only prove that: $H = \{x \in \mathbb{R}^n : a_i^\top x \leq b_i\}$ is closed.

iii.

iv.

v.

2 Polyhedra

i. Show that if $P \subseteq \mathbb{R}^n$ is a polyhedron, and $A \in \mathbb{R}^{m \times n}$, then $A(P)$ is a polyhedron. Hint: you may use the fact that

$$P \subseteq \mathbb{R}^{m+n} \text{ is a polyhedron } \Rightarrow \{x \in \mathbb{R}^n : (x, y) \in P \text{ for some } y \in \mathbb{R}^m\} \text{ is a polyhedron.}$$

ii. Show that if $Q \subseteq \mathbb{R}^m$ is a polyhedron, and $A \in \mathbb{R}^{m \times n}$, then $A^{-1}(Q)$ is a polyhedron.

Proofs:

i.

ii.

iii.

iv.

v.

3 Convex functions

(a) Let f be twice differentiable, with $\mathrm{dom}(f)$ convex. Prove that:

$$f \text{ is convex } \iff (\nabla f(x) - \nabla f(y))^T (x - y) \geq 0, \text{ for all } x, y.$$

(b) A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *coercive* provided that $f(x) \to \infty$ as $\|x\|_2 \to \infty$. A key fact about coercive functions is that they attain their infimums. Prove that a twice differentiable, strongly convex function is coercive and hence attains its infimum. Hint: use Q3 part (b.iv).

(b) Prove that the maximum of a convex function over a bounded polyhedron must occur at one of the vertices.

Proofs:

i.

ii.

iii.

iv.

v.

4 Lipschitz gradients

Let f be convex and twice continuously differentiable. We will show that the following statements are equivalent.

i. ∇f is Lipschitz with constant L;

ii. $(\nabla f(x) - \nabla f(y))^T (x - y) \le L\|x - y\|_2^2$ for all x, y;

iii. $\nabla^2 f(x) \preceq LI$ for all x;

iv. $f(y) \le f(x) + \nabla f(x)^T (y - x) + \frac{L}{2}\|y - x\|_2^2$ for all x, y.

Proofs:

i.

ii.

iii.

iv.

v.

5 Strongly Convex Functions

Let f be convex and twice continuously differentiable. We will show that the following statements are equivalent.

i. f is strongly convex with constant m;

ii. $(\nabla f(x) - \nabla f(y))^T (x - y) \ge m\|x - y\|_2^2$ for all x, y;

iii. $\nabla^2 f(x) \succeq mI$ for all x;

iv. $f(y) \ge f(x) + \nabla f(x)^T (y - x) + \frac{m}{2}\|y - x\|_2^2$ for all x, y.

Proofs:

i.

ii.

iii.

iv.

v.