Tutorial 1

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This file contains properties and tricks regarding:

- Convex Vs Closed Sets
- Convex functions
- Properties of Lipchitz Gradients
- Properties of Strong convexity.

1 Convex sets

(a) Closed sets and convex sets.

i. A polyhedron $\{x \in \mathbb{R}^n : Ax \leq b\}$, for some $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, is both convex and closed.

ii. Show that if $S_i \subseteq \mathbb{R}^n$, $i \in I$ is a collection of convex sets, then their intersection $\cap_{i \in I} S_i$ is also convex. Show that the same statement holds if we replace "convex" with "closed".

iii. Given an example of a closed set in $\mathbb{R}^2$ whose convex hull is not closed.

iv. Let $A \in \mathbb{R}^{m \times n}$. Show that if $S \subseteq \mathbb{R}^m$ is convex then so is $A^{-1}(S) = \{x \in \mathbb{R}^n : Ax \in S\}$, which is called the preimage of S under the map $A : \mathbb{R}^n \to \mathbb{R}^m$. Show that the same statement holds if we replace "convex" with "closed".

v. Let $A \in \mathbb{R}^{m \times n}$. Show that if $S \subseteq \mathbb{R}^n$ is convex then so is $A(S) = \{Ax : x \in S\}$, called the image of S under A.

vi. Give an example of a matrix $A \in \mathbb{R}^{m \times n}$ and a set $S \subseteq \mathbb{R}^n$ that is closed and convex but such that $A(S)$ is not closed.

(b, 4 pts) Polyhedra.

i. Show that if $P \subseteq \mathbb{R}^n$ is a polyhedron, and $A \in \mathbb{R}^{m \times n}$, then $A(P)$ is a polyhedron. Hint: you may use the fact that

$$P \subseteq \mathbb{R}^{m+n} \text{ is a polyhedron } \Rightarrow \{x \in \mathbb{R}^n : (x, y) \in P \text{ for some } y \in \mathbb{R}^m\} \text{ is a polyhedron.}$$

ii. Show that if $Q \subseteq \mathbb{R}^m$ is a polyhedron, and $A \in \mathbb{R}^{m \times n}$, then $A^{-1}(Q)$ is a polyhedron.

2 Convex functions (14 points)

(a) Let f be twice differentiable, with $\mathrm{dom}(f)$ convex. Prove that:

$$f \text{ is convex } \iff (\nabla f(x) - \nabla f(y))^T(x-y) \geq 0, \text{ for all } x, y.$$

(b) A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *coercive* provided that $f(x) \to \infty$ as $\|x\|_2 \to \infty$. A key fact about coercive functions is that they attain their infimums. Prove that a twice differentiable, strongly convex function is coercive and hence attains its infimum. Hint: use Q3 part (b.iv).

(b) Prove that the maximum of a convex function over a bounded polyhedron must occur at one of the vertices.

3 Lipschitz gradients

Let f be convex and twice continuously differentiable. We will show that the following statements are equivalent.

i. ∇f is Lipschitz with constant L;

ii. $(\nabla f(x) - \nabla f(y))^T(x-y) \leq L\|x-y\|_2^2$ for all x, y;

iii. $\nabla^2 f(x) \preceq LI$ for all x;

iv. $f(y) \leq f(x) + \nabla f(x)^T(y-x) + \frac{L}{2}\|y-x\|_2^2$ for all x, y.

4 Strong convexity

Let f be convex and twice continuously differentiable. We will show that the following statements are equivalent.

i. f is strongly convex with constant m;

ii. $(\nabla f(x) - \nabla f(y))^T(x-y) \geq m\|x-y\|_2^2$ for all x, y;

iii. $\nabla^2 f(x) \succeq mI$ for all x;

iv. $f(y) \geq f(x) + \nabla f(x)^T(y-x) + \frac{m}{2}\|y-x\|_2^2$ for all x, y.