Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 9

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Exercise 1.

(a) *Let's say w.l.o.g that $a^\star$ satifies the first literal (and only the first) of every clause, then $\neg a^\star$ would satify the second literal of each clause, and hence, $\neg a^\star$ would also be a satisfied assigment.*

(b) *We now have two competing targets $a^\star$ and $\neg a^\star$, and most importantly, their respective Hamming distance is complementary, i.e.,*

$$H(a, a^\star) = n - H(a, \neg a^\star)$$

Hence, when analyzing the algorithm, we will focus now on how each iteration of the algo changes the following measure:

$$\alpha = \min\left(H(a, a^\star), H(a, \neg a^\star)\right)$$

Since, α is at most $\lfloor \frac{n}{2} \rfloor$, the "diameter" of the original Markov Chains is cut by half. In the case where n is even, we get a fair Random Walk, and hence, we need at most $n^2/4$ steps.

Exercise 2.
(a) *The new recurrence relations could be summed up as follows:*

$$\begin{cases} E_0 &= 0 \\ E_i &= 1 + \frac{1}{2}(E_{i+1} + E_{i-1}) \quad \textit{for } i \in \{1, \dots, n-1\} \\ E_n &= 1 + \frac{1}{2}(E_n + E_{n-1}) \end{cases}$$

The last implies that: $E_n = 2 + E_{n-1} \implies D_n = 2$.
The second line implies that:

$$D_k = 2 + D_{k-1} \textit{ for } k \in \{1, \dots, n-1\}$$

By summing each equation from $k = 1$ up to $k = i \leq n-1$, we get:

$$D_1 = 2i + D_{i+1} \quad \textit{for} \quad i \in \{1, \dots, n-1\}$$

Particularly when $i = n - 1$

$$D_1 = 2(n-1) + D_n = 2n = E_1$$

And hence,

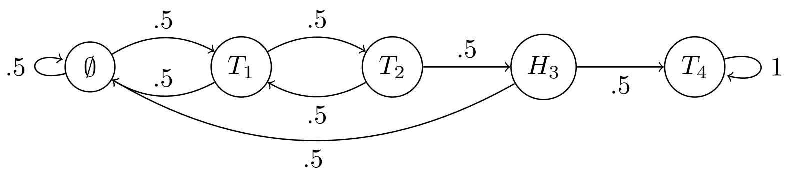
$$D_{i+1} = 2(n-i), \quad \text{for } i \in \{1, \ldots, n-1\}$$

(b) *For* $k \in \{1, \ldots, n\}$*, we have:*

$$\begin{aligned} E_k &= \sum_{i=1}^{k} D_i \\ &= \sum_{i=1}^{k} 2(n-(k-1)) \\ &= k(2n-k)+k \end{aligned}$$

Exercise 3.

(a) *The figure below represents the Markov Chain of interest:*



(b) *We refer to* E_0 *as the expected running time starting the the state* $\emptyset$*,*($E_0 \to \emptyset$) *and respectively:* $E_1 \to T_1, E_2 \to T_2, E_3 \to T_3, E_4 \to T_4$.

We have the following system:

$$\begin{cases} E_0 &= 1 + \frac{1}{2}\left(E_0 + E_1\right) \\ E_1 &= 1 + \frac{1}{2}\left(E_0 + E_2\right) \\ E_2 &= 1 + \frac{1}{2}\left(E_1 + E_3\right) \\ E_3 &= 1 + \frac{1}{2}\left(E_0 + E_4\right) \\ E_4 &= 0 \end{cases} \implies \begin{cases} E_0 &= 20 \\ E_1 &= 18 \\ E_2 &= 14 \\ E_3 &= 9 \\ E_4 &= 0 \end{cases}$$

We get 12 Steps in expectation.

(c) *Let's say this output has emerged:* $T[TTH_1T]HTH_2$*, the approach in* ***(a)*** *would detect the pattern between brackets, while the approach proposed in* ***(c)*** *would fail, as it would only check if the pattern has emerged looking back from* H_1 *and* H_2*.*

The expected running time is the Expectation of a geometric RV of $p = \frac{1}{2^4}$*, which is equal to* $\frac{1}{p} = 2^4 = 16$*, hence we would need 16 trials. As each trial requires* 4 *steps, we would need in expectation* $16 * 4 = 64$ *steps.*