Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 8

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Exercise 1.
(a) *The algorithm described briefly as follows, has a* ***cubic*** *runtime.*

for $1 \le i \le n, 1 \le j \le n$ ***do***
    $\alpha_{i,j} \leftarrow sum([A_{i,k}B_{k,j} : k \in \{1, \dots, n\}])$
    $\alpha_{i,j} \leftarrow \alpha_{i,j} - c_{i,j}$
    if $\alpha_{i,j} \neq 0$ ***then***
        return 0
    end if
end for
return 1

(b) *Computing x, y, and z requires $\mathcal{O}(n^2)$ each, and computing $t = y - z$ requires $\mathcal{O}(n)$, hence, the asymptotic runtime of this RA is $\mathcal{O}(n^2)$.*

(c) *If $r \neq 0$, the event $\mathcal{E}$ implies the following:*

- $r \in Ker(D)$
- *r is orthogonal to all rows of D*
- *Since $D \neq 0$, there exists at least one row $d_i \neq 0$ s.t: $d_i^\top r = 0$*

We have:

$$\begin{aligned}\mathbb{P}[\mathcal{E}] &= \mathbb{P}[Dr = 0 | r = 0]\mathbb{P}[r = 0] + \mathbb{P}[Dr = 0 | r \neq 0]\mathbb{P}[r \neq 0] \\ &= \frac{1}{3^n} + \mathbb{P}[Dr = 0 | r \neq 0](1 - \frac{1}{3^n}) \qquad \textit{[1]}\end{aligned}$$

For some $k \le n$, we let, $d_1, \dots, d_k$ be the rows of D that are not equal to 0. We have:

$$\begin{aligned}\mathbb{P}[Dr = 0 | r \neq 0] &= \mathbb{P}[d_1^\top r = 0, \dots, d_k^\top r = 0 | r \neq 0] \\ &\le \mathbb{P}[d_1^\top r = 0 | r \neq 0] \qquad \textit{[2]}\end{aligned}$$

Let $d_{1,j}$ be the last element of d_1 not equal to 0, we have: $d_1^\top r = 0$ ***if and only if*** *after $j-1$ picks of $r_1, \dots, r_{j-1}$, the j^{th} pick (i.e., r_j) is chosen s.t: $-d_{1,j}r_j = \sum_{i=1}^{j-1} d_{1,i}r_i$.*
And hence, knowing the values $r_1, \dots, r_{j-1}$, we conclude the following:

$$\mathbb{P}[d_1^\top r = 0 | r \neq 0] = \mathbb{P}\left[r_j = \sum_{i=1}^{j-1} \frac{-d_{1,i}}{d_{1,j}} r_i | r \neq 0 \right] \leq \frac{1}{3}$$

*Using **[2]**, we get $\mathbb{P}[Dr = 0 | r \neq 0] \leq \frac{1}{3}$, anf from **[1]**, we conclude that:*

$$\mathbb{P}[\mathcal{E}] \leq \frac{1}{3^n} + \frac{1}{3}(1 - \frac{1}{3^n}) \leq \frac{1}{3}$$

(d) *We can reduce the probability by amplification.*

Exercise 2.

(a) *We have:*

$$\begin{aligned}
\sum_{i \geq 1} \mathbb{P}[X \geq i] &= \sum_{i \geq 1} \sum_{j=i}^{\infty} \mathbb{P}[X = j] \\
&= \sum_{j \geq 1} \sum_{i=1}^{j} \mathbb{P}[X = j] \\
&= \sum_{j \geq 1} \mathbb{P}[X = j] \sum_{i=1}^{j} 1 \\
&= \sum_{j \geq 1} j \mathbb{P}[X = j] = \mathbb{E}[X]
\end{aligned}$$

If X can only take negative values, then:

$$\begin{aligned}
\mathbb{E}[X] &= -\mathbb{E}[-X] \\
&= -\sum_{j \geq 1} \mathbb{P}[-X \geq j] \\
&= -\sum_{j \geq 1} \mathbb{P}[X \leq -j]
\end{aligned}$$

(b) *Let $\mathbb{P}[X \geq 0] = p_X$, and $\mathbb{P}[Y \geq 0] = p_Y$, we have: $p_X \leq p_Y$, we suppose that $0 < p_X < 1$ and $0 < p_Y < 1$:*

$$\begin{aligned}
\mathbb{E}[Y] - \mathbb{E}[X] &= p_Y \mathbb{E}[Y | Y > 0] + (1 - p_Y)\mathbb{E}[Y | Y < 0] - p_X \mathbb{E}[X | X > 0] - (1 - p_X)\mathbb{E}[X | X < 0] \\
&= p_Y \mathbb{E}[Y | Y > 0] - p_X \mathbb{E}[X | X > 0] + (1 - p_Y)\mathbb{E}[Y | Y < 0] - (1 - p_X)\mathbb{E}[X | X < 0] \qquad \textbf{\textit{Eq.}}\ \#42
\end{aligned}$$

We have:
for $i \geq 1$:

$$
\begin{aligned}
\mathbb{P}[X \geq i] \leq \mathbb{P}[Y \geq 1] \implies & \sum_{k \geq 1} \mathbb{P}[X \geq k] \leq \sum_{k \geq 1} \mathbb{P}[Y \geq k] \\
\implies & \sum_{k \geq 1} \frac{\mathbb{P}[X \geq k]}{p_Y} \leq \sum_{k \geq 1} \frac{\mathbb{P}[Y \geq k]}{p_Y} \\
\implies & p_X \sum_{k \geq 1} \frac{\mathbb{P}[X \geq k]}{p_X} \leq p_Y \sum_{k \geq 1} \frac{\mathbb{P}[Y \geq k]}{p_Y} \qquad \textit{(as } p_X \leq p_Y \textit{)} \\
\implies & p_X \mathbb{E}[X|X > 0] \leq p_Y \mathbb{E}[Y|Y > 0] \quad \textbf{\textit{[1]}} \quad \textit{(using the result of the past question)}
\end{aligned}
$$

On the other side, we have:

$$
\mathbb{E}[X|X < 0] = -\sum_{i \geq 1} \frac{\mathbb{P}[X \leq -i]}{1 - p_X} \implies (1 - p_X)\mathbb{E}[X|X < 0] = -\sum_{i \geq 1} \mathbb{P}[X \leq -i]
$$

Similarily:

$$
(1 - p_Y)\mathbb{E}[Y|Y < 0] = -\sum_{i \geq 1} \mathbb{P}[Y \leq -i]
$$

Hence,

$$
\begin{aligned}
(1 - p_Y)\mathbb{E}[Y|Y < 0] - (1 - p_X)\mathbb{E}[X|X < 0] &= \sum_{i \geq 1} \mathbb{P}[X \leq -i] - P[Y \leq -i] \\
&= \sum_{i \geq 1} \mathbb{P}[Y \geq i] - P[X \geq i] \\
&\geq 0 \qquad \textit{[2]}
\end{aligned}
$$

*By using the results in **[1]** and **[2]** in **Eq.** #42, we conclude that* $\mathbb{E}[X] \leq \mathbb{E}[Y]$.