Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 5

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Exercise 1.
(a) *We have established in the notes that the probability of not making any mistakes after* $n-t$ *contractions is at least* $\frac{t(t-1)}{n(n-1)}$*, in this case, when* $t=n-1$*, we would get that this probability is* $\frac{n-2}{n}=1-\frac{2}{n}$*, which is in fact equal to* q_n*.*

(b) *We denote the probability that* S_1 *(resp* S_2*) is correct as* P_1 *(resp.* P_2*), we have:*

$$\begin{aligned}
P_1 &= \mathbb{P}[\{\textit{Output of L5 is correct}\}|\{\textit{Contraction at L4 is correct}\}]\mathbb{P}[\{\textit{Contraction at L4 is correct}\}] \\
&= P(n-1)q_n \\
P_2 &= P(n)
\end{aligned}$$

The algorithm is succesful if either the return at line 7 was a succes (i.e., S_1 *is a success) or the return at line 10 was a succes, which yiled the following:*

$$\begin{aligned}
p(n) &= \mathbb{P}[\{\textit{Entered L7}\}]\mathbb{P}[\{\textit{Succes}\}|\{\textit{entered L7}\}] + \mathbb{P}[\{\textit{Didn't Enter L7}\}]\mathbb{P}[\{\textit{Succes}\}|\{\textit{Didnt enter L7}\}] \\
&= q_n\mathbb{P}[\{\textit{Succes}\}|\{\textit{entered L7}\}] + (1-q_n)\mathbb{P}[\{\textit{Succes}\}|\{\textit{Didnt enter L7}\}] \\
&= q_n\mathbb{P}[\{S_1 \textit{ is succesful}\}] + (1-q_n)\mathbb{P}[\{\textit{The best of } S_1,\, S_2 \textit{ was succesful}\}] \\
&= q_n\mathbb{P}[\{S_1 \textit{ is succesful}\}] + (1-q_n)(1-\mathbb{P}[\{S_1 \textit{ and } S_2 \textit{ failed}\}]) \\
&= q_nP_1 + (1-q_n)[1-(1-P_1)(1-P_2)] \qquad \textit{(We will just keep re-arranging after now)} \\
&= q_n^2P(n-1) + (1-q_n)[1-(1-q_nP(n-1))(1-P(n))] \\
&= q_n^2P(n-1) + (1-q_n)[P(n) + q_nP(n-1) - q_nP(n-1)P(n)] \\
&= q_nP(n-1) + (1-q_n)P(n) - q_n(1-q_n)P(n-1)P(n)
\end{aligned}$$

$$\implies$$

$$q_nP(n) = q_nP(n-1) - q_n(1-q_n)P(n-1)P(n)$$

$$\implies$$

$$P(n) = P(n-1) - (1-q_n)P(n-1)P(n) \qquad (q_n \neq 0 \textit{ since } n \neq 2 \textit{ as we would verify line 1 if } n=2)$$

(c) *By dividing the equation that we derived in the last question by* $P(n)P(n-1)$*, we get the following, for* $k \in \{3, \ldots, n\}$*:*

$$\frac{1}{P(k-1)} - \frac{1}{P(k)} = -\frac{2}{k}$$
$$\Longrightarrow$$
$$\sum_{k=3}^{n} \frac{1}{P(k-1)} - \sum_{k=3}^{n} \frac{1}{P(k)} = -\sum_{k=3}^{n} \frac{2}{k}$$
$$\frac{1}{P(2)} - \frac{1}{P(n)} = -\sum_{k=3}^{n} \frac{2}{k}$$
$$\Longrightarrow$$
$$P(n) = \frac{1}{\sum_{k=3}^{n} \frac{2}{k} - 1}$$

Comparisons of the probability of succes with FastCut:

- *GeoContraction has a* $\Theta(\frac{1}{\log(n)})$ *(Assuming the bounds are exact like suggested in **(b)**)*
- *FastCut had a* $\Omega(\frac{1}{\log(n)})$

Exercise 2.

Let X_t *represent the number of walks that are left to reach home at step* t*. Obviously* $X_t \in \{0, \ldots, n\}$*, and we are interested in computing* $\mathbb{E}[T]$ *where* $T = \inf\{t \geq 0 | X_t = 0\}$*.*

We have the following:

$$\begin{aligned}
\mathbb{E}[X_t - X_{t+1} | X_t = s] &= \sum_{X_t, X_{t+1}} \mathbb{P}[(X_t, X_{t+1}) | X_t = s](X_t - X_{t+1}) \\
&= \sum_{X_{t+1}} \mathbb{P}[(s, X_{t+1}) | X_t = s](s - X_{t+1})
\end{aligned}$$

If $s = n$ *(we are at the bar),* X_{t+1} *has two options* $\{n, n-1\}$

$$\begin{aligned}
\mathbb{E}[X_t - X_{t+1} | X_t = s] &= (n - (n-1))\mathbb{P}[(n, n-1)] \\
&= \frac{1}{5}
\end{aligned}$$

If $0 < s < n$*,* X_{t+1} *has two options* $\{s-1, s+1\}$

$$\begin{aligned}
\mathbb{E}[X_t - X_{t+1} | X_t = s] &= (s - (s-1))\mathbb{P}[(s, s-1)] + (s - (s+1))\mathbb{P}[(s, s+1)] \\
&= \frac{3}{5} + \frac{2}{5} \\
&= 1
\end{aligned}$$

Hence, we get, for all $t > 0, s \neq n$*:*

$$\frac{1}{5} \leq \mathbb{E}[X_t - X_{t+1} | X_t = s] \leq 1$$

Which yields the following bounds, assuming $X_0 = n$*:*

$$n \leq \mathbb{E}[T] \leq 5n$$

Exercise 3.
(a)
(b)

Exercise 4.
(a)
(b)
(c)
(d)