Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 9

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Exercise 1.
(a) *As $\mathbb{E}[X] = \frac{n}{2}$, we have:*

$$\mathbb{P}[X \geq \frac{3}{4}n] = \mathbb{P}[X \geq (1 + \frac{1}{2})\frac{n}{2}] = \mathbb{P}[X \geq (1 + \frac{1}{2})\mathbb{E}[X])]$$

And similarily:

$$\mathbb{P}[X \leq \frac{1}{4}n] = \mathbb{P}[X \leq (1 - \frac{1}{2})\mathbb{E}[X])]$$

Using the Chernoff bounds, (Ineq 2 then 3, from Slides 4), we conclude that (for $\delta = 1/2 < 1$):

$$\mathbb{P}[X \leq \frac{1}{4}n] \leq e^{-\frac{n}{24}} \quad \text{and} \quad \mathbb{P}[X \geq \frac{3}{4}n] \leq e^{-\frac{n}{24}}$$

(b) *Using the the simplified Chernoff bounds:*

$$\begin{aligned}\mathbb{P}[X \geq n/2 + 2\sqrt{n}] &= \mathbb{P}[X \geq \frac{n}{2}(1 + \frac{4}{\sqrt{n}})] \\ &\leq e^{-\frac{n}{2}\left(\frac{4}{\sqrt{n}}\right)^2 \frac{1}{3}} \quad \textit{(Requires } \frac{4}{\sqrt{n}} < 1 \implies n > 16\textit{)} \\ &= e^{-8/3}\end{aligned}$$

Using the additive Chernoff (No extra requirements needed):

$$\begin{aligned}\mathbb{P}[X \geq n/2 + 2\sqrt{n}] &\leq e^{-2\left(2\sqrt{n}\right)^2 \frac{1}{n}} \\ &= e^{-8}\end{aligned}$$

(c) *We let $n_1 = \lfloor n/2 - \sqrt{n}/4 \rfloor$ and $n_2 = \lfloor n/2 + \sqrt{n}/4 \rfloor$, we have:*

$$
\begin{aligned}
\mathbb{P}[X \in [n/2 - \sqrt{n}/4, n/2 + \sqrt{n}/4]] &\leq \mathbb{P}[X \in [n_1, n_2]] \\
&= \sum_{k=n_1}^{n_2} \mathbb{P}[X = k] \\
&= \sum_{k=n_1}^{n_2} \binom{n}{k} \frac{1}{2^n} \\
&\leq \frac{1}{2^n} \sum_{k=n_1}^{n_2} 2^n \frac{\sqrt{2}}{\sqrt{n}} \quad \textit{(Using the hint)} \\
&= \frac{\sqrt{2}}{\sqrt{n}} \sum_{k=n_1}^{n_2} 1 = \frac{\sqrt{2}}{\sqrt{n}} (n_2 - n_2 + 1) \\
&\leq \frac{\sqrt{2}}{\sqrt{n}} \left(n/2 + \sqrt{n}/4 - (n/2 - \sqrt{n}/4 - 1) \right) \ \ \textit{(using: } \lfloor x \rfloor - \lfloor y \rfloor \leq x - (y-1)\textit{)} \\
&= \frac{\sqrt{2}}{\sqrt{n}} (1 + \sqrt{n}/2) \\
&= \frac{\sqrt{2}}{\sqrt{n}} + \frac{1}{\sqrt{2}}
\end{aligned}
$$

For $n_0 = 24$*, we have for* $n \geq n_0 : \frac{\sqrt{2}}{\sqrt{n}} + \frac{1}{\sqrt{2}} \leq \frac{\sqrt{2}}{\sqrt{24}} + \frac{1}{\sqrt{2}} \approx 0.995$

Exercise 2.

Let X_i *be the indicator random variable telling whether the i-th queried person is voting for Alice or not, we have* $A = \sum_{i=1}^{k} X_i$*. As* X_i*'s are drawn randomly with replacement, they are considered i.i.d realisations of the true distribution, i.e.,* $X_i \sim Bern(p)$*.*

We also have: $\mathbb{E}[A] = kp$*, hence,* $\hat{p} = A/k$ *is an unbiased estimate of* p*.*

We have:

$$
\begin{aligned}
\mathbb{P}[|\hat{p} - p| > \epsilon] &= \mathbb{P}[|k\hat{p} - kp| > k\epsilon] \\
&= \mathbb{P}\left[|A - \mathbb{E}[A]| > \mathbb{E}[A] \frac{k\epsilon}{\mathbb{E}[A]} \right] \\
&= \mathbb{P}\left[|A - \mathbb{E}[A]| > \mathbb{E}[A] \frac{\epsilon}{p} \right]
\end{aligned}
$$

We have proved in class the following for $0 < \delta < 1$*:*

$$
\mathbb{P}[X \geq (1+\delta)\mathbb{E}[X]] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}} \right)^{\mathbb{E}[X]} \quad \textit{And} \quad \mathbb{P}[X \leq (1-\delta)\mathbb{E}[X]] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}} \right)^{\mathbb{E}[X]}
$$

Using the inequality $\frac{2\delta}{2+\delta} \leq \log(1+\delta)$*, we can easily upper bound the past inequalities that we haave proved in class to the following versions of Chernoff:*

$$
\mathbb{P}[X \geq (1+\delta)\mathbb{E}[X]] \leq \exp(-\frac{\delta^2 \mathbb{E}[X]}{2+\delta}) \quad \textit{And} \quad \mathbb{P}[X \leq (1-\delta)\mathbb{E}[X]] \leq \exp(-\frac{\delta^2 \mathbb{E}[X]}{2})
$$

And hence:

$$\mathbb{P}\left[|X - \mathbb{E}[X]| \geq \mu\mathbb{E}[X]\right] \leq 2\exp\left(-\frac{\delta^2\mathbb{E}[X]}{2+\delta}\right)$$

Which yields:

$$\begin{aligned}\mathbb{P}[|\hat{p} - p| > \epsilon] &= \mathbb{P}\left[|A - \mathbb{E}[A]| > \mathbb{E}[A]\frac{\epsilon}{p}\right] \\ &\leq 2\exp\left(-\frac{(\epsilon/p)^2\mathbb{E}[X]}{2+(\epsilon/p)}\right) \\ &\leq 2\exp\left(-\frac{\epsilon^2 k}{2p+\epsilon}\right)\end{aligned}$$

Exercise 3.

Let the RV X represent the sum of the n rolls, we have established previously that: $\mathbb{E}[X] = \frac{7n}{2}$

Using Hoeffding's inequality leads to:

$$\begin{aligned}\mathbb{P}[X \geq 4n] &= \mathbb{P}[X \geq E[X] + \frac{n}{2}] \\ &\leq \exp\left(-\frac{2(n/2)^2}{\sum_{i=1}^{n}(6-1)^2}\right) \\ &= e^{-n/50}\end{aligned}$$

This tail that we obtained using Hoeffding's inequality decreases way faster than what we have obtained with Marokov and Chebychev's inequalities (respectively $\frac{7}{8}$ and $\frac{35}{3n}$).

Exercise 4.

Let's first establish a few results on $d(u,v)$ for two random vertices of the hypercube. $d(u,v)$ represents the number of bits v_i in v that are diffrent than u_i, hence, $d(u,v) \in \{0,\ldots,n\}$, and $\mathbb{P}[d(u,v) = k] = \frac{\binom{n}{k}}{2^n}$.(Why: How many ways can we pick k positions so we can flip their bits in v divided by the number of all configurations.)

By noticing that $\sum_{i=1}^{n} k\binom{n}{k} = n2^{n-1}$, we can conclude that:

$$\mathbb{E}[d(u,v)] = \sum_{i=1}^{n} k\frac{\binom{n}{k}}{2^n} = \frac{n}{2}$$

Let's prove the following result, for any two random vertices $u, v \in V$, $\epsilon > 0$:

$$\mathbb{P}\left[(1-\epsilon)\frac{n}{2} \leq d(u,v) \leq (1+\epsilon)\frac{n}{2}\right] \geq 1 - 2e^{-\frac{n\epsilon^2}{6}}$$

Proof:
Let's define the events E_1 and E_2 as follows:

$$E_1 = \{(1-\epsilon)\frac{n}{2} \leq d(u,v)\} = \{(1-\epsilon)\mathbb{E}[d(u,v)] \leq d(u,v)\}$$

And:

$$E_2 = \{(1+\epsilon)\frac{n}{2} \geq d(u,v)\} = \{(1+\epsilon)\mathbb{E}[d(u,v)] \geq d(u,v)\}$$

We are looking for to bound the probablity: $\mathbb{P}[E_1 \cap E_2]$

$$\begin{aligned} \mathbb{P}[E_1 \cap E_2] &= 1 - \mathbb{P}[\overline{E_1} \cup \overline{E_2}] \\ &\geq 1 - (\mathbb{P}[\overline{E_1}] + \mathbb{P}[\overline{E_2}]) \\ &\geq 1 - 2e^{-\mathbb{E}[d(u,v)]\epsilon^2/3} \\ &\geq 1 - 2e^{-\frac{n\epsilon^2}{6}} \end{aligned}$$

Here we used the union bound followed by Chernoff (Ineq 2. Slide 4 to bound $\mathbb{P}[\overline{E_2}]$*, and Ineq 3 for* $\mathbb{P}[\overline{E_1}]$*), assumig* $0 < \epsilon < 1$*.*

Let $V = \{v_1, \ldots, v_{2^n}\}$*,* ***we uniformely random pick n vertices from V*** *. Now, let's define the RVs* $\{X_i\}$*,* $\{Y_{i,j}\}$ *and* $V(v_1, \ldots, v_{2^n})$ *as follows:*

$$X_i = \begin{cases} 1 & \textit{if } v_i \textit{ was picked} \\ 0 & \textit{Otherwise.} \end{cases} \qquad Y_{i,j} = \begin{cases} 1 & \textit{if} \quad (1-\epsilon)\frac{n}{2} \leq d(v_i, v_j) \leq (1+\epsilon)\frac{n}{2} \\ 0 & \textit{Otherwise.} \end{cases}$$

And finally:

$$S(v_1, \ldots, v_{2^n}) = \sum_{i<j} X_i X_j Y_{i,j}$$

We need to proof that:

Proof: *We have:*

$$\begin{aligned}
\mathbb{E}[S(v_1,\dots,v_{2^n})] &= \sum_{i<j} \mathbb{E}[X_i X_j Y_{i,j}] \\
&= \sum_{i<j} \mathbb{E}[X_i]\mathbb{E}[X_j]\mathbb{E}[Y_{i,j}] \quad \textit{(every rv is independent from the other)} \\
&= \mathbb{E}[X_1]^2 \sum_{i<j} \mathbb{E}[Y_{i,j}] \\
&\geq \mathbb{E}[X_1]^2 (1-2e^{-\frac{n\epsilon^2}{6}}) \sum_{i<j} 1 \\
&= \mathbb{P}[\{\textit{Probb. to pick } v_2\}]^2 (1-2e^{-\frac{n\epsilon^2}{6}}) \frac{2^{2n}-2^n}{2} \\
&= \left(\frac{n}{2^n}\right)^2 (1-2e^{-\frac{n\epsilon^2}{6}}) \frac{2^{2n}-2^n}{2} = \frac{n^2}{2}(1-2e^{-\frac{n\epsilon^2}{6}})(1-\frac{1}{2^n}) \\
&= \frac{n(n-1)}{2}\frac{n}{n-1}(1-2e^{-\frac{n\epsilon^2}{6}})(1-\frac{1}{2^n}) \\
&= \frac{n(n-1)}{2}(1+\frac{1}{n-1})(1-2e^{-\frac{n\epsilon^2}{6}})(1-\frac{1}{2^n})
\end{aligned}$$

fix this:

For n big enough, $(1+\frac{1}{n-1})(1-2e^{-\frac{n\epsilon^2}{6}})(1-\frac{1}{2^n})$ is guarenteed to be ≥ 1

Hence, $\forall \epsilon > 0$, and with n big enough, $\mathbb{E}[S(v_1,\dots,v_{2^n})] \geq \frac{n(n-1)}{2}$, hence there exist a configuration of X, Y s.t $v() = \frac{n(n-1)}{2}$, which can only happen if all the n picks had $Y_{i,j} = 1$