Randomised Algorithms

Winter term 2022/2023, Exercise Sheet No. 6

Hand-out: Mon, 21. Nov.
Hand-in: Sun, 27. Nov.

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Exercise 1 [4 points]

Consider the following way to distribute n gifts to n children around a Christmas tree in multiple rounds. In each round, we allow the children to run around and reach out for the gifts as they wish, and we assume each child will pick a gift uniformly at random among the remaining gifts. However, if two or more children try to take the same gift then we put the gift back to the tree and ask them to try again the next round. Otherwise if a gift is reached by only one child then the child is allowed to take it and he/she will not participate in future rounds. Use drift analysis to give an upper bound on the expected number of rounds to fully distribute n gifts.
Hint: Apply drift analysis to the stochastic process $(X_t)_{t\geq 0}$ where X_t is the number of remaining gifts at round t. To estimate the expected number of gifts successfully distributed in each round, focus on the event that a specific gift reached by exactly one child, then use the linearity of expectation on the remaining gifts. You can define for convenience that $(1-1/s)^{s-1} := 1$ if $s = 1$, thus the bound $(1-1/s)^{s-1} \geq 1/e$ can be useful.

Exercise 2 [4 points]

Consider a single player playing a board game of "snakes and ladders", but without snakes and ladders. We have a board with cells labelled $0, 1, 2, \ldots, 100$. The goal is to reach cell 100, starting with a token initially placed on cell 0. In each turn you roll a fair 6-sided die and advance your token by the resulting score. However, if your token would exceed 100 it will remain put, for example when the token is at cell 99, a roll of 1 is required to complete the game. Use drift analysis to find the best possible upper and lower bounds on the expected time to complete the game.
Hint: If Z_t denotes the token's cell at time t, apply drift analysis to the stochastic process $(X_t)_{t\geq 0}$ where $X_t := 100 - Z_t$. Pay attention to the states close to the target.

Exercise 3 [6 points]

Let X be the sum of n independent rolls of a fair six-sided die.

(a) Calculate $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(b) Show an upper bound on $\mathrm{Prob}(X \geq 4n)$ using Markov's inequality.

(c) Show an upper bound on $\mathrm{Prob}(X \geq 4n)$ using Chebyshev's inequality.

(d) Which of the above inequalities can you apply to derive a probability bound for the lower tail, $\mathrm{Prob}(X \leq 3n)$?

Exercise 4 [6 points]

In the lecture on LazySelect, the analysis of Error 2 uses the statement $\mathrm{Prob}(a < S(k_\ell)) = O(n^{-1/4})$ without proof, only noting that this can be shown similarly to the analysis of Error 1 with k_ℓ in place of k. Now prove this statement. Assume for simplicity that k is sufficiently large so that $\ell = \max(\lfloor kn^{-1/4} - \sqrt{n}\rfloor, 1) = \lfloor kn^{-1/4} - \sqrt{n}\rfloor$ holds.