UNIVERSITY OF PASSAU
Faculty of Computer Science and Mathematics

Randomised Algorithms

Winter term 2022/2023, Exercise Sheet No. 5

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Exercise 1 [6 points]

Recently, another algorithm for the MinCut problem was proposed that starts an additional recursive call with some probability.

GeoContract(G, n)

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1:  if n = 2 then
2:      Return one of the nodes of G which is a trival solution.
3:  else
4:      G' := result of contracting a random edge of G.
5:      S_1 := GeoContract(G', n − 1).
6:      With probability q_n = 1 − 2/n
7:          Return S_1.
8:      Otherwise
9:          S_2 := GeoContract(G, n).
10:         Return the best solution out of {S1, S2}.
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(a) Estimate the probability of not making any mistake when randomly contracting a graph from size n to $n-1$ (ie. line 4 of the above algorithm) and compare that probability to the above choice of q_n.

Hint: We have produced the formula for the probability of not making any mistake at step i of the most basic randomised contraction algorithm in the lecture, the question here only involves a specific value of i.

(b) Let $P(n)$ be the probability that GeoContract successfully constructs a minimum cut for a graph G of size n. Write the recurrence relation between $P(n)$ and $P(n-1)$, assuming the bounds on probabilities are exact for simplification. Expand and simplify this relation to show that it is equivalent to:

$$P(n) = P(n-1) - (1 - q_n)P(n)P(n-1).$$

(c) Solve the above recurrence relation to find a closed-form formula for $P(n)$, and compare it with this success probability of FastCut from the lecture.

Hint: It can be easier to establish the relation between $1/P(n)$ and $1/P(n-1)$ so that the harmonic sum can be used.

Exercise 2 [4 points]

Suppose the drunk person from the video is on his way home from a bar which is n steps away from his home. If he reaches the bar again (distance n) he will hesitate to go home, so with probability $4/5$ he does not move and with probability $1/5$ he makes a step forward in the direction of his home. When he is on the way, he is less reluctant and takes a step forward with probability $3/5$ and a step backward with probability $2/5$.
Use the additive drift theorem to analyse the expected time for the person to arrive home.

Exercise 3 [4 points]

Consider random process $(X_t)_{t\geq 0}$ on state space $\{-1, 0, 1\}$ and given the current state X_t the distribution of the next state is:

$$X_{t+1} = \begin{cases} -1 & \text{with probability } 1/2 \\ 1 & \text{with probability } 1/2. \end{cases}$$

Let $X_0 = 1$ and define $T := \inf\{t \geq 0 \mid X_t \leq 0\}$.

(a) Compute $\mathrm{E}(T)$ using a geometric-distribution argument.

(b) If we apply the additive drift theorem on the process as it is (without adjusting it), which condition of theorem will be violated? Which results will be obtained by this direct application and by adjusting the process? Discuss the correctness by comparing them with the exact answer found in (a).

Exercise 4 **[$6 + 2 = 8$ points]**

Apply appropriate drift theorems from the lecture to derive the smallest (but still correct) upper bounds on the expected time to reach the target state 0 in the following scenarios.
The state space is always $\{0, 1, \ldots, n\}$, and the initial state is always $X_0 = n$.

(a) $\mathrm{Prob}(X_{t+1} = i \mid X_t = s) = \frac{1}{s}$ for all $i \in \{0, \ldots, s-1\}$ and all $s \in \{1, \ldots, n\}$.
(that is, any smaller state than the current one is chosen uniformly at random).

(b) $\mathrm{E}(X_t - X_{t+1} \mid X_t = s) \geq \frac{1}{s}$ for all $s \in \{1, \ldots, n\}$.

(c) $\mathrm{E}(X_t - X_{t+1} \mid X_t = s) \geq \sqrt{s}$ for all $s \in \{1, \ldots, n\}$.

(d) **(bonus +2 points)** Find an example where the bound in (b) is tight, explain your answer.