

Randomised Algorithms  
Winter term 2022/2023, Exercise Sheet No. 9

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**Exercise 1.**

(a) As  $\mathbb{E}[X] = \frac{n}{2}$ , we have:

$$\mathbb{P}[X \geq \frac{3}{4}n] = \mathbb{P}[X \geq (1 + \frac{1}{2})\frac{n}{2}] = \mathbb{P}[X \geq (1 + \frac{1}{2})\mathbb{E}[X]]$$

And similarly:

$$\mathbb{P}[X \leq \frac{1}{4}n] = \mathbb{P}[X \leq (1 - \frac{1}{2})\mathbb{E}[X]]$$

Using the Chernoff bounds, (Ineq 2 then 3, from Slides 4), we conclude that (for  $\delta = 1/2 < 1$ ):

$$\mathbb{P}[X \leq \frac{1}{4}n] \leq e^{-\frac{n}{24}} \quad \text{and} \quad \mathbb{P}[X \geq \frac{3}{4}n] \leq e^{-\frac{n}{24}}$$

(b) Using the the simplified Chernoff bounds:

$$\begin{aligned} \mathbb{P}[X \geq n/2 + 2\sqrt{n}] &= \mathbb{P}[X \geq \frac{n}{2}(1 + \frac{4}{\sqrt{n}})] \\ &\leq e^{-\frac{n}{2}(\frac{4}{\sqrt{n}})^2 \frac{1}{3}} \quad (\text{Requires } \frac{4}{\sqrt{n}} < 1 \implies n > 16) \\ &= e^{-8/3} \end{aligned}$$

Using the additive Chernoff (No extra requirements needed):

$$\begin{aligned} \mathbb{P}[X \geq n/2 + 2\sqrt{n}] &\leq e^{-2(2\sqrt{n})^2 \frac{1}{n}} \\ &= e^{-8} \end{aligned}$$

(c) We let  $n_1 = \lfloor n/2 - \sqrt{n}/4 \rfloor$  and  $n_2 = \lfloor n/2 + \sqrt{n}/4 \rfloor$ , we have:

$$\begin{aligned}
\mathbb{P}[X \in [n/2 - \sqrt{n}/4, n/2 + \sqrt{n}/4]] &\leq \mathbb{P}[X \in [n_1, n_2]] \\
&= \sum_{k=n_1}^{n_2} \mathbb{P}[X = k] \\
&= \sum_{k=n_1}^{n_2} \binom{n}{k} \frac{1}{2^n} \\
&\leq \frac{1}{2^n} \sum_{k=n_1}^{n_2} 2^n \frac{\sqrt{2}}{\sqrt{n}} \quad (\text{Using the hint}) \\
&= \frac{\sqrt{2}}{\sqrt{n}} \sum_{k=n_1}^{n_2} 1 = \frac{\sqrt{2}}{\sqrt{n}} (n_2 - n_1 + 1) \\
&\leq \frac{\sqrt{2}}{\sqrt{n}} (n/2 + \sqrt{n}/4 - (n/2 - \sqrt{n}/4 - 1)) \quad (\text{using: } \lfloor x \rfloor - \lfloor y \rfloor \leq x - (y - 1)) \\
&= \frac{\sqrt{2}}{\sqrt{n}} (1 + \sqrt{n}/2) \\
&= \frac{\sqrt{2}}{\sqrt{n}} + \frac{1}{\sqrt{2}}
\end{aligned}$$

For  $n_0 = 24$ , we have for  $n \geq n_0$  :  $\frac{\sqrt{2}}{\sqrt{n}} + \frac{1}{\sqrt{2}} \leq \frac{\sqrt{2}}{\sqrt{24}} + \frac{1}{\sqrt{2}} \approx 0.995$

### Exercise 2.

Let  $X_i$  be the indicator random variable telling whether the  $i$ -th queried person is voting for Alice or not, we have  $A = \sum_{i=1}^k X_i$ . As  $X_i$ 's are drawn randomly with replacement, they are considered i.i.d realisations of the true distribution, i.e.,  $X_i \sim \text{Bern}(p)$ .

We also have:  $\mathbb{E}[A] = kp$ , hence,  $\hat{p} = A/k$  is an unbiased estimate of  $p$ .

We have:

$$\begin{aligned}
\mathbb{P}[|\hat{p} - p| > \epsilon] &= \mathbb{P}[|k\hat{p} - kp| > k\epsilon] \\
&= \mathbb{P}\left[|A - \mathbb{E}[A]| > \mathbb{E}[A] \frac{k\epsilon}{\mathbb{E}[A]}\right] \\
&= \mathbb{P}\left[|A - \mathbb{E}[A]| > \mathbb{E}[A] \frac{\epsilon}{p}\right] \\
&\leq 2 \exp\left(-\mathbb{E}[A] \left(\frac{\epsilon}{p}\right)^2 \frac{1}{3}\right) \quad (\text{using Chernoff bounds}) \\
&= 2 \exp\left(-kp \left(\frac{\epsilon}{p}\right)^2 \frac{1}{3}\right) \\
&= 2 \exp\left(-\frac{\epsilon^2 k}{3p}\right) \leq 2 \exp\left(-\frac{\epsilon^2 k}{3}\right)
\end{aligned}$$

For  $k = 5000$  and  $\epsilon = 0.05$ , we get an upperbound of: 0.031

### Exercise 3.

Let the RV  $X$  represent the sum of the  $n$  rolls, we have established previously that:  $\mathbb{E}[X] = \frac{7n}{2}$

Using Hoeffding's inequality leads to:

$$\begin{aligned}
\mathbb{P}[X \geq 4n] &= \mathbb{P}[X \geq E[X] + \frac{n}{2}] \\
&\leq \exp\left(-\frac{2(n/2)^2}{\sum_{i=1}^n (6-1)^2}\right) \\
&= e^{-n/50}
\end{aligned}$$

This tail that we obtained using Hoeffding's inequality decreases way faster than what we have obtained with Marokov and Chebychev's inequalities (respectively  $\frac{7}{8}$  and  $\frac{35}{3n}$ ).

#### Exercise 4.

Let's first establish a few results on  $d(u, v)$  for two random vertices of the hypercube.  $d(u, v)$  represents the number of bits  $v_i$  in  $v$  that are different than  $u_i$ , hence,  $d(u, v) \in \{0, \dots, n\}$ , and  $\mathbb{P}[d(u, v) = k] = \frac{\binom{n}{k}}{2^n}$ . (Why: How many ways can we pick  $k$  positions so we can flip their bits in  $v$  divided by the number of all configurations.)

By noticing that  $\sum_{k=0}^n \binom{n}{k} = 2^n$ , we can conclude that:

$$\mathbb{E}[d(u, v)] = \sum_{k=0}^n k \frac{\binom{n}{k}}{2^n} = \frac{n}{2}$$

Let's prove the following result, for any two random vertices  $u, v \in V$ ,  $\epsilon > 0$ :

$$\mathbb{P}\left[(1 - \epsilon)\frac{n}{2} \leq d(u, v) \leq (1 + \epsilon)\frac{n}{2}\right] \geq 1 - 2e^{-\frac{n\epsilon^2}{6}}$$

#### **Proof:**

Let's define the events  $E_1$  and  $E_2$  as follows:

$$E_1 = \{(1 - \epsilon)\frac{n}{2} \leq d(u, v)\} = \{(1 - \epsilon)\mathbb{E}[d(u, v)] \leq d(u, v)\}$$

And:

$$E_2 = \{(1 + \epsilon)\frac{n}{2} \geq d(u, v)\} = \{(1 + \epsilon)\mathbb{E}[d(u, v)] \geq d(u, v)\}$$

We are looking for to bound the probability:  $\mathbb{P}[E_1 \cap E_2]$

$$\begin{aligned}
\mathbb{P}[E_1 \cap E_2] &= 1 - \mathbb{P}[\overline{E_1} \cup \overline{E_2}] \\
&\geq 1 - (\mathbb{P}[\overline{E_1}] + \mathbb{P}[\overline{E_2}]) \\
&\geq 1 - 2e^{-\mathbb{E}[d(u, v)]\epsilon^2/3} \\
&\geq 1 - 2e^{-\frac{n\epsilon^2}{6}}
\end{aligned}$$

Here we used the union bound followed by Chernoff (Ineq 2. Slide 4 to bound  $\mathbb{P}[\overline{E_2}]$ , and Ineq 3 for  $\mathbb{P}[\overline{E_1}]$ ), assumig  $0 < \epsilon < 1$ .

Let  $V = \{v_1, \dots, v_{2^n}\}$ , we uniformly random pick  $n$  vertices from  $V$ . Now, let's define the RVs  $\{X_i\}$ ,  $\{Y_{i,j}\}$  and  $V(v_1, \dots, v_{2^n})$  as follows:

$$X_i = \begin{cases} 1 & \text{if } v_i \text{ was picked} \\ 0 & \text{Otherwise.} \end{cases} \quad Y_{i,j} = \begin{cases} 1 & \text{if } (1-\epsilon)\frac{n}{2} \leq d(v_i, v_j) \leq (1+\epsilon)\frac{n}{2} \\ 0 & \text{Otherwise.} \end{cases}$$

And finally:

$$S(v_1, \dots, v_{2^n}) = \sum_{i < j} X_i X_j Y_{i,j}$$

We need to proof that:

**Proof:** We have:

$$\begin{aligned} \mathbb{E}[S(v_1, \dots, v_{2^n})] &= \sum_{i < j} \mathbb{E}[X_i X_j Y_{i,j}] \\ &= \sum_{i < j} \mathbb{E}[X_i] \mathbb{E}[X_j] \mathbb{E}[Y_{i,j}] \quad (\text{every rv is independent from the other}) \\ &= \mathbb{E}[X_1]^2 \sum_{i < j} \mathbb{E}[Y_{i,j}] \\ &\geq \mathbb{E}[X_1]^2 (1 - 2e^{-\frac{n\epsilon^2}{6}}) \sum_{i < j} 1 \\ &= \mathbb{P}[\{\text{Probb. to pick } v_2\}]^2 (1 - 2e^{-\frac{n\epsilon^2}{6}}) \frac{2^{2n} - 2^n}{2} \\ &= \left(\frac{n}{2^n}\right)^2 (1 - 2e^{-\frac{n\epsilon^2}{6}}) \frac{2^{2n} - 2^n}{2} = \frac{n^2}{2} (1 - 2e^{-\frac{n\epsilon^2}{6}}) \left(1 - \frac{1}{2^n}\right) \\ &= \frac{n(n-1)}{2} \frac{n}{n-1} (1 - 2e^{-\frac{n\epsilon^2}{6}}) \left(1 - \frac{1}{2^n}\right) \\ &= \frac{n(n-1)}{2} \left(1 + \frac{1}{n-1}\right) (1 - 2e^{-\frac{n\epsilon^2}{6}}) \left(1 - \frac{1}{2^n}\right) \end{aligned}$$

**fix this:**

For  $n$  big enough,  $(1 + \frac{1}{n-1})(1 - 2e^{-\frac{n\epsilon^2}{6}})(1 - \frac{1}{2^n})$  is guarenteed to be  $\geq 1$

Hence,  $\forall \epsilon > 0$ , and with  $n$  big enough,  $\mathbb{E}[S(v_1, \dots, v_{2^n})] \geq \frac{n(n-1)}{2}$ , hence there exist a configuration of  $X, Y$  s.t  $v() = \frac{n(n-1)}{2}$ , which can only happen if all the  $n$  picks had  $Y_{i,j} = 1$