Randomised Algorithms

Winter term 2022/2023, Exercise Sheet No. 2

Hand-out: Mon, 24. Oct.
Hand-in: Sun, 30. Oct.

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Exercise 1 [4 points]

Give an example of two random variables X, Y for which $\mathrm{E}(\min(X, Y)) \neq \min(\mathrm{E}(X), \mathrm{E}(Y))$.

Exercise 2 [6 points]

Recall that a discrete random variable X is a function from the countable sample space S to the real line, eg. $X: S \to \mathbb{R}$ and $X(s) \in \mathbb{R}$ for any $s \in S$, and the expectation is the sum $\mathrm{E}(X) := \sum_{s \in S} X(s)\mathrm{Prob}(\{s\})$.

(a) Let X, Y be two discrete random variables with finite expectations. If $X(s) \leq Y(s)$ for all $s \in S$ (this can be shortly denoted as $X \leq Y$), then what can we say about the relation between $\mathrm{E}(X)$ and $\mathrm{E}(Y)$?

(b) Use the answer in (a) to show that $\mathrm{E}(\min(X, Y)) \leq \mathrm{E}(X)$, and that $\mathrm{E}(\min(X, Y)) \leq \mathrm{E}(Y)$. From these, what can we say about the relation between $\mathrm{E}(\min(X, Y))$ and $\min(\mathrm{E}(X), \mathrm{E}(Y))$?

(c) Let X be a positive discrete random variable (ie. $X: S \to \mathbb{R}^+$) and $a > 0$, define $Y := a \cdot I_{\{X \geq a\}}$, where I_A denotes the indicator variable of event A. Show that $Y \leq X$, then use the answer in (a) to prove that

$$\mathrm{Prob}(X \geq a) \leq \mathrm{E}(X)/a$$

this is called Markov's inequality.

Exercise 3 [4 points]

Derive from the statement "$\mathrm{E}(Z) \geq m/2$" for the algorithm "RSAM" that there exists at least one assignment of literals that satisfies at least $m/2$ clauses.

Exercise 4 [6 points]

You're in line to board a plane. The first passenger to board is drunk and disorientated. Without looking at his seat reservation, he uniformly chooses a seat at random and sits down. To keep the peace, the flight attendants instruct each passenger to take their reserved seat if it is available, and otherwise uniformly select a free seat at random. The plane has 100 seats and is fully booked. You are the last person to board. What is the probability that your reserved seat is still available?
Hint: Consider which events (that is, which choices of seats) are relevant to the decision and which events will postpone the decision.