Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 7

Authors:
Ben Ayad, Mohamed Ayoub
Kamzon, Noureddine

December 1, 2022

Exercise 1.

Exercise 2.

Exercise 3.

Exercise 4.

The case where $n = 1$, is trivial, we will start our induction from $n = 2$.

Let's say $A = [a_1, a_2]$, we have two permutations in total, $A_1 = A$ and $A_2 = [a_2, a_1]$. In this case, the main loop of the algorithm has two runs, but the second run (and generally when $i = n$) doesn't not change the outcome of the list. Hence, the list is only changed in the first step ($i = 1$). Which yields two possibilities with equal probability, we either pick $j = 2$ (the algorithm swipes the two elements, and outputs, A_2), or we pick $j = 1$, and nothing changes to the list, and the algorithm outputs A_1.

Now, we suppose that the algorithm is correct for some $n \geq 2$, and prove that the algorithm is correct for an input of size $n + 1$:

Let, T be a list of size $n + 1$, and let $L = [L_1, L']$ be a random permutation of T, where L_1 is the first element of L, and L', is random permutaion of $T/\{L_1\}$ (i.e., the list T minus the element L_1). We need to prove that $FastRandomPermutation(T)$, has a probability of $\frac{1}{(n+1)!}$ of outputing L.

For the first step, the algorithm picks at uniformely random an element of T, and swipes it with the first elemenet of T (T_1), Hence, we have get a probability of $\frac{1}{n+1}$ of actually picking L_1. For the rest, we are left with a list of size n, $T/\{L_1\}$, and since L' is a valid permutation of $T/\{L_1\}$, we get by induction, that the probability pof picking L' in the steps $i \in \{2, \dots, n+1\}$ is $\frac{1}{n!}$.

By the the multiplication theorem of probability, we conclude that the probability of picking L as a permutaion is $\frac{1}{n+1}\frac{1}{n!} = \frac{1}{(n+1)!}$. This completes our proof by induction.