Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 9

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Exercise 1.
Let's first establish a few results on $d(u,v)$ for two random vertices of the hypercube. $d(u,v)$ represents the number of bits v_i in v that are diffrent than u_i, hence, $d(u,v) \in \{0,\dots,n\}$, and $\mathbb{P}[d(u,v)=k] = \frac{\binom{n}{k}}{2^n}$.(Why: How many ways can we pick k positions so we can flip their bits in v divided by the number of all configurations.)

By noticing that $\sum_{i=1}^{n} k\binom{n}{k} = n2^{n-1}$, we can conclude that:

$$\mathbb{E}[d(u,v)] = \sum_{i=1}^{n} k\frac{\binom{n}{k}}{2^n} = \frac{n}{2}$$

Let's prove the following result, for any two random vertices $u, v \in V$, $\epsilon > 0$:

$$\mathbb{P}\left[(1-\epsilon)\frac{n}{2} \leq d(u,v) \leq (1+\epsilon)\frac{n}{2}\right] \geq 1 - 2e^{-\frac{n\epsilon^2}{6}}$$

Proof:
Let's define the events E_1 and E_2 as follows:

$$E_1 = \{(1-\epsilon)\frac{n}{2} \leq d(u,v)\} = \{(1-\epsilon)\mathbb{E}[d(u,v)] \leq d(u,v)\}$$

And:

$$E_2 = \{(1+\epsilon)\frac{n}{2} \geq d(u,v)\} = \{(1+\epsilon)\mathbb{E}[d(u,v)] \geq d(u,v)\}$$

We are looking for to bound the probablity: $\mathbb{P}[E_1 \cap E_2]$

$$\begin{aligned}
\mathbb{P}[E_1 \cap E_2] &= 1 - \mathbb{P}[\overline{E_1} \cup \overline{E_2}] \\
&\geq 1 - (\mathbb{P}[\overline{E_1}] + \mathbb{P}[\overline{E_2}]) \\
&\geq 1 - 2e^{-\mathbb{E}[d(u,v)]\epsilon^2/3} \\
&\geq 1 - 2e^{-\frac{n\epsilon^2}{6}}
\end{aligned}$$

Here we used the union bound followed by Chernoff (Ineq 2. Slide 4 to bound $\mathbb{P}[\overline{E_2}]$, and Ineq 3 for $\mathbb{P}[\overline{E_1}]$), assumig $0 < \epsilon < 1$.

Let $V = \{v_1, \dots, v_{2^n}\}$, ***we uniformely random pick n vertices from V*** . *Now, let's define the RVs* $\{X_i\}$, $\{Y_{i,j}\}$ *and* $V(v_1, \dots, v_{2^n})$ *as follows:*

$$X_i = \begin{cases} 1 & \textit{if } v_i \textit{ was picked} \\ 0 & \textit{Otherwise.} \end{cases} \qquad Y_{i,j} = \begin{cases} 1 & \textit{if} \quad (1-\epsilon)\frac{n}{2} \leq d(v_i, v_j) \leq (1+\epsilon)\frac{n}{2} \\ 0 & \textit{Otherwise.} \end{cases}$$

And finally:

$$S(v_1, \dots, v_{2^n}) = \sum_{i<j} X_i X_j Y_{i,j}$$

We have:

$$\begin{aligned}
\mathbb{E}[S(v_1, \dots, v_{2^n})] &= \sum_{i<j} \mathbb{E}[X_i X_j Y_{i,j}] \\
&= \sum_{i<j} \mathbb{E}[X_i]\mathbb{E}[X_j]\mathbb{E}[Y_{i,j}] \quad \textit{(every rv is independent from the other)} \\
&= \mathbb{E}[X_1]^2 \sum_{i<j} \mathbb{E}[Y_{i,j}] \\
&\geq \mathbb{E}[X_1]^2 (1 - 2e^{-\frac{n\epsilon^2}{6}}) \sum_{i<j} 1 \\
&= \mathbb{P}[\{\textit{Probb. to pick } v_2\}]^2 (1 - 2e^{-\frac{n\epsilon^2}{6}}) \frac{2^{2n} - 2^n}{2} \\
&= \left(\frac{n}{2^n}\right)^2 (1 - 2e^{-\frac{n\epsilon^2}{6}}) \frac{2^{2n} - 2^n}{2} = \frac{n^2}{2}(1 - 2e^{-\frac{n\epsilon^2}{6}})(1 - \frac{1}{2^n}) \\
&= \frac{n(n-1)}{2} \frac{n}{n-1} (1 - 2e^{-\frac{n\epsilon^2}{6}})(1 - \frac{1}{2^n}) \\
&= \frac{n(n-1)}{2} (1 + \frac{1}{n-1})(1 - 2e^{-\frac{n\epsilon^2}{6}})(1 - \frac{1}{2^n})
\end{aligned}$$

fix this:

For n *big enough,* $(1 + \frac{1}{n-1})(1 - 2e^{-\frac{n\epsilon^2}{6}})(1 - \frac{1}{2^n})$ *is guarenteed to be* ≥ 1

Hence, $\forall \epsilon > 0$, *and with* n *big enough,* $\mathbb{E}[S(v_1, \dots, v_{2^n})] \geq \frac{n(n-1)}{2}$, *hence there exist a configuration of* X, Y *s.t* $v() = \frac{n(n-1)}{2}$, *which can only happen if all the* n *picks had* $Y_{i,j} = 1$