Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 5

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Exercise 1.
(a) *We have established in the notes that the probability of not making any mistakes after* $n - t$ *contractions is at least* $\frac{t(t-1)}{n(n-1)}$*, in this case, when* $t = n - 1$*, we would get that this probability is* $\frac{n-2}{n} = 1 - \frac{2}{n}$*, which is in fact equal to* q_n*.*

(b) *We denote the probability that* S_1 *(resp* S_2*) is correct as* P_1 *(resp.* P_2*), we have:*

$$\begin{aligned} P_1 &= \mathbb{P}[\{\textit{Contraction at L4 is correct}\}]\mathbb{P}[\{\textit{Output of L5 is correct}\}|\{\textit{Contraction at L4 is correct}\}] \\ &= q_n P(n-1) \\ P_2 &= P(n) \end{aligned}$$

The algorithm is succesful if either the return at line 7 was a succes (i.e., S_1 *is a success) or the return at line 10 was a succes, which yiled the following:*

$$\begin{aligned} p(n) &= \mathbb{P}[\{\textit{Entered L7}\}]\mathbb{P}[\{\textit{Succes}\}|\{\textit{entered L7}\}] + \mathbb{P}[\{\textit{Didn't Enter L7}\}]\mathbb{P}[\{\textit{Succes}\}|\{\textit{Didnt enter L7}\}] \\ &= q_n\mathbb{P}[\{\textit{Succes}\}|\{\textit{entered L7}\}] + (1-q_n)\mathbb{P}[\{\textit{Succes}\}|\{\textit{Didnt enter L7}\}] \\ &= q_n\mathbb{P}[\{S_1 \textit{ is succesful}\}] + (1-q_n)\mathbb{P}[\{\textit{The best of } S_1,\, S_2 \textit{ was succesful}\}] \\ &= q_n\mathbb{P}[\{S_1 \textit{ is succesful}\}] + (1-q_n)(1-\mathbb{P}[\{S_1 \textit{ and } S_2 \textit{ failed}\}]) \\ &= q_n P_1 + (1-q_n)[1-(1-P_1)(1-P_2)] \qquad \textit{(We will just keep re-arranging after now)} \\ &= q_n^2 P(n-1) + (1-q_n)[1-(1-q_nP(n-1))(1-P(n))] \\ &= q_n^2 P(n-1) + (1-q_n)[P(n) + q_nP(n-1) - q_nP(n-1)P(n)] \\ &= q_n P(n-1) + (1-q_n)P(n) - q_n(1-q_n)P(n-1)P(n) \end{aligned}$$

$$\Longrightarrow$$

$$q_nP(n) = q_nP(n-1) - q_n(1-q_n)P(n-1)P(n)$$

$$\Longrightarrow$$

$$P(n) = P(n-1) - (1-q_n)P(n-1)P(n) \qquad (\textit{We would verify line 1 if } n = 2, \textit{ hence } q_n \textit{ would never equal } 0\ )$$

(c) *By dividing the equation that we derived in the last question by* $P(k)P(k-1)$*, we get the following, for* $k \in \{3, \ldots, n\}$*:*

$$\frac{1}{P(k-1)} - \frac{1}{P(k)} = -\frac{2}{k}$$

$$\Longrightarrow$$

$$\sum_{k=3}^{n} \frac{1}{P(k-1)} - \sum_{k=3}^{n} \frac{1}{P(k)} = -\sum_{k=3}^{n} \frac{2}{k}$$

$$\frac{1}{P(2)} - \frac{1}{P(n)} = -\sum_{k=3}^{n} \frac{2}{k}$$

$$\Longrightarrow$$

$$P(n) = \frac{1}{\sum_{k=3}^{n} \frac{2}{k} + 1}$$

Comparisons of the probability of succes with FastCut:

- *GeoContraction has a* $\Theta(\frac{1}{\log(n)})$ *(Assuming the bounds are exact like suggested in* ***(b)*** *)*
- *FastCut had a* $\Omega(\frac{1}{\log(n)})$

Exercise 2.

Let X_t *represent the number of walks that are left to reach home at step* t*. Obviously* $X_t \in \{0, \ldots, n\}$*, and we are interested in computing* $\mathbb{E}[T]$ *where* $T = \inf\{t \geq 0 | X_t = 0\}$*.*

We have the following:

$$\begin{aligned}\mathbb{E}_{(X_t, X_{t+1})}[X_t - X_{t+1} | X_t = s] &= \mathbb{E}_{X_{t+1}}[s - X_{t+1} | X_t = s] \\ &= s - \mathbb{E}[X_{t+1} | X_t = s]\end{aligned}$$

If $s = n$ *(we are at the bar),* X_{t+1} *has two options* $\{n, n-1\}$

$$\begin{aligned}\mathbb{E}[X_t - X_{t+1} | X_t = s] &= n - n\mathbb{P}[X_{t+1} = n | X_t = s] - (n-1)\mathbb{P}[X_{t+1} = n-1 | X_t = s] \\ &= \frac{1}{5}\end{aligned}$$

If $0 < s < n$*,* X_{t+1} *has two options* $\{s-1, s+1\}$

$$\begin{aligned}\mathbb{E}[X_t - X_{t+1} | X_t = s] &= s - (s-1)\mathbb{P}[X_{t+1} = s-1 | X_t = s] - (s+1)\mathbb{P}[X_{t+1} = s+1 | X_t = s] \\ &= \frac{3}{5} + \frac{2}{5} \\ &= 1\end{aligned}$$

Hence, we get, for all $t > 0, s \neq 0$*:*

$$\frac{1}{5} \leq \mathbb{E}[X_t - X_{t+1} | X_t = s] \leq 1$$

Which yields the following bounds, assuming $X_0 = n$*:*

$$n \leq \mathbb{E}[T] \leq 5n$$

Exercise 3.
(a) *As the state 0 is never reacheable and $X_0 = 1$, T could be formulated as follows:*

$$T = \inf\{t \geq 1 | X_t = -1\}$$

i.e, the first time we reach the state $s = -1$ (our definition of "succcess", speaking from a geometric distribution terminology). By noting that each X_t is a Bernoulli experiment with $p = 1/2$, we conclude that $\mathbb{E}[T] = 1/p = 2$.

(b) *The state we are interested to reach in the definition of T is -1, in the theorem the state of interest is $s = 0$, and X_t only takes positive values in the theorem statement, again, here X_t can take the value -1.*

We have for $s \neq 0$: $\mathbb{E}[X_{t+1} | X_t = s] = 0$, and hence:

$$\mathbb{E}[X_t - X_{t+1} | X_t = s] = s - \mathbb{E}[X_{t+1} | X_t = s] = s$$

Which yields, $\mathbb{E}[X_t - X_{t+1} | X_t = s] \leq 1$ for all $s \neq 0$. using the additive drift theorem would lead to conclude that $1 \leq E[T]$

To make it work, *we define the random process (Y_t) as $Y_t = X_t + 1$, clearly $Y_t \in \{0, 1, 2\}$, and*

$$\begin{aligned} T &= \inf\{t \geq 0 | X_t = -1\} \\ &= \inf\{t \geq 0 | X_t + 1 = 0\} \\ &= \inf\{t \geq 0 | Y_t = 0\} \end{aligned}$$

The way we defined $(Y_t)_{t \geq 0}$, and how we formulated T using (Y_t), matches the theorem's assumptions. Moreover we have: $[s \neq 0 \wedge \mathbb{P}(Y_t = s) > 0 \implies s = 2]$, and hence we only need to verify the theorem's conditions for $s = 2$:

$$\begin{aligned} \mathbb{E}[Y_t - Y_{t+1} | Y_t = 2] &= 2 - \mathbb{E}[Y_{t+1} | Y_t = 2] \qquad \textit{((}Y_{t+1} \textit{ can only move to 0 or 2)} \\ &= 2 - 2\mathbb{P}[Y_{t+1} = 2 | Y_t = 2] \\ &= 2 - 2\frac{1}{2} \\ &= 1 \end{aligned}$$

And hence, using the additive drift theorem, we get:

$$\frac{\mathbb{E}[Y_0]}{1} \leq \mathbb{E}[T] \leq \frac{\mathbb{E}[Y_0]}{1} \implies \mathbb{E}[T] = 2, \textit{since } \mathbb{E}[Y_0] = 2$$

*Which is the same as the answer that we got in **(a)**.*

Exercise 4.
(a) *Let's first compute $\mathbb{E}[X_t - X_{t+1} | X_t = s]$ for some $0 < s \leq n$:*

$$\begin{aligned} \mathbb{E}[X_t - X_{t+1} | X_t = s] &= s - \sum_{i=0}^{s-1} \mathbb{P}[X_{t+1} = i | X_t = s] i \\ &= s - \frac{1}{s} \sum_{i=0}^{s-1} i \\ &= \frac{s+1}{2} \end{aligned}$$

Hence, we have $\mathbb{E}[X_t - X_{t+1}|X_t = s] \geq \frac{s+1}{2}$*, for all* $0 < s \leq n$*, with* $h(s) : s \mapsto \frac{s+1}{2}$ *being monotonically increasing, we have verified all assumptions of the Variable Drift Theorem, hence, we can conclude the following:*

$$\begin{aligned}
\mathbb{E}[T] &\leq \frac{1}{h(1)} + \int_1^n \frac{1}{h(x)}\, dx \\
&= 1 + \int_1^n \frac{2}{x+1}\, dx \\
&= 1 + 2\int_2^{n+1} (\ln(x))'\, dx \\
&= 1 + 2(\ln(n+1) - \ln(2)) \\
&\leq 2\ln(n+1) \qquad (1 - 2\ln(2) \approx -0.39)
\end{aligned}$$

Using the Additive Drift Theorem (since $\mathbb{E}[X_t - X_{t+1}|X_t = s] \geq 1$*, for all* $0 < s \leq n$*) would lead to conclude that* $\mathbb{E}[T] \leq n$*, which is less tight than the gap that we obtained using the Variable Drift Theorem.*

(b) *The function* $h(s)\colon s \mapsto \frac{1}{s}$ *isn't monotonically increasing, hence we can't use the Variable Drift Theorem.*

We have, for $0 < s \leq n$*,* $\quad \mathbb{E}[X_t - X_{t+1}|X_t = s] \geq \frac{1}{n}$*, hence, using the Additive Drift Theorem, yields the following:* $\mathbb{E}[T] \leq n^2$

(c) *We have,* $h(s) : s \mapsto \sqrt{s}$ *is monotonically increasing for all* $0 < s \leq n$*, using the Variable Drift Analysis, we get:*

$$\begin{aligned}
\mathbb{E}[T] &\leq \frac{1}{h(1)} + \int_1^n x^{-\frac{1}{2}}\, dx \\
&\leq 1 + 2\int_1^n (\sqrt{x})'\, dx \\
&\leq 2\sqrt{n} - 1
\end{aligned}$$

(d) *Consider the random process* $(X_t)_{t\geq 0}$ *on state space* $\{0, 1, \ldots, n\}$*, where knowing that* $X_t = s$ *for* $s > 0$*,* X_{t+1} *is defined as follows:*

$$X_{t+1} = \begin{cases} s-1 & \textit{with probability } \frac{1}{s} \\ s & \textit{with probability } 1 - \frac{1}{s} \end{cases}$$

For $s > 0$ *and* $t \geq 1$*:*

$$\begin{aligned}
\mathbb{E}[X_t - X_{t+1}|X_t = s] &= s - \mathbb{E}[X_{t+1}|X_t = s] \\
&= s - [(s-1)\frac{1}{s} + s(1 - \frac{1}{s})] \\
&= s - [1 - 1/s + s - 1] \\
&= \frac{1}{s}
\end{aligned}$$