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Randomised Algorithms

Winter term 2022/2023, Exercise Sheet No. 10

Hand-out: Mon, 19. Jan.
Hand-in: Sun, 25. Jan.

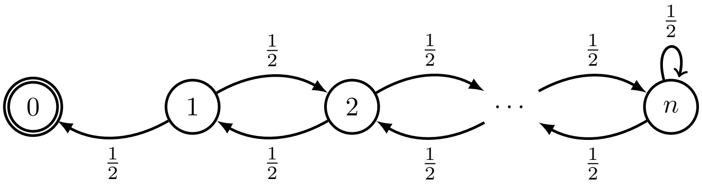
Exercise 1 [6 points]

Recall that the random walk algorithm for 2-SAT takes at most n^2 steps in expectation to find a satisfying assignment, if one exists. Assume we have a 2-SAT instance for which we know that there exists a satisfying assignment a^* that satisfies exactly one literal in every clause.

(a) Show that in addition to a^* there exists another satisfied assignment.

(b) Use this additional information to improve the leading constant in the quadratic expected running time of the algorithm on the instance.

Exercise 2 [6 points]



Consider the above variant of a fair random walk, here the state n only has probability $1/2$ of reflecting and otherwise it self-loops. Examine the steps of the analysis in the lecture for the standard random walk with a full reflecting state to spot the differences with the analysis of this variant by answering the following questions (you do not need to repeat all the arguments from the lecture).

(a) What are the new recurrence relations on E_i, ie. the expected time to reach 0 starting from state i, and the new recurrence relations on D_i where $D_i := E_i - E_{i-1}$?

(b) Solve the recurrence relations to find the closed-form expression for E_i.

(c) Use the above result to show that the expected number of games to finish a fair Gambler's Ruin process with two players is still $\ell_1\ell_2$ even when $\ell_1 + \ell_2$ is an odd number, ie. $\ell_1 + \ell_2 = 2n + 1$ for some $n \in \mathbb{N}$.

Exercise 3 [8 points]

Consider repeatedly and independently tossing a fair coin until the specific pattern TTHT is observed for the first time in the most recent four tosses. In this exercise, we will compute first exactly the expected number of tosses for this process by a Markov chain analysis, then discuss how to bound this expected number from above.

(a) Construct a Markov chain that describes the matching status of the process: the starting state is the empty pattern (nothing has been tossed, or currently the first toss is still wrong), the next state is when the first toss (T) is correct, and the third state is when the first and second tosses (TT) are correct, and so on. Draw the relevant transition arcs and describe their associated probabilities.

(b) Build the system of linear equations that links the expected waiting times E_i to reach the final state (ie. all consecutive tosses are correct) from starting state i. Solve this system to find E_0.

(c) Can we consider a simpler argument that in each time step after the first three tosses, the most recent four tosses have some fixed probability of generating the pattern? What is expected waiting time in that case, for example assuming the standard argument with a geometric distribution? Explain your answer.