Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 5

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Exercise 1.

(a) *We have established in the notes that the probability of not making any mistakes after $n-t$ contractions is at least $\frac{t(t-1)}{n(n-1)}$, in this case, when $t = n-1$, we would get that this probability is $\frac{n-2}{n} = 1 - \frac{2}{n}$, which is in fact equal to q_n.*

(b) *We denote the probability that S_1 (resp S_2) is correct as P_1 (resp. P_2), we have:*

$$\begin{aligned} P_1 &= \mathbb{P}[\{\textit{Output of L5 is correct}\}|\{\textit{Contraction at L4 is correct}\}]\mathbb{P}[\{\textit{Contraction at L4 is correct}\}] \\ &= P(n-1)q_n \\ P_2 &= P(n) \end{aligned}$$

The algorithm is succesful if either the return at line 7 was a succes (i.e., S_1 is a success) or the return at line 10 was a succes, which yiled the following:

$$\begin{aligned} p(n) &= \mathbb{P}[\{\textit{Entered L7}\}]\mathbb{P}[\{\textit{Succes}\}|\{\textit{entered L7}\}] + \mathbb{P}[\{\textit{Didn't Enter L7}\}]\mathbb{P}[\{\textit{Succes}\}|\{\textit{Didnt enter L7}\}] \\ &= q_n\mathbb{P}[\{\textit{Succes}\}|\{\textit{entered L7}\}] + (1-q_n)\mathbb{P}[\{\textit{Succes}\}|\{\textit{Didnt enter L7}\}] \\ &= q_n\mathbb{P}[\{S_1 \textit{ is succesful}\}] + (1-q_n)\mathbb{P}[\{\textit{The best of } S_1,\ S_2 \textit{ was succesful}\}] \\ &= q_n\mathbb{P}[\{S_1 \textit{ is succesful}\}] + (1-q_n)(1-\mathbb{P}[\{S_1 \textit{ and } S_2 \textit{ failed}\}]) \\ &= q_nP_1 + (1-q_n)[1-(1-P_1)(1-P_2)] \qquad \textit{(We will just keep re-arranging after now)} \\ &= q_n^2P(n-1) + (1-q_n)[1-(1-q_nP(n-1))(1-P(n))] \\ &= q_n^2P(n-1) + (1-q_n)[P(n) + q_nP(n-1) - q_nP(n-1)P(n)] \\ &= q_nP(n-1) + (1-q_n)P(n) - q_n(1-q_n)P(n-1)P(n) \end{aligned}$$

$$\Longrightarrow$$

$$q_nP(n) = q_nP(n-1) - q_n(1-q_n)P(n-1)P(n)$$

$$\Longrightarrow$$

$$P(n) = P(n-1) - (1-q_n)P(n-1)P(n) \qquad (q_n \neq 0 \textit{ since } n \neq 2 \textit{ as we would verify line 1 if } n = 2)$$

(c) *By dividing the equation that we derived in the last question by $P(n)P(n-1)$, we get the following, for $k \in \{3, \ldots, n\}$:*

$$\frac{1}{P(k-1)} - \frac{1}{P(k)} = -\frac{2}{k}$$
$$\Longrightarrow$$
$$\sum_{k=3}^{n} \frac{1}{P(k-1)} - \sum_{k=3}^{n} \frac{1}{P(k)} = -\sum_{k=3}^{n} \frac{2}{k}$$
$$\frac{1}{P(2)} - \frac{1}{P(n)} = -\sum_{k=3}^{n} \frac{2}{k}$$
$$\Longrightarrow$$
$$P(n) = \frac{1}{\sum_{k=3}^{n} \frac{2}{k} - 1}$$

Comparisons of the probability of succes with FastCut:

- *GeoContraction has a* $\Theta(\frac{1}{\log(n)})$ *(Assuming the bouds are exact like suggested in* ***(b)*** *)*
- *FastCut had a* $\Omega(\frac{1}{\log(n)})$

Exercise 2.
(a)
(b)
(c)

Exercise 3.
(a)
(b)

Exercise 4.
(a)
(b)
(c)
(d)