

Randomised Algorithms

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Exercise 1.

(a) The algorithm described briefly as follows, has a **cubic** runtime.

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1: for  $1 \leq i \leq n, 1 \leq j \leq n$  do
2:    $\alpha_{i,j} \leftarrow \text{sum}([A_{i,k}B_{k,j} : k \in \{1, \dots, n\}])$ 
3:    $\alpha_{i,j} \leftarrow \alpha_{i,j} - c_{i,j}$ 
4:   if  $\alpha_{i,j} \neq 0$  then
5:     return 0
6:   end if
7: end for
8: return 1

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(b) Computing x , y , and z requires $\mathcal{O}(n^2)$ each, and computing $t = y - z$ requires $\mathcal{O}(n)$, hence, the asymptotic runtime of this RA is $\mathcal{O}(n^2)$.

(c) If $r \neq 0$, the event \mathcal{E} implies the following:

- $r \in \text{Ker}(D)$
- r is orthogonal to all rows of D
- Since $D \neq 0$, there exists at least one row $d_i \neq 0$ s.t: $d_i^\top r = 0$
- $\sum_{i=1}^n r_i D_{.,i} = 0$

We have:

$$\begin{aligned}
 \mathbb{P}[\mathcal{E}] &= \mathbb{P}[Dr = 0 | r = 0] \mathbb{P}[r = 0] + \mathbb{P}[Dr = 0 | r \neq 0] \mathbb{P}[r \neq 0] \\
 &= \frac{1}{3^n} + \mathbb{P}[Dr = 0 | r \neq 0] \left(1 - \frac{1}{3^n}\right) \quad [1]
 \end{aligned}$$

For some $k \leq n$, we let, d_1, \dots, d_k be the rows of D that are not equal to 0. We have:

$$\begin{aligned}
 \mathbb{P}[Dr = 0 | r \neq 0] &= \mathbb{P}[d_1^\top r = 0, \dots, d_k^\top r = 0 | r \neq 0] \\
 &\leq \mathbb{P}[d_1^\top r = 0 | r \neq 0] \quad [2]
 \end{aligned}$$

Let $d_{1,j}$ be the last element of d_1 not equal to 0, we have: $d_1^\top r = 0$ **if and only if** after $j - 1$ picks of r_1, \dots, r_{j-1} , the j^{th} pick (i.e., r_j) is chosen s.t: $-d_{1,j}r_j = \sum_{i=1}^{j-1} d_{1,i}r_i$. Which yields the following:

$$\begin{aligned}
\mathbb{P}[d_1^\top r = 0 | r \neq 0] &= \mathbb{P}\left[r_j = \sum_{i=1}^{j-1} \frac{-d_{1,i}}{d_{1,j}} r_i | r \neq 0\right] \\
&= \frac{\mathbb{P}\left[r_j = \sum_{i=1}^{j-1} \frac{-d_{1,i}}{d_{1,j}} r_i, r \neq 0\right]}{\mathbb{P}[r \neq 0]} \quad (\text{We refer to } r' = [r_1, r_2, \dots, r_{j-1}]) \\
&= \frac{\mathbb{P}\left[r_j = \sum_{i=1}^{j-1} \frac{-d_{1,i}}{d_{1,j}} r_i, r \neq 0, r' \neq 0\right] \mathbb{P}[r' \neq 0] + \mathbb{P}\left[r_j = \sum_{i=1}^{j-1} \frac{-d_{1,i}}{d_{1,j}} r_i, r \neq 0, r' = 0\right] \mathbb{P}[r' = 0]}{\mathbb{P}[r \neq 0]}
\end{aligned}$$

For the second term, we have: $(r' = 0 \implies r_j = 0)$ and $(r' \neq 0 \text{ and } r = 0 \implies r_j \neq 0)$, hence the second term is equal to 0.

$$\begin{aligned}
\mathbb{P}[d_1^\top r = 0 | r \neq 0] &\leq \mathbb{P}\left[r_j = \sum_{i=1}^{j-1} \frac{-d_{1,i}}{d_{1,j}} r_i, r \neq 0, r' \neq 0\right] \frac{\mathbb{P}[r' \neq 0]}{\mathbb{P}[r \neq 0]} \\
&\leq \frac{1}{3} \frac{\mathbb{P}[r' \neq 0]}{\mathbb{P}[r \neq 0]} \\
&= \frac{1}{3} \frac{3^{n-1} - 1}{3^{n-1}} \frac{3^n}{3^n - 1} \\
&= \frac{3^{n-1} - 1}{3^n - 1}
\end{aligned}$$

Using [2], we get $\mathbb{P}[Dr = 0 | r \neq 0] \leq \frac{3^{n-1} - 1}{3^n - 1}$, and from [1], we conclude that:

$$\mathbb{P}[\mathcal{E}] \leq \frac{1}{3^n} + \frac{3^{n-1} - 1}{3^n - 1} \left(1 - \frac{1}{3^n}\right) = \frac{1}{3}$$

(d) We can reduce the probability by amplification.

Exercise 2.

(a) We consider the following Algorithm T_k , we choose $k = \lceil \sqrt{2n} \rceil + 1$.

- 1: We generate k images, as follows: $e_1 = h(1), \dots, e_k = h(k)$
- 2: **if** A collision happens **then**
- 3: **return** $s = n$
- 4: **end if**
- 5: **return** $s = n^2$

(b) We will refer to the probability of T_k being correct as $\mathbb{P}[T]$, we have:

$$\begin{aligned}
\mathbb{P}[T] &= \mathbb{P}[s = n] \mathbb{P}[T | s = n] + \mathbb{P}[s = n^2] \mathbb{P}[T | s = n^2] \\
&= \frac{1}{2} (\mathbb{P}[T | s = n] + \mathbb{P}[T | s = n^2]) \\
&= \frac{1}{2} (\mathbb{P}[A \text{ collision happening in } T_k | s = n] + \mathbb{P}[No \text{ collisions happening in } T_k | s = n^2])
\end{aligned}$$

From the lecture notes, we have, $\mathbb{P}[A \text{ collision happening in } T_k | s = n] > \frac{1}{2}$, also:

$$\begin{aligned}
 \mathbb{P}[\text{No collisions happening in } T_k | s = n^2] &= \prod_{i=2}^k \left(1 - \frac{i-1}{n^2}\right) \\
 &\geq \left(1 - \frac{k-1}{n^2}\right)^k \quad (\text{using the Bernoulli Inequality}) \\
 &\geq 1 - \frac{k(k-1)}{n^2} \\
 &> \frac{1}{2} \quad (\text{as } \frac{n}{2} > k)
 \end{aligned}$$

Which finally yields:

$$\begin{aligned}
 \mathbb{P}[T] &= \frac{1}{2} (\mathbb{P}[A \text{ collision happening in } T_k | s = n] + \mathbb{P}[\text{No collisions happening in } T_k | s = n^2]) \\
 &> \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2}\right) \\
 &= \frac{1}{2}
 \end{aligned}$$

Exercise 3.

(a) We have:

$$\begin{aligned}
 \sum_{i \geq 1} \mathbb{P}[X \geq i] &= \sum_{i \geq 1} \sum_{j=i}^{\infty} \mathbb{P}[X = j] \\
 &= \sum_{j \geq 1} \sum_{i=1}^j \mathbb{P}[X = j] \\
 &= \sum_{j \geq 1} \mathbb{P}[X = j] \sum_{i=1}^j 1 \\
 &= \sum_{j \geq 1} j \mathbb{P}[X = j] = \mathbb{E}[X]
 \end{aligned}$$

If X can only take negative values, then:

$$\begin{aligned}
 \mathbb{E}[X] &= -\mathbb{E}[-X] \\
 &= -\sum_{j \geq 1} \mathbb{P}[-X \geq j] \\
 &= -\sum_{j \geq 1} \mathbb{P}[X \leq -j]
 \end{aligned}$$

(b) Let $\mathbb{P}[X > 0] = p_X$, and $\mathbb{P}[Y > 0] = p_Y$, we have: $p_X \leq p_Y$, we suppose that $0 < p_X < 1$ and $0 < p_Y < 1$:

$$\begin{aligned}
 \mathbb{E}[Y] - \mathbb{E}[X] &= p_Y \mathbb{E}[Y|Y > 0] + (1 - p_Y) \mathbb{E}[Y|Y < 0] - p_X \mathbb{E}[X|X > 0] - (1 - p_X) \mathbb{E}[X|X < 0] \\
 &= p_Y \mathbb{E}[Y|Y > 0] - p_X \mathbb{E}[X|X > 0] + (1 - p_Y) \mathbb{E}[Y|Y < 0] - (1 - p_X) \mathbb{E}[X|X < 0] \quad \text{Eq. \#42}
 \end{aligned}$$

We have, for $i \geq 1$:

$$\begin{aligned}
 \mathbb{P}[X \geq i] \leq \mathbb{P}[Y \geq 1] &\implies \sum_{k \geq 1} \mathbb{P}[X \geq k] \leq \sum_{k \geq 1} \mathbb{P}[Y \geq k] \\
 &\implies p_X \sum_{k \geq 1} \frac{\mathbb{P}[X \geq k]}{p_X} \leq p_Y \sum_{k \geq 1} \frac{\mathbb{P}[Y \geq k]}{p_Y} \\
 &\implies p_X \mathbb{E}[X|X > 0] \leq p_Y \mathbb{E}[Y|Y > 0] \quad (\text{using the result of the past question}) \\
 &\implies p_Y \mathbb{E}[Y|Y > 0] - p_X \mathbb{E}[X|X > 0] \geq 0 \quad [1]
 \end{aligned}$$

On the other side, we have:

$$\mathbb{E}[X|X < 0] = -\sum_{i \geq 1} \frac{\mathbb{P}[X \leq -i]}{1 - p_X} \implies (1 - p_X) \mathbb{E}[X|X < 0] = -\sum_{i \geq 1} \mathbb{P}[X \leq -i]$$

Similarly:

$$(1 - p_Y) \mathbb{E}[Y|Y < 0] = -\sum_{i \geq 1} \mathbb{P}[Y \leq -i]$$

Hence,

$$\begin{aligned}
(1 - p_Y)\mathbb{E}[Y|Y < 0] - (1 - p_X)\mathbb{E}[X|X < 0] &= \sum_{i \geq 1} \mathbb{P}[X \leq -i] - P[Y \leq -i] \\
&= \sum_{i \geq 1} \mathbb{P}[Y \geq i] - P[X \geq i] \\
&\geq 0 \quad [2]
\end{aligned}$$

By using the results in [1] and [2] in **Eq.** #42, we conclude that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.