# Randomised Algorithms
# Winter term 2022/2023, Exercise Sheet No. 9


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**Exercise 1.**
***(a)*** *As* $\mathbb{E}[X] = \frac{n}{2}$*, we have:*

$$\mathbb{P}[X \geq \frac{3}{4}n] = \mathbb{P}[X \geq (1 + \frac{1}{2})\frac{n}{2}] = \mathbb{P}[X \geq (1 + \frac{1}{2})\mathbb{E}[X])]$$

*And similarily:*

$$\mathbb{P}[X \leq \frac{1}{4}n] = \mathbb{P}[X \leq (1 - \frac{1}{2})\mathbb{E}[X])]$$

*Using the Chernoff bounds, (Ineq 2 then 3, from Slides 4), we conclude that (for* $\delta = 1/2 < 1$*):*

$$\mathbb{P}[X \leq \frac{1}{4}n] \leq e^{-\frac{n}{24}} \quad \textit{and} \quad \mathbb{P}[X \geq \frac{3}{4}n] \leq e^{-\frac{n}{24}}$$

***(b)*** *Using the the simplified Chernoff bounds:*

$$\begin{aligned}\mathbb{P}[X \geq n/2 + 2\sqrt{n}] &= \mathbb{P}[X \geq \frac{n}{2}(1 + \frac{4}{\sqrt{n}})] \\ &\leq e^{-\frac{n}{2}\left(\frac{4}{\sqrt{n}}\right)^2\frac{1}{3}} \quad \textit{(Requires } \frac{4}{\sqrt{n}} < 1 \implies n > 16\textit{)} \\ &= e^{-8/3}\end{aligned}$$

*Using the additive Chernoff (No extra requirements needed):*

$$\begin{aligned}\mathbb{P}[X \geq n/2 + 2\sqrt{n}] &\leq e^{-2\left(\frac{4}{\sqrt{n}}\right)^2\frac{1}{n}} \\ &= e^{-32/n^2}\end{aligned}$$

***(c)*** *We let* $n_1 = \lfloor n/2 - \sqrt{n}/4 \rfloor$ *and* $n_2 = \lfloor n/2 + \sqrt{n}/4 \rfloor$*, we have:*

$$
\begin{aligned}
\mathbb{P}[X \in [n/2 - \sqrt{n}/4, n/2 + \sqrt{n}/4]] &\leq \mathbb{P}[X \in [n_1, n_2]] \\
&= \sum_{k=n_1}^{n_2} \mathbb{P}[X = k] \\
&= \sum_{k=n_1}^{n_2} \binom{n}{k} \frac{1}{2^n} \\
&\leq \frac{1}{2^n} \sum_{k=n_1}^{n_2} 2^n \frac{\sqrt{2}}{\sqrt{n}} \quad \textit{(Using the hint)} \\
&= \frac{\sqrt{2}}{\sqrt{n}} \sum_{k=n_1}^{n_2} 1 = \frac{\sqrt{2}}{\sqrt{n}}(n_2 - n_2 + 1) \\
&\leq \frac{\sqrt{2}}{\sqrt{n}} \left(n/2 + \sqrt{n}/4 - (n/2 - \sqrt{n}/4 - 1)\right) \quad \textit{(using: } \lfloor x \rfloor - \lfloor y \rfloor \leq x - (y-1)\textit{)} \\
&= \frac{\sqrt{2}}{\sqrt{n}}(1 + \sqrt{n}/2) \\
&= \frac{\sqrt{2}}{\sqrt{n}} + \frac{1}{\sqrt{2}}
\end{aligned}
$$

*For $n_0 = 24$, we have for $n \geq n_0$ :* $\frac{\sqrt{2}}{\sqrt{n}} + \frac{1}{\sqrt{2}} \leq \frac{\sqrt{2}}{\sqrt{24}} + \frac{1}{\sqrt{2}} \approx 0.995$

**Exercise 2.**

**Exercise 3.**
*Let the RV $X$ represent the sum of the $n$ rolls, we have established previously that:* $\mathbb{E}[X] = \frac{7n}{2}$

*Using Hoeffding's inequality leads to:*

$$
\begin{aligned}
\mathbb{P}[X \geq 4n] &= \mathbb{P}[X \geq E[X] + \frac{n}{2}] \\
&\leq \exp\left(-\frac{2(n/2)^2}{\sum_{i=1}^{n}(6-1)^2}\right) \\
&= e^{-n/50}
\end{aligned}
$$

*This tail that obtained using Hoeffding's inequality decreases way faster than what we have obtained with Marokov and Chebychev's inequalities (respectively $\frac{7}{8}$ and $\frac{35}{3n}$).*

**Exercise 4.**
*Let's first establish a few results on $d(u,v)$ for two random vertices of the hypercube. $d(u,v)$ represents the number of bits $v_i$ in $v$ that are diffrent than $u_i$, hence, $d(u,v) \in \{0, \ldots, n\}$, and $\mathbb{P}[d(u,v) = k] = \frac{\binom{n}{k}}{2^n}$.(Why: How many ways can we pick k positions so we can flip their bits in v divided by the number of all configurations.)*

*By noticing that $\sum_{i=1}^{n} k\binom{n}{k} = n2^{n-1}$, we can conclude that:*

$$
\mathbb{E}[d(u,v)] = \sum_{i=1}^{n} k \frac{\binom{n}{k}}{2^n} = \frac{n}{2}
$$

*Let's prove the following result, for any two random vertices $u, v \in V$, $\epsilon > 0$:*

$$\mathbb{P}\left[(1-\epsilon)\frac{n}{2} \leq d(u,v) \leq (1+\epsilon)\frac{n}{2}\right] \geq 1 - 2e^{-\frac{n\epsilon^2}{6}}$$

***Proof:***
*Let's define the events $E_1$ and $E_2$ as follows:*

$$E_1 = \{(1-\epsilon)\frac{n}{2} \leq d(u,v)\} = \{(1-\epsilon)\mathbb{E}[d(u,v)] \leq d(u,v)\}$$

*And:*

$$E_2 = \{(1+\epsilon)\frac{n}{2} \geq d(u,v)\} = \{(1+\epsilon)\mathbb{E}[d(u,v)] \geq d(u,v)\}$$

*We are looking for to bound the probablity:* $\mathbb{P}[E_1 \cap E_2]$

$$\begin{aligned}\mathbb{P}[E_1 \cap E_2] &= 1 - \mathbb{P}[\overline{E_1} \cup \overline{E_2}] \\ &\geq 1 - (\mathbb{P}[\overline{E_1}] + \mathbb{P}[\overline{E_2}]) \\ &\geq 1 - 2e^{-\mathbb{E}[d(u,v)]\epsilon^2/3} \\ &\geq 1 - 2e^{-\frac{n\epsilon^2}{6}}\end{aligned}$$

*Here we used the union bound followed by Chernoff (Ineq 2. Slide 4 to bound $\mathbb{P}[\overline{E_2}]$, and Ineq 3 for $\mathbb{P}[\overline{E_1}]$), assumig $0 < \epsilon < 1$.*

*Let $V = \{v_1, \ldots, v_{2^n}\}$, **we uniformely random pick n vertices from V** . Now, let's define the RVs $\{X_i\}$, $\{Y_{i,j}\}$ and $V(v_1, \ldots, v_{2^n})$ as follows:*

$$X_i = \begin{cases} 1 & \text{if } v_i \text{ was picked} \\ 0 & \text{Otherwise.} \end{cases} \qquad Y_{i,j} = \begin{cases} 1 & \text{if } \quad (1-\epsilon)\frac{n}{2} \leq d(v_i, v_j) \leq (1+\epsilon)\frac{n}{2} \\ 0 & \text{Otherwise.} \end{cases}$$

*And finally:*

$$S(v_1, \ldots, v_{2^n}) = \sum_{i<j} X_i X_j Y_{i,j}$$

*We need to proof that:*

***Proof:*** *We have:*

$$
\begin{aligned}
\mathbb{E}[S(v_1,\dots,v_{2^n})] &= \sum_{i<j} \mathbb{E}[X_i X_j Y_{i,j}] \\
&= \sum_{i<j} \mathbb{E}[X_i]\mathbb{E}[X_j]\mathbb{E}[Y_{i,j}] \quad \textit{(every rv is independent from the other)} \\
&= \mathbb{E}[X_1]^2 \sum_{i<j} \mathbb{E}[Y_{i,j}] \\
&\geq \mathbb{E}[X_1]^2 (1 - 2e^{-\frac{n\epsilon^2}{6}}) \sum_{i<j} 1 \\
&= \mathbb{P}[\{\textit{Probb. to pick } v_2\}]^2 (1 - 2e^{-\frac{n\epsilon^2}{6}}) \frac{2^{2n} - 2^n}{2} \\
&= \left(\frac{n}{2^n}\right)^2 (1 - 2e^{-\frac{n\epsilon^2}{6}}) \frac{2^{2n} - 2^n}{2} = \frac{n^2}{2} (1 - 2e^{-\frac{n\epsilon^2}{6}})(1 - \frac{1}{2^n}) \\
&= \frac{n(n-1)}{2} \frac{n}{n-1} (1 - 2e^{-\frac{n\epsilon^2}{6}})(1 - \frac{1}{2^n}) \\
&= \frac{n(n-1)}{2} (1 + \frac{1}{n-1})(1 - 2e^{-\frac{n\epsilon^2}{6}})(1 - \frac{1}{2^n})
\end{aligned}
$$

***fix this:***

*For $n$ big enough, $(1 + \frac{1}{n-1})(1 - 2e^{-\frac{n\epsilon^2}{6}})(1 - \frac{1}{2^n})$ is guarenteed to be $\geq 1$*

*Hence, $\forall \epsilon > 0$, and with $n$ big enough, $\mathbb{E}[S(v_1,\dots,v_{2^n})] \geq \frac{n(n-1)}{2}$, hence there exist a configuration of $X, Y$ s.t $v() = \frac{n(n-1)}{2}$, which can only happen if all the $n$ picks had $Y_{i,j} = 1$*