Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 9

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Exercise 1.

(a) *Let's say w.l.o.g that $a^\star$ satifies the first literal (and only the first) of every clause, then $\neg a^\star$ would satify the second literal of each clause, and hence, $\neg a^\star$ would also be a satisfied assigment.*

(b) *We now have two competing "$a^\star$", and most importantly, the Hamming distance is complementary, i.e.,*

$$H(a, a^\star) = n - H(a, \neg a^\star)$$

Exercise 2.
(a) *The new recurrence relations could be summed up as follows:*

$$\begin{cases} E_0 & = 0 \\ E_i & = 1 + \frac{1}{2}(E_{i+1} + E_{i-1}) \quad \textit{for } i \in \{1, \ldots, n-1\} \\ E_n & = 1 + \frac{1}{2}(E_n + E_{n-1}) \end{cases}$$

The last implies that: $E_n = 2 + E_{n-1} \implies D_n = 2$.
The second line implies that:

$$D_k = 2 + D_{k-1} \textit{ for } k \in \{1, \ldots, n-1\}$$

By summing each equation from $k = 1$ up to $k = i \leq n - 1$, we get:

$$D_1 = 2i + D_{i+1} \quad \textit{for} \quad i \in \{1, \ldots, n-1\}$$

Particularly when $i = n - 1$

$$D_1 = 2(n-1) + D_n = 2n = E_1$$

And hence,

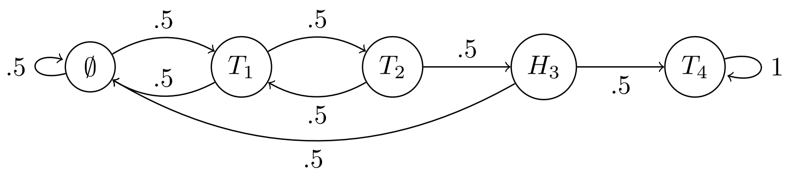
$$D_{i+1} = 2(n-i), \quad \textit{for } i \in \{1, \ldots, n-1\}$$

(b) *For $k \in \{1, \ldots, n\}$, we have:*

$$\begin{aligned} E_k &= \sum_{i=1}^{k} D_i \\ &= \sum_{i=1}^{k} 2(n - (k-1)) \\ &= k(2n-k) + k \end{aligned}$$

Exercise 3.

(a) *The figure below represents the Markov Chain of interest:*



(b) *We refer to* E_0 *as the expected running time starting the the state* $\emptyset$,*(*$E_0 \rightarrow \emptyset$*) and respectively:* $E_1 \rightarrow T_1, E_2 \rightarrow T_2, E_3 \rightarrow T_3, E_4 \rightarrow T_4$.

We have the following system:

$$\begin{cases} E_0 &= 1 + \frac{1}{2}\left(E_0 + E_1\right) \\ E_1 &= 1 + \frac{1}{2}\left(E_0 + E_2\right) \\ E_2 &= 1 + \frac{1}{2}\left(E_1 + E_3\right) \\ E_3 &= 1 + \frac{1}{2}\left(E_0 + E_4\right) \\ E_4 &= 0 \end{cases} \implies \begin{cases} E_0 &= 1 + \frac{1}{2}\left(E_0 + E_1\right) \\ E_1 &= 1 + \frac{1}{2}\left(E_0 + E_2\right) \\ E_2 &= 1 + \frac{1}{2}\left(E_1 + E_3\right) \\ E_3 &= 1 + \frac{1}{2}\left(E_0 + E_4\right) \\ E_4 &= 0 \end{cases} \implies \begin{cases} E_0 &= 1 + \frac{1}{2}\left(E_0 + E_1\right) \\ E_1 &= 1 + \frac{1}{2}\left(E_0 + E_2\right) \\ E_2 &= 1 + \frac{1}{2}\left(E_1 + E_3\right) \\ E_3 &= 1 + \frac{1}{2}\left(E_0 + E_4\right) \\ E_4 &= 0 \end{cases}$$

solve this lol

(c) *Let's say this output has emerged:* $T[TTH_1T]HTH_2$*, the approach in* ***(a)*** *would detect the pattern between brackets, while the approach proposed in* ***(c)*** *would fail, as it would only check if the pattern has emerged looking back from* H_1 *and* H_2.

The expected running time is the Expectation of a geometric RV of $p = \frac{1}{2^4}$*, which is equal to* $\frac{1}{p} = 2^4 = 16$*, hence we wold need 16 trials, as each trial requires* 4 *runs, we would need in expectation* $16 * 4 = 64$ *runs.*