Randomised Algorithms

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Exercise 1 [4 points]

Consider running the random QuickSort algorithm on the following input:

$$[11, 19, 1, 8, 7, 10, 15, 9, 4, 18]$$

(a) Explain what will happen to the comparison between 8 and 15, in the case the pivot is 7? Same question for the pivot being 15, and being 10?

(b) Overall, what is the probability that 8 and 15 are compared?

Exercise 2 [6 points]

Let $\mathcal{A}$ be a Monte Carlo algorithm with two-sided errors that runs in polynomial time $T(n)$ and outputs a correct result for a decision problem L with probability at least $1/2 + 1/\sqrt{n}$, where n is size of the input.

(a) Consider the probability amplification with $\mathcal{A}_t$ algorithm like in the lecture, ie. run $\mathcal{A}$ t times then output the decision based on the majority vote, give an upper bound for the error probability of $\mathcal{A}_t$.
You can use the result from the lecture without repeating its arguments in answering this question.

(b) Find t such that the running time of $\mathcal{A}_t$ remains polynomial while its error probability is reduced to no more than $1/2 - \varepsilon$ for some constant $\varepsilon > 0$, and use this to prove that $L \in \text{BPP}$.

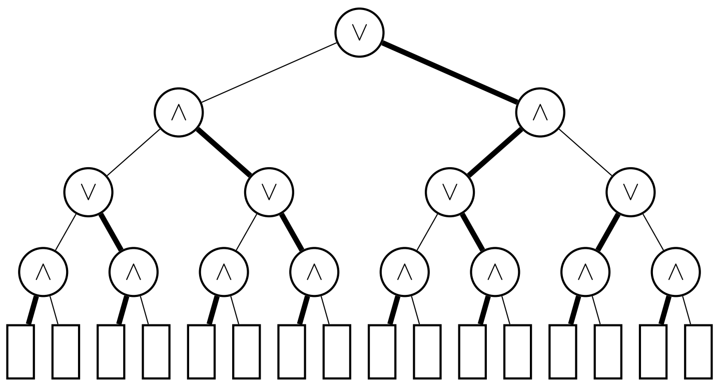
Hint: The inequality $1 + x \leq e^x$ for all $x \in \mathbb{R}$ might be useful.

(c) Suppose now that our original algorithm $\mathcal{A}$ only has probability $1/2 + 1/n$ to output a correct answer, is L still in BPP? Give a short justification for your answer.

Exercise 3 [6 points]

(a) Outline an algorithm that allows the construction of a worst input for a deterministic algorithm, denoted $\mathcal{A}(T_k)$, to evaluate a complete OR-AND binary tree T_k (the root is an OR) with k levels of $\vee$ operators and k levels of $\wedge$ operators.

(b) Execute your algorithm on the following tree, on which each bold edge indicates the first child that will be examined by $\mathcal{A}(T_k)$ during the evaluation of a parent node.



Exercise 4 [4 points]

Prove the following inequality for all matrices M of real numbers:

$$\max_i \min_j M_{ij} \leq \min_j \max_i M_{ij}.$$