Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 6

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Exercise 1.

Let $(X_t)_{t\geq 0}$ *be the stochastic process where* X_t *is the number of remaining gifts at round* t*.*
At step t*, we let* $\{g_1, \ldots, g_{X_t}\}$ *be an arbitrary ordering of the remaing gifts, we define the RVs* $(Y_i^t)_{1\leq i\leq X_t}$ *as follows:*

$$Y_i^t = \begin{cases} 1 & \textit{if the gift } g_i \textit{ was picked by one and only one child in the next round } t+1 \\ 0 & \textit{otherwise} \end{cases}$$

Also, with $X_t = s$*, we have,* $\mathbb{P}[Y_i^t = 1] = \binom{s}{1}\frac{1}{s}(\frac{s-1}{s})^{s-1} = (\frac{s-1}{s})^{s-1}$

Most importantly, we have $X_{t+1} = X_t - \sum_{i=1}^{i=X_t} Y_i^t$*, hence, we can write the following, for* $s > 0$*:*

$$\begin{aligned}
\mathbb{E}[X_t - X_{t+1}|X_t = s] &= \mathbb{E}[\sum_{i=1}^{i=X_t} Y_i^t|X_t = s] \\
&= \mathbb{E}[\sum_{i=1}^{i=s} Y_i^t|X_t = s] \\
&= \sum_{i=1}^{s} \mathbb{E}[Y_i^t] \\
&= \sum_{i=1}^{s} \mathbb{P}[Y_i^t = 1] \\
&= \sum_{i=1}^{s} (1 - \frac{1}{s})^{s-1} \\
&\geq \frac{s}{e}
\end{aligned}$$

With $h(s) : s \mapsto \frac{s}{e}$ *being monotonicaly increasing, and by applying the Variable Drift Theorem, we conclude the following:*

$$
\begin{aligned}
\mathbb{E}[T|X_0 = n] &\leq \frac{1}{h(1)} + \int_1^n \frac{1}{h(x)}\,dx \\
&= e + e\int_1^n \frac{1}{s}\,dx \\
&= e(1+\ln(n))
\end{aligned}
$$

Exercise 2.

We define the stochastic process $(X_t)_{t\geq 0}$ *as:* $X_t = 100 - Z_t$*, where* Z_t *represents the token's cell at time-step* t*. Let's first derive some preperties regarding the distribution of* X_t*.*

We have for $s \in \{6, \ldots, 100\}$*, and* $k \in \{1, \ldots, 6\}$*:* $\mathbb{P}[X_{t+1} = s - k | X_t = s] = \frac{1}{6}$

And for $s \in \{1, \ldots, 5\}$*, and* $k \in \{0, \ldots, s\}$

$$
\mathbb{P}[X_{t+1} = k|X_t = s] = \begin{cases} \frac{1}{6} & \textit{if } k < s \\ \frac{6-s}{6} & \textit{if } k = s \end{cases}
$$

Hence we have, for $5 < s \leq 100$*:*

$$
\begin{aligned}
\mathbb{E}[X_{t+1}|X_t = s] &= \sum_{i=s-6}^{s-1} i\mathbb{P}[X_{t+1} = i|X_t = s] \\
&= \sum_{i=s-6}^{s-1} i\frac{1}{6} \\
&= \frac{1}{6}\sum_{i=s-6}^{s-1} i \\
&= \frac{1}{6}(6s-21) = s - \frac{7}{2} \\
&\Longrightarrow \\
\mathbb{E}[X_t - X_{t+1}|X_t = s] &= s - (s - \frac{7}{2}) \\
&= \frac{7}{2}
\end{aligned}
$$

Hence we have, for $0 < s \leq 5$:

$$\begin{aligned}
\mathbb{E}[X_{t+1}|X_t = s] &= \sum_{i=0}^{s} i\mathbb{P}[X_{t+1} = i|X_t = s] \\
&= \sum_{i=0}^{s-1} i\mathbb{P}[X_{t+1} = i|X_t = s] + s\mathbb{P}[X_{t+1} = s|X_t = s] \\
&= \sum_{i=0}^{s-1} i\frac{1}{6} + \frac{s(6-s)}{6} \\
&= \frac{s(s-1)}{12} + \frac{s(6-s)}{6} \\
&= \frac{11s - s^2}{12} \\
&\implies \\
\mathbb{E}[X_t - X_{t+1}|X_t = s] &= s - (\frac{11s - s^2}{12}) \\
&= \frac{s^2 + s}{12} \\
&\implies \\
\frac{1^2+1}{12} \leq \mathbb{E}[X_t - X_{t+1}|X_t = s] &\leq \frac{5^2+5}{12} \\
&\implies \\
\frac{1}{6} \leq \mathbb{E}[X_t - X_{t+1}|X_t = s] &\leq \frac{5}{2}
\end{aligned}$$

Hence, for all $0 < s \leq 100$:

$$\frac{1}{6} \leq \mathbb{E}[X_t - X_{t+1}|X_t = s] \leq \frac{7}{2}$$

Using the Additive Drift Theorem, we conclude that:

$$\frac{100 * 2}{7} \leq \mathbb{E}[T|X_0 = 100] \leq 100 * 6 \implies \frac{200}{7} \leq \mathbb{E}[T|X_0 = 100] \leq 600$$

Exercise 3.

(a) *Let Y_i be the RV associated with the value of the i-th roll, we have: $X = \sum_{i=1}^{i=n} Y_i$, Let's first compute $\mathbb{E}[Y_i]$ and $var[Y_i]$:*

$$\mathbb{E}[Y_i] = \sum_{i=1}^{6} \frac{i}{6} = 7/2$$

And for the $var[Y_i]$, we have:

$$\begin{aligned}
\mathbb{E}[Y_i^2] &= \sum_{k=1}^{k=6} k^2.\mathbb{P}[Y_i = k] \\
&= \frac{1}{6}\sum_{k=1}^{k=6} k^2 \\
&= \frac{91}{6} \\
&\implies \\
var[Y_i] &= \frac{91}{6} - (\frac{7}{2})^2 = \frac{35}{12}
\end{aligned}$$

Now let's compute $\mathbb{E}[X]$:

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[\sum_{i=1}^{i=n} Y_i] \\
&= \sum_{i=1}^{i=n} \mathbb{E}[Y_i] \\
&= \frac{7n}{2}
\end{aligned}$$

Now let's compute $var(X)$, by noting that each roll Y_i is independent, we get:

$$\begin{aligned}
var[X] &= var[\sum_{i=1}^{i=n} Y_i] \\
&= n.var[Y_1] \\
&= \frac{35n}{12}
\end{aligned}$$

(b)

Using Markov's inequality, we have:

$$\begin{aligned}
\mathbb{P}[X \geq 4n] &\leq \frac{\mathbb{E}[X]}{4n} \\
\mathbb{P}[X \geq 4n] &\leq \frac{\frac{7n}{2}}{4n} \\
\mathbb{P}[X \geq 4n] &\leq \frac{7}{8}
\end{aligned}$$

(c)

We have:

$$\begin{aligned}
\mathbb{P}[X \geq 4n] &= \mathbb{P}[X - \mathbb{E}[X] \geq 4n - \frac{7n}{2}] \\
&= \mathbb{P}[X - \mathbb{E}[X] \geq \frac{n}{2}] \\
&\leq \mathbb{P}[|X - \mathbb{E}[X]| \geq \frac{n}{2}] \qquad \textit{[Line \#17]} \\
&\leq \frac{var[X]}{\frac{n^2}{2^2}} \qquad \textit{(Using Chebychev)} \\
&= \frac{35}{3n}
\end{aligned}$$

(d) *From the past question,* ***[Line #17]****, we get the following:*

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq \frac{n}{2}] \leq \frac{35}{3n}$$

We also have:

$$\begin{aligned}
\mathbb{P}[X \leq 3n] &= \mathbb{P}[X - \mathbb{E}[X] \leq -\frac{n}{2}] \\
&\leq \mathbb{P}[|X - \mathbb{E}[X]| \geq \frac{n}{2}]
\end{aligned}$$

And hence, we conclude that:

$$\mathbb{P}[X \leq 3n] \leq \frac{35}{3n}$$

Exercise 4.

Suppose $|P| > 4n^{\frac{3}{4}} + 2$*, and let* $k_l = k - 2n^{\frac{3}{4}}$ *and* $k_h = k + 2n^{\frac{3}{4}}$*. As established in the notes, for error 2 to occur we must either have* $a \geq S(k_l)$ *or* $b \leq S(k_h)$*. Now, suppose* $a > S(k_l)$*, (other case follows the same logic).*

We define the indicator RV X_i *as follows:* $X_i = \begin{cases} 1 & \textit{if the i-th drawn element is } \leq S(k_l) \\ 0 & \textit{otherwize} \end{cases}$

We have:

$\mathbb{E}[X_i] = \mathbb{P}[\{\textit{The i-th drawn element is } \leq S(k_l)\}] = \frac{k_l}{n}$.

And: $var[X_i] = \mathbb{E}[X_i^2] - \mathbb{E}[X_i]^2 = \frac{k_l}{n}(1 - \frac{k_l}{n}) \leq \frac{1}{4}$ *(Using:* $(\frac{k}{n} - \frac{1}{2})^2 \geq 0$*).*

Let $X = \sum_{i=1}^{\frac{3}{4}} X_i$ *be the number of sampled elements that are* $\leq S(k_l)$*. We have:*

$$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{n^{3/4}} X_i] = n^{3/4}\mathbb{E}[X_1] = \frac{k_l}{n^{1/4}}$$

And

$$var[X] = var[\sum_{i=1}^{n^{3/4}} X_i] = \sum_{i=1}^{n^{3/4}} var[X_i] \leq \frac{n^{3/4}}{4}$$

Now, we have the following:

$$
\begin{aligned}
\mathbb{P}[S(k_l) < a] &= \mathbb{P}[X < l] \\
&= \mathbb{P}[X - \mathbb{E}[X] < l - \mathbb{E}[X]] \\
&= \mathbb{P}[X - \mathbb{E}[X] < \frac{k - k_l}{n^{1/4}} - \sqrt{n}] \\
&= \mathbb{P}[X - \mathbb{E}[X] < \frac{2n^{3/4}}{n^{1/4}} - \sqrt{n}] \\
&= \mathbb{P}[X - \mathbb{E}[X] < -\sqrt{n}] \qquad \textit{Line Error} \\
&\leq \mathbb{P}[|X - \mathbb{E}[X]| > \sqrt{n}] \\
&\leq \frac{var[X]}{\sqrt{n}^2} \\
&\leq \frac{n^{3/4}}{4n} = \mathcal{O}(n^{\frac{-1}{4}})
\end{aligned}
$$

Remark on Line Error: *That line is an error, should be* $+\sqrt{n}$*, only discovered it by deadline time, so I'm sorry, I'll just submit draft as it is now.*