Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 5

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Exercise 1.

Let $(X_t)_{t\geq 0}$ be the stochastic process where X_t is the number of remaining gifts at round t. At step t, we let $\{g_1, \ldots, g_{X_t}\}$ be an arbitrary ordering of the remaing gifts, we define the RVs $(Y_i^t)_{1\leq i\leq X_t}$ as follows:

$$Y_i^t = \begin{cases} 1 & \textit{if the gift } g_i \textit{ was picked by one and only one child in the next round } t+1 \\ 0 & \textit{otherwise} \end{cases}$$

Also, with $X_t = s$, we have, $\mathbb{P}[Y_i^t = 1] = \binom{s}{1}\frac{1}{s}(\frac{s-1}{s})^{s-1} = (\frac{s-1}{s})^{s-1}$

Most importantly, we have $X_{t+1} = X_t - \sum_{i=1}^{i=X_t} Y_i^t$, hence, we can write the following, for $s > 0$:

$$\begin{aligned}
\mathbb{E}[X_t - X_{t+1}|X_t = s] &= \mathbb{E}[\sum_{i=1}^{i=X_t} Y_i^t|X_t = s] \\
&= \mathbb{E}[\sum_{i=1}^{i=s} Y_i^t|X_t = s] \\
&= \sum_{i=1}^{s} \mathbb{E}[Y_i^t] \\
&= \sum_{i=1}^{s} \mathbb{P}[Y_i^t = 1] \\
&= \sum_{i=1}^{s} (1 - \frac{1}{s})^{s-1} \\
&\geq \frac{s}{e}
\end{aligned}$$

With $h(s) : s \mapsto \frac{s}{e}$ being monotonicaly increasing, and by applying the Variable Drift Theorem, we conclude the following:

$$
\begin{aligned}
\mathbb{E}[T|X_0 = n] &\leq \frac{1}{h(1)} + \int_1^n \frac{1}{h(x)}\,dx \\
&= e + e\int_1^n \frac{1}{s}\,dx \\
&= e(1 + \ln(n))
\end{aligned}
$$

Exercise 2.

We define the stochastic process $(X_t)_{t\geq 0}$ as: $X_t = 100 - Z_t$, where Z_t represents the token's cell at time-step t. Let's first derive some preperties regarding the distribution of X_t.

We have for $s \in \{6, \dots, 100\}$, and $k \in \{1, \dots, 6\}$: $\mathbb{P}[X_{t+1} = s - k|X_t = s] = \frac{1}{6}$

And for $s \in \{1, \dots, 5\}$, and $k \in \{0, \dots, s\}$

$$
\mathbb{P}[X_{t+1} = k|X_t = s] = \begin{cases} \frac{1}{6} & \textit{if } k < s \\ \frac{6-s}{6} & \textit{if } k = s \end{cases}
$$

Hence we have, for $5 < s \leq 100$:

$$
\begin{aligned}
\mathbb{E}[X_{t+1}|X_t = s] &= \sum_{i=s-6}^{s-1} i\mathbb{P}[X_{t+1} = i|X_t = s] \\
&= \sum_{i=s-6}^{s-1} i\frac{1}{6} \\
&= \frac{1}{6}\sum_{i=s-6}^{s-1} i \\
&= \frac{1}{6}(6s - 21) = s - \frac{7}{2} \\
&\implies \\
\mathbb{E}[X_t - X_{t+1}|X_t = s] &= s - (s - \frac{7}{2}) \\
&= \frac{7}{2}
\end{aligned}
$$

Hence we have, for $0 < s \leq 5$:

$$\begin{aligned}
\mathbb{E}[X_{t+1}|X_t = s] &= \sum_{i=0}^{s} i\mathbb{P}[X_{t+1} = i|X_t = s] \\
&= \sum_{i=0}^{s-1} i\mathbb{P}[X_{t+1} = i|X_t = s] + s\mathbb{P}[X_{t+1} = s|X_t = s] \\
&= \sum_{i=0}^{s-1} i\frac{1}{6} + \frac{s(6-s)}{6} \\
&= \frac{s(s-1)}{12} + \frac{s(6-s)}{6} \\
&= \frac{11s - s^2}{12} \\
&\implies \\
\mathbb{E}[X_t - X_{t+1}|X_t = s] &= s - (\frac{11s - s^2}{12}) \\
&= \frac{s^2 + s}{12} \\
&\implies \\
\frac{1^2+1}{12} \leq \mathbb{E}[X_t - X_{t+1}|X_t = s] &\leq \frac{5^2+5}{12} \\
&\implies \\
\frac{1}{6} \leq \mathbb{E}[X_t - X_{t+1}|X_t = s] &\leq \frac{5}{2}
\end{aligned}$$

Hence, for all $0 < s \leq 100$:

$$\frac{1}{6} \leq \mathbb{E}[X_t - X_{t+1}|X_t = s] \leq \frac{7}{2}$$

Using the Additive Drift Theorem, we conclude that:

$$\frac{100 * 2}{7} \leq \mathbb{E}[T|X_0 = 100] \leq 100 * 6 \implies \frac{200}{7} \leq \mathbb{E}[T|X_0 = 100] \leq 600$$

Exercise 3.

(a) *Let Y_i be the RV associated with the value of the i-th roll, we have: $X = \sum_{i=1}^{i=n} Y_i$, Let's first compute $\mathbb{E}[Y_i]$ and $var[Y_i]$:*

$$\mathbb{E}[Y_i] = \sum_{i=1}^{6} \frac{i}{6} = 7/2$$

And for the $var[Y_i]$, we have:

$$\begin{aligned}
\mathbb{E}[Y_i^2] &= \sum_{k=1}^{k=6} k^2.\mathbb{P}[Y_i = k] \\
&= \frac{1}{6}\sum_{k=1}^{k=6} k^2 \\
&= \frac{91}{6} \\
&\implies \\
var[Y_i] &= \frac{91}{6} - (\frac{7}{2})^2 = \frac{35}{12}
\end{aligned}$$

Now let's compute $\mathbb{E}[X]$:

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[\sum_{i=1}^{i=n} Y_i] \\
&= \sum_{i=1}^{i=n} \mathbb{E}[Y_i] \\
&= \frac{7n}{2}
\end{aligned}$$

Now let's compute $var(X)$, by noting that each roll Y_i is independent, we get:

$$\begin{aligned}
var[X] &= var[\sum_{i=1}^{i=n} Y_i] \\
&= n.var[Y_1] \\
&= \frac{35n}{12}
\end{aligned}$$

(b)

Using Markov's inequality, we have:

$$\begin{aligned}
\mathbb{P}[X \geq 4n] &\leq \frac{\mathbb{E}[X]}{4n} \\
\mathbb{P}[X \geq 4n] &\leq \frac{\frac{7n}{2}}{4n} \\
\mathbb{P}[X \geq 4n] &\leq \frac{7}{8}
\end{aligned}$$

(c)

We have:

$$
\begin{aligned}
\mathbb{P}[X \geq 4n] &= \mathbb{P}[X - \mathbb{E}[X] \geq 4n - \frac{7n}{2}] \\
&= \mathbb{P}[X - \mathbb{E}[X] \geq \frac{n}{2}] \\
&\leq \mathbb{P}[|X - \mathbb{E}[X]| \geq \frac{n}{2}] \qquad \textit{[Line \#17]} \\
&\leq \frac{var[X]}{\frac{n^2}{2^2}} \qquad \textit{(Using Chebychev)} \\
&= \frac{35}{3n}
\end{aligned}
$$

(d) *From the past question,**[Line #17]**, we get the following:*

$$\mathbb{P}[|X - \mathbb{E}[X]| \geq \frac{n}{2}] \leq \frac{35}{3n}$$

We also have:

$$
\begin{aligned}
\mathbb{P}[X \leq 3n] &= \mathbb{P}[X - \mathbb{E}[X] \leq -\frac{n}{2}] \\
&\leq \mathbb{P}[|X - \mathbb{E}[X]| \geq \frac{n}{2}]
\end{aligned}
$$

And hence, we conclude that:

$$\mathbb{P}[X \leq 3n] \leq \frac{35}{3n}$$