Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 2

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Exercise 1.

Let X be a Bernoulli RV with $p = 0.5$, and let $Y = 1 - X$. We have $\mathbb{E}[X] = \mathbb{E}[Y] = 0.5$, and $\min(X, Y) = 0$. Therefore, $\min(\mathbb{E}[X], \mathbb{E}[Y]) = 0.5$ and $\mathbb{E}[\min(X, Y)] = 0$, clearly not equal.

Exercise 2.

(a)

$$\begin{aligned} \textit{We have: } \forall s\colon X(s) \le Y(s) &\implies \forall s\colon X(s)\mathbb{P}[s] \le Y(s)\mathbb{P}[s] \\ &\implies \sum_s X(s)\mathbb{P}[s] \le \sum_s Y(s)\mathbb{P}[s] \\ &\implies \mathbb{E}[X] \le \mathbb{E}[Y] \end{aligned}$$

(b) *We have $\min(X, Y) \le X$ and $\min(X, Y) \le Y$, using **(a)**, we get: $\mathbb{E}[\min(X, Y)] \le \mathbb{E}[X]$ and $\mathbb{E}[\min(X, Y)] \le \mathbb{E}[X]$. We can conclude that $\mathbb{E}[\min(X, Y)] \le \min(\mathbb{E}[X], \mathbb{E}[Y])$.*

(c) *For $s \in S$ s.t $X(s) \ge a$, we have: $Y(s) = a$ and hence, $Y(s) = a \le X(s)$. For $s \in S$ s.t $X(s) < a$, we have: $Y(s) = 0$ and hence, $Y(s) \le X(s)$, since X is always positive. Hence, $Y \le X$.*

Let's compute $\mathbb{E}[Y]$:

Y can only take two values $\{0, a\}$. Particularly, we have $\mathbb{P}[Y = a] = \mathbb{P}[I_{\{X \le a\}} = 1] = \mathbb{P}[X \le a]$. And hence $\mathbb{E}[Y] = a\mathbb{P}[X \le a]$

*Finally, we have $a\mathbb{P}[X \le a] = \mathbb{E}[Y] \le \mathbb{E}[X]$ (Using **(a)**, since $Y \le X$), dividing the inequality by a completes the proof.*

Exercise 3.

*If all assignments $\alpha \in \{0, 1\}^n$ satisfied strictly less than $\frac{m}{2}$ clauses, we would have, $\forall \alpha\colon Z(\alpha) < \frac{m}{2}$, using **Exercise 2 (a)**, we would get $\mathbb{E}[Z] < \frac{m}{2}$ (we don't lose the strict inequality, as we are still summing over a finite set), and hence the contradiction. Therefore at least one assignment satisfy $\frac{m}{2}$ clauses.*

Second attempt: *I'm not sure if the first attempt count as a derivation from the statement $\mathbb{E}[Z] \ge \frac{m}{2}$. We can also derive it the next way:*

Let W_1 (resp. W_2) be the set of evaluations that lead to $Z(\alpha) \geq \frac{m}{2}$ (resp. lead to $Z(\alpha) < \frac{m}{2}$). Each evaluation α has an equal probability of $\frac{1}{2^n}$. We need to prove that $W_2 \neq \emptyset$.

$$\begin{aligned}
\mathbb{E}[Z] &= \sum_{\alpha \in \{0,1\}^n} Z(\alpha)\mathbb{P}(\alpha) \\
&= \sum_{\alpha \in W_1} Z(\alpha)\mathbb{P}(\alpha) + \sum_{\alpha \in W_2} Z(\alpha)\mathbb{P}(\alpha) \\
&= \frac{1}{2^n}(\sum_{\alpha \in W_1} Z(\alpha) + \sum_{\alpha \in W_2} Z(\alpha)) \\
&\geq \frac{m}{2} \\
&= \frac{1}{2^n}(\sum_{\alpha \in W_1} \frac{m}{2} + \sum_{\alpha \in W_2} \frac{m}{2})
\end{aligned}$$

By re-arranging and dropping the $\frac{1}{2^n}$, we get the following inequality:

$$\sum_{\alpha \in W_1} (Z(\alpha) - \frac{m}{2}) + \sum_{\alpha \in W_2} (Z(\alpha) - \frac{m}{2}) \geq 0$$

The second term is a strictly negative number (assuming $W_2 \neq \emptyset$). As the sum of both terms is positive, the first term has to be strictly positive, and hence $W_1 \neq \emptyset$

Exercise 4.

By toying around with simple use cases, $n = 2$ and $n = 3$, we conjecture that the probability would equal $1/2$.

Let X_i be the random variable associated with the seat that the i^{th} passenger chose. At the end of the experiment, we end up with a chain of RVs, $X_1, \ldots, X_n$, and we are interested in computing $\mathbb{P}[X_n = n]$.

First, we notice that X_n can only take the values 1 or n, if any value $i \notin \{1, n\}$ was left, then it should've also been available at the step i, when the i^{th} passenger was making his choice, and he will then deterministically take his seat.

Let Ω be the set of all possible realizations of $X_1, \ldots, X_n$, and let W_1 (resp. W_2) be the set of realizations that lead to $X_n = n$ (resp. $X_n = 1$). We have $W_1 \neq \emptyset$ since $(1, 2, \ldots, n) \in W_1$, similarly, $W_2 \neq \emptyset$ since $(n, 2, 3, \ldots, n-1, 1) \in W_2$. More importantly, W_1 and W_2 partition the space Ω, since X_n can either take the values n or 1.

We have $\mathbb{P}[X_n = n] = \frac{|W_1|}{|\Omega|}$, therefore, by proving that $|W_1| = |W_2|$ we get our conjectured answer. From now on, the problem is more combinatorial.

We define the following function f:

$$\begin{aligned}
f \colon W_1 &\to \{1, \ldots, n\}^n \\
(x_1, \ldots, x_{n-1}, n) &\mapsto (y_1, \ldots, y_{n-1}, 1) \\
&\quad \textit{s.t for } i < n, \textit{ if } x_i = 1 \textit{ then } y_i = n \textit{ else } y_i = x_i
\end{aligned}$$

The function f flips the position of 1 and n, and keeps the remaining positions intact.

It remains to prove the following:

(a) *f is well defined*

(b) *$f(W_1) \subset W_2$, giving rise to $\hat{f}\colon W_1 \to W_2$ s.t: $\hat{f}(X) = f(X)$*

(c) *$\hat{f}$ is a bijection, and hence $|W_1| = |W_2|$.*

(a) *Let $A = (a_1, \dots, a_n)$, and $B = (b_1, \dots, b_n)$ be two **equal** elements of W_1, let's prove that $f(A) = f(B)$. Let i be the position of 1 in A (and B). We have, by definition of f: $f(A)_n = f(B)_n = 1$. Also, $\forall j \neq i\colon f(A)_j = A_j = B_j = f(B)_j$. Finally, $f(A)_i = n = f(B)_i$ (since $A_i = B_i = 1$), and hence $f(A) = f(B)$. Which completes the proof that f **is well defined.***

(b) *Let $(x_1, \dots, x_n) \in W_1$ we need to prove that $(y_1, \dots, y_n) = f(x_1, \dots, x_n)$ is a feasible event of the experiment, and hence conclude that $(y_1, \dots, y_n) \in W_2$, since $y_n = 1$. Let i be the position of 1 in X.*

Steps** 1 **to** $i-1$**: *$(y_1, \dots, y_{i-1})$ is an exact replica of $(x_1, \dots, x_{i-1})$, therefore it's feasible to have this outcome.*

At step** i**: *As the passenger i chose the seat 1, it means that the i^{th} seat was already taken, implying that he is going to pick uniformly from the available seats. As the n^{th} seat was still available (it was only taken at the last step $x_n = n$), the sequence $(y_1, \dots, y_i)$ is perfectly feasible.*

Steps from** $i+1$ **to** $n-1$**: *We start with the step $i+1$, now $(y_1, \dots, y_i)$ is feasible, let Y_{i+1} be the RV associated with next choice, knowing that the sequence of seats $(y_1, \dots, y_i)$ was already selected.*

*First, we notice that x_{i+1} is still available for Y_{i+1} to grab. **[Proof:** We have $x_{i+1} \notin \{x_1, \dots, x_i\}$ and $x_{i+1} \neq n$ (since $x_n = n$), therefore, $x_{i+1} \notin \{y_1, \dots, y_i\} = \{x_1, \dots, x_{i-1}\} \cup \{n\}$. **Proof end]**. So, if $x_{i+1} = i+1$, then $Y_{i+1} = y_{i+1} = x_{i+1} = i+1$ is the only possible outcome, hence, the sequence $(y_1, \dots, y_{i+1})$ is valid, otherwise (if $x_{i+1} \neq i+1$), then the seat $i+1$ have already been taken, i.e., $i+1 \in \{x_1, \dots, x_{i-1}\} \subset \{y_1, \dots, y_i\}$, and Y_{i+1} could pick any available seat, at uniformly random, since x_{i+1} is available to grab as we already established, $Y_{i+1} = y_{i+1} = x_{i+1}$ is a feasible outcome. And hence, $(y_1, \dots, y_{i+1})$ is a feasible sequence. The same reasoning is applied for the next steps, which proves that the sequence $(y_1, \dots, y_{n-1})$ is feasible.*

Step** n**: *The only left choice for Y_n to take is 1, since the n^{th} seat was already taken at step i. Therefore, the whole sequence $(y_1, \dots, y_n)$ is feasible, i.e., $(y_1, \dots, y_n) \in W_2$.*

(c) *$\hat{f}$ is clearly injective.*

Let $Y \in W_2$, and let's construct X as follows, For all $i < n$: $x_i = \delta(y_i \neq n)y_i + \delta(y_i = n)1$ and $x_n = n$. Following a similar line of thought to the past question, we can prove that X is a feasible event of the experiment, i.e., $X \in W_1$. Additionally, by construction of X, we have $\hat{f}(X) = Y$. This proves that $\hat{f}$ is surjective, and we can conclude that $\hat{f}$ is a bijection.

Since $\hat{f}$ is bijective, we get $|W_1| = |W_2|$, and hence:

$$
\begin{aligned}
\mathbb{P}[X_n = n] &= \frac{|W_1|}{|\Omega|} \\
&= \frac{|W_1|}{|W_1| + |W_2|} \quad \textit{(}W_1,\ W_2 \textit{ partition } \Omega\textit{)} \\
&= \frac{1}{2}
\end{aligned}
$$