Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 5

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Exercise 1.

(a) *We have established in the notes that the probability of not making any mistakes after $n - t$ contractions is at least $\frac{t(t-1)}{n(n-1)}$, in this case, when $t = n - 1$, we would get that this probability is $\frac{n-2}{n} = 1 - \frac{2}{n}$, which is in fact equal to q_n.*

(b) *We denote the probability that S_1 (resp S_2) is correct as P_1 (resp. P_2), we have:*

$$P_1 = \mathbb{P}[\{Output\ of\ L5\ is\ correct\}|\{Contraction\ at\ L4\ is\ correct\}]\mathbb{P}[\{Contraction\ at\ L4\ is\ correct\}]$$
$$= P(n-1)q_n$$
$$P_2 = P(n)$$

The algorithm is succesful if either the return at line 7 was a succes (i.e., S_1 is a success) or the return at line 10 was a succes, which yiled the following:

$$p(n) = \mathbb{P}[\{Entered\ L7\}]\mathbb{P}[\{Succes\}|\{entered\ L7\}] + \mathbb{P}[\{Didn't\ Enter\ L7\}]\mathbb{P}[\{Succes\}|\{Didnt\ enter\ L7\}]$$
$$= q_n\mathbb{P}[\{Succes\}|\{entered\ L7\}] + (1 - q_n)\mathbb{P}[\{Succes\}|\{Didnt\ enter\ L7\}]$$
$$= q_n\mathbb{P}[\{S_1\ is\ succesful\}] + (1 - q_n)\mathbb{P}[\{The\ best\ of\ S_1,\ S_2\ was\ succesful\}]$$
$$= q_n\mathbb{P}[\{S_1\ is\ succesful\}] + (1 - q_n)(1 - \mathbb{P}[\{S_1\ and\ S_2\ failed\}])$$
$$= q_nP_1 + (1 - q_n)[1 - (1 - P_1)(1 - P_2)] \qquad (We\ will\ just\ keep\ re\text{-}arranging\ after\ now)$$
$$= q_n^2P(n-1) + (1 - q_n)[1 - (1 - q_nP(n-1))(1 - P(n))]$$
$$= q_n^2P(n-1) + (1 - q_n)[P(n) + q_nP(n-1) - q_nP(n-1)P(n)]$$
$$= q_nP(n-1) + (1 - q_n)P(n) - q_n(1 - q_n)P(n-1)P(n)$$
$$\implies$$
$$q_nP(n) = q_nP(n-1) - q_n(1 - q_n)P(n-1)P(n)$$
$$\implies$$
$$P(n) = P(n-1) - (1 - q_n)P(n-1)P(n) \qquad (We\ would\ verify\ line\ 1\ if\ n = 2,\ hence\ q_n\ would\ never\ equal\ 0\ )$$

(c) *By dividing the equation that we derived in the last question by $P(n)P(n - 1)$, we get the following, for $k \in \{3, \ldots, n\}$:*

$$\frac{1}{P(k-1)} - \frac{1}{P(k)} = -\frac{2}{k}$$

$$\Longrightarrow$$

$$\sum_{k=3}^{n} \frac{1}{P(k-1)} - \sum_{k=3}^{n} \frac{1}{P(k)} = -\sum_{k=3}^{n} \frac{2}{k}$$

$$\frac{1}{P(2)} - \frac{1}{P(n)} = -\sum_{k=3}^{n} \frac{2}{k}$$

$$\Longrightarrow$$

$$P(n) = \frac{1}{\sum_{k=3}^{n} \frac{2}{k} + 1}$$

Comparisons of the probability of succes with FastCut:

- GeoContraction has a $\Theta(\frac{1}{\log(n)})$ (Assuming the bounds are exact like suggested in **(b)**)

- FastCut had a $\Omega(\frac{1}{\log(n)})$

Exercise 2.

Let X_t represent the number of walks that are left to reach home at step t. Obviously $X_t \in \{0, \ldots, n\}$, and we are interested in computing $\mathbb{E}[T]$ where $T = \inf\{t \geq 0 | X_t = 0\}$.

We have the following:

$$\begin{aligned}
\mathbb{E}_{(X_t, X_{t+1})}[X_t - X_{t+1} | X_t = s] &= \mathbb{E}_{X_{t+1}}[s - X_{t+1} | X_t = s] \\
&= s - \mathbb{E}[X_{t+1} | X_t = s] \\
&= s - \sum_{X_{t+1}} \mathbb{P}[X_{t+1} | X_t = s] X_{t+1}
\end{aligned}$$

If $s = n$ (we are at the bar), X_{t+1} has two options $\{n, n-1\}$

$$\mathbb{E}[X_t - X_{t+1} | X_t = s] = n - n\mathbb{P}[X_{t+1} = n | X_t = s] - (n-1)\mathbb{P}[X_{t+1} = n-1 | X_t = s]$$

$$= \frac{1}{5}$$

If $0 < s < n$, X_{t+1} has two options $\{s-1, s+1\}$

$$\mathbb{E}[X_t - X_{t+1} | X_t = s] = s - (s-1)\mathbb{P}[X_{t+1} = s-1 | X_t = s] - (s+1)\mathbb{P}[X_{t+1} = s+1 | X_t = s]$$

$$= \frac{3}{5} + \frac{2}{5}$$

$$= 1$$

Hence, we get, for all $t > 0, s \neq 0$:

$$\frac{1}{5} \leq \mathbb{E}[X_t - X_{t+1} | X_t = s] \leq 1$$

Which yields the following bounds, assuming $X_0 = n$:

$$n \leq \mathbb{E}[T] \leq 5n$$

Exercise 3.
(a) As the state 0 is never reacheable and $X_0 = 1$, T could be formulated as follows:

$$T = \inf\{t \geq 1 | X_t = -1\}$$

i.e, the first time we reach the state $s = -1$ (our definition of "succcess", speaking from a geometric distribution terminology). By noting that each X_t is a Bernoulli experiment with $p = 1/2$, we conclude that $\mathbb{E}[T] = 1/p = 2$.

(b) The state we are interested to reach in the definition of T is -1, in the theorem the state of interest is $s = 0$, and X_t only takes positive values in the theorem statement, again, here X_t can take the value -1.

We have for $s \neq 0$: $\mathbb{E}[X_{t+1} | X_t = s] = 0$, and hence:

$$\mathbb{E}[X_t - X_{t+1} | X_t = s] = s - \mathbb{E}[X_{t+1} | X_t = s] = s$$

Which yields, $\mathbb{E}[X_t - X_{t+1} | X_t = s] \leq 1$ for all $s \neq 0$. using the additive drift theorem would lead to conclude that $1 \leq E[T]$

To make it work, we define the random process (Y_t) as $Y_t = X_t + 1$, clearly $Y_t \in \{0, 1, 2\}$, and

$$\begin{aligned} T &= \inf\{t \geq 0 | X_t = -1\} \\ &= \inf\{t \geq 0 | X_t + 1 = 0\} \\ &= \inf\{t \geq 0 | Y_t = 0\} \end{aligned}$$

The way we defined $(Y_t)_{t \geq 0}$, and how we formulated T using (Y_t), matches the theorem's assumptions. Moreover we have: $[s \neq 0 \wedge \mathbb{P}(Y_t = s) > 0 \implies s = 2]$, and hence we only need to verify the theorem's conditions for $s = 2$:

$$\begin{aligned} \mathbb{E}[Y_t - Y_{t+1} | Y_t = 2] &= 2 - \mathbb{E}[Y_{t+1} | Y_t = 2] \qquad ((Y_{t+1} \text{ can only move to 0 or 2}) \\ &= 2 - 2\mathbb{P}[Y_{t+1} = 2 | Y_t = 2] \\ &= 2 - 2\frac{1}{2} \\ &= 1 \end{aligned}$$

And hence, using the additive drift theorem, we get:

$$\frac{\mathbb{E}[Y_0]}{1} \leq \mathbb{E}[T] \leq \frac{\mathbb{E}[Y_0]}{1} \implies \mathbb{E}[T] = 2, since \; \mathbb{E}[Y_0] = 2$$

Which is the same as the answer that we got in **(a)**.

Exercise 4.
(a) Let's first compute $\mathbb{E}[X_t - X_{t+1} | X_t = s]$ for some $0 < s \leq n$:

$$\begin{aligned} \mathbb{E}[X_t - X_{t+1} | X_t = s] &= s - \sum_{i=0}^{s-1} \mathbb{P}[X_{t+1} = i | X_t = s]i \\ &= s - \frac{1}{s} \sum_{i=0}^{s-1} i \\ &= \frac{s+1}{2} \end{aligned}$$

Hence, we have $\mathbb{E}[X_t - X_{t+1}|X_t = s] \geq \frac{s+1}{2}$, for all $0 < s \leq n$, with $h(s) : s \mapsto \frac{s+1}{2}$ being monotonically increasing, we have verified all assumptions of the Variable Drift Theorem, hence, we can conclude the following:

$$\mathbb{E}[T] \leq \frac{1}{h(1)} + \int_1^n \frac{1}{h(x)} \, dx$$

$$= 1 + \int_1^n \frac{2}{s+1} \, dx$$

$$= 1 + 2 \int_2^{n+1} (\ln(x))' \, dx$$

$$= 1 + 2(\ln(n+1) - \ln(2))$$

$$\leq 2\ln(n+1) \qquad (1 - 2\ln(2) \approx -0.39)$$

Using the Additive Drift Theorem (since $\mathbb{E}[X_t - X_{t+1}|X_t = s] \geq 1$, for all $0 < s \leq n$) would lead to conclude that $\mathbb{E}[T] \leq n$, which is less tight than the gap that we obtained using the Variable Drift Theorem.

(b) The function $h(s) : s \mapsto \frac{1}{s}$ isn't monotonically increasing, hence we can't use the Variable Drift Theorem.

We have, for $0 < s \leq n$, $\mathbb{E}[X_t - X_{t+1}|X_t = s] \geq \frac{1}{n}$, hence, using the Additive Drift Theorem, yields the following: $\mathbb{E}[T] \leq n^2$

(c) Blabla

$$\mathbb{E}[T] = \frac{1}{h(1)} + \int_1^n x^{-\frac{1}{2}} \, dx$$

$$= 1 + \frac{1}{2} \int_1^n \sqrt{x} \, dx$$

$$= \frac{1}{2}(\sqrt{n} + 1)$$

(d) Consider the random process $(X_t)_{t \geq 0}$ on state space $\{0, 1, \ldots, n\}$, where knowing that $X_t = s$ for $s > 0$, X_{t+1} is defined as follows:

$$X_{t+1} = \begin{cases} s - 1 & \text{with probability } \frac{1}{s} \\ s & \text{with probability } 1 - \frac{1}{s} \end{cases}$$

For $s > 0$ and $t \geq 1$:

$$\mathbb{E}[X_t - X_{t+1}|X_t = s] = s - \mathbb{E}[X_{t+1}|X_t = s]$$

$$= s - [(s-1)\frac{1}{s} + s(1 - \frac{1}{s})]$$

$$= s - [1 - 1/s + s - 1]$$

$$= \frac{1}{s}$$