Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 8

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Exercise 1.

*(a) The algorithm described briefly as follows, has a **cubic** runtime.*

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1: for 1 ≤ i ≤ n, 1 ≤ j ≤ n do
2:     α_{i,j} ← sum([A_{i,k}B_{k,j} : k ∈ {1,...,n}])
3:     α_{i,j} ← α_{i,j} − c_{i,j}
4:     if α_{i,j} ≠ 0 then
5:         return 0
6:     end if
7: end for
8: return 1
```

(b) Computing x, y, and z requires $\mathcal{O}(n^2)$ each, and computing $t = y - z$ requires $\mathcal{O}(n)$, hence, the asymptotic runtime of this RA is $\mathcal{O}(n^2)$.

(c) If $r \neq 0$, the event $\mathcal{E}$ implies the following:

- *$r \in Ker(D)$*

- *r is orthogonal to all rows of D*

- *Since $D \neq 0$, there exists at least one row $d_i \neq 0$ s.t: $d_i^\top r = 0$*

We have:

$$\mathbb{P}[\mathcal{E}] = \mathbb{P}[Dr = 0|r = 0]\mathbb{P}[r = 0] + \mathbb{P}[Dr = 0|r \neq 0]\mathbb{P}[r \neq 0]$$

$$= \frac{1}{3^n} + \mathbb{P}[Dr = 0|r \neq 0](1 - \frac{1}{3^n}) \qquad [1]$$

For some $k \leq n$, we let, $d_1, \ldots, d_k$ be the rows of D that are not equal to 0. We have:

$$\mathbb{P}[Dr = 0|r \neq 0] = \mathbb{P}[d_1^\top r = 0, \ldots, d_k^\top r = 0|r \neq 0]$$

$$\leq \mathbb{P}[d_1^\top r = 0|r \neq 0] \qquad [2]$$

*Let $d_{1,j}$ be the last element of d_1 not equal to 0, we have: $d_1^\top r = 0$ **if and only if** after $j - 1$ picks of $r_1, \ldots, r_{j-1}$, the j^{th} pick (i.e., r_j) is chosen s.t: $-d_{1,j}r_j = \sum_{i=1}^{j-1} d_{1,i}r_i$.*

And hence, knowing the values $r_1, \ldots, r_{j-1}$, we conclude the following:

$$\mathbb{P}[d_1^\top r = 0 | r \neq 0] = \mathbb{P}\left[r_j = \sum_{i=1}^{j-1} \frac{-d_{1,i}}{d_{1,j}} r_i | r \neq 0 \right] \leq \frac{1}{3}$$

Using [2], we get $\mathbb{P}[Dr = 0 | r \neq 0] \leq \frac{1}{3}$, anf from [1], we conclude that:

$$\mathbb{P}[\mathcal{E}] \leq \frac{1}{3^n} + \frac{1}{3}(1 - \frac{1}{3^n}) \leq \frac{1}{3}$$

(d) We can reduce the probability by amplification.

Exercise 2.

(a) We condider the following Algorithm T_k, we choose $k = [\sqrt{2n}] + 1$.

1: We generate k images, as follows: $e_1 = h(1), \ldots, e_k = h(k)$
*2: **if** A collision happens **then***
*3:   **return** $s = n$*
*4: **end if***
*5: **return** $s = n^2$*

(b) We will refer to the probability of T_k being correct as $\mathbb{P}[T]$, we have:

$$\mathbb{P}[T] = \mathbb{P}[s = n]\mathbb{P}[T | s = n] + \mathbb{P}[s = n^2]\mathbb{P}[T | s = n^2]$$
$$= \frac{1}{2}(\mathbb{P}[T | s = n] + \mathbb{P}[T | s = n^2])$$
$$= \frac{1}{2}(\mathbb{P}[A \text{ collision happening in } T_k | s = n] + \mathbb{P}[No \text{ collisions happening in } T_k | s = n^2])$$

From the lecture notes, we have, $\mathbb{P}[A \text{ collision happening in } T_k | s = n] > \frac{1}{2}$, also:

$$\mathbb{P}[No \text{ collisions happening in } T_k | s = n^2]) = \prod_{i=2}^{k}(1 - \frac{i-1}{n^2})$$
$$\geq (1 - \frac{k-1}{n^2})^k \qquad \text{(using the Bernoulli Inequality)}$$
$$\geq 1 - \frac{k(k-1)}{n^2}$$
$$> \frac{1}{2} \qquad (as \; \frac{n}{2} > k)$$

Which finally yields:

$$\mathbb{P}[T] = \frac{1}{2}(\mathbb{P}[A \text{ collision happening in } T_k | s = n] + \mathbb{P}[No \text{ collisions happening in } T_k | s = n^2])$$
$$> \frac{1}{2}(\frac{1}{2} + \frac{1}{2})$$
$$= \frac{1}{2}$$

Exercise 3.

(a) We have:

$$\sum_{i\geq 1} \mathbb{P}[X \geq i] = \sum_{i\geq 1} \sum_{j=i}^{\infty} \mathbb{P}[X = j]$$

$$= \sum_{j\geq 1} \sum_{i=1}^{j} \mathbb{P}[X = j]$$

$$= \sum_{j\geq 1} \mathbb{P}[X = j] \sum_{i=1}^{j} 1$$

$$= \sum_{j\geq 1} j\mathbb{P}[X = j] = \mathbb{E}[X]$$

If X can only take negative values, then:

$$\mathbb{E}[X] = -\mathbb{E}[-X]$$

$$= -\sum_{j\geq 1} \mathbb{P}[-X \geq j]$$

$$= -\sum_{j\geq 1} \mathbb{P}[X \leq -j]$$

(b) Let $\mathbb{P}[X \geq 0] = p_X$, and $\mathbb{P}[Y \geq 0] = p_Y$, we have: $p_X \leq p_Y$, we suppose that $0 < p_X < 1$ and $0 < p_Y < 1$:

$$\mathbb{E}[Y] - \mathbb{E}[X] = p_Y \mathbb{E}[Y|Y > 0] + (1 - p_Y)\mathbb{E}[Y|Y < 0] - p_X \mathbb{E}[X|X > 0] - (1 - p_X)\mathbb{E}[X|X < 0]$$
$$= p_Y \mathbb{E}[Y|Y > 0] - p_X \mathbb{E}[X|X > 0] + (1 - p_Y)\mathbb{E}[Y|Y < 0] - (1 - p_X)\mathbb{E}[X|X < 0] \qquad \textbf{\textit{Eq. \#42}}$$

We have:
for $i \geq 1$:

$$\mathbb{P}[X \geq i] \leq \mathbb{P}[Y \geq 1] \implies \sum_{k\geq 1} \mathbb{P}[X \geq k] \leq \sum_{k\geq 1} \mathbb{P}[Y \geq k]$$

$$\implies \sum_{k\geq 1} \frac{\mathbb{P}[X \geq k]}{p_Y} \leq \sum_{k\geq 1} \frac{\mathbb{P}[Y \geq k]}{p_Y}$$

$$\implies p_X \sum_{k\geq 1} \frac{\mathbb{P}[X \geq k]}{p_X} \leq p_Y \sum_{k\geq 1} \frac{\mathbb{P}[Y \geq k]}{p_Y} \qquad (as\ p_X \leq p_Y)$$

$$\implies p_X \mathbb{E}[X|X > 0] \leq p_Y \mathbb{E}[Y|Y > 0] \quad \textbf{\textit{[1]}} \quad (using\ the\ result\ of\ the\ past\ question)$$

On the other side, we have:

$$\mathbb{E}[X|X < 0] = -\sum_{i\geq 1} \frac{\mathbb{P}[X \leq -i]}{1 - p_X} \implies (1 - p_X)\mathbb{E}[X|X < 0] = -\sum_{i\geq 1} \mathbb{P}[X \leq -i]$$

3

Similarily:

$$(1 - p_Y)\mathbb{E}[Y|Y < 0] = -\sum_{i \geq 1} \mathbb{P}[Y \leq -i]$$

Hence,

$$
\begin{aligned}
(1 - p_Y)\mathbb{E}[Y|Y < 0] - (1 - p_X)\mathbb{E}[X|X < 0] &= \sum_{i \geq 1} \mathbb{P}[X \leq -i] - P[Y \leq -i] \\
&= \sum_{i \geq 1} \mathbb{P}[Y \geq i] - P[X \geq i] \\
&\geq 0 \qquad [2]
\end{aligned}
$$

*By using the results in [1] and [2] in **Eq.** #42, we conclude that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.*