Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 4

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Exercise 1.

(a) *Hey*

(b) *Hey*

Exercise 2.

Let $C = \{x_1, \ldots, x_N\}$ be a random cut of the graph, where $\{x_i\}_{1 \leq i \leq N}$ representes the edges. We are obviously interested in $\mathbb{E}[N]$, i.e., the expected number of edges in a cut. Let $E = \{e_1, \ldots, e_{|E|}\}$ and let the RV X_i be the indicator of edge e_i in C.

Clearly $N = \sum_{i=1}^{|E|} X_i$, and hence, $\mathbb{E}[N] = \sum_{i}^{|E|} \mathbb{E}[X_i]$

Now we prove that $\mathbb{E}(X_i) = 1/2$. Suppose the edge e_i connects the vertices A and B.

$$\begin{aligned}
\mathbb{E}[X_i] &= \mathbb{P}[X_i = 1] \\
&= \mathbb{P}[\{A \text{ random cut contains } e_i\}] \\
&= \mathbb{P}[\{A \text{ cut contains one and only one of } A \text{ or } B\ \}]
\end{aligned}$$

Each cut is defined by a split of vertices S_1/S_2, where S_1 selects $j \in \{1, \ldots, |V| - 1\}$ vertices at random from V. Each vertex has 1/2 probability to be in S_1 (resp. S_2).

$$\begin{aligned}
\mathbb{P}[\{(A, B) \in (S_1, S_2) \lor (A, B) \in (S_2, S_1)\}] &= \mathbb{P}[\{(A, B) \in (S_1, S_2)\}] + \mathbb{P}[\{(A, B) \in (S_2, S_1)\}] \\
&= \mathbb{P}[\{A \in S_1 \land B \in S_2\}] + \mathbb{P}[\{A \in S_1 \land B \in S_1\}] \\
&= \mathbb{P}[\{A \in S_1\}]\mathbb{P}[\{B \in S_2\}] + \mathbb{P}[\{A \in S_1\}]\mathbb{P}[\{B \in S_1\}] \\
&= 1/21/2 + 1/21/2 = 1/2
\end{aligned}$$