Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 5

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Exercise 1.

Let $(X_t)_{t\geq 0}$ be the stochastic process where X_t is the number of remaining gifts at round t. At step t, we let $\{g_1, \dots, g_{X_t}\}$ be an arbitrary ordering of the remaing gifts, we define the RVs $(Y_i^t)_{1\leq i\leq X_t}$ as follows:

$$Y_i^t = \begin{cases} 1 & \textit{if the gift } g_i \textit{ was picked by one and only one child in the next round } t+1 \\ 0 & \textit{otherwise} \end{cases}$$

Also, with $X_t = s$, we have, $\mathbb{P}[Y_i^t = 1] = \binom{s}{1}\frac{1}{s}(\frac{s-1}{s})^{s-1} = (\frac{s-1}{s})^{s-1}$

Most importantly, we have $X_{t+1} = X_t - \sum_{i=1}^{i=X_t} Y_i^t$, hence, we can write the following, for $s > 0$:

$$\begin{aligned}
\mathbb{E}[X_t - X_{t+1} | X_t = s] &= \mathbb{E}[\sum_{i=1}^{i=X_t} Y_i^t | X_t = s] \\
&= \mathbb{E}[\sum_{i=1}^{i=s} Y_i^t | X_t = s] \\
&= \sum_{i=1}^{s} \mathbb{E}[Y_i^t] \\
&= \sum_{i=1}^{s} \mathbb{P}[Y_i^t = 1] \\
&= \sum_{i=1}^{s} (1 - \frac{1}{s})^{s-1} \\
&\geq \frac{s}{e}
\end{aligned}$$

With $h(s) : s \mapsto \frac{s}{e}$ being monotonicaly increasing, and by applying the Variable Drift Theorem, we conclude the following:

$$
\begin{aligned}
\mathbb{E}[T|X_0 = n] &\leq \frac{1}{h(1)} + \int_1^n \frac{1}{h(x)}\, dx \\
&= \frac{1}{e} + e\int_1^n \frac{1}{s}\, dx \\
&= \frac{1}{e} + e\ln(n)
\end{aligned}
$$