Randomised Algorithms
Winter term 2022/2023, Exercise Sheet No. 3

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Exercise 1.

(a) *Comparisons between 15 and 8:*

- *Pivot is 7: Decision will be postponed.*
- *Pivot is 15: They will be compared.*
- *Pivot is 10: They will be imediately seperated and hence never be compared.*

(b) *The probability of 8 and 15 being compared, is the probability that the first pivot to be selected from the set $\{$Numbers in the input between 8 and 15$\}$ is either 8 or 15:*
$\frac{2}{|\{\text{Numbers in the input between 8 and 15}\}|} = \frac{2}{|\{8,11,19,10,15\}|} = \frac{2}{5}$

Exercise 2.

We assume for questions (a) and (b) that $\frac{1}{2} + \frac{1}{\sqrt{n}} < 1$, which yields that $n > 4$.

(a) *Based on the lecture notes, for a letter x of size n, we would have: $\mathbb{P}[\{A_t \text{ failing}\}] \leq (1 - \frac{4}{n})^{\frac{t}{2}}$*

(b) *Let's say we run $\mathcal{A}$, $2n^k$ times, for a positive integer k. The probablity of $\mathcal{A}_{2n^k}$ failing is:*

$$\begin{aligned}
\mathbb{P}[\{\mathcal{A}_{2n^k} \text{ failing}\}] &\leq (1 - \frac{4}{n})^{n^k} \\
&\leq e^{-(1-\frac{4}{n})n^k} \qquad \textit{(using the hint)} \\
&= e^{4n^{k-1} - n^k} \\
&\leq e^{-2n^k} \qquad \textit{(since } 4n^{k-1} \leq n^k\textit{)}
\end{aligned}$$

For k sufficiently large, the RHS is guarenteed to converge to 0, and hence $\forall \epsilon$ there exists k s.t: the RHS is less than $1/2 - \epsilon$

For $1/2 > \epsilon > 0$

$$
\begin{aligned}
&e^{-2n^k} \leq \frac{1}{2} - \epsilon \\
&\implies -2n^k \leq \ln(1/2 - \epsilon) \\
&\implies n^k \geq -\ln(1/2 - \epsilon)/2 \\
&\implies k\ln(n) \geq \ln(-\ln(1/2 - \epsilon)/2) = \alpha \\
&\implies k \geq \frac{\alpha}{\ln(n)}
\end{aligned}
$$

Therefore, any positive k verifying the past inequality would work. As $\mathcal{A}_{2n^k}$ still runs in polynomial time ($2n^kT(n)$), $L \in BPP$

(c) *The probability of $\mathcal{A}$ failing for an input of size n is the same of the probability of failing in the first case (when the probability is at least $\frac{1}{2} + \frac{1}{\sqrt{n}}$) for an input size of n^2. Hence, using **(b)**, $L \in BPP$.*

Exercise 3.

(a) *We start from Top to Bottom, we assign 1 to the root, and follow these two startegies to assign the levels below until we reach the leaves:*

If the parent is $\vee$*:*

- *First child:* 0
- *Second child: Parent Value*

If the parent is $\wedge$*:*

- *First child:* 1
- *Second child: Parent Value*

(b) *The following figures captures the algorithm:*

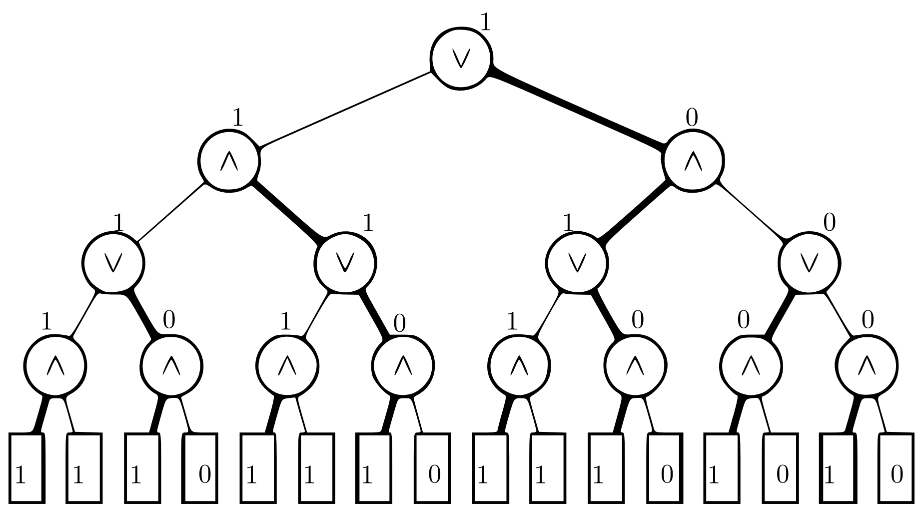

Figure 1: graph-incscape

Exercise 4.

We have for $i, j \in \{1, \dots, n\}$*:*

$$\min_i M_{i,j} \leq M_{i,j} \leq \max_j M_{i,j}$$

Hence, for $i \in \{1, \dots, n\}$, and as the RHS is independent of j:

$$\max_j \min_i M_{i,j} \leq \max_j M_{i,j}$$

Notice that the LHS is a constant (independent of both i and j), and the past inequality is verified $\forall i$. We finally get:

$$\max_j \min_i M_{i,j} \leq \min_i \max_j M_{i,j}$$