Digital Control

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1 1st exercise

1.1 Dynamical System Description

Before developing the observers or the controller it is crucial that we understand the system dynamics. This system can be described as follows:

Equations:

$$\dot{x} = \begin{bmatrix} 0.8815 & 0.4562 \\ -0.4562 & 0.7903 \end{bmatrix} \cdot x + \begin{bmatrix} 0.1185 & 0.1185 \\ 0.4562 & 0.4562 \end{bmatrix} \cdot \begin{bmatrix} u \\ w \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot x + \begin{bmatrix} 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u \\ w \end{bmatrix}$$

Notice that both the manipulated variable and the disturbance are added to the same matrix, this will help with acquiring the step response of the disturbance.

1.2 K and L_p matrices

To control the system, a $y = -K \cdot x$ controller is added along with a predictor estimator observer, $\hat{x}(k+1) = \Phi\hat{x}(k) + \Omega u(k) + Lp[y(k) - H\hat{x}(k)]$. To start with, a controllability and observability check is in order. MATLAB code:

```
% controllability check
controllability = ctrb(Phi, G);
ctrb_rank = rank(controllability);

% observability check
observability = obsv(Phi, H);
obsv_rank= rank(observability);
```

Both of these matrices, controllability and observability are full rank, enabling the placement of the poles for the controller and the observer anywhere.

In this case, the location of the poles is given in the problem statement, $z = 0.6 \pm 0.6i$ for the controller and $z = 0.3 \pm 0.3i$ for the observer. Solving the respective equations here,

$(z+p_1)(z+p_2) = det(zI - \Phi + \Gamma \cdot K)$
$(z+p_3)(z+p_4) = det(zI - \Phi + L_p \cdot H)$

The results from these two are the following matrices,

$$K = \begin{bmatrix} 1.2321 & 0.7142 \end{bmatrix}$$

and

$$L_p = \begin{bmatrix} 1.0718 \\ 0.268 \end{bmatrix}$$

1.3 Disturbance step response and closed loop poles

With the addition of the controller and the observer, the systems' dynamics are now described by the following equation.

$$\begin{bmatrix} \boldsymbol{x}(k+1) \\ \overline{\boldsymbol{x}}(k+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi} & -\boldsymbol{\Gamma K} \\ \boldsymbol{L_p H} & \boldsymbol{\Phi} - \boldsymbol{\Gamma K} - \boldsymbol{L_p H} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}(k) \\ \overline{\boldsymbol{x}}(k) \end{bmatrix}$$

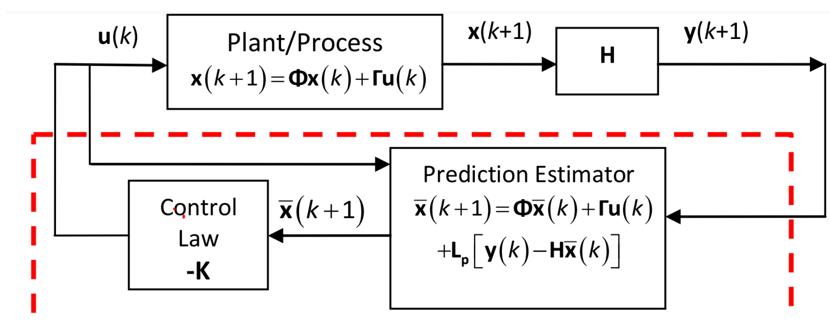

Figure 1: Closed loop schematic.

The step response here results in a steady-state error equal to about 0.078. The closed loop poles are given by this equation:

$det(z \cdot I - \Phi + \Gamma \cdot K + Lp \cdot H) = 0.$

Note that the discretized time step is 0.01 seconds.

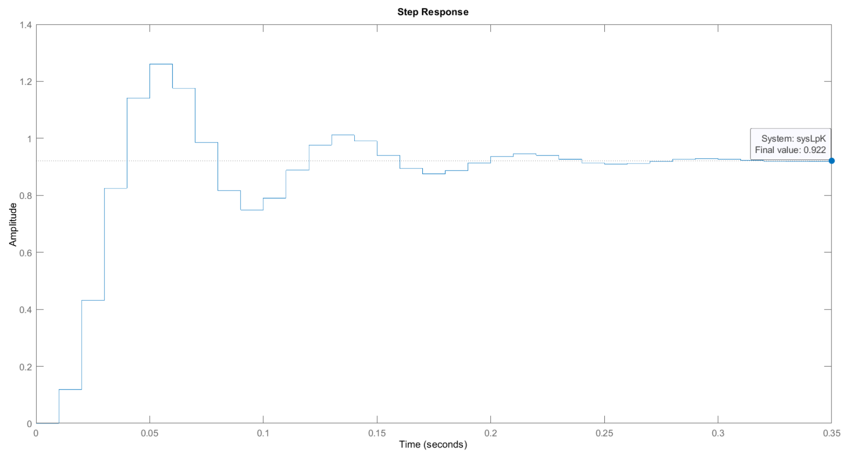

Figure 2: Disturbance step response

1.4 Integration term addition

In an attempt to reduce the steady-state error in our system, an integration term was added to the state representation. To keep it simple, the set point was chosen to be zero, and therefore, $\dot{x}_I = x - x_{sp} = x$. The model is now described as follows.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_I \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0.8815 & 0 & 0.4562 \\ 0 & 1 & 0 \\ -0.4562 & 0 & 0.7903 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_I \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.1185 & 0.1185 \\ 0 & 0 \\ 0.452 & 0.4562 \end{bmatrix} \cdot \begin{bmatrix} u \\ w \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \cdot x + \begin{bmatrix} 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} u \\ w \end{bmatrix}$$

However, this system is not controllable since it is not full rank, the rank is still 2. To navigate around this, the zeros on the 2nd row of the Γ matrix are turned into 0.0001, matching the precision of the other numbers.

1.5 4th question

To obtain the response of the closed loop system described in the previous sub-section, first, there is a need to incorporate another pole to the controller to match the dimensions of the Φ matrix. This pole is placed at $z = 0.9$, as dictated by the problem statement once again. The response to an initial condition perturbation can be seen below. The system, with the addition of the integral action, has zero steady state error.

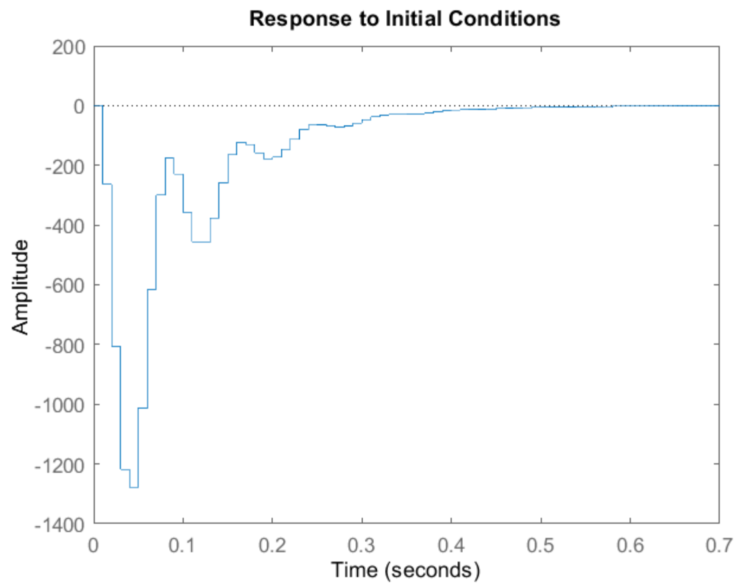

Figure 3: Initial conditions perturbation.

2 2nd exercise

2.1 Dynamical System Description

In this exercise, we are tasked to develop a controller that balances a double inverted pendulum on a cart. The manipulated variable is a force applied to the cart, u, while the controlled variables are the angles of the two pendulums θ_1 and θ_2. The system was linearized around the zero position, and, as described in the relevant paper, "Optimal Control of a Double Inverted Pendulum on a Cart", the following equations stand true.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

where

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{D}(\mathbf{0})^{-1}\frac{\partial \mathbf{G}(\mathbf{0})}{\partial \theta} & \mathbf{0} \end{pmatrix} \\
\mathbf{B} &= \begin{pmatrix} \mathbf{0} \\ \mathbf{D}(\mathbf{0})^{-1}\mathbf{H} \end{pmatrix}
\end{aligned}$$

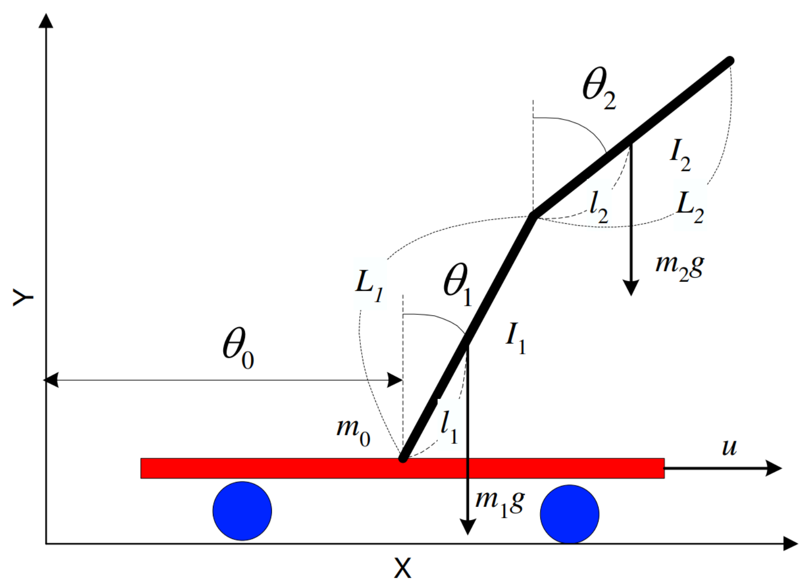

Figure 4: Double inverted pendulum on a cart

2.2 L_p and L_c design and initial condition response

Before designing the observers, one must first find out if the system is observable. To do so, the rank of the observability matrix is assessed. If the matrix is full rank, then the system is observable and the poles of the observers can be placed anywhere in the complex plane. Defining the system in Matlab, the observability matrix turns out to be,

$$OBSV = \begin{bmatrix} 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \\ 2.5500 & 0.4625 & 0.2100 & 0 & 0 & 0 \\ 0.4625 & 0.2125 & 0.1050 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.5500 & 0.4625 & 0.2100 \\ 0 & 0 & 0 & 0.4625 & 0.2125 & 0.1050 \\ 6.7605 & 1.2997 & 0.6017 & 0 & 0 & 0 \\ 1.2997 & 0.2701 & 0.1283 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6.7605 & 1.2997 & 0.6017 \\ 0 & 0 & 0 & 1.2997 & 0.2701 & 0.1283 \end{bmatrix}$$

This matrix is full rank, and in turn, the system is observable. As stated before, the poles of the predictor estimator can be placed anywhere in the complex plane. The current estimator is the product of the multiplication of L_p with the Φ matrix, $L_c = \Phi^{-1} \cdot L_p$. In real systems, the noise effects that the measurements are subject to would mandate that the gain of the observers should not be excessive to avoid their amplification, for this system though, noise is not taken into account.

Choosing the observer poles to be at $observer_poles = [-5, -10, -15, -20, -25, -30]$ and using the place command in Matlab, we end up with the two following initial perturbation responses.

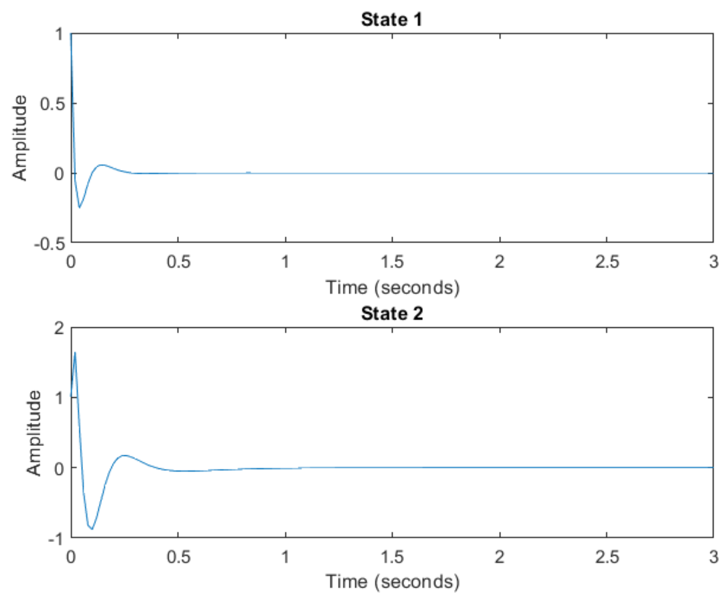

Figure 5: Initial perturbation response with prediction estimator.

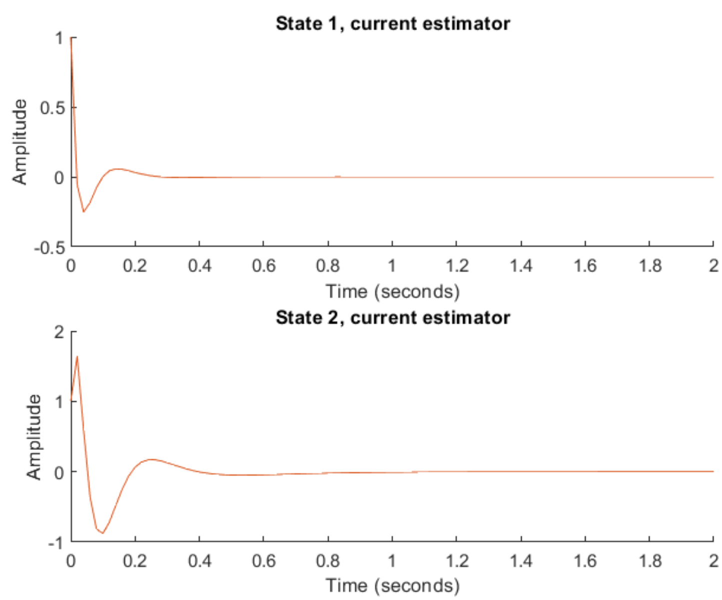

Figure 6: Initial perturbation response with current estimator.

Although the two matrices are not identical, their respective systems' responses appear quite similar. For reference below are the two matrices.

$$L_p = \begin{bmatrix} 9.4004e+01 & -5.6993e-03 \\ -1.5771e+02 & 1.6997e+01 \\ -2.6406e+09 & -4.6498e+03 \\ 3.4509e+06 & 4.7259e+00 \\ 1.6129e+06 & 3.3024e+03 \\ 1.6119e+11 & 1.8175e+03 \end{bmatrix}$$

and

$$L_c = \begin{bmatrix} -2.2499e+05 & -2.9652e-01 \\ -1.1044e+05 & -4.9148e+01 \\ -5.8643e+09 & -4.6863e+03 \\ 2.1316e+07 & 2.4486e+01 \\ 1.0544e+07 & 3.3122e+03 \\ 1.6119e+11 & 1.8254e+03 \end{bmatrix}$$

2.3 Nonlinear initial condition response

To acquire the response of the nonlinear system, we have to include the error in the response. To do the system must be reformulated into the following set of equations

$$\begin{bmatrix} \boldsymbol{e}(k+1) \\ \boldsymbol{x}(k+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi} - \boldsymbol{L_p H} & 0 \\ \boldsymbol{\Gamma K} & \boldsymbol{\Phi} - \boldsymbol{\Gamma K} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}(k) \\ \boldsymbol{x}(k) \end{bmatrix}$$

2.4 Additional state feedback controller

To add a state feedback controller, following the same logic as in the first exercise, one must first choose the desired pole locations. Here they were chosen as, $controller_poles = [-1, -2, -2+1i, -2-1i, -4+1i, -4-1i]$. The following responses are acquired for the linearized system. The figures below appear to be in continuous time but are in fact in a discrete domain. A slight steady state error can be observed.

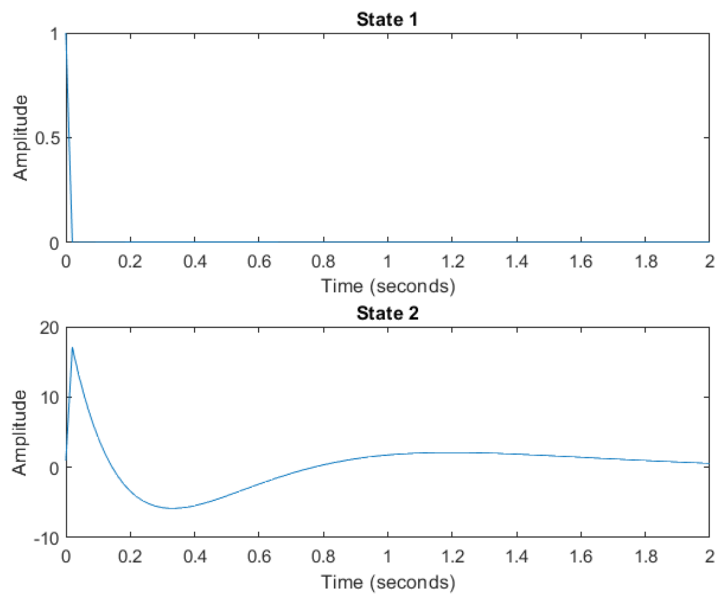

Figure 7: Initial perturbation response with controller in place

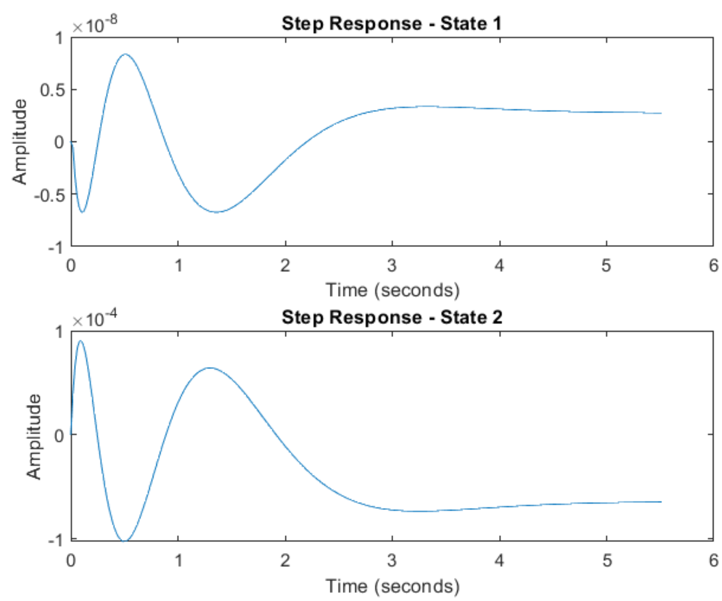

Figure 8: Step response with controller.

As far as the nonlinear system is concerned, acquiring the output requires computation of the error, in addition to the state response.

$$\begin{bmatrix} \boldsymbol{e}(k+1) \\ \boldsymbol{x}(k+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi} - \boldsymbol{L_p H} & 0 \\ \boldsymbol{\Gamma K} & \boldsymbol{\Phi} - \boldsymbol{\Gamma K} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}(k) \\ \boldsymbol{x}(k) \end{bmatrix}$$

Once the error is computed it is added to the each state and the real system output is calculated. This is a complicated process that I could not perform.