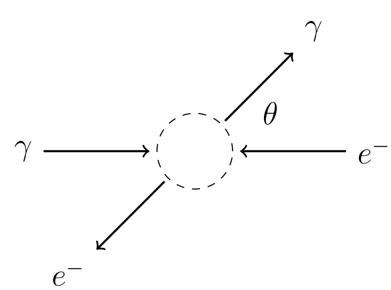
Compton Scattering

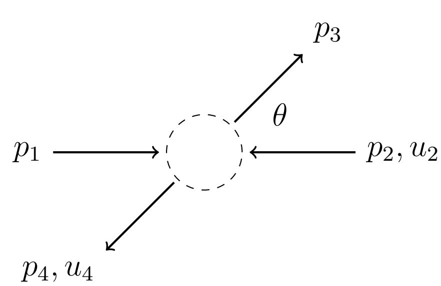
Contents

0.1 Theory

Compton scattering occurs when a high energy photon such as a gamma ray hits an electron. In typical Compton scattering experiments the incident electron is at rest with zero velocity. However, it is easier to develop a theory using the center of mass frame in which the photon and the electron have equal and opposite momentum. The following diagram shows the photon and electron scattering through angle θ in the center of mass frame.



Here is the same diagram with momentum and spinor labels.



Here are the momentum vectors for center of mass coordinates.

$$p_1 = \begin{pmatrix} \omega \\ 0 \\ 0 \\ \omega \end{pmatrix} \qquad p_2 = \begin{pmatrix} E \\ 0 \\ 0 \\ -\omega \end{pmatrix} \qquad p_3 = \begin{pmatrix} \omega \\ \omega\sin\theta\cos\phi \\ \omega\sin\theta\sin\phi \\ \omega\cos\theta \end{pmatrix} \qquad p_4 = \begin{pmatrix} E \\ -\omega\sin\theta\cos\phi \\ -\omega\sin\theta\sin\phi \\ -\omega\cos\theta \end{pmatrix}$$

Symbol ω is the photon energy and $E = \sqrt{\omega^2 + m^2}$ where m is electron mass. The spinors are

$$u_{21} = \begin{pmatrix} E+m \\ 0 \\ -\omega \\ 0 \end{pmatrix} \qquad u_{41} = \begin{pmatrix} E+m \\ 0 \\ p_4^z \\ p_4^x + ip_4^y \end{pmatrix}$$

$$u_{22} = \begin{pmatrix} 0 \\ E+m \\ 0 \\ \omega \end{pmatrix} \qquad u_{42} = \begin{pmatrix} 0 \\ E+m \\ p_4^x - ip_4^y \\ -p_4^z \end{pmatrix}$$

The last digit in a spinor subscript is 1 for spin up and 2 for spin down. Note that the spinors are not individually normalized. Instead, a combined spinor normalization constant $N = (E+m)^2$ will be used where needed.

This is the probability density for Compton scattering. The formula is from Feynman diagrams. Symbol s_j selects the spin of spinor j. Symbol e is electron charge. Symbols s and u are Mandelstam variables $s = (p_1 + p_2)^2$ and $u = (p_1 - p_4)^2$. Symbol $q_1 = p_1 + p_2$ and $q_2 = p_2 - p_3$.

$$|\mathcal{M}(s_2, s_4)|^2 = \frac{e^4}{N} \left| -\frac{\bar{u}_4 \gamma^\mu (\not{q}_1 + m)\gamma^\nu u_2}{s - m^2} - \frac{\bar{u}_4 \gamma^\nu (\not{q}_2 + m)\gamma^\mu u_2}{u - m^2} \right|^2$$

Let

$$a_1 = \bar{u}_4 \gamma^\mu (\not{q}_1 + m)\gamma^\nu u_2 \qquad a_2 = \bar{u}_4 \gamma^\nu (\not{q}_2 + m)\gamma^\mu u_2$$

Then

$$\begin{aligned}
|\mathcal{M}(s_2, s_4)|^2 &= \frac{e^4}{N} \left| -\frac{a_1}{s - m^2} - \frac{a_2}{u - m^2} \right|^2 \\
&= \frac{e^4}{N} \left(-\frac{a_1}{s - m^2} - \frac{a_2}{u - m^2} \right) \left(-\frac{a_1}{s - m^2} - \frac{a_2}{u - m^2} \right)^* \\
&= \frac{e^4}{N} \left(\frac{a_1 a_1^*}{(s - m^2)^2} + \frac{a_1 a_2^*}{(s - m^2)(u - m^2)} + \frac{a_1^* a_2}{(s - m^2)(u - m^2)} + \frac{a_2 a_2^*}{(u - m^2)^2} \right)
\end{aligned}$$

The expected probability density $\langle |\mathcal{M}|^2 \rangle$ is computed by summing $|\mathcal{M}|^2$ over all spin and polarization states and then dividing by the number of inbound states. There are four inbound states. The sum over polarizations is already accomplished by contraction of aa^* over μ and ν.

$$\begin{aligned}
\langle |\mathcal{M}|^2 \rangle &= \frac{1}{4} \sum_{s_2=1}^{2} \sum_{s_4=1}^{2} |\mathcal{M}(s_2, s_4)|^2 \\
&= \frac{e^4}{4} \sum_{s_2=1}^{2} \sum_{s_4=1}^{2} \frac{1}{N} \left(\frac{a_1 a_1^*}{(s - m^2)^2} + \frac{a_1 a_2^*}{(s - m^2)(u - m^2)} + \frac{a_1^* a_2}{(s - m^2)(u - m^2)} + \frac{a_2 a_2^*}{(u - m^2)^2} \right)
\end{aligned}$$

Use the Casimir trick to replace sums over spins with matrix products.

$$f_{11}=\frac{1}{N}\sum_{\text{spins}}a_1a_1^*=\operatorname{Tr}\Big((\not p_2+m)\gamma^\mu(\not q_1+m)\gamma^\nu(\not p_4+m)\gamma_\nu(\not q_1+m)\gamma_\mu\Big)$$
$$f_{12}=\frac{1}{N}\sum_{\text{spins}}a_1a_2^*=\operatorname{Tr}\Big((\not p_2+m)\gamma^\mu(\not q_2+m)\gamma^\nu(\not p_4+m)\gamma_\mu(\not q_1+m)\gamma_\nu\Big)$$
$$f_{22}=\frac{1}{N}\sum_{\text{spins}}a_2a_2^*=\operatorname{Tr}\Big((\not p_2+m)\gamma^\mu(\not q_2+m)\gamma^\nu(\not p_4+m)\gamma_\nu(\not q_2+m)\gamma_\mu\Big)$$

Hence

$$\langle|\mathcal{M}|^2\rangle=\frac{e^4}{4}\left(\frac{f_{11}}{(s-m^2)^2}+\frac{f_{12}}{(s-m^2)(u-m^2)}+\frac{f_{12}^*}{(s-m^2)(u-m^2)}+\frac{f_{22}}{(u-m^2)^2}\right)\tag{1}$$

Run “compton-scattering-1.txt” to verify the Casimir trick for Compton scattering.

These formulas compute probability densities from dot products. Recall that $a\cdot b=a^\mu g_{\mu\nu}b^\nu$.

$$\begin{aligned}
f_{11}&=-16(p_1\cdot p_1)(p_2\cdot p_4)+32(p_1\cdot p_2)(p_1\cdot p_4)+32(p_1\cdot p_4)(p_2\cdot p_2)+16(p_2\cdot p_2)(p_2\cdot p_4)\\
&\quad+64m^2(p_1\cdot p_1)+64m^2(p_1\cdot p_2)-64m^2(p_1\cdot p_4)-48m^2(p_2\cdot p_4)+64m^4\\
f_{12}&=-32(p_1\cdot p_2)(p_2\cdot p_4)+32(p_1\cdot p_3)(p_2\cdot p_4)-32(p_2\cdot p_2)(p_2\cdot p_4)+32(p_2\cdot p_3)(p_2\cdot p_4)\\
&\quad+32m^2(p_1\cdot p_2)-16m^2(p_1\cdot p_3)+16m^2(p_1\cdot p_4)\\
&\quad+48m^2(p_2\cdot p_2)-32m^2(p_2\cdot p_3)+48m^2(p_2\cdot p_4)-16m^2(p_3\cdot p_4)-32m^4\\
f_{22}&=16(p_2\cdot p_2)(p_2\cdot p_4)-32(p_2\cdot p_2)(p_3\cdot p_4)+32(p_2\cdot p_3)(p_3\cdot p_4)-16(p_2\cdot p_4)(p_3\cdot p_3)\\
&\quad-64m^2(p_2\cdot p_3)-48m^2(p_2\cdot p_4)+64m^2(p_3\cdot p_3)+64m^2(p_3\cdot p_4)+64m^4
\end{aligned}$$

In Mandelstam variables $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$, $u=(p_1-p_4)^2$ the formulas are

$$\begin{aligned}
f_{11}&=-8su+24sm^2+8um^2+8m^4\\
f_{12}&=8sm^2+8um^2+16m^4\\
f_{22}&=-8su+8sm^2+24um^2+8m^4
\end{aligned}\tag{2}$$

In a typical Compton scattering experiment where $E\gg m$ the approximation $m=0$ can be used. For the momentum vectors given above and for $m=0$, the probability density in the center of mass frame is

$$\langle|\mathcal{M}|^2\rangle=2e^4\left(\frac{1+\cos\theta}{2}+\frac{2}{1+\cos\theta}\right)$$

Run “compton-scattering-2.txt” to verify.

0.2 Lab frame

Compton scattering experiments are typically done in the “lab” frame where the electron is at rest. The following Lorentz boost Λ transforms momentum vectors from the center of mass frame to the lab frame.

$$\Lambda=\begin{pmatrix}E/m&0&0&\omega/m\\0&1&0&0\\0&0&1&0\\\omega/m&0&0&E/m\end{pmatrix},\qquad\Lambda p_2=\begin{pmatrix}m\\0\\0\\0\end{pmatrix}$$

Mandelstam variables are invariant under a boost.

$$\begin{aligned} s &= (p_1 + p_2)^2 = (\Lambda p_1 + \Lambda p_2)^2 \\ t &= (p_1 - p_3)^2 = (\Lambda p_1 - \Lambda p_3)^2 \\ u &= (p_1 - p_4)^2 = (\Lambda p_1 - \Lambda p_4)^2 \end{aligned}$$

In the lab frame, let ω_L be the angular frequency of the incident photon and let ω'_L be the angular frequency of the scattered photon.

$$\begin{aligned} \omega_L &= \Lambda p_1 \cdot (1,0,0,0) = \frac{\omega^2}{m} + \frac{\omega E}{m} \\ \omega'_L &= \Lambda p_3 \cdot (1,0,0,0) = \frac{\omega^2 \cos\theta}{m} + \frac{\omega E}{m} \end{aligned}$$

It follows that

$$\begin{aligned} s &= (p_1 + p_2)^2 = 2m\omega_L + m^2 \\ t &= (p_1 - p_3)^2 = 2m(\omega'_L - \omega_L) \\ u &= (p_1 - p_4)^2 = -2m\omega'_L + m^2 \end{aligned}$$

Compute $\langle|\mathcal{M}|^2\rangle$ using equations (1) and (2) and the above s, t, and u that involve ω_L and ω'_L.

$$\langle|\mathcal{M}|^2\rangle = 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \left(\frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1 \right)^2 - 1 \right)$$

From the Compton formula

$$\frac{1}{\omega'_L} - \frac{1}{\omega_L} = \frac{1 - \cos\theta_L}{m}$$

we have

$$\cos\theta_L = \frac{m}{\omega_L} - \frac{m}{\omega'_L} + 1$$

Hence

$$\langle|\mathcal{M}|^2\rangle = 2e^4 \left(\frac{\omega_L}{\omega'_L} + \frac{\omega'_L}{\omega_L} + \cos^2\theta_L - 1 \right)$$

Run “compton-scattering-3.txt” to verify.

0.3 Cross section

Now that we have derived $\langle|\mathcal{M}|^2\rangle$ we can investigate the angular distribution of scattered photons. For simplicity let us drop the L subscript from lab variables. From now on the symbols ω, ω', and θ will be lab frame variables.

The differential cross section for Compton scattering is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2} \left(\frac{\omega'}{\omega} \right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + \cos^2\theta - 1 \right)$$

From the Compton equation we have

$$\omega' = \frac{m\omega}{m + \omega(1 - \cos\theta)}$$

Use the Compton equation to eliminate ω' in $d\sigma$.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}\left(\frac{m}{m+\omega(1-\cos\theta)}\right)^2\left(\frac{m+\omega(1-\cos\theta)}{m} + \frac{m}{m+\omega(1-\cos\theta)} + \cos^2\theta - 1\right)$$

We can integrate $d\sigma$ to obtain a cumulative distribution function.

Let

$$I(\xi) = 2\pi\int_0^\xi \frac{d\sigma}{d\Omega}\sin\theta\,d\theta, \quad 0 \le \xi \le \pi$$

The factor 2π is from integrating over azimuth ϕ. The cumulative distribution function is

$$F(\theta) = \frac{I(\theta)}{I(\pi)}, \quad 0 \le \theta \le \pi$$

Hence

$$P(\theta_1 \le \theta \le \theta_2) = F(\theta_2) - F(\theta_1)$$

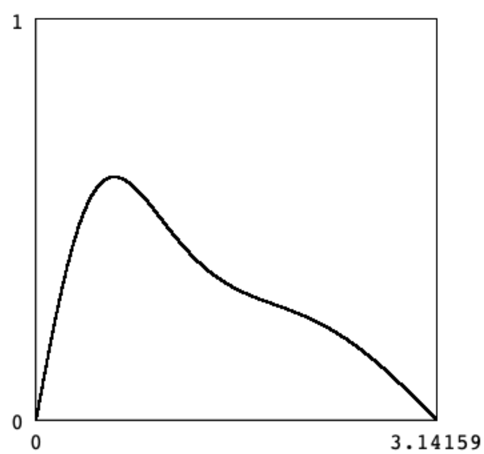
The probability density is

$$f(\theta) = \frac{dF(\theta)}{d\theta} = \frac{2\pi}{I(\pi)}\left(\frac{d\sigma}{d\Omega}\right)\sin\theta, \quad 0 \le \theta \le \pi$$

The following angular distribution is for $\omega = m$.

θ_1	θ_2	$P(\theta_1 \le \theta \le \theta_2)$
0°	45°	0.35
45°	90°	0.34
90°	135°	0.22
135°	180°	0.09

Run "compton-scattering-4.txt" to plot $f(\theta)$.



Plot of $f(\theta)$ for $\omega = m$.

0.4 Thomson scattering

When ω is much smaller than the electron mass m we have

$$\frac{m}{m+\omega(1-\cos\theta)} \approx 1$$

Hence for $\omega \ll m$ the differential cross section is approximately

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}(1+\cos^2\theta)$$

which is the formula for Thomson scattering.

0.5 LEP data

The following Compton scattering data is from the paper "Compton Scattering of Quasi-Real Virtual Photons at LEP."

x	y
-0.74	13380
-0.60	7720
-0.47	6360
-0.34	4600
-0.20	4310
-0.07	3700
0.06	3640
0.20	3340
0.33	3500
0.46	3010
0.60	3310
0.73	3330

The data are for the center of mass frame and have the following relationship with the differential cross section formula.

$$x = \cos\theta \qquad y = \frac{d\sigma}{d\cos\theta}$$

This is the differential cross section formula for Compton scattering in the center of mass frame with high energy approximation $m = 0$.

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left(\frac{1+\cos\theta}{2} + \frac{2}{1+\cos\theta}\right)$$

To compute predicted values $\hat{y}$ from the above formula, use $s = 40$ to approximate the QED values in the paper. Multiply the result by $(\hbar c)^2$ to convert to SI and multiply by 10^{40} to convert square meters to picobarns.

$$\hat{y} = \frac{\pi\alpha^2}{s}\left(\frac{1+x}{2} + \frac{2}{1+x}\right) \times (\hbar c)^2 \times 10^{40}$$

The following table includes the predicted cross section $\hat{y}$.

x	y	$\hat{y}$
-0.74	13380	12739
-0.60	7720	8468
-0.47	6360	6577
-0.34	4600	5472
-0.20	4310	4723
-0.07	3700	4259
0.06	3640	3936
0.20	3340	3691
0.33	3500	3532
0.46	3010	3420
0.60	3310	3338
0.73	3330	3291

The coefficient of determination R^2 measures how well predicted values fit the real data.

$$R^2 = 1 - \frac{\sum(y-\hat{y})^2}{\sum(y-\bar{y})^2} = 0.97$$

The result indicates that the model $d\sigma$ explains 97% of the variance in the data.

0.6 Eigenmath notes

Here are a few notes on how the Eigenmath scripts work.

To convert a_1 and a_2 to Eigenmath code, it is instructive to write a_1 and a_2 in full component form.

$$a_1^{\mu\nu} = \bar{u}_{4\alpha}\gamma^{\mu\alpha}{}_{\beta}(\not{q}_1 + m)^{\beta}{}_{\rho}\gamma^{\nu\rho}{}_{\sigma}u_2^{\sigma} \qquad a_2^{\nu\mu} = \bar{u}_{4\alpha}\gamma^{\nu\alpha}{}_{\beta}(\not{q}_2 + m)^{\beta}{}_{\rho}\gamma^{\mu\rho}{}_{\sigma}u_2^{\sigma}$$

Transpose γ tensors to form inner products over α and ρ.

$$a_1^{\mu\nu} = \bar{u}_{4\alpha}\gamma^{\alpha\mu}{}_{\beta}(\not{q}_1 + m)^{\beta}{}_{\rho}\gamma^{\rho\nu}{}_{\sigma}u_2^{\sigma} \qquad a_2^{\nu\mu} = \bar{u}_{4\alpha}\gamma^{\alpha\nu}{}_{\beta}(\not{q}_2 + m)^{\beta}{}_{\rho}\gamma^{\rho\mu}{}_{\sigma}u_2^{\sigma}$$

Convert transposed γ to Eigenmath code.

$$\gamma^{\alpha\mu}{}_{\beta} \quad \rightarrow \quad \texttt{gammaT = transpose(gamma)}$$

Then to compute a_1 we have

$$a_1 = \bar{u}_{4\alpha}\gamma^{\alpha\mu}{}_{\beta}(\not{q}_1 + m)^{\beta}{}_{\rho}\gamma^{\rho\nu}{}_{\sigma}u_2^{\sigma}$$

$\rightarrow$ `a1 = dot(u4bar[s4],gammaT,qslash1 + m I,gammaT,u2[s2])`

where s_2 and s_4 are spin indices. Similarly for a_2 we have

$$a_2 = \bar{u}_{4\alpha}\gamma^{\alpha\nu}{}_{\beta}(\not{q}_2 + m)^{\beta}{}_{\rho}\gamma^{\rho\mu}{}_{\sigma}u_2^{\sigma}$$

$\rightarrow$ `a2 = dot(u4bar[s4],gammaT,qslash2 + m I,gammaT,u2[s2])`

In component notation the product $a_1 a_1^*$ is

$$a_1 a_1^* = a_1^{\mu\nu} a_1^{*\mu\nu}$$

To sum over μ and ν it is necessary to lower indices with the metric tensor. Also, transpose a_1^* to form an inner product with ν.

$$a_1 a_1^* = a_1^{\mu\nu} a_{1\nu\mu}^*$$

Convert to Eigenmath code. The dot function sums over ν and the contract function sums over μ.

$a_1 a_1^* \quad \rightarrow$ `a11 = contract(dot(a1,gmunu,transpose(conj(a1)),gmunu))`

Similarly for $a_2 a_2^*$ we have

$a_2 a_2^* \quad \rightarrow$ `a22 = contract(dot(a2,gmunu,transpose(conj(a2)),gmunu))`

The product $a_1 a_2^*$ does not require a transpose because $a_1 a_2^* = a_1^{\mu\nu} a_2^{*\nu\mu}$.

$a_1 a_2^* \quad \rightarrow$ `a12 = contract(dot(a1,gmunu,conj(a2),gmunu))`

In component notation, a trace operator becomes a sum over an index, in this case α.

$$\begin{aligned} f_{11} &= \operatorname{Tr}\Big((\not p_2 + m)\gamma^\mu(\not q_1 + m)\gamma^\nu(\not p_4 + m)\gamma_\nu(\not q_1 + m)\gamma_\mu\Big) \\ &= (\not p_2 + m)^\alpha{}_\beta \gamma^{\mu\beta}{}_\rho (\not q_1 + m)^\rho{}_\sigma \gamma^{\nu\sigma}{}_\tau (\not p_4 + m)^\tau{}_\delta \gamma_\nu{}^\delta{}_\eta (\not q_1 + m)^\eta{}_\xi \gamma_\mu{}^\xi{}_\alpha \end{aligned}$$

As before, transpose γ tensors to form inner products.

$$f_{11} = (\not p_2 + m)^\alpha{}_\beta \gamma^{\beta\mu}{}_\rho (\not q_1 + m)^\rho{}_\sigma \gamma^{\sigma\nu}{}_\tau (\not p_4 + m)^\tau{}_\delta \gamma^\delta{}_{\nu\eta} (\not q_1 + m)^\eta{}_\xi \gamma^\xi{}_{\mu\alpha}$$

To convert to Eigenmath code, use an intermediate variable for the inner product.

$T^{\alpha\mu\nu}{}_{\nu\mu\alpha} \quad \rightarrow$ `T = dot(P2,gammaT,Q1,gammaT,P4,gammaL,Q1,gammaL)`

Now sum over the indices of T. The innermost contract sums over ν then the next contract sums over μ. Finally the outermost contract sums over α.

$f_{11} \quad \rightarrow$ `f11 = contract(contract(contract(T,3,4),2,3))`

Follow suit for f_{22}. For f_{12} the order of the rightmost μ and ν is reversed.

$$f_{12} = \operatorname{Tr}\Big((\not p_2 + m)\gamma^\mu(\not q_2 + m)\gamma^\nu(\not p_4 + m)\gamma_\mu(\not q_1 + m)\gamma_\nu\Big)$$

The resulting inner product is $T^{\alpha\mu\nu}{}_{\mu\nu\alpha}$ so the contraction is different.

$f_{12} \quad \rightarrow$ `f12 = contract(contract(contract(T,3,5),2,3))`

The innermost contract sums over ν followed by sum over μ then sum over α.

0.7 References

L3 Collaboration. "Compton Scattering of Quasi-Real Virtual Photons at LEP."
`arxiv.org/abs/hep-ex/0504012`