Garrett Kegel
MAT 189
ID:912604756

An Examination of the Intersection of Isometries on the 2-D Bounded Plane and the Moving Sofa Problem

Introduction

The Moving Sofa Problem solution has two bounds that describe the maximal planar area of a polygon that can rotate around the inner corner of a 2-D hallway shaped polygon remaining within the boundary and completing a rotation angle γ such that $\frac{\pi}{3} \leq \gamma \leq \frac{\pi}{2}$.

In particular, Kallus et al. defines a moving sofa constant μ which represents the largest possible area that can rotate around the inner corner of a 2-D hallway, L-shaped polygon with the length between the walls being unit length (Kallus 2). According to the research done by Dan Romik of the University of California, Davis and Yoav Kallus of the Santa Fe Institute, the largest area polygon that can rotate around the corner successfully is μ = 2.37 (Kallus 2). In their paper, they mention that Joseph Gerver most likely discovered the true maximal area being μ_g = 2.2195 (Kallus 2) and Gerver also proved the existence of the maximal area in his paper discovering an implicitly defined eighteen-sided polygonal shape that can successfully rotate around the corner as seen in Figure 1 (Gerver 2). Upon examining both of the proposed maximal sofa shape areas, these two values μ and μ_g give us a range of values that this area could possibly be (Kallus 2). In fact, Kallus et al. shows us that the maximal area that will be our solution is the maximal sofa shape area that is within the bounds of μ and μ_g (Kallus 2). In Gerver's paper, he shows us that this maximal area which Kallus et al. has

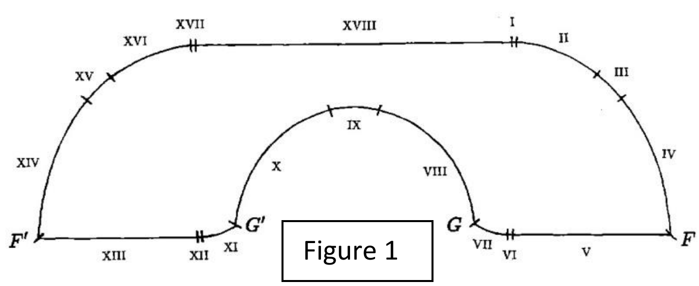

Figure 1

bounded, is the intersection of the collection of isometries of the hallway that have area corresponding to some value within the range of μ_g and μ (Gerver 2). Since there is an uncountably infinite number of values that are within the range of μ_g and μ, we can restrict the domain of a real valued function to consider a bijection counting function that assigns to each isometry an area that is within the bounds μ_g and μ. If we consider the intersection of this countable set of areas, then we will find an approximation to μ_{MS}, where μ_{MS} is the actual maximum sofa shape area (Kallus 2).

With respect to the earlier research stated above, we examine a few interesting aspects of this problem, these being the maximum area of a convex polygon that can rotate around the inner corner of an L shaped hallway; a moving sofa variant called The Ambidextrous Sofa Problem i.e., Conway's Car which is a modified sofa shape that can rotate around two inner corners; and a computer-assisted proof scheme deriving an upper bound to the sofa area. We will show that the maximum area of a convex polygon that can rotate around a corner of a hallway will be the area of a half circle with unit radius which we will make reference to the work of de Berg et al. who developed a method for "Computing the Maximum Overlap of Two Convex Polygons under Translations" (de Berg 9). From this, we continue on to examine The Ambidextrous Moving Sofa Problem in which we make reference to Phillip Gibbs article (Gibbs 6) and we also reference the work of Kiyoshi Maruyama in which an algorithm is described involving shaving off the edges of a proposed sofa shape (Maruyama 9). From this, we then examine the work of Yoav Kallus and Dan Romik in which they describe a computer-assisted proof scheme of deriving an improved upper bound to the sofa area (Kallus 1).

Discussion

The Maximal Area of a Convex Polygon that can Rotate Around the Corner

To show that the maximal area of a convex polygon that is a solution to the moving sofa problem, let P be the closed version of the hallway with six vertices and let Q be the sofa with a yet to be determined number of vertices denoted by n. We want to overlap the set of all possible placements of Q over P, which is defined to be the *configuration space* (de Berg 3). This configuration space is the set of all points that reside in both P and Q even after Q has been translated and rotated around the corner of P. This will result in a collection of combinatorically distinct Q shapes which are each different from one another. However, the union of these combinatorically distinct Q shapes will be a circle with the third quarter removed as seen in Figure 2. The checkerboard pattern in Figure 2 is the intersection of the semicircle. Starting from the red semicircle, the semicircle is rotated around the origin. Once this rotation is complete, the semicircle will be oriented in the direction of the blue semicircle. According to de Berg et al., since this semicircle is convex with two vertices and the hallway has six vertices, the complexity has order eight (de Berg 3). If we transform this semicircle to attempt to improve the maximal area that can rotate around the corner, then we force the shape to abandon its convexity, which allows for the semicircle to be elongated with a hole cut out, much like the shape in Figure 1.

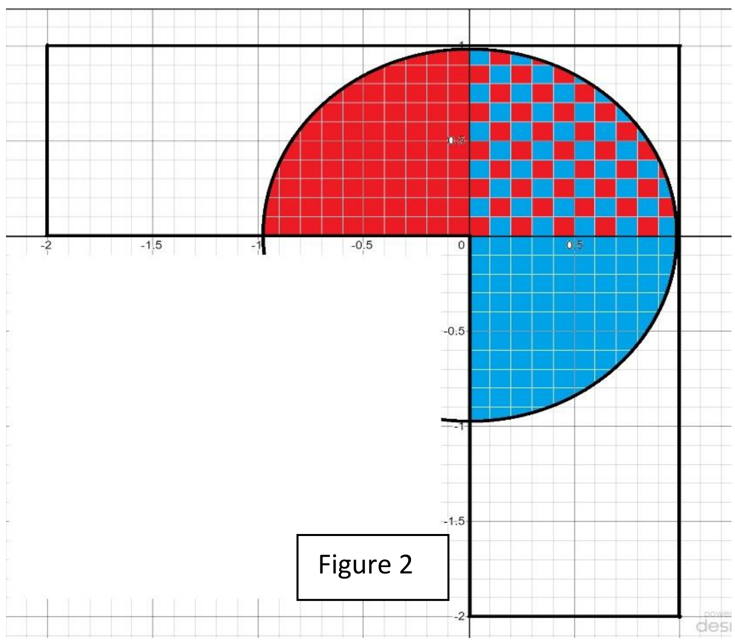
Figure 2

The Ambidextrous Moving Sofa, a Variant of the Original Problem

In the variant of the problem, we want to find the maximal area of a 2D polygon that can rotate around two corners in a connected hallway. As stated in Phillip Gibbs article about a computational method of calculating the sofa area, the article claims that the area of the shape that can pass between two continuous curves such that the minimum distance of any two points on both curves is at least one, to be at least $A = \frac{\pi}{2}$ (Gibbs 6). To calculate this optimal area, we turn to the work done by Kiyoshi Maruyama who found an algorithm used in finding this shape and described this technique using simple paper-cuts (Maruyama 9). As seen in Figure 3, the technique involves taking successive paper planes and carving the edges off until the shape becomes able to rotate around the corner of the hallway (Maruyama 10). Each plane, indicated by $S_0(x_1)$ and $S_0(x_2)$, is superimposed on each other to form an intersection as indicated in part (b) of figure 3. By using Maruyama's method, one can achieve a good approximation of a sofa area that is some percentage of the lower bound, which in this case is $\frac{\pi}{2}$. This ambidextrous sofa is shown in figure 4, which was computed by Maruyama and has an area of 1.27, which is about 80% of the lower bound, making this a decent approximation technique (Maruyama 17).

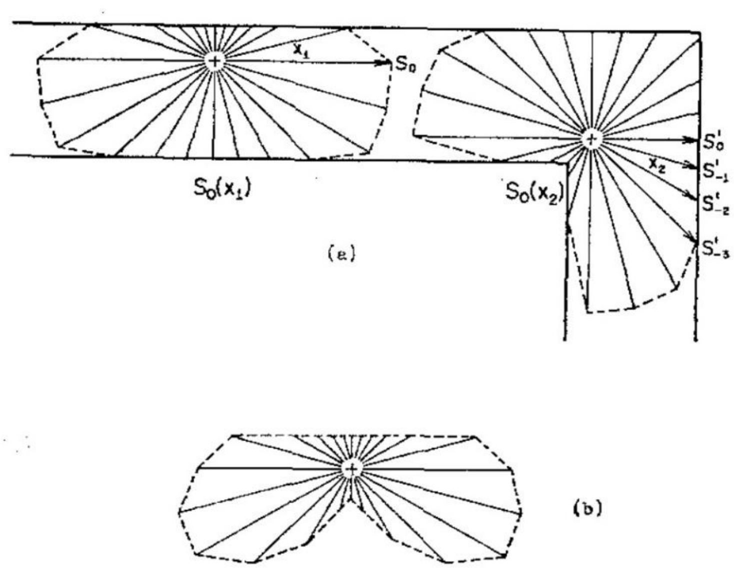

Figure 3

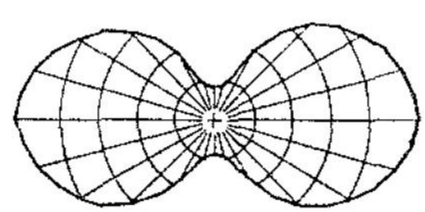

Figure 4

Improving an Upper Bound Using a Computer-Assisted Proof Scheme

Yoav Kallus and Dan Romik implemented a computer-assisted proof scheme coded in C++, using exact rational arithmetic to compute an upper bound of 2.37, further improving the range of possible values of the area of the sofa shape (Kallus 3). Kallus et al. continues by showing that a hallway is a bivariate function with domain (α, u) in $[0,\frac{\pi}{2}]$ x R^2 (Kallus 4). Each of these L shaped hallways is rotated by an angle α and each vector u is a translation vector (Kallus 5). If we consider the intersection of the hallway, the L shape, and another set called the Butterfly set, along with a vector of α's and a two-vector β in $(\alpha_k, \frac{\pi}{2}]$ to be used in the definition of the Butterfly set, we take the supremum of this intersection to find an upper bound on the maximal area corresponding to the chosen Butterfly set. Kallus et al. then proves that there exists a box in R^{2k} that contains a vector u which corresponds to the maximal area of the intersection (Kallus 5). We can use this vector u to compute an approximation to the maximal area for k iterations.

To better understand the problem, the hallway is chosen to rotate around a fixed sofa shape in which the initial L_0 shape is translated and each of these translations is superimposed to form an intersection. In particular, this intersection is the maximal sofa shape corresponding to a continuous path that the translated hallway moves along (Kallus 6). Along this path of motion, there is a collection of points corresponding to the rotation path β which determines the Butterfly Set of rotation area (Kallus 7). Also, we want to find the maximal area of all the connected components of the hallway along its path of motion and rotation path. Then, we take the intersection of these components and find an k approximation for this area.

Kallus and Romik use a computational strategy which is a variant of the geometric branch and bound optimization method (Kallus 13). This optimization technique involves separating the box that was discussed previously into sub-boxes to find an upper bound (Kallus 13). The technique then continues iteratively into further sub-boxes of sub-boxes dividing the sub-boxes by two with each iteration. This computed quantity decreases as the number of iterations k increases.

Conclusion

To conclude, the Moving Sofa Problem is a very complicated problem masquerading as a simple problem. The exact area of the maximal shape is not known, and the furthest research gives a range of values with an upper bound that decreases towards Gerver’s moving sofa bound as the number of iterations increase as shown by Kallus and Romik (Kallus 2). In addition, the maximal area polygon that can rotate around the corner of the hallway is a half-circle using a technique of overlapping polygons (de Berg 5). Also, we briefly examined the Ambidextrous Sofa shape from the point of view of Kiyoshi Maruyama who provided an algorithm for computing the shape (Maruyama 9). This is a truly impressive problem and will require more effort to discover new methods of computing the maximal area of the sofa shape.

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