Optimizing the Brachistochrone using a Genetic Algorithm

Vivek Gopalakrishnan

July 11, 2018

1 Defining the Cost Function

Find the time required for a particle to travel a path $y = f(x)$ with endpoints $(x_0, f(x_0), (x_n, f(x_n))$. Assume a uniform gravitational field and that there are no nonconservative forces. Additionally, assume the particle traveling the path starts at rest.

Because we are assuming a uniform gravitational field, the total energy of the system as a function of x is

$$E(x) = K + U_g = \frac{1}{2}mv(x)^2 + mgh(x) = \frac{1}{2}mv(x)^2 + mgf(x)\,. \tag{1}$$

Additionally, because the system starts at rest, $K(x_0) = 0$. By combining these equations, we can derive the following:

$$\begin{aligned} \frac{1}{2}mv(x)^2 + mgf(x) &= mgf(x_0) \\ v(x)^2 &= 2g(f(x_0) - f(x))\,, \\ v(x) &= \sqrt{2g(f(x_0) - f(x))}\,. \end{aligned} \tag{2}$$

To find the differential displacement along the curve (ds) for a differential change in the x direction (dx), we can use the arc length formula:

$$ds = dx\sqrt{1 + f'(x)^2}\,. \tag{3}$$

The time required to travel a differential portion of $f(x)$ is given by $dt = \frac{ds}{v(x)}$. Thus, the total time T required is as follows:

$$\begin{aligned} T(f(x)) = \int dt = \int_{x_0}^{x_n} \frac{ds}{v(x)} &= \int_{x_0}^{x_n} \sqrt{\frac{1 + f'(x)^2}{2g(f(x_0) - f(x))}}dx \\ &\propto \int_{x_0}^{x_n} \sqrt{\frac{1 + f'(x)^2}{f(x_0) - f(x)}}dx\,. \end{aligned} \tag{4}$$