Optimizing the Brachistochrone using a Genetic Algorithm

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1 Defining the Cost Function

Find the time required for a particle to travel a path $y = f(x)$ with endpoints $(x_0, f(x_0)), (x_n, f(x_n))$. Assume a uniform gravitational field and that there are no nonconservative forces. Additionally, assume the particle traveling the path starts at rest.

Because we are assuming a uniform gravitational field, the total energy of the system as a function of x is

$$E(x) = K + U_g = \frac{1}{2}mv(x)^2 + mgh(x) = \frac{1}{2}mv(x)^2 + mgf(x)\,. \tag{1}$$

Additionally, because the system starts at rest, $K(x_0) = 0\,.$ Because we assume energy is conserved throughout the system, $E(x) = E(x_0),$ By combining these equations, we can derive the following:

$$\frac{1}{2}mv(x)^2 + mgf(x) = mgf(x_0) \tag{2a}$$

$$v(x)^2 = 2g(f(x_0) - f(x)) \tag{2b}$$

$$v(x) = \sqrt{2g(f(x_0) - f(x))}\,. \tag{2c}$$

To find the differential displacement along the curve (ds) for a differential change in the x direction (dx), we can use the arc length formula:

$$ds = dx\sqrt{1 + f'(x)^2}\,. \tag{3}$$

The time required to travel a differential portion of $f(x)$ is given by $dt = \frac{ds}{v(x)}$. Thus, the total time T required is as follows:

$$\begin{aligned} T(f(x)) = \int dt = \int_{x_0}^{x_n} \frac{ds}{v(x)} &= \int_{x_0}^{x_n} \sqrt{\frac{1 + f'(x)^2}{2g(f(x_0) - f(x))}}dx \\ &\propto \int_{x_0}^{x_n} \sqrt{\frac{1 + f'(x)^2}{f(x_0) - f(x)}}dx\,. \end{aligned} \tag{4}$$