CQF FitchLearning

TUTORIAL

Statistical Essentials for VaR and ES

Dr Richard Diamond, CQF ARPM
Fellow of the Higher Education Academy (UK)

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Knowledge | Skills | Conduct

Learning outcomes

- understand the first principles: **inverse percentile** for Normal Distribution, **conditional expectation** for ES (CVaR)
- be able to read probability notation
- Introduce the backtesting of VaR (EXCEL) including the market risk tools of **RiskMetrics** and **Historical Simulation**.
- PYTHON please refer to well written Python Labs on the topic.
- EXTRA material, such as *Cornish-Fisher approximation* and *introduction to EVT* covered if time and participation interest permit.

How to use

- We re-visit the first principles in probability and expectation, but the tutorial is not a substitute to the core lectures and Primer. To re-explain a concept or formula might not be suitable to the flow. Please do print VaR and ES lecture Solutions and work with pen and paper if necessary. We will not be working with long derivations by-step, only key mathematical steps will be noted.

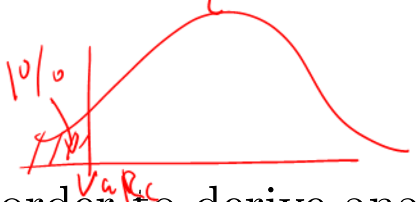
- The tutorial/lab is not a set program of content. Frequent presentation changes between slides/computation to be expected. It is not a lecture flow.

- Historically, tutorials are delivered ‘from the desk’ and typically review a computation (Excel, Python, R etc). The teaching is by presenting an example – limited to its own scope.

VaR as a percentile

Fully Analytical VaR for Efficient Markets – Assume that P&L of an investment portfolio is a random variable that follows Normal distribution $X \sim N(\mu, \sigma^2)$. Use the definition of *VaR as a percentile*,

left tail



-2.33

$$\Pr\left(x \leq \mathrm{VaR}(X)\right) = 1 - c \quad = 1 - 0.99 = 0.01$$

specific value of R.V. X

in order to derive analytical expression for VaR calculation.

Random Variable

$\boxed{\textbf{Solution:}}$ We start with probability for the P&L (loss) X exceeding VaR(X) threshold and convert X to a Standard Normal variable ϕ. The probability of loss $x < 0$ being worse than VaR < 0 is

$$\begin{aligned}
\Pr\left(x \leq \mathrm{VaR}(X)\right) &= 1 - c \\
\mathrm{VaR}_c(X) &= \inf\{x \,|\, \Pr(X > x) \leq 1 - c\} = \inf\{x \,|\, \underbrace{F_X(x)}_{CDF_X(x)} \geq c\}
\end{aligned}$$

Realisation of P&L or random variable X

VaR as a percentile (cont).

If P&L X is a random variable then VaR(X) is also a random variable. In order to use the well-known Normal Distribution functions, we have to work with the Standard Normal variable

$$
\begin{aligned}
\Pr(\phi \leq \frac{\mathrm{VaR}(X) - \mu}{\sigma}) &= 1 - c \qquad \Longrightarrow \\
\mathrm{VaR}(X) &= \mu + \Phi^{-1}(1 - c) \times \sigma
\end{aligned}
$$

Inverse CDF is a percentile function.

$x \to \phi$ "phi"

$$\Pr[x \leq \mathrm{VaR}(x)] = 1 - c$$

$$\cancel{\phi^{-1}}\left[\cancel{\phi}\left(\phi \leq \frac{\mathrm{VaR}(x) - \mu}{\sigma}\right)\right] = \phi^{-1}[1 - c]$$

ICDF

CDF

$$\mathrm{VaR}(x) = \mu + \underbrace{\phi^{-1}[1-c]}_{\text{negative}} \times \sigma$$

Parametric / Analytical VaR

$$\cancel{\Phi^{-1}}\left[\cancel{\Phi}\left(\frac{VaR(x)-\mu}{\sigma}\right)\right] = \Phi^{-1}[1-c]$$

$F_X(x) = \Phi(\;)$

inverse of CDF

$$VaR(x) - \mu = \Phi^{-1}[1-c] \times \sigma$$

$$VaR(x) = \mu + \underbrace{\Phi^{-1}[1-c] \times \sigma}$$

"-2,33"

99% VaR

$$\Phi^{-1}[1-0.99] = \Phi^{-1}[0.01] = -2,32635$$

Factor

Standardised Percentile

→ NORMSINV(0.01)

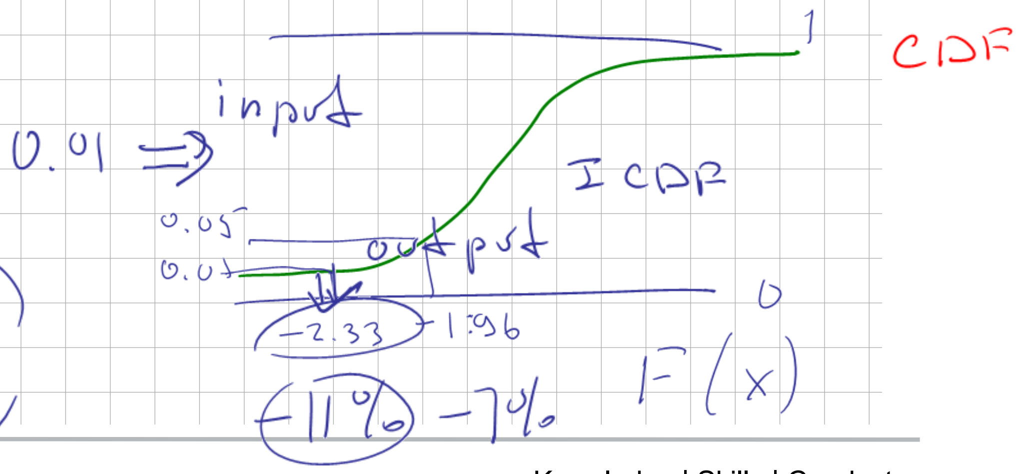
Percentile function

CDF $F_X(x)$

PDF $f_X(x)$

CDF

$$\int \ldots e^{\frac{x^2}{2}}$$

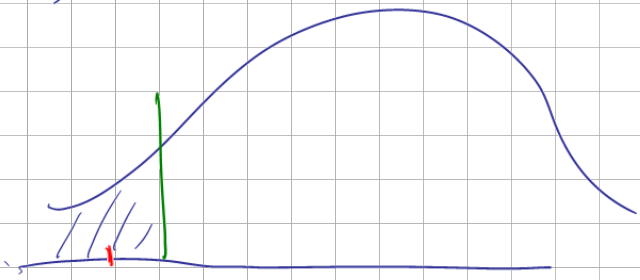


Empirical VaR / Historical Simulation

$P\&L(R_t^{GM})$

VaR

$P\&L_{returns} \sim N(\mu, \sigma^2)$



-2.33

Standardised residuals, Z

-2.7
-2.5
-2.33
-2.18
-1.98

?

VaR

$Z_t = \frac{X_t - \mu}{\sigma}$

Returns (sorted) X

-0.17
-0.155
-0.14
-0.12
-0.10
...

1% of $N_{obs} = 300$

inf{ }

$\hat{\sigma}, \hat{\mu}$

estimates

PERCENTILE(0.01)

$VaR = \mu - 2.32635 \times \sigma$

↳ Analytical VaR

$-2.33 \times \sigma$

$VaR = -0.14$

$\Rightarrow ES_{99\%} = -\frac{(0.17 + 0.155 + 0.14)}{3}$

↳ Analytical ES

$-2.67 \times \sigma$

(-2.6652)

$ES = \mu - 2.6652 \times \sigma$

↳ Historic Simulation $X \to Z$

↳ Monte Carlo Simulation

$Z_{sim} \to X$

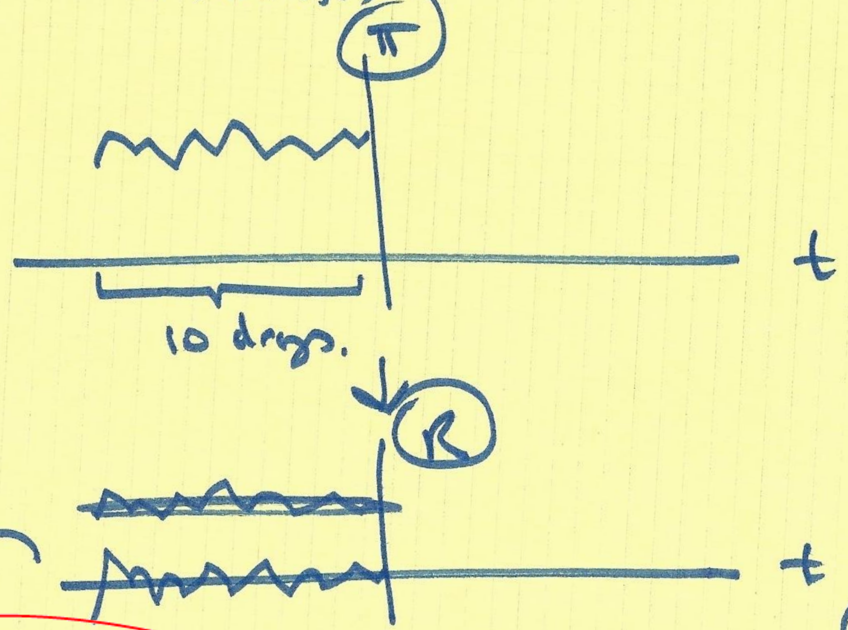
Problem 1: Historical Simulation Compute the 1-day VAR at 90% confidence (both in percent and monetary terms) for a portfolio of £3 million whose recent daily returns have been: +1%, 0%, -1%, -2%, +1%, +3%, -1%, 0%, -3%, 0%

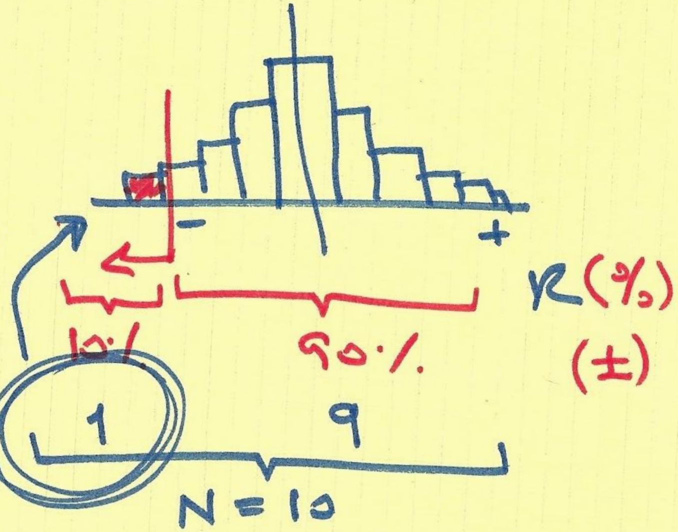
① VaR Historical Simulation

Π : $\boxed{3m£}$

VaR(1d, 90%)

VaR(1d, X)





R	sorted R
+1%	−3%
0%	−2%
−1%	−1%
−2%	−1%
+1%	0%
+3%	0%
−1%	0%
0%	+1%
−3%	+1%
0%	+3%

N=10

$\rightarrow$ VaR(1d, 90%) = +3% (%)

$\rightarrow$ VaR(1d, 90%) = (−3%)(3m£) = + 90,000 £ (£)

Expected Shortfall via average of percentiles (model-free definition and result)

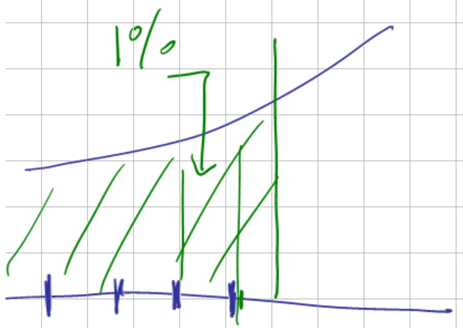
$E[X|\ldots]$

ES

$$ES(X) = E[X \mid \underbrace{X \le VaR_c(X)}]$$

mean $E[X]$ but here

mean of the tail



$$\underbrace{ES(X) = \frac{1}{1-c}\int_0^{1-c=0.01} VaR_u \, du}_{\underbrace{\frac{\sum VaR_u}{0.01}}_{\text{averaging}}}$$

Return

Average is ES

$du = 0.001$ (10 obs)

or

0.0001 (100 obs)

ES is

average worst outcome

mean of the tail values

Avg = -2.6652

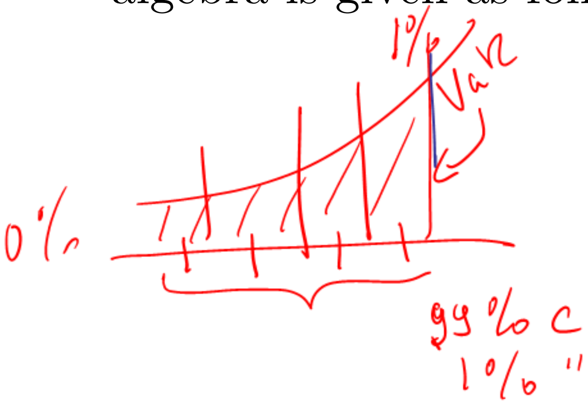
$VaR_{99\%} \approx -2,33$

$VaR_{99.1\%} \approx -2.33$

$VaR_{99.2\%} \approx -2.34$

Expected Shortfall via average of percentiles (model-free definition and result)

What about Expected Shortfall? The universal definition of ES in terms of expectations algebra is given as follows:



$$
\begin{aligned}
\mathrm{ES}_c(X) &= \mathbb{E}[\,X \mid X \le \mathrm{VaR}_c(X)] \\
\mathrm{ES}_c(X) &= \frac{1}{1-c}\int_0^{1-c \,\doteq\, 0.01} \mathrm{VaR}_u(X)\,du
\end{aligned}
$$

$t \to s$

$c \to u$

The actual ES calculation formula will vary depending on the distribution of P&L X, a random variable. Derive ES calculation formula for the case of Normal Distribution using the result $\mathrm{VaR}(X) = \mu + \Phi^{-1}(1-c) \times \sigma$.

$N(0,1)$

— $\phi(x)$ or Normal PDF

$\Phi^{-1}[\underset{1-c}{0.01}] = -2.33$

$\Phi^{-1}[\underset{c}{0.99}] = 2.33$

$$= \mu - \sigma\,\frac{\phi(\Phi^{-1}(\overset{1-c}{\cdot}))}{1-c}.$$

$\mathrm{ES}_{99\%}$ $\qquad$ $\mathrm{VaR}_{99\%}$

$-2.6652 < -2.32635$

The result has a quirk of ICDF being inside PDF but this is simply $\phi(\underline{-2.32635})$ = pdf(Fact

Recap

- Analytical approach: distribution represented by a Factor
  - Inverse Normal CDF gives the value of Factor (eg, -2.33). That depends on the desired percentile eg, 99%, 90%.
  - Multiply Factor x Volatility. Voilà!   The same recipe applies to t-distribution and other **Parametric (Analytical) VaR**

- Empirical approach: the distribution is represented by percentiles applied to <u>the sorted column</u> of returns
  - Manually search for value (return) that corresponds to 99% of observations
  - 1 year period is 252 days, translates into a tail of 2.5 observations for 99%
  - **Historical Simulation VaR**

VaR Backtesting DEMO

Imagine that each morning you calculate 99%/10day VaR from available prior data only. Once ten days pass you compare that VaR number to the realised return and check if your prediction about the worst loss was breached. You are given a dataset of FTSE 100 index levels, continue in Excel.

- Calculate Value at Risk for each day t (starting on Day 21) as follows:

~~GARCH~~ or EWMA

$$\text{VaR} = \mu_{10D} + \text{Factor} \times \sigma_{10D} \quad \dagger$$

where Factor is a percentile of the Standard Normal Distribution that 'cuts' 1% on the tail.

In Excel, you will have a final column with VaR_t as a percentage since calculation is done on returns.

VaR Backtesting: standard deviation timescale is DAILY

C.1 Calculate the rolling 99%/10day Value at Risk for an investment in the market index using a sample standard deviation of log-returns, as follows:

- The rolling standard deviation for a sample of 21 is computed for days 1-21, 2-22, ..., there must be 21 observations in the sample. So, you have a time series of σ_t.
- Scale standard deviation to reflect a ten days move $\sigma_{10D} = \sqrt{10 \times \sigma^2}$ (we can add variances) and scale an ~~average daily return as $\mu_{10D} = \mu \times 10$ where μ is a mean return of all data given.~~

- What is the difference between <u>10-day period (timescale)</u> and the number of returns we use to compute the std. deviation, eg 21 obs.?

42 obs

$\sigma = \sqrt{\frac{\sum_{i=1}^{21}(r-\mu)^2}{n-1}}$

$n = 21$ $\frac{1}{20}$

$n = 42$ $\frac{1}{41}$

σ^2 & $\bar{r}^2$

VaR Breaches and Control

Breaches	Prob, cumulative=FALSE
0	0.0000
1	0.0000
2	0.0002
3	0.0009
4	0.0031
5	0.0080
6	0.0173
7	0.0321
8	0.0517
9	0.0737
10	0.0943
11	0.1092
12	0.1154
13	0.1121
14	0.1008
15	0.0841
16	0.0656
17	0.0479
18	0.0329
19	0.0213
20	0.0131

Binomial Distribution is a useful tool in market risk and credit defaults.

If we set VaR percentile level at 95%, what is the number of breaches can be expected per year (252 days)?

VaR Backtesting: what to improve?

- Did we drop the mean, mu, from computation? Why?

Small, mo robust prediction

- Volatility estimation – is it responsive enough? Why ARCH-filtered volatility would make the breach count **worse**?

21 obs but not smaller for small samples (territory)

GARCH (1,1) $\sigma \to \bar{\sigma}$ tend to be overestimated

- Is Normal Percentile an adequate predictor for P&L in returns? What are the alternatives – see the EXTRA on Cornish-Fisher VaR.

RiskMetrics (EWMA)

$$\sigma_t^2 = \underbrace{\overset{0.6}{\lambda\sigma_{t-1}^2}}_{\text{represents past MAs}} + \overset{0.4}{(1-\lambda)}\underbrace{u_{t-1}^2}_{\text{1MA}}$$

$$\sum^{t-2} u^2 \to \sigma_{t-1}^2$$

u instead of r

Low λ

Low λ leads to more weight for the $(1-\lambda)u_{n-1}^2$ term, so the model is very responsive to the previous day's returns, i.e. news from the market.

High λ

High λ leads to a slow response to new information.

Example: The RiskMetrics database made available by by JP Morgan in 1994 uses the EWMA model with $\lambda = 0.94$ for updating daily estimation of variance across a range of markets.

Volatility Filtering (Risk Metrics, ARCH)

EWMA

$\cancel{0.94}$ 0.72

$$\sigma^2_{t+1} = \alpha\,\sigma^2_t + (1-\alpha)\,r^2_t$$

forecast

past vol²
(all / nearly all obs avail.)

past ret²

"Moving Avges"

$r_t \equiv u_t$
or
$r_t \equiv \varepsilon_t$

GARCH

$$\sigma^2_{t+1} = \beta\,\sigma^2_t + \alpha\,r^2_t + \omega$$

$$\left[\ \bar{\sigma}^2 = \frac{\omega}{1-\alpha-\beta}\ \right]$$

long term average variance (const)

Time Series Q&A

- Period of backtesting: 200+ comparisons (VaR_t vs 10D Return).
  - however, we are interested in a responsive forecast of *short-term volatility;* $\sigma_t \times \sqrt{10}$
  - therefore, a 'rolling window' of 21-42 observations (returns), eg a rolling 42-day window over 200 days.

- Since we assume Normal Factor, the sample size to be close or above 25 (don't want to go into estimation from small samples).
  - Can we improve on Factor? Yes, it's possible to use **Cornish-Fisher** expansion for Value at Risk.

Cornish-Fisher approximation for VaR

$VaR_{99\%}$ ... $+(-2.326 +$ ... $)$

$$VaR_{0.975} = \underbrace{-\frac{1}{2}\sigma^2 T}_{\text{"}\mu\text{" remember GBM solution}} + (-1.96 + 0.474\,S - 0.0687\,K - 0.146\,S^2)\sigma\sqrt{T}.$$

- Coming form regulatory practice, you might see Value at Risk computed such as above. (Minor point: we got rid of the average return mu).
- Cornish-Fisher expansion transforms our Factor -1.96 to be adjusted by Skew, Kurtosis, and Skew^{2}.
- Empirical skew, kurtosis are distribution parameters that can be computed from a rolling window **longer** than 42 obs (ie, different from sigma). 100-200 obs are more appropriate for such parameters.

REF Cornish-Fisher VaR

https://www.riskconcile.com/wp-content/uploads/2020/03/cornish_fisher_2.html

Cornish-Fisher approximation

$$VaR_{0.975} = -\underbrace{\frac{1}{2}}_{\text{"N"}}\sigma^2 T + (-1.96 + 0.474\,S - 0.0687\,K - 0.146\,S^2)\sigma\sqrt{T}.$$

Skew $\neq 0$

Whoa! Where does that come from?

Kurtosis $\neq 3$

- The factor, which is the adjusted inverse percentile $CF^{-1}(p)$ comes from:

$$\left[\; CF^{-1}(p) = \Phi^{-1}(p) + \frac{\text{Skew}}{6}\left(\Phi^{-1}(p)^2 - 1\right) + \frac{\text{Kurt}}{24}\left(\Phi^{-1}(p)^3 - 3\Phi^{-1}(p)\right) - \frac{\text{Skew}^2}{36}\left(2\Phi^{-1}(p)^3 - 5\Phi^{-1}(p)\right) \;\right]$$

Basel II and legacy backtesting

- Independence of breaches in VaR: Christoffersen’s 1998 Exceedance Independence Test

  Measures the dependence between consecutive days only

REF Christoffersen’s 1998 Exceedance Independence Test

https://www.value-at-risk.net/backtesting-independence-tests/

REF Statistical tests for VaR backtesting

https://www.mathworks.com/help/risk/overview-of-var-backtesting.html

END OF TUTORIAL

- it is sufficient outcome if you can follow the logic of techniques presented today (eg, how to backtest and count breaches in VaR, why using RiskMetrics/EWMA);
- please work though the VaR and ES Market Risk lecture exercise solutions.