1. $(2+x)^{-2} = \dfrac{1}{(x+2)^2} = \dfrac{1}{[2(\frac{x}{2}+1)]^2} = \dfrac{1}{2^2(\frac{x}{2}+1)^2} = \dfrac{1}{4(1+\frac{x}{2})^2}$

$= \dfrac{1}{4}\left(1+\dfrac{x}{2}\right)^{-2}$

$\left(1+\dfrac{x}{2}\right)^{-2} = (1+y)^{+n} = 1+ny+\dfrac{n(n-1)y^2}{2!}+\dfrac{n(n-1)(n-2)y^3}{3!}$. Sub $y=\frac{x}{2}$; $n=-2$

$= 1+(-2)\left(\dfrac{x}{2}\right)+\dfrac{(-2)(-3)}{2!}\left(\dfrac{x}{2}\right)^2+\dfrac{(-2)(-3)(-4)}{3!}\left(\dfrac{x}{2}\right)^3$

$\simeq 1-x+\dfrac{3}{4}x^2-\dfrac{1}{2}x^3$. (first four terms of the expansion).

$\dfrac{a}{1-r}$ a: first term, r: common ratio $\simeq$ Sum of the infinite terms in Geometric series.

$= a+ar+ar^2+ar^3+ar^4+ar^5+\dots$ If $a=1$

$\dfrac{1}{1-r} = 1+r+r^2+r^3+r^4+r^5+\dots$ Converges only if $|r|<1$; diverges if $|r|=1$; $|r|>1$

$r=\dfrac{x}{2}$; $\dfrac{1}{(1+\frac{x}{2})^2}$ converges if $\left|\frac{x}{2}\right|<1 \Rightarrow |x|<2$.

$\left(1+\dfrac{x}{2}\right)^{-2} = 1-x+\dfrac{3}{4}x^2-\dfrac{1}{2}x^3$

$\dfrac{1}{4}\left(1+\dfrac{x}{2}\right)^{-2} = \dfrac{1}{4}\left(1-x+\dfrac{3}{4}x^2-\dfrac{1}{2}x^3\right) = \dfrac{1}{4}-\dfrac{x}{4}+\dfrac{3}{16}x^2-\dfrac{1}{8}x^3$;

$\dfrac{1}{(x+2)^2} = \dfrac{1}{4(1+\frac{x}{2})^2} = \dfrac{1}{4}-\dfrac{x}{4}+\dfrac{3}{16}x^2-\dfrac{1}{8}x^3$

$\dfrac{1+x^2}{(2+x)^2} = (1+x^2)\left[\dfrac{1}{(2+x)^2}\right] = (1+x^2)\left[\dfrac{1}{4}-\dfrac{x}{4}+\dfrac{3}{16}x^2-\dfrac{1}{8}x^3\right]$

$= \dfrac{1}{4}-\dfrac{x}{4}+\dfrac{3}{16}x^2-\dfrac{1}{8}x^3+\dfrac{1}{4}x^2-\dfrac{1}{4}x^3+\dfrac{3}{16}x^4-\dfrac{1}{8}x^5$

$= \dfrac{1}{4}-\dfrac{1}{4}x+\dfrac{7}{16}x^2-\dfrac{3}{8}x^3$ (Ignoring the terms x^4 & x^5).

If $x=0.1$ ~~$x^4=0.0001$~~; ~~$x^5=0.00001$~~; $x^3=0.001$

2. $f(x)=\ln(1+x)$; Recall $\ln y = \int \frac{1}{t}\,dt$

$\ln(1+x) = \int \dfrac{1}{1+x}\,dx = \int \dfrac{1}{(1-(-x))}\,dx = \int\left[(1+(-x)+(-x)^2+(-x)^3+(-x)^4+\dots\right]dx$

$\sum^{\infty} (-1)^n \int x^n\,dx$

$$\ln(1+x) = \int \frac{1}{1+x}\,dx = \int \frac{1 \to a}{(1-\underset{r}{-x})}\,dx = \int \left[(1 + (-x) + (-x) + \ldots \right.$$

$$= \int \sum_{n=0}^{\infty} (-x)^n\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^n\,dx = \sum_{n=0}^{\infty} (-1)^n \int x^n\,dx$$

$$|-x| < 1 \Rightarrow |x| < 1 \Rightarrow -1 < x < 1 \Rightarrow (-1, 1).$$

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} + c.$$

Sub $x = 0$; $\ln 1 = 0 + c \Rightarrow c = 0.$

$$\Rightarrow \ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}. = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Replacing x with $-x$

$$\ln(1-x) = -x - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \cdots = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\ln(1+x) - \ln(1-x) = \left(x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} + \cdots\right) - \left(-x - \cancel{\frac{x^2}{2}} - \frac{x^3}{3} - \cancel{\frac{x^4}{4}} - \cdots\right)$$

$$= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots$$

$$\ln(1+x) - \ln(1-x) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right] = 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$$

$$\ln\frac{1+x}{1-x} = 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$$

Mclaurin series expansion for $\ln\left(\frac{1+x}{1-x}\right)$ using known series.

5 a) $f = 2x^4y^3 - xy^2 + 3y + 1 \leftarrow V$

$$f_x = \frac{\partial}{\partial x}\left(2x^4y^3 - xy^2 + 3y + 1\right) = \frac{\partial}{\partial x}(2x^4y^3) - \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial x}(3y) + \frac{\partial}{\partial x}(1)$$

$$= 2y^3\frac{\partial}{\partial x}(x^4) - y^2\frac{\partial}{\partial x}(x) + 0 + 0.$$

$$= 2y^3(4x^3) - y^2 = 8x^3y^3 - y^2. //$$

$$f_y = \frac{\partial}{\partial y}\left(2x^4y^3 - xy^2 + 3y + 1\right) = \frac{\partial}{\partial y}(2x^4y^3) - \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial y}(1)$$

$$= 2x^4\frac{\partial}{\partial y}(y^3) - x\frac{\partial}{\partial y}(y^2) + 3 + 0. = 2x^4(3y^2) - x(2y) + 3$$

$$= 6x^4y^2 - 2xy + 3. //$$

$$\frac{d}{dx}(e^x) = e^x \ln e = e^x.$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

base $\to$ #
exp $\to$ var;

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$ power rule.

$b \to v.$
$e \to$ #

5c) $f = (y^2+z^2)^x$

$$f_x = \frac{\partial}{\partial x}(y^2+z^2)^x = (y^2+z^2)^x \ln(y^2+z^2)$$

$$f_y = \frac{\partial}{\partial y}(y^2+z^2)^x = x(y^2+z^2)^{x-1}\frac{\partial}{\partial y}(y^2+z^2)$$

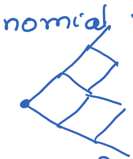
$$x(y^2+z^2)^{x-1}\,(2y+0)$$

$$f_y = \frac{\partial}{\partial y}(y^2+z^2)^{x} = x(y^2+z^2) \quad \frac{\partial}{\partial y}(\ )=$$

$$= x(y^2+z^2)^{x-1}\,(2y+0)$$

$$= 2xy\,(y^2+z^2)^{x-1}.$$

6. Binomial model. $\longrightarrow$ BS Model. $\rightarrow$ $\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0.$ $\rightarrow$ soln.

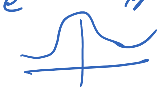


$\longrightarrow$

Heat eqn $\rightarrow$ diffusion eqn

linear parabolic PDE $\rightarrow$ ~~SV~~, ~~NF~~, SOS.

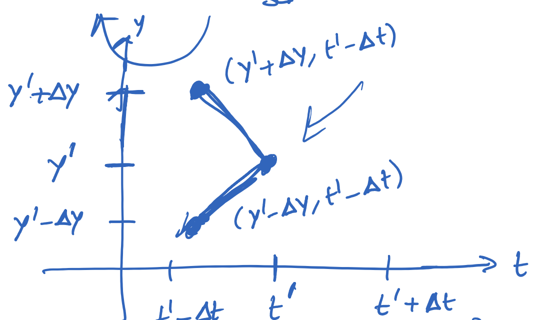
$$\frac{1}{\sqrt{2\pi\sigma^2 t}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2 t}}$$



$$e^{-r\Delta t}\{p f_u + (1-p) f_d\}$$

$S \to Su \rightarrow \max(K-Su, 0) = f_u$

$S \to Sd \rightarrow \max(K-Sd, 0) = f_d.$



$$p(y', t') = \frac{1}{2}p(y'+\Delta y, t'-\Delta t) + \frac{1}{2}p(y'-\Delta y, t'-\Delta t)$$

$$f'(x) \simeq \frac{f(x+h)-f(x)}{h}; \quad f(x+h) = f(x) + h f'(x)$$

TS 1V.

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x).$$

TS $\rightarrow$ 1D.

$$f(x+h, y+h) = f(x,y) + h f_x + h f_y + \frac{h^2}{2!}f_{xx} + \frac{h^2}{2!}f_{yy} + \frac{h^2}{2!}f_{xy}$$

TS 2V

$$p(\underbrace{y'}_{x}\underbrace{+\Delta y}_{h}, \underbrace{t'}_{y}\underbrace{-\Delta t}_{h}) = p(y', t') + \frac{\partial p}{\partial t}(-\Delta t) + \frac{\partial p}{\partial y}(+\Delta y) + \frac{1}{2!}\frac{\partial^2 p}{\partial y^2}(\Delta y)^2 + \cancel{\frac{1}{2!}\frac{\partial^2 p}{\partial t^2}(-\Delta t)^2}$$

$$+ \cancel{\frac{1}{2!}\frac{\partial^2 p}{\partial y \partial t}(\Delta y)(-\Delta t)} + O(h^3).$$

$$p(y' \pm \Delta y, t'-\Delta t) = p(y', t') - \frac{\partial p}{\partial t}\Delta t \pm \cancel{\frac{\partial p}{\partial y}\Delta y} + \frac{1}{2!}\frac{\partial^2 p}{\partial y^2}(\Delta y)^2 \rightarrow (-\Delta y)^2 = (\Delta y)^2$$

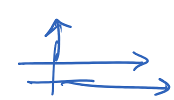
$$p(y' - \Delta y, t'-\Delta t) = p(y', t') - \frac{\partial p}{\partial t}\Delta t - \cancel{\frac{\partial p}{\partial y}\Delta y} + \frac{1}{2!}\frac{\partial^2 p}{\partial y^2}(\Delta y)^2$$

$$p(y'+\Delta y, t'-\Delta t) + p(y'-\Delta y, t'-\Delta t) = 2p(y', t') - 2\Delta t\frac{\partial p}{\partial t} + (\Delta y)^2\frac{\partial^2 p}{\partial y^2}$$

$$\frac{1}{2}\left[\text{LHS}\right] = p(y', t') - \frac{\partial p}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 p}{\partial y^2}(\Delta y)^2.$$

$$p(y', t') = p(y', t') - \frac{\partial p}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 p}{\partial y^2}(\Delta y)^2.$$

$$-p(y', t') \qquad -p(y', t')$$



$$-\frac{\partial p}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 p}{\partial y^2}(\Delta y)^2 = 0.$$

$\frac{(\Delta y)^2}{\Delta t} \rightarrow c.$

As $\Delta t \to 0$;

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{(\Delta y)^2}{\Delta t}\frac{\partial^2 p}{\partial y^2}.$$

$\rightarrow$ Constant $= 1$ $\quad e^{-\frac{y^2}{2t}}$

$\frac{\partial}{\partial t} \quad = \quad 2\,(\Delta t)\,.$

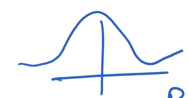
$\rightarrow$ Constant $= 1$

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2} \rightarrow p = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{y^2}{2t}}$$

$(\Delta y)^2 \cong (\Delta t)$ ✓

$\Delta y \cong \sqrt{\Delta t}\,.$

$$p(y,t) = \frac{1}{\sqrt{t}} f(\eta)\,; \quad \eta = \frac{y}{\sqrt{t}}\,; \quad \frac{d\eta}{dt} = -\frac{1}{2}t^{-3/2}y\,; \quad \frac{d\eta}{dy} = t^{-1/2}.$$



$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial t}\left(t^{-1/2} f(\eta)\right) = -\frac{1}{2}t^{-3/2} f(\eta) + t^{-1/2}\frac{df}{d\eta}\cdot\frac{d\eta}{dt}.$$

$$= -\frac{1}{2}t^{-3/2} f(\eta) - \frac{1}{2}t^{-1/2}t^{-3/2}y\frac{df}{d\eta}$$

$$= -\frac{1}{2}t^{-3/2}\left[f(\eta) + t^{-1/2}y\frac{df}{d\eta}\right].$$

$p \to t,\ f;\ f \to \eta$.

$$\eta = \frac{y}{\sqrt{t}} = y\,t^{-1/2}.$$

$$= -\frac{1}{2}t^{-3/2}\left[f(\eta) + \eta\frac{df}{d\eta}\right] = -\frac{1}{2}t^{-3/2}\frac{d}{d\eta}\left[\eta f\right] \rightarrow \text{LFODE}$$

IF.

$\frac{d\eta}{d\eta} f + \eta\frac{df}{d\eta}$

$f + \eta\frac{df}{d\eta}.$

$p \to t,\ f;\ f \to \eta;\ \eta \to t,\ y.$

$$p = t^{-1/2} f(\eta) \quad ; \quad \eta = y\,t^{-1/2}\,; \quad \frac{d\eta}{dy} = t^{-1/2}$$

$$\frac{\partial p}{\partial y} = t^{-1/2}\frac{df}{d\eta}\cdot\frac{d\eta}{dy} = t^{-1/2}\,t^{-1/2}\frac{df}{d\eta} = t^{-1}\frac{df}{d\eta}$$

$$\frac{\partial^2 p}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial y}\left(t^{-1}\frac{df}{d\eta}\right) = t^{-1}\frac{d^2 f}{d\eta^2}\cdot\frac{d\eta}{dy} = t^{-1}\frac{d^2 f}{d\eta^2}t^{-1/2}$$

$$= t^{-3/2}\frac{d^2 f}{d\eta^2}.$$

Recall $\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2}.$

$$-\cancel{\frac{1}{2}}\,\cancel{t^{-3/2}}\frac{d}{d\eta}(\eta f) = \cancel{\frac{1}{2}}\,\cancel{t^{-3/2}}\frac{d^2 f}{d\eta^2}.$$

$$-\frac{d}{d\eta}(\eta f) = \frac{d^2 f}{d\eta^2}. \quad \text{IBS wrt } \eta$$

$$-\int \frac{d}{d\eta}(\eta f)\,d\eta = \int \frac{d^2 f}{d\eta^2}\,d\eta$$

$$-\eta f = \frac{df}{d\eta} + k_1$$

As $\eta \to \infty$ $f \to 0$; $\frac{df}{d\eta} \to 0$;

$\frac{y}{\sqrt{t}} \to \infty.$

$t \to 0$;

$e^{-\frac{y^2}{2t}}$

$t \to 0.$ $\frac{y^2}{2t} \to \infty.$

$-\frac{y^2}{2t} \to -\infty.$

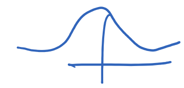
$e^{-y^2/2t} \to 0.$

$$0 = 0 + k_1 \Rightarrow k_1 = 0.$$

$$-\eta f = \frac{df}{d\eta}$$

$$df \quad -\eta\, d\eta$$

$- \eta | t - \quad d\eta$ $\qquad e^{-\eta^2/2} \to 0.$



$$\frac{df}{f} = -\eta \, d\eta$$

IBS;

$$\ln f = -\frac{\eta^2}{2} + k_2.$$

$$e^{\ln f} = e^{-\eta^2/2 + k_2}.$$

$$f = e^{-\frac{\eta^2}{2}} e^{k_2} = C e^{-\frac{\eta^2}{2}}$$

What is my C?

$$\int_{\mathbb{R}} f(\eta)\, d\eta = 1; \qquad C\int_{\mathbb{R}} C e^{-\eta^2/2}\, d\eta = 1;$$

$$C \int_{\mathbb{R}} e^{-\eta^2/2}\, d\eta = 1; \qquad C \int_{\mathbb{R}} e^{-\left(\frac{\eta}{\sqrt{2}}\right)^2} d\eta = 1; \qquad u = \frac{\eta}{\sqrt{2}}. \quad du = \frac{d\eta}{\sqrt{2}}.$$

$\eta \to \infty; u \to \infty.$
$\eta \to -\infty; u \to -\infty.$

$$C \int_{\mathbb{R}} e^{-u^2} \sqrt{2}\, du = 1; \qquad C\sqrt{2} \int_{\mathbb{R}} e^{-u^2} du = 1. \to \sqrt{\pi} \quad \text{IBS by polar coordinates.}$$

C III
Multi variable.

$$C\sqrt{2}\sqrt{\pi} = 1;$$

$$C = \frac{1}{\sqrt{2\pi}}; \qquad \eta = \frac{y}{\sqrt{t}}.; \quad \eta^2 = \frac{y^2}{t}.$$

$$f(\eta) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\eta^2}{2}}$$

$$f(t, y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2t}}.$$

$$p(t, y) = t^{-1/2} f(\eta) = \frac{1}{\sqrt{t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2t}} = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}.$$

PDF for ND with $\mu = 0$, $\sigma = 1$.

More generally speaking

$$p(t, y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(y-\mu t)^2}{2\sigma^2 t}}$$

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2}.$$

Options $\to$ BOPM $\xrightarrow{\text{As } \Delta t \to 0.}$ BS OPM $\xrightarrow{\text{1 hr.}}$ Heat eqn / Diff eqn. $\to$ soln = PDF of ND. $\frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(y-\mu t)^2}{2\sigma^2 t}}.$

$F(x, y) = x \sin y$, $\frac{dy}{dx}$.

$$\frac{d}{dx}(x \sin y) = (x)' \sin y + x (\sin y)' = 1 \cdot \sin y + x \cdot \cos y \cdot y' = 0 .$$

$$\begin{tikzcd} & F \arrow[dl, dash] \arrow[dr, dash] & \\ x & & y \arrow[d] \\ & & x \end{tikzcd}$$

$$\Rightarrow \sin y + x \cos y \; y' = 0 .$$

$$\Rightarrow y' = \frac{-\sin y}{x \cos y} \quad \begin{matrix} \to F_x \\ \to F_y . \end{matrix}$$

$$\Rightarrow y' = -\frac{\tan y}{x} .$$