Calibration and Data Analysis

I II

In this lecture. . .

Part I
Current

- the theoretical yield curve and the market yield curve, should they be the same?

Calibration

Continuation from other night.

- how calibration works, the pros and cons

Part II
historical

- how to analyze short-term interest rates to determine the best model

Data centred analysis

- how to analyze the slope of the yield curve to get information about the market price of interest rate risk

By the end of this lecture you will

already done in mod 3

- know the meaning of 'calibration'

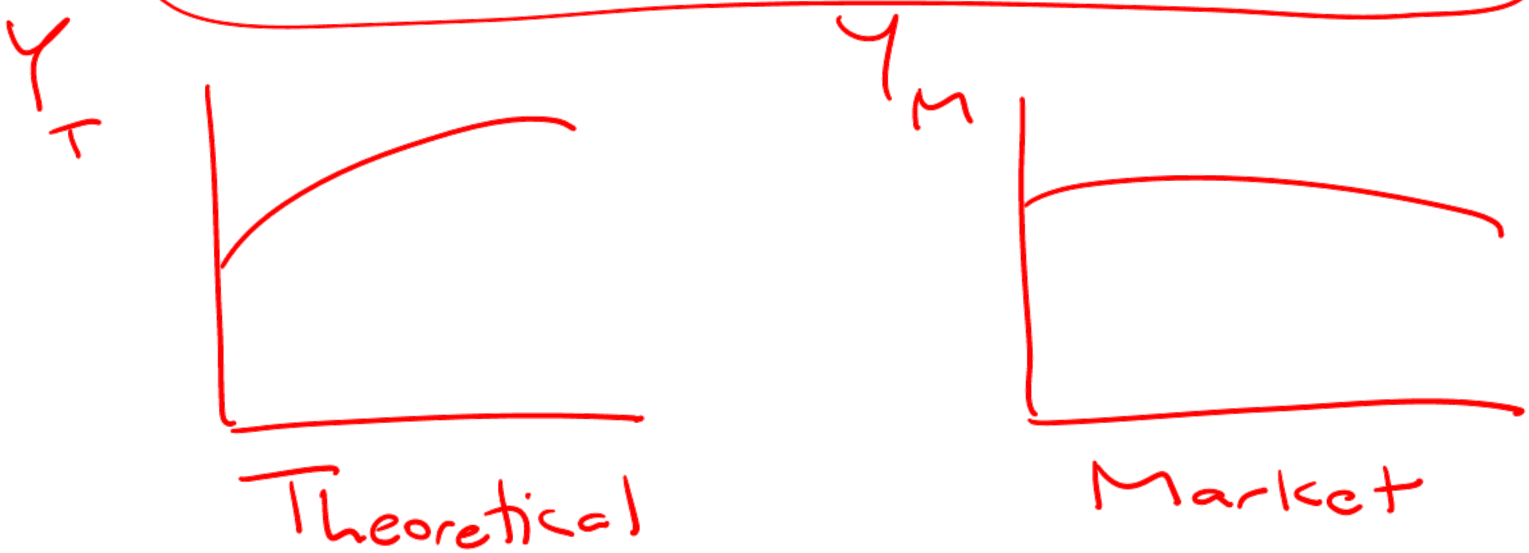
fitting the yield curve

- appreciate the pros and cons of calibration

Paul's favourite

- be able to analyze data to find a good model

λ MPOR



ZCB $Z = e^{A - rB}$ theoretical price

Introduction

In some texts $Z = e^{A+rB}$

We have seen the theory behind one-factor (and multi-factor) interest rate models. $\boxed{6.2}$

These models will have as an output a 'theoretical' yield curve.

This theoretical yield curve will not be the same as the yield curve seen in the market.

Is this good or bad?!

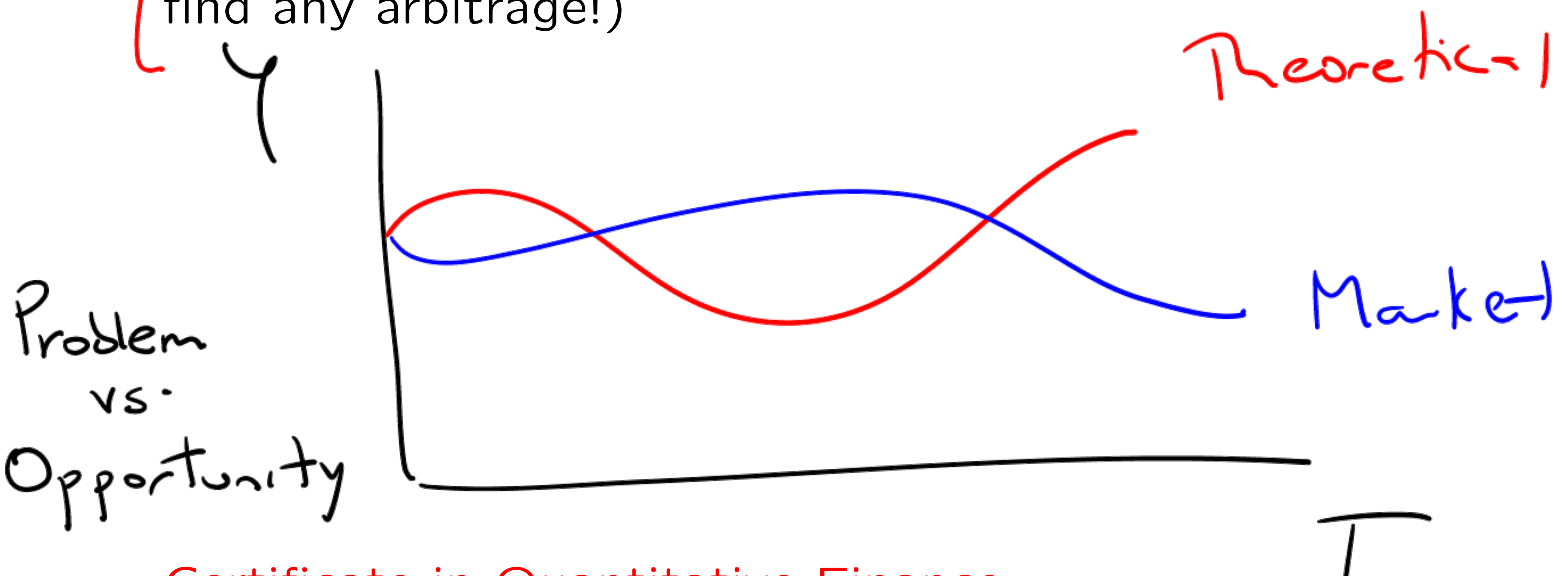
① Affine solutions $Z = e^{A - rB}$ (closed form sol^{n})

② PDE B.P.E (computational?)

③ M.C numerical schemes

1 **It is bad if. . .** your job is to price exotic, structured products which must be hedged with simple instruments. (How can you be expected to price exotics 'correctly' if you can't even match simple instruments?!)

2 **It is good if. . .** you believe your theoretical model and you are looking for arbitrage opportunities among simple instruments. (If your model output matched market prices then you'll never find any arbitrage!)



① People like calibration

$\rightarrow$ Theoretical prices $=$ market prices $\leftarrow$

Just calculated $e^{A-rB} = Z_M$

$$SN(d_1) - Ee^{-r(T-t)}N(d_2) = V_M(S,t)$$

$$\sigma$$

② $\underline{\text{Theory}} \longrightarrow \underline{\text{ZCB prices}}$

③ parameter $\underline{\underline{f}} \longrightarrow \begin{cases} f(t) \leftarrow \\ f_t \leftarrow \end{cases}$

$$dr = (\eta - \gamma r)\,dt + \sigma\,dX$$

$\gamma(t)$

$\hookrightarrow \sigma(t)$

$\eta(t)$

$$\sigma_{imp} \longrightarrow \sigma(S,t)$$

$$\underbrace{\sigma^2 = \frac{1}{T-t}\int_t^T \sigma_s^2\,ds}_{[t,T]}$$

So in using any model we have to decide how to choose the parameters.

Should the parameters be chosen to match

1 • the market yield curve? Or Calibration current

2 • historical interest rate data? Data analysis historical

I The former is **calibration** to a snapshot of the market at one instant in time.

t^* today's date = ~~17~~ 06/06/2023

II The latter is fitting to time series data.

data analysis

Let's start with calibration.

HWI, II

fitting the yield curve

Calibration

① Because of this need to correctly price liquid instruments, the idea of **yield curve fitting** or **calibration** has become popular.

When stochastic models are used in practice they are almost always fitted.

To match a theoretical yield curve to a market yield curve requires a model with enough degrees of freedom. (You are matching a curve, i.e. an 'infinite' number of points, so you need infinite degrees of freedom!)

② This is done by making one or more 'parameters' time dependent.

③
- This functional dependence on time is then carefully chosen to make an output of the model, the price of zero-coupon bonds, exactly match the market prices for these instruments.

$$Z_M = e^{A - rB}$$

(Legacy model)

Ho & Lee

$$\boxed{\eta_i = \eta(t_i)}$$

discretise time t

The Ho & Lee spot interest rate model is the *simplest that can be used to fit the yield curve*.

Now we don't necessarily say this model or idea is great, but we will go through the mathematics to see how it can be done.

Recap: In the Ho & Lee model the process for the *risk-neutral* spot rate is

fn. of time

Const. vol.

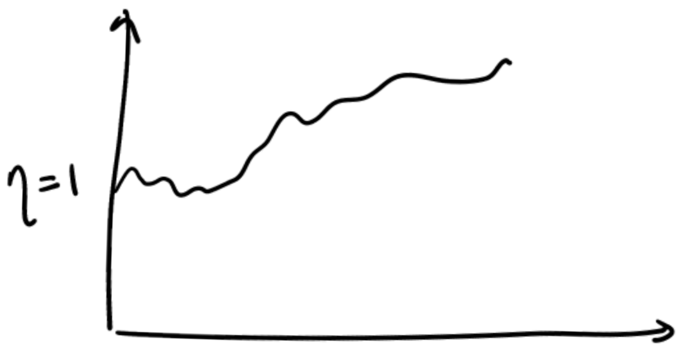
$$dr = \eta(t)dt + c\,dX.$$

time series

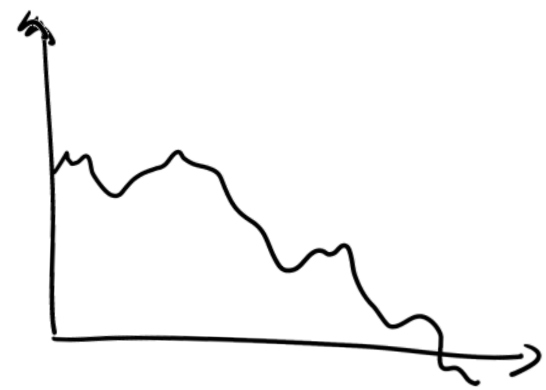
The standard deviation of the spot rate process, c, is constant, the drift rate η is time dependent.

Behaviour of Ho and Lee

If $\eta \sim O(1) \Rightarrow$ B.M with drift



η -ve



Differentiating under the integral sign

Leibniz rule

$$\frac{\partial}{\partial x}\int_a^x F(s,x)\,ds = F(x,x) + \int_a^x \frac{\partial F}{\partial x}(s,x)\,ds$$

As we've seen in an earlier CQF lecture, for this model the solution of the bond pricing partial differential equation for a zero-coupon bond is simply

$$\frac{\partial Z}{\partial t} = (\dot{A} - r\dot{B})Z \qquad \frac{\partial Z}{\partial r} = -BZ \qquad \frac{\partial^2 Z}{\partial r^2} = B^2 Z$$

Affine Idⁿ

$$① \quad Z(r,t;T) = e^{A(t;T) - r\overbrace{(T-t)}^{B(t;T)}},$$

where

$$\frac{\partial V}{\partial t} + \frac{1}{2}\underset{w^2}{c^2}\frac{\partial^2 V}{\partial r^2} + \underset{(u-\lambda w)}{\eta(t)}\frac{\partial V}{\partial r} - rV = 0$$

$$ⓐ \quad A(t;T) = -\int_t^T \eta(s)(T-s)ds + \frac{1}{6}c^2(T-t)^3.$$

$$ⓑ \quad B(t;T) = (T-t)$$

(Note that the variables are r and t, but we are also explicitly referring to the parameter T, the bond maturity.)

actual eqⁿ which is written in 2 parts

$$\underbrace{\dot{A} + \frac{1}{2}c^2 B^2 - B\eta(t)}\; - r\big(\underbrace{\dot{B}+1}\big) = 0 + 0 \times r$$

$$\frac{dB}{dt} = -1 \quad ①$$

Certificate in Quantitative Finance

$$\dot{A} + \frac{1}{2}c^2B^2 - B\eta(t) = 0 \quad ②$$

$$\int_t^T dA = \int_t \left(B\eta(s) - \frac{1}{2}c^2B^2\right)ds$$

Ho and Lee $\quad dr = \eta(t)\,dt + c\,dX$

from previous page

$$Z = e^{A - rB}$$

$$\frac{\partial Z}{\partial t} = (\dot{A} - r\dot{B})\,Z$$

B.P.E:

$$\frac{\partial V}{\partial t} + \frac{1}{2}c^2\frac{\partial^2 V}{\partial r^2} + \eta(t)\frac{\partial V}{\partial r} - rV = 0$$

$$\int_t^T dB = -\int_t^T d\tau$$

$$\frac{\partial Z}{\partial r} = -BZ$$

$$\frac{\partial^2 Z}{\partial r^2} = B^2 Z$$

$$Z_M = e^{\overbrace{-\int_t^T \eta(s)\,ds}^{A(t;T)} \; - \; r\underbrace{(T-t)}_{B(t;T)}}$$

inverse problem

Known from market

$B(t;T) = T-t$

Working forwards: If we know $\eta(t)$ then the above gives us the theoretical value of zero-coupon bonds of all maturities. I.e. start with model $(\eta(t))$ and find answer (Z).

An inverse problem: But what if we know Z from the market, but don't know the unobservable η? Turn this relationship around and ask the question

- 'What functional form must we choose for $\eta(t)$ to make the theoretical value of the discount rates for all maturities equal to the market values?'

That is calibration!

(What about the parameter c?) → some statistical technique

fixed $t^* = 6/06/2023$ we calibrate : vary T
(look at yield curve)

Suppose we want to calibrate our model today, time t^*. Today's spot interest rate is r^* and the discount factors *in the market* are $Z_M(t^*; T)$.

$$\boxed{dr = \eta(t)dt + c\, dX}$$

Call the special, calibrated, choice for η, $\eta^*(t)$.

t^* – today

$Z(r, t; T)$
t → t^* fixed
T → vary

$r^* =$ today's spot rate

$\eta^*(t) = \eta(t)$ using above technology

Assume cts. dist^n of maturities

To match the market and theoretical bond prices, we must solve

Inside A is $\eta(t)$

Screen

$$-\int_{t^*}^{T} \eta^{(*)}(s)\,ds$$

$$Z_M(t^*;T) = e^{A(t^*;T) - r^*(T-t^*)}$$

natural

market price

Theoretical price

Taking logarithms of this and rearranging slightly we get

which is what we want.

$$\int_{t^*}^{T} \eta^*(s)(T-s)ds = \left[-\log(Z_M(t^*;T)) - r^*(T-t^*) + \frac{1}{6}c^2(T-t^*)^3.\right] \tag{1}$$

Eq^n (1) Integral eq^n

We know everything on the right-hand side. So this is an **integral equation** for $\eta^*(t)$.

Leibniz Rule

Need to diff. (1) twice to "kill-off" $\int$ & hence obtain $\eta(t)$

Suppose, we have already diff the r.h.,

(Luckily for us, it is quite easy to solve!)

[Leibnitz Rule – diffⁿ under integral sign]

Observe what happens if we differentiate the integral term with respect to T.

$$\frac{\partial}{\partial x}\int_a^x F(s,x)\,ds = F(x,x) + \int_a^x \frac{\partial F(s,x)}{\partial x}\,ds. \quad (\dagger)$$

First differentiate once with respect to T using $(\dagger)$

$$\frac{d}{dT}\int_{t^*}^{T}\eta^*(s)(T-s)ds = \int_{t^*}^{T}\eta^*(s)ds.$$

$$= \underbrace{\eta^*(T)(T-T)}_{=0} + \int_{t^*}^{T}\underbrace{\frac{\partial}{\partial T}\left(\eta^*(s)(T-s)\right)}_{\eta^*\times 1}ds = \int_{t^*}^{T}\eta^*(s)ds$$

Differentiate again

$$\frac{d^2}{dT^2}\int_{t^*}^{T}\eta^*(s)(T-s)ds = \eta^*(T).$$

$$= \eta^*(T) + \int_{t^*}^{T}\frac{\partial}{\partial T}\eta^*(s)\,ds = \eta^*(T)+0$$

Now diff RHS : $\frac{\partial^2}{\partial T^2} = \left(-\frac{\partial^2}{\partial T^2}\log Z_M(t^*;T) + c^2(T-t^*)\right)$

Now replace T by t

So, differentiating (1) twice with respect to T we get

divided differences.

numerical diff.

$$\eta^*(t) = c^2(t - t^*) - \left(\frac{\partial^2}{\partial t^2}\log(Z_M(t^*;t)).\right)$$

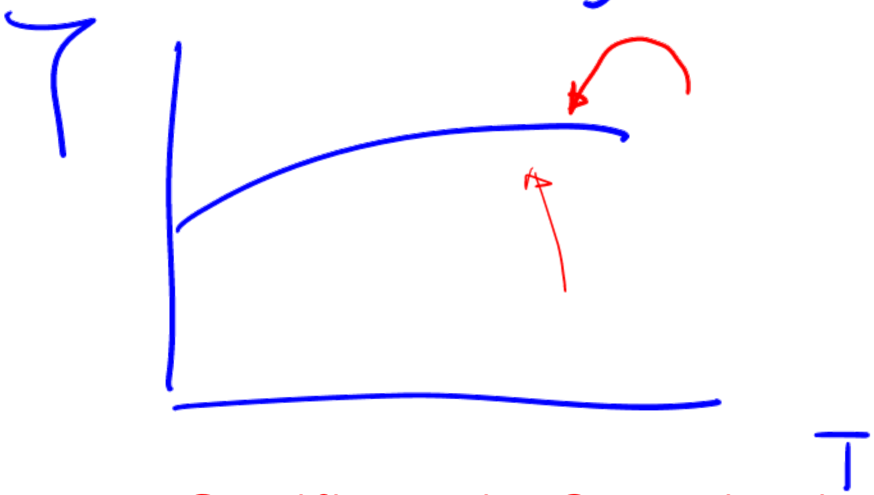
The solution!

$$dr = \eta(t)\,dt + c\,dX$$

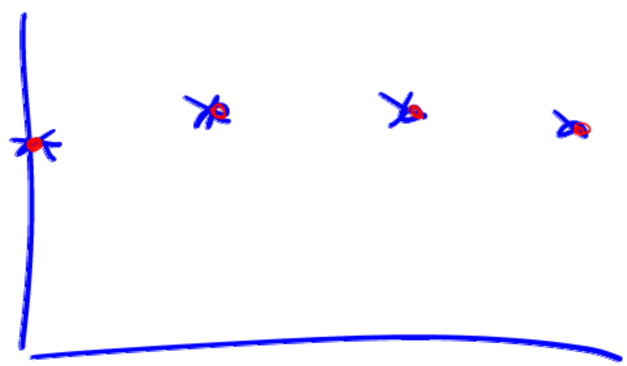
price more complex products.

With this choice for the time-dependent parameter $\eta(t)$ the theoretical and actual market prices of zero-coupon bonds are the same.

Theory



Market



Numerical Analysis Burden & Faires

Notes: Yield Curve Fitting

Complex

- Now that we know $\eta(t)$ we can price other fixed income instruments.

- We say that our prices are **consistent with the yield curve**.

- The same idea can be applied to other spot interest rate models. HW I $dr = (\eta(t) - \gamma(t) r)\,dt + \sqrt{\beta(t)}\,dX$

  HW II CIR with time dep.

- This is an inverse problem, and will typically be sensitive to input data (the Z).

$\boxed{\sigma(s,t)}$

Another calibrated model:
The extended Vasicek model of Hull & White

Most one-factor models have the potential for fitting, the more tractable the model the easier the fitting. If the model is not at all tractable then we can always resort to numerical methods.

The next easiest model to fit is the Vasicek model. The Vasicek model has the following stochastic differential equation for the risk-neutral spot rate

$\eta(t)$

HWI

(simple form)

$$dr = (\eta - \gamma r)dt + c\,dX.$$

Hull & White extend this to include a time-dependent parameter

$$dr = (\eta(t) - \gamma r)dt + c\,dX.$$

Assuming that γ and c have been estimated statistically, say, we choose $\eta = \eta^*(t)$ at time t^* so that our theoretical and the market prices of bonds coincide.

$\eta \longrightarrow \eta(t)$

$\boxed{\gamma, c \in \mathbb{R}}$

$Z = e^{A - rB}$

$V_t + \frac{1}{2}c^2 V_{rr} + (\eta(t) - \gamma r)V_r - rV = 0$

Again, as covered in an earlier CQF lecture, under this risk-neutral process the value of a zero-coupon bond is

Exercise: soln to follow

$$Z(r,t;T) = e^{A(t;T) - rB(t;T)},$$

where

fn of t

$$A(t;T) = -\int_t^T \eta(s)B(s;T)\,ds$$

$$+\frac{c^2}{2\gamma^2}\left(T - t + \frac{2}{\gamma}e^{-\gamma(T-t)} - \frac{1}{2\gamma}e^{-2\gamma(T-t)} - \frac{3}{2\gamma}\right).$$

and

$$\boxed{B(t;T) = \frac{1}{\gamma}\left(1 - e^{-\gamma(T-t)}\right).}$$

$\ln Z_M = \ln e^{A - rB}$

As before, to fit the yield curve at time t^* we must make $\eta^*(t)$ satisfy

Use Leibniz twice

$\frac{d}{dT}$

$$
\begin{aligned}
&- \int_{t^*}^{T} \eta^*(s) B(s;T) ds \\
&+ \frac{c^2}{2\gamma^2}\left(T - t^* + \frac{2}{\gamma}e^{-\gamma(T-t^*)} - \frac{1}{2\gamma}e^{-2\gamma(T-t^*)} - \frac{3}{2\gamma}\right) \\
&= \log(Z_M(t^*;T)) + r^* B(t^*,T). \qquad (2)
\end{aligned}
$$

This is an integral equation for $\eta^*(t)$ if we are given all of the other parameters and functions, such as the market prices of bonds, $Z_M(t^*;T)$.

Exercise

Equation (2) is easy to solve by differentiating the equation twice with respect to T. This gives

$$\eta^*(t) = -\frac{\partial^2}{\partial t^2}\log(Z_M(t^*;t)) - \gamma\frac{\partial}{\partial t}\log(Z_M(t^*;t)) + \frac{c^2}{2\gamma}\left(1 - e^{-2\gamma(t-t^*)}\right). \tag{3}$$

Note: (Please don't get the idea from this that all models are easy to calibrate or all integral equations are easy to solve!)

Very lucky

Recall in the derivation of the B.P.E :

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - rV = a(r,t)\frac{\partial V}{\partial r} \rightarrow \frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - a(r,t)\frac{\partial V}{\partial r} - rV = 0$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r} = a(r,t)\frac{\partial V}{\partial r} + rV + u\frac{\partial V}{\partial r}$$

Recall $$dV = \underbrace{\left(\frac{\partial V}{\partial t} + u\frac{\partial V}{\partial r} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2}\right)}_{\text{replace with lhs of B.P.E}} dt + w\frac{\partial V}{\partial r}dX$$

Itô IV on $V(r,t)$

$$dV = \left((a+u)\frac{\partial V}{\partial r} + rV\right)dt + w\frac{\partial V}{\partial r}dX$$

$$dV - rV\,dt = (a+u)\frac{\partial V}{\partial r}dt + w\frac{\partial V}{\partial r}dX$$

$$= w\frac{\partial V}{\partial r}\left[\left(\frac{a+u}{w}\right)dt + dX\right] = w\frac{\partial V}{\partial r}\left[dX + \overbrace{\left(\frac{a+u}{w}\right)}^{\text{excess}}dt\right]$$

Let $\frac{a+u}{w} = \lambda \rightarrow \boxed{a = \lambda w - u}$

PART II

And now for the opposite approach, analyze historical data. . . no reliance on a single 'snapshot.'

Unhedged bond $dV = w\frac{\partial V}{\partial r}dX + \left(\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r}\right)dt$

B.P.E: $$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - a(r,t)\frac{\partial V}{\partial r} - rV = 0$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + u\frac{\partial V}{\partial r} = u\frac{\partial V}{\partial r} + a(r,t)\frac{\partial V}{\partial r} + rV$$

$$dV = w\frac{\partial V}{\partial r}dX + \left(\frac{\partial V}{\partial r}(u+a) + rV\right)dt$$

$$dV - rV\,dt = w\frac{\partial V}{\partial r}dX + \frac{\partial V}{\partial r}(u+a)\,dt$$

$$= w\frac{\partial V}{\partial r}\left[\underline{\underline{dX}} + \underbrace{\left(\frac{u+a}{w}\right)}dt\right]$$

$$\frac{u+a}{w} = \lambda \longrightarrow \boxed{a(r,t) = \lambda w = u}$$

Data analysis to find the 'best' model for spot rate

The one (or more)-factor models for the spot interest rate that we have seen were all chosen for their nice properties; for most of them we were able to find simple closed-form solutions of the bond-pricing equation. thus far: e^{A-rB}

Clearly, this means that the models are not necessarily a good description of reality.

Let's recap these models quickly. . .

Popular one-factor spot-rate models

The real spot rate r satisfies the stochastic differential equation

Real

$$dr = u(r,t)dt + w(r,t)dX. \quad (4)$$

Risk-neutral models

Model	$u(r,t) - \lambda(r,t)w(r,t)$	$w(r,t)$
Vasicek	$a - br$	c const.
CIR	$a - br$ linear	sq. rt. $cr^{1/2}$
Ho & Lee	$a(t)$	c
Hull & White I	$a(t) - b(t)r$	$c(t)$
Hull & White II	$a(t) - b(t)r$	$c(t)r^{1/2}$
General affine	$a(t) - b(t)r$	$(c(t)r - d(t))^{1/2}$

Here $\lambda(r,t)$ denotes the market price of risk. The function $u - \lambda w$ is the risk-adjusted drift.

For all of these models the zero-coupon bond value is of the form $Z(r,t;T) = e^{A(t,T)-rB(t,T)}$.

The time-dependent coefficients in all of these models allow for the fitting of the yield curve and other interest-rate instruments.

In what follows, ignore previous models

From now on in this lecture we will see how to deduce a model for the spot rate from data; it is therefore unlikely to be nice and tractable!

No more $u(r,t)$, $w(r,t)$

but $u(r)$, $w(r)$

The method

The method that we use assumes that

- the model is time homogeneous drop time dep.

$$dr = u(r)\,dt + w(r)\,dX$$

- the spot rate is well behaved (i.e. doesn't wander too far away, go to zero or infinity for example)

The downside to the resulting model is that we cannot find closed-form solutions for contract values, the risk-neutral drift and the volatility don't have a sufficiently nice structure.

$$dr = (\eta(t) - \gamma(t) r)\,dt + \sqrt{\alpha(t) r + \beta(t)}\,dX$$

Source of his data

In the figure are shown the US one-month LIBOR rates, daily, for the period 1985–1997, and is the data that we use in our analysis.

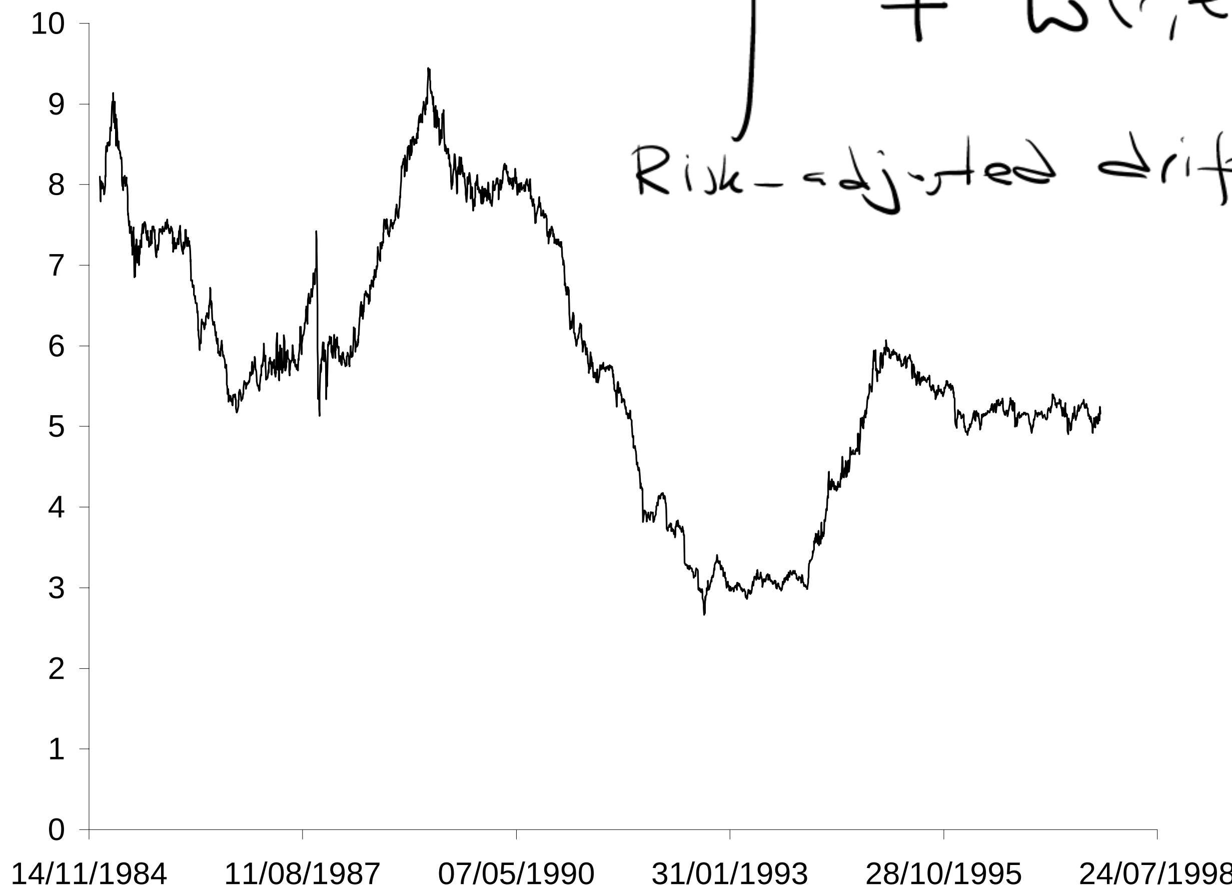
The ideas that we introduce can be applied to any currency, but here we use US data for illustration.

Aside: This method isn't specific to interest rates, it has also been used to model the gold price, equity and index volatility, and the rate of inflation.

R-N spot

$$dr = \left(u(r,t) - \lambda(r,t)\,w(r,t)\right)dt + w(r,t)\,dX$$

Risk-adjusted drift



(Extra lecture: "Fear & greed in F.I markets".)

Real $dr = u\,dt + w\,dX$ $dr = (u - \lambda w)\,dt + w\,dX$

R-N

risk-adjusted drift

There are three key stages:

$w(r)$

1. By differencing spot rate time series data we determine the volatility dependence on the spot rate $w(r)$. diffusion/vol

$u(r)$

2. By examining the [steady-state probability density function] for the spot rate we determine the functional form of the drift rate $u(r)$. Module 1.5

$\lambda(r)$

3. We examine the slope of the yield curve to determine the market price of risk $\lambda(r)$. MP.oR

Starting point: $\qquad$ C.I.R $\quad c\sqrt{r}$

1. The volatility structure

ν, β

Our first observation is that many popular models take the form

$\times$	dt	dX
dt	0	0
dX	0	dt

SDE of choice

$$dr = u(r)dt + \underbrace{\nu r^\beta}_{w(r)} dX. \qquad (5)$$

Power law in vol.

$dX^2 \Rightarrow dt$

Examples of such models are the Ho & Lee ($\beta = 0$), Vasicek ($\beta = 0$) and Cox, Ingersoll & Ross ($\beta = 1/2$) models.

$\longrightarrow dr^2 = \nu^2 r^{2\beta} dt \quad (*)$

$\nu, \beta \longrightarrow w(r)$

$$\mathbb{E}\left[dr^2\right] = \nu^2 r^{2\beta} dt \Rightarrow \sqrt{\frac{\mathbb{E}(\delta r^2)}{\delta t}} = \nu r^\beta$$

Using our US spot rate data we can estimate the best value for β using a very simple bucketing technique.

From the time-series data divide the changes in the interest rate, δr, into buckets covering a range of r values. e.g 7% - 8%

From previous slide take logs of ⊛

$$\log(\underbrace{\nu^2 r^{2\beta}\delta t}_{\nu^2\delta t\, r^{2\beta}}) = \log \nu^2 \delta t + \log r^{2\beta} \longleftarrow \text{property of logs}$$

$$= \boxed{\log \nu^2 \delta t} + 2\beta \log r \quad \text{RHS}$$

LHS : $\log(\mathbb{E}[\delta r^2]) \leftarrow y$

gradient $= m = 2\beta$

intercept $= c = \log \nu^2 \delta t$

RHS now becomes : $\underbrace{2\beta}_{m}\,\underbrace{\log r}_{x} + \underbrace{\log \nu^2 \delta t}_{c}$

$y = mx + c$

As an example

Look at all fr's in 7%-8% range

Just examine the δrs associated with each bucket.

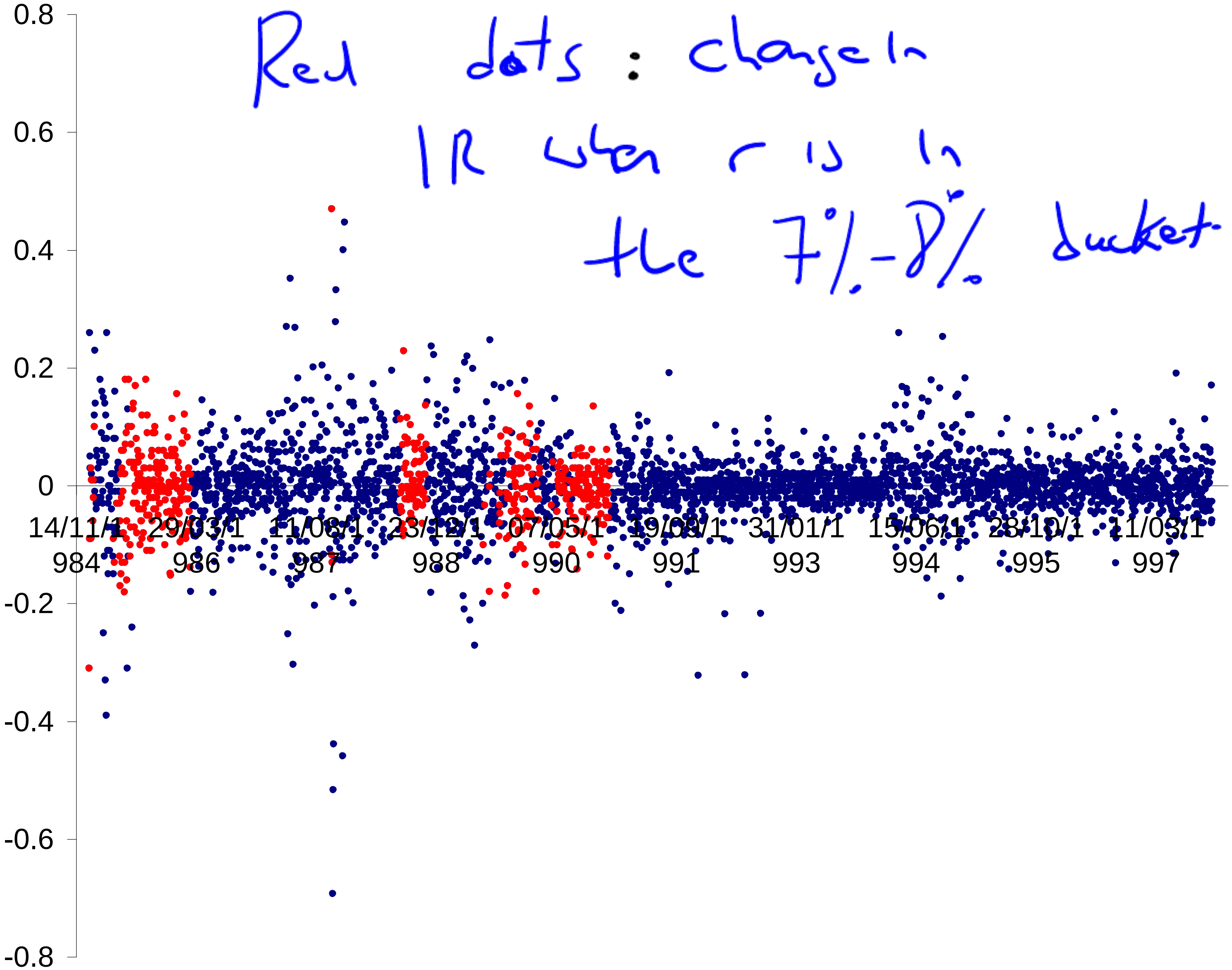
Move on to next bucket & repeat

i.e 8% – 9% bucket, 9% – 10% bucket

Calculate $\boxed{\delta r\text{'s each bucket / range}}$

$$\delta t = \frac{1}{252}$$

Red dots: change in IR when r is in the 7%-8% bucket.



Then calculate the average value of $(\delta r)^2$, for each bucket.

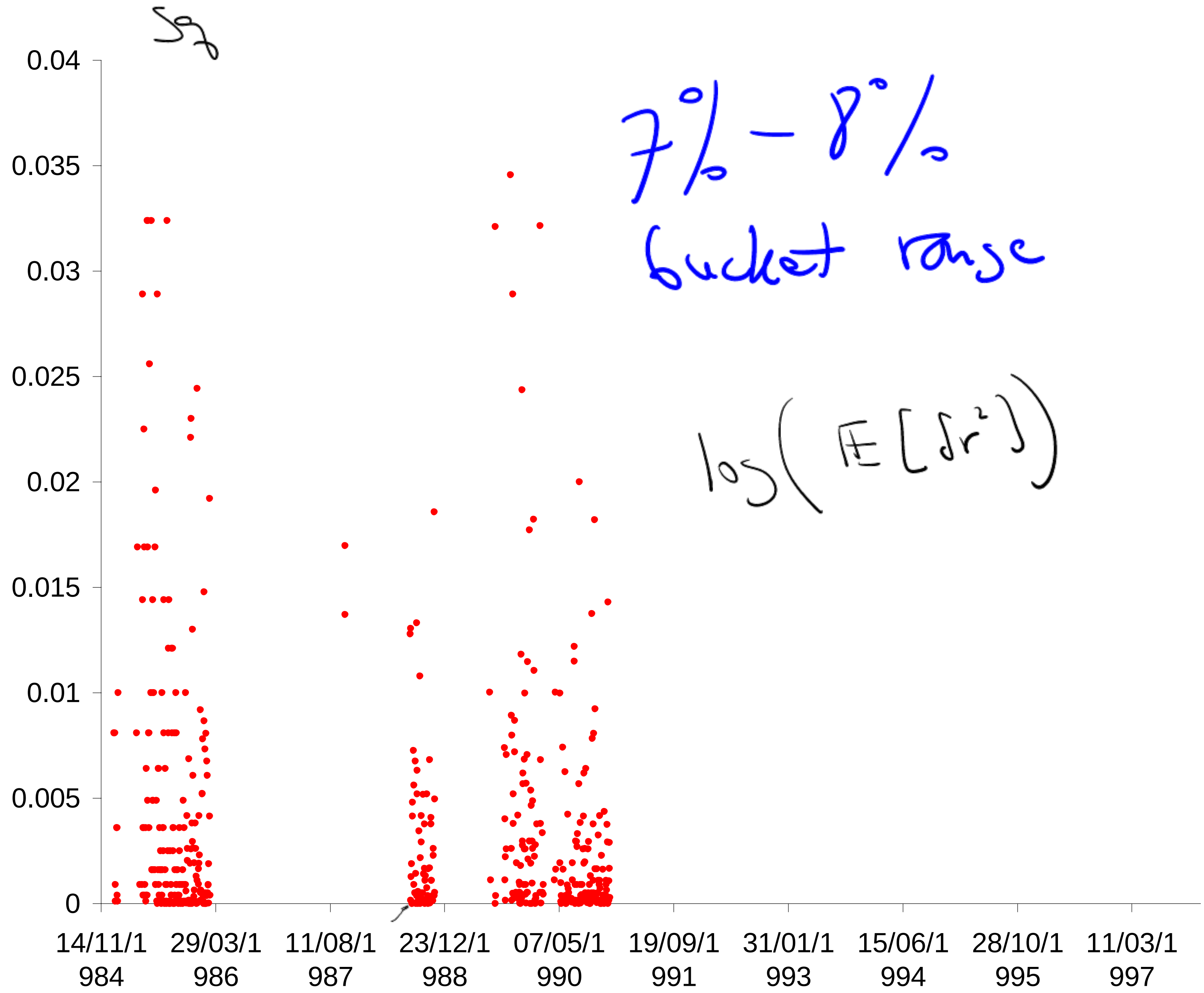
$$\mathbb{E}\left[\delta r^2\right] = \underbrace{\nu^2 r^{2\beta}\,\delta t}$$

$$\boxed{\delta t = \frac{1}{252}}$$

$$\delta r = u(r)\,\delta t + \nu r^\beta dX$$

SDE in discrete time have δ

$$\delta r^2 = \cancel{u^2 \delta t^2}_{\to 0} + \cancel{2u\nu r^\beta \delta t\, \delta X}_{\to 0} + \boxed{\nu^2 r^{2\beta}\,\delta t}$$


0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
14/11/1 984
29/03/1 986
11/08/1 987
23/12/1 988
07/05/1 990
19/09/1 991
31/01/1 993
15/06/1 994
28/10/1 995
11/03/1 997
7% - 8%
bucket range
log(E[δr²])

If the model (5) is correct we would expect

LHS is from data.

Average of δr^2

$$\underbrace{E[(\delta r)^2]} = \nu^2 r^{2\beta}\delta t$$

↳ Assumed model.

to leading order in the time step δt, which for our data is one day.

$\log E[\delta r^2] = \log \nu^2 \delta t + \log r^{2\beta}$

Now plot $\log(E[(\delta r)^2])$ against $\log r$ using the data.

$= \boxed{2\beta} \log r + \log \nu^2 \delta t$

The slope of this 'line' gives an estimate for 2β and where the line crosses the vertical axis can be used to find ν.

$y = mx + c$

$\log_{10}$

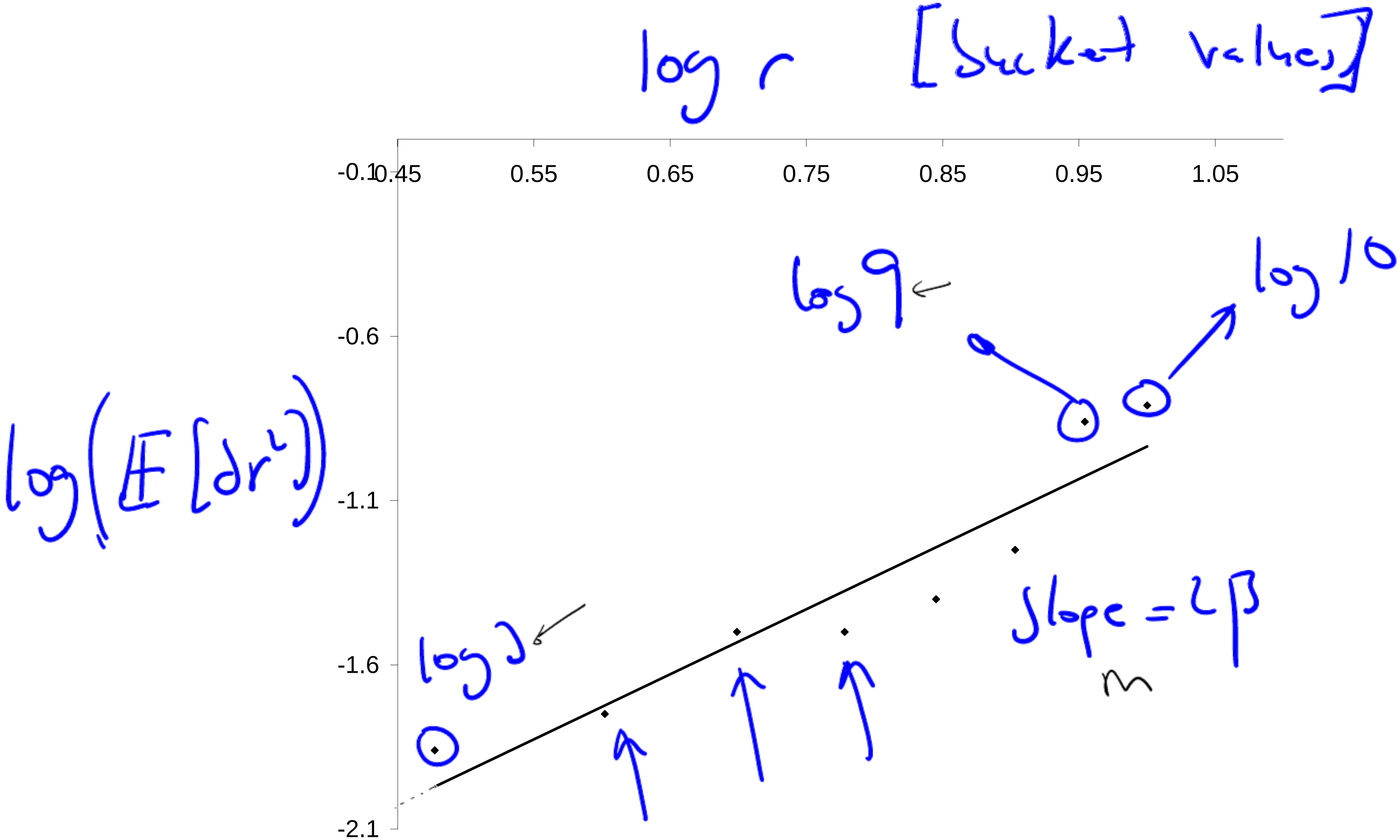
$\frac{\text{slope}}{2} = \beta$

$m = 2\beta$

$x = \log r$

$c = \log \nu^2 \delta t$

$\log \nu^2 \delta t$ $\quad 2 \log \nu \delta t \quad \delta t = 1/252$



Estimation of β

We can see that the line is very straight.

$dS = \mu S dt + \sigma S dX$

From this calculation it is estimated that

log-normal?

$w(r)$

r^1

$$\beta = 1.13 \quad \text{and} \quad \nu = 0.126.$$

This confirms that the spot rate randomness increases as the spot rate increases, approximately linearly.

(And this rules out Vasicek, Ho & Lee, etc. etc.)

νr^β

$w(r) = \underbrace{\nu r^{2\beta}}_{} = 0.126\, r$

2. The drift structure

Model $dr = a\,dt + b\,dX$

$\mathbb{E}[dr] = a\,dt$

It is statistically harder to estimate the drift term from the data; this term is smaller than the volatility term and thus subject to larger relative errors.

Our approach to finding the drift function is via the empirical and analytical determination of the steady-state probability density function for r.

$\mathbb{V}[dr] = b^2\,dt$?

$dr = \underbrace{u(r)}_{A}dt + \boxed{\underbrace{\nu r^\beta}_{B}} dX$

If r satisfies the s.d.e. (5) then the probability density function $p(r,t)$ for r satisfies the Fokker–Planck equation

transi pdf

module 1

=0

F.K.E

Lecture 1.5

$$\frac{\partial p}{\partial t} = \frac{1}{2}\nu^2\frac{\partial^2}{\partial r^2}(r^{2\beta}p) - \frac{\partial}{\partial r}(\boxed{u(r)}p). \qquad (6)$$

$p = p(r) \Rightarrow \partial p/\partial t = 0$

$P_\infty = P_r$

The steady state $p_\infty(r)$ will satisfy the time-independent version of (6):

$$\frac{1}{2}\nu^2\frac{d^2}{dr^2}(r^{2\beta}p_\infty) - \frac{d}{dr}(u(r)p_\infty) = 0. \qquad (7)$$

[Up till now if $u(r)$ is known the above ODE (7) can be solved $\rightarrow P_\infty(r)$]

By integrating (7) we find the relationship between the steady-state probability density function and the drift function:

Some algebra needed

known:

$$u(r) = \nu^2 \beta r^{2\beta-1} + \nu^2 \frac{1}{2} r^{2\beta} \frac{d}{dr} (\log p_\infty). \quad \boxed{7a}$$

refresh your memory by obtaining 7a from 7

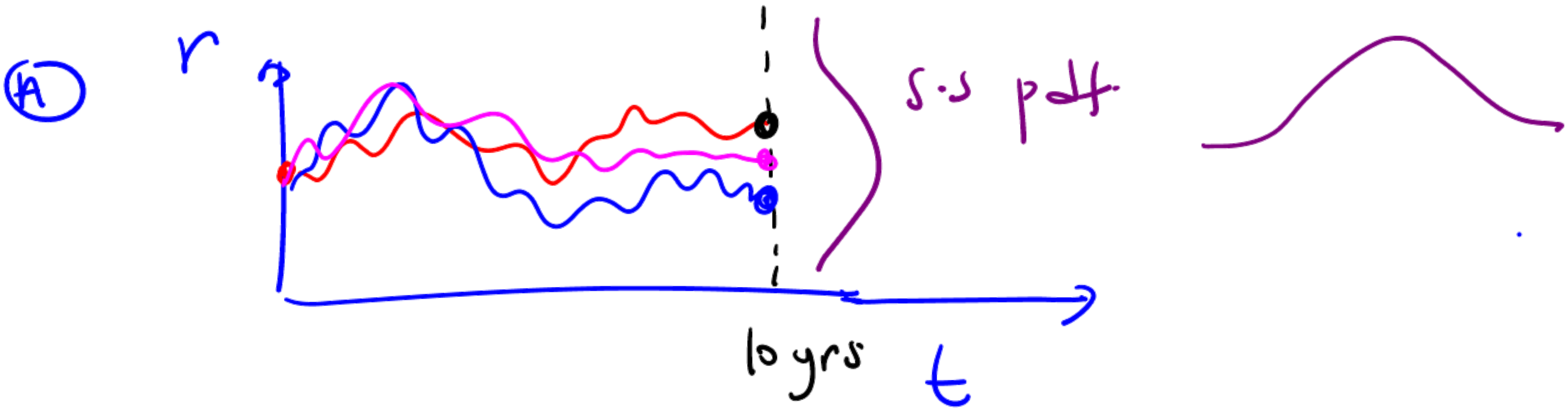
If we know one we can find the other.

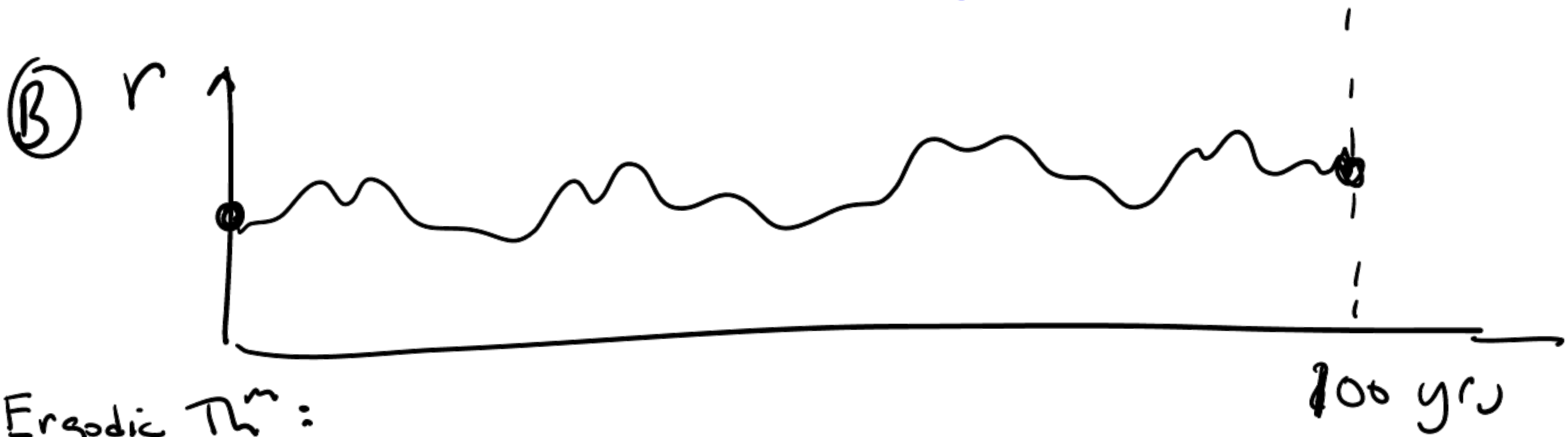
See model take home end of mod (non-assessed) exam

Only works in this case for $p_\infty(r)$

Do we know $p_\infty(r)$?

Yes ∵ Ergodic Th$^{\underline{m}}$





Ergodic Th^m:

1) If SDE time homog. ; (2) steady state pdf exists (must)

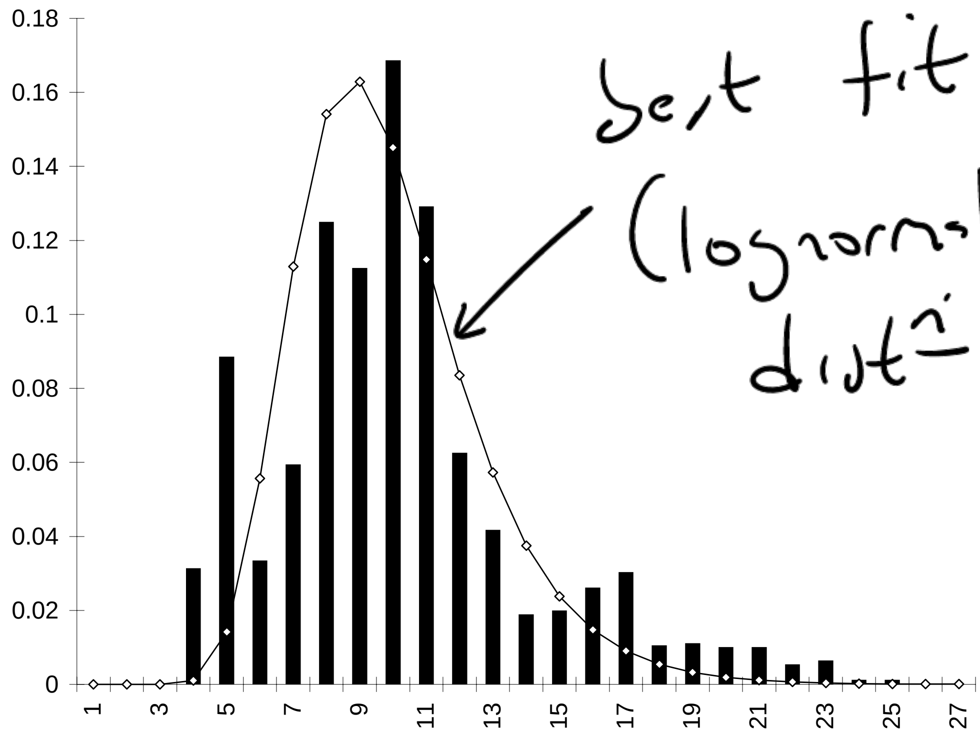
THEN

A) dist^n at end of many paths ≡

B) dist^n at every point in time for one very long path.

We can determine a plausible functional form for $p_\infty(r)$ from one-month US LIBOR rates.

pdf



best fit (lognormal distⁿ)

r

Paul has assumed a particular form of pdf.

This figure shows empirical data, the bars, and a fitted function, the line.

(M.L.E)

Our choice for $p_\infty(r)$ is

$$p_\infty(r) = \frac{1}{ar\sqrt{2\pi}} \exp\left(-\frac{1}{2a^2}(\log(r/\bar{r}))^2\right) \quad \text{assumed}$$

where $a = 0.4$ and $\bar{r} = 0.08$. From this we find that for the US market

(7a) becomes: $$\overbrace{u(r) = \nu^2 r^{2\beta-1}\left(\beta - \frac{1}{2} - \frac{1}{2a^2}\log(r/\bar{r})\right)}.$$

People assume mean reversion to look like $(\eta - \gamma r)$ or $-\gamma(r-\bar{r})$:

Mean reversion:
$r > \bar{r}$ log is +ve so −ve trend in $u(r)$
$r < \bar{r}$ log is −ve so +ve trend in $u(r)$

Advantages of working with the probability density function to find the drift function:

- more stable than other methods

- easy to see whether the probability density function ‘makes sense’

- spot rate cannot go to zero or infinity if probability density function zero there

The slope of the yield curve and the market price of risk

Now we have found $w(r)$ and $u(r)$, it only remains for us to find $\lambda(r)$.

We shall again allow λ to have a spot-rate dependence, but not a time dependence.

Slope is $\frac{1}{2}(u - \lambda w)$

$\lambda = \lambda(r)$

Note: There is no information about the market price of risk in the spot-rate process!

ZCB

Such information is contained within instruments of finite (not infinitesimal) maturity.

We will examine the short end of the curve for this information.

λ will come from bond

T.S.E :

$$f(x) = \sum_{0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

asymptotic analysis

Power $(T-t)$

Let us expand $Z(r,t;T)$ in a Taylor series about $t = T$, this is the short end of the yield curve.

We know that zero-coupon bonds satisfy (B.P.E)

$$\frac{\partial Z}{\partial t} + \frac{1}{2}w(r,t)^2\frac{\partial^2 Z}{\partial r^2} + (u(r,t) - \lambda(r,t)w(r,t))\frac{\partial Z}{\partial r} - rZ = 0.$$

power series

Look for a solution for small times to maturity of the form

$$Z \sim \underbrace{A(r)}_{=1} + a(r)(T-t) + b(r)(T-t)^2 + \cdots = \sum_{n=0}^{\infty} f_n(r)(T-t)^n$$

Power series

$$Z \sim 1 + a(r)(T-t) + b(r)(T-t)^2 + \cdots.$$

$A(r) = 1$ at $t = T$ redemption $Z = 1$

$$\boxed{Z = \sum_{n=0}^{\infty} a_n (T-t)^n}$$

Subst in BPE as written above and equate powers of $(T-t)$.

$A(r) = 1 \qquad a(r) = -r \qquad \delta(r) = \frac{1}{2}(r^2 - u + \lambda w)$

Put this form into the bond pricing equation and equate powers of $(T-t)$ and you will find that

$r^2 - (u - \lambda w)$

$\to e^{-r(T-t)}$

Curvature

$$Z(r, t; T) \sim 1 - r(T-t) + \frac{1}{2}(T-t)^2(r^2 - u + \lambda w) + \ldots \quad \text{as } t \to T.$$

V. close to maturity

$(T-t) \ll 1$ Redemption

little further along is worth $1 - r(T-t)$

This is just a simple Taylor series approximation to the solution for a zero-coupon bond, *for any one-factor model*!

power

Take log of Z & $\div$ thro by $-(T-t)$

(steps to follow)

$Y = -\frac{1}{T-t} \log Z$

T.S.E : $\log(1+x)$

From this we can find the shape of the yield curve near the short end:

Intercept. gradient.

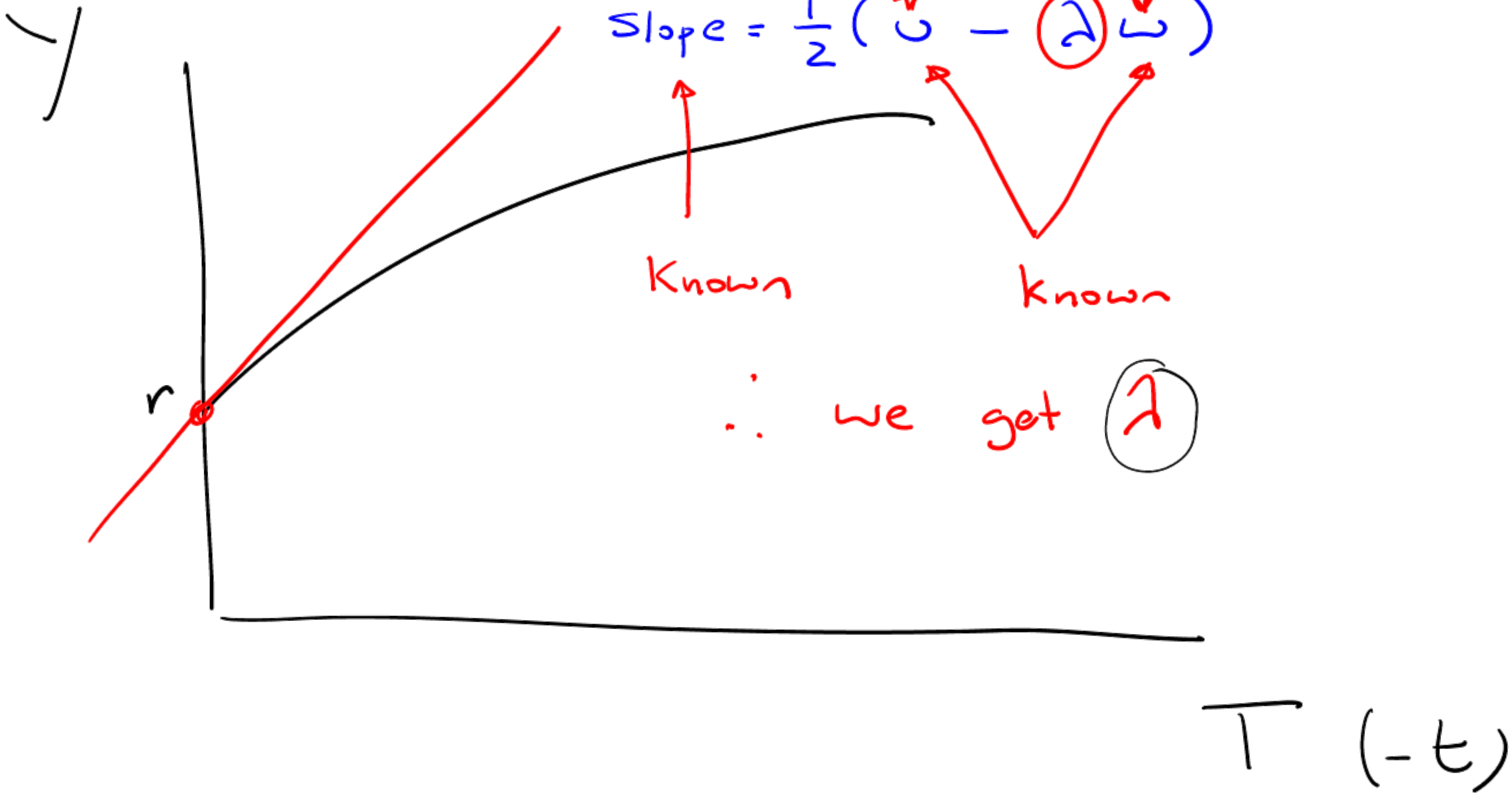
$$Y = -\frac{\ln Z}{T-t} \sim \underbrace{r}_{c} + \underbrace{\tfrac{1}{2}(u-\lambda w)}\underbrace{(T-t)}_{x} + \dots \quad \text{as } t \to T. \tag{8}$$

$y = c + mx$

The first term says that the short end of the yield curve is r (obvious!), and the second term says that the slope of the yield curve at the short end in this one-factor model is simply $(u - \lambda w)/2$.

We can use this result together with time-series data to determine the form for $u - \lambda w$ empirically. . . and since we have functional forms for $u(r)$ and $w(r)$ that means we can find $\lambda(r)$.

And so, the parameter λ as a function of r is . . .



$$\text{Slope} = \frac{1}{2}\left(u - \lambda w\right)$$

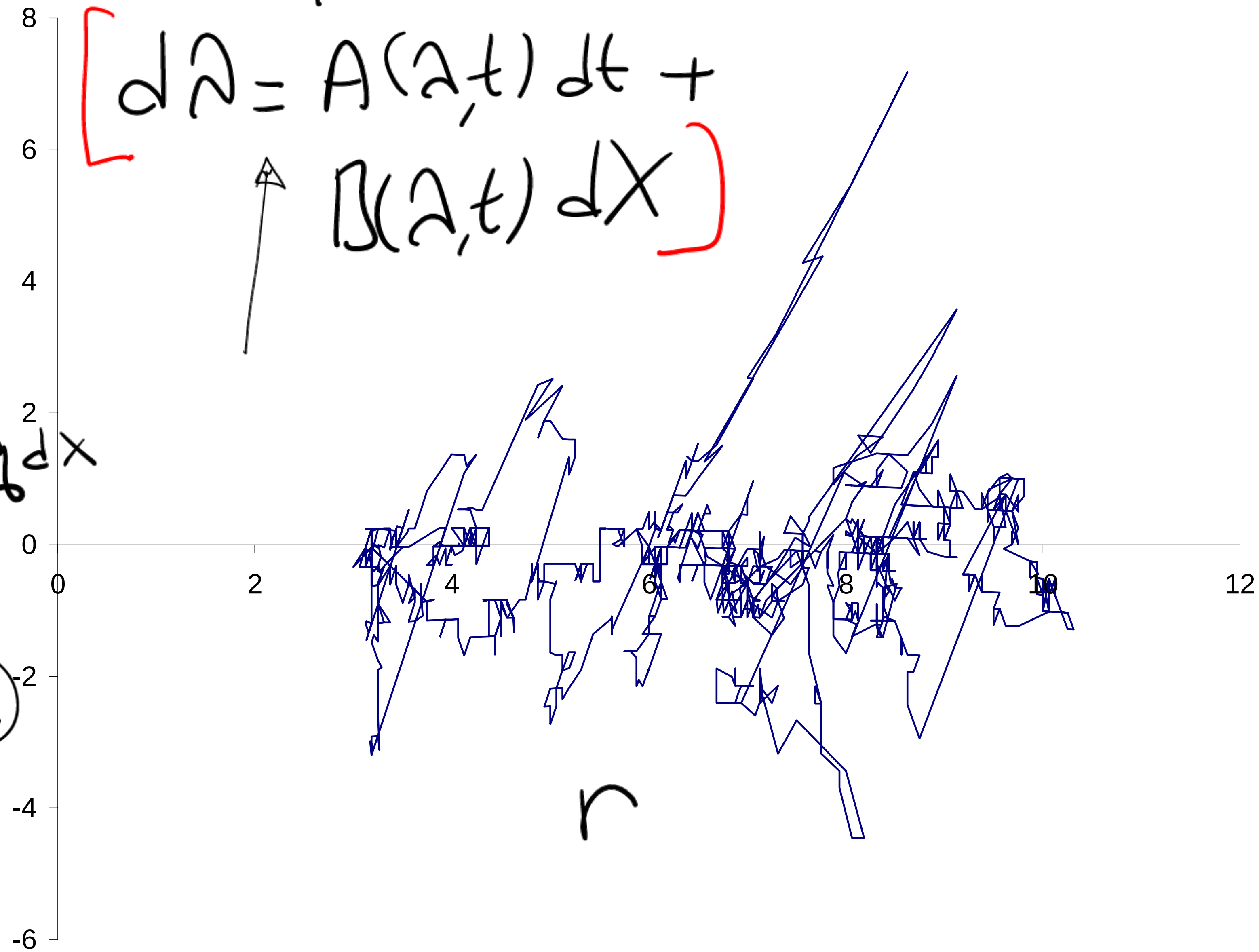
Known

Known

$\therefore$ we get λ

A complete mess!!

$$\left[d\lambda = A(\lambda,t)\,dt + B(\lambda,t)\,dX \right]$$



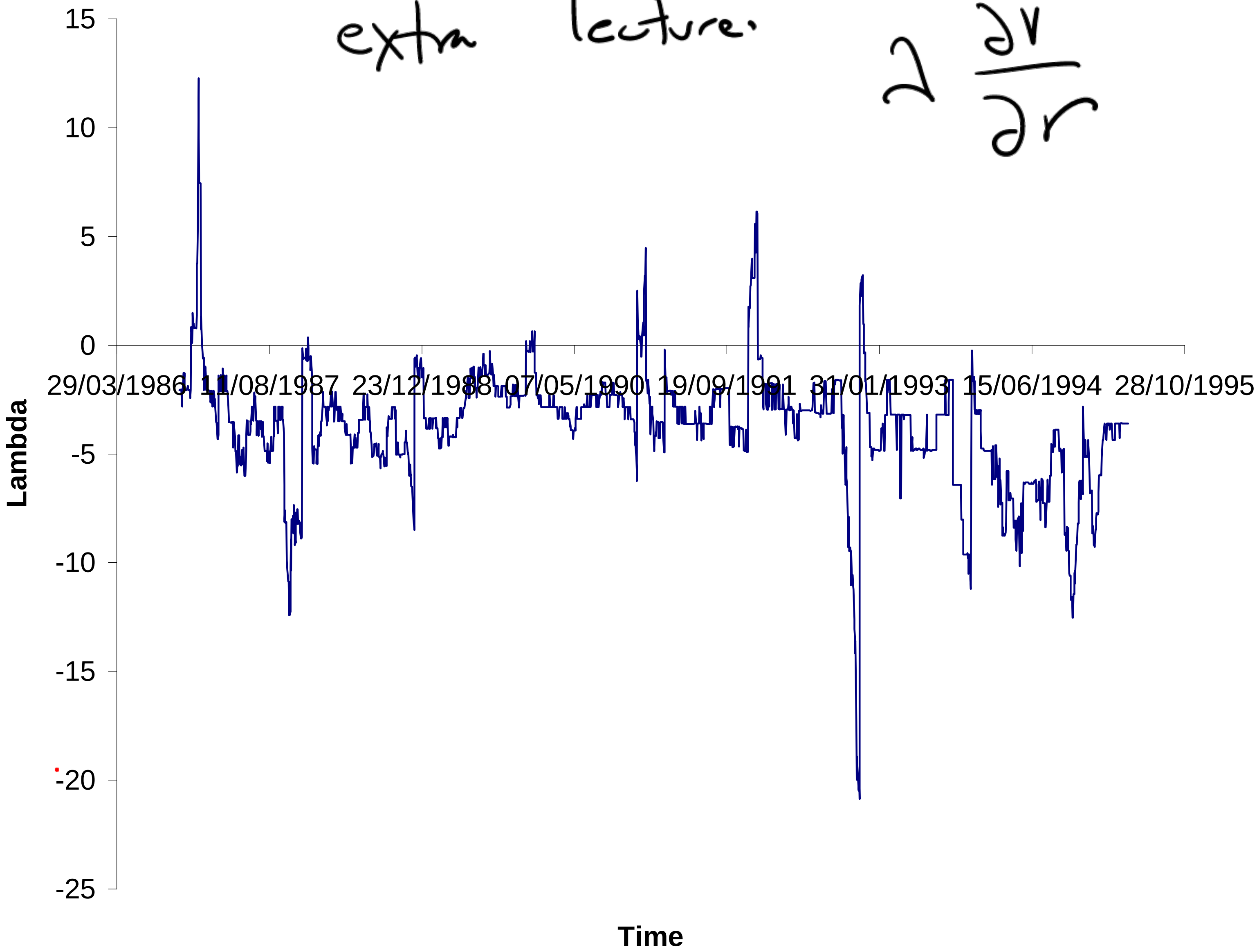
$$d\ell = p\,dt + q\,dX$$

stoch MPOR

A mess!

For more discussion see extra lecture.

$$\lambda \frac{\partial V}{\partial r}$$



Possible conclusion from this:

- it is 'easy' to model the spot interest rate!

- but difficult to model the $\underbrace{\text{market price of interest rate risk!}}_{\text{real challenge}}$

~~X~~ $\lambda \in \mathbb{R}$?

$\lambda = \lambda(t)$ ~~X~~

λ stochastic ?

$$d\lambda = a(\lambda, t)\, dt + \delta(\lambda, t)\, dX$$

2 factor model

$V(r, \lambda, t)$

Summary

Please take away the following important ideas

- Spot interest rate models are usually calibrated to match market data, in particular the forward curve (HJM)

I

- This calibration is in practice always inconsistent → unstable

- There are simple methods for examining interest rate data to find good models ∃ form $dr = u(r)\,dt + \nu r^{\beta}\,dX$

- Thanks for your interest – this was my last CQF lecture !