

Calculus A (II) One-to-One Tutoring

Introduction to Integrals

Chang, Yung-Hsuan

May 2, 2024

Definition 4.1 (Antiderivative).

For a given function $f(x)$, the function $F(x)$ is called an antiderivative of f if $\frac{dF}{dx}(x) = f(x)$.

Example 4.2.

Find an antiderivative of the following functions:

- | | | | |
|------------|------------------------------|-----------------|---------------------|
| 1. 0; | 5. $x^n, n \in \mathbb{N}$; | 9. e^x ; | 13. $\sin kx$; |
| 2. 1; | 6. $\frac{1}{x}$; | 10. $\sin 2x$; | 14. $\cos \ell x$; |
| 3. x ; | 7. $\sin x$; | 11. $\cos 2x$; | 15. e^{rx} ; and |
| 4. x^3 ; | 8. $\cos x$; | 12. e^{2x} ; | 16. a^x . |

Remark 4.3.

We use the long crowbar \int and the infinitesimal dx to clamp the integrand (usually, a function $f(x)$ in this case) $\int f(x) dx$ to indicate the “family” of antiderivatives of $f(x)$. Then we call the family of antiderivatives to be “(indefinite) integrals.” It is called indefinite since the difference between each member in the family is a constant but not deemed to be zero.

Remark 4.4.

If the crowbar in an integral is with subscript and superscript, then the integral is called a “definite integral.”

Theorem 4.5 (Fundamental Theorem of Calculus).

Let f be a continuous function on an interval $[a, b]$ and let F be an antiderivative of f . Then

1. $\frac{dF}{dx}(x) = f(x)$; and
2. $\int_a^b f(x) dx = F(b) - F(a)$.

Example 4.6.

Evaluate the following indefinite integrals:

- | | | |
|--------------------------------------------------|--------------------------------------|-----------------------------|
| 1. $\int 2x dx$; | 6. $\int x^n dx, n \in \mathbb{N}$; | 11. $\int \sin 2x dx$; |
| 2. $\int x dx$; | 7. $\int \frac{1}{x} dx$; | 12. $\int \cos 2x dx$; |
| 3. $\int 4x^3 dx$; | 8. $\int \sin x dx$; | 13. $\int e^{2x} dx$; |
| 4. $\int x^3 dx$; | 9. $\int \cos x dx$; | 14. $\int \sin kx dx$; and |
| 5. $\int (n+1) \cdot x^n dx, n \in \mathbb{N}$; | 10. $\int e^x dx$; | 15. $\int \cos \ell x dx$. |

Exercise 4.7.

Use Example 3.6, evaluate the following definite integrals:

- | | | |
|---------------------------------|-------------------------------|---------------------------------------|
| 1. $\int_0^{\sqrt{2}} x dx$; | 4. $\int_0^{\pi} \sin x dx$; | 7. $\int_0^{\pi} \sin 2x dx$; |
| 2. $\int_0^{\sqrt{2}} x^3 dx$; | 5. $\int_0^{\pi} \cos x dx$; | 8. $\int_0^{\pi} \cos 2x dx$; and |
| 3. $\int_1^2 \frac{1}{x} dx$; | 6. $\int_0^1 e^x dx$; | 9. $\int_0^{\frac{1}{2}} e^{2x} dx$. |

Theorem 4.8 (Properties of Definite Integrals).

Let f, g be integrable on $[a, b]$. Then the following are true:

1. $\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx$;
2. $\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx$ for $c \in [a, b]$;
3. $\int_a^b \alpha f(x) + g(x) \, dx = \alpha \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$;
4. if $f(x) \geq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx;$$

5. $|f|$ is integrable;
6. $\left| \int_a^b f(x) \, dx \right| \leq \int_a^b |f(x)| \, dx$; and
7. fg is integrable.

Theorem 4.9 (Substitution Rule).

Suppose that f is continuous on $[c, d] \supseteq [a, b]$ and $\frac{d\phi}{dt}(t)$ are continuous in $[\alpha, \beta]$ with $\phi(t) \in [c, d]$ for all $t \in [\alpha, \beta]$, where $\alpha = \phi(a)$ and $\beta = \phi(b)$. Then

$$\int_a^b f(x) \, dx = \int_{\alpha}^{\beta} f(\phi(t)) \frac{d\phi}{dt}(t) \, dt.$$

Remark 4.10.

The substitution rule comes from the chain rule.

Example 4.11.

Evaluate $\int \frac{2x+1}{x^2+x+1} \, dx$.

Example 4.12.

Evaluate $\int x e^{x^2} dx$.

Example 4.13.

Evaluate $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$.

Example 4.14.

Evaluate $\int \frac{e^x}{1 + e^{2x}} dx$.

Example 4.15.

Evaluate $\int \frac{\ln x}{x} dx$.

Theorem 4.16 (Integration by Parts).

If the functions f and g are differentiable and their derivatives $\frac{df}{dx}$ and $\frac{dg}{dx}$ are continuous on $[a, b]$, then

$$\int_a^b f(x) \frac{dg}{dx}(x) dx = [f(x)g(x)]_a^b - \int_a^b g(x) \frac{df}{dx}(x) dx.$$

Remark 4.17 (DI Method).

For convenience, the DI method is much quicker than think about f and g . Choose the most-easily-integrable function h_1 to be under I, and the other h_2 to be under D. Then use the following table to compute.

	D	I
+	$h_2(x)$	$h_1(x)$
−	$\frac{dh_2}{dx}(x)$	$\int h_1(x) dx$
+	$\frac{d^2h_2}{dx^2}(x)$	$\iint h_1(x) dx dx$
−	$\frac{d^3h_2}{dx^3}(x)$	$\iiint h_1(x) dx dx dx$
	\vdots	\vdots
$(-1)^k$	$\frac{d^k h_2}{dx^k}(x)$	$\overbrace{\int \cdots \int}^k h_1(x) \overbrace{dx \cdots dx}^k$

Then

$$\begin{aligned}
\int h_2(x) h_1(x) dx &= h_2(x) \cdot \int h_1(x) dx \\
&\quad - \frac{dh_2}{dx}(x) \cdot \iint h_1(x) dx \\
&\quad + \frac{d^2h_2}{dx^2}(x) \cdot \iiint h_1(x) dx dx \\
&\quad - \cdots \\
&\quad + (-1)^k \int \frac{d^k h_2}{dx^k}(x) \cdot \overbrace{\int \cdots \int}^k h_1(x) \overbrace{dx \cdots dx}^k dx.
\end{aligned}$$

Example 4.18.

Evaluate $\int \ln x dx$.

Example 4.19.

Evaluate $\int x^2 e^x dx$.

Example 4.20.

Evaluate $\int e^x \sin x dx$.

Theorem 4.21 (Iterated Integral).

If $f(x, y)$ is bounded on the rectangle $K = \{(x, y) \in \mathbb{R}^2 \mid x \in [a, b] \text{ and } y \in [c, d]\}$ and

$$F(y) := \int_a^b f(x, y) \, dx$$

is well-defined, then

$$\int_K f(x, y) \, dA = \int_c^d \left(\int_a^b f(x, y) \, dx \right) dy.$$

Example 4.22.

Evaluate the following integrals:

1. $\iint_{K_1} \sin x \cos y \, dx \, dy$, $K_1 = [0, \frac{\pi}{3}] \times [0, \frac{\pi}{6}]$;
2. $\iint_{K_2} e^x \sin y \, dx \, dy$, $K_2 = [0, 1] \times [0, \alpha]$;
3. $\iint_{K_3} x^{2n-1} y^m e^{x^n y^{m+1}} \, dx \, dy$; and
4. $\iint_{K_4} \frac{y^2}{x^2 y^2 + 1} \, dx \, dy$, $K_4 = [0, 1] \times [0, 1]$.

Example 4.23.

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \text{ and } x + y \leq 2\}$. Evaluate

$$\iint_D x + y \, dx \, dy.$$