Calculus A (II) One-to-One Tutoring

Introduction to Integrals

Chang, Yung-Hsuan

April 25, 2024

Definition 3.1 (Antiderivative).

For a given function $f(x)$, the function $F(x)$ is called an antiderivative of f if

$$\frac{\mathrm{d}F}{\mathrm{d}x}(x) = f(x).$$

Example 3.2.

Find an antiderivative of the following functions:

1. 0;
2. 1;
3. x;
4. x^3;
5. x^n, $n \in \mathbb{N}$;
6. $\dfrac{1}{x}$;
7. $\sin x$;
8. $\cos x$;
9. e^x;
10. $\sin 2x$;
11. $\cos 2x$;
12. e^{2x};
13. $\sin kx$;
14. $\cos \ell x$;
15. e^{rx}; and
16. a^x.

Remark 3.3.

We use the long crowbar $\int$ and the infinitesimal $\mathrm{d}x$ to clamp the integrand (usually, a function $f(x)$ in this case)

$$\int f(x)\,\mathrm{d}x$$

to indicate the "family" of antiderivatives of $f(x)$. Then we call the family of antiderivatives to be "(indefinite) integrals." It is called indefinite since the difference between each member in the family is a constant but not deemed to be zero.

Remark 3.4.

If the crowbar in an integral is with subscript and superscript, then the integral is called a "definite integral."

Theorem 3.5 (Fundamental Theorem of Calculus).

Let f be a continuous function on an interval $[a, b]$ and let F be an antiderivative of f. Then

1. $\dfrac{\mathrm{d}F}{\mathrm{d}x}(x) = f(x)$; and
2. $\displaystyle\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$.

Example 3.6.

Evaluate the following indefinite integrals:

1. $\displaystyle\int x \, \mathrm{d}x$;
2. $\displaystyle\int x^3 \, \mathrm{d}x$;
3. $\displaystyle\int x^n \, \mathrm{d}x$, $n \in \mathbb{N}$;
4. $\displaystyle\int \frac{1}{x} \, \mathrm{d}x$;
5. $\displaystyle\int \sin x \, \mathrm{d}x$;
6. $\displaystyle\int \cos x \, \mathrm{d}x$;
7. $\displaystyle\int e^x \, \mathrm{d}x$;
8. $\displaystyle\int \sin 2x \, \mathrm{d}x$;
9. $\displaystyle\int \cos 2x \, \mathrm{d}x$;
10. $\displaystyle\int e^{2x} \, \mathrm{d}x$;
11. $\displaystyle\int \sin kx \, \mathrm{d}x$; and
12. $\displaystyle\int \cos \ell x \, \mathrm{d}x$.

Exercise 3.7.

Use Example 3.6, evaluate the following definite integrals:

1. $\displaystyle\int_0^{\sqrt{2}} x \,\mathrm{d}x$;
2. $\displaystyle\int_0^{\sqrt{2}} x^3 \,\mathrm{d}x$;
3. $\displaystyle\int_1^{2} \frac{1}{x} \,\mathrm{d}x$;
4. $\displaystyle\int_0^{\pi} \sin x \,\mathrm{d}x$;
5. $\displaystyle\int_0^{\pi} \cos x \,\mathrm{d}x$;
6. $\displaystyle\int_0^{1} e^x \,\mathrm{d}x$;
7. $\displaystyle\int_0^{\pi} \sin 2x \,\mathrm{d}x$;
8. $\displaystyle\int_0^{\pi} \cos 2x \,\mathrm{d}x$; and
9. $\displaystyle\int_0^{\frac{1}{2}} e^{2x} \,\mathrm{d}x$.

Theorem 3.8 (Properties of Definite Integrals).

Let f, g be integrable on $[a,b]$. Then the following are true:

1. $\displaystyle\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$;
2. $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$ for $c \in [a,b]$;
3. $\displaystyle\int_a^b \alpha f(x) + g(x)\,\mathrm{d}x = \alpha\int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$;
4. if $f(x) \geq g(x)$ for all $x \in [a,b]$, then
$$\int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x;$$
5. $|f|$ is integrable;
6. $\displaystyle\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$; and
7. fg is integrable.

Theorem 3.9 (Substitution Rule).

Suppose that f is continuous on $[c,d] \supseteq [a,b]$ and $\dfrac{\mathrm{d}\phi}{\mathrm{d}t}(t)$ are conitnuous in $[\alpha,\beta]$ with $\phi(t) \in [c,d]$ for all $t \in [\alpha,\beta]$, where $\alpha = \phi(a)$ and $\beta = \phi(b)$. Then

$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(\phi(t))\frac{\mathrm{d}\phi}{\mathrm{d}t}(t)\,\mathrm{d}t.$$

Remark 3.10.

The substitution rule comes from the chain rule.

Example 3.11.

Evaluate $\displaystyle\int \frac{2x+1}{x^2+x+1}\,\mathrm{d}x$.

Example 3.12.

Evaluate $\displaystyle\int xe^{x^2}\,\mathrm{d}x$.

Example 3.13.

Evaluate $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x$.

Example 3.14.

Evaluate $\displaystyle\int \frac{e^x}{1+e^{2x}}\,\mathrm{d}x$.

Example 3.15.

Evaluate $\displaystyle\int \frac{\ln x}{x}\,\mathrm{d}x$.

Theorem 3.16 (Integration by Parts).

If the functions f and g are differentiable and their derivatives $\frac{\mathrm{d}f}{\mathrm{d}x}$ and $\frac{\mathrm{d}g}{\mathrm{d}x}$ are continuous on $[a, b]$, then

$$\int_a^b f(x)\frac{\mathrm{d}g}{\mathrm{d}x}(x)\,\mathrm{d}x = [f(x)g(x)]_a^b - \int_a^b g(x)\frac{\mathrm{d}f}{\mathrm{d}x}(x)\,\mathrm{d}x.$$

Remark 3.17 (DI Method).

For convenience, the DI method is much quicker than think about f and g. Choose the most-easily-integrable function h_1 to be under I, and the other h_2 to be under D. Then use the following table to compute.

$$\begin{array}{ccc}
 & \text{D} & \text{I} \\
+ & h_2(x) & h_1(x) \\
- & \frac{\mathrm{d}h_2}{\mathrm{d}x}(x) & \int h_1(x)\,\mathrm{d}x \\
+ & \frac{\mathrm{d}^2h_2}{\mathrm{d}x^2}(x) & \iint h_1(x)\,\mathrm{d}x\mathrm{d}x \\
- & \frac{\mathrm{d}^3h_2}{\mathrm{d}x^3}(x) & \iiint h_1(x)\,\mathrm{d}x\mathrm{d}x\mathrm{d}x \\
 & \vdots & \vdots \\
(-1)^k & \frac{\mathrm{d}^kh_2}{\mathrm{d}x^k}(x) & \overbrace{\int\cdots\int}^{k} h_1(x)\,\overbrace{\mathrm{d}x\cdots\mathrm{d}x}^{k}
\end{array}$$

Then

$$\begin{aligned}
\int h_2(x)h_1(x)\,\mathrm{d}x = {} & h_2(x)\cdot\int h_1(x)\,\mathrm{d}x \\
& - \frac{\mathrm{d}h_2}{\mathrm{d}x}(x)\cdot\iint h_1(x)\,\mathrm{d}x \\
& + \frac{\mathrm{d}^2h_2}{\mathrm{d}x^2}(x)\cdot\iiint h_1(x)\,\mathrm{d}x\mathrm{d}x \\
& - \cdots \\
& + (-1)^k\int\frac{\mathrm{d}^kh_2}{\mathrm{d}x^k}(x)\cdot\overbrace{\int\cdots\int}^{k} h_1(x)\,\overbrace{\mathrm{d}x\cdots\mathrm{d}x}^{k}\,\mathrm{d}x.
\end{aligned}$$

Example 3.18.

Evaluate $\displaystyle\int \ln x \, \mathrm{d}x$.

Example 3.19.

Evaluate $\displaystyle\int x^2 e^x \, \mathrm{d}x$.

Example 3.20.

Evaluate $\displaystyle\int e^x \sin x \, \mathrm{d}x$.

Theorem 3.21 (Iterated Integral).

If $f(x, y)$ is bounded on the rectangle $K = \{(x, y) \in \mathbb{R}^2 \mid x \in [a, b] \text{ and } y \in [c, d]\}$ and

$$F(y) \coloneqq \int_a^b f(x, y)\, \mathrm{d}x$$

is well-defined, then

$$\int_K f(x, y)\, \mathrm{d}A = \int_c^d \left(\int_a^b f(x, y)\, \mathrm{d}x \right) \mathrm{d}y.$$

Example 3.22.

Evaluate the following integrals:

1. $\displaystyle\iint_{K_1} \sin x \cos y\, \mathrm{d}x\, \mathrm{d}y,\ K_1 = [0, \tfrac{\pi}{3}] \times [0, \tfrac{\pi}{6}]$;
2. $\displaystyle\iint_{K_2} e^x \sin y\, \mathrm{d}x\, \mathrm{d}y,\ K_2 = [0, 1] \times [0, \alpha]$;
3. $\displaystyle\iint_{K_3} x^{2n-1} y^m e^{x^n y^{m+1}}\, \mathrm{d}x\, \mathrm{d}y$; and
4. $\displaystyle\iint_{K_4} \frac{y^2}{x^2 y^2 + 1}\, \mathrm{d}x\, \mathrm{d}y,\ K_4 = [0, 1] \times [0, 1]$.

Example 3.23.

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, \text{ and } x + y \leq 2\}$. Evaluate

$$\iint_D x + y \,\mathrm{d}x\,\mathrm{d}y.$$