Calculus A (II) One-to-One Tutoring

Practices on Integrals and Fibini's Theorem

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Example 5.1.

Evaluate the following indefinite integral: $\displaystyle\int \frac{1}{1-x^2}\,\mathrm{d}x$.

Example 5.2.

Evalutate the following indefinite integral: $\displaystyle\int \frac{1}{a^2+x^2}\,\mathrm{d}x$.

Example 5.3.

Evaluate the following indefinite integral: $\displaystyle\int \frac{x^5}{x^3+1}\,\mathrm{d}x$.

Example 5.4.

Evaluate the following indefinite integral: $\displaystyle\int \frac{x^2+1}{x}\,\mathrm{d}x$.

Example 5.5.

Evaluate the following indefinite integral: $\displaystyle\int \cos x \cdot e^{\sin x}\,\mathrm{d}x$.

Example 5.6.

Evaluate the following indefinite integral: $\displaystyle\int \frac{\sin(\ln t)}{t}\,\mathrm{d}t$.

Example 5.7.

Evaluate the following indefinite integral: $\displaystyle\int x(\ln x)^2\,\mathrm{d}x$.

Example 5.8.

Evaluate the following indefinite integral: $\displaystyle\int x\sin x\,\mathrm{d}x$.

Example 5.9.

Evaluate the following indefinite integral: $\displaystyle\int x\ln x\,\mathrm{d}x$.

Example 5.10.

Evaluate the following indefinite integral: $\displaystyle\int \sin x\cos x\ln(\cos x)\,\mathrm{d}x$.

Example 5.11.

Evaluate the following indefinite integral: $\displaystyle\int e^x\cos x\,\mathrm{d}x$.

Example 5.12.

Evaluate $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2} \sin x\cos y\,\mathrm{d}x\,\mathrm{d}y$.

Example 5.13.

Evaluate $\displaystyle\int_1^2\int_0^{\pi} y\sin(xy)\,\mathrm{d}y\,\mathrm{d}x$.

Example 5.14.

Evaluate $\iint_D 2y\,\mathrm{d}A$, where D is the region bounded by the line $y = x$ and the parabola $y = 3x - x^2$.

Example 5.15.

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Example 5.16.

Evaluate $\iint_D xy\,\mathrm{d}A$, where D is the region bounded by the line $y = x-1$ and the parabola $y^2 = 2x+6$.

Example 5.17.

Evaluate $\displaystyle\int_0^4 \int_0^{\sqrt{x}} \frac{y}{x^2+1}\,\mathrm{d}y\,\mathrm{d}x$.

Example 5.18.

Evaluate $\displaystyle\int_0^3 \int_0^{y} e^{-y^2}\,\mathrm{d}x\,\mathrm{d}y$.

Example 5.19.

Evaluate $\displaystyle\int_0^1 \int_1^{x} \sin(y^2)\,\mathrm{d}y\,\mathrm{d}x$.

Example 5.20.

Evaluate $\displaystyle\int_0^1 \int_{3y}^{3} e^{x^2}\,\mathrm{d}x\,\mathrm{d}y$.

Example 5.21.

Evaluate $\displaystyle\int_0^1 \int_y^1 \frac{\sin x}{x}\, \mathrm{d}x\, \mathrm{d}y$.

Question.

How can we use the substitution rule for multiple integrals?

Definition 5.22.

The Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} := \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Theorem 5.23 (Change of Variables in a Double Integral).

Suppose that T is a C^1 transformation whose Jacobian is nonzero and that T maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that R and S are type I or type II plane regions. Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x, y)\, \mathrm{d}A = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \mathrm{d}u\, \mathrm{d}v.$$

Remark 5.24.

Notice the correspondence between Theorem 5.23 and the substitution rule in one-dimensional integrals:

$$\int_a^b f(x)\, \mathrm{d}x = \int_\alpha^\beta f(\phi(t)) \frac{\mathrm{d}\phi}{\mathrm{d}t}(t)\, \mathrm{d}t.$$

Example 5.25.

Use the change of variables $\begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$ to evaluate the integral $\iint_R y \, \mathrm{d}A$, where R is the region bounded by the x-axis and the parabolas $y^2 = 4 - 4x$, and $y^2 = 4 + 4x$ with $y \geq 0$.

Example 5.26.

Use the transformation $\begin{cases} u = x - y \\ v = x + y \end{cases}$ to evaluate $\iint_R \frac{x - y}{x + y} \, \mathrm{d}A$, where R is the square with vertices $(0, 2)$, $(1, 1)$, $(2, 2)$, and $(1, 3)$.

Example 5.27.

Evaluate the integral $\iint_R e^{\frac{x+y}{x-y}} \, \mathrm{d}A$, where R is the trapezoidal region with vertices $(1, 0)$, $(2, 0)$, $(0, -2)$, and $(0, -1)$.