Calculus A II One-to-One Tutoring

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Example 3.1.

Compute all the first partial derivatives of $f(x, y) = x^3 + x^2y^2 - 2y^2$.

Recall 3.2 (Chain Rule).

If g is differentiable at x and f is differentiable at $g(x)$, then $(f \circ g)(x)$ is differentiable at x and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((f \circ g)(x)\right) = \frac{\mathrm{d}f}{\mathrm{d}x}(g(x)) \cdot \frac{\mathrm{d}g}{\mathrm{d}x}(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

Theorem 3.3 (Chain Rule).

Let $z = f(x, y), x = x(t), y = y(t)$. Then

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t}.$$

Example 3.4.

Let $z = x^2y + 3xy^4$ with $(x, y) = (\sin(2t), \cos(t))$. Compute $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

Example 3.5.

Let $u = x^4y + y^2z^3$ with $(x, y, z) = (rse^t, rs^2e^{-t}, r^2s\sin(t))$. Compute $\dfrac{\partial u}{\partial s}$.

Example 3.6.

Compute all first partial derivatives for $f(x, y) = e^{\sqrt{x^2+y}}$.

Example 3.7.

Let $z = x^2 + y^2$ with $(x, y) = (t - \cos t, 1 - \sin t)$. Compute $\dfrac{\mathrm{d}z}{\mathrm{d}t}$.

Example 3.8.

Compute all first partial derivatives for $f(x, y) = \arcsin\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right)$ with $xy > 0$.

Example 3.9.

Let $z = e^{xy}$ with $(x, y) = \left(\ln \sqrt{u^2 + v^2}, \arctan \frac{v}{u}\right)$. Compute $\frac{\partial z}{\partial u}$.

Definition 3.10 (Directional Derivative).

Let f be a real-valued function of two variables. The directional derivative of f at (x_0, y_0) for the direction $\hat{\mathbf{u}} = (u_1, u_2)$ (a unit vector) is defined when the following limit exists:

$$D_{\hat{\mathbf{u}}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}.$$

Remark 3.11.

In the previous definition, if $\hat{\mathbf{u}} = (1, 0)$, then

$$D_{\hat{\mathbf{u}}} f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0).$$

Example 3.12.

Determine the directional derivative of

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0); \\ 0, & \text{if } (x, y) = (0, 0), \end{cases}$$

for the direction $(1, 1)$ at $(0, 0)$.

Example 3.13.

Determine the directional derivative of $f(x,y) = xy^2 \arctan(z)$ for the direction $(1,1,1)$ at $(2,1,1)$.

Definition 3.14 (Limit).

The limit

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

if for all $\varepsilon > 0$ (the greek letter ε stands for "error"), there exists a $\delta > 0$ (the greek letter δ stands for "difference") such that $\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ implies $|f(x,y) - f(x_0,y_0)| < \varepsilon$.

Remark 3.15.

If there exist two path $C_1 : y = f_1(x)$ and $C_2 : y = f_2(x)$ such that

$$\lim_{\substack{(x,y)\to(x_0,y_0)\\(x,y)\in C_1}} f(x,y) \neq \lim_{\substack{(x,y)\to(x_0,y_0)\\(x,y)\in C_2}} f(x,y),$$

then the limit $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist.

Example 3.16.

Determine the existance of $\lim_{(x,y)\to(0,0)} f(x,y)$ for each of the following functions:

1. $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$;
2. $f(x,y) = \dfrac{xy}{x^2 + y^2}$;
3. $f(x,y) = \dfrac{x^2 y}{x^2 + y^2}$; and
4. $f(x,y) = \dfrac{x^{\frac{2}{3}} \sin y}{x^2 + y^2}$.

Example 3.17.

Determine $\lim_{(x,y)\to(0,0)} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}$.

Definition 3.18 (Continuity).

We say a function f is continuous at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0).$$

We say a function is continuous if it is continuous at every point in its domain.

Example 3.19.

Determine the conitnuity of the function $f(x, y) = \dfrac{x^2 y}{x^2 + y^2}$ at $(0, 0)$.