Homework 1 of Computational Mathematics

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Problem 1. Use the intermediate value theorem and Rolle's theorem to show the graph of

$$f(x) = x^3 + 2x + k$$

crosses the x-axis exactly once, regardless of the value of the constant k.

Proof. Let $k \in \mathbb{R}$. We first show that $f(x) = 0$ has a root. Suppose $k = 0$. Then, we can factorize $f(x) = x(x^2 + 2)$. Thus, it is clear that $f(x) = 0$ has only one root $x = 0$. Suppose $k > 0$. Choose $x_1 = 0$ and $x_2 = -k$. Then, $f(x_1) = k > 0$ and $f(x_2) = -k^3 - k < 0$. By the intermediate value theorem, there exists a root between x_1 and x_2. Suppose $k < 0$. Choose $x_3 = 0$ and $x_4 = -k$. Then, $f(x_3) = k < 0$ and $f(x_4) = -k^3 - k > 0$. By the intermediate value theorem, there exists a root between x_3 and x_4. We now show that there is only one root. Observe that $\dfrac{df}{dx}(x) = 3x^2 + 2 > 0$ for all $x \in \mathbb{R}$ regardless of the value of k. For the sake of contradiction, we suppose there are two distinct roots α and β such that $f(\alpha) = f(\beta) = 0$. By Rolle's theorem, there exists an x_0 between α and β such that $\dfrac{df}{dx}(x_0) = 0$, a contradiction. Hence, there is only one root. $\square$

Problem 2. Find $\max_{a \le x \le b} |f(x)|$ for the following functions and intervals.

a. $f(x) = \dfrac{2 - e^x + 2x}{3}$, $[0, 1]$

b. $f(x) = \dfrac{4x - 3}{x^2 - 2x}$, $[0, 5, 1]$

c. $f(x) = 2x\cos(2x) - (x - 2)^2$, $[2, 4]$

d. $f(x) = 1 + e^{-\cos(x-1)}$, $[1, 2]$

Solution.

a.

Problem 3. Find the second Taylor polynomial $P_2(x)$ for the function $f(x) = e^x \cos(x)$ about $x_0 = 0$.

  a. Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5) - P_2(0.5)|$ using the error formula, and compare it to the actual error.

  b. Find a bound for the error $|f(0.5) - P_2(0.5)|$ in using $P_2(x)$ to approximate $f(x)$ on the interval $[0, 1]$.

  c. Approximate $\int_0^1 f(x)\, \mathrm{d}x$ using $\int_0^1 P_2(x)\, \mathrm{d}x$.

  d. Find an upper bound for the error in (c) using $\int_0^1 |P_2(x)|\, \mathrm{d}x$.

Solution. By Taylor's theorem, the second Taylor polynomial about 0 is

$$P_2(x) = f(0) + \frac{1}{1!} \cdot \frac{\mathrm{d}f}{\mathrm{d}x}(0) \cdot x + \frac{1}{2!} \cdot \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(0) \cdot x^2$$

$$= 1 + x.$$

Problem 4. Let $f(x) = \dfrac{1}{1-x}$ and $x_0 = 0$. Find the n-th Taylor polynomial $P_n(x)$ for $f(x)$ about x_0. Find a value of n necessary for $P_n(x)$ to approximate $f(x)$ to within 10^{-6} on $[0, 0.5]$.

Solution.

Problem 5. Find the largest interval in which p^* must lie to approximate p with relative error at most 10^{-4} for each value of p.

  a. π

  b. e

  c. $\sqrt{2}$

  d. $\sqrt[3]{7}$

Solution.

Problem 6. Let
$$f(x) = \frac{e^x - e^{-x}}{x}.$$

a. Find $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{x}$.

b. Use three-digit rounding arithmatic to evaluate $f(x)$.

c. Replace each exponential function with its Maclaurin polynomial, and repeat part (b).

d. The actual value is $f(0.1) = 2.003335000$. Find the relative error for the values obtained in part (b) and (c).

Solution.

Problem 7. Use the 64-bit long real format to find the decimal equivalent of the following floating-point machine numbers.

a. 0   10000001010   1001001100000000000000000000000000000000000000000000

b. 1   10000001010   1001001100000000000000000000000000000000000000000000

c. 0   01111111111   0101001100000000000000000000000000000000000000000000

d. 0   01111111111   0101001100000000000000000000000000000000000000000001

Solution.

a. Let r_a be the desired floating-point number. Then,

$$r_a = (-1)^0 \, 2^{c-1023} \, (1 + m)$$

$$\approx 7.563656083721675640807711666125830654 \times 10^{-124},$$

where $c = 1 + 128 + 512 = 614$ and $m = \dfrac{1}{2^{52}} + \dfrac{1}{2^{49}} + \dfrac{1}{2^{46}} + \dfrac{1}{2^{45}} = \dfrac{147}{2^{45}}$.

Problem 8. The two-by-two system

$$\begin{cases} ax + by = e; \\ cx + dy = f, \end{cases}$$

where a, b, c, d, e, f are given, can be solved for x and y as follows:

1. Set $m = \dfrac{c}{a}$, provided $a \neq 0$;

2. $d_1 = d - mb$;

3. $f_1 = f - me$;

4. $y = \dfrac{f_1}{d_1}$; and then

5. $x = \dfrac{e - by}{a}$.

Solve the following linear systems using four-digit rounding arithmetic.

a. $\begin{cases} 1.130x - 6.990y = 14.20 \\ 1.013x + 6.099y = 14.22 \end{cases}$

b. $\begin{cases} 8.110x - 12.20y = -0.1370 \\ -18.11x + 112.2y = -0.1376 \end{cases}$

Solution.

Problem 9. Suppose one calculates in two-digit rounding arithmetic. A rectangular parallelepiped has sides of length 3 cm, 4 cm, and 5 cm, measured to the nearest centimeter. What are the best upper and lower bounds for the volume of this parallelepiped? What are the best upper and lower bounds for the surface area?

Solution.

Problem 10. Find the rate of convergence of the following.

a. $\left\{ \sin\left(\dfrac{1}{n^2}\right) \right\}_{n=1}^{\infty}$

b. $\left\{ \left(\sin\left(\dfrac{1}{n}\right) \right)^2 \right\}_{n=1}^{\infty}$

c. $\dfrac{\sin h}{h}$ as $h \to 0$

d. $\dfrac{1 - e^h}{h}$ as $h \to 0$

Solution.

Problem 11. Suppose that as x approaches zero,

$$F_1(x) = L_1 + O(x^\alpha) \qquad \text{and} \qquad F_2(x) = L_2 + O(x^\beta).$$

Let c_1 and c_2 be nonzero constants, and define

$$F(x) = c_1 \, F_1(x) + c_2 \, F_2(x) \qquad \text{and} \qquad G(x) = F_1(c_1 x) + F_2(c_2 x).$$

Show that if $\gamma = \min\{\alpha, \beta\}$, then as x approaches zero,

a. $F(x) = c_1 L_1 + c_2 L_2 + O(x^\gamma)$

b. $G(x) = L_1 + L_2 + O(x^\gamma)$.

Proof.