

# Homework 1 of Computational Mathematics

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**Problem 1.** Use the intermediate value theorem and Rolle's theorem to show the graph of

$$f(x) = x^3 + 2x + k$$

crosses the  $x$ -axis exactly once, regardless of the value of the constant  $k$ .

**Proof.** Let  $k \in \mathbb{R}$ . We first show that  $f(x) = 0$  has a root. Suppose  $k = 0$ . Then, we can factorize  $f(x) = x(x^2 + 2)$ . Thus, it is clear that  $f(x) = 0$  has only one root  $x = 0$ . Suppose  $k > 0$ . Choose  $x_1 = 0$  and  $x_2 = -k$ . Then,  $f(x_1) = k > 0$  and  $f(x_2) = -k^3 - k < 0$ . By the intermediate value theorem, there exists a root between  $x_1$  and  $x_2$ . Suppose  $k < 0$ . Choose  $x_3 = 0$  and  $x_4 = -k$ . Then,  $f(x_3) = k < 0$  and  $f(x_4) = -k^3 - k > 0$ . By the intermediate value theorem, there exists a root between  $x_3$  and  $x_4$ . We now show that there is only one root. Observe that  $\frac{df}{dx}(x) = 3x^2 + 2 > 0$  for all  $x \in \mathbb{R}$  regardless of the value of  $k$ . For the sake of contradiction, we suppose there are two distinct roots  $\alpha$  and  $\beta$  such that  $f(\alpha) = f(\beta) = 0$ . By Rolle's theorem, there exists an  $x_0$  between  $\alpha$  and  $\beta$  such that  $\frac{df}{dx}(x_0) = 0$ , a contradiction. Hence, there is only one root.  $\square$

**Problem 2.** Find  $\max_{a \leq x \leq b} |f(x)|$  for the following functions and intervals.

a.  $f(x) = \frac{2 - e^x + 2x}{3}, [0, 1]$

b.  $f(x) = \frac{4x - 3}{x^2 - 2x}, [0.5, 1]$

c.  $f(x) = 2x \cos(2x) - (x - 2)^2, [2, 4]$

d.  $f(x) = 1 + e^{-\cos(x-1)}, [1, 2]$

**Solution.** Note that we can first look for extrema for original functions and then find  $\max_{a \leq x \leq b} |f(x)|$ .

a. We first look for critical points. Set  $\frac{df}{dx}(x) = 0$  on  $[0, 1]$ . Then,

$$\frac{-e^x + 2}{3} = 0$$

$$\iff x = \ln 2.$$

Then, we have  $f(\ln 2) = \frac{2 \ln 2}{3} \approx 0.462$ ,  $f(0) = \frac{1}{3} \approx 0.333$ , and  $f(1) = \frac{4 - e}{3} \approx 0.427$ . Hence,

$$\max_{0 \leq x \leq 1} |f(x)| = \frac{2 \ln 2}{3}.$$

b. Observe that the derivative  $\frac{df}{dx}(x) = \frac{-4x^2 + 6x - 6}{(x^2 - 2x)^2}$  has no roots on  $\mathbb{R}$ . Hence, the function is monotonic and thus achieves extrema at end points. We have  $|f(0.5)| = \frac{4}{3}$  and  $|f(1)| = 1$ . Hence,

$$\max_{0.5 \leq x \leq 1} |f(x)| = \frac{4}{3}.$$

c. We first look for critical points. Set  $\frac{df}{dx}(x) = 0$  on  $[2, 4]$ . Then,

$$2 \cos(2x) - 4x \sin(2x) - 2(x - 2) = 0 \tag{2.1}$$

Let  $x^*$  satisfies (2.1). Then,  $f(x^*) \approx 4.981$ . We have  $f(x^*) \approx 4.981$ ,  $f(2) \approx -2.615$ , and  $f(4) \approx -5.164$ . Hence,  $\max_{2 \leq x \leq 4} |f(x)| = 4 - 8 \cos(8)$ .

d. We first look for critical points. Set  $\frac{df}{dx}(x) = 0$  on  $[1, 2]$ . Then,

$$e^{-\cos(x-1)} \cdot \sin(x-1) = 0$$

$$\iff \sin(x-1) = 0$$

$$\implies x = 1.$$

Then, we have  $f(1) = 1 + e^{-1} \approx 1.368$ ,  $f(2) = 1 + e^{-\cos 1} \approx 1.583$ . Hence,  $\max_{1 \leq x \leq 2} |f(x)| = 1 + e^{-\cos 1}$ . □

**Problem 3.** Find the second Taylor polynomial  $P_2(x)$  for the function  $f(x) = e^x \cos(x)$  about  $x_0 = 0$ .

- Use  $P_2(0.5)$  to approximate  $f(0.5)$ . Find an upper bound for error  $|f(0.5) - P_2(0.5)|$  using the error formula, and compare it to the actual error.
- Find a bound for the error  $|f(0.5) - P_2(0.5)|$  in using  $P_2(x)$  to approximate  $f(x)$  on the interval  $[0, 1]$ .
- Approximate  $\int_0^1 f(x) dx$  using  $\int_0^1 P_2(x) dx$ .
- Find an upper bound for the error in (c) using  $\int_0^1 |P_2(x)| dx$ .

**Solution.** By Taylor's theorem, we have

$$\begin{aligned} e^x \cos x &= f(0) + \frac{1}{1!} \cdot \frac{df}{dx}(0) \cdot x + \frac{1}{2!} \cdot \frac{d^2f}{dx^2}(0) \cdot x^2 + \frac{1}{3!} \cdot \frac{d^3f}{dx^3}(\xi(x)) \cdot x^3 \\ &= 1 + x + \frac{1}{6} \cdot (-2e^x (\sin \xi(x) + \cos \xi(x))) \cdot x^3 \\ &= 1 + x - \frac{\sqrt{2}}{3} \cdot e^{\xi(x)} \sin \left( \xi(x) + \frac{\pi}{4} \right) \cdot x^3 \end{aligned}$$

Hence, the second Taylor polynomial about 0 is

$$P_2(x) = 1 + x.$$

- The value of the second Taylor polynomial  $P_2(0.5) = 1.5$ , and the true value  $f(0.5) \approx 1.446889$ .

We can estimate the upper bound of  $|f(0.5) - P_2(0.5)|$  by

$$\begin{aligned} |f(0.5) - P_2(0.5)| &= \left| \frac{\sqrt{2}}{3} \cdot e^{\xi(0.5)} \sin \left( \xi(0.5) + \frac{\pi}{4} \right) \cdot (0.5)^3 \right| \\ &\leq \left| \frac{\sqrt{2}}{3} \cdot e^{0.5} \sin \left( 0.5 + \frac{\pi}{4} \right) \cdot (0.5)^3 \right| \\ &\leq 0.093222005. \end{aligned}$$

The actual error is  $|e^{0.5} \cos(0.5) - 1.5| \approx 0.0531111$  and is smaller than the upper bound.

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**Problem 4.** Let  $f(x) = \frac{1}{1-x}$  and  $x_0 = 0$ . Find the  $n$ -th Taylor polynomial  $P_n(x)$  for  $f(x)$  about  $x_0$ . Find a value of  $n$  necessary for  $P_n(x)$  to approximate  $f(x)$  to within  $10^{-6}$  on  $[0, 0.5]$ .

**Solution.** Notice that  $f(x) = \frac{1}{1-x}$  is the infinite sum of  $\{x^k\}_{k=0}^{\infty}$ . Hence, the  $n$ -th Taylor polynomial is  $P_n(x) = 1 + x + \cdots + x^n$ . We would like to find an  $n \in \mathbb{N}$  such that  $|P_n(x) - f(x)| < 10^{-6}$  for all  $x \in [0, 0.5]$ . Observe that  $f(x) > P_n(x)$  for all  $x \in [0, 0.5]$ , and  $\frac{d}{dx}(f(x) - P_n(x)) \geq 0$  for all  $x \in [0, 0.5]$ .

Hence,

$$\begin{aligned}\sup_{x \in [0, 0.5]} |P_n(x) - f(x)| &= f(0.5) - P_n(0.5) \\ &= 2 - \sum_{i=0}^n (0.5)^i \\ &= 2 - \frac{1 - 0.5^{n+1}}{1 - 0.5} \\ &= 0.5^n.\end{aligned}$$

Set  $0.5^n < 10^{-6}$ . Then, we obtain that  $n$  must be greater than 19.5. Hence, we choose  $n = 20$ . □

**Problem 5.** Find the largest interval in which  $p^*$  must lie to approximate  $p$  with relative error at most  $10^{-4}$  for each value of  $p$ .

- a.  $\pi$
- b.  $e$
- c.  $\sqrt{2}$
- d.  $\sqrt[3]{7}$

**Solution.** The relative of  $p^*$  for  $p$  is  $\left| \frac{p^* - p}{p} \right|$ . Set

$$\left| \frac{p^* - p}{p} \right| \leq 10^{-4}.$$

Then,

$$|p^* - p| \leq p \cdot 10^{-4}$$

$$p - p \cdot 10^{-4} \leq p^* \leq p + p \cdot 10^{-4}.$$

Hence, the answer for each is

- a.  $[\pi - \pi \cdot 10^{-4}, \pi + \pi \cdot 10^{-4}]$ ;
- b.  $[e - e \cdot 10^{-4}, e + e \cdot 10^{-4}]$ ;
- c.  $[\sqrt{2} - \sqrt{2} \cdot 10^{-4}, \sqrt{2} + \sqrt{2} \cdot 10^{-4}]$ ; and
- d.  $[\sqrt[3]{7} - \sqrt[3]{7} \cdot 10^{-4}, \sqrt[3]{7} + \sqrt[3]{7} \cdot 10^{-4}]$ .

□

**Problem 6.** Let

$$f(x) = \frac{e^x - e^{-x}}{x}.$$

- Find  $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x}$ .
- Use three-digit rounding arithmetic to evaluate  $f(0.1)$ .
- Replace each exponential function with its third Maclaurin polynomial, and repeat part (b).
- The actual value is  $f(0.1) = 2.003335000$ . Find the relative error for the values obtained in part (b) and (c).

**Solution.**

- Using l'Hôpital's rule,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{x} &\stackrel{L'}{=} \lim_{x \rightarrow 0} \frac{e^x + e^x}{1} \\ &= 2. \end{aligned}$$

- Plug in the value  $x = 0.1$ , by calculator,  $f(0.1) = 2.00333500039688\dots$ . Using three-digit rounding arithmetic,  $f(0.1) \stackrel{\mathbf{R}_3}{\approx} 2.00$ , where  $\stackrel{\mathbf{R}_3}{\approx}$  denotes three-digit rounding arithmetic.

- Plug in the value  $x = 0.1$  with substitution  $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ , we have

$$\begin{aligned} f(0.1) &\stackrel{\mathbf{M}_3}{\approx} \frac{1 + 0.1 + \frac{0.01}{2} + \frac{0.001}{6} - 1 + 0.1 - \frac{0.01}{2} + \frac{0.001}{6}}{0.1} \\ &= 2.00\bar{3} \\ &\stackrel{\mathbf{R}_3}{\approx} 2.00, \end{aligned}$$

where  $\stackrel{\mathbf{M}_3}{\approx}$  denotes approximation with its third Maclaurin polynomial.

- The relative error in part (b) is

$$\left| \frac{2.003335000 - 2.00}{2.003335000} \right| \approx 0.0164740595;$$

the relative error in part (c) is

$$\left| \frac{2.003335000 - 2.00}{2.003335000} \right| \approx 0.0164740595.$$

□

**Problem 7.** Use the 64-bit long real format to find the decimal equivalent of the following floating-point machine numbers.

- [illegible]

**Solution.**

- a. Let  $r_a$  be the desired floating-point number. Then,

$$r_a = (-1)^0 2^{c-1023} (1+m)$$

$$\approx 7.5636560837216756408077116661258306544 \times 10^{-124},$$

where  $c = 1 + 128 + 512 = 644$  and  $m = \frac{1}{2^{52}} + \frac{1}{2^{49}} + \frac{1}{2^{46}} + \frac{1}{2^{45}} = \frac{147}{2^{45}}$ .



**Problem 8.** The two-by-two system

$$\begin{cases} ax + by = e; \\ cx + dy = f, \end{cases}$$

where  $a, b, c, d, e, f$  are given, can be solved for  $x$  and  $y$  as follows:

1. Set  $m = \frac{c}{a}$ , provided  $a \neq 0$ ;
2.  $d_1 = d - mb$ ;
3.  $f_1 = f - me$ ;
4.  $y = \frac{f_1}{d_1}$ ; and then
5.  $x = \frac{e - by}{a}$ .

Solve the following linear systems using four-digit rounding arithmetic.

$$\begin{aligned} \text{a. } & \begin{cases} 1.130x - 6.990y = 14.20 \\ 1.013x + 6.099y = 14.22 \end{cases} \\ \text{b. } & \begin{cases} 8.110x - 12.20y = -0.1370 \\ -18.11x + 112.2y = -0.1376 \end{cases} \end{aligned}$$

**Solution.**

- a.
  1. Provided that  $a \neq 0$ , set  $m = \frac{1.013}{1.130} \overset{\mathbf{R}_4}{\approx} 0.8965$ ;
  2.  $d_1 = 6.099 - 0.8965 \cdot (-6.990) \overset{\mathbf{R}_4}{\approx} 12.37$ ;
  3.  $f_1 = 14.22 - 0.8965 \cdot 14.20 \overset{\mathbf{R}_4}{\approx} 1.490$ ;
  4.  $y = \frac{12.37}{1.490} \overset{\mathbf{R}_4}{\approx} 8.302$ ; and then
  5.  $x = \frac{14.20 - 6.990 \cdot 8.302}{1.130} \overset{\mathbf{R}_4}{\approx} -38.79$ .
- b.
  1. Provided that  $a \neq 0$ , set  $m = \frac{8.110}{-18.11} \overset{\mathbf{R}_4}{\approx} -0.4478$ ;
  2.  $d_1 = 112.2 - (-0.4478) \cdot (-12.20)$

**Problem 9.** Suppose one calculates in two-digit rounding arithmetic. A rectangular parallelepiped has sides of length 3 cm, 4 cm, and 5 cm, measured to the nearest centimeter. What are the best upper and lower bounds for the volume of this parallelepiped? What are the best upper and lower bounds for the surface area?

**Solution.**

**Problem 10.** Find the rate of convergence of the following.

- a.  $\left\{ \sin \left( \frac{1}{n^2} \right) \right\}_{n=1}^{\infty}$
- b.  $\left\{ \left( \sin \left( \frac{1}{n} \right) \right)^2 \right\}_{n=1}^{\infty}$
- c.  $\frac{\sin h}{h}$  as  $h \rightarrow 0$
- d.  $\frac{1 - e^h}{h}$  as  $h \rightarrow 0$

**Solution.**

- a. Notice that  $\left\{ \sin \left( \frac{1}{n^2} \right) \right\}_{n=1}^{\infty} \rightarrow 0$ . Choose  $\{\beta_n\} = \left\{ \frac{1}{n^2} \right\}$ , which converges to 0. Choose  $K = 1$ .

Then,

$$\begin{aligned} \left| \sin \left( \frac{1}{n^2} \right) - 0 \right| &= \left| \sin \left( \frac{1}{n^2} \right) \right| \\ &< 1 \cdot \frac{1}{n^2} \end{aligned}$$

for all  $n \in \mathbb{N}$ . Hence,  $\left\{ \sin \left( \frac{1}{n^2} \right) \right\}_{n=1}^{\infty}$  converges to 0 with a rate of convergence  $\mathcal{O} \left( \frac{1}{n^2} \right)$ .

- b. Notice that  $\left\{ \left( \sin \left( \frac{1}{n} \right) \right)^2 \right\}_{n=1}^{\infty} \rightarrow 0$ . Choose  $\{\beta_n\} = \left\{ \frac{1}{n^2} \right\}$ , which converges to 0. Choose  $K = 1$ .

Then,

$$\begin{aligned} \left| \left( \sin \left( \frac{1}{n} \right) \right)^2 - 0 \right| &= \left( \sin \left( \frac{1}{n} \right) \right)^2 \\ &< \left( \frac{1}{n} \right)^2 \\ &= 1 \cdot \frac{1}{n^2} \end{aligned}$$

for all  $n \in \mathbb{N}$ . Hence,  $\left\{ \left( \sin \left( \frac{1}{n} \right) \right)^2 \right\}_{n=1}^{\infty}$  converges to 0 with a rate of convergence  $\mathcal{O} \left( \frac{1}{n^2} \right)$ .

- c. Notice that  $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$ . Choose  $G(h) = h^2$ , of which the limit is 0 as  $h \downarrow 0$ . Choose  $K = 1$ . We

first Taylor expand  $\sin h$  about  $x_0 = 0$ :

$$\sin h \approx h - \frac{h^3}{6} + \mathcal{O}(h^5).$$

Then,

$$\begin{aligned} \left| \frac{\sin h}{h} - 1 \right| &= \left| \frac{-\frac{h^3}{6} + \mathcal{O}(h^5)}{h} \right| \\ &< 1 \cdot h^2 + \mathcal{O}(h^4) \end{aligned}$$

for  $h \in (0, 1)$ . Note that  $h^2 + \mathcal{O}(h^4) = \mathcal{O}(h^2)$ . Hence,  $\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$  with a rate of convergence  $\mathcal{O}(h^2)$ .

d. Notice that  $\lim_{h \rightarrow 0} \frac{1 - e^h}{h} = -1$ . Choose  $G(h) = h$ , of which the limit is 0 as  $h \downarrow 0$ . Choose  $K = 1$ .

We first Taylor expand  $e^h$  about  $x_0 = 0$ :

$$e^h \approx 1 + \frac{h}{1!} + \frac{h^2}{2!} + \mathcal{O}(h^3).$$

Then,

$$\begin{aligned} \left| \frac{1 - e^h}{h} - (-1) \right| &= \left| \frac{1 - e^h + h}{h} \right| \\ &= \frac{\frac{h^2}{2} + \mathcal{O}(h^3)}{h} \\ &< 1 \cdot h + \mathcal{O}(h^2) \end{aligned}$$

for  $h \in (0, 1)$ . Hence,  $\lim_{h \rightarrow 0} \frac{1 - e^h}{h} = -1$  with a rate of convergence  $\mathcal{O}(h)$ . □

**Problem 11.** Suppose that as  $x$  approaches zero,

$$F_1(x) = L_1 + O(x^\alpha) \quad \text{and} \quad F_2(x) = L_2 + O(x^\beta).$$

Let  $c_1$  and  $c_2$  be nonzero constants, and define

$$F(x) = c_1 F_1(x) + c_2 F_2(x) \quad \text{and} \quad G(x) = F_1(c_1 x) + F_2(c_2 x).$$

Show that if  $\gamma = \min\{\alpha, \beta\}$ , then as  $x$  approaches zero,

a.  $F(x) = c_1 L_1 + c_2 L_2 + O(x^\gamma)$

b.  $G(x) = L_1 + L_2 + O(x^\gamma).$

**Proof.**