Homework 2 of Computational Mathematics

Chang, Yung-Hsuan
111652004
Department of Applied Mathematics

March 30, 2024

Problem 1. Use a fixed-point iteration method to determine a solution accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1, 2]$. Use $p_0 = 1$.

Solution. We want to find the solution to $f(x) = x$ with $f(x) = \sqrt{1 + \frac{1}{x}}$. Then, by calculator,

$$p_1 = \sqrt{2} = 1.414,$$

$$p_2 = 1.307,$$

$$p_3 = 1.329,$$

$$p_4 = 1.324,$$

which is accurate to within 10^{-2} for $x^3 - x - 1 = 0$ on $[1, 2]$. $\square$

Problem 2. Use Theorem 2.1 to find a bound for the number of iterations needed to achieve an approximation with accuracy 10^{-3} to the solution of $x^3 + x - 4 = 0$ lying in the interval $[1, 4]$. Find an approximation to the root with this degree of accuracy.

Solution. Let $f(x) = x^3 + x - 4$. Since $f \in C[1, 4]$ and $f(1) \cdot f(4) = (-2) \cdot 64 < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approaches to a zero p of f with

$$|p_n - p| \leq \frac{4-1}{2^n}, \quad \text{when } n \geq 1.$$

Then,

$$\frac{3}{2^n} \leq 10^{-3} \implies n \geq \log_2(3000) > 11.$$

111652004_CM_HW2.tex U   111652004_CM_HW2_P2.py U

HW2 > 111652004_CM_HW2_P2.py > ...

```python
end_point_1 = 1
end_point_2 = 4
tolerance = 0.001
maximum_number_of_iterations = 50
p_0 = 2.5
i = 1

def f(x):
    return x**3 + x - 4

FA = f(end_point_1)

while i <= maximum_number_of_iterations:
    p = end_point_1 + (end_point_2 - end_point_1) / 2
    FP = f(p)
    if (p == 0 or (end_point_2 - end_point_1) / 2 < tolerance):
        print(f"p = {p} with {i} iterations.")
        exit(0)
    i = i + 1
    if FA * FP > 0:
        end_point_1 = p
        FA = FP
    else:
        end_point_2 = p

print(f"Method failed after {maximum_number_of_iterations}.")
```

PROBLEMS OUTPUT DEBUG CONSOLE TERMINAL PORTS GITLENS COMMENTS

```
PS E:\Eiken\Visual Studio Code Git Sync\CM_HW> & C:/Users/yungh/AppData/Local/Microsoft/WindowsApps/python3.11.exe "
p = 1.378662109375 with 12 iterations.
```

Using the Bisection method, by Python, $p = 1.37866$. $\square$

Problem 3. The following four methods are preposed to compute $21^{1/3}$. Rank them in order, based on their apparent speed of convergence, assuming $p_0 = 1$.

a. $p_n = \dfrac{20p_{n-1} + \frac{21}{{p_{n-1}}^2}}{21}$

b. $p_n = p_{n-1} - \dfrac{{p_{n-1}}^3 - 21}{3{p_{n-1}}^2}$

c. $p_n = p_{n-1} - \dfrac{{p_{n-1}}^4 - 21p_{n-1}}{{p_{n-1}}^2 - 21}$

d. $p_n = \sqrt{\dfrac{21}{p_{n-1}}}$

Solution. For a, choose $g_1(x) = \dfrac{20x + \frac{21}{x^2}}{21} = \dfrac{20x}{21} + \dfrac{1}{x^2}$. Then $g_1{}'(x) = \dfrac{20}{21} - \dfrac{2}{x^3}$. Hence

$$g_1{}'(\sqrt[3]{21}) = \frac{20}{21} - \frac{2}{21} \approx 0.86.$$

For b, choose $g_2(x) = x - \dfrac{x^3 - 21}{3x^2} = \dfrac{2x}{3} + \dfrac{7}{x^2}$. Then $g_2{}'(x) = \dfrac{2}{3} - \dfrac{14}{x^3}$. Hence

$$g_2{}'(\sqrt[3]{21}) = \frac{2}{3} - \frac{1}{3} \approx 0.33.$$

For c, choose $g_3(x) = x - \dfrac{x^4 - 21x}{x^2 - 21} = \dfrac{x^3 - x^4}{x^2 - 21}$. Then $g_3{}'(x) = \dfrac{-2x^5 + x^4 + 84x^3 - 63x^2}{x^4 - 42x^2 + 441}$. Hence

$$g_3{}'(\sqrt[3]{21}) \approx 5.7.$$

For d, choose $g_4(x) = \sqrt{\dfrac{21}{x}}$. Then $g_4{}'(x) = \dfrac{-\sqrt{21}}{2x^{\frac{3}{2}}}$. Hence

$$g_4{}'(\sqrt[3]{21}) = \frac{1}{2}.$$

To sum up, the apparent speed of convergence in order is b, d, a, and c does not converge (the derivative at $\sqrt[3]{21}$ is greater than 1.) $\square$

Problem 4. Use Theorem 2.3 to show that $g(x) = 2^{-x}$ has a unique fixed point on $\left[\frac{1}{3}, 1\right]$. Use fixed-point iteration to find an approximation to the fixed point accurate to within 10^{-4}. Use corollary 2.5 to estimate the number of iterations required to achieve 10^{-4} accuracy, and compare this theoratical estimate the the number actually needed.

Solution. We know that $g \in C\left[\frac{1}{3}, 1\right]$ and $g(x) \in [0.5, 0.9637] \subseteq \left[\frac{1}{3}, 1\right]$. Then g has at least a fixed point in $[a, b]$. Moreover, $g'(x)$ exists on $\left(\frac{1}{3}, 1\right)$. Choose $k = 0.7$. Then

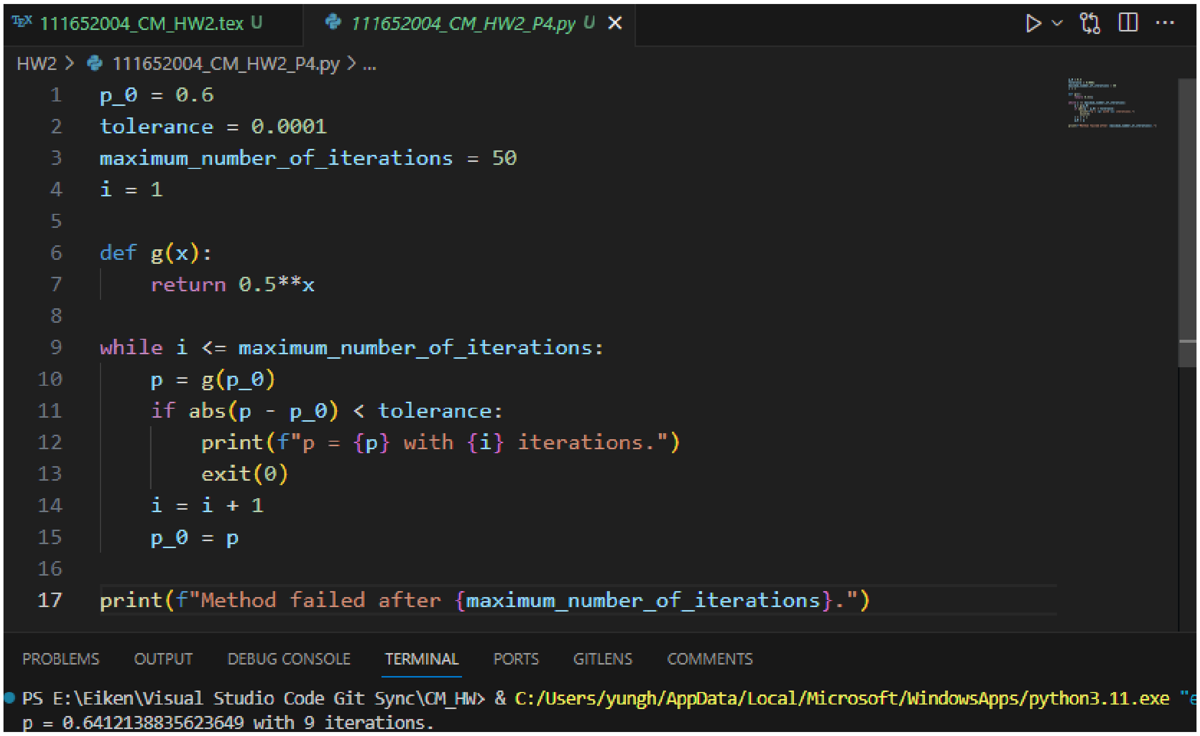
$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}x}2^{-x}\right| &= \ln 2 \cdot 2^{-x} \\ &< \ln 2 \cdot 2^{-0} \\ &= \ln 2 \\ &< k\end{aligned}$$

for all $x \in (0, \infty)$. Hence, $g'(x)$ exists on $\left(\frac{1}{3}, 1\right)$ and a positive $0 < k < 1$ exsits with $|g'(x)| \leq k$ for all $x \in \left(\frac{1}{3}, 1\right)$. Then there exists exactly one fixed point in $\left[\frac{1}{3}, 1\right]$. It is known the assumption of Theorem 2.4 holds, i.e., $g'(x)$ exists on $\left(\frac{1}{3}, 1\right)$ and a positive $0 < k < 1$ exsits with $|g'(x)| \leq k$ for all $x \in \left(\frac{1}{3}, 1\right)$. By Corollary 2.5,

$$|p_n - p| \leq 0.7^n \max\{0.6 - \frac{1}{3}, 1 - 0.6\}.$$

Then,

$$0.7^n \max\{0.6 - \frac{1}{3}, 1 - 0.6\} \leq 10^{-4} \implies n > 23.$$



```python
p_0 = 0.6
tolerance = 0.0001
maximum_number_of_iterations = 50
i = 1

def g(x):
    return 0.5**x

while i <= maximum_number_of_iterations:
    p = g(p_0)
    if abs(p - p_0) < tolerance:
        print(f"p = {p} with {i} iterations.")
        exit(0)
    i = i + 1
    p_0 = p

print(f"Method failed after {maximum_number_of_iterations}.")
```

```
PS E:\Eiken\Visual Studio Code Git Sync\CM_HW> & C:/Users/yungh/AppData/Local/Microsoft/WindowsApps/python3.11.exe "e
p = 0.6412138835623649 with 9 iterations.
```

By Python, the number of iteration actually needed is 9, which is smaller due to my overestimation for error. □

Problem 5. Let A be a given positive constant and $g(x) = 2x - Ax^2$.

a. Show that if fixed-point iteration converges to a nonzero limit, then the limit is $p = \dfrac{1}{A}$, so the inverse of a number can be found using only multiplications and subtractions.

b. Find an interval about $\dfrac{1}{A}$ for which fixed-point iteration converges, provided p_0 is in that interval.

Solution.

a. Suppose that fixed-point iteration converges to a nonzero limit, say p. We want to show that $p = \dfrac{1}{A}$. For the sake of contradiction, suppose $p \neq \dfrac{1}{A}$. Then, by the definition of convergence, we have

$$p = 2p - Ap^2.$$

By the fact that p is nonzero, we further have

$$1 = 2 - Ap,$$

which implies $Ap = 1$, a contradiction.

b. We have $g(x) = 2x - Ax^2$ with $g \in C\left[\dfrac{2}{3A}, \dfrac{4}{3A}\right]$ and then

$$\begin{aligned}
g(x) &= -A\left(x + \frac{1}{A}\right)^2 + \frac{1}{A} \\
&\leq -A \cdot \left(\frac{1}{3A}\right)^2 + \frac{1}{A} \\
&= \frac{1}{A} - \frac{1}{9A} \\
&= \frac{8}{9A}.
\end{aligned}$$

Hence $g(x) \in \left[\dfrac{2}{3A}, \dfrac{4}{3A}\right]$ for all $x \in \left[\dfrac{2}{3A}, \dfrac{4}{3A}\right]$. In addition, $g'(x) = 2 - 2Ax$ exists on $\left(\dfrac{2}{3A}, \dfrac{4}{3A}\right)$. Then $|g'(x)| < \dfrac{2}{3}$ for all $x \in \left(\dfrac{2}{3A}, \dfrac{4}{3A}\right)$. By the fix-point theorem, the sequence defined by

$$p_n = g(g_{n-1}), \quad n \geq 1$$

converges to the unique fixed point $\dfrac{1}{A}$ in $\left[\dfrac{2}{3A}, \dfrac{4}{3A}\right]$. $\square$

Problem 6. Show that if A is any positive number, then the sequence defined by

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}}, \quad \text{for } n \geq 1,$$

converges to $\sqrt{A}$ whenever $x_0 > 0$.

Solution. Let $A > 0$. Let $x_0 > 0$. Let $k \in \mathbb{N}$. Suppose $x_k > 0$. Then $x_{k+1} = \frac{1}{2}x_k + \frac{A}{2x_k} > 0$. Thus $x_k > 0$ for all $k \in \mathbb{N}$. By the AM-GM inequality, we have

$$x_n = \frac{1}{2}x_{n-1} + \frac{A}{2x_{n-1}} \geq \sqrt{A} \tag{6.1}$$

for all $n \in \mathbb{N}$, and the equation does not hold provided that $x_{n-1} = \sqrt{A}$. We now separate three cases for the initial x_0. First, suppose that $x_0 = \sqrt{A}$. Then it is obvious that $x_k = \sqrt{A}$ for all $k \in \mathbb{N}$. Hence $x_k \to \sqrt{A}$ as $k \to \infty$. We now deal with $x_0 \neq \sqrt{A}$. Suppose $x_0 \neq \sqrt{A}$. Then by (6.1), $x_k > \sqrt{A}$ for all $k \in \mathbb{N}$. Thus

$$\begin{aligned}
x_{k+1} - x_k &= \frac{1}{2}x_k + \frac{A}{2x_k} - x_k \\
&= \frac{A}{2x_k} - \frac{1}{2}x_k \\
&= \frac{1}{2x_k}\left(A - {x_k}^2\right) \\
&< 0
\end{aligned}$$

for all $k \in \mathbb{N}$, which implies that $\{x_n\}$ is decreasing. Since $\{x_n\}$ is bounded and monotone, by the monotone convergence theorem, $\{x_n\}$ converges. Say the limit is L. Then by the recursive relation, $L = \frac{1}{2}L + \frac{A}{2L}$, which implies $L = \pm\sqrt{A}$. Since $\{x_n\} \subseteq [\sqrt{A}, \infty)$, the limit is $\sqrt{A}$. □

Problem 7. Let $f(x) = -x^3 - \cos x$. With $p_0 = -1$ and $p_1 = 0$, find p_3.

a. Use the Secant method.

b. Use the method of False Position.

Solution.

Problem 8. Problems involving the amount of money required to pay off a mortgage over a fixed period of time involve the formula

$$A = \frac{P}{i}\left(1 - (1+i)^{-n}\right),$$

known as an *ordinary annuity equation.* In this equation, A is the amount of the mortgage, P is the amount of each payment, and i is the interest rate per period for the n payment periods. Suppose that a 30-year home mortgage in the amount of \$135,000 is needed and that the borrower can afford house payments of at most \$1000 per month. What is the maximal interest rate the borrower can afford to pay?

Solution.

Problem 9.

a. Show that for any positive integer k, the sequence defined by $p_n = \dfrac{1}{n^k}$ converges linearly to $p = 0$.

b. Show that the sequence $p_n = 10^{-2^n}$ converges quadratically to 0.

Solution.

a. Let $k \in \mathbb{N}$. It is clear that $\dfrac{1}{n^k} \to 0$ as $n \to \infty$. Choose $\alpha = 1$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{\left|\frac{1}{(n+1)^k} - 0\right|}{\left|\frac{1}{n^k} - 0\right|^1} &= \left(\lim_{n\to\infty} \frac{n}{n+1}\right)^k \\ &= 1\end{aligned}$$

Hence the sequence is lienarly convergent.

b. It is clear that $10^{-2^n} \to 0$ as $n \to \infty$. Choose $\alpha = 2$. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{\left|10^{-2^{n+1}} - 0\right|}{\left|10^{-2^n} - 0\right|^2} &= \lim_{n\to\infty} 10^{-2^{n+1}+2^{n+1}} \\ &= 1.\end{aligned}$$

Since the asymptotic error constant is $\lambda = 1$, the sequence is quadratically convergent. $\square$

Problem 10.

a. The following sequences are linearly convergent. Generate the first five terms of the sequence $\{\hat{p_n}\}$ using Aitken's Δ^2 method.

$$p_0 = 0.5, \quad p_n = \cos(p_{n-1}), \quad n \geq 1$$

b. Use Steffensen's method to find, to an accuracy of 10^{-4}, the root of $x^3 - x - 1 = 0$ that lies in $[1, 2]$.

Solution.

Problem 11. Given a polynomial $P(x) = x^3 - 5x^2 + 8x - 6$, do the following:

a. Evaluate $P(2)$, $P'(2)$, $P(4)$, and $P'(4)$ by Horner's method.

b. Find the root of $P(x)$ with error less than 0.00001 between $[2, 4]$ by using the Newton method with initial point $x_0 = 2$ and $x_0 = 4$. Determine which initial point may lead to the root.

c. Deflate $P(x)$ into a quadartic ppolynomial by using the results in (b) and find the complex roots of $P(x)$.

d. Perform one step of Muller's Method starting from initial $(0, P(0))$, $(1, P(1))$ and $(2, P(2))$.

e. Implement a MATLAB code of Muller's Method to find the complex root within error less than 0.00001 and compare with the answer you find in (c).

Solution.