Homework 5 of Computational Mathematics

Chang, Yung-Hsuan

111652004

Department of Applied Mathematics

May 21, 2024

Problem 1. Show that each of the following initial-value problems has a unique solution and find the solution. Can Theorem 5.4 be applied in each case?

a. $y' = t^{-2}(\sin 2t - 2ty), \quad 1 \le t \le 2, \quad y(1) = 2$; and

b. $y' = -y + t\sqrt{y}, \quad 2 \le t \le 3, \quad y(2) = 2$.

Solution.

a. The domain of $f(t, y) = \dfrac{\sin 2t - 2ty}{t^2}$ is $D = [1, 2] \times \mathbb{R}$. It is clear that f is continuous on D. Fix a $t \in [1, 2]$. Then, for $y_1, y_2 \in \mathbb{R}$,

$$\begin{aligned} |f(t, y_1) - f(t, y_2)| &= \frac{2t|y_1 - y_2|}{t^2} \\ &\le 2|y_1 - y_2|, \end{aligned}$$

which implies f satisfies a Lipschitz condition in the variable y on D with a Lipschitz constant 2. By Theorem 5.4, the initial-value problem has a unique solution. Using algebra, we have

$$\begin{aligned} t^2 y' + 2ty &= \sin 2t \\ t^2 y &= -\frac{1}{2}\cos 2t + C \\ \overset{y(1)=2}{\Longrightarrow} \qquad y &= \frac{-\cos 2t + 4 + \cos 2}{2t^2}. \end{aligned}$$

b.

Problem 2. For each choice of $f(t, y)$ given in parts (a)-(d):

i. Does f satisfy a Lipschitz condition on $D = \{(t, y) \mid 0 \leq t \leq 1, -\infty < y < \infty\}$?

ii. Can Theorem 5.6 be used to show that the initial-value problem

$$y' = f(t, y), \quad 0 \leq t \leq 1, \quad y(0) = 1$$

is well-posed?

a. $f(t, y) = e^{t-y}$; and

b. $f(t, y) = \dfrac{1+y}{1+t}$.

Solution.

a. Fix a $t \in [0, 1]$. Then, for $y_1, y_2 \in \mathbb{R}$ with $y_1 > y_2$,

$$\begin{aligned} \left|e^{t-y_1} - e^{t-y_2}\right| &\geq \left|e^{-y_1} - e^{-y_2}\right| \\ &\geq e^{-y_2} \end{aligned}$$

is unbounded, which implies that f does not satisfy a Lipschitz condition on D in the variable y.

We cannot use Theorem 5.6 here since f does not satisfy a Lipschitz condition.

b. Fix a $t \in [0, 1]$. Then, for $y_1, y_2 \in \mathbb{R}$,

$$\left|\frac{1+y_1}{1+t} - \frac{1+y_2}{1+t}\right| \leq |y_1 - y_2|,$$

which implies f satisfies a Lipschitz condition in the variable y on D with Lipschitz constant 1.

Since f is continuous on D, by Theorem 5.6, the initial-value problem is well-posed. $\square$

Problem 3. Use Euler's method to approximate the solutions for each of the following initial-value problems.

a. $y' = \dfrac{2-2ty}{t^2+1}, \quad 0 \le t \le 1, \quad y(0) = 1$ with $h = 0.1$; and

b. $y' = \dfrac{y^2}{1+t}, \quad 1 \le t \le 2, \quad y(1) = -\dfrac{1}{\ln 2}$ with $h = 0.1$.

Show that the actual solutions are indeed $y(t) = \dfrac{2t+1}{t^2+1}$ and $y(t) = \dfrac{-1}{ln(t+1)}$, respectively. Plot the errors between your numerical solutions and the exact solutions. Draw your conclusion regarding to the order of error with respect to the time step dt.

Solution.

Problem 4. Given the initial-value problem

$$y' = -y + t + 1, \quad 0 \le t \le 5, \quad y(0) = 1$$

with exact solution $y(t) = e^{-t} + t$.

a. Approximate $y(5)$ using Euler's method with $h = 0.2$, $h = 0.1$, and $h = 0.05$.

b. Determine the optimal value of h to use in computing $y(5)$, assuming $\delta = 10^{-6}$ and that Eq. (5.14) is valid.

Solution.

Problem 5. Use Taylor's method of order two to approximate the solutions for each of the following initial-value problems.

a. $y' = \dfrac{1+y}{1-t}, \quad 1 \le t \le 2, \quad y(1) = 2$ with $h = 0.5$; and

b. $y' = -y + t\sqrt{y}, \quad 2 \le t \le 3, \quad y(2) = 2$ with $h = 0.25$.

Solution.

Problem 6. Given the initial-value problem

$$y' = \frac{2}{t}y + t^2 e^t, \quad 1 \le t \le 2, \quad y(1) = 0$$

with exact solution $y(t) = t^2(e^t - e)$.

a. Use Taylor's method of order two with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

b. Use the answers generated in part (a) and linear interpolation to approximate y at the following values, and compare them to the actual values of y.

   i. $y(1.04)$;

   ii. $y(1.55)$; and

   iii. $y(1.97)$.

c. Use Taylor's method of order four with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

d. Use the answers generated in part (c) and piecewise cubic Hermite interpolation to approximate y at the following values, and compare them to the actual values of y.

   i. $y(1.04)$;

   ii. $y(1.55)$; and

   iii. $y(1.97)$.

Solution.

Problem 7. Use the modified Euler method to approximate the solutions to each of the following initial-value problems, and compare the results to the actual values.

a. $y' = \dfrac{y^2}{1+t}, \quad 1 \le t \le 2, \quad y(1) = -\dfrac{1}{\ln 2}$ with $h = 0.1$; actual solution $y = \dfrac{-1}{\ln(t+1)}$; and

b. $y' = \dfrac{y^2+y}{t}, \quad 1 \le t \le 3, \quad y(1) = -2$ with $h = 0.2$; actual solution $y(t) = \dfrac{2t}{1-2t}$.

Solution.

Problem 8. Show that the difference method

$$\omega_0 = \alpha$$

$$\omega_{i+1}\omega_i + a_1\, f(t_i, \omega_i) + a_2\, f(t_i + \alpha_2, \omega_1 + \delta_2\, f(t_i, \omega_i)),$$

for each $i = 0, 1, 2, \ldots, N-1$, cannot have local truncation error $\mathcal{O}(h^3)$ for any choice of constants a_1, a_2, α_2, and δ_2.

Solution.

Problem 9. Use the Runge-Kutta Fehlberg algorithm with tolerance $TOL = 10^{-4}$ to approximate the solution to the following initial-value problems.

a. $y' = \left(\frac{y}{t}\right)^2 + \frac{y}{t}$, $\quad 1 \le t \le 1.2$, $\quad y(1) = 1$ with $hmax = 0.05$ and $hmin = 0.02$; and

b. $y' = \sin t + e^{-t}$, $\quad 0 \le t \le 1$, $\quad y(0) = 0$ with $hmax = 0.25$ and $hmin = 0.02$.

Solution.

Problem 10. Use each of the Adams-Bashforth methods to approximate the solutions to the following initial-value problems. In each case use starting values obtained from the Runge-Kutta method of order four. Compare the results to the actual values.

a. $y' = \dfrac{y}{t} - \left(\dfrac{y}{t}\right)^2, \quad 1 \le t \le 2, \quad y(1) = 1$ with $h = 0.1$; actual solution $y(t) = \dfrac{t}{1 + \ln(t)}$; and

b. $y' = -5y + 5t^2 + 2t, \quad 0 \le t \le 1, \quad y(0) = \dfrac{1}{3}$ with $h = 0.1$; actual solution $y(t) = -3 + \dfrac{2}{1 + e^{-2t}}$.

Solution.

Problem 11. The initial-value problem

$$y' = e^y, \quad 0 \le t \le 0.2, \quad y(0) = 1$$

has solution $y(t) = 1 - \ln(1 - et)$. Applying the three-step Adams-Moulton method to this problem is equivalent to finding the fixed point ω_{i+1} of

$$g(\omega) = \omega_i + \frac{h}{24}\left(9e^{\omega} + 19e^{\omega_i} - 5e^{\omega_{i-1}} + e^{\omega_{i-2}}\right).$$

a. With $h = 0.01$, obtain ω_{i+1} by functional iteration for $i = 2, \ldots, 19$ using exact starting values ω_0, ω_1, and ω_2. At each step use ω_i to initially approximate ω_{i+1}.

b. Will Newton's method speed the convergence over functional iteration?

Solution.

Problem 12. Derive the Adams-Bashforth three-step method by the following method. Set

$$y(t_{i+1}) = t(t_i) + ah\, f(t_i, y(t_i)) + bh\, f(t_{i-1}, y(t_{i-1})) + ch\, f(t_{i-2}, y(t_{i-2})).$$

Expand $y(t_{i+1})$, $f(t_{i-2}, y(t_{i-2}))$, and $f(t_{i-1}, y(t_{i-1}))$ in Taylor series about $(t_i, y(t_i))$, and equate the coefficients of h, h^2, and h^3 to obtain a, b, and c.

Solution.

Problem 13. Use the Adams variable step-size predictor-corrector algorithm with $TOL = 10^{-4}$ to approximate the solutions to the following initial-value problems:

a. $y' = \sin t + e^{-t}, \quad 0 \le t \le 1, \quad y(0) = 0$ with $hmax = 0.2$ and $hmin = 0.01$; and

b. $y' = -ty + \dfrac{4t}{y}, \quad 0 \le t \le 1, \quad y(0) = 1$ with $hmax = 0.2$ and $hmin = 0.01$.

Solution.

Problem 14. Let $P(t)$ be the number of individuals in a population at time t, measured in years. If the average birth rate b is constant and the average death rate d is proportional to the size of the population (due to overcrowding), then the growth rate of the population is given by the logistic equation

$$\frac{\mathrm{d}P}{\mathrm{d}t}(t) = b\,P(t) - k(P(t))^2,$$

where $d = k\,P(t)$. Suppose $P(0) = 50976$, $b = 2.9 \times 10^{-2}$, and $k = 1.4 \times 10^{-7}$. Find the population after 5 years using the extrapolation method (based on the Euler method and the midpoint method) with times step $h = 0.1$. Justify the order of truncation error from your numerical answers.

Solution.

Problem 15. Suppose the swinging pendulum described in the lead example of this chapter is 2 ft long and that $g = 32.17$ ft/s^2. With $h = 0.1$ s, compare the angle θ obtained for the following two initial-value problems at $t = 0, 1, 2$.

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\sin\theta = 0, \quad \theta(0) = \frac{\pi}{6}, \quad \theta'(0) = 0.$$

You shall use Adams fourth order predictor-corrector algorithm to obtain your numerical answer.

Solution.

Problem 16. Consider the differential equation

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha.$$

a. Show that

$$y'(t_i) = \frac{-3y(t_y) + 4y(t_{i+1} - y(t_{i+2}))}{2h} + \frac{h^2}{3} y'''(\xi_1)$$

for some $\xi \in (t_i, t_{i+2})$.

b. Part (a) suggests the difference method

$$\omega_{i+2} = 4\omega_{i+1} - 3\omega_i - 2h\, f(t_i, \omega_i), \quad \text{for } i = 0, 1, 2 \ldots, N-2.$$

Use this method to solve

$$y' - 1 - y, \quad 0 \leq t \leq 1, \quad y(0) = 0$$

with $h = 0.1$. Use the starting values $\omega_0 = 0$ and $\omega_1 = y(t_1) = 1 - e^{-0.1}$.

c. Repeat part (b) with $h = 0.01$ and $\omega_1 = 1 - e^{-0.01}$.

d. Analyze this method for consistency, stability, and convergence.

Solution.

Problem 17. Given the multistep method

$$\omega_{i+1} = -\frac{3}{2}\omega_i + 3\omega_{i-1} - \frac{1}{2}\omega_{i-2} + 3g\, f(t_i, \omega_i), \quad \text{for } i = 2, 3, \ldots, N-1$$

with starting values ω_0, ω_1, and ω_2:

a. Find the local truncation error.

b. Comment on consistency, stability, and convergence.

Solution.

Problem 18. Discuss consistency, stability, and convergence for the implicit trapezoidal method

$$\omega_{i+1} = \omega_i + \frac{h}{2}\left(f(t_{i+1}, \omega_{i+1}) + f(t_i, \omega_i)\right), \quad \text{for } i = 0, 1, 2, \ldots, N-1$$

with $\omega_0 = \alpha$ applied to the differential equation

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Solution.

Problem 19. Show that the fourth-order Runge-Kutta method,

$$k_1 = h\,f(t_i, \omega_i),$$

$$k_2 = h\,f(t_i + h/2, \omega_i + k_1/2),$$

$$k_3 = h\,f(t_i + h/2, \omega_i + k_2/2),$$

$$k_4 = h\,f(t_i + h, \omega_i + k_3),$$

$$\omega_{i+1} = \omega_i + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

when applied to the differential equation $y' = \lambda y$, can be written in the form

$$\omega_{i+1} = \left(1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3 + \frac{1}{24}(h\lambda)^4\right)\omega_i.$$

Solution.