Homework 1 of Computational Mathematics

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Problem 1. Use the intermediate value theorem and Rolle's theorem to show the graph of

$$f(x) = x^3 + 2x + k$$

crosses the x-axis exactly once, regardless of the value of the constant k.

Proof. Let $k \in \mathbb{R}$. We first show that $f(x) = 0$ has a root. Suppose $k = 0$. Then, we can factorize $f(x) = x(x^2 + 2)$. Thus, it is clear that $f(x) = 0$ has only one root $x = 0$. Suppose $k > 0$. Choose $x_1 = 0$ and $x_2 = -k$. Then, $f(x_1) = k > 0$ and $f(x_2) = -k^3 - k < 0$. By the intermediate value theorem, there exists a root between x_1 and x_2. Suppose $k < 0$. Choose $x_3 = 0$ and $x_4 = -k$. Then, $f(x_3) = k < 0$ and $f(x_4) = -k^3 - k > 0$. By the intermediate value theorem, there exists a root between x_3 and x_4. We now show that there is only one root. Observe that $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) = 3x^2 + 2 > 0$ for all $x \in \mathbb{R}$ regardless of the value of k. For the sake of contradiction, we suppose there are two distinct roots α and β such that $f(\alpha) = f(\beta) = 0$. By Rolle's theorem, there exists an x_0 between α and β such that $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x_0) = 0$, a contradiction. Hence, there is only one root. $\square$

Problem 2. Find $\max\limits_{a \le x \le b} |f(x)|$ for the following functions and intervals.

    a. $f(x) = \dfrac{2 - e^x + 2x}{3}$, $[0, 1]$

    b. $f(x) = \dfrac{4x - 3}{x^2 - 2x}$, $[0.5, 1]$

    c. $f(x) = 2x \cos(2x) - (x - 2)^2$, $[2, 4]$

    d. $f(x) = 1 + e^{-\cos(x-1)}$, $[1, 2]$

Solution. Note that we can first looking for extrema for original functions and then find $\max\limits_{a \le x \le b} |f(x)|$.

a. We first look for critical points. Set $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x)(x) = 0$ on $[0, 1]$. Then,

$$\frac{-e^x + 2}{3} = 0$$

$$\Longleftrightarrow \qquad x = \ln 2.$$

Then, we have $f(\ln 2) = \dfrac{2 \ln 2}{3} \approx 0.462$, $f(0) = \dfrac{1}{3} \approx 0.333$, and $f(1) = \dfrac{4 - e}{3} \approx 0.427$. Hence,

$\max\limits_{0 \le x \le 1} |f(x)| = \dfrac{2 \ln 2}{3}$.

b. Observe that the derivative $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) = \dfrac{-4x^2 + 6x - 6}{(x^2 - 2x)^2}$ has no roots on $\mathbb{R}$. Hence, the function is

monotonic and thus achieves extrema at end points. We have $|f(0.5)| = \dfrac{4}{3}$ and $|f(1)| = 1$. Hence,

$\max\limits_{0.5 \le x \le 1} |f(x)| = \dfrac{4}{3}$.

c. We first look for critical points. Set $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) = 0$ on $[2, 4]$. Then,

$$2 \cos(2x) - 4x \sin(2x) - 2(x - 2) = 0 \tag{2.1}$$

Let $x^* \in [2, 4]$ satisfies (2.1). Then, $f(x^*) \approx 4.981$. We have $f(x^*) \approx 4.981$, $f(2) \approx -2.615$, and

$f(4) \approx -5.164$. Hence, $\max\limits_{2 \le x \le 4} |f(x)| = 4 - 8 \cos(8)$.

d. We first look for critical points. Set $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) = 0$ on $[1, 2]$. Then,

$$e^{-\cos(x-1)} \cdot \sin(x - 1) = 0$$

$$\Longleftrightarrow \qquad \sin(x - 1) = 0$$

$$\Longrightarrow \qquad x = 1.$$

Then, we have $f(1) = 1 + e^{-1} \approx 1.368$, $f(2) = 1 + e^{-\cos 1} \approx 1.583$. Hence, $\max\limits_{1 \le x \le 2} |f(x)| = $

$1 + e^{-\cos 1}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Problem 3. Find the second Taylor polynomial $P_2(x)$ for the function $f(x) = e^x \cos(x)$ about $x_0 = 0$.

    a. Use $P_2(0.5)$ to approximate $f(0.5)$. Find an upper bound for error $|f(0.5) - P_2(0.5)|$ using the error formula, and compare it to the actual error.

    b. Find a bound for the error $|f(0.5) - P_2(0.5)|$ in using $P_2(x)$ to approximate $f(x)$ on the interval $[0, 1]$.

    c. Approximate $\int_0^1 f(x)\, \mathrm{d}x$ using $\int_0^1 P_2(x)\, \mathrm{d}x$.

    d. Find an upper bound for the error in (c) using $\int_0^1 |R_2(x)|\, \mathrm{d}x$.

Solution. By Taylor's theorem, we have

$$e^x \cos x = f(0) + \frac{1}{1!} \cdot \frac{\mathrm{d}f}{\mathrm{d}x}(0) \cdot x + \frac{1}{2!} \cdot \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(0) \cdot x^2 + \frac{1}{3!} \cdot \frac{\mathrm{d}^3 f}{\mathrm{d}x^3}(\xi(x)) \cdot x^3$$

$$= 1 + x + \frac{1}{6} \cdot \left(-2 e^{\xi(x)} \left(\sin \xi(x) + \cos \xi(x) \right) \right) \cdot x^3$$

$$= 1 + x - \frac{\sqrt{2}}{3} \cdot e^{\xi(x)} \sin\left(\xi(x) + \frac{\pi}{4} \right) \cdot x^3$$

Hence, the second Taylor polynomial about 0 is

$$P_2(x) = 1 + x.$$

    a. The value of the second Taylor polynomial $P_2(0.5) = 1.5$, and the true value $f(0.5) \approx 1.446889$.

    We can estimate the upper bound of $|f(0.5) - P_2(0.5)|$ by

$$|f(0.5) - P_2(0.5)| = \left| \frac{\sqrt{2}}{3} \cdot e^{\xi(0.5)} \sin\left(\xi(0.5) + \frac{\pi}{4} \right) \cdot (0.5)^3 \right|$$

$$\leq \left| \frac{\sqrt{2}}{3} \cdot e^{0.5} \sin\left(0.5 + \frac{\pi}{4} \right) \cdot (0.5)^3 \right|$$

$$\leq 0.093222005.$$

    The actual error is $\left| e^{0.5} \cos(0.5) - 1.5 \right| \approx 0.0531111$ and is smaller than the upper bound.

    b. For $t \in [0, 1]$, the maximum of e^t is $e^1 = e$, the maximum of $\sin\left(t + \frac{\pi}{4} \right)$ is $\sin\left(\frac{\pi}{4} + \frac{\pi}{4} \right) = 1$, and the maximum of t^3 is $1^3 = 1$. Hence, the maximum of $\frac{\sqrt{2}}{3} \cdot e^{\xi(x)} \sin\left(\xi(x) + \frac{\pi}{4} \right) \cdot x^3$ on $x \in [0, 1]$ is $\frac{\sqrt{2}}{3} \cdot e \cdot 1 \cdot 1^3 \approx 1.281410342720.$

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c.

$$\int_0^1 P_2(x)\,\mathrm{d}x = \int_0^1 1 + x\,\mathrm{d}x$$
$$= \left[x + \frac{x^2}{2} \right]_0^1$$
$$= \frac{3}{2}.$$

d. The error is approximately

$$\left| \int_0^1 f(x)\,\mathrm{d}x - \int_0^1 P_2(x)\,\mathrm{d}x \right| \le \int_0^1 |R_2(x)|\,\mathrm{d}x$$
$$= \frac{\sqrt{2}}{3} \int_0^1 \left| e^x \sin\left(x + \frac{\pi}{4} \right) \cdot x^3 \right|\,\mathrm{d}x$$
$$\overset{\mathcal{M}}{\approx} 0.2625,$$

where $\overset{\mathcal{M}}{\approx}$ suggests that the value is computed and approximated by MATLAB. See Figure 3.d.1.
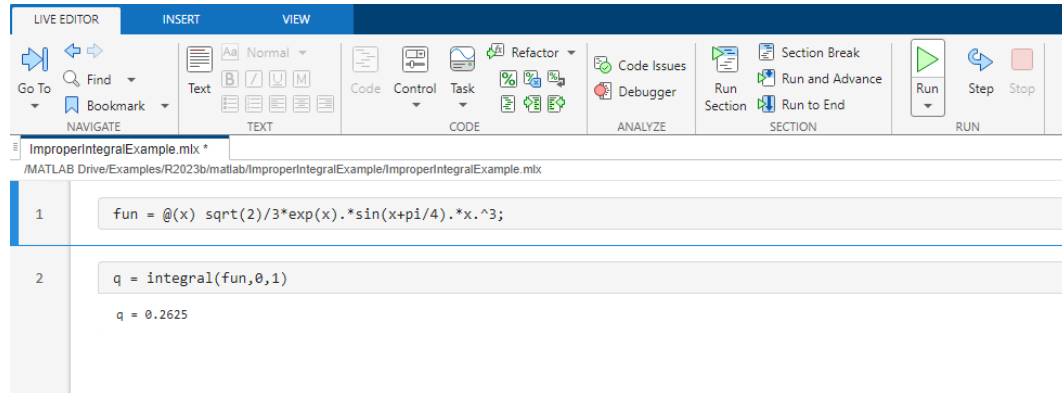

Figure 3.d.1.

Hence, an upper bound is 0.27. □

Problem 4. Let $f(x) = \dfrac{1}{1-x}$ and $x_0 = 0$. Find the n-th Taylor polynomial $P_n(x)$ for $f(x)$ about x_0. Find a value of n necessary for $P_n(x)$ to approximate $f(x)$ to within 10^{-6} on $[0, 0.5]$.

Solution. Notice that $f(x) = \dfrac{1}{1-x}$ is the infinite sum of $\{x^k\}_{k=0}^{\infty}$. Hence, the n-th Taylor polynomial is $P_n(x) = 1 + x + \cdots + x^n$. We would like to find an $n \in \mathbb{N}$ such that $|P_n(x) - f(x)| < 10^{-6}$ for all $x \in [0, 0.5]$. Observe that $f(x) > P_n(x)$ for all $x \in [0, 5]$, and $\dfrac{\mathrm{d}}{\mathrm{d}x}(f(x) - P_n(x)) \geq 0$ for all $x \in [0, 0.5]$. Hence,

$$\sup_{x \in [0, 0.5]} |P_n(x) - f(x)| = f(0.5) - P_n(0.5)$$

$$= 2 - \sum_{i=0}^{n}(0.5)^i$$

$$= 2 - \frac{1 - 0.5^{n+1}}{1 - 0.5}$$

$$= 0.5^n.$$

Set $0.5^n < 10^{-6}$. Then, we obtain that the integer n must be greater than 19. Hence, we choose $n = 20$ as the desired exponent. $\qquad\square$

Problem 5. Find the largest interval in which p^* must lie to approximate p with relative error at most 10^{-4} for each value of p.

   a. π

   b. e

   c. $\sqrt{2}$

   d. $\sqrt[3]{7}$

Solution. The relative of p^* for p is $\left|\dfrac{p^* - p}{p}\right|$. Set

$$\left|\frac{p^* - p}{p}\right| \le 10^{-4}.$$

Then,

$$|p^* - p| \le p \cdot 10^{-4}$$

$$p - p \cdot 10^{-4} \le p^* \le p + p \cdot 10^{-4}$$

whenever $p \ge 0$. Hence, by the fact that all p^*'s are greater than 0, the desired interval for each is

   a. $\left[\pi - \pi \cdot 10^{-4}, \pi + \pi \cdot 10^{-4}\right]$;

   b. $\left[e - e \cdot 10^{-4}, e + e \cdot 10^{-4}\right]$;

   c. $\left[\sqrt{2} - \sqrt{2} \cdot 10^{-4}, \sqrt{2} + \sqrt{2} \cdot 10^{-4}\right]$; and

   d. $\left[\sqrt[3]{7} - \sqrt[3]{7} \cdot 10^{-4}, \sqrt[3]{7} + \sqrt[3]{7} \cdot 10^{-4}\right]$. $\square$

Problem 6. Let
$$f(x) = \frac{e^x - e^{-x}}{x}.$$

a. Find $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{x}$.

b. Use three-digit rounding arithmatic to evaluate $f(0.1)$.

c. Replace each exponential function with its third Maclaurin polynomial, and repeat part (b).

d. The actual value is $f(0.1) = 2.003335000$. Find the relative error for the values obtained in part (b) and (c).

Solution.

a. Using l'Hôpital's rule,
$$\lim_{x \to 0} \frac{e^x - e^{-x}}{x} \overset{L'}{=} \lim_{x \to 0} \frac{e^x + e^{-x}}{1}$$
$$= 2.$$

b. Plug in the value $x = 0.1$, by calculator, $f(0.1) \overset{\mathbf{R}_3}{\approx} \dfrac{1.11 - 0.905}{0.10} = 2.05$, where $\overset{\mathbf{R}_3}{\approx}$ denotes three-digit rounding arithmatic.

c. Plug in the value $x = 0.1$ with substitution $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6}$, we have
$$f(0.1) \overset{\mathbf{M}_3}{\approx} \frac{1 + 0.1 + \dfrac{0.01}{2} + \dfrac{0.001}{6} - 1 + 0.1 - \dfrac{0.01}{2} + \dfrac{0.001}{6}}{0.1}$$
$$= \frac{0.1 + 0.1 + 0.000167 + 0.000167}{0.1}$$
$$\overset{\mathbf{R}_3}{\approx} 2.00,$$

where $\overset{\mathbf{M}_3}{\approx}$ denotes approximation with its third Maclaurin polynomial.

d. The relative error in part (b) is
$$\left| \frac{2.003335000 - 2.05}{2.003335000} \right| \approx 0.02329365783;$$

the relative error in part (c) is
$$\left| \frac{2.003335000 - 2.00}{2.003335000} \right| \approx 0.001664724073.$$

$\square$

Problem 7. Use the 64-bit long real format to find the decimal equivalent of the following floating-point machine numbers.

  a. 0 10000001010 1001001100000000000000000000000000000000000000000000

  b. 1 10000001010 1001001100000000000000000000000000000000000000000000

  c. 0 01111111111 0101001100000000000000000000000000000000000000000000

  d. 0 01111111111 0101001100000000000000000000000000000000000000000001

Solution.

  a. Let r_a be the desired floating-point number. Then,

$$r_a = (-1)^0 \, 2^{c-1023} \, (1+m)$$

$$= 2^{11} \cdot 1.57421875$$

$$= 3224,$$

where $c = 1024 + 8 + 2 = 1034$ and $m = \dfrac{1}{2^1} + \dfrac{1}{2^4} + \dfrac{1}{2^7} + \dfrac{1}{2^8} = \dfrac{147}{256} = 0.57421875$.

  b. Let r_b be the desired floating-point number. Observe that $r_b = -r_a$. Thus, $r_b = -3224$.

  c. Let r_c be the desired floating-point number. Then,

$$r_c = (-1)^0 \, 2^{c-1023} \, (1+m)$$

$$= 2^0 \cdot 1.32421875$$

$$= 1.32421875,$$

where $c = 512 + 256 + \cdots + 2 + 1 = 1023$ and $m = \dfrac{1}{2^2} + \dfrac{1}{2^4} + \dfrac{1}{2^7} + \dfrac{1}{2^8} = \dfrac{83}{256} = 0.32421875$.

  d. Let r_d be the desired floating-point number. Then,

$$r_d = (-1)^0 \, 2^{c-1023} \, (1+m)$$

$$= 2^0 \cdot 1.3242187500000002220446049250313080847263336181640625$$

$$= 1.3242187500000002220446049250313080847263336181640625,$$

where $c = 512+256+\cdots+2+1 = 1023$ and $m = \dfrac{1}{2^2}+\dfrac{1}{2^4}+\dfrac{1}{2^7}+\dfrac{1}{2^8}+\dfrac{1}{2^{52}} = \dfrac{2^{50} + 2^{48} + 2^{45} + 2^{44} + 1}{2^{52}} =$

0.3242187500000002220446049250313080847263336181640625. $\square$

Problem 8. The two-by-two system

$$\begin{cases} ax + by = e; \\ cx + dy = f, \end{cases}$$

where a, b, c, d, e, f are given, can be solved for x and y as follows:

1. Set $m = \dfrac{c}{a}$, provided $a \neq 0$;

2. $d_1 = d - mb$;

3. $f_1 = f - me$;

4. $y = \dfrac{f_1}{d_1}$; and then

5. $x = \dfrac{e - by}{a}$.

Solve the following linear systems using four-digit rounding arithmetic.

a.
$$\begin{cases} 1.130x - 6.990y = 14.20 \\ 1.013x - 6.099y = 14.22 \end{cases}$$

b.
$$\begin{cases} 8.110x + 12.20y = -0.1370 \\ -18.11x + 112.2y = -0.1376 \end{cases}$$

Solution.

a.   1. Provided that $a \neq 0$, set $m = \dfrac{1.013}{1.130} \overset{\mathbf{R}_4}{\approx} 0.8965$;

2. $d_1 = -6.099 - 0.8965 \cdot (-6.990) \overset{\mathbf{R}_4}{\approx} 0.1680$;

3. $f_1 = 14.22 - 0.8965 \cdot 14.20 \overset{\mathbf{R}_4}{\approx} 1.490$;

4. $y = \dfrac{1.490}{0.1680} \overset{\mathbf{R}_4}{\approx} 8.869$; and then

5. $x = \dfrac{14.20 - (-6.990) \cdot 8.869}{1.130} \overset{\mathbf{R}_4}{\approx} 67.42$.

b.   1. Provided that $a \neq 0$, set $m = \dfrac{-18.11}{8.110} \overset{\mathbf{R}_4}{\approx} -2.233$;

2. $d_1 = 112.2 - (-2.233) \cdot 12.20 \overset{\mathbf{R}_4}{\approx} 139.4$;

3. $f_1 = -0.1376 - (-2.233) \cdot (-0.1370) \overset{\mathbf{R}_4}{\approx} -0.4435$;

4. $y = \dfrac{-0.4435}{139.4} \overset{\mathbf{R}_4}{\approx} -0.003181$; and then

5. $x = \dfrac{-0.1370 - 12.20 \cdot (-0.003181)}{8.110} \overset{\mathbf{R}_4}{\approx} -0.01211$. $\square$

Problem 9. Suppose one calculates in two-digit rounding arithmetic. A rectangular parallelepiped has sides of length 3 cm, 4 cm, and 5 cm, measured to the nearest centimeter. What are the best upper and lower bounds for the volume of this parallelepiped? What are the best upper and lower bounds for the surface area?

Solution. The true values of the three sides are in $[2.5, 3.5)$ cm, $[3.5, 4.5)$ cm, and $[4.5, 5.5)$ cm, respectively. Hence, for the volume, the best upper bound is $3.5 \cdot 4.5 \cdot 5.5 \overset{\mathbf{R}_2}{\approx} 88$, and the best lower bound is $2.5 \cdot 4.5 \cdot 3.5 \overset{\mathbf{R}_2}{\approx} 39$; for the surface area, the best upper bound is $2 \cdot (3.5 \cdot 4.5 + 3.5 \cdot 5.5 + 4.5 \cdot 5.5) = 120$, and the best lower bound is $2 \cdot (2.5 \cdot 3.5 + 2.5 \cdot 4.5 + 3.5 \cdot 4.5) = 72$. $\qquad\square$

Problem 10. Find the rate of convergence of the following.

a. $\lim\limits_{n\to\infty} \sin\left(\dfrac{1}{n^2}\right)$

b. $\lim\limits_{n\to\infty} \left(\sin\left(\dfrac{1}{n}\right)\right)^2$

c. $\lim\limits_{h\to0} \dfrac{\sin h}{h}$

d. $\lim\limits_{h\to0} \dfrac{1 - e^h}{h}$

Solution.

a. Notice that $\lim\limits_{n\to\infty} \sin\left(\dfrac{1}{n^2}\right) = 0$. Choose $\{\beta_n\} = \left\{\dfrac{1}{n^2}\right\}$, which converges to 0. Choose $K = 1$. Then,

$$\left|\sin\left(\frac{1}{n^2}\right) - 0\right| = \left|\sin\left(\frac{1}{n^2}\right)\right|$$

$$< 1 \cdot \frac{1}{n^2}$$

for all $n \in \mathbb{N}$. Hence, $\left\{\sin\left(\dfrac{1}{n^2}\right)\right\}_{n=1}^{\infty}$ converges to 0 with a rate of convergence $\mathcal{O}\left(\dfrac{1}{n^2}\right)$.

b. Notice that $\lim\limits_{n\to\infty} \left(\sin\left(\dfrac{1}{n}\right)\right)^2 = 0$. Choose $\{\beta_n\} = \left\{\dfrac{1}{n^2}\right\}$, which converges to 0. Choose $K = 1$. Then,

$$\left|\left(\sin\left(\frac{1}{n}\right)\right)^2 - 0\right| = \left(\sin\left(\frac{1}{n}\right)\right)^2$$

$$< \left(\frac{1}{n}\right)^2$$

$$= 1 \cdot \frac{1}{n^2}$$

for all $n \in \mathbb{N}$. Hence, $\left\{\left(\sin\left(\dfrac{1}{n}\right)\right)^2\right\}_{n=1}^{\infty}$ converges to 0 with a rate of convergence $\mathcal{O}\left(\dfrac{1}{n^2}\right)$.

c. Notice that $\lim\limits_{h\to0} \dfrac{\sin h}{h} = 1$. Choose $G(h) = h^2$, of which the limit is 0 as $h \downarrow 0$. Choose $K = 1$. We first Taylor expend $\sin h$ about $x_0 = 0$:

$$\sin h \approx h - \frac{h^3}{3!} + \frac{h^5}{5!} - \cdots .$$

Then,

$$\left|\frac{\sin h}{h} - 1\right| = \left|-\frac{h^2}{3!} + \frac{h^4}{5!} - \cdots\right|$$

$$< 1 \cdot h^2 + \mathcal{O}(h^4)$$

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for $h \in (0, 1)$. Note that $h^2 + \mathcal{O}(h^4) = \mathcal{O}(h^2)$. Hence, $\lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1$ with a rate of convergence $\mathcal{O}(h^2)$.

d. Notice that $\lim\limits_{h \to 0} \dfrac{1 - e^h}{h} = -1$. Choose $G(h) = h$, of which the limit is 0 as $h \downarrow 0$. Choose $K = 1$.

We first Taylor expend e^h about $x_0 = 0$:

$$e^h \approx 1 + \frac{h}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + \cdots .$$

Then,

$$\left| \frac{1 - e^h}{h} - (-1) \right| = \left| \frac{1 - e^h + h}{h} \right|$$

$$= \frac{h}{2!} + \frac{h^2}{3!} + \cdots$$

$$< 1 \cdot h + \mathcal{O}(h^2)$$

for $h \in (0, 1)$. Hence, $\lim\limits_{h \to 0} \dfrac{1 - e^h}{h} = -1$ with a rate of convergence $\mathcal{O}(h)$. $\square$

Definition A ($\mathcal{O}$-Notation). Suppose f and g are real-valued functions defined on the same set of nonnegative real numbers. We say f is of order at most g, written $f(h) = \mathcal{O}(g(h))$, if and only if there exists an $M > 0$ and a small $\delta > 0$ such that $|f(h)| \leq M|g(h)|$ for all $h \in [0, \delta)$.

Theorem B. For h approaching zero,

  a. $c\,\mathcal{O}(h^r) = \mathcal{O}(h^r)$ for all $c \neq 0$ for all $r \in \mathbb{R}$;

  b. $\mathcal{O}((ch)^r) = \mathcal{O}(h^r)$ for all $c \neq 0$ for all $r \in \mathbb{R}$; and

  c. $\mathcal{O}(h^a) + \mathcal{O}(h^b) = \mathcal{O}(h^c)$, where $c = \min\{a, b\}$.

Note that the "=" here means inclusion ($\subseteq$) instead of implying two sets are equal.

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  a. Let $c \in \mathbb{R} \setminus \{0\}$. Let $f \in c\,\mathcal{O}(h^r)$. Then, there exists an $M_0 > 0$ and exists a small $\delta > 0$ such that

$$|f(h)| \leq |c|M_0|h^r|$$

  for all $h \in [0, \delta)$. Choose $M = |c|M_0 + 1$. Then,

$$|f(h)| \leq M|h^r|,$$

  which implies $f(h) \in \mathcal{O}(h^r)$. That is, $c\,\mathcal{O}(h^r) = \mathcal{O}(h^r)$.

  b. Let $c \in \mathbb{R} \setminus \{0\}$. Let $f \in \mathcal{O}((ch)^r)$. Then, there exists an $M_0 > 0$ and exists a small $\delta > 0$ such that

$$|f(h)| \leq M_0|(ch)^r| = M_0|c|^r|h^r|$$

  for all $h \in [0, \delta)$. Choose $M = |c|^r M_0 + 1$. Then,

$$|f(h)| \leq M|h^r|,$$

  which implies $f(h) \in \mathcal{O}(h^r)$. That is, $\mathcal{O}((ch)^r) = \mathcal{O}(h^r)$.

  c. With loss of generality, suppose $a > b$. Let $f \in \mathcal{O}(h^a)$. Let $g \in \mathcal{O}(h^b)$. Then, it is clear that

$$g(h) \geq f(h) \tag{B.1}$$

  for all $h \in [0, \delta)$ for some small $\delta \in (0, 1)$. Hence,

$$f(h) + g(h) \leq 2 \cdot g(h) \tag{B.2}$$

$$= 2\,\mathcal{O}(h^b) \tag{B.3}$$

$$= \mathcal{O}(h^b), \tag{B.4}$$

where (B.2) follows from (B.1), (B.3) follows from assumption, and (B.4) follows from Theorem B.a. If it is the case $a = b$, apply Theorem B.a. $\qquad\square$

Problem 11. Suppose that as x approaches zero,

$$F_1(x) = L_1 + \mathcal{O}(x^\alpha) \qquad \text{and} \qquad F_2(x) = L_2 + \mathcal{O}(x^\beta). \tag{11.1}$$

Let c_1 and c_2 be nonzero constants, and define

$$F(x) = c_1\, F_1(x) + c_2\, F_2(x) \qquad \text{and} \qquad G(x) = F_1(c_1 x) + F_2(c_2 x). \tag{11.2}$$

Show that if $\gamma = \min\{\alpha, \beta\}$, then as x approaches zero,

a. $F(x) = c_1 L_1 + c_2 L_2 + \mathcal{O}(x^\gamma)$

b. $G(x) = L_1 + L_2 + \mathcal{O}(x^\gamma)$.

Proof.

a. The proof is based on Theorem B.

$$F(x) = c_1\, F_1(x) + c_2\, F_2(x) \tag{11.3}$$

$$= c_1(L_1 + \mathcal{O}(x^\alpha)) + c_2(L_2 + \mathcal{O}(x^\beta)) \tag{11.4}$$

$$= c_1 L_1 + c_2 L_2 + \mathcal{O}(x^\alpha) + \mathcal{O}(x^\beta) \tag{11.5}$$

$$= c_1 L_1 + c_2 L_2 + \mathcal{O}(x^\gamma), \tag{11.6}$$

where (11.3) follows from definition provided in (a), (11.4) follows from (11.1), (11.5) follows from

Theorem B.a, and (11.6) follows from Theorem B.c.

b. The proof is based on Theorem B.

$$G(x) = F_1(c_1 x) + F_2(c_2 x) \tag{11.7}$$

$$= L_1 + \mathcal{O}((c_1 x)^\alpha) + L_2 + \mathcal{O}((c_2 x)^\beta) \tag{11.8}$$

$$= L_1 + L_2 + \mathcal{O}(x^\alpha) + \mathcal{O}(x^\beta) \tag{11.9}$$

$$= L_1 + L_2 + \mathcal{O}(x^\gamma), \tag{11.10}$$

where (11.7) follows from definition provided in (b), (11.8) follows from (11.1), (11.9) follows from

Theorem B.b, and (11.10) follows from Theorem B.c. $\qquad\square$