November 12, 2024

Intermediate Macroeconomics

Abstract

This is a summary note of the course "ECO_2F002_EP Intermediate Macroeconomics" instructed by Dr. Mehdi SENOUCI and Dr. Gauthier VERMANDEL at École polytechnique in the first semester in 2024.

Contents

Preface

The following content mostly follows the lecture notes provided by the instructors, but figures are mainly re-produced; there are also some personal remarks.

The texts that are **bold** are indicating keywords, while the texts that are *italic* are indicating new nouns. If you see a blue text, that means it is a link; you can click it and go to its place.

1 The Solow Model

Kaldor in 1961 published "Capital Accumulation and Economics Growth," in which he stylized that there are five facts in processes of economics growth,

1. output per worker grows at a roughly constant rate that does not diminish over time (roughly 2%),
2. capital per worker grows over time,
3. the capital/output ratio is roughly constant (everything stays in the same window),
4. the rate of return to capital is roughly constant,
5. shares of capital and labor in net income are roughly constant, and
6. appreciable differences in the rate of growth of labor productivity and of total output in different societies. (Note: this is of non-regularity.)

The Solow model, proposed by Solow in 1956, replicates the first five Kaldor facts with minimal assumptions.

1.1 The Production Function

We say that a single consumption $Y(t)$ of good (output) is being produced by number $L(t)$ of labors and capital $K(t)$ via a production function in the form of

$$Y(t) = F_t(K(t), L(t)).$$

The general scenario is that the production function F_t might change over time due to technical change. Solow eliminates this concern by specifically assuming

$$Y(t) = F(K(t), A(t)L(t)).$$

This function F need to some conditions, which we will discuss later.

The function $A(t)$ captures "**labor-augmenting** technological progress." Overtime, labor becomes more productive. That is, when $A(t)$ increases, if we fix the amount $L(t)$ of labors-hours, there will be more output produced.

We assume that all population is comprised of labor force with no unemployment. In addition, the economy is closed without government. The same final good is used for either consumption $C(t)$ or investment $I(t)$, i.e., $Y(t) = C(t) + I(t)$.

Below are the conditions that $F : \mathbb{R}^2 \to \mathbb{R}$ needs to satisfy.

1. To be strictly positive and strictly diminishing marginal products MP_k and MP_L. That is,

$$\frac{\partial F}{\partial K} > 0, \quad \frac{\partial F}{\partial L} > 0, \quad \frac{\partial^2 F}{\partial K^2} < 0, \quad \text{and} \quad \frac{\partial^2 F}{\partial L^2} < 0.$$

2. Inputs are essential. That is, $F(0, L) = 0 \quad \text{and} \quad F(K, 0) = 0$.

3. To be homogeneous of degree one in $K(t)$ and $L(t)$. That is,

$$F(\alpha \cdot K(t), A(t)(\alpha \cdot L(t))) = \alpha \cdot F(K(t), A(t)L(t)).$$

There is another underlying assumption that fixed-quantity inputs (such as land or natural resources) are assumed to be unimportant, or replaceable by man-made capital.

The most comprehensive production function is the constant elasticity of substitution production function (CES production function),

$$Y(t) = \left(\alpha \cdot (K(t))^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(A(t)L(t))^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}. \tag{1.1.1}$$

We can derive many other productions functions via the constant elasticity of substitution production function. In (1.1.1), if we take $\sigma \to 1$, we will get the Cobb-Douglas production function,

$$Y(t) = K(t)^{\alpha}(A(t)L(t))^{1-\alpha};$$

if we take $\sigma \to 0$, we will get the Leontief production function,

$$Y(t) = \min\{\alpha \cdot K(t), (1-\alpha) \cdot A(t)L(t)\};$$

if we take $\sigma \to \infty$, we will get the linear production function,

$$Y(t) = \alpha \cdot K(t) + (1-\alpha) \cdot A(t)L(t).$$

1.2 Evolution of Inputs

We assume, by Kaldor facts, labor $L(t)$ and productivity $A(t)$ grow at constant. That is,

$$g_L(t) := \frac{\dot{L}(t)}{L(t)} = n \geq 0 \qquad \text{and} \qquad g_A(t) := \frac{\dot{A}(t)}{A(t)} = g \geq 0.$$

In previous equations, $g_X(t)$ denotes the *growth rate* of variable $X(t)$. Note that the *rate of change* of $X(t)$ is $\dot{X}(t)$. Do not be confused by these two nouns.

Aside from these assumptions, we say that a fraction $s \in (0, 1]$ of output is invested. That is,

$$I(t) = s \cdot Y(t) \qquad \text{and} \qquad C(t) = (1 - s) \cdot Y(t).$$

In addition, capital depreciates at a rate of $\delta \geq 0$.

Hence, the dynamics of capital stock is given by

$$\dot{K}(t) = s \cdot Y(t) - \delta \cdot K(t), \tag{1.2.1}$$

in which suggests the rate of change is increased by investment and is decreased by depreciation.

1.3 The Balanced Growth Path

The *balanced growth path* (BGP) is the situation when all variables grow at constant rates. Over time, the economy tends to converge to the balanced growth path.

Not only they grow at constant rates, output and capital grow at the same rate as well. Using (1.2.1), we have

$$\frac{\dot{K}(t)}{K(t)} = s \cdot \frac{Y(t)}{K(t)} - \delta := c_0, \tag{1.3.1}$$

where c_0 is merely a constant that we set to follow the assumption "all variables grow at constant rates." Hence, working on the second equation in (1.3.1), we have

$$\frac{Y(t)}{K(t)} = \frac{c_0 + \delta}{s}.$$

Log-differentiating the equation above, since the right-hand side of the equation is a constant, we obtain

$$g_Y(t) - g_K(t) = 0. \tag{1.3.2}$$

We first set variables to be per unit of efficient labor,

$$k(t) := \frac{K(t)}{A(t)L(t)} \quad \text{and} \quad y(t) := \frac{Y(t)}{A(t)L(t)}.$$

Using constant returns to scale, we have

$$\begin{aligned} y(t) &= \frac{Y(t)}{A(t)L(t)} \\ &= \frac{F(K(t), A(t)L(t))}{A(t)L(t)} \\ &= F(k(t), 1) \\ &:= f(k(t)), \end{aligned} \tag{1.3.3}$$

where (1.3.3) is called the *intensive form.*

Out goal is to characterize the evolution of $k(t)$. Since

$$k(t) = \frac{K(t)}{A(t)L(t)},$$

we log-differentiate and have

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)},$$

where

$$\frac{\dot{A}(t)}{A(t)} = g \quad \text{and} \quad \frac{\dot{L}(t)}{L(t)} = n$$

by assumption. Using (1.2.1), we further have

$$\frac{\dot{k}(t)}{k(t)} = \frac{s \cdot Y(t) - \delta \cdot K(t)}{K(t)} - g - n$$

$$= s \cdot \frac{y(t)}{k(t)} - \delta - g - n.$$

Therefore, we have the *law of motion of capital per unit of efficient labor,*

$$\dot{k}(t) = s \cdot y(t) - (\delta + g + n) \cdot k(t). \tag{1.3.4}$$

The rate of change in $k(t)$ is the difference of two terms:

1. how much the economy actually saves and
2. how much the economy needs to save to keep $k(t)$ constant.

The economy needs to save *more* in order to maintain $k(t)$ if the following happens:

1. the greater is the rise in labor productivity (higher g), since more productive labor needs to be endowed with more capital,
2. the more rapid is population growth (higher n), since more workers need to be endowed with capital, or
3. the more quickly the existing capital depreciates (higher δ).

1.4 The Steady State

We say that the economy reaches a *steady state* if $\dot{k}(t) = 0$ after some time, so that, from (1.3.4), we have

$$s \cdot f(k(t)) = (\delta + g + n) \cdot k(t).$$

Let k_s denote the steady-state capital per effective labor if there is any. When we are writing down a solution, we have to ask some questions.

1. Does the solution exist?
2. If it exists, is it unique?
3. Does the economy converge to the steady state?

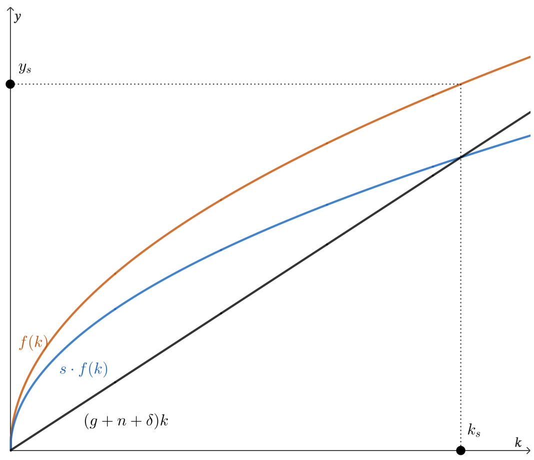

Figure 1.4.1. The steady state at the intersection of $y = s \cdot f(k(t))$ and $y = (g + n + \delta) \cdot k(t)$.

There are, strictly speaking, two steady states in Figure 1.4.1, $(0, 0)$ and (k_s, y_s). However, the trivial steady state at the origin is unstable. Consider (1.2.1), the first derivative (speed) $\dot{k}(t)$ is determined by the difference $s \cdot f(k(t)) - (g + n + \delta) \cdot k(t)$ of two functions. Thus, when $f(k(t)) > (g + n + \delta) \cdot k(t)$, we will have $\dot{k}(t) > 0$; when $f(k(t)) < (g + n + \delta) \cdot k(t)$, we will have $\dot{k}(t) < 0$. Hence, we can see that on $(0, k_s)$, a point will move horizontally to its right since $\dot{k} > 0$; conversely, on (k_s, ∞), a point will move horizontally to its left since $\dot{k} < 0$.

We have to make sure that the non-trivial steady state exists. If the function f satisfy the *Inada conditions,*

$$\lim_{k \to 0} f'(k) = \infty \qquad \text{and} \qquad \lim_{k \to \infty} f'(k) = 0,$$

then the non-trivial steady state must exist. The first equation ensures that the slope near 0 must be greater than $(g + n + \delta)$; the second equation ensures that the slope of the function f must be less than $(g + n + \delta)$, so that there will be an intersection.

We will have **only one** non-trivial solution since f is concave (due to diminishing MP_K). Most importantly, the economy converges to the non-trivial solution.

1.5 Convergence to the Steady State

If the initial state of capital is less than the steady state, capital will accumulate.

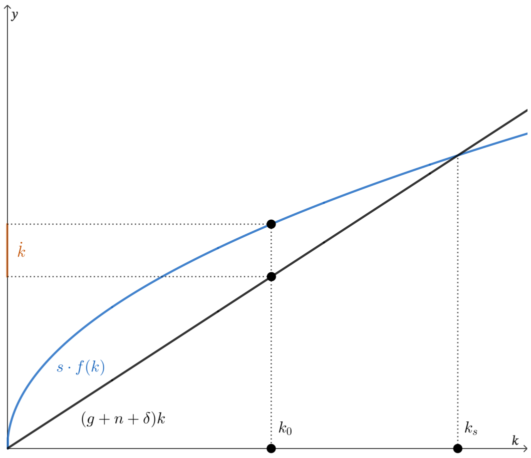

Figure 1.5.1. Accumulation of capital

More importantly, the growth will slow down as the economy gets closer to k_s from the left.

There must be some impact of the change in one of the parameters, s, g, n, or δ. We have two conceptually distinct but closely related questions.

1. (*Comparative statics*) How would the steady state **shift**?
2. (*Transitional dynamics*) How the economy will **move** there?

We will focus on the change in the saving rate s. The saving rate is the result of the households' choice, and it may be affected by the government policy.

Comparative Statics

Suppose the economy is initially at the steady state k_s. As some time t_0, saving rate permanently increases from s_0 to $\tilde{s}$. The investment $s \cdot Y(t)$ exceeds break-even investment, this suggests the increment of k. Hence, the economy moves to the new steady state with higher k and y, say at $(\tilde{k_s}, \tilde{y_s})$. Evidently, steady-state output per efficient labor increases. See Figure 1.5.2. However, does the growth rate become permanently larger? The answer is negative. Once the economy reaches the new steady state, capital per capita k and output per capita y still grow at a rate g.

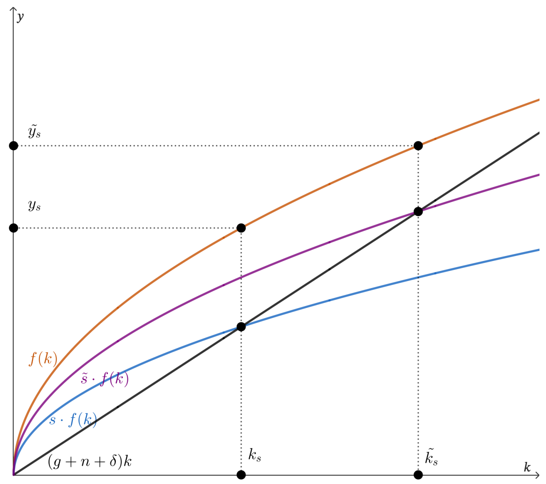

Figure 1.5.2. The rise in s increases k_s and y_s.

Transitional Dynamics (Illustrations)

The movement of capital will take time, but the movement of saving rate is immediate.

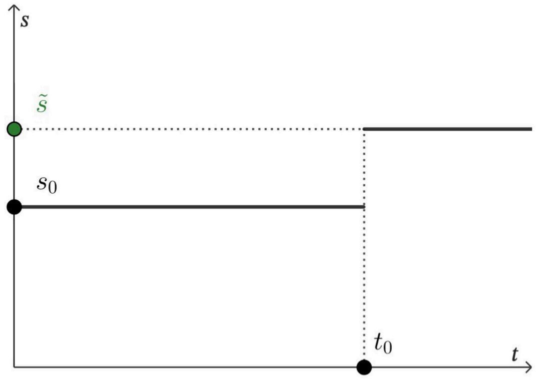

Figure 1.5.3. The time path for the saving rate s.

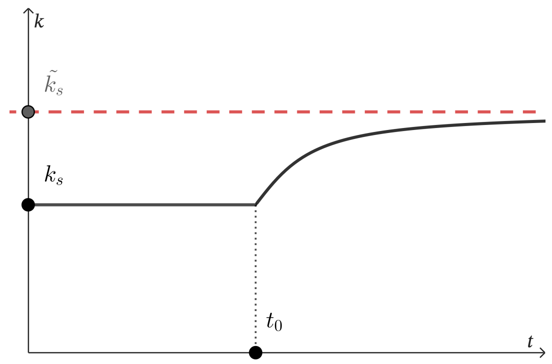

Figure 1.5.4. The time path for capital per efficient labor k.

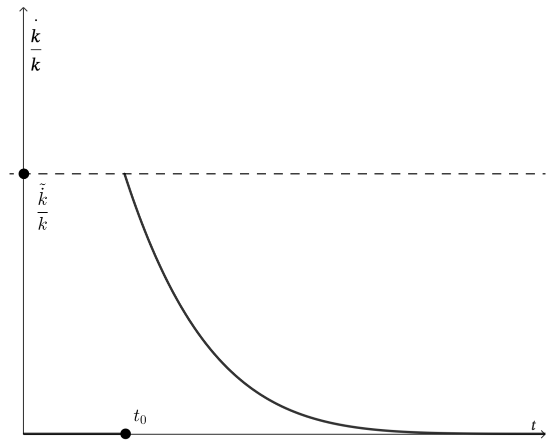

Figure 1.5.5. The time path for the growth rate of k.

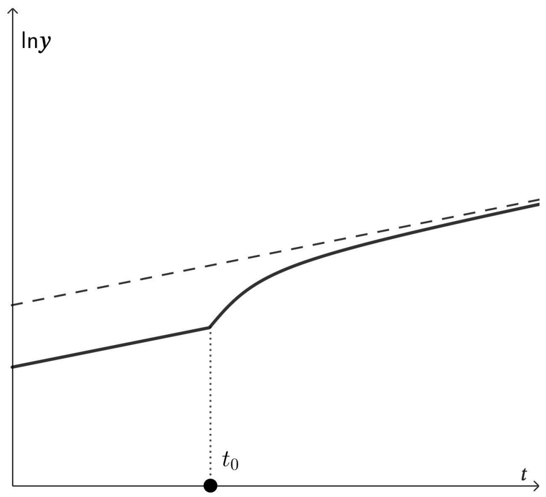

Figure 1.5.6. The time path for the logarithm of output per capita $\ln y$.

1.6 The Golden Rule

In terms of welfare, what matters is per-capita consumption rather than per-capita output. How can we achieve the maximum consumption per efficient labor in the steady state by adjusting the saving rate? Evidently, it should neither be too high nor too low. (In particular, we have $c = 0$ when either $s = 0$ or $s = 1$.) There is an inverse U-shaped relationship between per-capita consumption c and saving rate s.

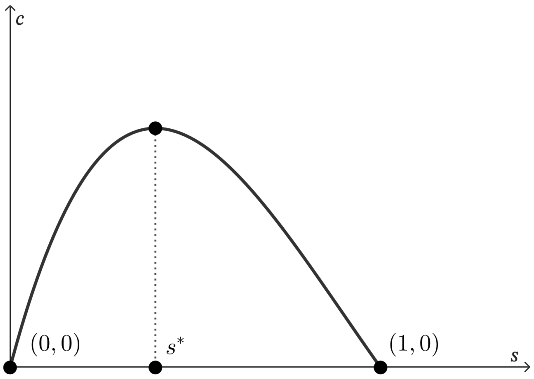

Figure 1.6.1. An illustration about the saving rate s^* maximizes c in the steady state.

Analytically, the goal is to find

$$\max_{k,s}(1-s)\cdot f(k) \tag{1.6.1}$$

under the constraint

$$s\cdot f(k)=(g+n+\delta)\cdot k. \tag{1.6.2}$$

Plug (1.6.2) into the function in (1.6.1), we just need to maximize the function

$$\tilde{f}(k)=f(k)-(g+n+\delta)\cdot k.$$

We look for a point k such that $\tilde{f}'(k)=0$. It is clear that the point k^* such that $f'(k^*)=g+n+\delta$ is what we are looking for. Hence, the *golden-rule saving rate* is found as

$$s^*=\frac{(g+n+\delta)k^*}{f(k^*)}.$$

See Figure 1.6.2 for visualization.

Depending on whether the economy initially operated below or above the golden-rule saving rate, the rise in the saving rate s may either **increase** or **reduce** the steady-state consumption per efficient labor.

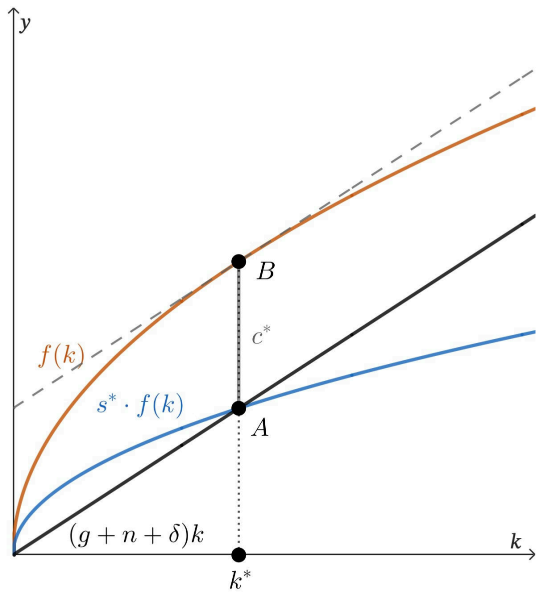

Figure 1.6.2. The special k^* maximizes the distance between $y = f(k)$ and $y = (g+n+\delta) \cdot k$.

The immediate impact of a rise in s is **always** a drop in c. However, as the economy moves to the new steady state with higher k_s, the consumption $c(t) = (1-s) \cdot f(k(t))$ gradually increases.

If initially the economy saves too much, then the reduction in s will increase c both in the short run and in the long run. When $s > s^*$, the economy is called *dynamically inefficient.*

When the saving rate s increases until a value $s_0 < s^*$, there will be an immediate decrease in consumption (from originally c_s to c'), but the consumption will gradually accumulate to a higher level ($\tilde{c_s}$).

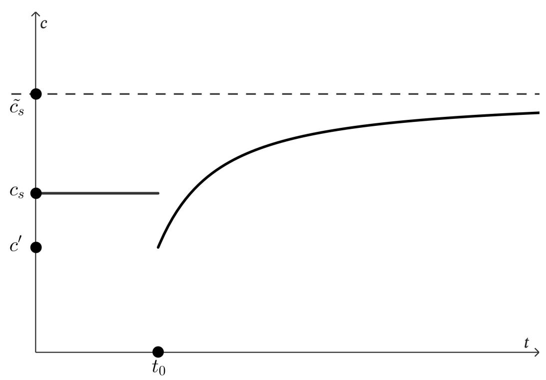

Figure 1.6.3. The time path for c when s increases and $s_0 < s^*$.

On the contrary, when the saving rate s increases until a value $s_0 > s^*$, there will be an immediate decrease in consumption (from originally c_s to c') as well, but the consumption will converge to a lower level ($\tilde{c_s}$).

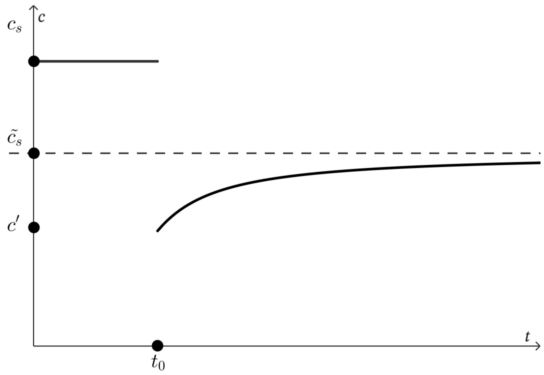

Figure 1.6.4. The time path for c when s increases and $s_0 > s^*$.

1.7 Conclusion

This model replicates all the first five Kaldor facts, but it is not at ease with the sixth Kaldor fact except if we invoke different productivity growth patterns across countries. It is indeed a good guide to time series, but a bad guide to cross-section since technology (non-rival) should not stop at national borders.

There are ways to enrich the model, including endogenizing the growth rate g (the endogenous growth theory, the next section) and endogenizing the saving rate s by introducing households' intertemporal optimization (the optimal growth model or the Ramsey–Cass–Koopmans model).

2 The Endogenous Growth Theory

The Solow model successfully theorize balanced growth by **exogenous** (to be a parameter in the model) and **steady** technical change, i.e., $g_A(t) = g$. The increment in A might come from innovation by chance (e.g., tarte Tatin) or purposeful innovation. Moreover, innovation carries a lot of economic value: higher productivity means higher GDP per capita. Knowledge (technology) A is a club good, which is non-rival and partly excludable. The use of knowledge by on does not preclude its use by anyone else, and those who did not contribute to its accumulation can still benefit from it.

The endogenous growth theory starts, on the contrary to the Solow model, with the idea of endogenizing productivity growth $g_A(t)$. There are many models, and we will cover

1. learn-by-doing, which states that technical change is a side-effect of some economic activities, and
2. investment in research and development, which is more often used nowadays.

2.1 The Learn-by-Doing Model

As we stated previously, this model consider technical change not a result of deliberate efforts but a **side-effect** of some economic activities.

We assume that all inputs are now engages in goods production with the Cobb-Douglas production function

$$Y(t) = (K(t))^{\alpha}(A(t)L(t))^{1-\alpha}. \tag{2.1.1}$$

In addition, the stock of knowledge

$$A(t) = B \cdot (K(t))^{\phi}, \tag{2.1.2}$$

is a function of the stock of capital, where B and ϕ are positive constants. Plug (2.1.2) into (2.1.1), we have

$$Y(t) = B^{1-\alpha}(K(t))^{\alpha+\phi\cdot(1-\alpha)}(L(t))^{1-\alpha}.$$

We assume $\delta = 0$ to simplify mathematical analysis. Thus, the growth

$$\dot{K}(t) = s \cdot Y(t) \tag{2.1.3}$$

of $K(t)$ becomes some fraction of the total production, and we obtain the dynamics of $K(t)$

$$\dot{K}(t) = s \cdot B^{1-\alpha}(K(t))^{\alpha+\phi\cdot(1-\alpha)}(L(t))^{1-\alpha}. \tag{2.1.4}$$

We further consider a *knife-edge condition,* which is a specific case, when $n = 0$ and $\phi = 1$. Then, $L(t)$ becomes a constant L, and the production function becomes

$$\begin{aligned} Y(t) &= B^{1-\alpha}K(t)L^{1-\alpha} \\ &= (BL)^{1-\alpha}K(t), \end{aligned}$$

which is linear in $K(t)$. Letting b denotes the constant $(BL)^{1-\alpha}$ under this knife-edge condition, we will see that, from (2.1.3), the growth rate of capital becomes

$$\frac{\dot{K}(t)}{K(t)} = sb.$$

This suggests that saving rate affects long-run growth.

2.2 The Model of Investment in Research and Development

In Section 2.1, we considered the accumulation of knowledge is a side-effect of economic activities. In this model, we instead consider the accumulation of knowledge the result of purposeful activities on behalf of innovative firms. Thus, it is reasonable to assume that there is a fraction a_L of labor is employed in research and development, and the rest $(1 - a_L)$ of labor is employed in production.

A Model without Capital

For simplicity, we first assume that there is no capital. We will put capital back later. The production is given by

$$Y(t) = (1 - a_L) \cdot A(t)L(t). \tag{2.2.1}$$

We assume that labor force grows at a constant rate

$$g_L(t) = n > 0.$$

Notice that from (2.2.1), log-differentiating yields

$$g_Y(t) = g_A(t) + n;$$

dividing both sides by $L(t)$ and then log-differentiating yield

$$g_{\frac{Y}{L}}(t) = g_A(t).$$

We assume the dynamics of knowledge to be

$$\dot{A}(t) = B \cdot (a_L \cdot L(t))^{\gamma} (A(t))^{\theta}, \tag{2.2.2}$$

where B, γ, and θ are positive constant. The power γ measures the extent of decreasing marginal returns to labor in research and development. The dynamics has the "standing on the shoulders of giants" effect when $\theta > 1$ and has the "fishing-out" effect when $\theta < 1$; we can imagine that earlier fundamental research facilitates subsequent discoveries when $\theta > 1$ and that obvious discoveries are made first, then subsequent ideas become increasingly difficult to discover when $\theta < 1$. The later is more to be assumed nowadays.

From (2.2.2), we divide both sides by $A(t)$ and obtain

$$\frac{\dot{A}(t)}{A(t)} = B \cdot (a_L \cdot L(t))^{\gamma} (A(t))^{\theta - 1}. \tag{2.2.3}$$

We use the substitution $g_{A(t)} = \dfrac{\dot{A}(t)}{A(t)}$ to show a magic trick. We log-differentiate (2.2.3) and have

$$\begin{aligned} \frac{\dot{g_A}(t)}{g_A(t)} &= \gamma \cdot \frac{\dot{L}(t)}{L(t)} + (\theta - 1) \cdot \frac{\dot{A}(t)}{A(t)} \\ &= \gamma n + (\theta - 1) \cdot g_A(t) \\ \dot{g_A}(t) &= \gamma n \cdot g_A(t) + (\theta - 1) \cdot (g_A(t))^2, \end{aligned} \tag{2.2.4}$$

which is a quadratic expression of $g_A(t)$. Thus, the dynamics of $g_A(t)$ can be represented by a phase diagram (a variable X on the x-axis and its derivative with respect to time $\dot{X}$ on the y-axis). The convergence crucially depends on the value of θ. We first solve (2.2.4). If $\theta \neq 1$, the solutions to the quadratic equation

$$\gamma n \cdot g_A(t) + (\theta - 1) \cdot (g_A(t))^2 = 0$$

are $g_A(t) = 0$ or $g_A(t) = \dfrac{\gamma n}{1-\theta}$.

When $\theta < 1$, the graph of $\dot{g_A}(t)$ will be a parabola with a positive leading coefficient; the growth of knowledge will converge to a certain level ${g_A}^*$.

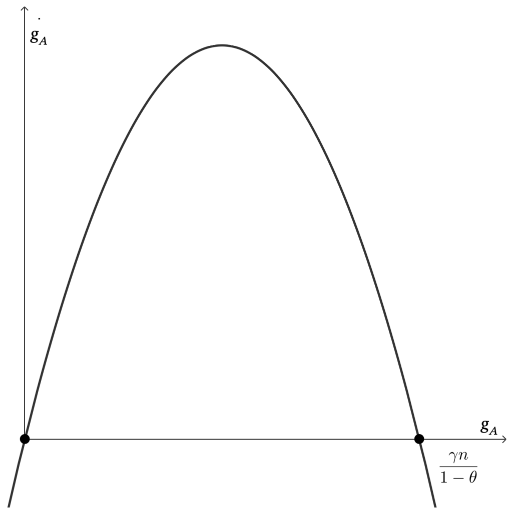

Figure 2.2.1. When $\theta < 1$, the growth $g_A(t)$ of knowledge converges to ${g_A}^*$.

When $\theta > 1$, the graph of $\dot{g_A}(t)$ will be a parabola with a negative leading coefficient; the growth is explosive.

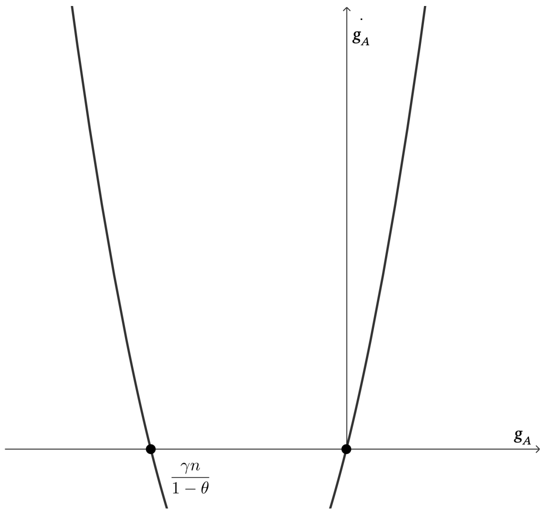

Figure 2.2.2. When $\theta > 1$, the growth $g_A(t)$ of knowledge diverges ($g_A(t) \to \infty$ as $t \to \infty$).

When $\theta = 1$, the graph of $\dot{g_A}(t)$ will be a line with slope γn. Since γ is a positive constant, the growth of knowledge stays constant if and only if $n = 0$.

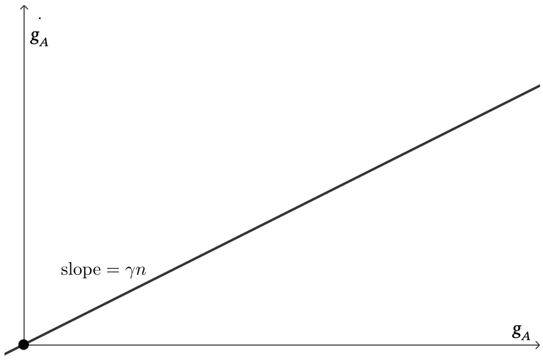

Figure 2.2.3. When $\theta = 1$, the convergence of growth rate of knowledge depends on n.

A Model with Capital

We use the Cobb-Douglas production function to introduce capital, assuming

$$Y(t) = ((1 - a_K) \cdot K(t))^{\alpha}((1 - a_L) \cdot A(t)L(t))^{1-\alpha}. \tag{2.2.5}$$

Do compare (2.2.5) with (2.2.1). We keep the assumption that there is no depreciation, and hence

$$\dot{K}(t) = s \cdot Y(t). \tag{2.2.6}$$

We also assume the dynamics of knowledge to be

$$\dot{A}(t) = B \cdot (a_K \cdot K(t))^{\beta}(a_L \cdot L(t))^{\gamma}(A(t))^{\theta}. \tag{2.2.7}$$

Do compare (2.2.7) with (2.2.2). You will find out that the model without capital is a special version with powers α and β being zero.

We now have two endogenous stock variables, $A(t)$ and $K(t)$. From (2.2.6), we plug in (2.2.5) and have

$$\frac{\dot{K}(t)}{K(t)} = s \cdot (1-a_K)^{\alpha} \left((1-a_L) \cdot \frac{A(t)L(t)}{K(t)} \right)^{1-\alpha}$$

$$= \underbrace{s \cdot (1-a_K)^{\alpha} (1-a_L)^{1-\alpha}}_{c_K} \left(\frac{A(t)L(t)}{K(t)} \right)^{1-\alpha}. \tag{2.2.8}$$

From (2.2.7), we plug in (2.2.5) and have

$$\frac{\dot{A}(t)}{A(t)} = \underbrace{B{a_k}^{\beta}{a_L}^{\gamma}}_{c_A} \cdot (K(t))^{\beta} (L(t))^{\gamma} (A(t))^{\theta-1}. \tag{2.2.9}$$

Log-differentiating (2.2.8) and (2.2.9), since c_K and c_A are constants, we have

$$\frac{\dot{g_K}(t)}{g_K(t)} = (1-\alpha) \cdot (g_A(t) + g_L(t) - g_K(t))$$

$$= (1-\alpha) \cdot (g_A(t) + n - g_K(t)), \tag{2.2.10}$$

and

$$\frac{\dot{g_A}(t)}{g_A(t)} = \beta \cdot g_K(t) + \gamma \cdot g_L(t) + (\theta - 1) \cdot g_A(t)$$

$$= \beta \cdot g_K(t) + \gamma n + (\theta - 1) \cdot g_A(t). \tag{2.2.11}$$

On the balanced growth path, by definition, $A(t)$ and $K(t)$ grow at constant rates, that means $g_A(t)$ and $g_K(t)$ are constants. Hence, $\dot{g_A}(t) = \dot{g_K}(t) = 0$. Therefore, in the (g_A, g_K) space, we can draw the graph of $\dot{g_K}(t) = 0$,

$$g_K = n + g_A \tag{2.2.12}$$

and the graph of $\dot{g_A}(t) = 0$,

$$g_K = -\frac{\gamma}{\beta} \cdot n + \frac{1-\theta}{\beta} \cdot g_A. \tag{2.2.13}$$

In order to make (2.2.12) and (2.2.13) to intersect in the first quadrant, we need to assume $\dfrac{1-\theta}{\beta} > 1$, which means decreasing returns to $A(t)$ and $K(t)$ in research and development. This assumption implies $\theta < 1$ immediately. Thus, the following about convergence is natural as we discussed previously.

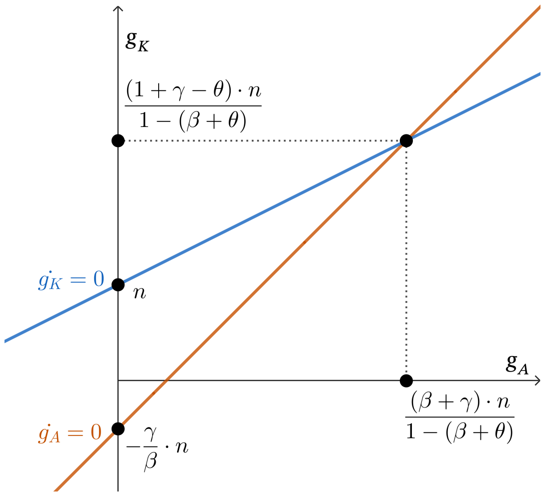

Figure 2.2.4. The intersection will be in the first quadrant under the assumption.

The growth rate will converge to the intersection. We cannot analytically prove this result, but we can use figures with expressions to represent the idea. We will see combining Figure 2.2.5 and Figure 2.2.6 do the work with any point in each section (divided by two unparallel lines, 4 sections in total) tends to the intersection $({g_A}^*, {g_K}^*) := \left(\dfrac{(1+\gamma-\theta)\cdot n}{1-(\beta+\theta)}, \dfrac{(\beta+\gamma)\cdot n}{1-(\beta+\theta)}\right)$.

Please see Figure 2.2.5 and Figure 2.2.6 along with the following expressions. We use algebra to simplify (2.2.10) and (2.2.11). The strategy that we fix one but another is because the quadratic term is thorny, and we hence fix it.

For (2.2.10), we fix a $g_{K_0} > n$, i.e., we focus on the horizontal line $g_K(t) = g_{K_0}$, and have

$$
\begin{aligned}
\dot{g_K}(t) &= ((1-\alpha)\cdot\left(g_A(t)+n-g_{K_0}\right)\cdot g_{K_0} \\
&= \underbrace{(1-\alpha)\cdot g_{K_0}}_{\mathcal{K}>0}\cdot\left(g_A+\underbrace{\left(n-g_{K_0}\right)}_{-\mathcal{N}<0}\right) \\
&= \mathcal{K}\cdot(g_A(t)-\mathcal{N})
\end{aligned} \tag{2.2.14}
$$

The constant $\mathcal{K}>0$ since $g_{K_0}>n$ and $\alpha\in(0,1)$; the constant $\mathcal{N}>0$ follows by assumption $g_{K_0}>n$.

For (2.2.11), we fix a $g_{A_0}>\dfrac{\gamma n}{1-\theta}$, i.e., we focus on the vertical line $g_A(t)=g_{A_0}$, and have

$$
\begin{aligned}
\dot{g_A}(t) &= \left(\beta\cdot g_K(t)+\gamma n+(\theta-1)\cdot g_{A_0}\right)\cdot g_{A_0} \\
&= \underbrace{(\theta-1)\cdot\left(g_{A_0}\right)^2+\gamma n\cdot g_{A_0}}_{\text{constant }-\Theta<0}+\underbrace{\beta\cdot g_{A_0}}_{\Gamma>0}\cdot g_K(t) \\
&= \Gamma\cdot g_K(t)-\Theta
\end{aligned} \tag{2.2.15}
$$

The constant $-\Theta<0$ since our assumption $g_{A_0}>\dfrac{\gamma n}{1-\theta}$ implies

$$0>\gamma n+(\theta-1)g_{A_0}$$

and thus

$$
\begin{aligned}
(\theta-1)\cdot\left(g_{A_0}\right)^2+\gamma n\cdot g_{A_0} &= g_{A_0}\cdot\left(\gamma n+(\theta-1)g_{A_0}\right) \\
&< 0.
\end{aligned}
$$

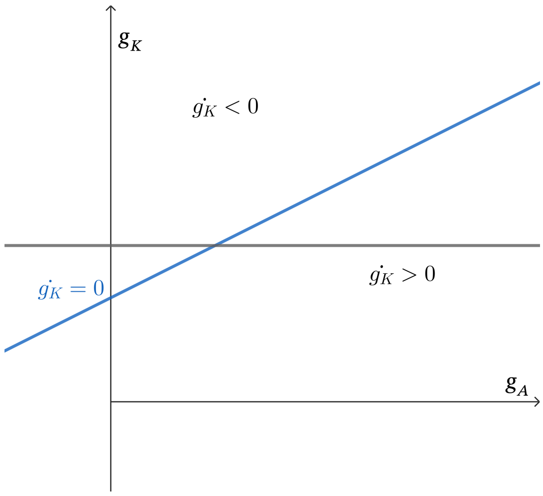

Figure 2.2.5. The intersection is at $\left(\mathcal{N}, g_{K_0}\right)$. By (2.2.14), if $g_A < \mathcal{N}$, then $\dot{g_K} < 0$; if $g_A > \mathcal{N}$, then $\dot{g_K} > 0$.

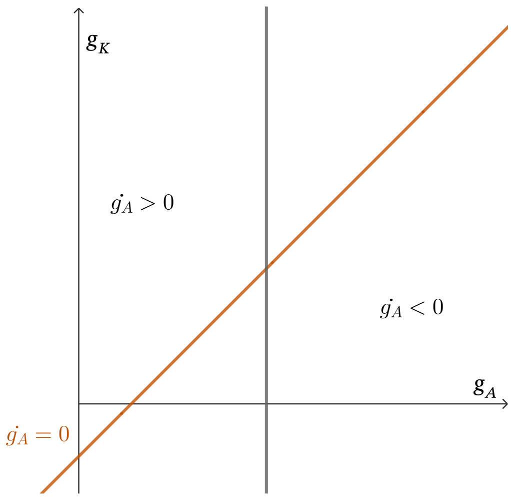

Figure 2.2.6. The intersection is at $\left(g_{A_0}, \dfrac{\Theta}{\Gamma}\right)$. By (2.2.15), if $g_K > \dfrac{\Theta}{\Gamma}$, then $\dot{g_A} > 0$; if $g_K < \dfrac{\Theta}{\Gamma}$, then $\dot{g_A} < 0$.

2.3 The Solow Model with Human Capital

The instructor Dr. SENOUCI did not cover this part in the second lecture. The following content is listed for future reference.

This model features **individual-level productivity** growth. Each individual makes oneself more productive by accumulating human capital.

Consider the Cobb-Douglas production function

$$Y(t) = (K(t))^{\alpha}(A(t)H(t))^{1-\alpha},$$

where $H(t)$ is *worker's productive services* or *effective labor.* That is, $H(t)$ stands for the union (sum) raw labor $L(t)$ and human capital HK. Each worker's HK depends on the years of education E with following relation

$$H(t) = e^{\phi E}L(t),$$

where ϕ is a positive constant. Let $k := \frac{K}{AH}$ and $y = \frac{Y}{AH}$, we will have dynamics

$$\dot{k} = sy - (n + g + \delta)k,$$

which is identical to (1.3.4).

Impact of a Change in the Years of Education

Assume each person lives for T years, studying for the first E years of one's life and working for the remaining $T - E$ years. How does an increase in E affect output per person?

Let $B(t)$ be the number people born at date t. We assume that the birth rate is with a growth rate n. That is, the ratio of the number $B(t-\tau)$ of people born at date $(t-\tau)$ to the number $B(t)$ of people born at date t is

$$B(t) = B(t-\tau) \cdot e^{n\tau}.$$

The total population $N(t)$ at date t is equal to those born in $[t-T, t]$, namely

$$\begin{aligned} N(t) &= \int_0^T B(t-\tau)\,\mathrm{d}\tau \\ &= B(t)\int_0^T e^{-n\tau}\,\mathrm{d}\tau \\ &= \frac{1-e^{-nT}}{n} \cdot B(t). \end{aligned} \tag{2.3.1}$$

Likewise, the number $L(t)$ of workers at date t is equal to those who born in $[t-T, t-E]$, namely

$$
\begin{aligned}
L(t) &= \int_E^T B(t-\tau)\,\mathrm{d}\tau \\
&= \frac{e^{-nE}-e^{-nT}}{n}\cdot B(t).
\end{aligned}
\tag{2.3.2}
$$

Therefore, from (2.3.1) and (2.3.2), the output per person at date t equals

$$
\begin{aligned}
\frac{Y(t)}{N(t)} &= \frac{Y(t)}{A(t)H(t)}\cdot\frac{A(t)H(t)}{N(t)} \\
&= y(t)\cdot\frac{A(t)e^{\phi E}L(t)}{N(t)} \\
&= A(t)y(t)\cdot\frac{e^{\phi E}\cdot\left(e^{-nE}-e^{-nT}\right)}{1-e^{-nT}}
\end{aligned}
\tag{2.3.3}
$$

The Golden-Rule Level of Education

An increase in E has ambiguous effect on the steady-state $\frac{Y(t)}{N(t)}$. It will make workers become more productive, i.e., $H(t)$ increases. However, it will also make workers work for less time, i.e., $T-E$ falls.

Therefore, there is the golden-rule level E^* of education that maximizes $\frac{Y(t)}{N(t)}$. To simplify the differentiated function, we differentiate $\frac{1-e^{-nT}}{A(t)y(t)}\cdot\frac{Y(t)}{N(t)}$ and have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{1-e^{-nT}}{A(t)y(t)}\cdot\frac{Y(t)}{N(t)}\right) &= \frac{\mathrm{d}}{\mathrm{d}E}\left(e^{\phi E}\cdot\left(e^{-nE}-e^{-nT}\right)\right) \\
&= e^{\phi E}\cdot\left(\phi\cdot e^{-nE}-\phi\cdot e^{-nT}-n\cdot e^{-nE}\right).
\end{aligned}
$$

Since $e^{\phi E}>0$, we set the first derivative to zero, eliminate $e^{\phi E}$, and have

$$\phi \cdot e^{-nE^*} - \phi \cdot e^{-nT} - n \cdot e^{-nE^*} = 0$$

$$e^{-nE^*} = \frac{\phi}{\phi - n} \cdot e^{-nT}$$

$$-nE^* = \ln \frac{\phi}{\phi - n} - nT$$

$$E^* = T - \frac{1}{n} \ln \frac{\phi}{\phi - n}.$$

2.4 Conclusion

Technological change can be theorized as a side-effect of economic activity or as an investment. We covered a learning-by-doing model, which depicts productivity growth as a side-effect of some rational economic activities, and two models with investment in research and development, which employ specialized workers in the lecture; a human-capital-based model is also referred.

The most up-to-date models involve investment in research and development or are based on human capital—naturally, these models are more complex than what we have covered so far. They feature individually rational decision-making. As Romer explains in 1986 and 1990, firms invest in research and development in the hope of outpacing their competitors, and in pure and perfect competition, no costly research and development is undertaken due to the high costs. Lucas, in 1988, also notes that individuals acquire human capital to increase future labor earnings, although this comes at the expense of reduced earnings today. These models are complex, and we are currently unable to treat them extensively.

3 Theory of Consumption

We have taken minimal assumptions about consumption; we assumed a constant saving rate s with consumption

$$C_t = (1 - s)Y_t$$

and investment

$$I_t = sY_t.$$

We now lay the foundations of the pure neoclassical theory of consumption.

We are interested in the rational allocation of income to consumption **at different rates**. The reason we are doing this is that we want to microfound the macroeconomic demand for consumption in dynamic models.

We want a theory that makes sense of the "Kuznets consumption puzzle:"

- over the long run, consumption per capita and output per capita have parallel trends, i.e., C/Y is nearly constant, and
- in the short-run, consumption per capita is positively correlated with output per capita, but less volatile; the change in per-capita disposable income is more than real per-capita consumption.

3.1 The Intertemporal Utility Function

Some consumer lives for $T + 1$ periods with $T \geq 1$. We let integers $0, 1, ..., T$ denote those periods. In addition, we use the notation $[0, T]$ to indicate the set $\{0, 1, ..., T\}$.

The consumer gets utility from consuming C_t at the end of each period t. The utility function U is written as

$$U(C_0, ..., C_T) = \sum_{t=0}^{T} \beta^t \cdot u(C_t), \tag{3.1.1}$$

where $\beta \in (0, 1)$ is called the *subjective discount factor,* and *instantaneous utility function* $u : \mathbb{R}^+ \to \mathbb{R}$ is twice continuously differentiable, strictly increasing, and strictly concave.

The form of U in (3.1.1) is the only function that satisfies *time consistency.* In addition, $\beta \in (0, 1)$ ensures that the agent puts more weight on the present than on the future: the future is discounted at a constant rate β. The lower β, the more impatient the agent is.

Moreover, strictly concave function u will make the consumer consume at all periods, as indifference curves are convex.

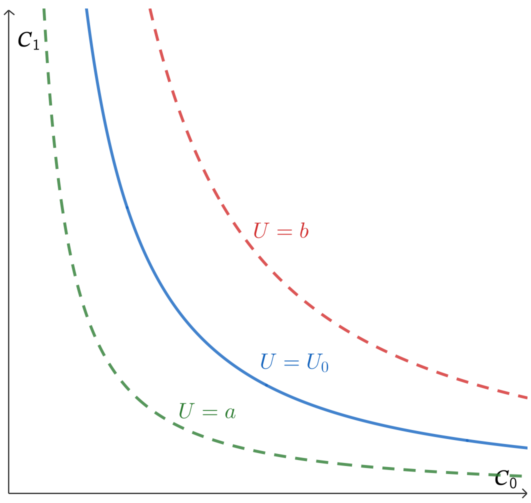

Figure 3.1.1. Indifference curves for $T = 1$ with $a < U_0 < b$.

For the instantaneous utility function u, there is a common function called the *constant intertemporal elasticity of substitution* (CIES) utility in the form of

$$u(C) = \frac{C^{1-\theta} - 1}{1 - \theta}, \tag{3.1.2}$$

where $\theta \in \mathbb{R}^+ \setminus \{1\}$. The constant θ is the intertemporal elasticity of substitution as well as the inverse of the relative risk aversion. Note that the log utility, also known as the Cobb-Douglas utility,

$$u(C) = \ln C$$

is the case for (3.1.2) as $\theta \to 1$.

Elasticity measures how responsive the quantity demanded or supplied of a good is to changes in its price or other factors. If the price elasticity of demand for a good is high, it means that a small change in price causes a large change in the quantity demanded, often because consumers can easily switch to alternatives. On the contrary, if the elasticity is low, demand is relatively stable, meaning that changes in price have little effect on the quantity demanded, often because the good is a necessity or lacks close substitutes.

3.2 The Intertemporal Budget Constraint

The goal of the agent is to maximize the intertemporal utility function U.

Assume that the agent earns an exogenous *stream of income* $(Y_t)_{t=0}^{T}$ with $Y_t \geq 0$ for all t and some are positive. Suppose the *real interest rate* between two consecutive periods is a constant $r > -1$.

The agent can freely borrow and lend at real rate r. It means that for any $t \in \{0, 1, ..., t-1\}$, the agent can trade one unit of C_t for $(1+r)$ units of C_{t+1}, and vice versa. Thus, we can know that $(1+r)$ is the price of current consumption in terms of the consumption in the next period. In general, the interest rate r_t may vary from one date to another, but we will abstract from that complication.

Let $(S_t)_{t=0}^{T}$ be the amount of savings at each period with

$$S_t = Y_t - C_t.$$

The amount of saving can be positive or negative. Let A_t be the amount of savings stock accumulated at the end of each period. Hence, we have

$$\begin{aligned}
A_0 &= S_0, \\
A_1 &= (1+r)A_0 + S_1, \\
A_2 &= (1+r)A_1 + S_2, \\
&\vdots \\
A_T &= (1+r)A_{T-1} + S_T.
\end{aligned}$$

Remember that r is the real interest rate. Hence, to save $S \geq 0$ at some period brings $(1+r)S$ extra resources one period after, and to incur debt $S \leq 0$ at some period leads to $(1+r)S$ less resources one period after.

We assume that the agent will not die in debt, and hence $A_T \geq 0$. In addition, the agent will try to consume as much as possible at date T, and hence $A_T \leq 0$. These imply $A_T = 0$. Therefore, at the end of period t, we multiply $\frac{1}{(1+r)^t}$ on both sides of each equation and have

$$\begin{aligned}
A_0 &= S_0, \\
\frac{1}{1+r} \cdot A_1 &= A_0 + \frac{1}{1+r} \cdot S_1, \\
\frac{1}{(1+r)^2} \cdot A_2 &= \frac{1}{1+r} \cdot A_1 + \frac{1}{(1+r)^2} \cdot S_2, \\
&\vdots \\
0 &= \frac{1}{(1+r)^{T-1}} \cdot A_{T-1} + \frac{1}{(1+r)^T} \cdot S_T.
\end{aligned}$$

Summing both sides yields

$$0 = \sum_{t=0}^{T} \frac{S_t}{(1+r)^t}.$$

Since $S_t = Y_t - C_t$, we have the *intertemporal budge constraint* (IBC)

$$\sum_{t=0}^{T} \frac{C_t}{(1+r)^t} = \sum_{t=0}^{T} \frac{Y_t}{(1+r)^t}. \tag{3.2.1}$$

Remember we have derived (3.2.1) by using that the agent dies with 0 wealth. This entails this relationship between the consumption and the income streams.

The left-hand side of (3.2.1) is the *present value* (PV) of the consumption stream $(C_t)_{t=0}^{T}$. It is the value in terms of date-0 consumption of everything that the agent will consume during one's lifetime. The right-hand side is the present value of the income stream. The value in terms of date-0 good of everything that the agent will earn during one's lifetime.

A consumption path is *feasible* if and only if it satisfies (3.2.1).

3.3 The Fisher Model

Now, let's focus on the case when $T = 1$. That is, an individual needs to allocate one's consumption stream over just two periods: "today" $(t = 0)$ and tomorrow $(t = 1)$. The intertemporal budget constraint is

$$C_0 + \frac{C_1}{1+r} = Y_0 + \frac{Y_1}{1+r}.$$

The point (Y_1, Y_2) is called the *initial endowment.*

Since there are only two periods, the intertemporal utility function becomes

$$U(C_0, C_1) = u(C_0) + \beta \cdot u(C_1).$$

In addition, there is the intertemporal budget constraint

$$C_0 + \frac{C_1}{1+r} = Y_0 + \frac{Y_1}{1+r}. \tag{3.3.1}$$

The relative price of consumption is given by $\dfrac{1}{1/(1+r)} = 1 + r$. By giving up $(1 + r)$ units of future consumption C_2, the individual can obtain 1 unit of current consumption C_1.

Optimization

The individual's optimality condition equates the marginal rate of substitution of C_0 for C_1 to their price ratio

$$\frac{u'(C_0)}{\beta \cdot u'(C_1)} = 1 + r.$$

Another mathematical derivation is that we set the first derivative of the function U under the intertemporal budget constraint (3.3.1) with respect to C_0 to be 0 and have

$$
\begin{aligned}
U'(C_0) &= u'(C_0) + \beta \cdot u'(C_1) \cdot \frac{\mathrm{d}C_1}{\mathrm{d}C_0} \\
&= 0 \\
\implies \quad u'(C_0) &= (1+r) \cdot \beta \cdot u'(C_1).
\end{aligned} \tag{3.3.2}
$$

Note that we have $\dfrac{\mathrm{d}C_1}{\mathrm{d}C_0} = -(1+r)$ from (3.3.1).

We can also utilize the optimality to derive the relation. Say the point $({C_0}^*, {C_1}^*)$ is optimal. If C_0 increases by $\mathrm{d}C$, then C_1 will decrease by $(1+r)\,\mathrm{d}C$ from (3.3.1). Since the point $({C_0}^*, {C_1}^*)$ is optimal, we should have the change of the intertemporal utility function is zero, i.e.,

$$
\begin{aligned}
\mathrm{d}U &= u'({C_0}^*)\,\mathrm{d}C - \beta \cdot u'({C_1}^*) \cdot (1+r)\,\mathrm{d}C \\
&= 0.
\end{aligned} \tag{3.3.3}
$$

If $\mathrm{d}U \neq 0$, then the point $({C_0}^*, {C_1}^*)$ will not be optimal. The last equation in (3.3.3) coincides (3.3.2).

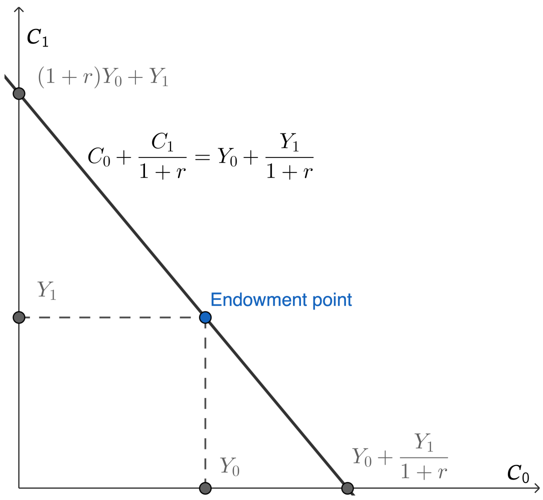

Figure 3.3.1. Consumer's endowment in the Fisher model.

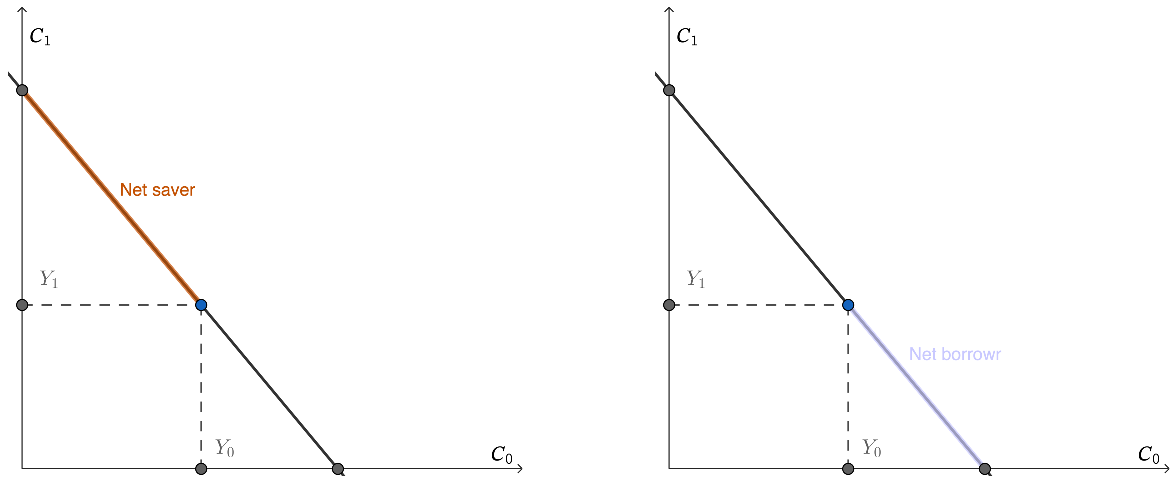

Figure 3.3.2. Consumer's choices correspond to difference status (saver or borrower).

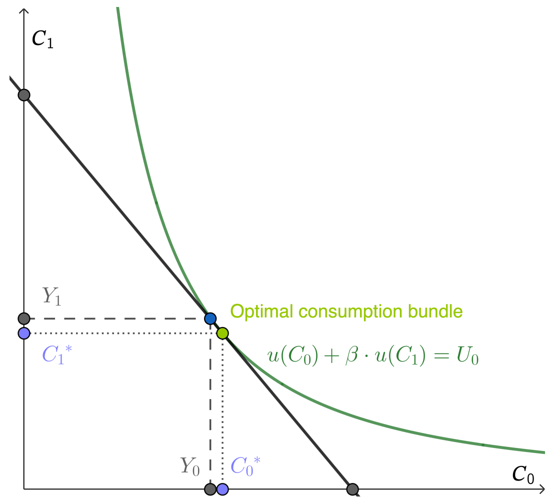

Figure 3.3.3. Consumer's optimum in the Fisher model. The amount ${C_0}^* - Y_0$ is saving S^*.

For given present value of lifetime income $Y_0 + \dfrac{Y_1}{1+r}$, the time pattern of income is not important for consumption, but critical for saving. The individual who is relatively rich now and expects to be poor in the future (high Y_0 and low Y_1) will be a *saver;* the one with a low Y_0 and a high Y_1 will be a *borrower.*

Analytical Solutions

If the instantaneous utility function u is the **Cobb-Douglas** utility function, then the optimal consumption path $({C_0}^*, {C_1}^*)$ must satisfy equations (3.3.1) and (3.3.2). Hence, we should solve

$$
\begin{cases}
{C_0}^* + \dfrac{{C_1}^*}{1+r} &= Y_0 + \dfrac{Y_1}{1+r}; \\[2ex]
\dfrac{1}{{C_0}^*} &= \beta \cdot (1+r) \cdot \dfrac{1}{{C_1}^*},
\end{cases}
$$

$$
\Longrightarrow \quad
\begin{cases}
{C_0}^* + \dfrac{{C_1}^*}{1+r} &= Y_0 + \dfrac{Y_1}{1+r}; \\[2ex]
{C_1}^* &= \beta \cdot (1+r) \cdot {C_0}^*,
\end{cases}
$$

$$
\Longrightarrow \quad
\begin{cases}
{C_0}^* = \dfrac{1}{1+\beta} \cdot \left(Y_0 + \dfrac{Y_1}{1+r}\right); \\[2ex]
{C_1}^* = \dfrac{\beta \cdot (1+r)}{1+\beta} \cdot \left(Y_0 + \dfrac{Y_1}{1+r}\right).
\end{cases}
$$

If the instantaneous utility function u is the **constant intertemporal elasticity in substitution** utility function, then the optimal consumption path $({C_0}^*, {C_1}^*)$ must satisfy equations (3.3.1) and (3.3.2). Hence, we should solve

$$
\begin{cases}
{C_0}^* + \dfrac{{C_1}^*}{1+r} = Y_0 + \dfrac{Y_1}{1+r}; \\[2ex]
({C_0}^*)^{-\theta} = \beta \cdot (1+r) \cdot ({C_1}^*)^{-\theta},
\end{cases}
$$

$$
\Longrightarrow \quad
\begin{cases}
{C_0}^* + \dfrac{{C_1}^*}{1+r} &= Y_0 + \dfrac{Y_1}{1+r}; \\[2ex]
{C_0}^* &= (\beta \cdot (1+r))^{-\frac{1}{\theta}} \cdot {C_1}^*,
\end{cases}
$$

$$
\Longrightarrow \quad
\begin{cases}
{C_0}^* = \dfrac{Y_0 + \dfrac{Y_1}{1+r}}{1 + \dfrac{(\beta \cdot (1+r))^{\frac{1}{\theta}}}{1+r}}; \\[4ex]
{C_1}^* = \dfrac{(\beta \cdot (1+r))^{\frac{1}{\theta}} \cdot \left(Y_0 + \dfrac{Y_1}{1+r}\right)}{1 + \dfrac{(\beta \cdot (1+r))^{\frac{1}{\theta}}}{1+r}}.
\end{cases}
$$

3.4 Ricardian Equivalence

Ricardian equivalence, in plain English, showcases that the timing of taxes has **no** impact on private agents' consumption. We can explain in another way: for a given public expense G, whether G is funded through taxation or deficit does not influence the saving/consumption behavior of private agents.

Suppose the government has to spend G today. There are two options: pay for it with taxes today or with a deficit today. However, a deficit today means higher taxes tomorrow for the private agents. Say the government switch from taxes today (option 1) to taxes tomorrow (option 2). The individuals react by saving today what they would have paid in taxes in option 1, to pay future taxes in option 2 without changing their intertemporal consumption profile, which is the same under option 1 and option 2.

We stick to the two-period model still. The following argument will hold for T periods as well. With taxes, the household's date-t disposable income is $Y_t - T_t$, where T_t is the taxes in date t; hence, the intertemporal budget constraint becomes

$$C_0 + \frac{C_1}{1+r} = (Y_0 - T_0) + \frac{Y_1 - T_1}{1+r}. \tag{3.4.1}$$

The government's budget constraints are

$$G_0 = T_0 + B \qquad \text{and} \qquad G_1 = T_1 - (1+r)B,$$

where B stands for government's borrowing at period 0.

We combine the two equations for the government's budget constraints and have

$$G_1 = T_1 - (1+r) \cdot (G_0 - T_0)$$

$$\Longrightarrow \qquad G_0 + \frac{G_1}{1+r} = T_0 + \frac{T_1}{1+r},$$

which is the government's intertemporal budget constraint.

Assume the government decides to cut current taxes with $\Delta T_0 < 0$ without altering the spending profile, so $\Delta G_0 = \Delta G_1 = 0$. It would have to raise borrowing $\Delta B = -\Delta T_0 > 0$. To repay debt, it would have to raise future taxes

$$\begin{aligned}\Delta T_1 &= (1+r)\Delta B \\ &= -(1+r)\Delta T_0.\end{aligned}$$

Would the consumer's lifetime disposable income change? **No**, it would not.

The right-hand side of (3.4.1) will change to

$$\begin{aligned}(Y_0 - T_0 - \Delta T_0) + \frac{Y_1 - T_1 - \Delta T_1}{1+r} &= (Y_0 - T_0) + \frac{Y_1 - T_1}{1+r} - \Delta T_0 - \frac{\Delta T_1}{1+r} \\ &= (Y_0 - T_0) + \frac{Y_1 - T_1}{1+r} - \Delta T_0 - \frac{-(1+r)\Delta T_0}{1+r} \\ &= (Y_0 - T_0) + \frac{Y_1 - T_1}{1+r},\end{aligned}$$

which did not change at all.

If the budget constraints are unchanged, the optimal intertemporal consumption profile $({C_1}^*, {C_2}^*)$ will not change either. What will change is individual's endowment point and savings.

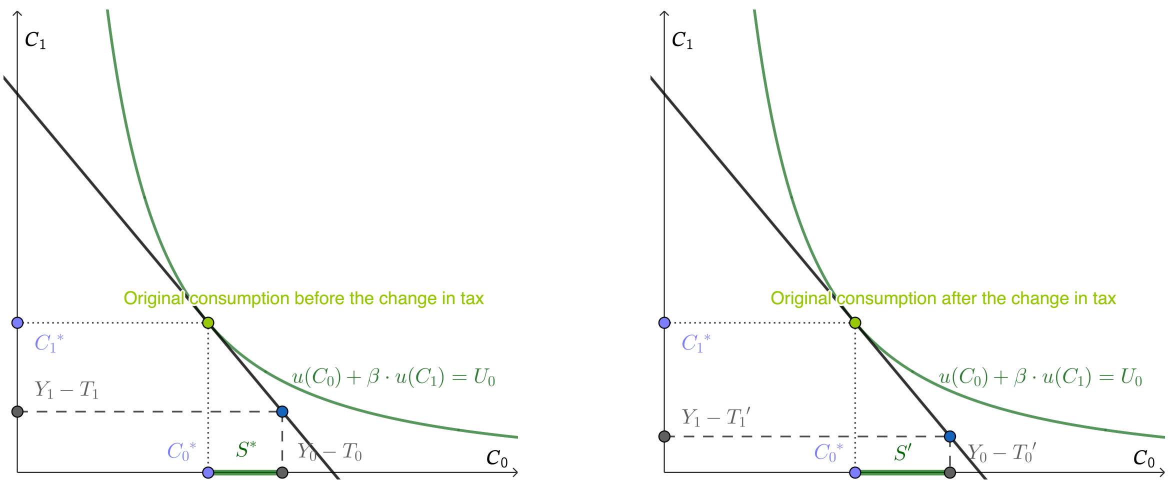

Figure 3.4.1. A deficit-financed tax cut does not affect $({C_0}^*, {C_1}^*)$.

In fact, the assumptions needed for Ricardian equivalence to hold is strict and nearly impossible to be achieved in the real world. Those assumptions are

1. consumers are forward-looking and have long horizons (as long as the one of the government),
2. financial markets are perfect (borrow and lend at market interest rates freely and same interest rate for households and the government),
3. there is no uncertainty, and
4. taxation is non-distortionary, i.e., the budget is financed by **lump-sum** taxes (based on a fixed amount).

3.5 The Life Cycle Theory

We sketch the life cycle theory qualitatively.

The households' goal is to maintain a stable standard of living. Consider a consumer who

- lives for T periods,
- works during the first t_R years of one's life,
- earns a wage $w > 0$ during the working life in $[0, t_R]$, and
- is retired and does not earn anything in $[t_R, T]$.

The consumer saves (accumulates wealth) during working life and dissaves during retirement.

How does the optimal consumption path look like?

Suppose there is no interest rate involved, and one's consumption is denoted C. Then, the optimal consumption C^* is to spend all one's money at period T. Hence, it should satisfy the equation

$$C^* \cdot T = w \cdot t_R.$$

The left-hand side is lifetime income, and the right-hand side is lifetime income. Since $T > t_R$, it follows that $w > C^*$. Hence, the consumer will save in one's first t_R periods, and dissave in one's last $T - t_R$ periods. The assets of the consumer will reach the peak at the end of period t_R.

3.6 Consumption under Uncertainty

So far, the only motive for saving was provision for future consumption, for oneself or one's descendants. However, the individual may also save to pursue other goals; for example, there is *precautionary motive* to incent one to save in order to insure oneself against future fluctuations of income.

Here is the general idea: future income fluctuations create downside risks for the consumer. If one has saved X, one can be certain to consume at least $(1+r)X$ in the future, even if income is zero. Therefore, individuals accumulate savings to buffer against risks and reach a target wealth level.

We will see that more uncertainty leads to more savings if and only if u' is convex, i.e., $u'''(C) > 0$ for all C.

Consider the following 2-period optimization problem,

$$\max_{C_0, \widetilde{C}_1} u(C_0) + \beta \cdot \mathbb{E}\Big(u\big(\widetilde{C}_1\big)\Big) \tag{3.6.1}$$

under the constraint

$$C_0 + \frac{\widetilde{C}_1}{1+r} = Y_0 + \frac{\widetilde{Y}_1}{1+r}, \tag{3.6.2}$$

where the period-0 income Y_0 is certain, period-1 income $\widetilde{Y}_1$ is a random variable distributed over $\big[\underline{Y}, \overline{Y}\big]$ with density function $f_{\widetilde{Y}_1}$. Hence, the consumer will choose some C_0 at date 0 and will consume

$$\widetilde{C}_1 = (1+r)\left(Y_0 + \frac{\widetilde{Y}_1}{1+r} - C_0\right)$$

at date 1. We can see that $\widetilde{C}_1$ is a random variable as well.

The consumer maximizes the *expected utility,* which is also known as the von Neumann-Morgenstern utility.

We plug (3.6.2) into the function in the right-hand side of (3.6.1) by eliminating $\widetilde{C_1}$ and have

$$
\begin{aligned}
g(C_0) &:= u(C_0) + \beta \cdot \mathbb{E}\Big(u\big(\widetilde{C_1}\big)\Big) \\
&= u(C_0) + \beta \cdot \mathbb{E}\left(u\left((1+r)\left(Y_0 + \frac{\widetilde{Y_1}}{1+r} - C_0\right)\right)\right) \\
&= u(C_0) + \beta \cdot \mathbb{E}\Big(u\Big((1+r)\cdot(Y_0 - C_0) + \widetilde{Y_1}\Big)\Big) \\
&= u(C_0) + \beta \cdot \int_{\underline{Y}}^{\overline{Y}} u\Big(f_{\widetilde{Y_1}}(y) + (1+r)\cdot(Y_0 - C_0)\Big)\ \mathrm{d}y.
\end{aligned}
$$

Suppose that we can change the order of the differentiation operator and the integral operator. We differentiate both sides with respect to C_0 and have

$$
\begin{aligned}
g'(C_0) &= u'(C_0) - \beta \cdot \int_{\underline{Y}}^{\overline{Y}} u'\Big(f_{\widetilde{Y_1}}(y) + (1+r)\cdot(Y_0 - C_0)\Big)\cdot(-(1+r))\ \mathrm{d}y \\
&= u'(C_0) - \beta\cdot(1+r)\cdot\mathbb{E}\Big(u'\big(\widetilde{C_1}\big)\Big). \qquad (3.6.3)
\end{aligned}
$$

Since we are looking for the maximum, we set (3.6.3) to be zero and obtain the condition

$$
u'(C_0) = \beta\cdot(1+r)\cdot\mathbb{E}\Big(u'\big(\widetilde{C_1}\big)\Big). \qquad (3.6.4)
$$

Assume that u' is convex. Recall Jensen's inequality, that is, for any positive random variable X and for any convex function f, $\mathbb{E}(f(X)) \geq f(\mathbb{E}(X))$. Plug $f = u'$ and $X = \widetilde{C_1}$ into Jensen's inequality, we will have

$$
\mathbb{E}\Big(u'\big(\widetilde{C_1}\big)\Big) \geq u'\Big(\mathbb{E}\big(\widetilde{C_1}\big)\Big). \qquad (3.6.5)
$$

Compare two situations, without uncertainty and with uncertainty. Call ${C_0}^*$ the solution for C_0 with no uncertainty over Y_0 and Y_1; call ${C_0}'$ the solution for C_0 with uncertainty over Y_1. Note that $C_0, Y_0,$ and Y_1 are variables like x or y, while ${C_0}^*$ and ${C_0}'$ are solutions (constant).

See Figure 3.6.1 and Figure 3.6.2 about the comparison between two levels of uncertainty. In Figure 3.6.1 and Figure 3.6.2, $\sigma_1 < \sigma_2$ and ${C_0}' > {C_0}''$. This suggests more uncertainty leads more saving.

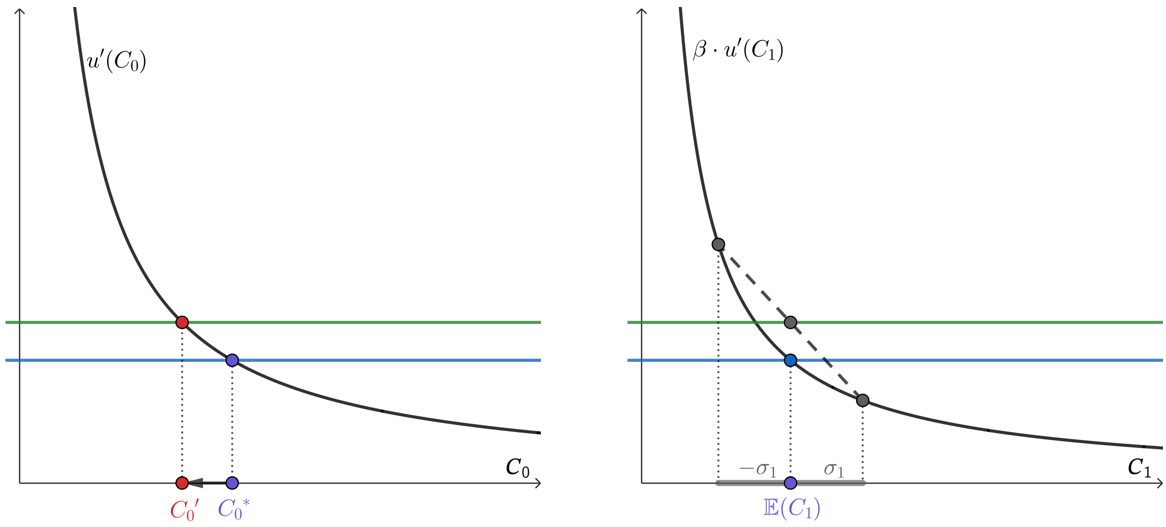

Figure 3.6.1. With convex u', a rise in $\mathbb{V}(Y_1)$ lowers optimal C_0. Before the rise of σ, the optimal first-period consumption is ${C_0}'$.

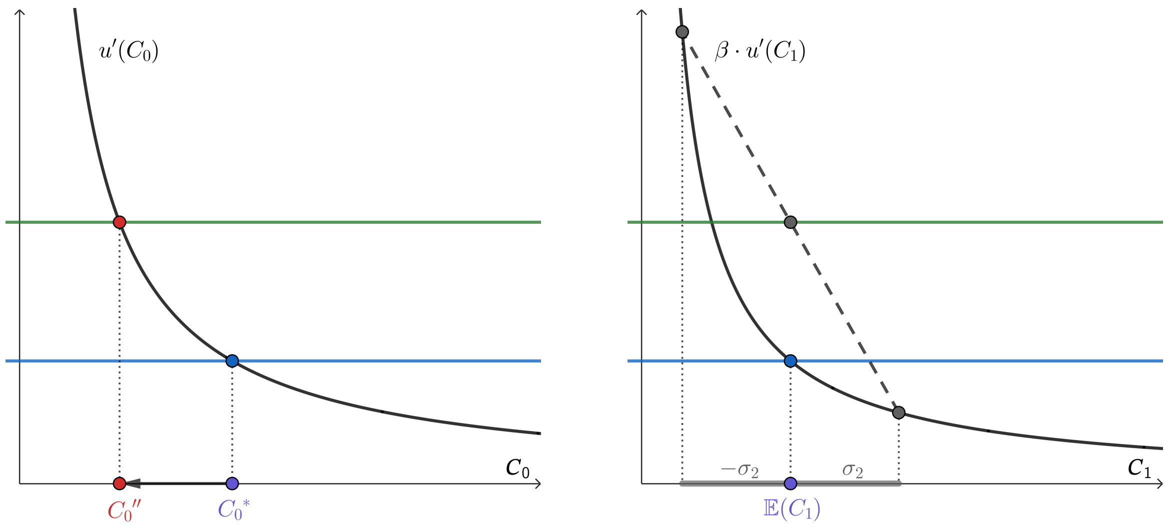

Figure 3.6.2. With convex u', a rise in $\mathbb{V}(Y_1)$ lowers optimal C_0. After the rise of σ, the optimal first-period consumption is ${C_0}''$.

4 Overlapping-Generations Models

We have taken assumptions regarding aggregated relationships, such as $S = sY$ in Section 1, and modeled individual behavior in Section 3. The next step is to aggregate individual behaviors; in order to do so, we need to assume a **demographic structure**. We specifically focus on the interactions between utility-maximizing agents on markets.

The *overlapping-generations* structure is one central demographic structure in macroeconomics. The other one is made of infinite-lives consumers (or dynasties) all born at the initial date.

4.1 Overlapping-Generations

The assumption of overlapping generations implies that at each date, different generations of individuals, of various ages, coexist. One period after, the oldest people die, the young gets older, and a new generation comes up.

Why do we study overlapping-generations models? In reality, generations do overlap: people are born and do die. This is a natural framework of analysis for inter-generational issues:

- causes and implications of life-cycle behavior,
- pension systems (fully-funded vs. pay-as-you-go),
- public dept,
- endogenous fertility, family macroeconomics, human capital, etc.

Simple overlapping-generations models generate new and important insights on inter-generational dynamics and welfare.

However, market equilibrium can be Pareto-suboptimal in overlapping-generations models even seemingly without any market distortion (imperfect competition, moral hazard, adverse selection, etc.). More importantly, we have a conclusion that the demographic structure itself can lead to some inefficiency.

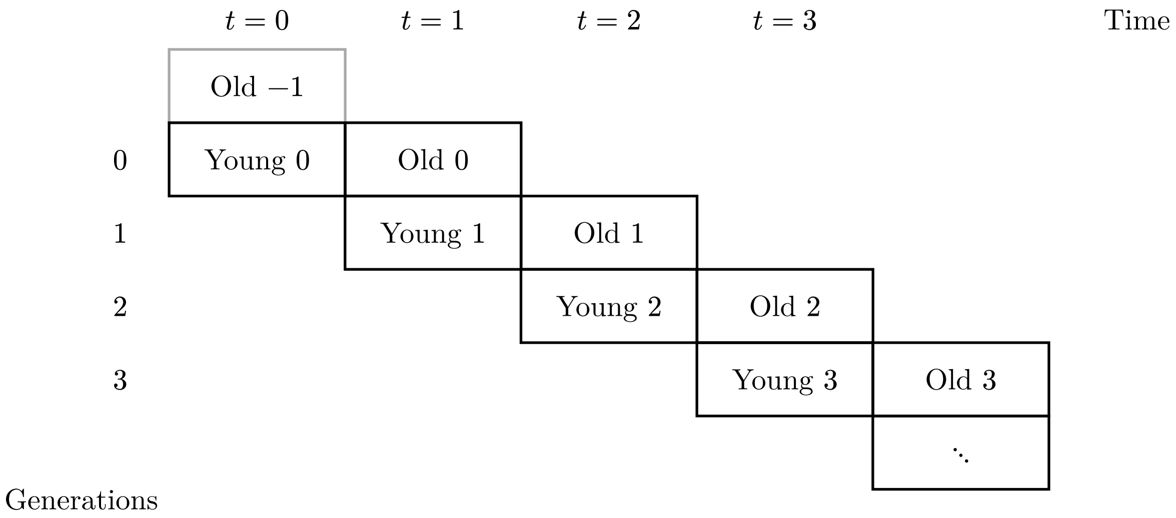

Figure 4.1.1. Two-Period Overlapping Generations

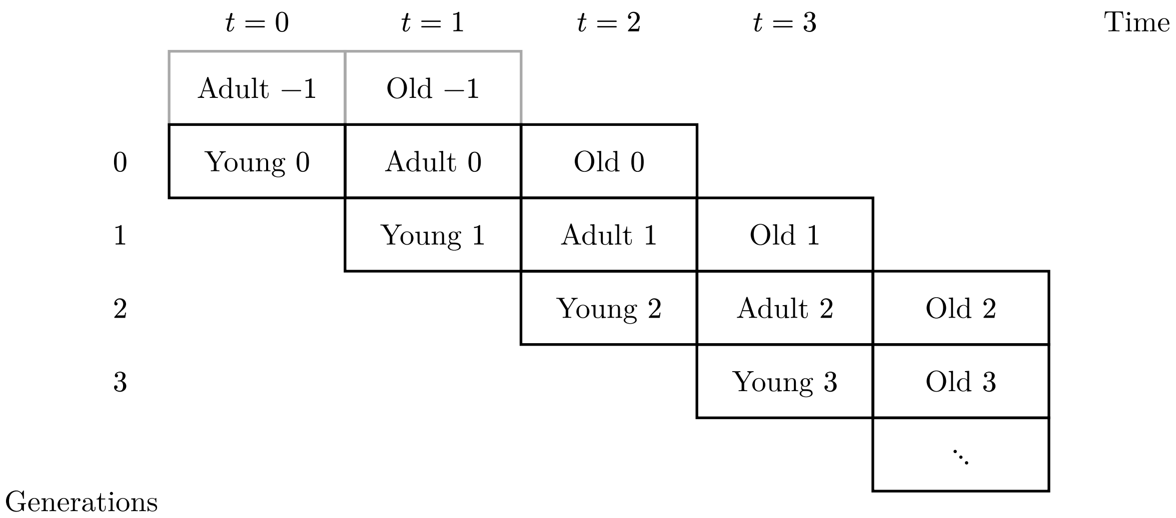

Figure 4.1.2. Three-Period Overlapping Generations

4.2 Overlapping-Generations in Endowment Economies

In this subsection, we will see a 2-period overlapping generations model given by Paul Samuelson in 1958.

Consider a 2-period overlapping generations structure. There is one perishable (non-storable) good. Each generation is $(1+n)$ times as big as the previous one with the relation

$$N_{t+1} = (1+n)N_t,$$

where N_t is the size of generation t, $N_0 = 1$. Take Figure 4.1.1 for example, the number of people in Young 2 will be $(1+n)$ times as big as the number of people in Young 1, and the number of people in Old 1 equals the number of people in Young 1.

In addition, we assume that all individuals have the utility function U over young-age consumption c^y and old-age consumption c^o with

$$U(c^y, c^o) = u(c^y) + \beta \cdot u(c^o),$$

where $u'' < 0$, $u' > 0$, and u satisfies the Inada condition.

The young is endowed with 1 unit of the perishable good, and the old is endowed with 0.

Credit Markets

It is natural that people start from credit markets, i.e., markets without money.

If facing an interest rate $r > -1$, the young born at date t wants to save to finance old-age consumption, since there will be no income in your old-age year.

However, there is actually no way to save. The good is perishable, the other young people want to save as well, and old people will not be able to repay in the next period. Hence, the old will just consume $c^o = 0$, while the young will consume $c^y = 1$, which is a suboptimal equilibrium.

The credit market cannot function on a bilateral basis under 2-period overlapping generation models, as two people of different generations never meet twice.

Remedy

Imagine that there exists an agree-on in fixed quantity M (say gold or useless paper "money"), initially held by the old of date $t = 0$.

Suppose that the old and every future generation believe they will be able to buy and sell money at price p_t in period t in terms of goods.

People might now be able to buy money for goods when young, sell money for goods when old. Does it happen at equilibrium? What is the equilibrium interest rate $\dfrac{p_{t+1}}{p_t}$?

Suppose you were born at date $t \geq 0$. If you know p_t and have perfect foresight over p_{t+1}, your program will be maximizing your utility U, i.e., looking for

$$\max_{m_t} u(1 - p_t m_t) + \beta \cdot u(p_{t+1} m_t),$$

where m_t stands for the price of money in terms of the good.

To maximize the function, we just set the first derivative to be zero and have

$$u'(1 - p_t m_t) \cdot (-p_t) + \beta \cdot u'(p_{t+1} m_t) \cdot p_{t+1} = 0$$

$$u'(1 - p_t m_t) = \beta \cdot u'\left(\frac{p_{t+1}}{p_t} \cdot p_t m_t\right) \cdot \frac{p_{t+1}}{p_t},$$

which implies

$$p_t m_t = f\left(\frac{p_{t+1}}{p_t}\right)$$

for some function f. This money demand function f is a saving function.

The demand-supply equilibrium at date t requires the total money demand equals the total money supply, i.e.,

$$\begin{aligned}\underbrace{p_t \cdot M}_{\text{total supply}} &= \underbrace{N_t \cdot p_t \cdot m_t}_{\text{total demend}} \\ &= \underbrace{N_0}_{1} \cdot (1+n)^t \cdot f\left(\frac{p_{t+1}}{p_t}\right) \\ &= (1+n)^t \cdot f\left(\frac{p_{t+1}}{p_t}\right). \qquad (4.2.1)\end{aligned}$$

Similarly, the demand-supply equilibrium at date $t+1$ requires

$$p_{t+1} \cdot M = (1+n)^{t+1} \cdot f\left(\frac{p_{t+2}}{p_{t+1}}\right). \qquad (4.2.2)$$

Consider a steady state when $\frac{p_{t+1}}{p_t} = 1 + r$ for all t. Then, (4.2.2) divided by (4.2.1) yields

$$1 + r = 1 + n.$$

The only possible equilibrium real interest rate (rate of return on money) is the population growth rate!

If $\dfrac{p_{t+1}}{p_t} = 1 + n$ for all t and p_0 is determined by $p_0 M = f(1+n)$, then supply-demand will be ensured at all dates on the money market (or equivalently consumption-goods), and the equilibrium is also welfare-maximizing for all generations: money restores optimality, as it restores inter-generational trade. Notice that, however, you need to be certain that future generations will all accept money as a means of savings.

We talked about money (non-productive) a lot in this section. What about an economy with a productive asset?

4.3 The Diamond Model

Consider a 2-period overlapping-generation model with population growth rate $n > 0$, and size of the cohort at time t is $L_t = (1+n)^t L_0$. The initial old (born at $t = -1$) are each endowed with capital $\overline{k}$. Consumers born at date t will have the utility

$$U({c^y}_t, {c^o}_{t+1}) = u({c^y}_t) + \beta \cdot u({c^o}_{t+1}).$$

The young has 1 unit of labor, and the old has 0. The young earns competitive wage w_t, consumes ${c^y}_t$, and saves s_t in the form of capital. Also, the young earns competitive interest rate r_{t+1} on s_t, consumes ${c^o}_{t+1} = \left(1 + r_{t+1}\right) \cdot s_t$ when old. We assume that capital does not depreciate.

The production function is $F(K, L)$, of which firms operate competitively. We assume that F is beautiful enough; that is, $F \in C^2$, is homogeneous of degree 1, is strictly increasing in K and L, and satisfies the Inada condition. The intensive form of f also satisfies $f'' < 0$ and $f' > 0$.

Date-t capital stock K_t is made of the savings of the old date t:

$$K_t = s_{t-1} L_{t-1} \ \text{ for } t \geq 1, \qquad \text{and } K_0 = \overline{k} L_{-1}.$$

The Optimal Steady State

We first consider the problem of a **benevolent social planner** who chooses the levels of consumption of each generation and of capital stock. The planner wants to provide the next generation with the same possibilities. To be more precise, nk_t capital must be accumulated at each period, so that the capital-labor ratio $\dfrac{K_t}{L_t}$ will be constant for all t. That is to say, the social planner has 0 discount rate, who values only the steady-state level well-being.

Hence, the optimization problem becomes to find

$$\max_{c^y, c^o, k} U(c^y, c^o) \tag{4.3.1}$$

under the constraint

$$F(K, L) = Lc^y + \frac{L}{1+n} \cdot c^o + nK, \tag{4.3.2}$$

where $+nK$ is to maintain the capital-labor ratio.

We can simplify (4.3.2) to make the intensive form appear, having

$$F(K, L) = Lc^y + \frac{L}{1+n} \cdot c^o + nK$$

$$F(k, 1) = c^y + \frac{c^o}{1+n} + nk$$

$$f(k) - nk = c^y + \frac{c^o}{1+n}. \tag{4.3.3}$$

The problem has two components: maximizing the net output $f(k) - nk$ with respect to k and maximizing U under the budget constraint.

We first maximize the net output. It is clear that we have

$$f'(k^*) = n. \tag{4.3.4}$$

We can now maximize U under (4.3.3) with (4.3.4), having that the optimal consumption bundle (c^{y*}, c^{o*}) will satisfy

$$\frac{\mathrm{d}}{\mathrm{d}c^y}(u(c^y) + \beta \cdot u(c^o)) = 0$$

$$\implies \quad u'(c^{y*}) + \beta \cdot u'(c^{o*}) \cdot \frac{\mathrm{d}}{\mathrm{d}c^y}((1+n) \cdot (f(k^*) - nk^* - c^y)) = 0$$

$$\implies \quad u'(c^{y*}) + \beta \cdot u'(c^{o*}) \cdot (-1-n) = 0$$

$$\implies \quad u'(c^{y*}) = \beta \cdot (1+n) \cdot u'(c^{o*}). \tag{4.3.5}$$

Decentralized dynamics

We now turn to the actual equilibrium, also known as the *decentralized dynamics.* We want to know what happens when the market work freely.

The competitive wage rate and interest rate are given by the condition that

$$w_t = \mathrm{MP}_{L_t} \qquad \text{and} \qquad r_t = \mathrm{MP}_{K_t}.$$

We can compute such by manipulating a coefficient, having

$$\begin{aligned}
\frac{\partial}{\partial L} F(K, L) &= \frac{\partial}{\partial L} L \cdot F\left(\frac{K}{L}, 1\right) \\
&= \frac{\partial}{\partial L} L \cdot f(k) \\
&= f(k) + L \cdot f'(k) \cdot \frac{\partial k}{\partial L} \\
&= f(k) + L \cdot f'(k) \cdot \left(-\frac{K}{L^2}\right) \\
&= f(k) - k \cdot f'(k)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial K} F(K, L) &= \frac{\partial}{\partial K} L \cdot f(k) \\
&= L \cdot f'(k) \cdot \frac{\partial k}{\partial K} \\
&= f'(k).
\end{aligned}$$

Hence, we have

$$w_t = f(k_t) - k_t \cdot f'(k_t) \qquad \text{and} \qquad r_{t+1} = f'(k_{t+1}).$$

Then, the price-taking agents want to find

$$\max_{c^y{}_t, c^o{}_{t+1}} U(c^y{}_t, c^o{}_{t+1}) \tag{4.3.6}$$

under the constraint

$$c^y{}_t + \frac{c^o{}_{t+1}}{1 + r_{t+1}} = w_t. \tag{4.3.7}$$

We just need to solve

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}c^o{}_{t+1}}\left(u(c^y{}_t) + \beta \cdot u(c^o{}_{t+1})\right) = 0 \\
\implies \quad & u'(c^y{}_t) \cdot \frac{\mathrm{d}c^y{}_t}{\mathrm{d}c^o{}_{t+1}} + \beta \cdot u'(c^o{}_{t+1}) = 0 \\
\implies \quad & \beta \cdot u'(c^o{}_{t+1}) = \frac{u'(c^y{}_t)}{1 + r_{t+1}} \\
\implies \quad & u'(c^y{}_t) = \beta \cdot (1 + r_{t+1}) \cdot u'(c^o{}_{t+1}) \\
\implies \quad & u'(w_t - s_t) = \beta \cdot (1 + r_{t+1}) \cdot u'((1 + r_{t+1}) \cdot s_t)
\end{aligned}$$

in which w_t and r_{t+1} will determine a saving function

$$s_t = \arg\max_{s_t} U(w_t - s_t, (1 + r_{t+1}) \cdot s_t) := s(w_t, r_{t+1}).$$

The model's dynamics are summarized by a sequence $(k_t)_{t=0}$. The sequence $(k_t)_{t=0}$ is an equilibrium sequence of capital-labor ratios if and only if

$$\begin{aligned}
\frac{K_{t+1}}{L_{t+1}} = s_t L_t \qquad \iff \qquad k_{t+1} &= \frac{s_t}{1+n} \\
&= \frac{s(f(k_t) - k_t \cdot f'(k_t), f'(k_{t+1}))}{1+n}.
\end{aligned}$$

Date-t savings must be consistent with date-t wages and date-$(t+1)$ interest rate, or there might be no equilibrium.

Note worthy that if $u = \ln$, then s does not depend on $f'(k_{t+1})$.

Decentralized Steady State

Suppose there is a steady-state capital-labor ratio $\hat{k}$, i.e.,

$$\hat{k} = \frac{s\Big(f\big(\hat{k}\big) - k_t \cdot f'\big(\hat{k}\big), f'\big(\hat{k}\big)\Big)}{1+n},$$

where $\hat{k}$ is determined by preferences (saving rate s), technology (intensive form f), and demography (population growth rate n).

We can now state a central theorem on overlapping-generation models about the steady-state interest rate.

Theorem. Let $\hat{r} = f'\big(\hat{k}\big)$ be the steady-state interest rate. If $\hat{r} < n$, then the decentralized steady state is dynamically inefficient, i.e., it is possible to increase the welfare of all generations. If $\hat{r} > n$, then the decentralized steady state is dynamically efficient, i.e., it is not possible to strictly increase the welfare of some generation without strictly decreasing the welfare of another generation.

Proof. Suppose $\hat{r} < n$. Let $\varepsilon > 0$. Let every young transfer ε to the old, so that each old receives $(1+n)\varepsilon$. This will benefit all generations since the market price of c^o at steady state is

$$\frac{1}{1+\hat{r}},$$

which is greater than $\dfrac{1}{1+n}$: the consumer gets more utility by having access to a small amount of c^o at less-than-market price. Suppose $\hat{r} > n$. That is, the case of capital under-accumulation. Then, any policy will hurt at least one generation: intra-period or inter-generational trade hurts at least some generation, and forced capital accumulation hurts the current generation. ∎

This should last forever, or the very last generation will be hurt. It is similar to the pension system, which left the young pay some for the old.

To illustrate graphically, see the following figures.

	$t = t_0$	$t_0 + 1$	$t_0 + 2$
	Old $t_0 - 1$		
Gen. t_0	Young t_0	Old t_0	
$t_0 + 1$		Young $t_0 + 1$	Old $t_0 + 1$
$t_0 + 2$			Young $t_0 + 2$

	$t = t_0$	$t_0 + 1$	$t_0 + 2$
	Gets $(1+n)\varepsilon$		
Gen. t_0	Lost ε $(\uparrow)$	Old t_0	
$t_0 + 1$		Young $t_0 + 1$	Old $t_0 + 1$
$t_0 + 2$			Young $t_0 + 2$

	$t = t_0$	$t_0 + 1$	$t_0 + 2$
	Gets $(1+n)\varepsilon$		
Gen. t_0	Lost ε $(\uparrow)$	Gets $(1+n)\varepsilon$	
$t_0 + 1$		Lost ε $(\uparrow)$	Old $t_0 + 1$
$t_0 + 2$			Young $t_0 + 2$

Figure 4.3.3. Illustration about the Central Theorem

You can see that everyone (each generation) is better-off since

$$-\varepsilon + (1+n)\varepsilon > -\varepsilon + (1+\hat{r})\varepsilon.$$

Remark. There should be some content about public debt in the original following section. However, it is too complicated for me to understand, and Dr. SENOUCI said we should consider those off-program. Hence, there is no content about this topic.

5 The IS-LM Model

In 1937, Hicks developed the IS-LM framework to formalize Keynes' ideas. The four letters are the first letter of investment, saving, liquidity preference, and money supply, respectively. In short, this model represents equilibrium in the goods (IS) and money (LM) markets.

Hicks interpreted Keynes within a general equilibrium framework: short-run macroeconomic equilibrium with **weak aggregate demand** and unemployment and emphasize of the role of interest rates in balancing investment and savings. This interpretation allowed Keynesian thought to be integrated into neoclassical economics.

In classical view of general equilibrium, markets are naturally clear through price adjustments, and full employment is achieved as long as wages and prices are flexible. However, Keyens' general theory rejects the idea that markets are always clear. This is because in the short run, wages and prices can be rigid, which lead to unemployment and output gaps $y^d - y^n$, where y^d is the output demanded, and y^n is the natural market demanded.

This model focuses on how output is determined in the short run. It looks at two things:

- the aggregated damand and its main compoments,
- money market and the role of interst rates.

The IS-LM model helps illustrate how stabilization policies (such as fiscal or monetary policy) can be used to influence the economy, aiming to stabilize output and employment levels.

In addition, it serves as a tool for analyzing the impact of fisical (government spending and taxation) and monetary (interest rate adjustments by the central bank) policies on the economy, and it helps illustrate the trade-offs policymakers face, such as the impact of monetary expansion on interest rates and investment.

As mentioned, we would like to find equilibria on the goods and money markets. This is a static model, so that time does not play an important role. We assume the following:

- economy consists of households, firms, and the government,
- a single homogeneous good Y is produced by perfectly competitive firms and consumed by households,
- (*normal rigidities*) prices are **fixed**, so that the firms are ready to supply any amount of Y at the given price level P, and
- (*closed economy*) there is no international trade in goods or assets.

5.1 The Goods Market

The IS curve represents the goods markets. The planned aggregate expenditures AE_p consist of

- consumption C: goods and services bought by households,
- investment I: spending on the acquisition of capital by firms, and
- government spending G.

Hence, the goods market equilibrium conditions requires, from the demand side,

$$C + I + G = Y,$$

where $C + I + G = \mathrm{AE}_p$. The **actual** expenditures are **always** equal to the income because every dollar spend by a buyer becomes income to the seller. Therefore, the equilibrium requires

$$\mathrm{AE}_{\text{actual}} = Y = \mathrm{AE}_p.$$

Consumption

The consumption C is positively depending on disposable income $Y - T$, where T are net taxes. We will write

$$C = C(Y - T),$$

where the derivative of C is called the marginal propensity to consume (MPC). The marginal propensity to consume C' is a positive decimal, and it measures how much out of one extra unit of income if spent on consumption. We can assume a specific functional form: linear consumption function

$$C(Y-T) = C_0 + c_y \cdot (Y-T), \tag{5.1.1}$$

where C_0 is the incompressible consumption refers to the minimum level of consumption that households need to maintain, and c_y is the marginal propensity to consume.

Investment

The investment is negatively depending on the interest rate r

$$I = I(r)$$

with $\frac{\mathrm{d}I}{\mathrm{d}r} < 0$. The intuition is that firms' borrowing costs rise when r increases, which makes it less profitable to invest, and with other conditions remaining the same. We can do something similar as we do in (5.1.1), to assume a specific linear investment function

$$I(r) = I_0 - i_r \cdot r, \tag{5.1.2}$$

where i_r measures interest sensitivety of investment demand by how much investment falls if r increases by 1 percentage point.

Government Spending and Taxation

The two parameters G and T are assumed to be exogenously fixed. We have other possibilities as well. For example,

- balanced budget $G = T$,
- procyclical expenditures $G'(Y) > 0$ or $T'(Y) > 0$, or
- countercyclical expenditures $G'(Y) < 0$ or $T'(Y) < 0$.

Taxes T can be lump-sum or proportional. The linear form of taxation is

$$T(Y) = T_0 + t_y \cdot Y, \tag{5.1.3}$$

where T_0 is lump-sum tax, and t_y is the rate for proportional tax.

The IS curve is the locus showing all combinations of Y and r such that the goods market is in equilibrium. In the linear economy, this gives us a negative relation between r and Y By

$$
\begin{aligned}
Y &= C + I + G \\
&= C_0 + c_y \cdot (Y - T) + I_0 - i_r \cdot r + G \\
\implies \quad Y &= -\frac{i_r}{1 - c_y} \cdot r + \frac{C_0 - c_y \cdot T - I_0 + G}{1 - c_y}. \qquad (5.1.4)
\end{aligned}
$$

The intuition of (5.1.4) is that an increase in r leads a decrease in investment, which leads a decrease in AE_p; as a result, Y decreases.

Note that when we are drawing an IS curve, we will put Y to be the horizonal axis and r to be the vertical axis, which is the contrary to the form of (5.1.4).

5.2 The Money Market

The LM curve represents the money market. We assume that there are two financial assets in the economy: money and bonds. Money does not pay interest and is more liquid (immediate); bonds pay a nomial interest $i > 0$ an is less liquid. Also, we have two motives for holding cash: transactions (depends on Y) and specilative (depends on i). Is is assumed that there is no money illusion, i.e., money is valued for its purchasing power.

We set the demand for real money balances is given by

$$
\frac{M^D}{P} = L(Y, i)
$$

for some function L, with M^D the demand for money. More income raises the need for more transactions $\frac{\partial L}{\partial Y} > 0$; interest rate i is the opportunity cost of holding cash $\frac{\partial L}{\partial i} < 0$. We can again use linear form to assume a particular form

$$
L(Y, i) = L_0 + I_y \cdot Y - I_i \cdot i, \qquad (5.2.1)
$$

where L_0 is the autonomous money demand, I_y is the income sensitivity of money demand, and I_r is the interest sensitivety of money demand.

The real money supply is assumed to be exogenously fixed at $\frac{M^S}{P}$. Since the price level P is fixed, supply of real money balances $\frac{M^S}{P}$ is independent of Y and r. Here we also assume that real and nominal rates are equal.

In practice in modern monetary system, money supply is controlled by the central bank and is fixed in the short run.

The LM curve is the locus showing all combinations of Y and r such that the money market is in equilibrium, i.e.,

$$\frac{M^D}{P} = \frac{M^S}{P}.$$

In a linear economy, this is a positive relation between r and Y since

$$\begin{aligned}
\frac{M^S}{P} &= \frac{M^D}{P} \\
&= L(Y, i) \\
&= L_0 + I_y \cdot Y - I_i \cdot i \\
&= L_0 + I_y \cdot Y - I_i \cdot r \\
\implies \qquad r &= \frac{I_Y}{I_i} \cdot Y + \frac{L_0 - M^S/P}{I_i}.
\end{aligned} \tag{5.2.2}$$

5.3 Equilibrium in the IS-LM model

Both IS and LM are in partial equilibrium for given r or Y. When both goods are markets are in equilibrium, there will be a unique pair (r^*, Y^*) solving the system

$$\begin{cases} Y = C(Y - T) + I(r) + G, \\ L(Y, r) = \frac{M^S}{P}, \end{cases}$$

corresponds to the equilibrium in the IS-LM model. That pair is exactly the intersection of the IS and the LM curves.

5.4 Policy

In this framework, we can analyze the government policy. The aim is to stablize output Y around its "natural" or **full-employment** (also called **potential**) level Y^p. We can see policies with different aspects:

- expansionary (increasing Y) or contractionary (decreasing Y),
- fiscal or monetary.

For fiscal policies, we have government spendings G and taxations T. Fisical policy shifts the IS curve: either direct (of G) or indirect (of T, through change in C) impact on aggregated demand. Recall (5.1.4).

For monetary policies, we have open-market operations (affecting M^S) and discount window (affecting r). Monetary policy shifts the LM curve: the change in M^S shifts the vertical money supply curve $\frac{M^D}{P} = \frac{M^S}{P}$, and hence for any Y, the interest rate r equilibrating the money market changes.

To access the overall effect of a policy, one needs to compute the magnitude of the **policy multiplier**, which is the change in output ΔY per given change in $G, T,$ or M^S.

We combine (5.1.4) and (5.2.2) to have the compact system

$$\begin{cases} r &= \dfrac{I_Y}{I_i} \cdot Y + \dfrac{L_0 - M^S/P}{I_i}, \\ \\ Y &= -\dfrac{i_r}{1-c_y} \cdot r + \dfrac{C_0 - c_y \cdot T - I_0 + G}{1-c_y}. \end{cases} \tag{5.4.1}$$

One can express (5.4.1) in variations for clarity purposes, as a function of policy instruments: $\Delta G, \Delta T,$ or ΔM^S to have

$$
\begin{cases}
\Delta r &= \dfrac{I_Y}{I_i} \cdot \Delta Y - \dfrac{1}{I_i} \cdot \dfrac{\Delta M^S}{P}, \\[2ex]
\Delta Y &= \dfrac{-i_r \cdot r - c_y \cdot \Delta T + \Delta G}{1 - c_y}.
\end{cases}
\tag{5.4.2}
$$

The resulting core insight from this model can be obtained from one equation derived from (5.4.2)

$$
\Delta Y = \frac{1}{1 - c_y + i_r \cdot I_y / I_i} \cdot \left(\Delta G - c_y \Delta T + \frac{i_r}{I_i} \cdot \frac{\Delta M^S}{P} \right). \tag{5.4.3}
$$

Fisical Policy Multiplier

Suppose the government conducts a fisical expansion[1] $\Delta G > 0$. How does the equilibrium Y^* change? From (5.4.3), the *fiscal policy multiplier* reads as

$$
\frac{\Delta Y}{\Delta G} = \frac{1}{1 - c_y + i_r \cdot I_y / I_i}, \tag{5.4.4}
$$

which indicates that the marginal propensity to consume c_y and crowding out effects i_r are key determinants.

Now, suppose the government conducts a tax decrement $\Delta T < 0$. How does the equilibrium Y^* change? From (5.4.3) again, the *tax multiplier* reads as

$$
\frac{\Delta Y}{\Delta G} = \frac{-c_y}{1 - c_y + i_r \cdot I_y / I_i}, \tag{5.4.5}
$$

which is simlar to the tax multiplier, attenuated by the marginal propensity to consume c_y.

Monetary Policy multiplier

Suppose the central bank conducts a monetary expation $\Delta M^S > 0$. How does the equilibrium Y^* change? From (5.4.3), the *monetary policy multiplier* reads as

$$
\frac{\Delta Y}{\Delta M^S} = \frac{1}{1 - c_y + i_r \cdot I_y / I_i} \cdot \frac{i_r}{I_i} \cdot \frac{1}{P}, \tag{5.4.6}
$$

which is similar to the tax multiplier, ambigously affected by i_r.

[1]It can be a fiscal contraction as well. The sign does not affect the formula. So do the following formulae.

Crowding Out Effect

From (5.4.4), the marginal propensity to consume c_y, the interest sensitivity of invesement demand i_r, and the slope of the LM curve $-\dfrac{I_y}{I_i}$ determine the magnitude of the fiscal policy multiplier. We can see that if $i_r \cdot \dfrac{I_y}{I_i}$ is large, the effect of the fiscal policy is reduced.

When the government increases spending (fiscal expansion), it inittially raises the aggregate demand, leading to an increase in output. As a result, households and firms demand more money for transactions. This increases in money demand affects the equilibrium in the money market. In order to meet the higher demand for money, the interest rate r increases because the supply of money is fixed in the short run. Hence, firms will reduce investment, and this reduction in investment partially offsets the initial increase in income caused by higher government spending. This effect is called the **crowding out effect**.

Increasing aggregate demand unintendedly reduces private investment. One can think about a policy mix, in which money supply could be used to offset the crowding out effect. Consider the variations in interest rate, assuming $\Delta r = 0$, yields

$$\frac{\Delta M^S}{P} = I_y \Delta Y.$$

The *policy mix multiplier* becomes

$$\Delta Y = \frac{1}{1 - c_y} \Delta G$$

with money supply $\dfrac{\Delta M^S}{P} = \dfrac{I_y}{1 - c_y} \cdot \Delta G$.

6 Unemployment

Unemployed are those who are not working, but able and willing to work and actively searching for job.

We have classified four generes of unemployment:

1. frictional, which occurs when people are temporarily unemployed while transitioning between (**searching** for) jobs,
2. structural, which arises from a **mismatch** between workers' skills or locations and the requirements of available jobs, e.g., elevator attendants,
3. classical (real wage), which occurs when wages are kept above the market-clearing level, often due to minimum wage laws, labor unions, or other wage-setting mechanisms (**real wage rigidities**), and
4. Keynesian (cyclical), which is caused by a decrease in aggregate demand in the economy, often during a recession.

The first three comprise the natural rate u_n of unemployment.

In practice, the natural rate of unemployment is a theoretical concept that cannot be directly observed. However, it represents the rate of unemployment to which the economy would converge in the absence of shocks and frictions. It serves as a **benchmark**, distinguishing the cyclical fluctuations in unemployment from its long-term, structural component.

The two components, cyclical unemployment and structural unemployment, require distinct policy responses; the former calls for stabilization measures, while the later necessitates longer-term reforms.

6.1 Walrasian Labor Market

The *Walrasian labor market* is based on the concept of general equilibrium, where wages and employment are determined by the interactions of labor supply and labor demand. It assumes that markets are **perfectly competitive**, and wages adjust to clear the market, i.e., there is no involuntary unemployment.

Since it is a perfect competitioin, workers and firms are all price-takers: no one can individually influence the wage level. In addition, wages adjust instantly to equate labor supply, and the market-clearning wage guarantees that all workers who want to work at the equilibrium wage can find employment.

6.2 Efficiency Wages

In fact, firms pay more than the reservation (equilibrium) wage w^*. In 1914, Henry Ford introduced the five dollar day which doubles the wage. The *efficiency wage* considerations were investigated by Raff and Summers in 1987; the increase in productivity and profits at Ford yielded less turnover.

However, the macroeconomic implication of setting a wage higher than reservation wage will imply unemployment from the Walrasian view of labor markets.

We start from the Walrasian labor market with inelastic labor supply. We assume that there are large numbers of firms, and they are perfectly competitive. The output is described by the production function F with

$$Y = F(e \cdot N^D),$$

where e is the *efficiency unit of labor* which is **increasing in wage**, and F is increasing and concave. We also assume that the demand for goods is stochastic Y, allowing firms to adjust supply through inputs e and N^D. As we said, the labor supply is inelastic and is set to be at $\overline{N}^S$.

The firm chooses number of workers to hire N and wage w to look for

$$\max_{N^D, w} F(e(w) \cdot N^D) - w \cdot N^D, \tag{6.2.1}$$

where $N^D \in \left[0, \overline{N}^S\right]$, and $w \geq 0$.

When we have $N^D < \overline{N}^S$, the first order conditions for (6.2.1) are

$$F'\left(e(w)\cdot N^D\right)\cdot e(w)-w=0 \qquad \text{and} \qquad F'\left(e(w)\cdot N^D\right)\cdot N^D\cdot e'(w)-N^D=0.$$

Combining the two equations above, we will have

$$\frac{w\cdot e'(w)}{e(w)}=1, \tag{6.2.2}$$

in which left-hand side is called the *the elasticity of effort with respect to the wage rate* ε_w^e. The *efficiency wage* $\hat{w}$ is also implicitly defined by (6.2.2).

In (6.2.2), the potimal condition equated the elasticity of effort with respect to the wage rate to unity. That is, firm wants to hire effective labors as cheaply as possible, which minimizes the cost per efficient unit $w/e(w)$.

When $N^D=\overline{N}^S$, we just solve

$$F'\left(e(w)\cdot\overline{N}^S\right)\cdot e(w)-w=0.$$

It can be easier to consider with some closed-form expressions. We suppose the production function is defined by

$$F(e\cdot N^D)=\left(e\cdot N^D\right)^\alpha, \tag{6.2.3}$$

and the efficiency unit of labor is defined by

$$e(w)=\begin{cases}\left(\dfrac{w-x}{x}\right)^\beta, & \text{if } w\geq x,\\[2ex] 0, & \text{if } w<x,\end{cases} \tag{6.2.4}$$

where $\beta\in(0,1)$, and x captures outside options. Then, we can find the reservation wage as a state-dependent relation of N^D and $\overline{N}^S$ with

$$w^*=\begin{cases}\dfrac{x}{1-\beta}, & \text{if } N^D<\overline{N}^S,\\[3ex] x\cdot\left(1+\left(\dfrac{x\cdot N^{1-\alpha}}{\alpha\beta}\right)^{\frac{1}{\beta-1+(\alpha-1)\cdot\beta}}\right), & \text{if } N^D<\overline{N}^S.\end{cases}$$

6.3 Generalization

For the efficiency we assumed previously, there can be more things in the parameter list. For example, the average wage paid by other firms $\overline{w}$, or the current level of unemployment u.

Correspondingly, we can assume the efficiency is given by

$$e(w, \overline{w}, u)$$

with $\dfrac{\partial e}{\partial \overline{w}} < 0$ and $\dfrac{\partial e}{\partial u} < 0$. Then, each firm chooses w and N with taking $\overline{w}$ and u as given.

We give an example with some closed-form expressions. Suppose the efficiency of labor is given by

$$e(w) = \begin{cases} \left(\dfrac{w-x}{x}\right)^{\beta}, & \text{if } w \geq x, \\ \\ 0, & \text{if } w < x, \end{cases} \tag{6.3.1}$$

where $\beta \in (0, 1)$, and

$$x = (1 - b \cdot u) \cdot \overline{w}$$

stands for the worker's outside opportunities. The symbol b characterizes the concern about unemployment, e.g., low skill workers may have $b > 1$, but computer scientists may have $b \approx 0$.

Then, (6.2.2) becomes

$$\beta \cdot \left(\frac{w-x}{x}\right)^{\beta-1} \cdot \frac{1}{x} \cdot \frac{w}{\left(\dfrac{w-x}{x}\right)^{\beta}} = 1. \tag{6.3.2}$$

Solving (6.3.2), we have

$$\hat{w} = \frac{x}{1-\beta} = \frac{1-bu}{1-\beta}\overline{w}.$$

This means that all firms are identical, and the wage is in equilibrium. Hence, the equilibrium (natural) unemployment rate becomes

$$u_n = \frac{\beta}{b}.$$