

# INTERMEDIATE MACROECONOMICS

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## ABSTRACT

This is a summary note of the course "ECO\_2F002\_EP Intermediate Macroeconomics" instructed by Dr. Mehdi SENOUCI and Dr. Gauthier VERMANDEL at École polytechnique in the first semester in 2024.

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# Preface

The following content mostly follows the lecture notes provided by the instructors, but figures are mainly re-produced; there are also some personal remarks.

The last section (Section 5) is about problem sets. I copied the majority of the answers as well as figures in files given by the instructors, Dr. Mehdi SENOUCI and Dr. Gauthier VERMANDEL, and the TA, PENG Lanxin.

The texts that are **bold** are indicating keywords, while the texts that are *italic* are indicating new nouns.

# 1 The Solow Model

Kaldor in 1961 published “Capital Accumulation and Economics Growth,” in which he stylized that there are five facts in processes of economics growth,

1. output per worker grows at a roughly constant rate that does not diminish over time (roughly 2%),
2. capital per worker grows over time,
3. the capital/output ratio is roughly constant (everything stays in the same window),
4. the rate of return to capital is roughly constant,
5. shares of capital and labor in net income are roughly constant, and
6. appreciable differences in the rate of growth of labor productivity and of total output in different societies. (Note: this is of non-regularity.)

The Solow model, proposed by Solow in 1956, replicates the first five Kaldor facts with minimal assumptions.

## 1.1 The Production Function

We say that a single consumption  $Y(t)$  of good (output) is being produced by number  $L(t)$  of labors and capital  $K(t)$  via a production function in the form of

$$Y(t) = F_t(K(t), L(t)).$$

The general scenario is that the production function  $F_t$  might change over time due to technical change. Solow eliminates this concern by specifically assuming

$$Y(t) = F(K(t), A(t)L(t)).$$

This function  $F$  need to some conditions, which we will discuss later.

The function  $A(t)$  captures “**labor-augmenting** technological progress.” Overtime, labor becomes more productive. That is, when  $A(t)$  increases, if we fix the amount  $L(t)$  of labors-hours, there will be more output produced.

We assume that all population is comprised of labor force with no unemployment. In addition, the economy is closed without government. The same final good is used for either consumption  $C(t)$  or investment  $I(t)$ , i.e.,  $Y(t) = C(t) + I(t)$ .

Below are the conditions that  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  needs to satisfy.

1. To be strictly positive and strictly diminishing marginal products  $MP_k$  and  $MP_L$ . That is,

$$\frac{\partial F}{\partial K} > 0, \quad \frac{\partial F}{\partial L} > 0, \quad \frac{\partial^2 F}{\partial K^2} < 0, \quad \text{and} \quad \frac{\partial^2 F}{\partial L^2} < 0.$$

2. Inputs are essential. That is,  $F(0, L) = 0$  and  $F(K, 0) = 0$ .
3. To be homogeneous of degree one in  $K(t)$  and  $L(t)$ . That is,

$$F(\alpha \cdot K(t), A(t)(\alpha \cdot L(t))) = \alpha \cdot F(K(t), A(t)L(t)).$$

There is another underlying assumption that fixed-quantity inputs (such as land or natural resources) are assumed to be unimportant, or replaceable by man-made capital.

The most comprehensive production function is the constant elasticity of substitution production function (CES production function),

$$Y(t) = \left( \alpha \cdot (K(t))^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(A(t)L(t))^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}. \quad (1.1.1)$$

We can derive many other production functions via the constant elasticity of substitution production function. In (1.1.1), if we take  $\sigma \rightarrow 1$ , we will get the Cobb-Douglas production function,

$$Y(t) = K(t)^\alpha (A(t)L(t))^{1-\alpha};$$

if we take  $\sigma \rightarrow 0$ , we will get the Leontief production function,

$$Y(t) = \min\{\alpha \cdot K(t), (1-\alpha) \cdot A(t)L(t)\};$$

if we take  $\sigma \rightarrow \infty$ , we will get the linear production function,

$$Y(t) = \alpha \cdot K(t) + (1-\alpha) \cdot A(t)L(t).$$

## 1.2 Evolution of Inputs

We assume, by Kaldor facts, labor  $L(t)$  and productivity  $A(t)$  grow at constant. That is,

$$g_L(t) := \frac{\dot{L}(t)}{L(t)} = n \geq 0 \quad \text{and} \quad g_A(t) := \frac{\dot{A}(t)}{A(t)} = g \geq 0.$$

In previous equations,  $g_X(t)$  denotes the *growth rate* of variable  $X(t)$ . Note that the *rate of change* of  $X(t)$  is  $\dot{X}(t)$ . Do not be confused by these two nouns.

Aside from these assumptions, we say that a fraction  $s \in (0, 1]$  of output is invested. That is,

$$I(t) = s \cdot Y(t) \quad \text{and} \quad C(t) = (1 - s) \cdot Y(t).$$

In addition, capital depreciates at a rate of  $\delta \geq 0$ .

Hence, the dynamics of capital stock is given by

$$\dot{K}(t) = s \cdot Y(t) - \delta \cdot K(t), \tag{1.2.1}$$

in which suggests the rate of change is increased by investment and is decreased by depreciation.

## 1.3 The Balanced Growth Path

The *balanced growth path* (BGP) is the situation when all variables grow at constant rates. Over time, the economy tends to converge to the balanced growth path.

Not only they grow at constant rates, output and capital grow at the same rate as well. Using (1.2.1), we have

$$\frac{\dot{K}(t)}{K(t)} = s \cdot \frac{Y(t)}{K(t)} - \delta := c_0, \tag{1.3.1}$$

where  $c_0$  is merely a constant that we set to follow the assumption “all variables grow at constant rates.” Hence, working on the second equation in (1.3.1), we have

$$\frac{Y(t)}{K(t)} = \frac{c_0 + \delta}{s}.$$

Log-differentiating the equation above, since the right-hand side of the equation is a constant, we obtain

$$g_Y(t) - g_K(t) = 0. \quad (1.3.2)$$

We first set variables to be per unit of efficient labor,

$$k(t) := \frac{K(t)}{A(t)L(t)} \quad \text{and} \quad y(t) := \frac{Y(t)}{A(t)L(t)}.$$

Using constant returns to scale, we have

$$\begin{aligned} y(t) &= \frac{Y(t)}{A(t)L(t)} \\ &= \frac{F(K(t), A(t)L(t))}{A(t)L(t)} \\ &= F(k(t), 1) \\ &:= f(k(t)), \end{aligned} \quad (1.3.3)$$

where (1.3.3) is called the *intensive form*.

Our goal is to characterize the evolution of  $k(t)$ . Since

$$k(t) = \frac{K(t)}{A(t)L(t)},$$

we log-differentiate and have  $\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - \frac{\dot{A}(t)}{A(t)} - \frac{\dot{L}(t)}{L(t)}$ , where  $\frac{\dot{A}(t)}{A(t)} = g$  and

$$\frac{\dot{L}(t)}{L(t)} = n$$

by assumption. Using (1.2.1), we further have

$$\begin{aligned} \frac{\dot{k}(t)}{k(t)} &= \frac{s \cdot Y(t) - \delta \cdot K(t)}{K(t)} - g - n \\ &= s \cdot \frac{y(t)}{k(t)} - \delta - g - n. \end{aligned}$$

Therefore, we have the *law of motion of capital per unit of efficient labor*,

$$\dot{k}(t) = s \cdot y(t) - (\delta + g + n) \cdot k(t). \quad (1.3.4)$$

The rate of change in  $k(t)$  is the difference of two terms:

1. how much the economy actually saves and
2. how much the economy needs to save to keep  $k(t)$  constant.

The economy needs to save *more* in order to maintain  $k(t)$  if the following happens:

1. the greater is the rise in labor productivity (higher  $g$ ), since more productive labor needs to be endowed with more capital,
2. the more rapid is population growth (higher  $n$ ), since more workers need to be endowed with capital, or
3. the more quickly the existing capital depreciates (higher  $\delta$ ).

## 1.4 The Steady State

We say that the economy reaches a *steady state* if  $\dot{k}(t) = 0$  after some time, so that, from (1.3.4), we have

$$s \cdot f(k(t)) = (\delta + g + n) \cdot k(t).$$

Let  $k_s$  denote the steady-state capital per effective labor if there is any. When we are writing down a solution, we have to ask some questions.

1. Does the solution exist?
2. If it exists, is it unique?
3. Does the economy converge to the steady state?

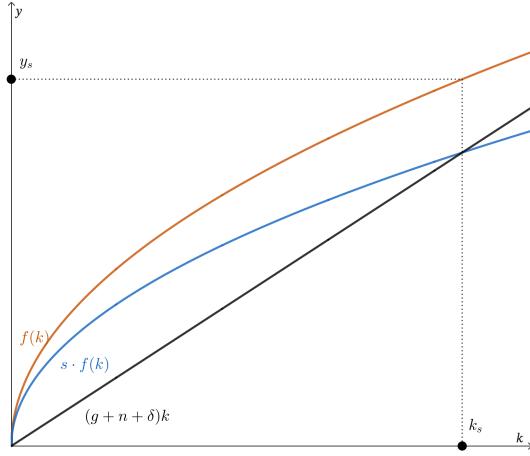


Figure 1.4.1. The steady state at the intersection of  $y = s \cdot f(k(t))$  and  $y = (g + n + \delta) \cdot k(t)$ .

There are, strictly speaking, two steady states in Figure 1.4.1,  $(0, 0)$  and  $(k_s, y_s)$ . However, the trivial steady state at the origin is unstable. Consider (1.2.1), the first derivative (speed)  $\dot{k}(t)$  is determined by the difference  $s \cdot f(k(t)) - (g + n + \delta) \cdot k(t)$  of two functions. Thus, when  $f(k(t)) > (g + n + \delta) \cdot k(t)$ , we will have  $\dot{k}(t) > 0$ ; when  $f(k(t)) < (g + n + \delta) \cdot k(t)$ , we will have  $\dot{k}(t) < 0$ . Hence, we can see that on  $(0, k_s)$ , a point will move horizontally to its right since  $\dot{k} > 0$ ; conversely, on  $(k_s, \infty)$ , a point will move horizontally to its left since  $\dot{k} < 0$ .

We have to make sure that the non-trivial steady state exists. If the function  $f$  satisfy the *Inada conditions*,

$$\lim_{k \rightarrow 0} f'(k) = \infty \quad \text{and} \quad \lim_{k \rightarrow \infty} f'(k) = 0,$$

then the non-trivial steady state must exist. The first equation ensures that the slope near 0 must be greater than  $(g + n + \delta)$ ; the second equation ensures that the slope of the function  $f$  must be less than  $(g + n + \delta)$ , so that there will be an intersection.

We will have **only one** non-trivial solution since  $f$  is concave (due to diminishing  $MP_K$ ). Most importantly, the economy converges to the non-trivial solution.

## 1.5 Convergence to the Steady State

If the initial state of capital is less than the steady state, capital will accumulate.

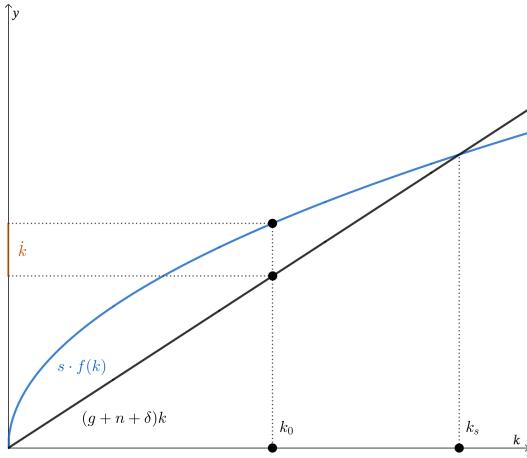


Figure 1.5.1. Accumulation of capital

More importantly, the growth will slow down as the economy gets closer to  $k_s$  from the left.

There must be some impact of the change in one of the parameters,  $s, g, n$ , or  $\delta$ . We have two conceptually distinct but closely related questions.

1. (*Comparative statics*) How would the steady state **shift**?
2. (*Transitional dynamics*) How the economy will **move** there?

We will focus on the change in the saving rate  $s$ . The saving rate is the result of the households' choice, and it may be affected by the government policy.

### Comparative Statics

Suppose the economy is initially at the steady state  $k_s$ . As some time  $t_0$ , saving rate permanently increases from  $s_0$  to  $\tilde{s}$ . The investment  $s \cdot Y(t)$  exceeds break-even investment, this suggests the increment of  $k$ . Hence, the economy moves to the new steady state with higher  $k$  and  $y$ , say at  $(\tilde{k}_s, \tilde{y}_s)$ . Evidently, steady-state output per efficient labor increases. See Figure 1.5.2. However, does the growth rate become permanently larger? The answer is negative. Once the economy reaches the new steady state, capital per capita  $k$  and output per capita  $y$  still grow at a rate  $g$ .

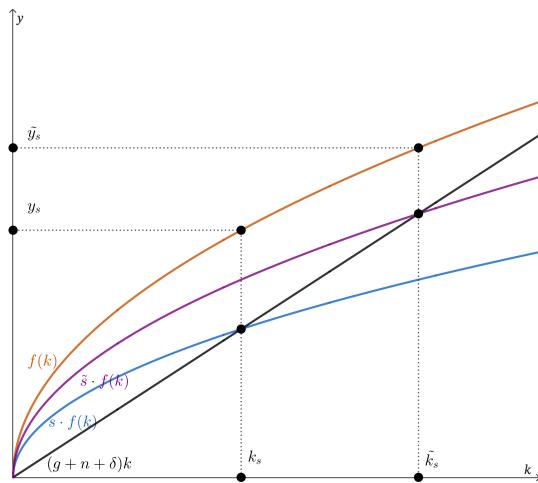


Figure 1.5.2. The rise in  $s$  increases  $k_s$  and  $y_s$ .

### Transitional Dynamics (Illustrations)

The movement of capital will take time, but the movement of saving rate is immediate.

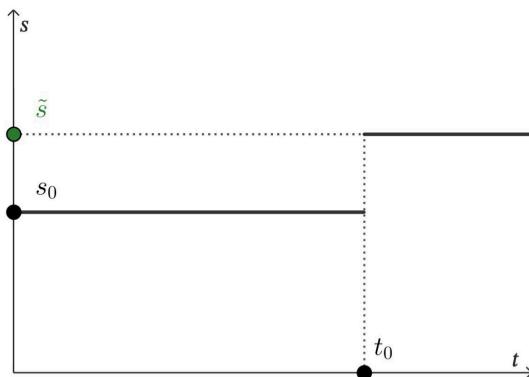


Figure 1.5.3. The time path for the saving rate  $s$ .

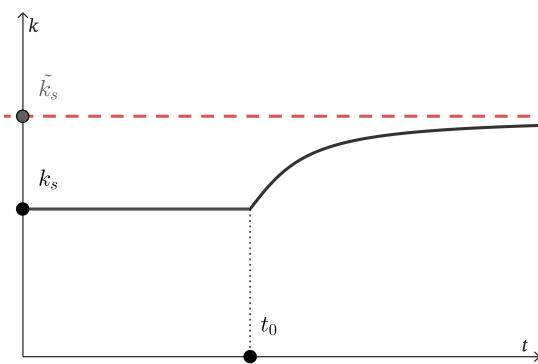


Figure 1.5.4. The time path for capital per efficient labor  $k$ .

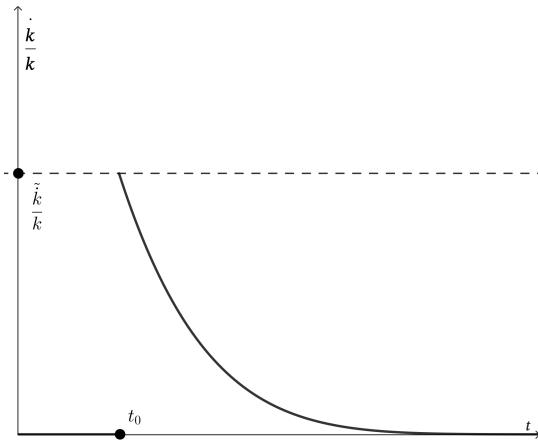


Figure 1.5.5. The time path for the growth rate of  $k$ .

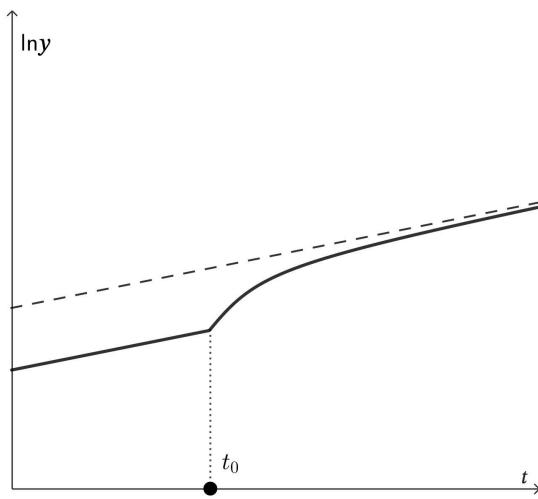


Figure 1.5.6. The time path for the logarithm of output per capita  $\ln y$ .

## 1.6 The Golden Rule

In terms of welfare, what matters is per-capita consumption rather than per-capita output. How can we achieve the maximum consumption per efficient labor in the steady state by adjusting the saving rate? Evidently, it should neither be too high nor too low. (In particular, we have  $c = 0$  when either  $s = 0$  or  $s = 1$ .) There is an inverse U-shaped relationship between per-capita consumption  $c$  and saving rate  $s$ .

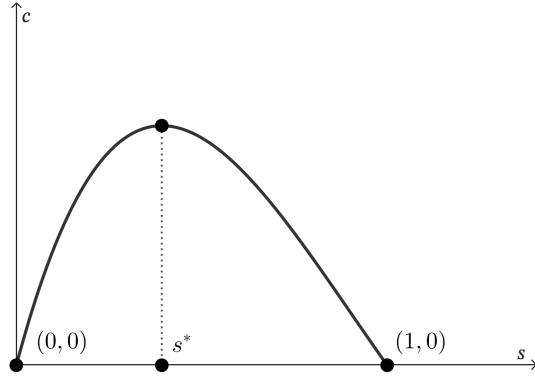


Figure 1.6.1. An illustration about the saving rate  $s^*$  maximizes  $c$  in the steady state.

Analytically, the goal is to find

$$\max_{k,s} (1-s) \cdot f(k) \quad (1.6.1)$$

under the constraint

$$s \cdot f(k) = (g + n + \delta) \cdot k. \quad (1.6.2)$$

Plug (1.6.2) into the function in (1.6.1), we just need to maximize the function

$$\tilde{f}(k) = f(k) - (g + n + \delta) \cdot k.$$

We look for a point  $k$  such that  $\tilde{f}'(k) = 0$ . It is clear that the point  $k^*$  such that  $f'(k^*) = g + n + \delta$  is what we are looking for. Hence, the *golden-rule saving rate* is found as

$$s^* = \frac{(g + n + \delta)k^*}{f(k^*)}.$$

See Figure 1.6.2 for visualization.

Depending on whether the economy initially operated below or above the golden-rule saving rate, the rise in the saving rate  $s$  may either **increase** or **reduce** the steady-state consumption per efficient labor.

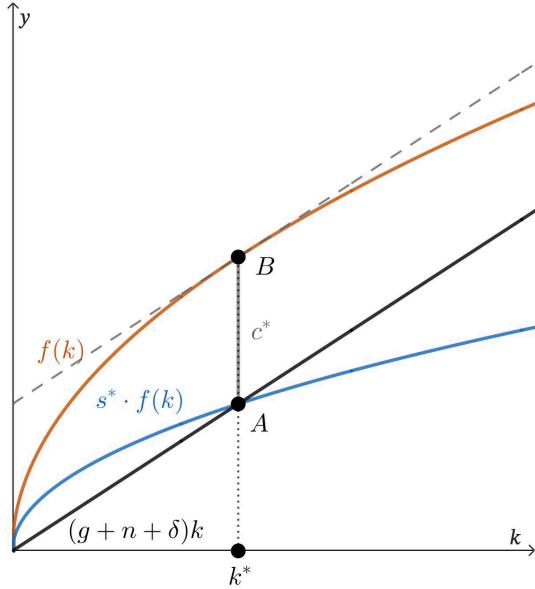


Figure 1.6.2. The special  $k^*$  maximizes the distance between  $y = f(k)$  and  $y = (g + n + \delta) \cdot k$ .

The immediate impact of a rise in  $s$  is **always** a drop in  $c$ . However, as the economy moves to the new steady state with higher  $k_s$ , the consumption  $c(t) = (1 - s) \cdot f(k(t))$  gradually increases.

If initially the economy saves too much, then the reduction in  $s$  will increase  $c$  both in the short run and in the long run. When  $s > s^*$ , the economy is called *dynamically inefficient*.

When the saving rate  $s$  increases until a value  $s_0 < s^*$ , there will be an immediate decrease in consumption (from originally  $c_s$  to  $c'$ ), but the consumption will gradually accumulate to a higher level ( $\tilde{c}_s$ ).

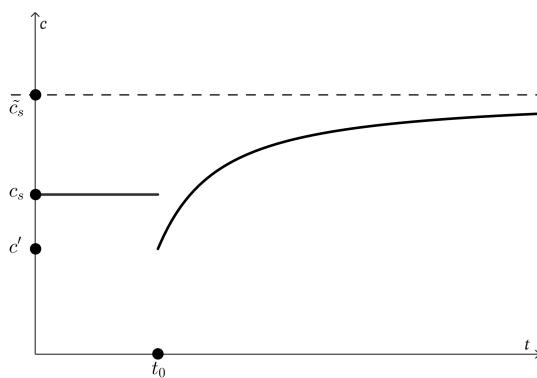


Figure 1.6.3. The time path for  $c$  when  $s$  increases and  $s_0 < s^*$ .

On the contrary, when the saving rate  $s$  increases until a value  $s_0 > s^*$ , there will be an immediate decrease in consumption (from originally  $c_s$  to  $c'$ ) as well, but the consumption will converge to a lower level ( $\tilde{c}_s$ ).

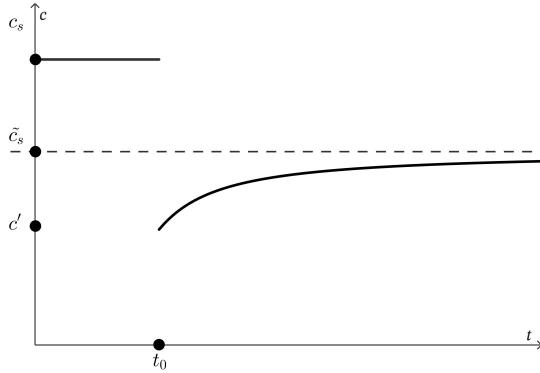


Figure 1.6.4. The time path for  $c$  when  $s$  increases and  $s_0 > s^*$ .

## 1.7 Conclusion

This model replicates all the first five Kaldor facts, but it is not at ease with the sixth Kaldor fact except if we invoke different productivity growth patterns across countries. It is indeed a good guide to time series, but a bad guide to cross-section since technology (non-rival) should not stop at national borders.

There are ways to enrich the model, including endogenizing the growth rate  $g$  (the endogenous growth theory, the next section) and endogenizing the saving rate  $s$  by introducing households' intertemporal optimization (the optimal growth model or the Ramsey–Cass–Koopmans model).

## 2 The Endogenous Growth Theory

The Solow model successfully theorize balanced growth by **exogenous** (to be a parameter in the model) and **steady** technical change, i.e.,  $g_A(t) = g$ . The increment in  $A$  might come from innovation by chance (e.g., tarte Tatin) or purposeful innovation. Moreover, innovation carries a lot of economic value: higher productivity means higher GDP per capita. Knowledge (technology)  $A$  is a club good, which is non-rival and partly excludable. The use of knowledge by one does not preclude its use by anyone else, and those who did not contribute to its accumulation can still benefit from it.

The endogenous growth theory starts, on the contrary to the Solow model, with the idea of endogenizing productivity growth  $g_A(t)$ . There are many models, and we will cover

1. learn-by-doing, which states that technical change is a side-effect of some economic activities, and
2. investment in research and development, which is more often used nowadays.

### 2.1 The Learn-by-Doing Model

As we stated previously, this model consider technical change not a result of deliberate efforts but a **side-effect** of some economic activities.

We assume that all inputs are now engaged in goods production with the Cobb-Douglas production function

$$Y(t) = (K(t))^\alpha (A(t)L(t))^{1-\alpha}. \quad (2.1.1)$$

In addition, the stock of knowledge

$$A(t) = B \cdot (K(t))^\phi, \quad (2.1.2)$$

is a function of the stock of capital, where  $B$  and  $\phi$  are positive constants. Plug (2.1.2) into (2.1.1), we have

$$Y(t) = B^{1-\alpha} (K(t))^{\alpha+\phi \cdot (1-\alpha)} (L(t))^{1-\alpha}.$$

We assume  $\delta = 0$  to simplify mathematical analysis. Thus, the growth

$$\dot{K}(t) = s \cdot Y(t) \quad (2.1.3)$$

of  $K(t)$  becomes some fraction of the total production, and we obtain the dynamics of  $K(t)$

$$\dot{K}(t) = s \cdot B^{1-\alpha} (K(t))^{\alpha+\phi \cdot (1-\alpha)} (L(t))^{1-\alpha}. \quad (2.1.4)$$

We further consider a *knife-edge condition*, which is a specific case, when  $n = 0$  and  $\phi = 1$ . Then,  $L(t)$  becomes a constant  $L$ , and the production function becomes

$$\begin{aligned} Y(t) &= B^{1-\alpha} K(t) L^{1-\alpha} \\ &= (BL)^{1-\alpha} K(t), \end{aligned}$$

which is linear in  $K(t)$ . Letting  $b$  denotes the constant  $(BL)^{1-\alpha}$  under this knife-edge condition, we will see that, from (2.1.3), the growth rate of capital becomes

$$\frac{\dot{K}(t)}{K(t)} = sb.$$

This suggests that saving rate affects long-run growth.

## 2.2 The Model of Investment in Research and Development

In Section 2.1, we considered the accumulation of knowledge is a side-effect of economic activities. In this model, we instead consider the accumulation of knowledge the result of purposeful activities on behalf of innovative firms. Thus, it is reasonable to assume that there is a fraction  $a_L$  of labor is employed in research and development, and the rest  $(1 - a_L)$  of labor is employed in production.

### A Model without Capital

For simplicity, we first assume that there is no capital. We will put capital back later. The production is given by

$$Y(t) = (1 - a_L) \cdot A(t) L(t). \quad (2.2.1)$$

We assume that labor force grows at a constant rate

$$g_L(t) = n > 0.$$

Notice that from (2.2.1), log-differentiating yields

$$g_Y(t) = g_A(t) + n;$$

dividing both sides by  $L(t)$  and then log-differentiating yield

$$g_{\frac{Y}{L}}(t) = g_A(t).$$

We assume the dynamics of knowledge to be

$$\dot{A}(t) = B \cdot (a_L \cdot L(t))^\gamma (A(t))^\theta, \quad (2.2.2)$$

where  $B$ ,  $\gamma$ , and  $\theta$  are positive constant. The power  $\gamma$  measures the extent of decreasing marginal returns to labor in research and development. The dynamics has the “standing on the shoulders of giants” effect when  $\theta > 1$  and has the “fishing-out” effect when  $\theta < 1$ ; we can imagine that earlier fundamental research facilitates subsequent discoveries when  $\theta > 1$  and that obvious discoveries are made first, then subsequent ideas become increasingly difficult to discover when  $\theta < 1$ . The later is more to be assumed nowadays.

From (2.2.2), we divide both sides by  $A(t)$  and obtain

$$\frac{\dot{A}(t)}{A(t)} = B \cdot (a_L \cdot L(t))^\gamma (A(t))^{\theta-1}. \quad (2.2.3)$$

We use the substitution  $g_{A(t)} = \frac{\dot{A}(t)}{A(t)}$  to show a magic trick. We log-differentiate (2.2.3) and have

$$\begin{aligned} \frac{\dot{g}_A(t)}{g_A(t)} &= \gamma \cdot \frac{\dot{L}(t)}{L(t)} + (\theta - 1) \cdot \frac{\dot{A}(t)}{A(t)} \\ &= \gamma n + (\theta - 1) \cdot g_A(t) \\ \dot{g}_A(t) &= \gamma n \cdot g_A(t) + (\theta - 1) \cdot (g_A(t))^2, \end{aligned} \quad (2.2.4)$$

which is a quadratic expression of  $g_A(t)$ . Thus, the dynamics of  $g_A(t)$  can be represented by a phase diagram (a variable  $X$  on the  $x$ -axis and its derivative with respect to time  $\dot{X}$  on the  $y$ -axis). The convergence crucially depends on the value of  $\theta$ . We first solve (2.2.4). If  $\theta \neq 1$ , the solutions to the quadratic equation

$$\gamma n \cdot g_A(t) + (\theta - 1) \cdot (g_A(t))^2 = 0$$

are  $g_A(t) = 0$  or  $g_A(t) = \frac{\gamma n}{1 - \theta}$ .

When  $\theta < 1$ , the graph of  $\dot{g}_A(t)$  will be a parabola with a positive leading coefficient; the growth of knowledge will converge to a certain level  $g_A^*$ .

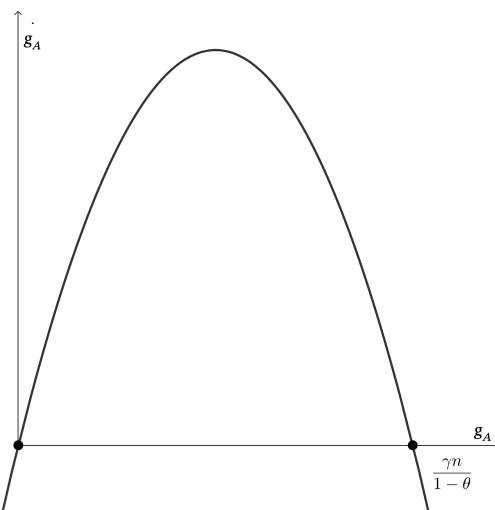


Figure 2.2.1. When  $\theta < 1$ , the growth  $g_A(t)$  of knowledge converges to  $g_A^*$ .

When  $\theta > 1$ , the graph of  $\dot{g}_A(t)$  will be a parabola with a negative leading coefficient; the growth is explosive.

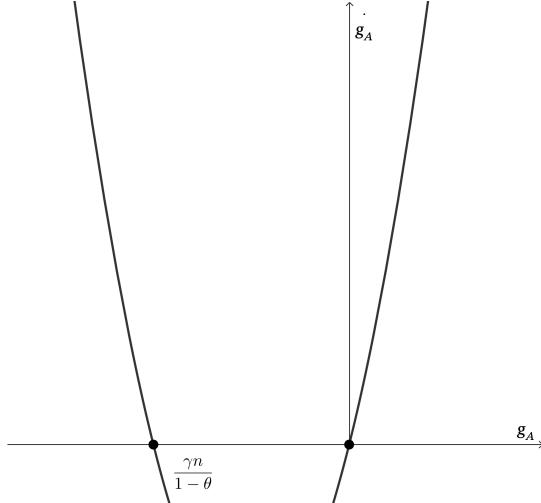


Figure 2.2.2. When  $\theta > 1$ , the growth  $g_A(t)$  of knowledge diverges ( $g_A(t) \rightarrow \infty$  as  $t \rightarrow \infty$ ).

When  $\theta = 1$ , the graph of  $g_A(t)$  will be a line with slope  $\gamma n$ . Since  $\gamma$  is a positive constant, the growth of knowledge stays constant if and only if  $n = 0$ .

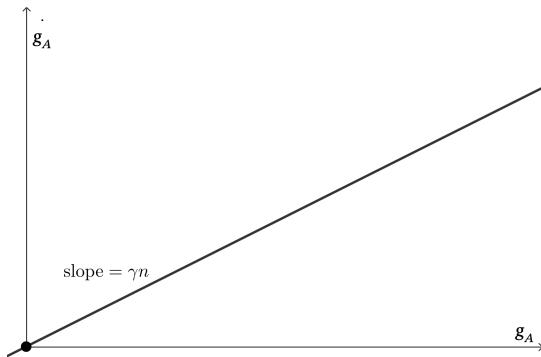


Figure 2.2.3. When  $\theta = 1$ , the convergence of growth rate of knowledge depends on  $n$ .

## A Model with Capital

We use the Cobb-Douglas production function to introduce capital, assuming

$$Y(t) = ((1 - a_K) \cdot K(t))^\alpha ((1 - a_L) \cdot A(t)L(t))^{1-\alpha}. \quad (2.2.5)$$

Do compare (2.2.5) with (2.2.1). We keep the assumption that there is no depreciation, and hence

$$\dot{K}(t) = s \cdot Y(t). \quad (2.2.6)$$

We also assume the dynamics of knowledge to be

$$\dot{A}(t) = B \cdot (a_K \cdot K(t))^\beta (a_L \cdot L(t))^\gamma (A(t))^\theta. \quad (2.2.7)$$

Do compare (2.2.7) with (2.2.2). You will find out that the model without capital is a special version with powers  $\alpha$  and  $\beta$  being zero.

We now have two endogenous stock variables,  $A(t)$  and  $K(t)$ . From (2.2.6), we plug in (2.2.5) and have

$$\begin{aligned}\frac{\dot{K}(t)}{K(t)} &= s \cdot (1 - a_K)^\alpha \left( (1 - a_L) \cdot \frac{A(t)L(t)}{K(t)} \right)^{1-\alpha} \\ &= \underbrace{s \cdot (1 - a_K)^\alpha (1 - a_L)^{1-\alpha}}_{c_K} \left( \frac{A(t)L(t)}{K(t)} \right)^{1-\alpha}. \end{aligned}\quad (2.2.8)$$

From (2.2.7), we plug in (2.2.5) and have

$$\frac{\dot{A}(t)}{A(t)} = \underbrace{Ba_k^\beta a_L^\gamma}_{c_A} \cdot (K(t))^\beta (L(t))^\gamma (A(t))^{\theta-1}. \quad (2.2.9)$$

Log-differentiating (2.2.8) and (2.2.9), since  $c_K$  and  $c_A$  are constants, we have

$$\begin{aligned}\frac{\dot{g}_K(t)}{g_K(t)} &= (1 - \alpha) \cdot (g_A(t) + g_L(t) - g_K(t)) \\ &= (1 - \alpha) \cdot (g_A(t) + n - g_K(t)),\end{aligned}\quad (2.2.10)$$

and

$$\begin{aligned}\frac{\dot{g}_A(t)}{g_A(t)} &= \beta \cdot g_K(t) + \gamma \cdot g_L(t) + (\theta - 1) \cdot g_A(t) \\ &= \beta \cdot g_K(t) + \gamma n + (\theta - 1) \cdot g_A(t).\end{aligned}\quad (2.2.11)$$

On the balanced growth path, by definition,  $A(t)$  and  $K(t)$  grow at constant rates, that means  $g_A(t)$  and  $g_K(t)$  are constants. Hence,  $\dot{g}_A(t) = \dot{g}_K(t) = 0$ . Therefore, in the  $(g_A, g_K)$  space, we can draw the graph of  $\dot{g}_K(t) = 0$ ,

$$g_K = n + g_A \quad (2.2.12)$$

and the graph of  $\dot{g}_A(t) = 0$ ,

$$g_K = -\frac{\gamma}{\beta} \cdot n + \frac{1-\theta}{\beta} \cdot g_A. \quad (2.2.13)$$

In order to make (2.2.12) and (2.2.13) to intersect in the first quadrant, we need to assume  $\frac{1-\theta}{\beta} > 1$ , which means decreasing returns to  $A(t)$  and  $K(t)$  in research and development. This assumption implies  $\theta < 1$  immediately. Thus, the following about convergence is natural as we discussed previously.

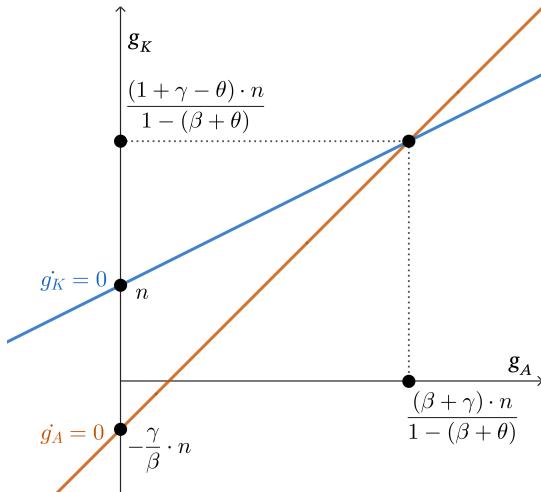


Figure 2.2.4. The intersection will be in the first quadrant under the assumption.

The growth rate will converge to the intersection. We cannot analytically prove this result, but we can use figures with expressions to represent the idea. We will see combining Figure 2.2.5 and Figure 2.2.6 do the work with any point in each section (divided by two unparallel lines, 4 sections in total) tends to the intersection  $(g_A^*, g_K^*) := \left( \frac{(1 + \gamma - \theta) \cdot n}{1 - (\beta + \theta)}, \frac{(\beta + \gamma) \cdot n}{1 - (\beta + \theta)} \right)$ .

Please see Figure 2.2.5 and Figure 2.2.6 along with the following expressions. We use algebra to simplify (2.2.10) and (2.2.11). The strategy that we fix one but another is because the quadratic term is thorny, and we hence fix it.

For (2.2.10), we fix a  $g_{K_0} > n$ , i.e., we focus on the horizontal line  $g_K(t) = g_{K_0}$ , and have

$$\begin{aligned}
\dot{g}_K(t) &= ((1 - \alpha) \cdot (g_A(t) + n - g_{K_0}) \cdot g_{K_0} \\
&= \underbrace{(1 - \alpha) \cdot g_{K_0}}_{\mathcal{K} > 0} \cdot \left( g_A + \underbrace{(n - g_{K_0})}_{-\mathcal{N} < 0} \right) \\
&= \mathcal{K} \cdot (g_A(t) - \mathcal{N})
\end{aligned} \tag{2.2.14}$$

The constant  $\mathcal{K} > 0$  since  $g_{K_0} > n$  and  $\alpha \in (0, 1)$ ; the constant  $\mathcal{N} > 0$  follows by assumption

$$g_{K_0} > n.$$

For (2.2.11), we fix a  $g_{A_0} > \frac{\gamma n}{1 - \theta}$ , i.e., we focus on the vertical line  $g_A(t) = g_{A_0}$ , and have

$$\begin{aligned}
\dot{g}_A(t) &= (\beta \cdot g_K(t) + \gamma n + (\theta - 1) \cdot g_{A_0}) \cdot g_{A_0} \\
&= \underbrace{(\theta - 1) \cdot (g_{A_0})^2 + \gamma n \cdot g_{A_0}}_{\text{constant } -\Theta < 0} + \underbrace{\beta \cdot g_{A_0} \cdot g_K(t)}_{\Gamma > 0} \\
&= \Gamma \cdot g_K(t) - \Theta
\end{aligned} \tag{2.2.15}$$

The constant  $-\Theta < 0$  since our assumption  $g_{A_0} > \frac{\gamma n}{1 - \theta}$  implies

$$0 > \gamma n + (\theta - 1)g_{A_0}$$

and thus

$$\begin{aligned}
(\theta - 1) \cdot (g_{A_0})^2 + \gamma n \cdot g_{A_0} &= g_{A_0} \cdot (\gamma n + (\theta - 1)g_{A_0}) \\
&< 0.
\end{aligned}$$

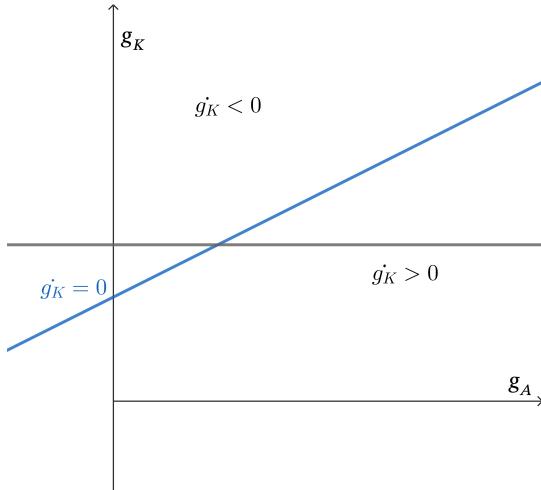


Figure 2.2.5. The intersection is at  $(\mathcal{N}, g_{K_0})$ . By (2.2.14), if  $g_A < \mathcal{N}$ , then  $\dot{g}_K < 0$ ; if  $g_A > \mathcal{N}$ , then  $\dot{g}_K > 0$ .

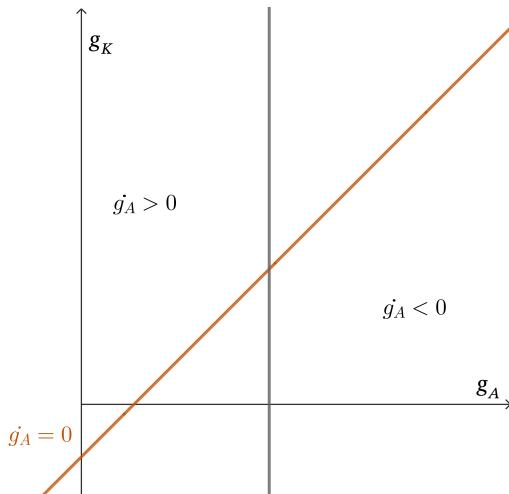


Figure 2.2.6. The intersection is at  $(g_{A_0}, \frac{\Theta}{\Gamma})$ . By (2.2.15), if  $g_K > \frac{\Theta}{\Gamma}$ , then  $\dot{g}_A > 0$ ; if  $g_K < \frac{\Theta}{\Gamma}$ , then  $\dot{g}_A < 0$ .

## 2.3 The Solow Model with Human Capital

The instructor Dr. SENOUCI did not cover this part in the second lecture. The following content is listed for future reference.

This model features **individual-level productivity** growth. Each individual makes oneself more productive by accumulating human capital.

Consider the Cobb-Douglas production function

$$Y(t) = (K(t))^\alpha (A(t)H(t))^{1-\alpha},$$

where  $H(t)$  is *worker's productive services* or *effective labor*. That is,  $H(t)$  stands for the union (sum) raw labor  $L(t)$  and human capital HK. Each worker's HK depends on the years of education  $E$  with following relation

$$H(t) = e^{\phi E} L(t),$$

where  $\phi$  is a positive constant. Let  $k := \frac{K}{AH}$  and  $y = \frac{Y}{AH}$ , we will have dynamics

$$\dot{k} = sy - (n + g + \delta)k,$$

which is identical to (1.3.4).

### **Impact of a Change in the Years of Education**

Assume each person lives for  $T$  years, studying for the first  $E$  years of one's life and working for the remaining  $T - E$  years. How does an increase in  $E$  affect output per person?

Let  $B(t)$  be the number people born at date  $t$ . We assume that the birth rate is with a growth rate  $n$ . That is, the ratio of the number  $B(t - \tau)$  of people born at date  $(t - \tau)$  to the number  $B(t)$  of people born at date  $t$  is

$$B(t) = B(t - \tau) \cdot e^{n\tau}.$$

The total population  $N(t)$  at date  $t$  is equal to those born in  $[t - T, t]$ , namely

$$\begin{aligned} N(t) &= \int_0^T B(t - \tau) d\tau \\ &= B(t) \int_0^T e^{-n\tau} d\tau \\ &= \frac{1 - e^{-nT}}{n} \cdot B(t). \end{aligned} \tag{2.3.1}$$

Likewise, the number  $L(t)$  of workers at date  $t$  is equal to those who born in  $[t - T, t - E]$ , namely

$$\begin{aligned}
L(t) &= \int_E^T B(t - \tau) d\tau \\
&= \frac{e^{-nE} - e^{-nT}}{n} \cdot B(t).
\end{aligned} \tag{2.3.2}$$

Therefore, from (2.3.1) and (2.3.2), the output per person at date  $t$  equals

$$\begin{aligned}
\frac{Y(t)}{N(t)} &= \frac{Y(t)}{A(t)H(t)} \cdot \frac{A(t)H(t)}{N(t)} \\
&= y(t) \cdot \frac{A(t)e^{\phi E} L(t)}{N(t)} \\
&= A(t)y(t) \cdot \frac{e^{\phi E} \cdot (e^{-nE} - e^{-nT})}{1 - e^{-nT}}
\end{aligned} \tag{2.3.3}$$

### The Golden-Rule Level of Education

An increase in  $E$  has ambiguous effect on the steady-state  $\frac{Y(t)}{N(t)}$ . It will make workers become more productive, i.e.,  $H(t)$  increases. However, it will also make workers work for less time, i.e.,  $T - E$  falls.

Therefore, there is the golden-rule level  $E^*$  of education that maximizes  $\frac{Y(t)}{N(t)}$ . To simplify the differentiated function, we differentiate  $\frac{1 - e^{-nT}}{A(t)y(t)} \cdot \frac{Y(t)}{N(t)}$  and have

$$\begin{aligned}
\frac{d}{dE} \left( \frac{1 - e^{-nT}}{A(t)y(t)} \cdot \frac{Y(t)}{N(t)} \right) &= \frac{d}{dE} (e^{\phi E} \cdot (e^{-nE} - e^{-nT})) \\
&= e^{\phi E} \cdot (\phi \cdot e^{-nE} - \phi \cdot e^{-nT} - n \cdot e^{-nE}).
\end{aligned}$$

Since  $e^{\phi E} > 0$ , we set the first derivative to zero, eliminate  $e^{\phi E}$ , and have

$$\phi \cdot e^{-nE^*} - \phi \cdot e^{-nT} - n \cdot e^{-nE^*} = 0$$

$$e^{-nE^*} = \frac{\phi}{\phi - n} \cdot e^{-nT}$$

$$-nE^* = \ln \frac{\phi}{\phi - n} - nT$$

$$E^* = T - \frac{1}{n} \ln \frac{\phi}{\phi - n}.$$

## 2.4 Conclusion

Technological change can be theorized as a side-effect of economic activity or as an investment.

We covered a learning-by-doing model, which depicts productivity growth as a side-effect of some rational economic activities, and two models with investment in research and development, which employ specialized workers in the lecture; a human-capital-based model is also referred.

The most up-to-date models involve investment in research and development or are based on human capital—naturally, these models are more complex than what we have covered so far. They feature individually rational decision-making. As Romer explains in 1986 and 1990, firms invest in research and development in the hope of outpacing their competitors, and in pure and perfect competition, no costly research and development is undertaken due to the high costs. Lucas, in 1988, also notes that individuals acquire human capital to increase future labor earnings, although this comes at the expense of reduced earnings today. These models are complex, and we are currently unable to treat them extensively.

### 3 Theory of Consumption

We have taken minimal assumptions about consumption; we assumed a constant saving rate  $s$  with consumption

$$C_t = (1 - s)Y_t$$

and investment

$$I_t = sY_t.$$

We now lay the foundations of the pure neoclassical theory of consumption.

We are interested in the rational allocation of income to consumption **at different rates**.

The reason we are doing this is that we want to microfound the macroeconomic demand for consumption in dynamic models.

We want a theory that makes sense of the “Kuznets consumption puzzle:”

- over the long run, consumption per capita and output per capita have parallel trends, i.e.,  $C/Y$  is nearly constant, and
- in the short-run, consumption per capita is positively correlated with output per capita, but less volatile; the change in per-capita disposable income is more than real per-capita consumption.

#### 3.1 The Intertemporal Utility Function

Some consumer lives for  $T + 1$  periods with  $T \geq 1$ . We let integers  $0, 1, \dots, T$  denote those periods.

In addition, we use the notation  $[0, T]$  to indicate the set  $\{0, 1, \dots, T\}$ .

The consumer gets utility from consuming  $C_t$  at the end of each period  $t$ . The utility function  $U$  is written as

$$U(C_0, \dots, C_T) = \sum_{t=0}^T \beta^t \cdot u(C_t), \quad (3.1.1)$$

where  $\beta \in (0, 1)$  is called the *subjective discount factor*, and *instantaneous utility function*  $u : \mathbb{R}^+ \rightarrow \mathbb{R}$  is twice continuously differentiable, strictly increasing, and strictly concave.

The form of  $U$  in (3.1.1) is the only function that satisfies *time consistency*. In addition,  $\beta \in (0, 1)$  ensures that the agent puts more weight on the present than on the future: the future is discounted at a constant rate  $\beta$ . The lower  $\beta$ , the more impatient the agent is.

Moreover, strictly concave function  $u$  will make the consumer consume at all periods, as indifference curves are convex.

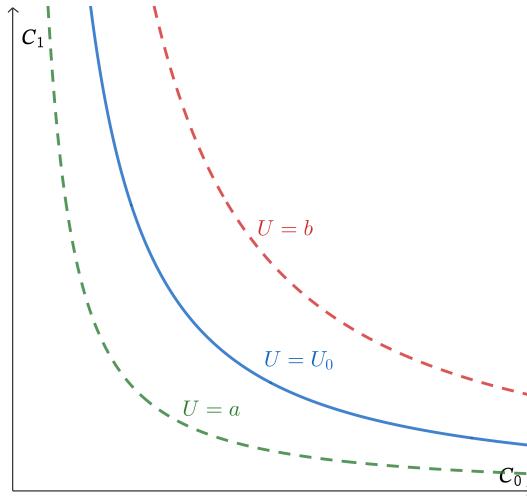


Figure 3.1.1. Indifference curves for  $T = 1$  with  $a < U_0 < b$ .

For the instantaneous utility function  $u$ , there is a common function called the *constant intertemporal elasticity of substitution* (CIES) utility in the form of

$$u(C) = \frac{C^{1-\theta} - 1}{1 - \theta}, \quad (3.1.2)$$

where  $\theta \in \mathbb{R}^+ \setminus \{1\}$ . The constant  $\theta$  is the intertemporal elasticity of substitution as well as the inverse of the relative risk aversion. Note that the log utility, also known as the Cobb-Douglas utility,

$$u(C) = \ln C$$

is the case for (3.1.2) as  $\theta \rightarrow 1$ .

Elasticity measures how responsive the quantity demanded or supplied of a good is to changes in its price or other factors. If the price elasticity of demand for a good is high, it means that a small change in price causes a large change in the quantity demanded, often because consumers

can easily switch to alternatives. On the contrary, if the elasticity is low, demand is relatively stable, meaning that changes in price have little effect on the quantity demanded, often because the good is a necessity or lacks close substitutes.

### 3.2 The Intertemporal Budget Constraint

The goal of the agent is to maximize the intertemporal utility function  $U$ .

Assume that the agent earns an exogenous *stream of income*  $(Y_t)_{t=0}^T$  with  $Y_t \geq 0$  for all  $t$  and some are positive. Suppose the *real interest rate* between two consecutive periods is a constant  $r > -1$ .

The agent can freely borrow and lend at real rate  $r$ . It means that for any  $t \in \{0, 1, \dots, T-1\}$ , the agent can trade one unit of  $C_t$  for  $(1+r)$  units of  $C_{t+1}$ , and vice versa. Thus, we can know that  $(1+r)$  is the price of current consumption in terms of the consumption in the next period. In general, the interest rate  $r_t$  may vary from one date to another, but we will abstract from that complication.

Let  $(S_t)_{t=0}^T$  be the amount of savings at each period with

$$S_t = Y_t - C_t.$$

The amount of saving can be positive or negative. Let  $A_t$  be the amount of savings stock accumulated at the end of each period. Hence, we have

$$A_0 = S_0,$$

$$A_1 = (1+r)A_0 + S_1,$$

$$A_2 = (1+r)A_1 + S_2,$$

⋮

$$A_T = (1+r)A_{T-1} + S_T.$$

Remember that  $r$  is the real interest rate. Hence, to save  $S \geq 0$  at some period brings  $(1+r)S$  extra resources one period after, and to incur debt  $S \leq 0$  at some period leads to  $(1+r)S$  less resources one period after.

We assume that the agent will not die in debt, and hence  $A_T \geq 0$ . In addition, the agent will try to consume as much as possible at date  $T$ , and hence  $A_T \leq 0$ . These imply  $A_T = 0$ . Therefore, at the end of period  $t$ , we multiply  $\frac{1}{(1+r)^t}$  on both sides of each equation and have

$$A_0 = S_0,$$

$$\frac{1}{1+r} \cdot A_1 = A_0 + \frac{1}{1+r} \cdot S_1,$$

$$\frac{1}{(1+r)^2} \cdot A_2 = \frac{1}{1+r} \cdot A_1 + \frac{1}{(1+r)^2} \cdot S_2,$$

⋮

$$0 = \frac{1}{(1+r)^{T-1}} \cdot A_{T-1} + \frac{1}{(1+r)^T} \cdot S_T.$$

Summing both sides yields

$$0 = \sum_{t=0}^T \frac{S_t}{(1+r)^t}.$$

Since  $S_t = Y_t - C_t$ , we have the *intertemporal budget constraint* (IBC)

$$\sum_{t=0}^T \frac{C_t}{(1+r)^t} = \sum_{t=0}^T \frac{Y_t}{(1+r)^t}. \quad (3.2.1)$$

Remember we have derived (3.2.1) by using that the agent dies with 0 wealth. This entails this relationship between the consumption and the income streams.

The left-hand side of (3.2.1) is the *present value* (PV) of the consumption stream  $(C_t)_{t=0}^T$ .

It is the value in terms of date-0 consumption of everything that the agent will consume during

one's lifetime. The right-hand side is the present value of the income stream. The value in terms of date-0 good of everything that the agent will earn during one's lifetime.

A consumption path is *feasible* if and only if it satisfies (3.2.1).

### 3.3 The Fisher Model

Now, let's focus on the case when  $T = 1$ . That is, an individual needs to allocate one's consumption stream over just two periods: "today" ( $t = 0$ ) and tomorrow ( $t = 1$ ). The intertemporal budget constraint is

$$C_0 + \frac{C_1}{1+r} = Y_0 + \frac{Y_1}{1+r}.$$

The point  $(Y_1, Y_2)$  is called the *initial endowment*.

Since there are only two periods, the intertemporal utility function becomes

$$U(C_0, C_1) = u(C_0) + \beta \cdot u(C_1).$$

In addition, there is the intertemporal budget constraint

$$C_0 + \frac{C_1}{1+r} = Y_0 + \frac{Y_1}{1+r}. \quad (3.3.1)$$

The relative price of consumption is given by  $\frac{1}{1/(1+r)} = 1+r$ . By giving up  $(1+r)$  units of future consumption  $C_2$ , the individual can obtain 1 unit of current consumption  $C_1$ .

#### Optimization

The individual's optimality condition equates the marginal rate of substitution of  $C_0$  for  $C_1$  to their price ratio

$$\frac{u'(C_0)}{\beta \cdot u'(C_1)} = 1+r.$$

Another mathematical derivation is that we set the first derivative of the function  $U$  under the intertemporal budget constraint (3.3.1) with respect to  $C_0$  to be 0 and have

$$U'(C_0) = u'(C_0) + \beta \cdot u'(C_1) \cdot \frac{dC_1}{dC_0}$$

$$= 0$$

$$\Rightarrow u'(C_0) = (1+r) \cdot \beta \cdot u'(C_1). \quad (3.3.2)$$

Note that we have  $\frac{dC_1}{dC_0} = -(1+r)$  from (3.3.1).

We can also utilize the optimality to derive the relation. Say the point  $(C_0^*, C_1^*)$  is optimal. If  $C_0$  increases by  $dC$ , then  $C_1$  will decrease by  $(1+r)dC$  from (3.3.1). Since the point  $(C_0^*, C_1^*)$  is optimal, we should have the change of the intertemporal utility function is zero, i.e.,

$$\begin{aligned} dU &= u'(C_0^*) dC - \beta \cdot u'(C_1^*) \cdot (1+r) dC \\ &= 0. \end{aligned} \quad (3.3.3)$$

If  $dU \neq 0$ , then the point  $(C_0^*, C_1^*)$  will not be optimal. The last equation in (3.3.3) coincides (3.3.2).

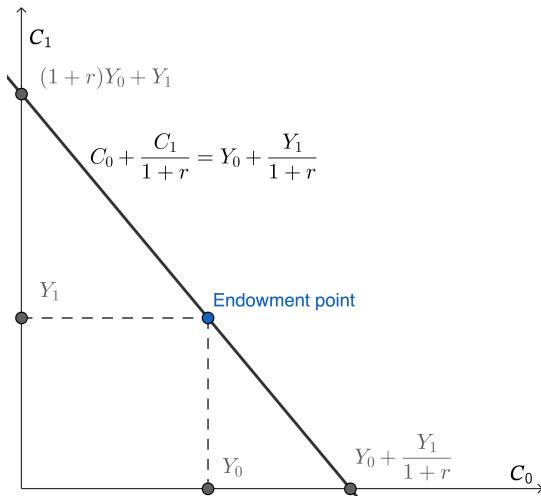


Figure 3.3.1. Consumer's endowment in the Fisher model.

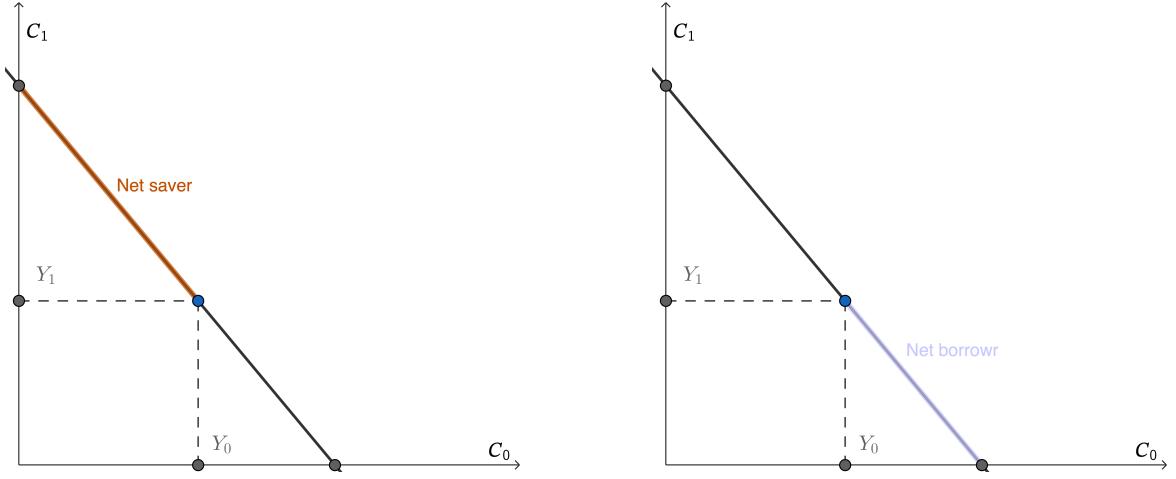


Figure 3.3.2. Consumer's choices correspond to difference status (saver or borrower).

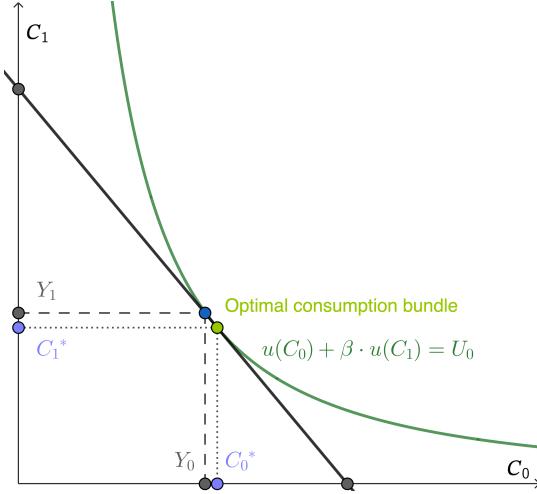


Figure 3.3.3. Consumer's optimum in the Fisher model. The amount  $C_0^* - Y_0$  is saving  $S^*$ .

For given present value of lifetime income  $Y_0 + \frac{Y_1}{1+r}$ , the time pattern of income is not important for consumption, but critical for saving. The individual who is relatively rich now and expects to be poor in the future (high  $Y_0$  and low  $Y_1$ ) will be a *saver*; the one with a low  $Y_0$  and a high  $Y_1$  will be a *borrower*.

### Analytical Solutions

If the instantaneous utility function  $u$  is the **Cobb-Douglas** utility function, then the optimal consumption path  $(C_0^*, C_1^*)$  must satisfy equations (3.3.1) and (3.3.2). Hence, we should solve

$$\begin{aligned}
& \left\{ \begin{array}{l} C_0^* + \frac{C_1^*}{1+r} = Y_0 + \frac{Y_1}{1+r}; \\ \frac{1}{C_0^*} = \beta \cdot (1+r) \cdot \frac{1}{C_1^*}, \end{array} \right. \\
\implies & \left\{ \begin{array}{l} C_0^* + \frac{C_1^*}{1+r} = Y_0 + \frac{Y_1}{1+r}; \\ C_1^* = \beta \cdot (1+r) \cdot C_0^*, \end{array} \right. \\
\implies & \left\{ \begin{array}{l} C_0^* = \frac{1}{1+\beta} \cdot \left( Y_0 + \frac{Y_1}{1+r} \right); \\ C_1^* = \frac{\beta \cdot (1+r)}{1+\beta} \cdot \left( Y_0 + \frac{Y_1}{1+r} \right). \end{array} \right.
\end{aligned}$$

If the instantaneous utility function  $u$  is the **constant intertemporal elasticity in substitution** utility function, then the optimal consumption path  $(C_0^*, C_1^*)$  must satisfy equations (3.3.1) and (3.3.2). Hence, we should solve

$$\begin{aligned}
& \left\{ \begin{array}{l} C_0^* + \frac{C_1^*}{1+r} = Y_0 + \frac{Y_1}{1+r}; \\ (C_0^*)^{-\theta} = \beta \cdot (1+r) \cdot (C_1^*)^{-\theta}, \end{array} \right. \\
\implies & \left\{ \begin{array}{l} C_0^* + \frac{C_1^*}{1+r} = Y_0 + \frac{Y_1}{1+r}; \\ C_0^* = (\beta \cdot (1+r))^{-\frac{1}{\theta}} \cdot C_1^*, \end{array} \right. \\
\implies & \left\{ \begin{array}{l} C_0^* = \frac{Y_0 + \frac{Y_1}{1+r}}{1 + \frac{(\beta \cdot (1+r))^{\frac{1}{\theta}}}{1+r}}; \\ C_1^* = \frac{(\beta \cdot (1+r))^{\frac{1}{\theta}} \cdot \left( Y_0 + \frac{Y_1}{1+r} \right)}{1 + \frac{(\beta \cdot (1+r))^{\frac{1}{\theta}}}{1+r}}. \end{array} \right.
\end{aligned}$$

### 3.4 Ricardian Equivalence

Ricardian equivalence, in plain English, showcases that the timing of taxes has **no** impact on private agents' consumption. We can explain in another way: for a given public expense  $G$ , whether  $G$  is funded through taxation or deficit does not influence the saving/consumption behavior of private agents.

Suppose the government has to spend  $G$  today. There are two options: pay for it with taxes today or with a deficit today. However, a deficit today means higher taxes tomorrow for the private agents. Say the government switch from taxes today (option 1) to taxes tomorrow (option 2). The individuals react by saving today what they would have paid in taxes in option 1, to pay future taxes in option 2 without changing their intertemporal consumption profile, which is the same under option 1 and option 2.

We stick to the two-period model still. The following argument will hold for  $T$  periods as well. With taxes, the household's date- $t$  disposable income is  $Y_t - T_t$ , where  $T_t$  is the taxes in date  $t$ ; hence, the intertemporal budget constraint becomes

$$C_0 + \frac{C_1}{1+r} = (Y_0 - T_0) + \frac{Y_1 - T_1}{1+r}. \quad (3.4.1)$$

The government's budget constraints are

$$G_0 = T_0 + B \quad \text{and} \quad G_1 = T_1 - (1+r)B,$$

where  $B$  stands for government's borrowing at period 0.

We combine the two equations for the government's budget constraints and have

$$\begin{aligned} G_1 &= T_1 - (1+r) \cdot (G_0 - T_0) \\ \implies G_0 + \frac{G_1}{1+r} &= T_0 + \frac{T_1}{1+r}, \end{aligned}$$

which is the government's intertemporal budget constraint.

Assume the government decides to cut current taxes with  $\Delta T_0 < 0$  without altering the spending profile, so  $\Delta G_0 = \Delta G_1 = 0$ . It would have to raise borrowing  $\Delta B = -\Delta T_0 > 0$ . To repay debt, it would have to raise future taxes

$$\Delta T_1 = (1 + r)\Delta B$$

$$= -(1 + r)\Delta T_0.$$

Would the consumer's lifetime disposable income change? **No**, it would not.

The right-hand side of (3.4.1) will change to

$$\begin{aligned} (Y_0 - T_0 - \Delta T_0) + \frac{Y_1 - T_1 - \Delta T_1}{1 + r} &= (Y_0 - T_0) + \frac{Y_1 - T_1}{1 + r} - \Delta T_0 - \frac{\Delta T_1}{1 + r} \\ &= (Y_0 - T_0) + \frac{Y_1 - T_1}{1 + r} - \Delta T_0 - \frac{-(1 + r)\Delta T_0}{1 + r} \\ &= (Y_0 - T_0) + \frac{Y_1 - T_1}{1 + r}, \end{aligned}$$

which did not change at all.

If the budget constraints are unchanged, the optimal intertemporal consumption profile  $(C_1^*, C_0^*)$  will not change either. What will change is individual's endowment point and savings.

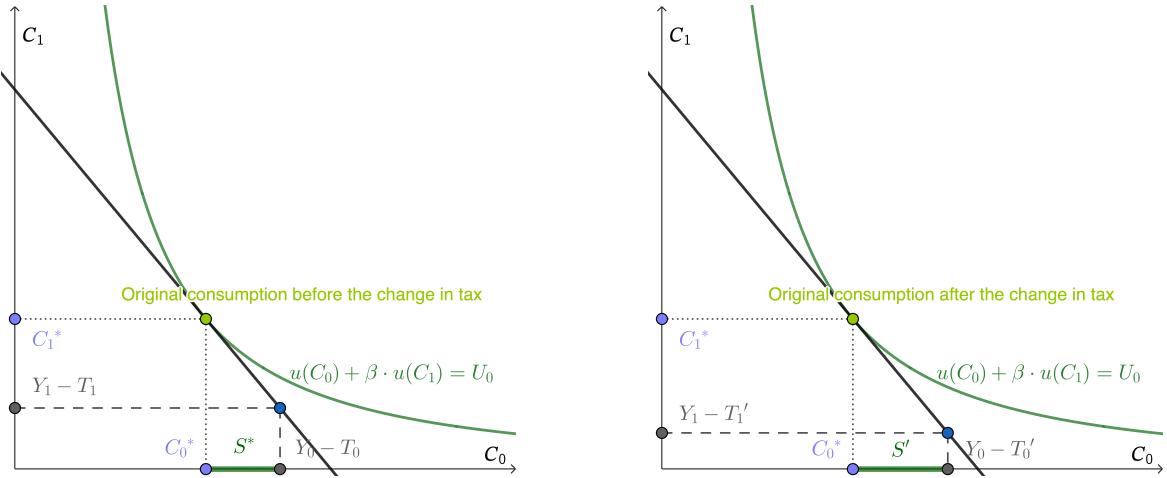


Figure 3.4.1. A deficit-financed tax cut does not affect  $(C_0^*, C_1^*)$ .

In fact, the assumptions needed for Ricardian equivalence to hold is strict and nearly impossible to be achieved in the real world. Those assumptions are

1. consumers are forward-looking and have long horizons (as long as the one of the government),
2. financial markets are perfect (borrow and lend at market interest rates freely and same interest rate for households and the government),
3. there is no uncertainty, and
4. taxation is non-distortionary, i.e., the budget is financed by **lump-sum** taxes (based on a fixed amount).

### 3.5 The Life Cycle Theory

We sketch the life cycle theory qualitatively.

The households' goal is to maintain a stable standard of living. Consider a consumer who

- lives for  $T$  periods,
- works during the first  $t_R$  years of one's life,
- earns a wage  $w > 0$  during the working life in  $[0, t_R]$ , and
- is retired and does not earn anything in  $[t_R, T]$ .

The consumer saves (accumulates wealth) during working life and dissaves during retirement.

How does the optimal consumption path look like?

Suppose there is no interest rate involved, and one's consumption is denoted  $C$ . Then, the optimal consumption  $C^*$  is to spend all one's money at period  $T$ . Hence, it should satisfy the equation

$$C^* \cdot T = w \cdot t_R.$$

The left-hand side is lifetime income, and the right-hand side is lifetime income. Since  $T > t_R$ , it follows that  $w > C^*$ . Hence, the consumer will save in one's first  $t_R$  periods, and dissave in one's last  $T - t_R$  periods. The assets of the consumer will reach the peak at the end of period  $t_R$ .

### 3.6 Consumption under Uncertainty

So far, the only motive for saving was provision for future consumption, for oneself or one's descendants. However, the individual may also save to pursue other goals; for example, there is *precautionary motive* to incent one to save in order to insure oneself against future fluctuations of income.

Here is the general idea: future income fluctuations create downside risks for the consumer. If one has saved  $X$ , one can be certain to consume at least  $(1 + r)X$  in the future, even if income is zero. Therefore, individuals accumulate savings to buffer against risks and reach a target wealth level.

We will see that more uncertainty leads to more savings if and only if  $u'$  is convex, i.e.,  $u'''(C) > 0$  for all  $C$ .

Consider the following 2-period optimization problem,

$$\max_{C_0, \tilde{C}_1} u(C_0) + \beta \cdot \mathbb{E}(u(\tilde{C}_1)) \quad (3.6.1)$$

under the constraint

$$C_0 + \frac{\tilde{C}_1}{1+r} = Y_0 + \frac{\tilde{Y}_1}{1+r}, \quad (3.6.2)$$

where the period-0 income  $Y_0$  is certain, period-1 income  $\tilde{Y}_1$  is a random variable distributed over  $[\underline{Y}, \bar{Y}]$  with density function  $f_{\tilde{Y}_1}$ . Hence, the consumer will choose some  $C_0$  at date 0 and will consume

$$\tilde{C}_1 = (1+r) \left( Y_0 + \frac{\tilde{Y}_1}{1+r} - C_0 \right)$$

at date 1. We can see that  $\tilde{C}_1$  is a random variable as well.

The consumer maximizes the *expected utility*, which is also known as the von Neumann-Morgenstern utility.

We plug (3.6.2) into the function in the right-hand side of (3.6.1) by eliminating  $\tilde{C}_1$  and have

$$\begin{aligned} g(C_0) &:= u(C_0) + \beta \cdot \mathbb{E}(u(\tilde{C}_1)) \\ &= u(C_0) + \beta \cdot \mathbb{E}\left(u\left((1+r)\left(Y_0 + \frac{\tilde{Y}_1}{1+r} - C_0\right)\right)\right) \\ &= u(C_0) + \beta \cdot \mathbb{E}(u((1+r) \cdot (Y_0 - C_0) + \tilde{Y}_1)) \\ &= u(C_0) + \beta \cdot \int_{\underline{Y}}^{\bar{Y}} u(f_{\tilde{Y}_1}(y) + (1+r) \cdot (Y_0 - C_0)) \, dy. \end{aligned}$$

Suppose that we can change the order of the differentiation operator and the integral operator.

We differentiate both sides with respect to  $C_0$  and have

$$\begin{aligned} g'(C_0) &= u'(C_0) - \beta \cdot \int_{\underline{Y}}^{\bar{Y}} u'\left(f_{\tilde{Y}_1}(y) + (1+r) \cdot (Y_0 - C_0)\right) \cdot(-(1+r)) \, dy \\ &= u'(C_0) - \beta \cdot (1+r) \cdot \mathbb{E}(u'(\tilde{C}_1)). \end{aligned} \tag{3.6.3}$$

Since we are looking for the maximum, we set (3.6.3) to be zero and obtain the condition

$$u'(C_0) = \beta \cdot (1+r) \cdot \mathbb{E}(u'(\tilde{C}_1)). \tag{3.6.4}$$

Assume that  $u'$  is convex. Recall Jensen's inequality, that is, for any positive random variable  $X$  and for any convex function  $f$ ,  $\mathbb{E}(f(X)) \geq f(\mathbb{E}(X))$ . Plug  $f = u'$  and  $X = \tilde{C}_1$  into Jensen's inequality, we will have

$$\mathbb{E}(u'(\tilde{C}_1)) \geq u'(\mathbb{E}(\tilde{C}_1)). \tag{3.6.5}$$

Compare two situations, without uncertainty and with uncertainty. Call  $C_0^*$  the solution for  $C_0$  with no uncertainty over  $Y_0$  and  $Y_1$ ; call  $C_0'$  the solution for  $C_0$  with uncertainty over  $Y_1$ . Note that  $C_0, Y_0$ , and  $Y_1$  are variables like  $x$  or  $y$ , while  $C_0^*$  and  $C_0'$  are solutions (constant).

See Figure 3.6.1 and Figure 3.6.2 about the comparison between two levels of uncertainty. In Figure 3.6.1 and Figure 3.6.2,  $\sigma_1 < \sigma_2$  and  $C_0' > C_0''$ . This suggests more uncertainty leads more saving.

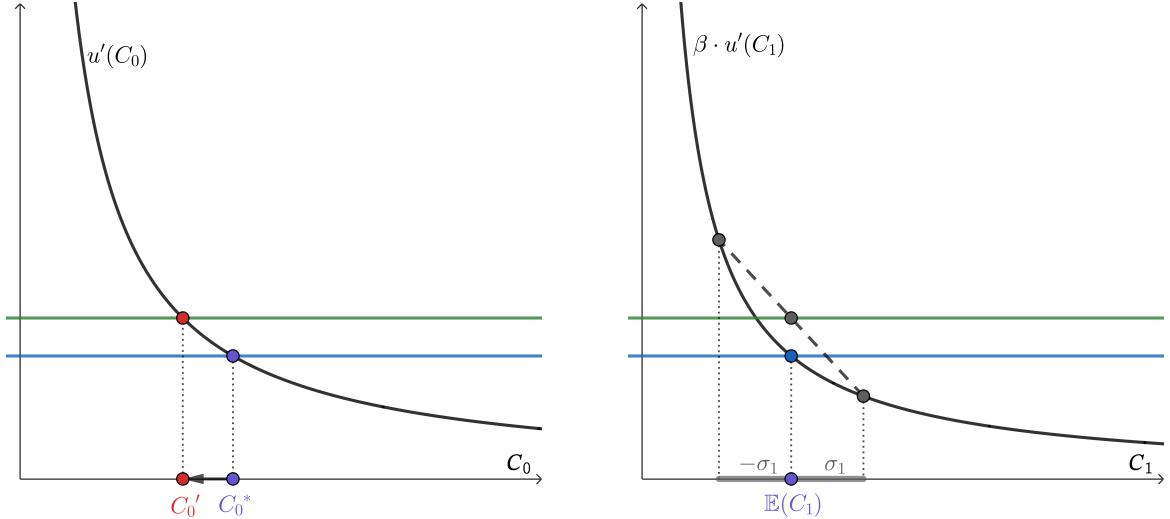


Figure 3.6.1. With convex  $u'$ , a rise in  $\mathbb{V}(Y_1)$  lowers optimal  $C_0$ . Before the rise of  $\sigma$ , the optimal first-period consumption is  $C_0'$ .

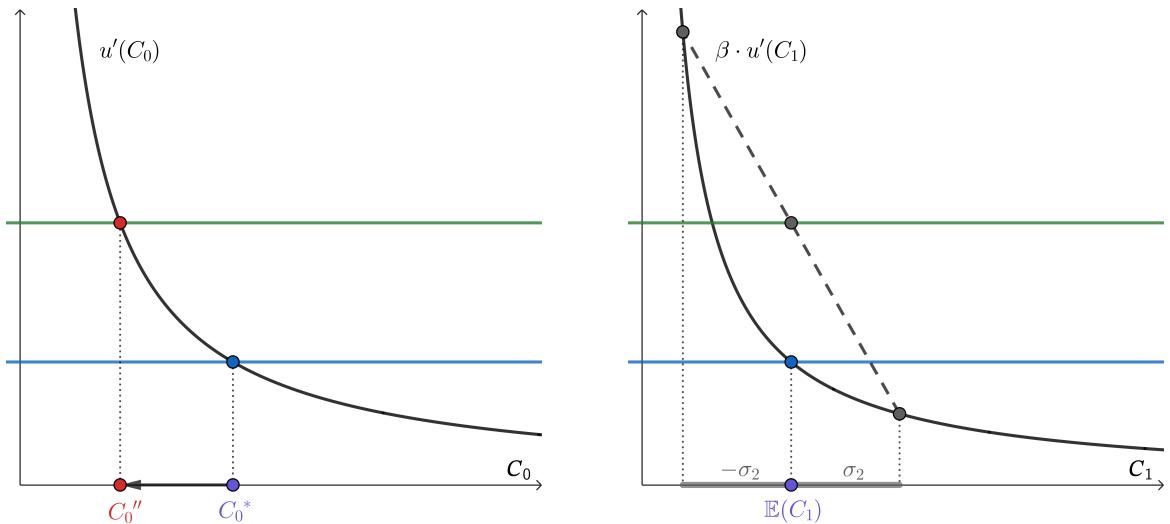


Figure 3.6.2. With convex  $u'$ , a rise in  $\mathbb{V}(Y_1)$  lowers optimal  $C_0$ . After the rise of  $\sigma$ , the optimal first-period consumption is  $C_0''$ .

## 4 Overlapping-Generations Models

We have taken assumptions regarding aggregated relationships, such as  $S = sY$  in Section 1, and modeled individual behavior in Section 3. The next step is to aggregate individual behaviors; in order to do so, we need to assume a **demographic structure**. We specifically focus on the interactions between utility-maximizing agents on markets.

The *overlapping-generations* structure is one central demographic structure in macroeconomics. The other one is made of infinite-lives consumers (or dynasties) all born at the initial date.

### 4.1 Overlapping-Generations

The assumption of overlapping generations implies that at each date, different generations of individuals, of various ages, coexist. One period after, the oldest people die, the young gets older, and a new generation comes up.

Why do we study overlapping-generations models? In reality, generations do overlap: people are born and do die. This is a natural framework of analysis for inter-generational issues:

- causes and implications of life-cycle behavior,
- pension systems (fully-funded vs. pay-as-you-go),
- public dept,
- endogenous fertility, family macroeconomics, human capital, etc.

Simple overlapping-generations models generate new and important insights on inter-generational dynamics and welfare.

However, market equilibrium can be Pareto-suboptimal in overlapping-generations models even seemingly without any market distortion (imperfect competition, moral hazard, adverse selection, etc.). More importantly, we have a conclusion that the demographic structure itself can lead to some inefficiency.

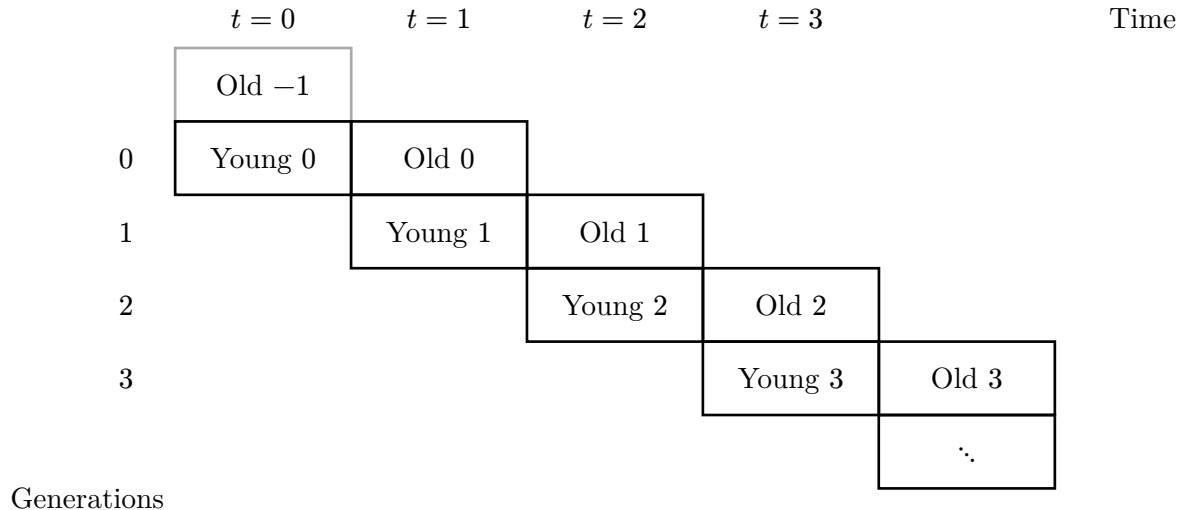


Figure 4.1.1. Two-Period Overlapping Generations

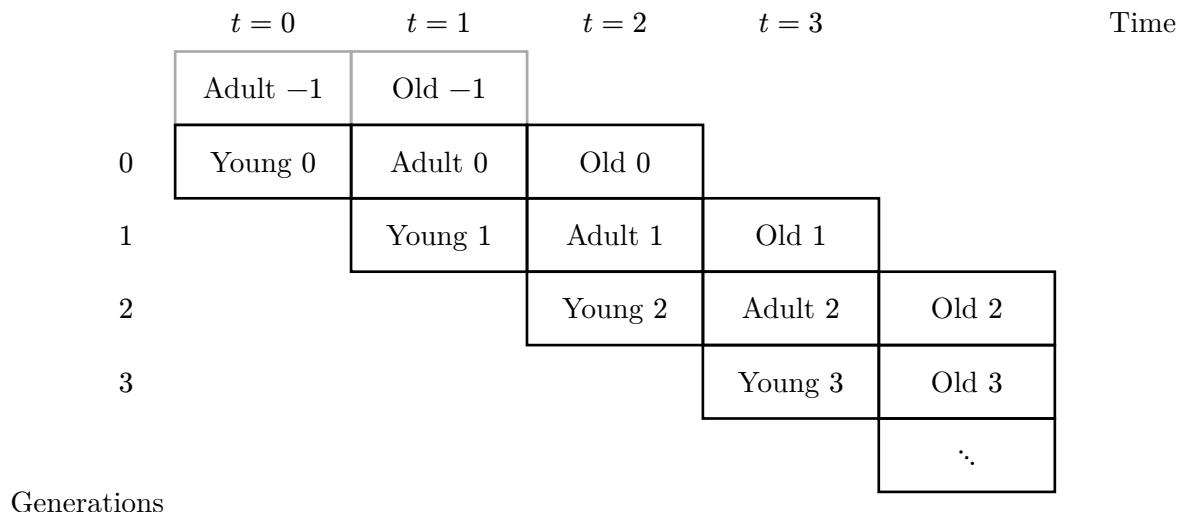


Figure 4.1.2. Three-Period Overlapping Generations

## 4.2 Overlapping-Generations in Endowment Economies

In this subsection, we will see a 2-period overlapping generations model given by Paul Samuelson in 1958.

Consider a 2-period overlapping generations structure. There is one perishable (non-storable) good. Each generation is  $(1 + n)$  times as big as the previous one with the relation

$$N_{t+1} = (1 + n)N_t,$$

where  $N_t$  is the size of generation  $t$ ,  $N_0 = 1$ . Take Figure 4.1.1 for example, the number of people in Young 2 will be  $(1 + n)$  times as big as the number of people in Young 1, and the number of people in Old 1 equals the number of people in Young 1.

In addition, we assume that all individuals have the utility function  $U$  over young-age consumption  $c^y$  and old-age consumption  $c^o$  with

$$U(c^y, c^o) = u(c^y) + \beta \cdot u(c^o),$$

where  $u'' < 0$ ,  $u' > 0$ , and  $u$  satisfies the Inada condition.

The young is endowed with 1 unit of the perishable good, and the old is endowed with 0.

### Credit Markets

It is natural that people start from credit markets, i.e., markets without money.

If facing an interest rate  $r > -1$ , the young born at date  $t$  wants to save to finance old-age consumption, since there will be no income in your old-age year.

However, there is actually no way to save. The good is perishable, the other young people want to save as well, and old people will not be able to repay in the next period. Hence, the old will just consume  $c^o = 0$ , while the young will consume  $c^y = 1$ , which is a suboptimal equilibrium.

The credit market cannot function on a bilateral basis under 2-period overlapping generation models, as two people of different generations never meet twice.

### Remedy

Imagine that there exists an agree-on in fixed quantity  $M$  (say gold or usless paper “money”), initially held by the old of date  $t = 0$ .

Suppose that the old and every future generation believe they will be able to buy and sell money at price  $p_t$  in period  $t$  in terms of goods.

People might now be able to buy money for goods when young, sell money for goods when old. Does it happen at equilibrium? What is the equilibrium interest rate  $\frac{p_{t+1}}{p_t}$ ?

Suppose you were born at date  $t \geq 0$ . If you know  $p_t$  and have perfect foresight over  $p_{t+1}$ , your program will be maximizing your utility  $U$ , i.e., looking for

$$\max_{m_t} u(1 - p_t m_t) + \beta \cdot u(p_{t+1} m_t),$$

where  $m_t$  stands for the price of money in terms of the good.

To maximize the function, we just set the first derivative to be zero and have

$$u'(1 - p_t m_t) \cdot (-p_t) + \beta \cdot u'(p_{t+1} m_t) \cdot p_{t+1} = 0$$

$$u'(1 - p_t m_t) = \beta \cdot u' \left( \frac{p_{t+1}}{p_t} \cdot p_t m_t \right) \cdot \frac{p_{t+1}}{p_t},$$

which implies

$$p_t m_t = f \left( \frac{p_{t+1}}{p_t} \right)$$

for some function  $f$ . This money demand function  $f$  is a saving function.

The demand-supply equilibrium at date  $t$  requires the total money demand equals the total money supply, i.e.,

$$\begin{aligned} \underbrace{p_t \cdot M}_{\text{total supply}} &= \underbrace{N_t \cdot p_t \cdot m_t}_{\text{total demand}} \\ &= \underbrace{N_0}_{1} \cdot (1+n)^t \cdot f \left( \frac{p_{t+1}}{p_t} \right) \\ &= (1+n)^t \cdot f \left( \frac{p_{t+1}}{p_t} \right). \end{aligned} \tag{4.2.1}$$

Similarly, the demand-supply equilibrium at date  $t+1$  requires

$$p_{t+1} \cdot M = (1+n)^{t+1} \cdot f \left( \frac{p_{t+2}}{p_{t+1}} \right). \tag{4.2.2}$$

Consider a steady state when  $\frac{p_{t+1}}{p_t} = 1+r$  for all  $t$ . Then, (4.2.2) divided by (4.2.1) yields

$$1+r = 1+n.$$

The only possible equilibrium real interest rate (rate of return on money) is the population growth rate!

If  $\frac{p_{t+1}}{p_t} = 1 + n$  for all  $t$  and  $p_0$  is determined by  $p_0 M = f(1 + n)$ , then supply-demand will be ensured at all dates on the money market (or equivalently consumption-goods), and the equilibrium is also welfare-maximizing for all generations: money restores optimality, as it restores inter-generational trade. Notice that, however, you need to be certain that future generations will all accept money as a means of savings.

We talked about money (non-productive) a lot in this section. What about an economy with a productive asset?

### 4.3 The Diamond Model

Consider a 2-period overlapping-generation model with population growth rate  $n > 0$ , and size of the cohort at time  $t$  is  $L_t = (1 + n)^t L_0$ . The initial old (born at  $t = -1$ ) are each endowed with capital  $\bar{k}$ . Consumers born at date  $t$  will have the utility

$$U(c^y_t, c^o_{t+1}) = u(c^y_t) + \beta \cdot u(c^o_{t+1}).$$

The young has 1 unit of labor, and the old has 0. The young earns competitive wage  $w_t$ , consumes  $c^y_t$ , and saves  $s_t$  in the form of capital. Also, the young earns competitive interest rate  $r_{t+1}$  on  $s_t$ , consumes  $c^o_{t+1} = (1 + r_{t+1}) \cdot s_t$  when old. We assume that capital does not depreciate.

The production function is  $F(K, L)$ , of which firms operate competitively. We assume that  $F$  is beautiful enough; that is,  $F \in C^2$ , is homogeneous of degree 1, is strictly increasing in  $K$  and  $L$ , and satisfies the Inada condition. The intensive form of  $f$  also satisfies  $f'' < 0$  and  $f' > 0$ .

Date- $t$  capital stock  $K_t$  is made of the savings of the old date  $t$ :

$$K_t = s_{t-1} L_{t-1} \quad \text{for } t \geq 1, \quad \text{and } K_0 = \bar{k} L_{-1}.$$

## The Optimal Steady State

We first consider the problem of a **benevolent social planner** who chooses the levels of consumption of each generation and of capital stock. The planner wants to provide the next generation with the same possibilities. To be more precise,  $nk_t$  capital must be accumulated at each period, so that the capital-labor ratio  $\frac{K_t}{L_t}$  will be constant for all  $t$ . That is to say, the social planner has 0 discount rate, who values only the steady-state level well-being.

Hence, the optimization problem becomes to find

$$\max_{c^y, c^o, k} U(c^y, c^o) \quad (4.3.1)$$

under the constraint

$$F(K, L) = Lc^y + \frac{L}{1+n} \cdot c^o + nK, \quad (4.3.2)$$

where  $+nK$  is to maintain the capital-labor ratio.

We can simplify (4.3.2) to make the intensive form appear, having

$$\begin{aligned} F(K, L) &= Lc^y + \frac{L}{1+n} \cdot c^o + nK \\ F(k, 1) &= c^y + \frac{c^o}{1+n} + nk \\ f(k) - nk &= c^y + \frac{c^o}{1+n}. \end{aligned} \quad (4.3.3)$$

The problem has two components: maximizing the net output  $f(k) - nk$  with respect to  $k$  and maximizing  $U$  under the budget constraint.

We first maximize the net output. It is clear that we have

$$f'(k^*) = n. \quad (4.3.4)$$

We can now maximize  $U$  under (4.3.3) with (4.3.4), having that the optimal consumption bundle  $(c^{y*}, c^{o*})$  will satisfy

$$\begin{aligned}
& \frac{d}{dc^y}(u(c^y) + \beta \cdot u(c^o)) = 0 \\
\implies & u'(c^{y*}) + \beta \cdot u'(c^{o*}) \cdot \frac{d}{dc^y}((1+n) \cdot (f(k^*) - nk^* - c^y)) = 0 \\
\implies & u'(c^{y*}) + \beta \cdot u'(c^{o*}) \cdot (-1-n) = 0 \\
\implies & u'(c^{y*}) = \beta \cdot (1+n) \cdot u'(c^{o*}). \quad (4.3.5)
\end{aligned}$$

### Decentralized dynamics

We now turn to the actual equilibrium, also known as the *decentralized dynamics*. We want to know what happens when the market work freely.

The competitive wage rate and interest rate are given by the condition that

$$w_t = \text{MP}_{L_t} \quad \text{and} \quad r_t = \text{MP}_{K_t}.$$

We can compute such by manipulating a coefficient, having

$$\begin{aligned}
\frac{\partial}{\partial L} F(K, L) &= \frac{\partial}{\partial L} L \cdot F\left(\frac{K}{L}, 1\right) \\
&= \frac{\partial}{\partial L} L \cdot f(k) \\
&= f(k) + L \cdot f'(k) \cdot \frac{\partial k}{\partial L} \\
&= f(k) + L \cdot f'(k) \cdot \left(-\frac{K}{L^2}\right) \\
&= f(k) - k \cdot f'(k)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial K} F(K, L) &= \frac{\partial}{\partial K} L \cdot f(k) \\
&= L \cdot f'(k) \cdot \frac{\partial k}{\partial K} \\
&= f'(k).
\end{aligned}$$

Hence, we have

$$w_t = f(k_t) - k_t \cdot f'(k_t) \quad \text{and} \quad r_{t+1} = f'(k_{t+1}).$$

Then, the price-taking agents want to find

$$\max_{c^y_t, c^o_{t+1}} U(c^y_t, c^o_{t+1}) \quad (4.3.6)$$

under the constraint

$$c^y_t + \frac{c^o_{t+1}}{1 + r_{t+1}} = w_t. \quad (4.3.7)$$

We just need to solve

$$\begin{aligned} & \frac{d}{dc^o_{t+1}} (u(c^y_t) + \beta \cdot u(c^o_{t+1})) = 0 \\ \implies & u'(c^y_t) \cdot \frac{dc^y_t}{dc^o_{t+1}} + \beta \cdot u'(c^o_{t+1}) = 0 \\ \implies & \beta \cdot u'(c^o_{t+1}) = \frac{u'(c^y_t)}{1 + r_{t+1}} \\ \implies & u'(c^y_t) = \beta \cdot (1 + r_{t+1}) \cdot u'(c^o_{t+1}) \\ \implies & u'(w_t - s_t) = \beta \cdot (1 + r_{t+1}) \cdot u'((1 + r_{t+1}) \cdot s_t) \end{aligned}$$

in which  $w_t$  and  $r_{t+1}$  will determine a saving function

$$s_t = \arg \max_{s_t} U(w_t - s_t, (1 + r_{t+1}) \cdot s_t) := s(w_t, r_{t+1}).$$

The model's dynamics are summarized by a sequence  $(k_t)_{t=0}$ . The sequence  $(k_t)_{t=0}$  is an equilibrium sequence of capital-labor ratios if and only if

$$\begin{aligned} \frac{K_{t+1}}{L_{t+1}} = s_t L_t & \implies k_{t+1} = \frac{s_t}{1 + n} \\ & = \frac{s(f(k_t) - k_t \cdot f'(k_t), f'(k_{t+1}))}{1 + n}. \end{aligned}$$

Date- $t$  savings must be consistent with date- $t$  wages and date- $(t + 1)$  interest rate, or there might be no equilibrium.

Note worthy that if  $u = \ln$ , then  $s$  does not depend on  $f'(k_{t+1})$ .

### Decentralized Steady State

Suppose there is a steady-state capital-labor ratio  $\hat{k}$ , i.e.,

$$\hat{k} = \frac{s(f(\hat{k}) - k_t \cdot f'(\hat{k}), f'(\hat{k}))}{1 + n},$$

where  $\hat{k}$  is determined by preferences (saving rate  $s$ ), technology (intensive form  $f$ ), and demography (population growth rate  $n$ ).

We can now state a central theorem on overlapping-generation models about the steady-state interest rate.

**Theorem.** Let  $\hat{r} = f'(\hat{k})$  be the steady-state interest rate. If  $\hat{r} < n$ , then the decentralized steady state is dynamically inefficient, i.e., it is possible to increase the welfare of all generations. If  $\hat{r} > n$ , then the decentralized steady state is dynamically efficient, i.e., it is not possible to strictly increase the welfare of some generation without strictly decreasing the welfare of another generation.

**Proof.** Suppose  $\hat{r} < n$ . Let  $\varepsilon > 0$ . Let every young transfer  $\varepsilon$  to the old, so that each old receives  $(1 + n)\varepsilon$ . This will benefit all generations since the market price of  $c^o$  at steady state is

$$\frac{1}{1 + \hat{r}},$$

which is greater than  $\frac{1}{1 + n}$ : the consumer gets more utility by having access to a small amount of  $c^o$  at less-than-market price. Suppose  $\hat{r} > n$ . That is, the case of capital under-accumulation. Then, any policy will hurt at least one generation: intra-period or inter-generational trade hurts at least some generation, and forced capital accumulation hurts the current generation. ■

This should last forever, or the very last generation will be hurt. It is similar to the pension system, which left the young pay some for the old.

To illustrate graphically, see the following figures.

	$t = t_0$	$t_0 + 1$	$t_0 + 2$
$t_0 + 1$	Old $t_0 - 1$	Young $t_0$	Old $t_0$
$t_0 + 2$	Young $t_0 + 1$	Old $t_0 + 1$	Young $t_0 + 2$

	$t = t_0$	$t_0 + 1$	$t_0 + 2$
$t_0 + 1$	Gets $(1 + n)\varepsilon$	Old $t_0$	Young $t_0 + 1$
$t_0 + 2$	Lost $\varepsilon (\uparrow)$	Old $t_0 + 1$	Young $t_0 + 2$

	$t = t_0$	$t_0 + 1$	$t_0 + 2$
$t_0 + 1$	Gets $(1 + n)\varepsilon$	Lost $\varepsilon (\uparrow)$	Old $t_0 + 1$
$t_0 + 2$	Lost $\varepsilon (\uparrow)$	Old $t_0 + 1$	Young $t_0 + 2$

Figure 4.3.3. Illustration about the Central Theorem

You can see that everyone (each generation) is better-off since

$$-\varepsilon + (1 + n)\varepsilon > -\varepsilon + (1 + \hat{r})\varepsilon.$$

## Public Debt

We first talk about the external debt.

## 5 Problem Sets

### 5.1 The Solow Model and the Golden Rule

1. Consider the Solow growth model

$$\begin{cases} Y = F(K, L); \\ \dot{K} = sY - \delta K; \\ g_L = n, \end{cases}$$

where  $Y$  is output,  $K$  is capital stock, and  $L$  is the labor force.

The function  $F(K, L)$  has constant returns to scale and strictly diminishing marginal returns to capital, and the per-worker production function is  $y = f(k) = F(k, 1)$ , where  $y = \frac{Y}{L}$  and  $k = \frac{K}{L}$ . Labor force and technology are both constant over time ( $n = 0$  and  $g = 0$ ).

Investment  $I$  equals to saving, which is a fraction  $s \in (0, 1)$  of income. Capital depreciates at rate  $\delta \in (0, 1)$ . Consumption per worker is  $c = (1 - s) \cdot f(k)$ .

- i. Prove that the evolution of the per-worker capital stock  $k$  over time is determined by

$$\dot{k} = s \cdot f(k) - \delta k.$$

**Proof.** By the definition of  $k$ , log-differentiating and simplifying yield

$$\begin{aligned} k &= \frac{K}{L} \\ \implies \frac{\dot{k}}{k} &= \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \\ \implies \dot{k} &= \left( s \cdot \frac{Y}{K} - \delta \right) \cdot k \\ &= s \cdot \frac{Y}{L} - \delta k \\ &= s \cdot f(k) - \delta k. \end{aligned}$$

■

- ii. Derive the condition determining the golden-rule level  $k^*$  of capital stock.

**Solution.** The golden-rule level  $k^*$  maximizes the consumption under steady state.

In steady state, we have

$$s \cdot f(k) = \delta k,$$

and the consumption per worker is

$$c = (1 - s) \cdot f(k).$$

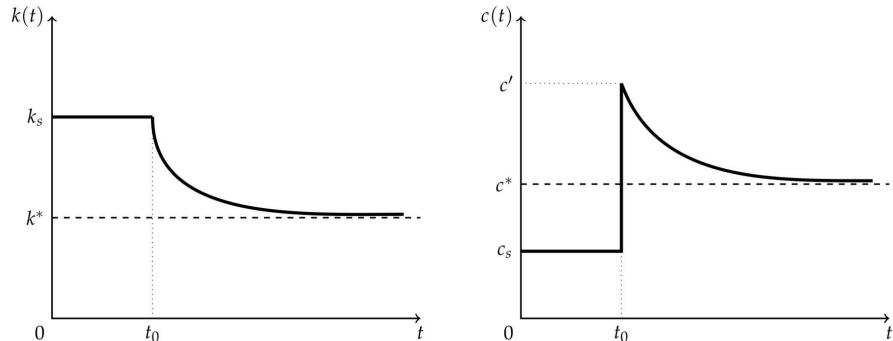
Hence, we need to maximize

$$c_{\text{steady state}} = f(k) - \delta k.$$

Since  $f$  is concave down,  $f(k) - \delta k$  is also concave down when  $k > 0$ , and there will be only one maximum with its derivative 0. Hence, the condition is  $f'(k^*) = \delta$ . ■

- iii. Considering an economy starting from a saving rate where the steady-state capital stock is above the golden rule, sketch graphs of  $k$  and  $c$  over time following an immediate decrease in the saving rate sufficient to reach the golden rule in the long run.

**Solutions.** If the economy begins in a steady state with more capital than required for the Golden rule then the saving rate must be reduced to reach the golden rule. This is because a lower saving rate implies a lower steady-state capital stock. There is a slow process of adjustment for capital where  $k$  gradually falls from a higher steady state to a lower steady state.



b. Assume that  $f(k) = \sqrt{k}$ ,  $\delta = 0.15$ , and  $s = 0.3$ .

- i. Find the steady-state capital stock  $k_s$  per worker.

**Solution.** The expression in the steady state is that

$$s \cdot f(k) = \delta k.$$

Hence, we simply have

$$0.3 \cdot \sqrt{k_s} = 0.15k_s$$

$$k_s = 4.$$

- ii. Calculate whether this is more, less, or exactly the right amount of capital required for the golden rule.

**Solution.** The expression for the golden-rule level is

$$f'(k^*) = \delta.$$

Hence, we have

$$\frac{1}{2\sqrt{k^*}} = 0.15$$

$$k^* = \frac{100}{9}.$$

The amount  $k_s = 4$  is less than the amount required for the golden rule. ■

- c. Assume that owners of capital save all of their income, while all labor income is consumed.

Will the economy have more, less, or exactly the right amount of capital in the long run

required for the golden rule? Explain your reasoning.

**Solution.** This means that  $s = 1$ . Regardless of the value of  $\delta$ , for the expression in the steady state, we always have

$$f(k_s) = \delta k_s. \quad (5.1.1)$$

Moreover, the condition we must have to achieve reach the golden rule is  $f'(k^*) = \delta$ , which is a direct consequence of (5.1.1). Therefore, the steady-state level of capital resulting from the saving behavior assumed in the question is exactly equal to the golden-rule level of capital. ■

2. Consider now the following Solow growth model with government spending.

Output  $Y$  is produced according to the production function  $Y = F(K, L)$ , where  $K$  is the capital stock, and  $L$  is the labor force. The function  $F(K, L)$  has constant returns to scale and diminishing marginal returns to capital, and the per-worker production function is  $y = f(k)$ , where  $y := \frac{Y}{L}$ , and  $k := \frac{K}{L}$ . The labor force being equal to the employment and total population grows at the a positive rate  $n$ . There is no technical progress ( $g = 0$ ).

The government purchases  $G = \gamma L$  units of consumption goods, where  $\gamma > 0$  is per-capita government spending. The government finances its purchases through lump-sum taxes, so that the government budget is balanced,  $G = T$ , where  $T$  denotes total taxes. Investment  $I$  equals saving being a constant fraction  $s \in (0, 1)$  of disposable income  $Y - T$ . Capital depreciates at rate  $\delta \in (0, 1)$ .

- a. Derive analytically the balanced growth path for capital,  $K$ , and output per capita,  $y$ , and show how the steady-state level of capital per worker,  $k$ , is determined. (Hint: use the condition  $C(t) + I(t) + G(t) = Y(t)$ .)

**Solution.** Using the dynamics of capital stock, we firstly have

$$\begin{aligned} \dot{K} &= I - \delta K \\ &= s \cdot (Y - T) - \delta K \\ &= s \cdot (Y - \gamma L) - \delta K \\ \implies \frac{\dot{K}}{K} &= s \cdot \frac{y - \gamma}{k} - \delta. \end{aligned} \tag{5.1.2}$$

By the definition of  $k$ , log-differentiating and plugging in (5.1.2) yield

$$\begin{aligned}\frac{\dot{k}}{k} &= \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \\ &= s \cdot \frac{y - \gamma}{k} - \delta - n \\ \implies \dot{k} &= s \cdot (y - \gamma) - (\delta + n) \cdot k \\ &= s \cdot (f(k) - \gamma) - (n + \delta) \cdot k.\end{aligned}$$

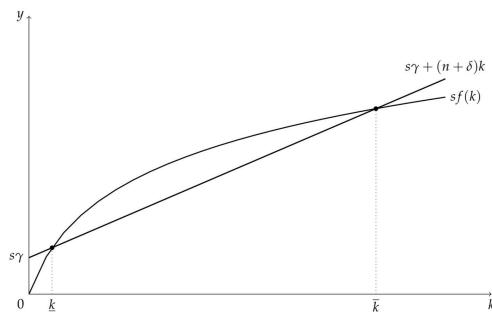
The steady state is when  $\dot{k} = 0$ , i.e.,

$$s \cdot f(k) = s\gamma + (n + \delta) \cdot k.$$

■

- b. Illustrate graphically that there can be two steady states, one with high capital per worker,  $\bar{k}$ , and one with low  $\underline{k}$ . Which steady state does the economy converge to? Why the other steady-state is unstable?

**Solution.** There will be two steady states.



If  $k < \underline{k}$ , actual investment is lower than break-even investment, and the capital stock per worker decreases over time and converges to zero. If  $k > \underline{k}$ , the economy will diverge from  $\underline{k}$  making the low steady state unstable. It will ultimately converge to its stable steady state  $\bar{k}$  even if there is too much capital and  $k > \bar{k}$ . Indeed, if  $\underline{k} < k < \bar{k}$ , actual investment exceeds break-even investment and the economy accumulates capital per worker. The

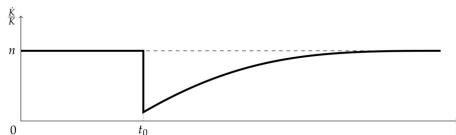
opposite takes place when  $k > \bar{k}$ , because break-even investment (or capital consumption line) is above the actual investment line and capital stock decreases. ■

- c. Ignore the unstable steady state and consider a permanent increase in per capita government spending,  $\gamma$ . Illustrate the paths of the following variables over time:
- The growth rate of capital.

**Solution.** An increase in  $\gamma$  lowers capital per worker and output per worker in the (new) steady state. When the economy reaches its new steady state, the growth rates of aggregate output, investment and consumption will equal the growth rate in population  $n$ , the growth rate of the aggregates in the previous steady state.

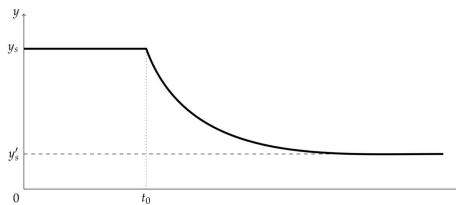
Since government in our example takes away from the public investable resources and doesn't invest itself, a higher level of government consumption will lead to lower values of capital per worker, output per worker and consumption per worker in the steady state.

The growth of  $K$  drops, but as the economy moves to the new steady state with lower  $k$ , it gradually moves back to  $n$ . The dynamics of  $g_K$  is illustrated below.



- Output per capita.

**Solution.** It follows the dynamics of  $k$ , since  $y = f(k)$ . It smoothly converges to the new (and lower) steady-state level  $y_s'$ . The dynamics of  $y$  is illustrated below.

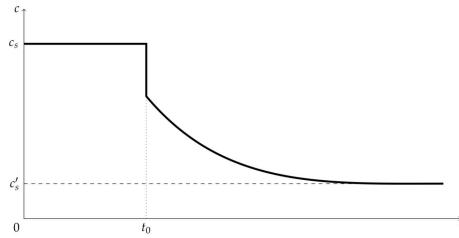


iii. Private consumption per capita.

**Solution.** Consumption depends on the disposable income,

$$\begin{aligned} C &= (1 - s)(Y - T) \\ &= (1 - s)(Y - G) \\ \implies c &= (1 - s)(y - \gamma). \end{aligned}$$

Thus, per-capita consumption sharply decreases in the short-run when the government increases its per-capita spending. Then it follows the transition path of  $y(t)$  being a constant fraction of the disposable income per capita. The dynamics of  $c$  is illustrated below.



3. Consider the Solow model of economic growth with no technological progress ( $g = 0$ ).

The production function is  $Y = F(K, L)$ , where both output  $Y$  and the capital stock  $K$  are measured in terms of the quantity of trucks and  $L$  is the labor force. Assume each worker requires a truck in order to produce any output (new trucks), but cannot use more than one truck at the same time. Assume also that it takes four workers (and four trucks) to produce one unit of output. It implies that the production function is

$$Y = \frac{1}{4} \min\{K, L\}.$$

The labor force is equal to employment and grows at a constant rate  $n$ , the capital stock depreciates at a constant rate  $\delta$ , and the saving rate is  $s$ . One unit of good can be turned into one truck.

Let  $y := \frac{Y}{L}$  and  $k := \frac{K}{L}$  denote output per person and capital per person respectively.

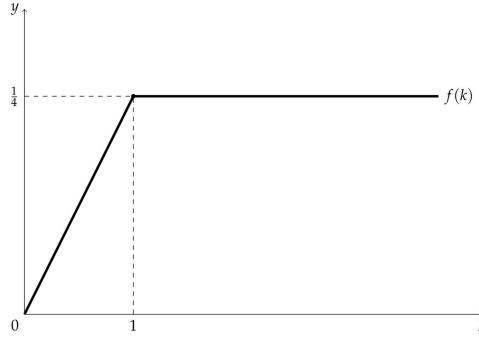
The per-person production function is  $y = f(k)$ .

- Write down the expression for the production function per-person and plot the corresponding Solow diagram in the  $(k, y)$  space.

**Solution.** This question is equivalently asking us to draw the graph of

$$y = \frac{1}{4} \cdot \min\{k, 1\}.$$

The graph is depicted below.



- Assume now that the population grows by 0.3% every year ( $n = 0.003$ ), while 0.7% of trucks become obsolete every year ( $\delta = 0.007$ ).

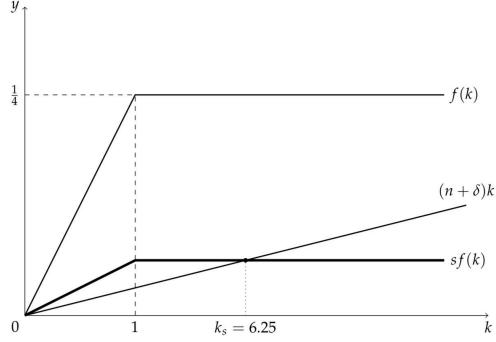
Suppose that the agents save 25% of their income ( $s = 0.25$ ). Find the steady-state capital stock per person,  $k_s$ , and output per person,  $y_s$ . Illustrate graphically.

**Solution.** The expression in the steady state is

$$s \cdot f(k_s) = (\delta + n) \cdot k_s.$$

Hence, we simply have

$$0.25 \cdot \frac{1}{4} \cdot \min\{k_s, 1\} = 0.01 \cdot k_s.$$



After drawing the equations, we can see that  $k_s = 6.25$ . ■

- c. Define the golden rule level of the capital per capita  $k^*$  on the balanced growth path and find its value for the parameters given in b.

**Solution.** The expression for the golden-rule level is

$$f'(k^*) = \delta + n.$$

However, since  $f$  is kinked at  $k = 1$ , we can only analyze the slope.

When  $k < 1$ , we have  $f'(k) = \frac{1}{4} > 0.01$ ; when  $k > 1$ , we have  $f'(k) = 0 < 0.01$ . hence,

the golden-rule level of capital per capital is 1. ■

- d. Justify whether the saving rate of 25% is too low or too high to reach the golden rule.

Determine the golden-rule saving rate  $s^*$ .

**Solution.** From Figure 5.1.5, it will be optimal to reduce the saving rate  $s$  until the increasing segment of the actual investment  $s \cdot f(k)$  starts to coincide with the break-even investment line  $(n + \delta) \cdot k$ . That is, the golden-rule saving rate has to satisfy

$$s^* \cdot f'(k) = n + \delta,$$

which implies

$$s^* \cdot \frac{1}{4} = 0.01$$

$$s^* = 0.04.$$

Hence, 25% is too high. ■

## 5.2 Endogenous Growth

1. Consider an economy where output is given by

$$Y(t) = (K(t))^\alpha (A(t)L(t))^{1-\alpha}. \quad (5.2.1)$$

Let the population be constant and equal to 1, i.e.,  $L(t) = 1$  for all  $t$ , and thus  $n = 0$ .

Suppose there is no depreciation, i.e.,  $\delta = 0$ , so that the capital accumulation is given by

$$\dot{K}(t) = s \cdot Y(t). \quad (5.2.2)$$

Assume that the knowledge accumulation occurs as a side-effect of goods production,

$$\dot{A}(t) = B \cdot Y(t). \quad (5.2.3)$$

- a. Find expressions for  $g_A(t)$  and  $g_K(t)$  in terms of  $A(t)$ ,  $K(t)$ , and the parameters.

**Solution.** We first find  $g_A(t)$ . Plugging (5.2.1) in (5.2.3), we have

$$\begin{aligned} \dot{A}(t) &= B \cdot (K(t))^\alpha (A(t)L(t))^{1-\alpha} \\ \implies g_A(t) &= B \cdot \left( \frac{K(t)}{A(t)} \right)^\alpha (L(t))^{1-\alpha} \\ &= B \cdot \left( \frac{K(t)}{A(t)} \right)^\alpha. \end{aligned} \quad (5.2.4)$$

We now find  $g_K(t)$ . Plugging (5.2.1) in (5.2.2), we have

$$\begin{aligned} \dot{K}(t) &= s \cdot (K(t))^\alpha (A(t)L(t))^{1-\alpha} \\ \implies g_K(t) &= s \cdot (K(t))^{\alpha-1} (A(t)L(t))^{1-\alpha} \\ &= s \cdot \left( \frac{A(t)}{K(t)} \right)^{1-\alpha}. \end{aligned} \quad (5.2.5)$$

- b. Sketch  $\dot{g}_A = 0$  and  $\dot{g}_K = 0$  lines in  $(g_A, g_K)$  space.

**Solution.** From (5.2.4) and (5.2.5), we log-differentiate and have

$$\begin{aligned}\frac{\dot{g}_A(t)}{g_A(t)} &= \alpha \cdot (g_K(t) - g_A(t)) \\ \Rightarrow \quad \dot{g}_A(t) &= \alpha \cdot (g_K(t) - g_A(t)) \cdot g_A(t)\end{aligned}$$

and

$$\begin{aligned}\frac{\dot{g}_K(t)}{g_K(t)} &= (1 - \alpha) \cdot (g_A(t) - g_K(t)) \\ \Rightarrow \quad \dot{g}_K(t) &= (1 - \alpha) \cdot (g_A(t) - g_K(t)) \cdot g_K(t).\end{aligned}$$

Hence, both  $\dot{g}_A = 0$  and  $\dot{g}_K = 0$  are the equation  $g_K - g_A = 0$ . However, we have to note the tendencies of  $\dot{g}_K$  and  $\dot{g}_A$ . For  $\dot{g}_A(t)$ , if we fix a  $g_A$ , the greater  $g_K$  is, the greater  $\dot{g}_A(t)$  is; hence, above the line is  $\dot{g}_A > 0$ , and below the line is  $\dot{g}_A < 0$ . For  $\dot{g}_K(t)$ , if we fix a  $g_K$ , the greater  $g_A$  is, the greater  $\dot{g}_K(t)$  is; hence, to the right of the line is  $\dot{g}_K > 0$ , and to the left of the line is  $\dot{g}_K < 0$ .

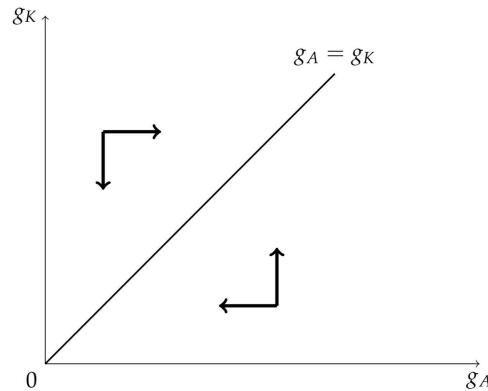


Figure 5.2.1. Phase diagram for  $g_A$  and  $g_K$ .

■

- c. Does the economy converge to a balanced growth path? If so, what are the growth rates of  $K$ ,  $A$ , and  $Y$  on the balanced growth path?

**Solution.** From Figure 5.2.1, we can see that the economy converges to the balanced

growth path. That is, the economy converges to some point lying on  $g_A = g_K$ . To determine the path, we need to combine the expressions for  $g_A(t)$  and  $g_K(t)$  derived in a and try to explicitly express  $g_K(t)$  from  $g_A(t)$ . We simply use (5.2.5) to divide (5.2.4) and have

$$\begin{aligned} \frac{g_K(t)}{g_A(t)} &= \frac{s}{B} \cdot \left( \frac{A(t)}{K(t)} \right)^{1-\alpha} \cdot \left( \frac{A(t)}{K(t)} \right)^\alpha \\ &= \frac{s}{B} \cdot \left( \frac{g_A(t)}{B} \right)^{-\frac{1}{\alpha}} \\ \implies g_K(t) &= s \cdot \left( \frac{B}{g_A(t)} \right)^{\frac{1}{\alpha}-1}. \end{aligned} \quad (5.2.6)$$

Plugging the equilibrium, on the balanced growth path,  $g_A = g_K = g^*$  into (5.2.6), we obtain

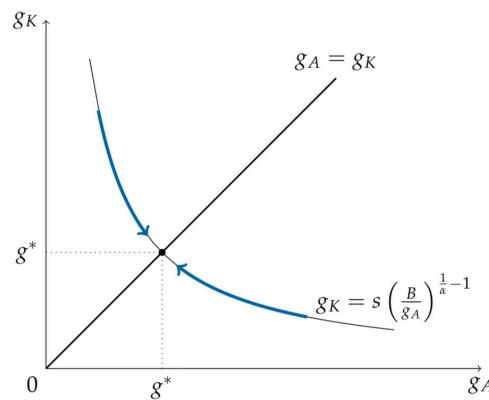
$$g^* = s^\alpha B^{1-\alpha}.$$

To find the growth rate of output on the balanced growth path, log-differentiating (5.2.1) yields

$$g_Y = \alpha g_K + (1 - \alpha) g_A.$$

Since we are focusing on the balanced growth path, we plug  $g_A = g_K = g^*$  in and have

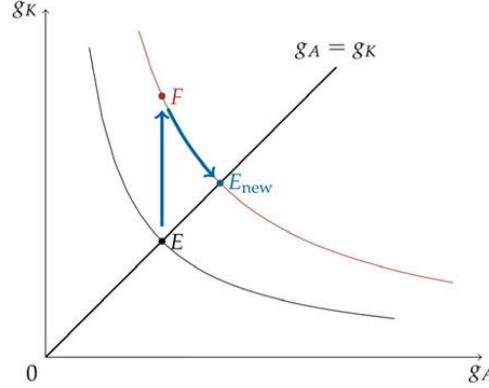
$$g_Y = g^*.$$



■

- d. How does an increase in  $s$  affect long-run growth?

**Solution.** The increase in  $s$  will shift neither  $\dot{g}_A = 0$  nor  $\dot{g}_K = 0$  since both will coincide with the line  $g_A = g_K$ . However, the hyperbola in (5.2.6) will shift outwards.



Since the change in  $s$  does not affect  $g_A$ , whereas it increases  $g_K$ , the economy will jump from point  $E$  to point  $F$ , where  $F$  is just above  $E$ . Then, the economy will gradually converge to the new steady state at point  $E_{\text{new}}$ . ■

2. Many economists have criticized endogenous growth models on the basis of the existence of scale effects by which large countries should have higher growth rates than small countries.

Consider a model where productivity growth is an externality caused by capital accumulation. With usual notations, consider the following model

$$\begin{cases} Y = AK^\alpha L^{1-\alpha}; \\ A = BK^\phi; \\ \dot{K} = sY; \\ g_L = n, \end{cases}$$

with  $n > 0$  and  $\alpha, s \in (0, 1)$ .

- a. Write  $g_K$  as a function of  $K, L$ , and the model's parameters.

**Solution.** We plug  $Y = AK^\alpha L^{1-\alpha}$  and  $A = BK^\phi$  into  $\dot{K} = sY$ , having

$$\begin{aligned}\dot{K} &= sBK^\phi K^\alpha L^{1-\alpha} \\ \implies g_K &= sBL^{1-\alpha}K^{\alpha+\phi-1}.\end{aligned}$$

■

- b. Derive an expression of  $g_Y$  depending on  $g_K$  and  $g_L$ . What inequality must hold so that there exists a balanced growth path, i.e., a regime where  $g_Y$  and  $g_K$  are strictly positive constants? How does  $g_y$  vary with  $n$ ?

**Solution.** We log-differentiate the first assumption and have

$$\begin{aligned}g_Y &= g_A + \alpha g_K + (1 - \alpha)g_L \\ &= \phi g_K + \alpha g_K + (1 - \alpha) \cdot g_L \\ &= (\phi + \alpha) \cdot g_K + (1 - \alpha) \cdot g_L.\end{aligned}$$

The sufficient condition that there exists a balanced growth path is to have the (1.3.2).

Hence, we must have

$$g_Y = \frac{(1 - \alpha) \cdot n}{1 - \phi - \alpha}. \quad (5.2.7)$$

This suggests that the denominator cannot be zero. Since growth rates are positive, the inequality

$$1 - \phi - \alpha > 0$$

must hold. We now derive the expression for  $g_y$ . From (5.2.7), since  $y = \frac{Y}{L}$ , we have

$$g_y = g_Y - n$$

$$= \frac{\phi n}{1 - \varphi - \alpha}.$$

Hence,  $g_y$  is increasing in  $n$ . More population growth means more people, thus more output, hence more capital accumulation and ultimately more productivity. ■

- c. Now replace the second equation in the model by  $A = B \cdot \left(\frac{K}{L}\right)^\phi$ . Write an expression for  $g_K$  along a balanced growth path. What value(s) of  $\phi + \alpha$  is consistent with the existence

of a balanced growth path? How does  $g_y$  vary with  $n$ ? Interpret the differences between the two models.

**Solution.** We plug  $Y = AK^\alpha L^{1-\alpha}$  and  $A = B \cdot \left(\frac{K}{L}\right)^\phi$  into  $\dot{K} = sY$ , having

$$\dot{K} = sB \cdot \left(\frac{K}{L}\right)^\phi K^\alpha L^{1-\alpha}$$

$$\Rightarrow g_K = sBL^{1-\phi-\alpha}K^{\alpha+\phi-1}$$

$$= sB \cdot \left(\frac{L}{K}\right)^{1-\phi-\alpha}.$$

The balanced growth path requires  $g_K$  to be a constant. Hence, either the variable  $\frac{L}{K}$  is constant or  $1 - \phi - \alpha = 0$ . For the first case,  $\frac{L}{K}$  being constant implies the growth rate of  $\frac{L}{K}$  is 0, which means  $g_K = g_L = n$ ; there is no growth in output  $y$  per capita. For the second case,  $\phi - \alpha = 1$  implies  $g_K = g_Y = sB$ ; the model has a balanced growth path where  $g_y = sB - n$ .

In the first (initially given) model, productivity depends on the aggregate stock of capital, that goes up when  $n$  goes up. In the second model, productivity depend on the capital-labor ratio, which decreases when population growth is higher (because more workers come up at each period); in this case population growth makes the growth rate of income per capita decrease. ■

3. Consider the continuous-time growth model with physical capital  $K_t$  and human capital  $H_t$

$$\begin{cases} Y_t = ((1 - a_K)K_t)^\alpha((1 - a_H)H_t)^{1-\alpha}; \\ \dot{K}_t = sY_t - \delta_K K_t; \\ \dot{H}_t = (a_K K_t)^\gamma (a_H H_t)^\phi (A_t L_t)^{1-\gamma-\phi} - \delta_H H_t; \\ \dot{L}_t = nL_t; \\ \dot{A}_t = gA_t, \end{cases}$$

with

$$a_K, a_H, \alpha, \gamma, \phi, \delta_K, \delta_H \in (0, 1), \quad \gamma + \phi > 1, \quad \text{and} \quad n, g > 0.$$

The fractions  $a_K$  and  $a_H$  respectively stand for the proportion of total physical and human capital stocks devoted to human capital accumulation rather than to output production. The fractions  $\delta_K$  and  $\delta_H$  are the depreciation rates of physical and human capital.

Notice that this model assumes that human capital is produced in its own sector with its own production function. Raw labor  $L$  is useful only as something to be educated, not as an input into the production of final goods. Similarly, knowledge  $A$  is useful only as something that can be conveyed to students, not as a direct input to goods production.

a. Define  $k := \frac{K}{AL}$  and  $h := \frac{H}{AL}$ . Derive equations for  $\dot{k}$  and  $\dot{h}$  as depending on  $k$  and  $h$ .

**Solution.** Log-differentiating the given definitions, we have

$$\begin{aligned} \frac{\dot{k}}{k} &= g_K - g_A - g_L \\ &= s \cdot \frac{Y}{K} - \delta_K - g - n \\ &= s \cdot (1 - a_K)^\alpha \left( (1 - a_H) \frac{H}{K} \right)^{1-\alpha} - \delta_K - g - n \\ &= s \cdot (1 - a_K)^\alpha (1 - a_H)^{1-\alpha} h^{1-\alpha} k^{\alpha-1} - \delta_K - g - n \\ \implies \dot{k} &= s \cdot ((1 - a_K)k)^\alpha ((1 - a_H)h)^{1-\alpha} - (\delta_K + g + n)k \end{aligned} \tag{5.2.8}$$

and

$$\begin{aligned} \frac{\dot{h}}{h} &= g_H - g_A - g_L \\ &= (a_K K)^\gamma a_H^\phi H^{\phi-1} (AL)^{1-\gamma-\phi} - \delta_H - g - n \\ \implies \dot{h} &= (a_K k)^\gamma (a_H h)^\phi - (\delta_H + g + n)h. \end{aligned} \tag{5.2.9}$$

- b. Derive an equation describing the set of combinations of  $k$  and  $h$  such that  $\dot{k} = 0$ . Sketch it in the  $(k, h)$  space. Do the same for  $\dot{h} = 0$  on the same graph.

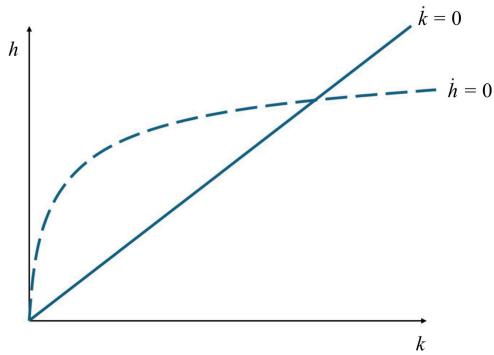
**Solution.** We need some patience and algebra sense to answer this question. We will use more subscripts with calligraphic capital letters to represent constants to avoid visual confusion. We first deal with  $\dot{k} = 0$ . In this case, we have

$$\begin{aligned} s \cdot ((1 - a_K)k)^\alpha ((1 - a_H)h)^{1-\alpha} &= (\delta_K + g + n)k \\ \underbrace{s \cdot (1 - a_K)^\alpha (1 - a_H)^{1-\alpha}}_{\mathcal{S}} h^{1-\alpha} &= \underbrace{(\delta_K + g + n)}_{\mathcal{K}} k^{1-\alpha} \\ h &= \left( \frac{\mathcal{K}}{\mathcal{S}} \right)^{\frac{1}{1-\alpha}} k, \end{aligned} \quad (5.2.10)$$

which shows a proportional relationship between  $h$  and  $k$ . We now deal with  $\dot{h} = 0$ . In this case, we have

$$\begin{aligned} (\delta_H + g + n)h &= (a_K k)^\gamma (a_H h)^\phi \\ \underbrace{(\delta_H + g + n)}_{\mathcal{H}} h^{1-\phi} &= \underbrace{a_K^\gamma a_H^\phi}_{\mathcal{A}} k^\gamma \\ h &= \left( \frac{\mathcal{A}}{\mathcal{H}} \right)^{\frac{1}{1-\phi}} k^{\frac{\gamma}{1-\phi}}. \end{aligned} \quad (5.2.11)$$

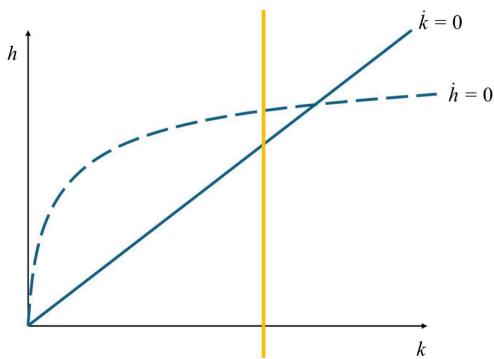
Since  $\gamma + \phi > 1$ , the graph of (5.2.11) will be a power function with power  $\frac{\gamma}{1-\phi} \neq 1$ . See the following figure for graphs of the equations.



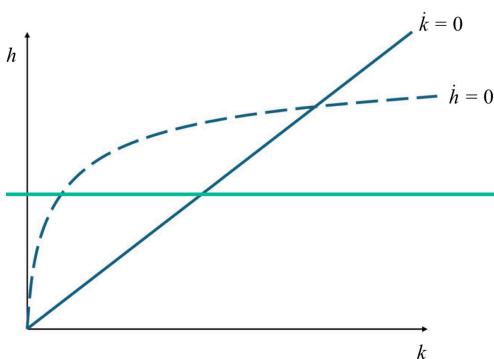
■

- c. Based on the graph of the previous question, explain why  $k$  and  $h$  tend to some limit for any initial conditions.

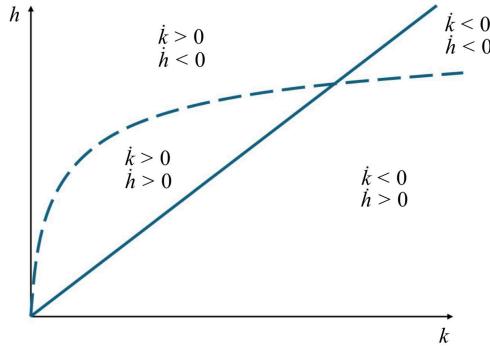
**Solution.** From (5.2.8), we fix  $k$  and find that the greater  $h$  is, the greater  $\dot{k}$  is; hence, above the line is  $\dot{k} > 0$ , and below the line is  $\dot{k} < 0$ . Use the following figure for assistance.



From (5.2.9), we fix  $h$  and find that the greater  $k$  is, the greater  $\dot{h}$  is; hence, to the right of the curve is  $\dot{h} > 0$ , and to the left of the curve is  $\dot{h} < 0$ . Use the following figure for assistance.



Combining both of them will suggest the following figure, which shows that any initial condition will converge to the intersection.



■

- d. Suppose that  $\dot{k} = \dot{h} = 0$ . What is the growth rate of physical capital per capita, the one of human capital per capita, and the one of output per capita?

**Solution.** By assumption, we have

$$g_K - g_L - g_A = 0 \quad \text{and} \quad g_H - g_L - g_A = 0.$$

Hence,

$$g_{\frac{K}{L}} = g_K - g_L$$

$$= g_A$$

$$= n$$

and

$$g_{\frac{H}{L}} = g_H - g_L$$

$$= g_A$$

$$= n.$$

Log-differentiating the first assumption yields

$$\begin{aligned} g_Y &= \alpha g_K + (1 - \alpha) g_H \\ &= \alpha(g_L + g_A) + (1 - \alpha)(g_L + g_A) \\ &= g + n. \end{aligned}$$

Hence,

$$g_y = g_Y - n$$

$$= g.$$

■

### 5.3 Theory of Consumption

1. Consider the two-period Fisher model of consumption choice. An individual receives income  $Y_1 > 0$  in the first period and income  $Y_2 > 0$  in the second period, and one can save or borrow at real interest rate  $r > 0$ . One optimally chooses consumption plan  $(C_1, C_2)$ . His utility function is  $U$ , which is strictly increasing in each argument, strictly quasi-concave, and continuously differentiable. We also assume that

$$\frac{\partial U}{\partial C_1}(C_1, C_2) \xrightarrow{C_1 \rightarrow 0} \infty \quad \text{for all } C_2 > 0 \quad \text{and} \quad \frac{\partial U}{\partial C_2}(C_1, C_2) \xrightarrow{C_2 \rightarrow 0} \infty \quad \text{for all } C_1 > 0.$$

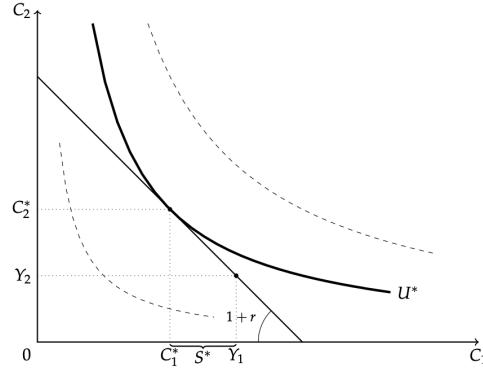
The individual maximizes  $U$  under the intertemporal budget constraint.

- a. Write down an equation for the individual's life-time budget constraint (i.e. the  $(C_1, C_2)$  that the individual can afford). Draw a diagram illustrating how the optimal consumption plan is determined.

**Solution.** By (3.2.1), the individual's lifetime budget constraint is

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r}. \quad (5.3.1)$$

The optimal consumption bundle  $(C_1, C_2)$  is determined by maximizing the utility function  $U$  such that  $U(C_1, C_2)$  attains the maximum under the budget constraint. The optimal consumption bundle  $(C_1, C_2)$  is the point where (5.3.1) is the tangent to  $U(C_1, C_2) = U_0$ .



■

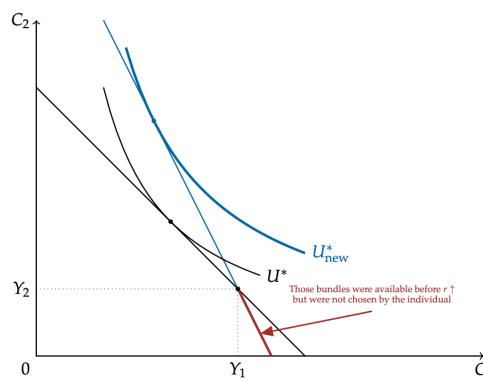
- b. Consider that the real interest rate rises. How does it affect the life-time budget constraint? Would a saver ever find it rational to start borrowing when interest rates rise?

**Solution.** Since  $Y_1 + \frac{Y_2}{1+r}$  is constant, we can simplify the (5.3.1) to

$$C_2 = -(1+r)C_1 + (1+r)Y_1 + Y_2.$$

Hence, if  $r$  rises, the slope of (5.3.1) will be smaller, i.e., the line will be steeper and pivot clockwise around the endowment point  $(Y_1, Y_2)$ .

After the rise in the interest rate, it is impossible for a rational saver to be a borrower since the bundles were available before the rise in the interest rate from  $r$  to  $r'$ , but they (illustrated in the following figure with yellow) were not chosen by the consumer.



■

- c. Could a saver respond to a rise in  $r$  by saving **less**? Explain.

**Solution.** Yes. A saver can respond to a rise in  $r$  by saving less. We should answer by

considering the *substitution effect* (can be seen on a unique indifference curve) and the *income effect* (can be seen between two parallel budget constraint).

The substitution effect is defined as the response to the new interest rate, after hypothetically giving (or taking away) an amount of income that allows the individual just to afford a consumption point on the original indifference curve. The substitution effect is found by locating the point on the original indifference curve where the gradient is equal to the new gradient of the budget constraint. Since the budget constraint becomes steeper owing to the higher interest rate, the substitution effect implies a movement up and to the left the indifference curve, which means a reduction in current consumption. Intuitively, high interest rates make current consumption relatively more expensive than future consumption by raising the return to lending, thus encouraging saving and discouraging the current spending.

The income effect is defined as the response to removing the hypothetical increase or decrease of income that was used to calculate the substitution effect. Thus, by definition, the sum of the income and substitution effects is equal to the overall response to the change in interest rates. The nature of the income effect depends on whether the individual was initially a borrower or a saver. In this question, the individual is a saver, so the initial consumption point is to the left of the endowment point.

As a result of the higher interest rate, the budget constraint pivots clockwise around the endowment point. This shows that there must be a hypothetical decrease in income to calculate the substitution effect. This corresponds to the dashed budget line with the same gradient as the new budget line, but in a lower position that allows a consumption point on the original indifference curve to be just affordable.

Calculating the income effect means obtaining the effect of removing this hypothetical decrease in income, that is, the effect of the parallel shift of the budget constraint from

the dashed line to the one passing through the endowment point. This is an upward shift, and given the assumption that consumption is a normal good, the response of current consumption to this shift is positive.

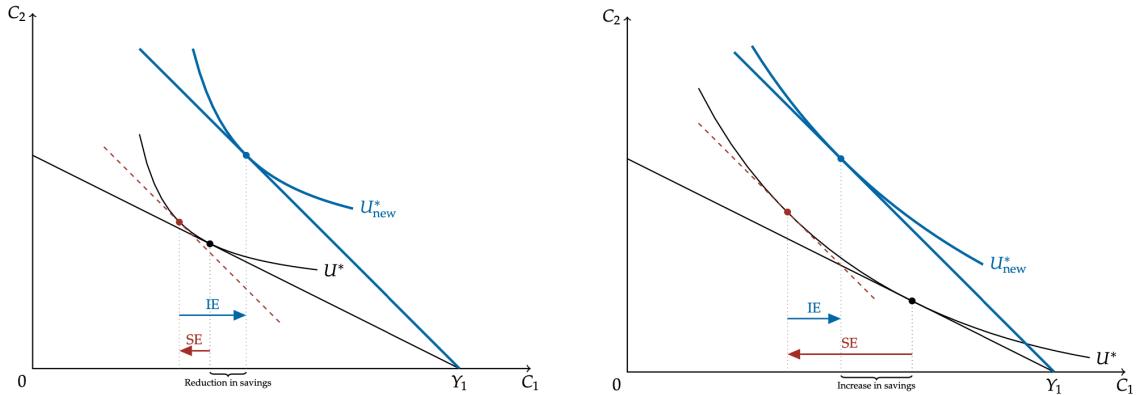
With a positive income effect and a negative substitution effect, the overall response of current consumption  $C_1$  (and thus saving  $S = Y_1 - C_1$ ) is ambiguous. When the income effect dominates, an increase in  $r$  leads to a reduction in savings; when the substitution effect dominates, an increase in  $r$  leads to a rise in savings.

A specific example is in (3.1.2). When  $\theta > 1$ , we can know that if  $Y_2 = 0$ , the optimal consumption in the first period

$$C_1^* = \frac{Y_0}{1 + \frac{(\beta \cdot (1+r))^{\frac{1}{\theta}}}{1+r}}$$

$$= \frac{Y_0}{1 + \beta^{\frac{1}{\theta}}} \cdot (1+r)^{\frac{1-\theta}{\theta}}$$

is decreasing in  $r$ .



2. Consider an individual who lives from 0 to  $T$ , and whose lifetime utility is given by

$$U(C_0, \dots, C_T) = \sum_{t=0}^T u(C(t)),$$

where  $u' > 0$  and  $u'' < 0$ . The integer  $T$  is greater than 2.

The individual's income is exogenously given by

$$Y(t) = \begin{cases} Y_0 + gt, & \text{if } t = 0, \dots, R; \\ 0, & \text{if } t = R + 1, \dots, T, \end{cases}$$

where  $g > 0$ . The retirement age,  $R \geq 1$ , satisfies  $R + 1 < T$ . The interest rate is zero, the individual has no initial wealth, and there is no uncertainty.

- a. What is the individual's lifetime budget constraint?

**Solution.** We have to make the lifetime consumption equal life time income. Hence, the budget constraint is

$$\begin{aligned} \sum_{t=0}^R Y_0 + gt &= \sum_{t=0}^T C_t \\ \Leftrightarrow (R+1)Y_0 + g \cdot \frac{R \cdot (R+1)}{2} &= \sum_{t=0}^T C_t. \end{aligned} \quad (5.3.2)$$

■

- b. Prove that the individual wishes constant  $C(t)$  for all dates  $t \in \{0, \dots, T\}$ .

**Proof.** Without loss of generality, assume  $C(t_1) > C(t_2)$  for some  $t_1 \neq t_2 \in \{0, \dots, T\}$ , and this sequence of consumption will yield optimal. Since  $u$  is strictly concave, we will have

$$u\left(\frac{C(t_1) + C(t_2)}{2}\right) > \frac{u(C(t_1)) + u(C(t_2))}{2}.$$

Hence, there is one more consumption bundle  $\left(\frac{C(t_1) + C(t_2)}{2}, \frac{C(t_1) + C(t_2)}{2}\right)$  at  $t_1$  and  $t_2$  on the budget constraint (since  $r = 0$ ) that yields a greater utility, a contradiction.

Therefore, the individual wishes constant consumption for all dates. ■

- c. What is the path of the individual's wealth as a function of  $t$ ?

**Solution.** From (5.3.2), we know that the constant consumption

$$C^* = \frac{R+1}{T+1} \cdot \left(Y_0 + \frac{gR}{2}\right).$$

Hence, we simply have

$$W(t) = \begin{cases} \sum_{\tau=0}^t (Y_0 + g\tau) - (t+1)C^*, & \text{if } t \leq R; \\ W(R) - (t+1)C^*, & \text{else.} \end{cases}$$

Since  $W(R) = (R+1)Y_0 + g \cdot \frac{R \cdot (R+1)}{2}$  and  $C^* = \frac{R+1}{T+1} \cdot \left( Y_0 + \frac{gR}{2} \right)$ , we can further simplify  $W(t)$  and have

$$W(t) = \begin{cases} (t+1) \cdot \left( Y_0 + \frac{gt}{2} - C^* \right), & \text{if } t \leq R; \\ (T-t)C^*, & \text{else.} \end{cases}$$

■

3. Consider an individual who lives for three periods.

In period 1, his or her objective function is

$$U_1(C_1, C_2, C_3) = \ln C_1 + \beta \cdot (\ln C_2 + \ln C_3),$$

where  $\beta \in (0, 1)$  is the discount factor. In period 2, it is

$$U_2(C_2, C_3) = \ln C_2 + \beta \ln(C_3).$$

Since the individual's period-3 choice problem is trivial (one should consume whatever is remaining), the period-3 objective function is irrelevant.

The individual has initial wealth  $W > 0$  and faces a real interest rate  $r = 0$ .

- a. Find the values of  $C_1$ ,  $C_2$ , and  $C_3$  under the following assumptions about how they are determined.

- i. (*Commitment*) The individual chooses  $C_1$ ,  $C_2$ , and  $C_3$  in period 1.

**Solution.** We want to maximize

$$U_1(C_1, C_2, C_3) \tag{5.3.3}$$

under the budget constraint

$$C_1 + C_2 + C_3 = W. \quad (5.3.4)$$

Hence, we substitute  $C_3$  by  $W - C_1 - C_2$  in (5.3.3) and have

$$\begin{aligned} U_1(C_1, C_2, C_3) &= U_1(C_1, C_2, W - C_1 - C_2) \\ &= \ln C_1 + \beta \cdot \ln C_2 + \beta \cdot \ln(W - C_1 - C_2). \end{aligned}$$

We first differentiate  $U_1$  with respect to  $C_1$  and to  $C_2$ , we have

$$\frac{\partial U_1}{\partial C_1}(C_1, C_2, C_3) = \frac{1}{C_1} - \frac{\beta}{W - C_1 - C_2}$$

and

$$\frac{\partial U_1}{\partial C_2}(C_1, C_2, C_3) = \frac{\beta}{C_2} - \frac{\beta}{W - C_1 - C_2}.$$

Setting the partial derivatives to zero, we solve the system

$$\begin{aligned} &\left\{ \begin{array}{l} \frac{1}{C_1} - \frac{\beta}{W - C_1 - C_2} = 0, \\ \frac{\beta}{C_2} - \frac{\beta}{W - C_1 - C_2} = 0; \end{array} \right. \\ \Rightarrow &\left\{ \begin{array}{l} \beta C_1 = W - C_1 - C_2, \\ C_2 = W - C_1 - C_2; \end{array} \right. \\ \Rightarrow &\left\{ \begin{array}{l} C_1^* = \frac{W}{1 + 2\beta}, \\ C_2^* = \frac{\beta W}{1 + 2\beta}, \\ C_3^* = \frac{\beta W}{1 + 2\beta}. \end{array} \right. \end{aligned}$$

■

ii. (*No commitment, naïvete*) The individual chooses  $C_1$  in period 1 to maximize the period-1 objective function, thinking he or she will also choose  $C_2$  to maximize this objective function. In fact, however, the individual chooses  $C_2$  to maximize the period-2 objective function.

**Solution.** Let  $\tilde{C}_1$  be the choice of the individual in period 1, which should be considered a constant. We now want to maximize

$$U_2(C_2, C_3) \quad (5.3.5)$$

under the budget constraint

$$\tilde{C}_1 + C_2 + C_3 = W. \quad (5.3.6)$$

Hence, we substitute  $C_3$  by  $W - \tilde{C}_1 - C_2$  in (5.3.5) and have

$$\begin{aligned} U_2(C_2, C_3) &= U_2(C_2, W - \tilde{C}_1 - C_2) \\ &= \ln \tilde{C}_2 + \beta \cdot \ln(W - \tilde{C}_1 - C_2). \end{aligned}$$

We first differentiate  $U_1$  with respect to  $C_2$ , we have

$$\frac{\partial U_1}{\partial C_2}(C_2, C_3) = \frac{1}{C_2} - \frac{\beta}{W - \tilde{C}_1 - C_2}.$$

Setting the partial derivative to zero, we have

$$\begin{aligned} \frac{1}{C_2} - \frac{\beta}{W - \tilde{C}_1 - C_2} &= 0 \\ \Rightarrow \qquad \qquad \qquad \beta C_2 &= W - \tilde{C}_1 - C_2 \\ \Rightarrow \qquad \qquad \qquad C_2 &= \frac{W - \tilde{C}_1}{1 + \beta}. \end{aligned}$$

Thus,  $C_3 = \frac{\beta \cdot (W - \tilde{C}_1)}{1 + \beta}$  by (5.3.6). In period 1, the individual chooses  $C_1$  just as she did under commitment since she (wrongly) believes she will choose  $C_2$  in the same way as under commitment. Thus, we have

$$\widetilde{C}_1 = C_1^*,$$

where  $C_1^*$  is the one in the last question. Therefore,

$$\begin{cases} C_1^* = \frac{W}{1+2\beta}, \\ C_2^* = \frac{2\beta W}{(1+\beta)(1+2\beta)}, \\ C_3^* = \frac{2\beta^2 W}{(1+\beta)(1+2\beta)}. \end{cases}$$

■

iii. (*No commitment, sophistication*) The individual chooses  $C_1$  in period 1 to maximize the period-1 objective function, realizing that he or she will choose  $C_2$  in period 2 to maximize the period-2 objective function.

**Solution.** From the previous question, we know that if one is maximizing the period-2 objective function, we must have

$$C_2^*(C_1) = \frac{W - C_1}{1 + \beta} \quad \text{and} \quad C_3^*(C_1) = \frac{\beta \cdot (W - C_1)}{1 + \beta}. \quad (5.3.7)$$

That is,  $C_2$  and  $C_3$  will be uniquely determined by  $C_1$ . Hence, by (5.3.7), we only need to maximize

$$\begin{aligned} & U_1(C_1, C_2^*(C_1), C_3^*(C_1)) \\ &= \ln C_1 + \beta \cdot (\ln C_2^*(C_1)) + \beta \cdot (\ln C_3^*(C_1)) \\ &= \ln C_1 + \beta \cdot \left( \ln \frac{W - C_1}{1 + \beta} \right) + \beta \cdot \left( \ln \frac{\beta \cdot (W - C_1)}{1 + \beta} \right) \\ &= \ln C_1 + \beta \ln(W - C_1) - \beta \ln(1 + \beta) + \beta \ln(W - C_1) + \beta \ln \frac{\beta}{1 + \beta}. \quad (5.3.8) \end{aligned}$$

Set the partial derivative of (5.3.8) with respect to  $C_1$  to be zero, we have

$$\begin{aligned} \frac{1}{C_1} - \frac{\beta}{W - C_1} - \frac{\beta}{W - C_1} &= 0 \\ \Rightarrow W - C_1 - 2\beta \cdot C_1 &= 0 \\ \Rightarrow C_1^* &= \frac{W}{1 + 2\beta}. \end{aligned}$$

We can hence see that the three-tuple  $(C_1^*, C_2^*, C_3^*)$  is the same as the one in the previous question. ■

- b. Use your answers to parts (a) and (b) to explain in what sense the individuals' preferences are time-inconsistent. Explain intuitively why sophistication does not produce different behavior than naïvete.

**Solution.** The individual's preferences are time-inconsistent because the optimal choice of period-2 consumption that is made in the first period is no longer the optimal choice once period 2 actually arrives.

This is illustrated by the fact that if the individual does not commit to period-2 consumption in the first period, then when period 2 arrives one chooses

$$\begin{aligned} \widetilde{C}_2 &= \frac{2\beta W}{(1 + \beta)(1 + 2\beta)} \\ &= \frac{2}{1 + \beta} \cdot \frac{\beta W}{1 + 2\beta} \\ &> \frac{\beta W}{1 + 2\beta} \\ &= C_2^*, \end{aligned}$$

where the inequality holds because

$$\begin{aligned} \frac{2}{1 + \beta} &> \frac{2}{1 + 1} \\ &= 1 \end{aligned}$$

with the discount factor  $\beta \in (0, 1)$ .

From the period-1 objective function we can see that in the first period the individual is indifferent between period-2 and period-3 consumption; they are both discounted by the discount factor  $\beta < 1$ .

But when period 2 actually arrives, we can see from the period-2 objective function that the individual then prefers period-2 consumption over period-3 consumption: his marginal rate of substitution proportionately increases by a factor  $\frac{1}{\beta}$ .

The key to the result that sophistication does not affect behavior is the assumption of log utility.

Think of a sophisticated individual contemplating a marginal decrease in  $C_1$ , relative to what a naïve individual would do. The naïve individual believes she will allocate the increase in saving equally between  $C_2$  and  $C_3$  and that marginal utility will be the same in the two future periods. The sophisticated individual realizes that she will, in fact, devote most of the increase in saving to  $C_2$  and that  $C_2$  will be high. The individual does not particularly value  $C_2$  thus marginal utility in period 2 will be low. This tends to make the increase in saving look relatively less attractive to the sophisticated individual than to the naïve individual.

But the sophisticated individual also realizes that some of the increase in saving will be devoted to  $C_3$ , which will be low. The individual values  $C_3$  as much as  $C_2$  and thus marginal utility will be high in period 3. This tends to make the increase in saving look relatively more attractive to the sophisticated individual than to the naïve individual.

With log utility, these two effects exactly offset each other. With a general utility function, a sophisticated individual can consume either more or less in the first period than a naïve individual. ■