MIT 18.100A Lecture Notes

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July 11, 2023

Contents

Preface

This book mostly refers to lecture notes of MIT 18.100A (2020 Fall) and Lebl's *Basic Analysis I: Introduction to Real Analysis* (Volume I).

I tried my best to explain thoughts and refer previous propositions to make proofs clearer. Simplicity is not the main target that mathematics beginners need to achieve.

There will be some annoying terminologies that one might struggle with or forget. I use hyper-reference in the book so that you may jump to the unfamiliar definitions or propositions.

Definitions, which are denoted with red background, are the most fundamental rules that we follow; we set the rule, and we use logic to induce propositions, lemmas, theorems, and corollaries, all of which are denoted with blue background but with different brightness.

Theorems are crucial statements, sometimes crucial enough to have a name. Propositions are less important than theorems. Lemmas are used to prove a theorem more clearly. Corollaries follow immediately from theorems or propositions. There will be a proof for each, and proofs end with a square "□".

Axioms, which are denoted with a yellow background, are the settings that enable definitions to be valid. Notations are also denoted with a yellow, lighter than the one of axioms, background since that is the way how we represent our thought and is not changed easily.

Examples are denoted with an orange background. Examples are items mentioned in the previous definition or theorems. Proofs end with a black square "■". Remarks are denoted with green background to notify you something you may know but forget.

One may struggle with some definitions, propositions, or theorems. It is always fine to take time. Nobody is born with the knowledge in the field of Real Analysis. One may just take time and ponder texts for a while. Quoting from the lecture notes of Basic Mathematics by Chih-Wen Weng, I hope that one may find interest in proofs and realize that proofs are just like a structured novel, in which any appearance of characters are at the most appropriate timing but with foreshadowing. The novel is all texts but a picture, but after one's reading, there must be an unforgettable image.

1 Sets, Induciton, and Cardinality

Definition 1.1 (Sets)**.**

A set is a collection of objects, called elements or members of the set. The empty set $\emptyset$ is the set with no elements.

Notation 1.2 (Symbols)**.**

Let S be a set. Then,

- $a \in S$ means a is an element in S;
- $\forall$ means for all;
- $\exists$ means there exists;
- $\nexists$ means there does not exist;
- $\exists!$ means there exists a unique;
- $:=$ means define;
- $\implies$ means implies;
- $\iff$ means if and only if.

Definition 1.3 (Set Relations)**.**

Let A and B be sets. Then,

1. A is a subset of B means every element of A is in B;
2. A and B are equal means $A \subseteq B$ and $B \subseteq A$;
3. A is a proper subset of B means A is a subset of B but A and B are not equal.

Notation 1.4 (Description of Set)**.**

There are many ways to describe a set. Commonly, we write $S = \{x \mid P(x)\}$ to mean S is the set with all elements x which satisfy $P(x)$. Other possible ways will be $S = \{x \in A \mid P(x)\}$, which means S is the set will all x in A which satisfy $P(x)$, or $S = \{a, b, c\}$, which means S is the set with members a, b, and c.

Definition 1.5.

Usually, we have 4 operations for sets. Let A and B be sets. Then,

1. the <u>union</u> of A and B is $A \cup B = \{x \mid x \in A \vee x \in B\}$;
2. the <u>intersection</u> of A and B is $A \cap B = \{x \mid x \in A \wedge x \in B\}$;
3. the <u>set difference</u> of A and B is $A \setminus B = \{x \mid x \in A \wedge x \notin B\}$;
4. the <u>complement</u> of A is $A^c = \{x \mid x \notin A\}$.

We call A and B are <u>disjoint</u> if $A \cap B = \emptyset$.

Theorem 1.6 (De Morgan's Laws)**.**

Let A, B, and C be sets. Then,

1. $(A \cup B)^c = A^c \cap B^c$;
2. $(A \cup B)^c = A^c \cup B^c$;
3. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$;
4. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

Proof.

1.
$$\begin{aligned}(A \cup B)^c &= \{x \mid x \notin A \cup B\} \\ &= \{x \mid x \notin A \wedge x \notin B\} \\ &= \{x \mid x \notin A\} \cap \{x \mid x \notin B\} \\ &= A^c \cap B^c.\end{aligned}$$

2.
$$\begin{aligned}(A \cap B)^c &= \{x \mid x \notin A \cap B\} \\ &= \{x \mid x \notin A \vee x \notin B\} \\ &= \{x \mid x \notin A\} \cup \{x \mid x \notin B\} \\ &= A^c \cup B^c.\end{aligned}$$

3.
$$
\begin{aligned}
A \setminus (B \cup C) &= \{x \mid x \in A \wedge x \notin (B \cup C)\} \\
&= \{x \mid x \in A \wedge (x \notin B \wedge x \notin C)\} \\
&= \{x \mid (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C)\} \\
&= (A \setminus B) \cap (A \setminus C).
\end{aligned}
$$

4.
$$
\begin{aligned}
A \setminus (B \cap C) &= \{x \mid x \in A \wedge x \notin (B \cap C)\} \\
&= \{x \mid x \in A \wedge (x \notin B \vee x \notin C)\} \\
&= \{x \mid (x \in A \wedge x \notin B) \vee (x \in A \wedge x \notin C)\} \\
&= (A \setminus B) \cup (A \setminus C).
\end{aligned}
$$
□

Axiom 1.7 (Well-ordering Property of $\mathbb{N}$)**.**

The well-ordering property of $\mathbb{N}$ states that there exists x in any $S \subseteq \mathbb{N}$ such that $s \geq x$ for any s in S.

Theorem 1.8 (Mathematical Induction)**.**

The concept was invented by Pascal in 1665. Let $P(n)$ be a statement holds for any $n \in \mathbb{N} \wedge n \geq i$. We need to do the following two:

1. (base base) check the statement when n is the initial value i (usually $n = 1$);
2. (inductive step) check the statement when $n = k + 1$ with the assumption that the statement holds when $n = k$.

Then, $P(n)$ is true for any $n \geq i, n \in \mathbb{N}$.

Proof. Let $S = \{n \in \mathbb{N} \wedge n \geq i \mid \neg P(n)\}$. We want to show that $S = \emptyset$ by contradiction. Suppose $S \neq \emptyset$. Since S is a subset of $\mathbb{N}$, by the well-ordering property of $\mathbb{N}$, there exists the smallest element m in S. Since $P(i)$ is true, we know that $m \neq i$. In addition, we know that m is the least element, so $m - 1$ is not in S, which implies that $P(m-1)$ is true. By the assumption, $P(m-1) \implies P(m)$, we have $P(m)$ is true. Thus, m is not in S, which yields a contradiction. Therefore, $S = \emptyset$ and $P(n)$ is true for any natural number $n \geq i$. □

Proposition 1.9.

For any real number $c \neq 1$ and for any $n \in \mathbb{N}$, $\displaystyle\sum_{i=0}^{n} c^i = \frac{1-c^{n+1}}{1-c}$.

Proof. We will prove this by induction on n.

(Base case) The left-hand side of the equation is $1+c$, and the right-hand side of the equation is $\dfrac{1-c^2}{1-c} = 1+c$.

Hence, the statement holds when $n = 1$.

(Inductive step) Assume that $\displaystyle\sum_{i=0}^{k} c^i = \frac{1-c^{k+1}}{1-c}$. Then, we have

$$\begin{aligned}
\sum_{i=0}^{k+1} c^i &= \sum_{i=0}^{k} c^i + c^{k+1} \\
&= \frac{1-c^{k+1}}{1-c} + c^{k+1} \\
&= \frac{1-c^{k+1}+c^{k+1}(1-c)}{1-c} \\
&= \frac{1-c^{(k+1)+1}}{1-c}.
\end{aligned}$$

Therefore, for any real number $c \neq 1$ and for any $n \in \mathbb{N}$, $\displaystyle\sum_{i=0}^{n} c^i = \frac{1-c^{n+1}}{1-c}$. □

Proposition 1.10.

For any real number $c \geq -1$ and for any $n \in \mathbb{N}$, $(1+c)^n \geq 1+nc$.

Proof. We will prove this by induction on n.

(Base case) The left-hand side of the equation is $1+c$, and the right-hand side of the equation is $(1+c)^1 = 1+c$.

Hence, the statement holds when $n = 1$.

(Inductive step) Assume that $(1+c)^m \geq 1+mc$. Then, we have

$$\begin{aligned}
(1+c)^{m+1} &= (1+c)^m \cdot (1+c) \\
&\geq (1+mc)(1+c) \\
&= 1+(m+1)c+mc^2 \\
&\geq 1+(m+1)c.
\end{aligned}$$

Therefore, for any real number $c \geq -1$ and for any $n \in \mathbb{N}$, $(1+c)^n \geq 1+nc$. □

Definition 1.11 (Function)**.**

If A and B are sets, a function $f : A \to B$ is a <u>mapping</u> that assigns each $x \in A$ to a unique element in B denoted $f(x)$.

Definition 1.12 (Injective)**.**

A function f is <u>injective</u> or <u>one-to-one</u> (1-1) if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Definition 1.13 (Surjective)**.**

A function f is <u>surjective</u> or <u>onto</u> if $f(A) = B$.

Definition 1.14 (Bijective)**.**

A function f is <u>bijective</u> if it is both surjective and bijective.

Remark 1.15.

If a function $f : A \to B$ is bijective, then $f^{-1} : B \to A$ is the function which assigns each $y \in B$ to the unique $x \in A$ such that $f(x) = y$. Note that $f(f^{-1}(x)) = x$ for all x in the domain of f.

Definition 1.16 (Cardinality)**.**

Suppose A and B are sets. If there exists an injective function $f : A \to B$, then $|A| \leq |B|$.

Definition 1.17 (Equivalence of Cardinality)**.**

Sets A and B have the same cardinality if there exists a bijection $f : A \to B$.

Notation 1.18.

$|A| = n$ if $|A| = |\{1, \ldots, n\}|$. In this case, we say A is finite; $|A| < |B|$ if $|A| \leq |B|$ but $|A| \neq |B|$.

Theorem 1.19 (Cantor-Bernstein-Schroeder Theorem)**.**

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Proof. We omit the proof here. $\square$

Definition 1.20.

If $|A| = |\mathbb{N}|$, then A is countably infinite. If A is finite or countably infinite, we say A is countable. Otherwise, we say A is uncountable.

Proposition 1.21.

1. $|\{2n \mid n \in \mathbb{N}\}| = |\mathbb{N}|$.
2. $|\{2n - 1 \mid n \in \mathbb{N}\}| = |\mathbb{N}|$.
3. $|\mathbb{Q}^+| = |\mathbb{N}|$.

Proof.

1. Let $f : \mathbb{N} \to \{2n \mid n \in \mathbb{N}\}$ as $f(n) = 2n$, then f is bijective. Thus, $|\{2n \mid n \in \mathbb{N}\}| = |\mathbb{N}|$.
2. Let $f : \mathbb{N} \to \{2n - 1 \mid n \in \mathbb{N}\}$ as $f(n) = 2n - 1$, then f is bijective. Thus, $|\{2n - 1 \mid n \in \mathbb{N}\}| = |\mathbb{N}|$.
3. We use the Cantor-Bernstein-Schroeder Theorem to prove this statement. Define $f_1 : \mathbb{N} \to \mathbb{Q}^+$ as $f_1(n) = n$, then f_1 is obviously injective. Thus, $|\mathbb{N}| \leq |\mathbb{Q}^+|$. Moreover, define $f_2 : \mathbb{Q}^+ \to \mathbb{N}$ as

$$f_2\left(\frac{p}{q}\right) = \begin{cases} p^2 - (q-1) & \text{if } p > q \\ 1 & \text{if } p = q \\ (q-1)^2 + p & \text{if } p < q \end{cases},$$

where p and q are coprime. Then, f_2 is injective. Thus, $|\mathbb{Q}^+| \leq |\mathbb{N}|$. Since, $|\mathbb{N}| \leq |\mathbb{Q}^+|$ and $|\mathbb{Q}^+| \leq |\mathbb{N}|$, by the Cantor-Bernstein-Schroeder Theorem, $|\mathbb{Q}^+| = |\mathbb{N}|$. $\square$

Definition 1.22 (Power Set)**.**

If A is a set, $\mathcal{P}(A) = \{B \mid B \subseteq A\}$. $\mathcal{P}(A)$ is called the power set of A, also denoted as $\mathbb{P}(A)$.

Example 1.23.

Let $A \subseteq \mathbb{R}$. Then,

1. if $A = \emptyset$ then $\mathcal{P}(A) = \{\emptyset\}$;
2. if $A = \{1\}$, then $\mathcal{P}(A) = \{\emptyset, \{1\}\}$;
3. if $A = \{1, 2\}$, then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Proof. This follows immediately by Definition 3.1. ■

Theorem 1.24 (Cantor's Theorem)**.**

If A is a set, then $|A| < |\mathcal{P}(A)|$.

Proof. Let A be a set. Define the function $f : A \to \mathcal{P}(A)$ by $f(x) = \{x\}$. f is injective as $\{x\} = \{y\}$ implies $x = y$. Thus, $|A| \leq |\mathcal{P}(A)|$. We are now showing $|A| \neq |\mathcal{P}(A)|$. We will do so through contradiction. Suppose that $|A| = |\mathcal{P}(A)|$. Then, there exists a surjection $g : A \to \mathcal{P}(A)$. Let $B \coloneqq \{x \in A \mid x \notin g(x)\} \in \mathcal{P}(A)$. Since g is surjective, there exists $b \in A$ such that $g(b) = B$. There are two cases:

1. $b \in B$, then $b \notin g(b) = B$, which implies $b \notin B$;
2. $b \notin B$, then $b \notin g(b) = B$, which implies $b \in B$.

We obtain contradictions in both case. Thus, g is not surjective and $|A| \neq |\mathcal{P}(A)|$. Therefore, $|A| < |\mathcal{P}(A)|$. □

Corollary 1.25.

$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots$.

Proof. This follows immediately from the Cantor's Theorem. □

Remark 1.26.

By Corollary 1.25, there are an infinite number of infinite sets.

Proposition 1.27.

For any $n \in \mathbb{N} \cup \{0\}$, $n < 2^n$.

Proof. We will prove this by induction on n.

(Base case) Trivial.

(Inductive step) We have $k < 2^k$ by assumption, then we multiply 2 on both sides of the equation, having $2k < 2^{k+1}$. Since we have the assumption that $k \geq 1$, we can have $2k \geq k+1$. Therefore, $k+1 < 2^{k+1}$.

Therefore, for any $n \in \mathbb{N} \cup \{0\}$, $n < 2^n$. $\square$

Theorem 1.28.

There exists a unique ordered field, denoted $\mathbb{R}$, containing $\mathbb{Q}$ with the least upper bound property.

Proof. The proof is omitted. We should now avoid using algebra and just assume is true. $\square$

Definition 1.29 (Ordered Set)**.**

An ordered set is a set S with a relation $<$ called an ”ordering” such that

1. for any x and y in S, either $x < y$, $y < x$, or $x = y$;
2. if $x < y$ and $y < z$, then $x < z$.

Example 1.30.

Number 4 will be a non-example.

1. $\mathbb{Z}$ is an ordered set, with the relation that $\forall m \in \mathbb{Z}\ \forall n \in \mathbb{Z}\ m > n \iff m - n \in \mathbb{N}$;
2. $\mathbb{Q}$ is an ordered set, with the relation that $\forall p \in \mathbb{Z}\ \forall q \in \mathbb{Z}\ p > q \iff \exists m \in \mathbb{N}\ \exists n \in \mathbb{N}\ p - q = \dfrac{m}{n}$;
3. $\mathbb{Q} \times \mathbb{Q}$ is an ordered set with the relation $(q, r) > (s, t) \iff q > s \vee (q = s \wedge r > t)$;
4. consider the set $\mathcal{P}(\mathbb{N})$. Let $A \in \mathcal{P}(\mathbb{N})$ and let $B \in \mathcal{P}(\mathbb{N})$. Let $A \prec B$ if $A \subseteq B$. This is **not** an ordered set since it doesn't satisfy the first property of an ordered set.

Proof. The proof is not related to the scope of this course thus omitted. $\blacksquare$

Remark 1.31.

In Example 1.30, the order on $\mathbb{Q} \times \mathbb{Q}$ is called dictionary order or lexicographic order.

2 Properties of the Real Numbers

Definition 2.1 (Bounded Above and Bounded Below)**.**

Let S be an ordered set and let $E \subseteq S$. If there exists $b \in S$ such that $x \leq b$ for all $x \in E$, then E is bounded above; if there exists $c \in S$ such that $x \geq c$ for all $x \in E$, then E is bounded below.

Definition 2.2 (Upper Bound and Lower Bound)**.**

Let S be an ordered set and let $E \subseteq S$. If there exists $b \in S$ such that $x \leq b$ for all $x \in E$, then b is an upper bound of E; if there exists $c \in S$ such that $x \geq c$ for all $x \in E$, then c is a lower bound of E.

Definition 2.3 (Least Upper Bound)**.**

We say that b_0 is the least upper bound or the supremum of E if the following two hold:

1. b_0 is an upper bound for E;
2. if b is an upper bound for E then $b_0 \leq b$.

Definition 2.4 (Greatest Lower Bound)**.**

We say that c_0 is the greatest lower bound or the infinimum (infimum) of E if the following two hold:

1. c_0 is a lower bound for E;
2. if c is a lower bound for E then $c_0 \geq c$.

Notation 2.5.

The least upper bound of E is denoted $\sup E$, and the greatest lower bound of E is denoted $\inf E$.

Example 2.6.

- If $S = \mathbb{Z}$ and $E = \{-2, -1, 0, 1, 2\}$, then $\inf E = -2$ and $\sup E = 2$.
- If $S = \mathbb{Q}$ and $E = \{q \in \mathbb{Q} \mid 0 \leq q < 1\}$, then $\inf E = 0 \in E$, but $\sup E = 1 \notin E$.
- If $S = \mathbb{Z}$ and $E = \mathbb{N}$, then $\inf E = 1$, but $\sup E$ does not exist.

Proof. The proof is omitted. ■

Definition 2.7 (Least Upper Bound Property)**.**

An ordered set S has the least upper bound property if every nonempty and bounded above $E \subseteq S$ has a supremum in S.

Example 2.8.

$-\mathbb{N} = \{-1, -2, -3, \cdots\}$ has the least upper bound property.

Proof. Trivial. ∎

Remark 2.9.

$E \subseteq S$ is bounded above if and only if $-E \subseteq \mathbb{N}$ is bounded below. By the well-ordering property of $\mathbb{N}$, $-E$ has a least element $x \in -E$, and thus $-x = \sup E$.

Remark 2.10.

The premise of Proposition 2.11 is false, i.e., $\sup\{q \in \mathbb{Q} \mid q > 0 \wedge q^2 < 2\} \notin \mathbb{Q}$. See Proposition 2.12.

Proposition 2.11.

If $x \in \mathbb{Q}$ and $x = \sup\{q \in \mathbb{Q} \mid q > 0 \wedge q^2 < 2\}$, then $x > 0$ and $x^2 = 2$.

Proof. Let $E = \{q \in \mathbb{Q} \mid q > 0 \wedge q^2 < 2\}$, and suppose $x \in \mathbb{Q}$ such that $x = \sup E$. Since $1 \in E$ and x is an upper bound for E, $1 \leq x$, which implies $x > 0$. Now we just need to prove that $x^2 = 2$.

We will do this by showing $x^2 \geq 2$ and $x^2 \leq 2$.

1. (Show that $x^2 \geq 2$) Suppose $x^2 < 2$. Define $h \coloneqq \min\left\{\frac{1}{2}, \frac{2 - x^2}{2(2x+1)}\right\} < 1$. If $x^2 < 2$ then $h > 0$. We now

prove that $x + h \in E$. Thus, x is not a supremum. By $h < 1$, we have

$$\begin{aligned}
(x+h)^2 &= x^2 + 2xh + h^2 \\
&< x^2 + h(2x+1) \\
&\leq x^2 + \frac{2-x^2}{2(2x+1)} \cdot (2x+1) \\
&= x^2 + \frac{2-x^2}{2} \\
&< 2 + \frac{2-2}{2} \\
&= 2.
\end{aligned}$$

Thus, $x + h \in E$. By $h > 0$ and $x + h \in E$, we have $x \neq \sup E$, which is a contradiction. Hence, $x^2 \geq 2$.

2. (Show that $x^2 \leq 2$) Suppose $x^2 > 2$. Define $h := \dfrac{x^2-2}{2x}$. If $x^2 > 2$, then $h > 0$ and $x - h = \dfrac{x^2+2}{2x} > 0$.

   We will show that $x - h$ is an upper bound for E. Thus, x is not a supremum. We have

$$\begin{aligned}
(x-h)^2 &= x^2 - 2xh + h^2 \\
&= x^2 - (x^2 - 2) + h^2 \\
&= 2 + h^2 \\
&> 2.
\end{aligned}$$

   Let $q \in E$.

$$\begin{aligned}
& q^2 < 2 < (x-h)^2 \\
\implies\ & (x-h)^2 - q^2 > 0 \\
\implies\ & ((x-h)+q)((x-h)+q) > 0 \\
\implies\ & (x-h) - q > 0.
\end{aligned}$$

   Thus, for all $q \in E$, $q < x - h < x$. This implies $x \neq \sup E$, which is a contradiction. Hence, $x^2 \leq 2$.

Therefore, $x^2 = 2$. □

Proposition 2.12.

The set $E = \{q \in \mathbb{Q} \mid q > 0 \land q^2 < 2\}$ does not have a supremum in $\mathbb{Q}$.

Proof. Suppose there exists $x \in \mathbb{Q}$ such that $x = \sup E$. Then, by our previous theorem, $x^2 = 2$. Thus, there exist $m \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $m > n$ and $x = \dfrac{m}{n}$, and therefore $nx \in \mathbb{N}$. Let $S = \{k \in \mathbb{N} \mid kx \in \mathbb{N}\}$. Then, $S \neq \emptyset$ since $n \in S$. By the well-ordering property of $\mathbb{N}$, S has a least element $k_0 \in S$. Let $k_1 = k_0x - k_0 \in \mathbb{Z}$. Then, $k_1 = k_0(x-1) > 0$ since $k_0 \in \mathbb{N}$ and $x > 1$. Therefore, $k_1 \in \mathbb{Z}^+$. Since $x^2 = 2$, we have $x < 2$. Thus, $k_1 = k_0(x-1) < k_0(2-1) = k_0$. So, since $k_1 \in \mathbb{N}$ and $k_1 < k_0$, we have $k_1 \notin S$ as k_0 is the least element of S. However,

$$\begin{aligned} xk_1 &= x(k_0(x-1)) \\ &= k_0x^2 - xk_0 \\ &= 2k_0 - xk - 0 \\ &= 2k_0 - k_1, \end{aligned}$$

which implies that $xk_1 \in \mathbb{N}$, further implies that $k_1 \in S$, which is a contraction. Thus, there does not exist $x \in \mathbb{Q}$ such that $x = \sup E$. $\square$

Definition 2.13 (Field)**.**

A set F is a field if it has two operations: addition $(+)$ and multiplication $(\cdot)$ with the following properties:

(A1) if x and y are both in F, then $x + y \in F$;

(A2) (*Commutativity*) if x and y are both in F, then $x + y = y + x$;

(A3) (*Associativity*) if x, y, and z are all in F, then $(x + y) + z = x + (y + z)$;

(A4) there exists an element in F, denoted 0, such that $0 + x = x = x + 0$;

(A5) for any x in F, there exists $y \in F$ such that $x + y = 0$. $-x$ denotes y;

(M1) if x and y are both in F, then $x \cdot y \in F$;

(M2) (*Commutativity*) if x and y are both in F, $x \cdot y = y \cdot x$;

(M3) (*Associativity*) if x, y, and z are all in F, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;

(M4) there exists an element in$F$, denoted 1, such that $1 \cdot x = x = x \cdot 1$ for all $x \in F$;

(M5) for any x in $F \setminus \{0\}$, there exists an element in F, denoted x^{-1}, such that $x \cdot x^{-1} = 1$;

(D) (*Distributativity*) if x, y, and z are all in F, then $(x + y)z = xz + yz$.

Example 2.14.

- $\mathbb{Z}_2 = \{0, 1\}$ with $1 + 1 = 0$.
- $\mathbb{Z}_3 = \{0, 1, 2\}$ with $c := a + b \pmod 3$, i.e., $2 + 1 = 3 = 0 \pmod 3$ and $2 \cdot 2 = 4 = 3 + 1 = 1 \pmod 3$.

Proof. The proof is omitted. ■

Proposition 2.15.

If x is in F and F is a field, then $0x = 0$.

Proof. Let $x \in F$. Then,

$$\begin{aligned} 0x &= (0 - 0)x \\ &= 0x - 0x \\ &= 0. \end{aligned}$$

□

Definition 2.16 (Ordered field)**.**

A field F is an <u>ordered field</u> if F is also an ordered set with ordering $<$, and the following two hold:

1. $\forall x \in F\ \forall y \in F\ \forall z \in F\ x < y \implies x + z < y + z$;
2. $x > 0 \wedge y > 0 \implies xy > 0$.

If $x > 0$, we say x is <u>positive</u>; if $x \geq 0$ we say x is <u>non-negative</u>.

Proposition 2.17.

If $x > 0$, then $-x < 0$ (and vice versa).

Proof. If $x \in F$ and $x > 0$, then $-x + x > -x$, which implies $0 > -x$. □

Proposition 2.18.

Let $x \in F$ and $y \in F$, where F is an ordered field. If $x > 0$ and $y < 0$ or $x < 0$ and $y > 0$, then $xy < 0$.

Proof. Suppose $x > 0$ and $y < 0$. $x > 0$ and $-y > 0$. Hence, $-xy = x(-y) > 0$. Thus, $xy < 0$. If $x < 0$ and $y > 0$, then $-x > 0$ and $y > 0$, which implies $-xy = (-x)y > 0$. Therefore, $xy < 0$. □

Theorem 2.19.

Let F be an ordered field with the least upper bound property. If $A \subseteq F$ is nonempty and bounded below, then $\inf A$ exists in F. That is to say, for an ordered field F, if F has the least upper bound property, then F has a greatest lower bound property.

Proof. Suppose $A \subseteq F$ is nonempty and bounded below. Let a be a lower bound of A. Let $B = \{-x \mid x \in A\} \subseteq F$, then $-a$ is an upper bound of B. Thus, there exists $\sup B$ since F has the least upper bound property. By $B = \{-x \mid x \in A\}$, we have $-x \le \sup B$ for all $x \in A$. Hence, $x \ge -\sup B$ for all $x \in A$. Thus, $-\sup B$ is a lower bound of A. Since $-a$ is an upper bound of B, $\sup B \le -a$, which implies $-\sup B \ge a$. By $x \ge \sup -B$ and $-\sup B \ge a$, we have $-\sup B$ is the greatest lower bound of A. □

Proposition 2.20.

There exists a unique r in $\mathbb{R}$ such that $r > 0$ and $r^2 = 2$.

Proof. We are going to show the statement by checking the existence and the uniqueness.

(Existence) Let $E = \{x \in \mathbb{R} \mid x > 0 \ \wedge \ x^2 = 2\}$. Since E is bounded above by 2, $\sup E$ exists. Let r denote $\sup E$. Since $1 \in E$ and r is an upper bound for E, $1 \le x$, which implies $x > 0$. Now we just need to prove that $x^2 = 2$. We will do this by showing $r^2 \ge 2$ and $r^2 \le 2$.

1. (Show that $r^2 \ge 2$) Suppose $r^2 < 2$. Define $h \coloneqq \min\left\{\frac{1}{2}, \frac{2-r^2}{2(2r+1)}\right\} < 1$. If $r^2 < 2$ then $h > 0$. We now prove that $r + h \in E$ and thus r is not a supremum. By $h < 1$, we have

$$
\begin{aligned}
(r+h)^2 &= r^2 + 2rh + h^2 \\
&< r^2 + h(2r+1) \\
&\le r^2 + \frac{2-r^2}{2(2r+1)} \cdot (2r+1) \\
&= r^2 + \frac{2-r^2}{2} \\
&< 2 + \frac{2-2}{2} \\
&= 2.
\end{aligned}
$$

Thus, $r + h \in E$. By $h > 0$ and $r + h \in E$, we have $r \neq \sup E$, which is a contradiction. Hence, $r^2 \ge 2$.

2. (Show that $r^2 \leq 2$) Suppose $r^2 > 2$. Define $h \coloneqq \dfrac{r^2-2}{2r}$. If $r^2 > 2$, then $h > 0$ and $r - h = \dfrac{r^2+2}{2r} > 0$.

   We will show that $r - h$ is an upper bound for E and thus r is not a supremum. We have

$$\begin{aligned}(r-h)^2 &= r^2 - 2rh + h^2\\ &= r^2 - (r^2-2) + h^2\\ &= 2 + h^2\\ &> 2.\end{aligned}$$

   Let $q \in E$.

$$\begin{aligned}& q^2 < 2 < (r-h)^2\\ \implies\quad & (r-h)^2 - q^2 > 0\\ \implies\quad & ((r-h)+q)((r-h)+q) > 0\\ \implies\quad & (r-h) - q > 0.\end{aligned}$$

   Thus, for all $q \in E$, $q < r - h < r$. This implies $r \neq \sup E$, which is a contradiction. Hence, $r^2 \leq 2$.

Thus, $r^2 = 2$.

(Uniqueness) Suppose there exists $r^* > 0$ with $r^{*2} = 2$. By $r + r^* > 0$, we have

$$\begin{aligned}r^2 - r^{*2} &= 0\\ (r+r^*)(r-r^*) &= 0\\ r - r^* &= 0.\end{aligned}$$

Thus, $r = r^*$. □

Theorem 2.21 (Archimedean Property)**.**

If $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $n > \dfrac{y}{x}$.

Proof. Suppose that $x \in \mathbb{R}^+$ and $y \in \mathbb{R}$. We will prove this by contradiction. Suppose that for any natural numbers n, $n \leq \dfrac{y}{x}$. In other words, $\mathbb{N}$ is bounded above by $\dfrac{y}{x}$. Hence, there exists $a \in \mathbb{R}$ such that $a = \sup \mathbb{N}$. Since $a - 1$ is not a upper bound for $\mathbb{N}$, there exists $m \in \mathbb{N}$ such that $a - 1 < m < a$. The inequality on the left implies that $a < m+1$, which means a is not an upper bound for $\mathbb{N}$, which is a contradiction. Therefore, there exists $n \in \mathbb{N}$ such that $n > \dfrac{y}{x}$. $\square$

Theorem 2.22 (Density of the Rationals)**.**

If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x < y$, then there exists $r \in \mathbb{Q}$ such that $x < r < y$.

Proof. Suppose $x \in \mathbb{R}$ and $y \in \mathbb{R}$. There are three cases:

1. If $0 \le x < y$, then, by the Archimedean property, there exists $n \in \mathbb{N}$ such that $n(y-x) > 1$. Consider the set $S = \{k \in \mathbb{N} \mid k > nx\}$, it is nonempty by the Archimedean property. By the well-ordering property of $\mathbb{N}$, S has a least element m. Thus, we have $m > nx$. Since m is the least element in S, then $m-1$ is not in S. Hence, we have $m-1 \le nx$. By $m-1 \le nx$ and $n(y-x) > 1$, we have $m \le nx+1 < ny$. Therefore, by $m > nx$ and $m < ny$, we have $x < \dfrac{m}{n} < y$.
2. If $x < 0 < y$, then $r = 0$.
3. If $x < y \le 0$, then we have $0 \le -y < -x$. We can apply 1. and have $y > -\dfrac{m}{n} > x$. $\square$

Example 2.23.

$\sup\left\{1 - \dfrac{1}{n} \mid n \in \mathbb{N}\right\} = 1.$

Proof.

We first show that 1 is an upper bound for $\left\{1 - \dfrac{1}{n} \mid n \in \mathbb{N}\right\}$. Since for any $n \in \mathbb{N}$, $1-\dfrac{1}{n} < 1$. Now we show that any upper bound for $\left\{1 - \dfrac{1}{n} \mid n \in \mathbb{N}\right\}$ is not less than 1. Suppose x is an upper bound for $\left\{1 - \dfrac{1}{n} \mid n \in \mathbb{N}\right\}$. For the sake of contradiction, we assume $x < 1$. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $n(1-x) > 1$, which implies $\dfrac{n-1}{n} > x$, then x is not an upper bound for $\left\{1 - \dfrac{1}{n} \mid n \in \mathbb{N}\right\}$. Hence, $x \ge 1$. Therefore, $\sup\left\{1 - \dfrac{1}{n} \mid n \in \mathbb{N}\right\} = 1$. $\blacksquare$

Theorem 2.24.

Suppose that $S \subseteq \mathbb{R}$ is nonempty and bounded above. Then, $x = \sup S$ if and only if the following two hold:

1. x is an upper bound for S;

2. for any positive ε, there exists $y \in S$ such that $x - \varepsilon < y \leq x$.

Proof.

($\Longrightarrow$) Assume $x = \sup S$, then $x \geq s$ for all $s \in S$, and $x \leq u$ for all $u \geq s$ for all $s \in S$. Thus, by $x \geq s$ for all $s \in S$, we know that x is an upper bound for S. Let $\varepsilon > 0$. Take $y = x - \dfrac{\varepsilon}{2}$, then $y \in S$, $y < x$, and $x - \varepsilon < y$. Thus, there exists $y \in S$ such that $x - \varepsilon < y \leq x$.

($\Longleftarrow$) Assume x is an upper bound for S and there exists $y \in S$ such that $x - \varepsilon < y \leq x$ for all $\varepsilon > 0$. Let u be an upper bound for S. For the sake of contradiction, assume that $x > u$. Take $\varepsilon = \dfrac{x-u}{2} > 0$, then $x - \varepsilon = \dfrac{x+u}{2} > \dfrac{u+u}{2} = u$. By assumption, there exists $y \in S$ such that $u < y \leq x$. However, u is an upper bound for S, i.e., $y \leq u$, which yields a contradiction. Hence, $x \leq u$.

Therefore, $x = \sup S$ if and only if x is an upper bound for S and there exists $y \in S$ such that $x - \varepsilon < y \leq x$ for any positive ε. $\square$

Notation 2.25.

For $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$, define

1. $x + A \coloneqq \{x + a \mid a \in A\}$
2. $xA \coloneqq \{xa \mid a \in A\}$.

Proposition 2.26.

If $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ is bounded above, then $x + A$ is bounded above and $\sup(x + A) = x + \sup A$.

Proof. Let u be the least upper bound for A, then we want to show $\sup(x + A) = x + \sup A$. Since u is the least upper bound for A, $u \geq a$ for all $a \in A$ and $u \leq v$ for all $v \geq a$ for all $a \in A$. $x + u \geq x + a$ for all $a \in A$ and $x + u \leq x + v$ for all $v \geq a$ for all $a \in A$. Thus, $\sup(x + A) = x + u$. Therefore, $\sup(x + A) = x + \sup A$. $\square$

Proposition 2.27.

If $x > 0$ and $A \subseteq \mathbb{R}$ is bounded above then xA is bounded above and $\sup(xA) = x \sup A$.

Proof. Let u be the least upper bound for A, then we want to show $\sup(xA) = x \sup A$. Since u is the least upper bound for A, $u \geq a$ for all $a \in A$ and $u \leq v$ for all $v \geq a$ for all $a \in A$. $xu \geq xa$ for all $a \in A$ and $xu \leq xv$ for all $v \geq a$ for all $a \in A$. Thus, $\sup(xA) = xu$. Therefore, $\sup(xA) = x \sup A$. $\square$

Proposition 2.28.

Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. If $x \leq y$ for all for all $x \in A$ and for all $y \in B$, then $\sup A \leq \inf B$.

Proof. Let $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$. Assume that $x \leq y$ for all $x \in A$ and for all $y \in B$. This implies that $\sup A \leq y$ for all $y \in B$. Since $B \subseteq \mathbb{R}$ is a set with an upper bound, $\inf B$ exists. Therefore, $\sup A \leq \inf B$. $\square$

Definition 2.29.

If $x \in \mathbb{R}$ we define $|x| \coloneqq \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

Theorem 2.30.

1. $|x| \geq 0$;
2. $|x| = 0 \iff x = 0$;
3. $\forall x \in \mathbb{R}\ |-x| = |x|$;
4. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ |xy| = |x||y|$;
5. $|x^2| = x^2 = |x|^2$;
6. $\forall x \in \mathbb{R}\ x \leq |x|$;
7. $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ |x| \leq y \iff -y \leq x \leq y$.

Proof.

1. For $x \geq 0$, we have $|x| = x \geq 0$. For $x \leq 0$, we have $|x| = -x \geq 0$. Therefore, $|x| \geq 0$ for all x.
2. $(\implies)$ If $x = 0$, then $|x| = 0$.

   $(\impliedby)$ If $|x| = 0$, then we have $x = -x$. Thus, $x = 0$.

   Therefore, $|x| = 0 \iff x = 0$.
3. For $x \geq 0$, $|-x| = -(-x) = x = |x|$. For $x \leq 0$, $|x| = -x = |-x|$. Therefore, $|x| = |-x|$ for all x.
4. We separate cases to discuss:
   (a) $x \geq 0 \wedge y \geq 0 \quad |xy| = xy = |xy|$;
   (b) $x \geq 0 \wedge y \leq 0 \quad |xy| = -xy = x(-y) = |x||y|$;
   (c) $x \leq 0 \wedge y \geq 0 \quad |xy| = -xy = (-x)y = |x||y|$;
   (d) $x \leq 0 \wedge y \leq 0 \quad |xy| = xy = (-x)(-y) = |x||y|$.
5. Take $y = x$ in 3.2, then we have $|x^2| = |x|^2$. Since $x^2 \geq 0$, we have $|x^2| = |x|^2 = x^2$.
6. For $x \geq 0$, $x = |x|$. For $x \leq 0$, $|x| = -x > x$. Therefore, $x \leq |x|$ for all $x \in \mathbb{R}$.

7. For $x \geq 0$, if $|x| \leq y$, then $-y \leq 0 \leq |x| = x \leq y$. For $x \leq 0$, if $|x| \leq y$, then $-y \leq 0 \leq -x \leq y$. Therefore, $|x| \leq y \iff -y \leq x \leq y$ for all $x \in \mathbb{R}$ for all $y \in \mathbb{R}$. □

Theorem 2.31 (Triangle Inequality)**.**

If $x \in \mathbb{R}$ and $y \in \mathbb{R}$, then $|x+y| \leq |x| + |y|$.

Proof. Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$. We have $x + y \leq |x| + |y|$ and $-(x+y) \leq |x| + |y|$. Hence, $-(|x|+|y|) \leq x + y \leq |x| + |y|$. Therefore, $|x+y| \leq |x| + |y|$. □

Theorem 2.32 (Reverse Triangle Inequality)**.**

If $x \in \mathbb{R}$ and $y \in \mathbb{R}$, then $||x| - |y|| \leq x - y$.

Proof. Let $x \in \mathbb{R}$ and let $y \in \mathbb{R}$. Since

$$\begin{aligned} |x| &= |x - y + y| \\ &\leq |x - y| + |y| \\ \implies |x| - |y| &\leq |x - y| \end{aligned}$$

and

$$\begin{aligned} |y| &= |y - x + x| \\ &\leq |x - y| + |x| \\ \implies |x| - |y| &\geq -|x - y|, \end{aligned}$$

by Theorem 2.30.7, we have $||x| - |y|| \leq |x - y|$. □

Definition 2.33 (Represented)**.**

Let $x \in (0,1]$ and $d_i \in \{0, 1, \dots, 9\}$. We say that x is <u>represented</u> by the digits $\{d_i \mid i \in \mathbb{N}\}$, i.e., $x = 0.d_1 d_2 \dots$, if $x = \sup \left\{ \sum_{i=1}^{n} d_i \cdot 10^{-i} \mid n \in \mathbb{N} \right\}$.

Theorem 2.34.

For every set of digits $\{d_i \mid i \in \mathbb{N}\}$, there exists a unique $x \in [0,1]$ such that $x = 0.d_1 d_2 \dots$ for all $n \in \mathbb{N}$.

Proof. Take an arbitrary infinite sequence of digits $0.d_i d_2 \dots$. Let $D_n = \sum_{i=1}^{n} \frac{d_i}{10^i}$, then we have $0 \leq D_n \leq 1$ for all $n \in \mathbb{N}$. Let $x = \sup_{n \in \mathbb{N}} D_n$, then $x \geq D_n$. Take $m \in \mathbb{N}$. If $m > n$, then

$$\begin{aligned} D_m - D_n &= \sum_{i=n+1}^{m} \frac{d_i}{10^i} \\ &\leq \sum_{i=n+1}^{m} \frac{9}{10^i} \\ &= \frac{9}{10^{n+1}} \cdot \frac{1 - \left(\frac{1}{10}\right)^{m-n}}{1 - \frac{1}{10}} \\ &< \frac{1}{10^n}. \end{aligned}$$

We take the supremum over D_m to find $x - D_n \leq \frac{1}{10^n}$. Therefore, for every set of digits $\{d_i \mid i \in \mathbb{N}\}$, there exists a unique $x \in [0, 1]$ such that $x = 0.d_1 d_2 \dots$ for all $n \in \mathbb{N}$. $\square$

Theorem 2.35.

For every $x \in (0, 1]$, there exists a unique sequence of digits $\{d_i \mid i \in \mathbb{N}\}$ such that $x = 0.d_1 d_2 \dots$ and $0.d_1 d_2 \dots d_n < x \leq 0.d_1 d_2 \dots d_n + 10^{-n}$ for all $n \in \mathbb{N}$.

Proof. Let $x \in (0, 1]$. We use some notation for convenience: x_n denotes $10^{n-1}x$, and x'_n denotes the units digit of x_n. Since we have $x'_2 \in (0, 10]$, there exists one and only a non-negative integer $d_1 < 10$ such that $d_1 < x'_2 \leq d_1 + 1$; of course $d_1 < x_2 \leq d_1 + 1$. There exists another one and only non-negative integer $d_2 < 10$ such that $d_2 \leq [x'_3] \leq d_2 + 1$. Since x'_3 is the units digit of x_3, it is an non-negative integer that is less than 10; hence, we have $10d_1 + d_2 < x_3 \leq 10d_1 + d_2 + 1$. Similarly, we will have a d_3 such that $100d_1 + 10d_2 + d_3 < x_4 \leq 100d_1 + 10d_2 + d_3 + 1$. By keep doing so, we will have a sequence $\{d_i\}_{i=1}^{n}$ such that $\sum_{i=1}^{n-1} 10^{n-i-1} d_i < x_n \leq \sum_{i=1}^{n-1} 10^{n-i-1} d_i + 1$ for all $n \in \mathbb{N} \setminus \{1\}$. Hence, we will have $\sum_{i=1}^{n-1} 10^{n-i-1} d_i < 10^{n-1} x \leq \sum_{i=1}^{n-1} 10^{n-i-1} d_i + 1$ for all $n \in \mathbb{N} \setminus \{1\}$. Since 10^{n-1} is never 0, we divide the inequality by 10^{n-1}, having $\sum_{i=1}^{n-1} 10^{-i} d_i < x \leq \sum_{i=1}^{n-1} 10^{-i} d_i + 10^{n-1}$ for all $n \in \mathbb{N} \setminus \{1\}$. For the case $n = 1$, showing $d_1 < x_2 \leq d_1 + 1$ is equivalent to showing $0.d_1 < x \leq 0.d_1 + 10^{-1}$. We can generalize the induction to infinity, having the result of $x = 0.d_1 d_2 \dots$. Therefore, both propositions are proved. $\square$

Proposition 2.36.

$(0, 1]$ is uncountable.

Proof. We prove this by contradiction. Suppose $(0, 1]$ is countable. Then, there exists a bijective function $f : \mathbb{N} \to (0, 1]$. We construct a y in $(0, 1]$ but not in the range of f. We write $f(n) = 0.d_{1,n}d_{2,n}d_{3,n}\ldots$. That means, the input of f is a natural number, and the output of f will be a sequence of natural numbers. Define $e_j = \begin{cases} 1 & \text{if } d_{j,j} \neq 1 \\ 2 & \text{if } d_{j,j} = 1 \end{cases}$. Let $y = 0.e_1e_2e_3\ldots$. Then, for any $n \in \mathbb{N}$, we have $0.e_1e_2e_3\ldots e_n < y \leq 0.e_1e_2e_3\ldots e_n + 10^{-n}$ as all e_j's are positive. Thus $0.e_1e_2e_3\ldots$ is the unique representation of y, which implies that for some m, $y = f(m)$. However, for any $n \in \mathbb{N}$, $d_{n,n} \neq e_n$. Thus, $f(n) \neq y$ for any $n \in \mathbb{N}$, which yields a contradiction. Therefore, $(0, 1]$ is uncountable. $\square$

Corollary 2.37.

$\mathbb{R}$ is uncountable.

Proof. Define $f : \mathbb{R} \to (0, 1]$ by $f(x) = \dfrac{\arctan(x)}{\pi} + \dfrac{1}{2}$. Since f is bijective, $\mathbb{R}$ and $(0, 1]$ have the same cardinality. Therefore, by Proposition 2.36, $\mathbb{R}$ is uncountable. $\square$

3 Sequences and Convergence

Definition 3.1 (Sequence of Real Numbers)**.**

A sequence of real numbers is a function $x : \mathbb{N} \to \mathbb{R}$. Usually, x_n denotes $x(n)$.

Notation 3.2.

A sequence is denoted by either $\{x_n\}$, $\{x_n\}_{n=1}^{\infty}$, or $x_1, x_2, \ldots$.

Definition 3.3 (Bounded)**.**

A sequence $\{x_n\}$ is bounded if there exists $M \geq 0$ such that $|x_n| \leq M$ for any $n \in \mathbb{N}$.

Definition 3.4 (Sequence Convergence)**.**

A sequence $\{x_n\}$ converges to x if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $|x_n - x| < \varepsilon$.

Proposition 3.5.

Let $x \in \mathbb{R}$ and let $y \in \mathbb{R}$. If $|x - y| < \varepsilon$ for any $\varepsilon > 0$, then $x = y$.

Proof. Suppose $x \neq y$. Take $\varepsilon = \dfrac{|x-y|}{2}$, then $|x - y| < \dfrac{|x-y|}{2}$, which yields a contradiction. $\square$

Theorem 3.6 (Uniqueness of the Limits of Convergent Sequences)**.**

If a sequence $\{x_n\}$ converges to x and y, then $x = y$.

Proof. We prove this using Proposition 3.5. Suppose $\{x_n\}$ converges to x and y, we want to show $|x - y| < \varepsilon$ for any $\varepsilon > 0$. On the one hand, since $\{x_n\}$ converges to x, we know that there is an $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $|x_n - x| < \dfrac{\varepsilon}{2}$. On the other hand, since $\{x_n\}$ converges to y, we know that there is an $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|x_n - y| < \dfrac{\varepsilon}{2}$. Take $N = \max\{N_1, N_2\}$, then $n > N$ implies $|x_n - x| < \dfrac{\varepsilon}{2}$ and $|x_n - y| < \dfrac{\varepsilon}{2}$. By the Triangle Inequality, $|x - y| < \varepsilon$. By the Proposition 3.5, $x = y$. $\square$

Notation 3.7.

If $\{x_n\}$ converges to $x \in \mathbb{R}$, then we write $\lim_{n\to\infty} x_n = x$.

Proposition 3.8.

If $x_n = c$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n = c$.

Proof. Let $\varepsilon > 0$. Take $N = 1$. Then, $n > N$ implies $|x_n - c| = 0 < \varepsilon$. Therefore, $\lim_{n\to\infty} x_n = c$. □

Example 3.9.

If $x_n = \dfrac{1}{n}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\varepsilon > 0$. Take $N = \left[\dfrac{1}{\varepsilon}\right] + 1$. Then, $n > N$ implies

$$\begin{aligned}\left|\frac{1}{n} - 0\right| &< \frac{1}{\left[\frac{1}{\varepsilon}\right] + 1} \\ &< \frac{1}{\frac{1}{\varepsilon}} \\ &= \varepsilon.\end{aligned}$$

Therefore, $\lim_{n\to\infty} \dfrac{1}{n} = 0$. ■

Example 3.10.

Let $x_n = \dfrac{1}{n^2 + 2n + 100}$ for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\varepsilon > 0$. Take $N = \left[\dfrac{1}{2\varepsilon}\right] + 1$. Then, $n > N$ implies

$$\begin{aligned}\left|\frac{1}{n^2 + 2n + 100} - 0\right| &< \frac{1}{2n} \\ &< \frac{1}{2 \cdot \left(\left[\frac{1}{2\varepsilon}\right] + 1\right)} \\ &< \frac{1}{2 \cdot \frac{1}{2\varepsilon}} \\ &= \varepsilon.\end{aligned}$$

Therefore, $\lim_{n\to\infty} \dfrac{1}{n^2 + 2n + 100} = 0$. ■

Negation 3.11 (Divergence)**.**

A sequence $\{x_n\}$ diverges if for all $x \in \mathbb{R}$, there exists $\varepsilon > 0$, for all $N \in \mathbb{N}$, $n \geq N$ implies $|x_n - x| \geq \varepsilon$.

Proposition 3.12.

The sequence $x_n = (-1)^n$ is divergent.

Proof. Suppose $\{(-1)^n\}$ converges to a real number x. Then, for any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $n > N_0$ implies $|x_n - x| < \varepsilon$. Take $\varepsilon = 1$ and take $m_1 = N_0 + 1$, we have $|(-1)^{N_0+1} - x| < 1$. Take $\varepsilon = 1$ and take $m_2 = N_0 + 2$, we have $|(-1) \cdot (-1)^{N_0+1} - x| < 1$. Since either $(-1)^{N_0+1} = 1$ or $(-1)^{N_0+1} = -1$, we separate two cases to discuss. If $(-1)^{N_0+1} = 1$, then x satisfies both $|1 - x| < 1$ and $|-1-x| < 1$. There is no such x satisfying the two inequalities. If $(-1)^{N_0+1} = -1$, then x satisfies both $|-1-x| < 1$ and $|1 - x| < 1$. Again, there is no such x satisfying the two inequalities. Therefore, $\lim_{n\to\infty} (-1)^n$ diverges. $\square$

Theorem 3.13.

If a sequence $\{x_n\}$ is convergent, then it is bounded.

Proof. Suppose $\lim_{n\to\infty} x_n = x$. Then, there exists $N \in \mathbb{N}$ such that $n > N$ implies $|x_n - x| < 1$. Let $M = |x_1| + |x_2| + \cdots + |x_N| + |x| + 1$. Then, $x_i < M$ for all $x \in \{1, 2, \ldots, N\}$. For $n > N$, since $|x_n - x| < 1$, we have

$$\begin{aligned} x_n &< 1 + x \\ |x_n| &< |1 + x| \\ &\leq 1 + |x|. \end{aligned}$$

Hence, $x_n < M$ for all $n > N$. Therefore, $\{x_n\}$ is bounded. $\square$

Remark 3.14.

The inverse of Theorem 3.13 is false. A counter example is $x_n = (-1)^n$.

Definition 3.15 (Monotone)**.**

A sequence $\{x_n\}$ is monotone increasing if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence $\{x_n\}$ is monotone decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{N}$. If $\{x_n\}$ is either monotone increasing or monotone decreasing, we say $\{x_n\}$ is monotone or monotonic.

Theorem 3.16 (Monotone Convergence Theorem)**.**

Let $\{x_n\}$ be a monotone increasing sequence. Then, $\{x_n\}$ is convergent if and only if $\{x_n\}$ is bounded. Moreover, $\lim_{n\to\infty} x_n = \sup\{x_n \mid n \in \mathbb{N}\}$.

Proof.

($\Longrightarrow$) By Theorem 3.13, a convergent sequence is bounded.

($\Longleftarrow$) Suppose $\{x_n\}$ is monotone increasing and bounded. Then, $\sup\{x_n \mid n \in \mathbb{N}\}$ exists in $\mathbb{R}$ by the least upper bound property of $\mathbb{R}$. Let x denote $\sup\{x_n \mid n \in \mathbb{N}\}$. We want to show $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. Since x is the least upper bound of $\{x_n\}$, there exists $M \in \mathbb{N}$ such that $x - \varepsilon < x_M < x$. Then, for any $n > M$, we have $x - \varepsilon < x_M \leq x_n < x < x + \varepsilon$. Thus, $-\varepsilon < x_M - x < \varepsilon$, which is $|x_n - x| < \varepsilon$ for all $n > M$. Therefore, $\{x_n\}$ is convergent. $\square$

Theorem 3.17 (Monotone Convergence Theorem)**.**

Let $\{x_n\}$ be a monotone decreasing sequence. Then, $\{x_n\}$ is convergent if and only if $\{x_n\}$ is bounded. Moreover, $\lim_{n\to\infty} x_n = \inf\{x_n \mid n \in \mathbb{N}\}$.

Proof.

($\Longrightarrow$) By Theorem 3.13, a convergent sequence is bounded.

($\Longleftarrow$) Suppose $\{x_n\}$ is monotone decreasing and bounded. Then, $\inf\{x_n \mid n \in \mathbb{N}\}$ exists in $\mathbb{R}$ by the greatest lower bound property of $\mathbb{R}$. Let x denote $\inf\{x_n \mid n \in \mathbb{N}\}$. We want to show $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. Since x is the greatest lower bound of $\{x_n\}$, there exists $M \in \mathbb{N}$ such that $x < x_M < x + \varepsilon$. Then, for any $n > M$, we have $x - \varepsilon < x < x_n \leq x_M < x + \varepsilon$. Thus, $-\varepsilon < x_M - x < \varepsilon$, which is $|x_n - x| < \varepsilon$ for all $n > M$. Therefore, $\{x_n\}$ is convergent. $\square$

Proposition 3.18.

If $c \in (0,1)$, then $\lim_{n\to\infty} c^n = 0$. If $c > 1$, then c_n is unbounded.

Proof. Suppose that $c \in (0,1)$. Since $c^n > 0$ and c^n is monotone decreasing, c^n converges by the Monotone Convergence Theorem. We are now showing $\lim_{n\to\infty} c^n = 0$. Let $\varepsilon > 0$. Take $N = [\log_c \varepsilon] + 1$. Then, $n > N$ implies

$$\begin{aligned} |c^n - 0| &= c^n \\ &= c^N \cdot c^{n-N} \\ &\le c^{n-N} \cdot \varepsilon \\ &< \varepsilon. \end{aligned}$$

Therefore, if $c \in (0,1)$, $\lim_{n\to\infty} c^n = 0$. Now suppose $c > 1$. Let $M > 0$. Take $n = [\log_c M] + 1$. Then,

$$\begin{aligned} c^n &= c^{[\log_c M]+1} \\ &> M. \end{aligned}$$

Hence, c^n is divergent. Therefore, if $c > 1$, then c_n is unbounded. □

Definition 3.19 (Subsequence)**.**

Informally, a subsequence is a sequence with entries coming from another given sequence. In other words, let $\{x_n\}$ be a sequence and let $\{m_k\}$ be a strictly increasing sequence of natural numbers. Then, the sequence $\{x_{m_k}\}_{k=1}^{\infty}$ is called a <u>subsequence</u> of $\{x_n\}$.

Theorem 3.20.

If a sequence $\{x_n\}$ converges to x, then any subsequence of $\{x_n\}$ converges to x.

Proof. Suppose $\{x_n\}$ converges to x. Then, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $|x_n - x| < \varepsilon$. Let $\{x_{m_k}\}$ be a subsequence of $\{x_n\}$. If $k > N$, since $\{m_k\}$ is a sequence of strictly increasing natural numbers, then $m_k \ge k > N$. Hence, for any $\varepsilon > 0$, $k > N$ implies $|x_{m_k} - x| < \varepsilon$. Therefore, $\lim_{k\to\infty} x_{m_k} = x$. □

Theorem 3.21 (Squeeze Theorem)**.**

Let $\{a_n\}$, $\{b_n\}$, and $\{x_n\}$ be sequences such that $a_n \le x_n \le b_n$ for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} a_n = x$ and $\lim_{n\to\infty} b_n = x$, then $\lim_{n\to\infty} x_n = x$.

Proof. Suppose $\lim_{n\to\infty} a_n = x$ and $\lim_{n\to\infty} b_n = x$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = x$, there exists $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $|a_n - x| < \varepsilon$. Since $\lim_{n\to\infty} b_n = x$, there exists $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|b_n - x| < \varepsilon$. $|a_n - x| < \varepsilon$ implies $x - \varepsilon < a_n$. $|b_n - x| < \varepsilon$ implies $b_n < x + \varepsilon$. Take $N = \max\{N_1, N_2\}$, then $n > N$ implies $x - \varepsilon < a_n \le x_n \le b_n < x + \varepsilon$. Hence, $|x_n - x| < \varepsilon$. Therefore, $\lim_{n\to\infty} x_n = x$. $\square$

Proposition 3.22.

$\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} x_n - x = 0$.

Proof. This can be checked directly via the definition and thus omitted. $\square$

Proposition 3.23.

$\lim_{n\to\infty} x_n = 0$ if and only if $\lim_{n\to\infty} |x_n| = 0$.

Proof.

($\implies$) Suppose $\lim_{n\to\infty} x_n = 0$. Then, there exists $N \in \mathbb{N}$ such that $n > N$ implies $|x_n - 0| < \varepsilon$. Thus, $n > N$ implies $||x_n| - 0| < \varepsilon$. Hence, $\lim_{n\to\infty} |x_n| = 0$.

($\impliedby$) Suppose $\lim_{n\to\infty} |x_n| = 0$. Then, there exists $N \in \mathbb{N}$ such that $n > N$ implies $||x_n| - 0| < \varepsilon$. Thus, $n > N$ implies $|x_n| - 0 < \varepsilon$. Hence, $\lim_{n\to\infty} x_n = 0$. $\square$

Example 3.24.

Let $x_n = \dfrac{n^2}{n^2 + n + 1}$ for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} x_n = 1$.

Proof. Since $0 \le \left|\dfrac{n^2}{n^2+n+1} - 1\right| \le \dfrac{1}{n}$, by the Squeeze Theorem, $\lim_{n\to\infty} \left|\dfrac{n^2}{n^2+n+1} - 1\right| = 0$. By Proposition 3.23, $\lim_{n\to\infty} \dfrac{n^2}{n^2+n+1} - 1 = 0$. By Proposition 3.22, $\lim_{n\to\infty} \dfrac{n^2}{n^2+n+1} = 1$. $\blacksquare$

Theorem 3.25.

Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences. If $x_n \le y_n$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} x_n \le \lim\limits_{n\to\infty} y_n$.

Proof. Let $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$. For the sake of contradiction, suppose that $y < x$. Then, there exists $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $|x_n - x| < \dfrac{x-y}{2}$, and there exists $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|y_n - y| < \dfrac{x-y}{2}$. Take $N = N_1 + N_2$, then

$$\begin{aligned} &\begin{cases} y_N & < \dfrac{x-y}{2} + y \\ x - \dfrac{x-y}{2} & < x_N \end{cases} \\ \iff &\begin{cases} y_N & < \dfrac{x+y}{2} \\ \dfrac{x+y}{2} & < x_N \end{cases} \\ \implies & y_N < x_N, \end{aligned}$$

which contradicts $x_n \le y_n$. Therefore, $\lim\limits_{n\to\infty} x_n \le \lim\limits_{n\to\infty} y_n$. □

Theorem 3.26.

Let $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$. Then,

1. $\lim\limits_{n\to\infty} x_n + y_n = x + y$;
2. $\lim\limits_{n\to\infty} c \cdot x_n = c \cdot x$ for all $c \in \mathbb{R}$;
3. $\lim\limits_{n\to\infty} x_n \cdot y_n = x \cdot y$;
4. if $y_n \neq 0$ for all $n \in \mathbb{N}$ and $y \neq 0$, then $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = \dfrac{x}{y}$.

Proof.

1. Suppose $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$. Let $\varepsilon > 0$. Then, there exist N_1 such that $n > N_1$ implies $|x_n - x| < \dfrac{\varepsilon}{2}$, and there exist N_2 such that $n > N_2$ implies $|y_n - y| < \dfrac{\varepsilon}{2}$. Take $N = \max\{N_1, N_2\}$. Then, $n > N$ implies

$$\begin{aligned} |(x_n + y_n) - (x + y)| &= |x_n + y_n - x - y| \\ &\le |x_n - x| + |y_n - y| \\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} x_n + y_n = x + y$.

2. Suppose $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. If $c = 0$, then the sequence converges to 0 by Proposition 3.8. If not, since $\lim_{n\to\infty} x_n = x$, we know that there exists $N \in \mathbb{N}$ such that $n > N$ implies $|x_n - x| < \dfrac{\varepsilon}{|c|}$. Then, $n > N$ implies $|cx_n - cx| < \varepsilon$. Therefore, $\lim_{n\to\infty} c \cdot x_n = c \cdot x$ for all $c \in \mathbb{R}$.

3. Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Thus, $\{y_n\}$ is bounded, i.e., there exists $M \geq 0$ such that $|y_n| \leq M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then, there exist N_1 such that $n > N_1$ implies $|x_n - x| < \dfrac{\varepsilon}{2(M+1)}$, and there exist N_2 such that $n > N_2$ implies $|y_n - y| < \dfrac{\varepsilon}{2(|x|+1)}$. Since $n > N$ implies
$$\begin{aligned}
|x_n y_n - xy| &= |(x_n - x)\cdot y_n + (y_n - y)\cdot x| \\
&\leq |(x_n - x)\cdot y_n| + |(y_n - y)\cdot x| \\
&= |y_n||x_n - x| + |x||y_n - y| \\
&\leq M|x_n - x| + |x||y_n - y| \\
&< \frac{M}{M+1}\cdot\frac{\varepsilon}{2} + \frac{|x|}{|x|+1}\cdot\frac{\varepsilon}{2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}$$
we obtain that $|x_n y_n - xy| < \varepsilon$. Therefore, $\lim_{n\to\infty} x_n \cdot y_n = x \cdot y$.

4. Suppose $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y \neq 0$. Then, there exist $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $|y_n - y| < \dfrac{|y|}{2}$. We show $\lim_{n\to\infty} \dfrac{1}{y_n} = \dfrac{1}{y}$, then we apply Theorem 3.26.3. By the Triangle Inequality, $|y| \leq |y_n - y| + |y_n|$. Thus, $n > N_1$ implies
$$\begin{aligned}
& |y_n - y| < \frac{|y|}{2} \\
\iff\ & |y_n - y| + |y_n| < \frac{|y|}{2} + |y_n| \\
\implies\ & |y| < \frac{|y|}{2} + |y_n| \\
\iff\ & \frac{|y|}{2} \leq |y_n|.
\end{aligned}$$
Let $\varepsilon > 0$. By assumption, we know that there exists $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|y_n - y| < \dfrac{y^2\varepsilon}{2}$. Take $N = \max\{N_1, N_2\}$, then $n > N$ implies

$$
\begin{aligned}
\left|\frac{1}{y_n}-\frac{1}{y}\right| &= \frac{|y-y_n|}{|y||y_n|} \\
&= \frac{|y_n-y|}{|y||y_n|} \\
&\leq \frac{|y_n-y|}{|y|\cdot\dfrac{|y|}{2}} \\
&= |y_n-y|\frac{2}{y^2} \\
&< \varepsilon.
\end{aligned}
$$

Therefore, $\lim_{n\to\infty}\frac{1}{y_n}=\frac{1}{y}$. Applying Theorem 3.26.3 completes the proof. □

Corollary 3.27.

Let $\lim_{n\to\infty} x_n = x$. Then, $\lim_{n\to\infty}(x_n)^k = x^k$ for all $k\in\mathbb{N}$.

Proof. This follows immediately from Theorem 3.26.3 by induction. □

Theorem 3.28.

Let x_n be a sequence of positive real numbers such that $\lim_{n\to\infty} x_n = x$. Then, $\lim_{n\to\infty}\sqrt{x_n}=\sqrt{x}$.

Proof. Suppose $x_n > 0$ for all $n\in\mathbb{N}$ and $\lim_{n\to\infty} x_n = x$. Since $\lim_{n\to\infty} x_n = x$, we know that there exists N_1 such that $n > N_1$ implies $|x_n - x| < \varepsilon^2$. If $x=0$, then $|\sqrt{x_n}-\sqrt{x}| = \sqrt{|x_n|} < \varepsilon$. If $x>0$, then by Theorem 3.13, $\{x_n\}$ is bounded, i.e., there exists $M\geq 0$ such that $|x_n|\leq M^2$ for all $n\in\mathbb{N}$. Let $\varepsilon>0$. Then, there exist N_2 such that $n>N_2$ implies $|x_n-x|<\frac{\varepsilon\sqrt{x}}{2}$. Take $N=\max\{N_1,N_2\}$, then $n>N$ implies

$$
\begin{aligned}
|\sqrt{x_n}-\sqrt{x}| &= \frac{|x_n-x|}{\sqrt{x_n}+\sqrt{x}} \\
&\leq \frac{|x_n-x|}{\sqrt{x}} \\
&< \frac{\varepsilon\sqrt{x}}{2\sqrt{x}} \\
&= \frac{\varepsilon}{2} \\
&< \varepsilon.
\end{aligned}
$$

Therefore, $\lim_{n\to\infty}\sqrt{x_n}=\sqrt{x}$. □

Proposition 3.29.

If $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} |x_n| = |x|$.

Proof. Suppose $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. Then, there exist N such that $n > N$ implies $|x_n - x| < \varepsilon$. Thus, by the Reverse Triangle Inequality,

$$\begin{aligned} ||x_n| - |x|| &\leq x_n - x \\ &\leq |x_n - x| \\ &< \varepsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} |x_n| = |x|$. □

Remark 3.30.

The inverse of Proposition 3.29 is false. A counter example is $x_n = (-1)^n$.

Proposition 3.31.

If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

Proof.Let $p > 0$ and let $\varepsilon > 0$. Take $N = \left[\frac{1}{\sqrt[p]{\varepsilon}}\right] + 1$. Then, $n > N$ implies

$$\begin{aligned} \left|\frac{1}{n^p} - 0\right| &< \frac{1}{\left(\left[\frac{1}{\sqrt[p]{\varepsilon}}\right] + 1\right)^p} \\ &< \frac{1}{\left(\frac{1}{\sqrt[p]{\varepsilon}}\right)^p} \\ &< \varepsilon. \end{aligned}$$

Therefore, if $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$. □

Proposition 3.32.

If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

Proof.Let $\varepsilon > 0$.

1. Suppose $p = 1$. Then, $\lim_{n\to\infty} \sqrt[n]{p} = 1$.

2. Suppose $p > 1$. Take $N = [\log_{\varepsilon+1} p] + 1$. Then, $n > N$ implies

$$\begin{aligned} |\sqrt[n]{p} - 1| &< \sqrt[N]{p} - 1 \\ &= p^{\frac{1}{[\log_{\varepsilon+1} p]+1}} - 1 \\ &< p^{\frac{1}{\log_{\varepsilon+1} p}} - 1 \\ &= p^{\log_p(\varepsilon+1)} - 1 \\ &= \varepsilon. \end{aligned}$$

Thus, $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$.

3. Suppose $0 < p < 1$. Since $\sqrt[n]{p} > 0$, the inequality $1 - \sqrt[n]{p} < 1$ holds for all $n \in \mathbb{N}$. Thus, we just need to show $1 - \sqrt[n]{p} < \varepsilon$ where $\varepsilon \in (0, 1)$. Let $\varepsilon \in (0, 1)$. Take $N = [\log_{1-\varepsilon} p] + 1$. Then, $n > N$ implies

$$\begin{aligned} |\sqrt[n]{p} - 1| &= 1 - \sqrt[n]{p} \\ &< 1 - \sqrt[N]{p} \\ &= 1 - p^{\frac{1}{[\log_{1-\varepsilon} p]+1}} \\ &< 1 - p^{\frac{1}{\log_{1-\varepsilon} p}} \\ &= 1 - p^{\log_p 1-\varepsilon} \\ &= \varepsilon. \end{aligned}$$

Thus, $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$.

Therefore, $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$ for all $p > 0$. □

Proposition 3.33.

If $p > 0$, then $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

Proof. We have the identity $n = (1 + (\sqrt[n]{n} - 1))^n$. Hence,

$$\begin{aligned} n &= \left(1 + \left(\sqrt[n]{n} - 1\right)\right)^n \\ &= \sum_{k=1}^{n} \binom{n}{k} \left(\sqrt[n]{n} - 1\right)^k \\ &\geq \binom{n}{2} \left(\sqrt[n]{n} - 1\right)^2 \\ &= \frac{n(n-1)}{2} \left(\sqrt[n]{n} - 1\right)^2. \end{aligned}$$

Thus, for all $n \in \mathbb{N}$, $0 \leq \sqrt[n]{n} - 1 \leq \sqrt{\dfrac{2}{n-1}}$. By the Squeeze Theorem, $\lim\limits_{n\to\infty} \sqrt[n]{n} - 1 = 0$. By Proposition 3.22, $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$. $\square$

4 Limit Superior, Limit Inferior, and Completeness of the Reals

Definition 4.1 (Limit Superior)**.**

Let $\{x_n\}$ be a bounded sequence. The limit superior of $\{x_n\}$ is defined, if the limit exists, by

$$\limsup_{n\to\infty} x_n \coloneqq \lim_{n\to\infty} \left(\sup\{x_k \mid k \geq n\}\right).$$

Definition 4.2 (Limit Inferior)**.**

Let $\{x_n\}$ be a bounded sequence. The limit inferior of $\{x_n\}$ is defined, if the limit exists, by

$$\liminf_{n\to\infty} x_n \coloneqq \lim_{n\to\infty} \left(\inf\{x_k \mid k \geq n\}\right).$$

Theorem 4.3.

Let $\{x_n\}$ be a bounded sequence. Let $a_n = \sup\{x_k \mid k \geq n\}$ and $b_n = \inf\{x_k \mid k \geq n\}$. Then, the following three hold:

1. $\{a_n\}$ is monotone decreasing and $\{b_n\}$ is monotone increasing;
2. $\{a_n\}$ and $\{b_n\}$ are bounded;
3. $\liminf\limits_{n\to\infty} x_n \leq \limsup\limits_{n\to\infty} x_n$.

Proof.

1. Since $\{x_k \mid k \geq n+1\} \subseteq \{x_k \mid k \geq n\}$, we have $\sup\{x_k \mid k \geq n+1\} \leq \sup\{x_k \mid k \geq n\}$. Hence, $a_{n+1} \leq a_n$. Thus, $\{a_n\}$ is monotone decreasing. Similarly, since $\{x_k \mid k \geq n+1\} \subseteq \{x_k \mid k \geq n\}$, we have $\inf\{x_k \mid k \geq n+1\} \geq \inf\{x_k \mid k \geq n\}$. Hence, $b_{n+1} \geq b_n$. Thus, $\{b_n\}$ is monotone increasing.
2. Since $\{x_n\}$ is bounded, there exists $r > 0$ such that $|x_n| \leq r$ for all $n \in \mathbb{N}$. Hence, $-r \leq b_n \leq a_n \leq r$ for all $n \in \mathbb{N}$. Thus, $\{a_n\}$ and $\{b_n\}$ are bounded.
3. By Theorem 3.25, $\liminf\limits_{n\to\infty} x_n \leq \limsup\limits_{n\to\infty} x_n$ as $\inf x_n \leq \sup x_n$. $\square$

Remark 4.4.

Theorem 4.3 says that limit superior and limit inferior of any sequence of real numbers always exist.

Example 4.5.

Let $x_n = (-1)^n$. Then, $\limsup_{n\to\infty} x_n = 1$ and $\liminf_{n\to\infty} x_n = -1$.

Proof. Since $\{x_n\} = \{1, -1\}$, $\sup\{x_n\} = 1$ and $\inf\{x_n\} = -1$ for all $n \in \mathbb{N}$. Therefore, $\limsup_{n\to\infty} x_n = 1$ and $\liminf_{n\to\infty} x_n = -1$. ■

Example 4.6.

Let $x_n = \dfrac{1}{n}$. Then, $\limsup_{n\to\infty} x_n = 0$ and $\liminf_{n\to\infty} x_n = 0$.

Proof. Compute by definition,

$$\begin{aligned}
\limsup_{n\to\infty} x_n &= \lim_{n\to\infty} \left(\sup \left\{ \frac{1}{k} \mid k \geq n \right\} \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \\
&= 0; \\
\liminf_{n\to\infty} x_n &= \lim_{n\to\infty} \left(\inf \left\{ \frac{1}{k} \mid k \geq n \right\} \right) \\
&= \lim_{n\to\infty} 0 \\
&= 0.
\end{aligned}$$

Therefore, $\limsup_{n\to\infty} x_n = 0$ and $\liminf_{n\to\infty} x_n = 0$. ■

Theorem 4.7.

If $\{x_n\}$ is a bounded sequence, then there exist subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$ and $\lim_{k\to\infty} x_{m_k} = \liminf_{n\to\infty} x_n$.

Proof. We first show that there exists a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$. Let $a_n = \sup\{x_k \mid k \geq n\}$. Then, there exists $n_1 \in \mathbb{N}$ such that $a_1 - 1 < x_{n_1} \leq a_1$. Since $a_{n_1+1} = \sup\{x_k \mid k \geq n_1 + 1\}$, there exists an natural number $n_2 > n_1$ such that $a_{n_1+1} - \dfrac{1}{2} < x_{n_2} \leq a_{n_1+1}$. Similarly, since $a_{n_2+1} = \sup\{x_k \mid k \geq n_2 + 1\}$, there exists an natural number $n_3 > n_2$ such that $a_{n_2+1} - \dfrac{1}{3} < x_{n_3} \leq a_{n_2+1}$. Continuing in this fashion, we obtain a strictly increasing sequence of positive integers $n_1, n_2, n_3, \ldots$ such that $a_{n_k+1} - \dfrac{1}{k+1} < x_{n_k} \leq a_{n_k+1}$. Since $\lim_{k\to\infty} a_{n_k+1} = \limsup_{n\to\infty} x_n$ and $\lim_{k\to\infty} \dfrac{1}{k+1}$, by the Squeeze Theorem, $\lim_{k\to\infty} x_{n_k} = \limsup_{n\to\infty} x_n$. We now show that there exists a subsequence $\{x_{m_k}\}$ such that $\lim_{k\to\infty} x_{m_k} = \liminf_{n\to\infty} x_n$. Let $b_n = \inf\{x_k \mid k \geq n\}$. Then, there exists $m_1 \in \mathbb{N}$ such that $a_1 \leq x_{m_1} < a_1 + 1$. Since $a_{m_1+1} = \sup\{x_k \mid k \geq m_1 + 1\}$, there exists an natural number $m_2 > m_1$ such that $a_{m_1+1} \leq x_{m_2} < a_{m_1+1} + \frac{1}{2}$. Similarly, since $a_{m_2+1} = \sup\{x_k \mid k \geq m_2 + 1\}$, there exists an natural number $m_3 > m_2$ such that $a_{m_2+1} \leq x_{m_3} < a_{m_2+1} + \frac{1}{3}$. Continuing in this fashion, we obtain a strictly increasing sequence of positive integers $m_1, m_2, m_3, \ldots$ such that $a_{m_k+1} \leq x_{m_k} < a_{m_k+1} + \frac{1}{k+1}$. Since $\lim_{k\to\infty} a_{m_k+1} = \liminf_{n\to\infty} x_n$ and $\lim_{k\to\infty} \frac{1}{k+1}$, by the Squeeze Theorem, $\lim_{k\to\infty} x_{m_k} = \liminf_{n\to\infty} x_n$. □

Theorem 4.8 (Bolzano-Weierstrass Theorem)**.**

Every bounded sequence has a convergent subsequence.

Proof. This follows immediately from Theorem 4.7. □

Theorem 4.9.

Let $\{x_n\}$ be a bounded sequence. Then, $\{x_n\}$ converges if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$.

Proof.

($\Longrightarrow$) Suppose $\lim_{n\to\infty} x_n = x$. Then, by Theorem 3.20 and Theorem 4.7, $\limsup_{n\to\infty} x_n = x$ and $\liminf_{n\to\infty} x_n = x$.

($\Longleftarrow$) Suppose $\limsup_{n\to\infty} x_n = x$ and $\liminf_{n\to\infty} x_n = x$. Since $\inf\{x_k \mid k \geq n\} \leq x_n \leq \sup\{x_k \mid k \geq n\}$ for all $n \in \mathbb{N}$, by the Squeeze Theorem, $\lim_{n\to\infty} x_n = x$. □

Definition 4.10 (Cauchy)**.**

A sequence $\{x_n\}$ is Cauchy if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ and $k > N$ imply $|x_n - x_k| < \varepsilon$.

Example 4.11.

Let $x_n = \frac{1}{n}$. Then, $\{x_n\}$ is Cauchy.

Proof. Let $\varepsilon > 0$. Take $N = \left[\frac{2}{\varepsilon}\right] + 1$, then $n > N$ and $k > N$ imply

$$\begin{aligned}\left|\frac{1}{n}-\frac{1}{k}\right| &\leq \left|\frac{1}{n}\right|+\left|\frac{1}{k}\right| \\ &\leq \frac{2}{N} \\ &< \frac{2}{\frac{2}{\varepsilon}} \\ &= \varepsilon.\end{aligned}$$

Therefore, $\left\{\frac{1}{n}\right\}$ is Cauchy. ■

Negation 4.12 (Not Cauchy)**.**

A sequence $\{x_n\}$ is not Cauchy if there exists $\varepsilon > 0$, for all $N \in \mathbb{N}$, there exists $n > N$ and $k > N$ such that $|x_n - x_k| \geq \varepsilon$.

Example 4.13.

Let $x_n = (-1)^n$. Then, $\{x_n\}$ is not Cauchy.

Proof. Take $\varepsilon = 1$. Let $N \in \mathbb{N}$. Then,

$$\begin{aligned}\left|(-1)^N - (-1)^{N+1}\right| &= 2 \\ &> 1.\end{aligned}$$

Therefore, $\{(-1)^n\}$ is not Cauchy. ■

Theorem 4.14.

If a sequence $\{x_n\}$ is Cauchy, then it is bounded.

Proof. Suppose $\{x_n\}$ is Cauchy, i.e., there exists $N \in \mathbb{N}$ such that $n > N$ and $k > N$ imply $|x_n - x_k| < 1$. Therefore, for all $n \geq N$, $|x_n - x_N| < 1$. Thus, by the Reverse Triangle Inequality, $n \geq N$ implies

$$\begin{aligned} 1 &> |x_n - x_N| \\ &> ||x_n| - |x_N|| \\ &\geq |x_n| - |x_N| \\ \implies 1 + |x_N| &> |x_n|.\end{aligned}$$

Hence, take $M = |x_1| + |x_2| + \cdots + |x_N| + 1$, then $|x_n| \leq M$ for all $n \in \mathbb{N}$. □

Theorem 4.15.

If $\{x_n\}$ is Cauchy and a subsequence $\{x_{n_k}\}$ converges, then $\{x_n\}$ converges.

Proof. Suppose that $\{x_n\}$ converges and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x$. Let $\varepsilon > 0$. Since $\lim_{k\to\infty} x_{n_k} = x$, there exists $M_1 \in \mathbb{N}$ such that $k > M_1$ implies $|x_{n_k} - x| < \frac{\varepsilon}{2}$. Since $\{x_n\}$ is Cauchy, there exists $M_2 \in \mathbb{N}$ such that $n > M_2$ and $m > M_2$ imply $|x_n - x_m| < \frac{\varepsilon}{2}$. Take $M = M_1 + M_2$. If $n \geq M$, then $n_M \geq M \geq M_1$ and $n \geq M_2$. Thus,

$$\begin{aligned} |x_n - x| &\leq |x_n - x_{n_M}| + |x_{n_M} - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore, $\{x_n\}$ converges. □

Theorem 4.16.

A sequence of real numbers $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ is convergent.

Proof.

($\implies$) Suppose $\{x_n\}$ is Cauchy. Then, $\{x_n\}$ is bounded by Theorem 4.14. Thus, $\{x_n\}$ has a convergent subsequence by the Bolzano-Weierstrass Theorem. By Theorem 4.15, $\{x_n\}$ is convergent.

($\impliedby$) Suppose that $\{x_n\}$ is convergent and $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, there exists $M \in \mathbb{N}$ such that $n \geq M$ implies $|x_n - x| < \frac{\varepsilon}{2}$. Then, $n > M$ and $k > M$ imply

$$\begin{aligned} |x_n - x_k| &\leq |x_n - x| + |x_k - x| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore, $\{x_n\}$ is Cauchy. □

Remark 4.17.

The reason that Cauchy is equivalent to convergent is the least upper bound property of $\mathbb{R}$.

Remark 4.18.

A sequence of rational numbers $\{x_n\}$ is Cauchy if $\{x_n\}$ is convergent. The inverse is not true. A counter-example is $\lim_{n\to\infty} x_n = \sqrt{2}$.

5 Infinite Series, Convergence, and Absolute Convergence

Definition 5.1 (Series)**.**

Given a sequence $\{x_n\}$, the series, denoted $\sum_{n=1}^{\infty} x_n$ or $\sum x_n$, is summation of $\{x_n\}$.

Definition 5.2 (Converge)**.**

We say $\sum x_n$ converges if the sequence $\left\{ s_m = \sum_{n=1}^{m} x_n \right\}_{m=1}^{\infty}$ converges. We call the terms of $\{s_m\}$ the partial sums. If $\lim_{m\to\infty} s_m = s$, we write $\sum x_n = s$ and treat $\sum x_n$ as a number.

Example 5.3.

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

Proof. We check directly. Since

$$\begin{aligned} s_m &= \sum_{n=1}^{m} \frac{1}{n(n+1)} \\ &= 1 - \frac{1}{m+1} \end{aligned}$$

and $\lim_{m\to\infty} 1 - \frac{1}{m+1} = 1$, the partial sum of $\left\{ \frac{1}{n(n+1)} \right\}$ converges to 1. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges. ■

Proposition 5.4 (Geometric Series)**.**

If $|r| < 1$, then $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$.

Proof. Since for any $m \in \mathbb{N}$, $s_m = \sum_{n=0}^{m} r^n = \frac{1 - r^{m+1}}{1-r}$ by Proposition 1.9. Since $|r| < 1$, $\lim_{m\to\infty} |r|^{m+1} = 0$. Therefore, $\lim_{m\to\infty} s_m = \frac{1}{1-r}$. □

Proposition 5.5.

Let $\{x_n\}$ be a sequence and let $M \in \mathbb{N}$. Then, $\sum_{n=1}^{\infty} x_n$ converges if and only if $\sum_{n=M}^{\infty} x_n$ converges.

Proof. Since $\sum_{n=1}^{M-1} x_n$ is a finite number, $\sum_{n=1}^{\infty} x_n$ converges if and only if $\sum_{n=M}^{\infty} x_n$ converges. □

Definition 5.6 (Cauchy)**.**

$\sum x_n$ is Cauchy if the sequence of partial sums of $\{x_n\}$ is Cauchy.

Theorem 5.7.

$\sum x_n$ is Cauchy if and only if $\sum x_n$ is convergent.

Proof.

($\implies$) Suppose $\sum x_n$ is Cauchy. Then, by Definition 5.6, the sequence of partial sums of $\{x_n\}$ is Cauchy. By Theorem 4.16, the sequence of partial sums of $\{x_n\}$ is convergent. By Definition 5.2, $\sum x_n$ is convergent.

($\impliedby$) Suppose $\sum x_n$ is convergent. Then, by Definition 5.2, the sequence of partial sums of $\{x_n\}$ is convergent. By Theorem 4.16, the sequence of partial sums of $\{x_n\}$ is Cauchy. By Definition 5.6, $\sum x_n$ is Cauchy. $\square$

Theorem 5.8.

$\sum x_n$ is Cauchy if and only if for any $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that $m > M$ and $\ell > m$ imply $\left|\sum_{i=m+1}^{\ell} x_i\right| < \varepsilon$.

Proof.

($\implies$) Suppose $\sum x_n$ is Cauchy. Then, there exists $N \in \mathbb{N}$ such that $n > N$ and $k > N$ imply $|s_n - s_k| < \varepsilon$. Let $\ell > m > N$. Then, $|s_m - s_\ell| < \varepsilon$.

($\impliedby$) Suppose there exists $M \in \mathbb{N}$ such that $m > M$ and $\ell > m$ imply $\left|\sum_{i=m+1}^{\ell} x_i\right| < \varepsilon$ for any $\varepsilon > 0$. Then, $\left|\sum_{i=1}^{\ell} x_i - \sum_{i=1}^{m} x_i\right| < \varepsilon$. Thus, $\left\{\sum_{i=1}^{n} x_i\right\}_n$ is Cauchy, which implies $\sum x_n$ is Cauchy. $\square$

Corollary 5.9.

If $\sum x_n$ is convergent, then $\lim_{n\to\infty} x_n = 0$.

Proof. Suppose $\sum x_n$ is convergent. Then, $\sum x_n$ is Cauchy by Theorem 5.7. Let $\varepsilon > 0$. By Theorem 5.8, take $\ell = m + 1$, then we know that there exists $M \in \mathbb{N}$ such that $m > M$ implies $|x_{m+1}| < \varepsilon$. Therefore, $\lim_{n\to\infty} x_n = 0$. $\square$

Remark 5.10.

The inverse of Corollary 5.9 is not true. See Proposition 5.14.

Remark 5.11.

If $\lim_{n\to\infty} x_n \neq 0$, then $\sum x_n$ is divergent.

Corollary 5.12.

If $|r| \geq 1$, then $\sum_{n=0}^{\infty} r^n$ diverges.

Proof. Suppose $|r| \geq 1$. Then, $\lim_{n\to\infty} r^n \neq 0$. Thus, by Corollary 5.9, $\sum_{n=0}^{\infty} r^n$ is not convergent. □

Corollary 5.13 (Geometric Series)**.**

$\sum_{n=0}^{\infty} r^n$ converges if and only if $|r| < 1$.

Proof. Corollary 5.10 and Proposition 5.4 complete the proof. □

Proposition 5.14 (Harmonic Series)**.**

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Proof. We are showing this by showing a subsequence of $\left\{\sum_{i=1}^{m} \frac{1}{i}\right\}_m$ is unbounded, and Theorem 3.20 completes the proof. Consider the subsequence of partial sums $\{s_{2^n}\}_n$, since

$$\begin{aligned}
\sum_{i=1}^{2^n} \frac{1}{i} &= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right) \\
&\geq 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right) \\
&= 1 + \sum_{j=1}^{n} \left(2^{j-1} \cdot \frac{1}{2^j}\right) \\
&= 1 + \sum_{j=1}^{n} \frac{1}{2} \\
&= 1 + \frac{n}{2},
\end{aligned}$$

we obtain that $\{s_{2^n}\}_n$ is divergent. By Theorem 3.20, the sequence of partial sums is divergent. Therefore, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. $\square$

Theorem 5.15.

Let $\alpha \in \mathbb{R}$ and let $\sum x_n$ and $\sum y_n$ be convergent series. Then, $\sum(\alpha x_n + y_n) = \alpha \sum x_n + \sum y_n$.

Proof. The partial sums satisfy $\displaystyle\sum_{n=1}^{m}(\alpha x_n + y_n) = \alpha \sum_{n=1}^{m} x_n + \sum_{n=1}^{m} y_n$. By linear properties of limits, it follows that $\displaystyle\lim_{m\to\infty} \sum_{n=1}^{m}(\alpha x_n + y_n) = \alpha \sum x_n + \sum y_n$. $\square$

Proposition 5.16.

If $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\sum x_n$ converges if and only if $\{s_m\}$ is bounded.

Proof. Since x_n are all non-negative, the sequence of partial sums is monotone increasing. By the Monotone Convergence Theorem, $\{s_m\}$ converges if and only if $\{s_m\}$ is bounded. Therefore, $\sum x_n$ converges if and only if $\{s_m\}$ is bounded. $\square$

Definition 5.17 (Absolute Convergence)**.**

$\sum x_n$ <u>converges absolutely</u> if $\sum |x_n|$ converges.

Lemma 5.18.

Let $\{x_n\}$ be a sequence. For any integer $m \geq 2$, $\displaystyle\left|\sum_{i=1}^{m} x_i\right| \leq \sum_{i=1}^{m} |x_i|$.

Proof. We will prove this by induction on m.

(Base case) $|x_1 + x_2| \leq |x_1| + |x_2|$ holds by Triangle Inequality.

(Inductive step) We have $\displaystyle\left|\sum_{i=1}^{k} x_i\right| \leq \sum_{i=1}^{k} |x_i|$ by assumption.

$$
\begin{aligned}
\left|\sum_{i=1}^{k+1} x_i\right| &= \left|\left(\sum_{i=1}^{k} x_i\right) + x_{k+1}\right| \\
&\leq \left|\left(\sum_{i=1}^{k} x_i\right)\right| + |x_{k+1}| \\
&\leq \sum_{i=1}^{k} |x_i| + |x_{k+1}| \\
&= \sum_{i=1}^{k+1} |x_i| .
\end{aligned}
$$

Therefore, $\left|\sum_{i=1}^{m} x_i\right| \leq \sum_{i=1}^{m} |x_i|$ for all integer $m \geq 2$. □

Theorem 5.19.

If $\sum |x_n|$ converges, then $\sum x_n$ converges.

Proof. Suppose $\sum |x_n|$ converges. Then, $\sum |x_n|$ is Cauchy by Theorem 5.7. Thus, there exists $N \in \mathbb{N}$ such that $n > N$ and $k > N$ imply $\left|\sum_{i=1}^{n} |x_i| - \sum_{i=1}^{k} |x_i|\right| < \varepsilon$. Without loss of generality, assume that $k > n$. Then, $\sum_{i=n+1}^{k} |x_i| < \varepsilon$. By Lemma 5.18,

$$
\begin{aligned}
\varepsilon &> \sum_{i=n+1}^{k} |x_i| \\
&\geq \left|\sum_{i=n+1}^{k} x_i\right| .
\end{aligned}
$$

Hence, $\sum x_n$ is Cauchy. Therefore, $\sum x_n$ converges. □

Theorem 5.20 (Comparison Test)**.**

If $0 \leq x_n \leq y_n$ for all $n \in \mathbb{N}$, then

1. if $\sum y_n$ converges, then $\sum x_n$ converges;
2. if $\sum x_n$ diverges, then $\sum y_n$ diverges.

Proof.

1. Suppose $0 \leq x_n \leq y_n$ and $\sum y_n$ converges. By the Monotone Convergence Theorem, $\sum_{i=1}^{n} y_i$ is bounded. Assume $M > \sum_{i=1}^{n} y_i$ for all $n \in \mathbb{N}$, then $M > \sum_{i=1}^{n} x_i$ for all $n \in \mathbb{N}$. Thus, $\sum_{i=1}^{n} x_i$ is bounded. Moreover, $\sum_{i=1}^{n} x_i$ is monotone increasing. Therefore, by the Monotone Convergence Theorem, it is convergent.

2. Suppose $0 \leq x_n \leq y_n$ and $\sum x_n$ diverges. Then, $\sum_{i=1}^{n} x_i > M$ for all $M \in \mathbb{R}$. Thus, $\sum_{i=1}^{n} y_i > M$ for all $M \in \mathbb{R}$, which implies $\sum_{i=1}^{n} y_i$ is unbounded. Therefore, $\sum y_n$ diverges. $\square$

Proposition 5.21 (p-Test)**.**
$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p < 1$.

Proof.

($\Longrightarrow$) We will prove by contradiction. Suppose $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. Suppose $p \geq 1$. Then, since $\frac{1}{n^p} > \frac{1}{n}$, by the Comparison Test, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, which yields a contradiction. Hence, $p < 1$.

($\Longleftarrow$) Assume $p < 1$. We compute directly to check whether the sequence of partial sums of $\left\{\frac{1}{n^p}\right\}$ is bounded or not. If it is bounded, by the Monotone Convergence Theorem, $\left\{\sum_{m=1}^{n} \frac{1}{m^p}\right\}_n$ is convergent, which means $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. Since $\frac{1}{m} > 0$ for all $m \in \mathbb{N}$,

$$
\begin{aligned}
\sum_{m=1}^{n} \frac{1}{m^p} < \sum_{m=1}^{2^n-1} \frac{1}{m^p}
&= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \cdots + \frac{1}{7^p}\right) + \cdots + \left(\frac{1}{(2^{n-1})^p} + \cdots + \frac{1}{(2^n - 1)^p}\right) \\
&< 1 + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \cdots + \frac{1}{4^p}\right) + \cdots + \left(\frac{1}{(2^{n-1})^p} + \cdots + \frac{1}{(2^{n-1})^p}\right) \\
&= 1 + \left(\frac{2}{2^p}\right) + \left(\frac{4}{4^p}\right) + \cdots + \left(\frac{2^{n-1}}{(2^{n-1})^p}\right) \\
&= 1 + (2^{1-p})^1 + (2^{1-p})^2 + \cdots + (2^{1-p})^{n-1} \\
&< 1 + (2^{1-p})^1 + (2^{1-p})^2 + \cdots + (2^{1-p})^{n-1} + \cdots \\
&= \frac{1}{1 - (2^{1-p})}.
\end{aligned}
$$

Hence, the partial sums of $\left\{\frac{1}{n^p}\right\}$ is bounded. Therefore, by the Monotone Convergence Theorem, $\left\{\sum_{m=1}^{n} \frac{1}{m^p}\right\}_n$ is convergent, which means $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. $\square$

Remark 5.22.

The proof of the p-Test may be proven by the Integral Test, which is not in this book.

Theorem 5.23 (d'Alembert's Ratio Test)**.**

Let $\{x_n\}$ be a sequence. Suppose $x_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|} = L$. Then,

1. if $L < 1$, then $\sum x_n$ converges absolutely;
2. if $L > 1$, then $\sum x_n$ diverges.

If the limit is 1, then this theorem doesn't apply.

Proof.

1. Suppose $0 \leq L < 1$. Let $r \in (L, 1)$. By assumption, there exists $N \in \mathbb{N}$ such that $n > N$ implies $\left|\frac{|x_{n+1}|}{|x_n|} - L\right| < r - L$. Thus,
$$\begin{aligned} 2L - r < \frac{|x_{n+1}|}{|x_n|} < r \\ \implies \qquad |x_{n+1}| < r|x_n|, \end{aligned}$$
which implies $|x_{k+N}| < r^k|x_N|$ for all $k \in \mathbb{N}$. Let $m > N$ and $m \in \mathbb{N}$. Then,
$$\begin{aligned} \sum_{k=1}^{m} |x_k| &= \sum_{k=1}^{N} |x_k| + \sum_{k=N+1}^{m} |x_k| \\ &< \sum_{k=1}^{N} |x_k| + \sum_{k=N+1}^{m} r^{k-N}|x_N| \\ &< \sum_{k=1}^{N} |x_k| + \sum_{k=0}^{\infty} r^k|x_N| \\ &= \sum_{k=1}^{N} |x_k| + \frac{|x_N|}{1-r}. \end{aligned}$$
Thus, $\sum_{k=1}^{m} |x_k|$ is bounded. Since it is monotone increasing, by the Monotone Convergence Theorem, $\sum_{k=1}^{m} |x_k|$ converges. Thus, $\sum |x_n|$ converges.
2. Suppose $L > 1$. By assumption, there exists $N \in \mathbb{N}$ such that $n > N$ implies $\left|\frac{|x_{n+1}|}{|x_n|} - L\right| < 1$. Thus,
$$\begin{aligned} -1 + L < \frac{|x_{n+1}|}{|x_n|} < 1 + L \\ \implies \quad |x_{n+1}| > (L-1)|x_n| > 0, \end{aligned}$$
which implies $\lim_{n\to\infty} |x_n| \neq 0$, further implies $\lim_{n\to\infty} x_n \neq 0$. Thus, $\sum x_n$ diverges. $\square$

Example 5.24.

The series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.

Proof. Since

$$\lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{n+1}{2n}$$

$$= \frac{1}{2},$$

by the Ratio Test, $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges. ■

Proposition 5.25.

The series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges absolutely for all $x \in \mathbb{R}$.

Proof. Since

$$\lim_{n\to\infty} \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \lim_{n\to\infty} \frac{|x|}{n+1}$$

$$= 0$$

for all $x \in \mathbb{R}$, $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges absolutely for all $x \in \mathbb{R}$. □

Theorem 5.26 (Cauchy Root Test)**.**

Let $\{x_n\}$ be a sequence. Suppose $\lim_{n\to\infty} \sqrt[n]{|x_n|} = L$. Then,

1. if $L < 1$, then $\sum x_n$ converges absolutely;
2. if $L > 1$, then $\sum x_n$ diverges.

If the limit is 1, then this theorem doesn't apply.

Proof.

1. Suppose $0 \leq L < 1$. Let $r \in (L, 1)$. By assumption, there exists $N \in \mathbb{N}$ such that $n > N$ implies $\left|\sqrt[n]{|x_n|} - L\right| < r - L$. Thus, for all $n > \mathbb{N}$,

$$
\begin{aligned}
& 2L - r < \sqrt[n]{|x_n|} < r \\
\implies & \sqrt[n]{|x_n|} < r \\
\iff & |x_n| < r^n,
\end{aligned}
$$

which implies $|x_{k+N}| < r^k$ for all $k \in \mathbb{N}$. Let $m > N$ and $m \in \mathbb{N}$. Then,

$$
\begin{aligned}
\sum_{k=1}^{m} |x_k| &= \sum_{k=1}^{N} |x_k| + \sum_{k=N+1}^{m} |x_k| \\
&< \sum_{k=1}^{N} |x_k| + \sum_{k=N+1}^{m} r^{k-N} \\
&< \sum_{k=1}^{N} |x_k| + \sum_{k=0}^{\infty} r^k \\
&= \sum_{k=1}^{N} |x_k| + \frac{1}{1-r}.
\end{aligned}
$$

Thus, $\sum_{k=1}^{m} |x_k|$ is bounded. Since it is monotone increasing, by the Monotone Convergence Theorem, $\sum_{k=1}^{m} |x_k|$ converges. Thus, $\sum |x_n|$ converges.

2. Suppose $L > 1$. By assumption, there exists $N \in \mathbb{N}$ such that $n > N$ implies $\left| \sqrt[n]{|x_n|} - L \right| < L - 1$. Thus,

$$
\begin{aligned}
& 1 < \sqrt[n]{|x_n|} < 2L - 1 \\
\implies & \sqrt[n]{|x_n|} > 1 > 0,
\end{aligned}
$$

which implies $\lim_{n\to\infty} |x_n| \neq 0$, further implies $\lim_{n\to\infty} x_n \neq 0$. Thus, $\sum x_n$ diverges. □

Example 5.27.

The series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges.

Proof. Since

$$
\begin{aligned}
\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} &= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \\
&= \frac{1}{e},
\end{aligned}
$$

by the Root Test, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges. ■

Lemma 5.28.

If $\{x_n\}$ is monotone and $\lim_{n\to\infty} x_{2n} = x$, then $\lim_{n\to\infty} x_n = x$.

Proof. Let $\varepsilon > 0$. Suppose that $\lim_{n\to\infty} x_{2n} = x$. Then, since $\{x_{2n}\}$ is bounded above, there exists $B_1 \in \mathbb{R}$ such that $x_{2n} \leq B_1$ for all $n \in \mathbb{N}$. Since $\{x_n\}$ is monotone increasing, we have $x_n \leq B_1$ for all $n \in \mathbb{N}$. Hence, $\{x_n\}$ is bounded above. By the Monotone Convergence Theorem, $\{x_n\}$ converges. By Theorem 3.20, $\lim_{n\to\infty} x_n = x$. Now, suppose $\{x_n\}$ is monotone decreasing. We will show that $\{x_n\}$ is bounded below. Since $\{x_{2n}\}$ is bounded below, there exists $B_2 \in \mathbb{R}$ such that $x_{2n} \geq B_2$ for all $n \in \mathbb{N}$. Since $\{x_n\}$ is monotone decreasing, we have $x_n \geq B_2$ for all $n \in \mathbb{N}$. Hence, $\{x_n\}$ is bounded below. By the Monotone Convergence Theorem, $\{x_n\}$ converges. By Theorem 4.16 and Theorem 4.15, $\lim_{n\to\infty} x_n = x$. □

Theorem 5.29 (Alternating Series Test)**.**

If $\{x_n\}$ is a monotone decreasing sequence such that $\lim_{n\to\infty} x_n = 0$, then $\sum(-1)^n x_n$ converges.

Proof. Suppose $\{x_n\}$ is monotone decreasing and $\lim_{n\to\infty} x_n = 0$. Let $s_n = \sum_{k=1}^{n}(-1)^k x_k$. Since $\{x_n\}$ is monotone decreasing, $-x_k + x_{k+1} \leq 0$. Hence,

$$\begin{aligned} s_{2n} &\geq s_{2n} - x_{2n+1} + x_{2n+2} \\ &= s_{2(n+1)} \end{aligned}$$

for all $n \in \mathbb{N}$. Thus, $\{s_{2n}\}$ is monotone decreasing. Moreover, since $x_{2n} - x_{2n+1} \geq 0$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} s_{2n} &= -x_1 + (x_2 - x_3) + (x_4 - x_5) + \cdots + (x_{2n-1} - x_{2n}) \\ &\geq -x_1. \end{aligned}$$

Thus, $\{s_{2n}\}$ is bounded below. Therefore, by the Monotone Convergence Theorem, $\{s_{2n}\}$ converges. By Lemma 5.28, $\{s_n\}$ converges. Therefore, $\sum(-1)^n x_n$ converges. □

Example 5.30.

The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges but not converges absolutely.

Proof. This follows immediately from the Alternating Series Test and Proposition 5.14. ■

Theorem 5.31 (Rearrangement)**.**

Suppose $\sum |x_n|$ converges and $\sum x_n = x$. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijective function. Then, $\sum |x_{\sigma(n)}|$ converges and $\sum x_{\sigma(n)} = x$.

Proof. Since $\left\{\sum_{k=1}^{n} |x_{\sigma(k)}|\right\}_n$ is monotone increasing, if we can show that it is bounded, then $\sum |x_{\sigma(n)}|$ is convergent by the Monotone Convergence Theorem. Since $\sum x_n$ converges absolutely, there exists $B \geq 0$ such that $\sum_{i=1}^{n} |x_i| \leq B$ for all $n \in \mathbb{N}$. Let $m \in \mathbb{N}$. Then, $\{\sigma(k)\}_{k=1}^{m}$ is a finite subset of $\mathbb{N}$. Thus, there exists $\ell \in \mathbb{N}$ such that $\{\sigma(k)\}_{k=1}^{m} \subseteq \{k\}_{k=1}^{\ell}$. Hence,

$$\begin{aligned}\sum_{k=1}^{m} |x_{\sigma(k)}| &\leq \sum_{k=1}^{\ell} |x_k| \\ &\leq B.\end{aligned}$$

Thus, by the Monotone Convergence Theorem, $\sum |x_{\sigma(n)}|$ is convergent. Now, we are showing $\sum x_{\sigma(n)} = x$. Let $\varepsilon > 0$ and let $\sum x_n = x$. On the one hand, there exists $M_1 \in \mathbb{N}$ such that $n > M_1$ implies $\left|\left(\sum_{i=1}^{n} x_i\right) - x\right| < \frac{\varepsilon}{2}$. On the other hand, since $\sum |x_n|$ converges, the sequence $\left\{\sum_{i=1}^{n} |x_i|\right\}_n$ is Cauchy by Theorem 4.16. Then, there exists $M_2 \in \mathbb{N}$ such that $m > M_2$ and $n > m$ imply $\sum_{i=m+1}^{n} |x_i| < \frac{\varepsilon}{2}$. Let $M_3 = \max\{M_1, M_2\}$, then $m > M_3$ and $n > m$ imply $\left|\left(\sum_{i=1}^{m} x_i\right) - x\right| < \frac{\varepsilon}{2}$ and $\sum_{i=m+1}^{n} |x_i| < \frac{\varepsilon}{2}$. Since $\{\sigma^{-1}(k)\}_{k=1}^{M_3}$ is a finite set, there exists $M \in \mathbb{N}$ such that $\{k\}_{k=1}^{M_3} \subseteq \{\sigma(k)\}_{k=1}^{M}$, which also implies $M \geq M_3$ implicitly. Thus, $n > M$ implies

$$\begin{aligned}\left|\left(\sum_{k=1}^{n} x_{\sigma(k)}\right) - x\right| &= \left|\left(\sum_{i \in \{\sigma(k)\}_{k=1}^{n}} x_i\right) - x\right| \\ &= \left|\left(\left(\sum_{i=1}^{M+1} x_i\right) + \left(\sum_{i \in \left(\{\sigma(k)\}_{k=1}^{n} \setminus \{k\}_{k=1}^{M+1}\right)} x_i\right)\right) - x\right| \\ &\leq \left|\left(\sum_{i=1}^{M+1} x_i\right) - x\right| + \sum_{i \in \left(\{\sigma(k)\}_{k=1}^{n} \setminus \{k\}_{k=1}^{M+1}\right)} |x_i| \\ &\leq \left|\left(\sum_{i=1}^{M+1} x_i\right) - x\right| + \sum_{i=M+2}^{\max\{\sigma(k)\}_{k=1}^{n}} |x_i| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

The second equation, which is in the second line, follows since $i \in \{\sigma(k)\}_{k=1}^{n}$ is a finite subset of $\mathbb{N}$. The first inequality, which is in the third line, follows from the Triangle Inequality and Lemma 5.18. The third inequality, which is in the penultimate line, follows from $M+2 > M_3$ and $\max\{\sigma(k)\}_{k=1}^{n} > M+2$. Therefore, $\sum x_{\sigma(n)} = x$. □

Remark 5.32.

Theorem 5.31 says that if a series is absolutely convergent, then rearrangement of the sequence will not change the value of the series.

Example 5.33.

The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \cdots = \ln 2$ is not absolutely convergent. There is a rearrangement that changes it value.

Proof. We have

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \cdots = \ln 2. \tag{5.33.1}$$

We multiply $\frac{1}{2}$ to equation 5.33.1, having

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \frac{1}{18} + \cdots = \frac{1}{2}\ln 2. \tag{5.33.2}$$

We further align equation 5.33.2, having

$$0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + 0 + \cdots = \frac{1}{2}\ln 2. \tag{5.33.3}$$

We add up equation 5.33.1 and equation 5.33.3, having the rearrangement

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots = \frac{3}{2}\ln 2. \tag{5.33.4}$$

Equation 5.33.4 is a rearrangement of equation 5.33.1, in which one negative term occurs after each pair of positive terms, but the two series have different values. ■

6 Limits of Functions and Sequential Limits

Definition 6.1 (Cluster Point)**.**

Let $S \subseteq \mathbb{R}$. $x \in \mathbb{R}$ is a cluster point of S if for all $\delta > 0$, the set $((x - \delta, x) \cup (x, x + \delta)) \cap S$ is not empty.

Remark 6.2.

The statement regarding cluster points Definition 6.1 is equivalent to the following: x is a cluster point of S if for all $\delta > 0$, there exists $y \in S$ such that $0 < |x - y| < \delta$.

Example 6.3.

Let $S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$. Then, 0 is a cluster point of S.

Proof. Let $\delta > 0$. $((-\delta, 0) \cup (0, \delta)) \cap \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$ is not empty since there exists $N = \left[\frac{1}{\delta} \right] + 1$ such that $n > N$ implies $\frac{1}{n} < \delta$. ■

Example 6.4.

Let $S = (0, 1)$. Then, $[0, 1]$ is the set of cluster points of S.

Proof. Let $\delta > 0$. Suppose $x \in (0, 1)$. Then, $((x - \delta, x) \cup (x, x + \delta)) \cap (0, 1))$ is not empty. Moreover, 0 and 1 are also cluster points of S. ■

Example 6.5.

Let $S = \mathbb{Q}$. Then, $\mathbb{R}$ is the set of cluster points of S.

Proof. Let $\delta > 0$. Suppose $x \in \mathbb{R}$. Then, $((x - \delta, x) \cup (x, x + \delta)) \cap \mathbb{Q}$ is not empty by Theorem 2.22. ■

Example 6.6.

Let $S = \{0\}$. Then, there are no cluster points of S.

Proof. Let $\delta > 0$. Let $E = ((x - \delta, x) \cup (x, x + \delta)) \cap \{0\}$. If $x = 0$, then $E = \emptyset$. If $x > 0$, then take $\delta = \frac{x}{2}$, $\left(\left(\frac{x}{2}, x \right) \cup \left(x, \frac{3x}{2} \right) \right) \cap \{0\} = \emptyset$. If $x < 0$, then take $\delta = -\frac{x}{2}$, $\left(\left(\frac{3x}{2}, x \right) \cup \left(x, \frac{x}{2} \right) \right) \cap \{0\} = \emptyset$. Therefore, there are no cluster points of S. ■

Example 6.7.

Let $S = \mathbb{Z}$. Then, there are no cluster points of S.

Proof. The proof is analogous to the one of Example 6.6, thus it is omitted. ■

Theorem 6.8.

Let $S \subseteq \mathbb{R}$. $x \in \mathbb{R}$ is a cluster point of S if and only if there exists a sequence $\{x_n\}$ of elements in $S \setminus \{x\}$ such that $\lim_{n\to\infty} x_n = x$.

Proof.

($\Longrightarrow$) Suppose x is a cluster point of S. Let $x_n \in \left(x - \dfrac{1}{n}, x\right) \cup \left(x, x + \dfrac{1}{n}\right) \cap S$ for all $n \in \mathbb{N}$. Note that x_n exists for all $n \in \mathbb{N}$ since x is a cluster point of S. Then, we have the inequality $0 < |x - x_n| < \dfrac{1}{n}$. By the Squeeze Theorem, $\lim_{n\to\infty} |x - x_n| = 0$. By Proposition 3.23 and Proposition 3.22, $\lim_{n\to\infty} x_n = x$.

($\Longleftarrow$) Suppose there exists a sequence $\{x_n\}$ of elements in $S \setminus \{x\}$ such that $\lim_{n\to\infty} x_n = x$. Then, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|x_{N+1} - x| < \varepsilon$, i.e., $x_{N+1} \in (x - \varepsilon, x) \cup (x, x + \varepsilon) \cap S$. □

Definition 6.9 (Function Convergence)**.**

Let $S \subseteq \mathbb{R}$ and let c be a cluster point of S. Define $f : S \to \mathbb{R}$. $f(x)$ converges to $L \in \mathbb{R}$ at c if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in 0 < |x - c| < \delta \cap S$ implies $|f(x) - L| < \varepsilon$.

Notation 6.10.

$f(x)$ converges to $L \in \mathbb{R}$ at c is denoted either $\lim_{x\to c} f(x) = L$, $f(x) \xrightarrow{x\to c} L$, or $f(x) \to L$ as $x \to c$.

Theorem 6.11.

Let $S \subseteq \mathbb{R}$. Define $f : S \to \mathbb{R}$. If $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, then $L_1 = L_2$.

Proof. Let $\varepsilon > 0$. Suppose $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$. Then, there exists $\delta_1 > 0$ such that $x \in 0 < |x - c| < \delta_1 \cap S$ implies $|f(x) - L_1| < \varepsilon$ and there exists $\delta_2 > 0$ such that $x \in 0 < |x - c| < \delta_2 \cap S$ implies $|f(x) - L_2| < \varepsilon$. Without loss of generality, suppose $L_1 > L_2$. Take $\varepsilon^* = \dfrac{L_1 - L_2}{2}$ and take $\delta^* = \min\{\delta_1, \delta_2\}$, any x in $(c - \delta, c) \cup (c, c + \delta) \cap S$. Then, $|f(x) - L_1| < \dfrac{L_1 - L_2}{2}$ and $|f(x) - L_2| < \dfrac{L_1 - L_2}{2}$ both hold, which is impossible. Therefore, $L_1 = L_2$. □

Example 6.12.

Let $f(x) = ax + b$. Then, $\lim_{x \to c} f(x) = ac + b$ for all $c \in \mathbb{R}$.

Proof. Let $\varepsilon > 0$. Take $\delta = \dfrac{\varepsilon}{1 + |a|}$, then $0 < |x - c| < \delta$ implies

$$\begin{aligned} |ax + b - ac + b| &= |a||x - c| \\ &< \frac{|a|}{1 + |a|}\varepsilon \\ &< \varepsilon. \end{aligned}$$

Therefore, $\lim_{x \to c} f(x) = ac + b$ for all $c \in \mathbb{R}$. ■

Example 6.13.

Let $f(x) = \sqrt{x}$. Then, $\lim_{x \to c} f(x) = \sqrt{c}$ for all $c \in \mathbb{R}^+$.

Proof. Let $\varepsilon > 0$. Take $\delta = \sqrt{c} \cdot \varepsilon$, then $0 < |x - c| < \delta$ implies

$$\begin{aligned} |\sqrt{x} - \sqrt{c}| &= \frac{x - c}{\sqrt{x} + \sqrt{c}} \\ &< \frac{x - c}{\sqrt{c}} \\ &< \varepsilon. \end{aligned}$$

Therefore, $\lim_{x \to c} f(x) = \sqrt{c}$ for all $c \in \mathbb{R}^+$. ■

Example 6.14.

Let $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$. Then, $\lim_{x \to 1} f(x) = 1$.

Proof. Let $\varepsilon > 0$. Take $\delta = 1$, then $0 < |x - 1| < \delta$ implies $|1 - 1| < \varepsilon$. Therefore, $\lim_{x \to 1} f(x) = 1$. ■

Theorem 6.15.

Let $S \subseteq \mathbb{R}$, let c be a cluster point of S, and let $f : S \to \mathbb{R}$. Then, $\lim_{x \to c} f(x) = L$ if and only if for every sequence $\{x_n\}$ in $S \setminus \{c\}$ such that $\lim_{n \to \infty} x_n = c$, $\lim_{n \to \infty} f(x_n) = L$.

Proof.

($\implies$) Let $\varepsilon > 0$. Suppose $\lim_{x \to c} f(x) = L$. Then, there exists $\delta > 0$ such that $0 < |x - c| < \delta$ implies

$|f(x) - L| < \varepsilon$. Let $\{x_n\}$ be a sequence in $S \setminus \{c\}$ such that $\lim_{n\to\infty} x_n = c$. Since $\lim_{n\to\infty} x_n = c$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $0 < |x_n - c| < \delta$. Then, $n > M$ implies $|f(x_n) - L| < \varepsilon$.

($\Longleftarrow$) Suppose for every sequence $\{x_n\}$ in $S \setminus \{c\}$ such that $\lim_{n\to\infty} x_n = c$, $\lim_{n\to\infty} f(x_n) = L$ and suppose $\lim_{x\to c} f(x) \neq L$. Then, there exists $\varepsilon^* > 0$ such that for all $\delta > 0$, there exists $x \in (c - \delta, c) \cup (c, c + \delta)$ such that $|f(x) - L| \geq \varepsilon^*$. Take $\delta = \dfrac{1}{n}$. Then, for all $n \in \mathbb{N}$, there exists x_n such that $0 < |x_n - c| < \dfrac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon^*$, which contradicts to the assumption $\lim_{n\to\infty} f(x_n) = L$. Therefore, $\lim_{x\to c} f(x) = L$. □

Example 6.16.

Let $f(x) = x^2$. Then, $\lim_{x\to c} f(x) = c^2$ for all $c \in \mathbb{R}$.

Proof. We obtain $\lim_{x\to c} x = c$ by Example 6.12. By Theorem 6.15, we know that for every sequence $\{x_n\}$ in $\mathbb{R} \setminus \{c\}$ such that $\lim_{n\to\infty} x_n = c$, $\lim_{n\to\infty} x_n = c$. Thus, by Corollary 3.27, $\lim_{n\to\infty} {x_n}^2 = c^2$. Therefore, by Theorem 6.15, $\lim_{x\to c} f(x) = c^2$ for all $c \in \mathbb{R}$. ■

Example 6.17.

Let $f(x) = \sin\left(\dfrac{1}{x}\right)$. Then, $\lim_{x\to 0} f(x)$ does not exist but $\lim_{x\to 0} x f(x) = 0$.

Proof. Let $x_n = \dfrac{2}{(2n-1)\pi}$, then $\lim_{n\to\infty} x_n = 0$. However, $f(x_n) = (-1)^{n+1}$, which does not converge by Example 4.13 and Theorem 4.16. We now show $\lim_{x\to 0} x f(x) = 0$. Suppose $x_n \neq 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = 0$. Then, $0 \leq \left| x_n \sin\left(\dfrac{1}{x_n}\right) \right| \leq |x_n|$. By the Squeeze Theorem, $\lim_{n\to\infty} \left| x_n \sin\left(\dfrac{1}{x_n}\right) \right| = 0$. Thus, by Proposition 3.23, $\lim_{n\to\infty} x_n \sin\left(\dfrac{1}{x_n}\right) = 0$. Therefore, by Theorem 6.15, $\lim_{x\to 0} x f(x) = 0$. ■

Theorem 6.18 (Squeeze Theorem)**.**

Let f, g, and h be $S \to \mathbb{R}$ functions such that $f(x) \leq g(x) \leq h(x)$ for all $x \in S$. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

Proof. By Theorem 6.15, $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = L$, where $\{x_n\}$ is a sequence in $S \setminus \{c\}$ such that $\lim_{n\to\infty} x_n = c$. Since $f(x) \leq g(x) \leq h(x)$ for all $x \in S$, $f(x_n) \leq g(x_n) \leq h(x_n)$ for all $n \in \mathbb{N}$. By the Squeeze Theorem, $\lim_{n\to\infty} g(x_n) = L$. By Theorem 6.15, $\lim_{x\to c} g(x) = L$. □

Theorem 6.19.

Let $S \subseteq \mathbb{R}$, let c be a cluster point of S, let $f : S \to \mathbb{R}$, and let $g : S \to \mathbb{R}$. If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, and $f(x) \le g(x)$ for all $x \in S$, then $\lim_{x \to c} f(x) \le \lim_{x \to c} g(x)$.

Proof. Let $\lim_{x \to c} f(x) = L_1$ and let $\lim_{x \to c} g(x) = L_2$. Let $\{x_n\}$ be a sequence such that $\lim_{n \to \infty} x_n = c$. Then, by Theorem 3.25, since $f(x_n) \le g(x_n)$, we obtain $\lim_{n \to \infty} f(x_n) \le \lim_{n \to \infty} g(x_n)$. Therefore, by Theorem 6.15, $\lim_{x \to c} f(x) \le \lim_{x \to c} g(x)$. □

Theorem 6.20.

Let $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Then,

1. $\lim_{x \to c} f(x) + g(x) = L + M$;
2. $\lim_{x \to c} c \cdot f(x) = c \cdot L$ for all $c \in \mathbb{R}$;
3. $\lim_{x \to c} f(x) \cdot g(x) = L \cdot M$;
4. if $g(x) \neq 0$ for all $x \in S$ and $\lim_{x \to c} g(x) \neq 0$, then $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$.

Proof.

1. Let $\varepsilon > 0$. Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Then, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $x \in (c - \delta_1, c) \cup (c, c + \delta_1)$ implies $|f(x) - L| < \dfrac{\varepsilon}{2}$ and $x \in (c - \delta_2, c) \cup (c, c + \delta_2)$ implies $|g(x) - M| < \dfrac{\varepsilon}{2}$. Take $\delta = \min\{\delta_1, \delta_2\}$, then $x \in (c - \delta, c) \cup (c, c + \delta)$ implies
$$\begin{aligned} |f(x) + g(x) - L - M| &\le |f(x) - L| + |g(x) - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$
Therefore, $\lim_{x \to c} f(x) + g(x) = L + M$.
2. Let $\varepsilon > 0$. Suppose $\lim_{x \to c} f(x) = L$. Then, there exists $\delta > 0$ such that $x \in (c - \delta, c) \cup (c, c + \delta)$ implies $|f(x) - L| < \dfrac{\varepsilon}{|c|}$. Then, $x \in (c - \delta, c) \cup (c, c + \delta)$ implies
$$\begin{aligned} |c \cdot f(x) - c \cdot L| &= |c||f(x) - L| \\ &< |c| \cdot \frac{\varepsilon}{|c|} \\ &= \varepsilon. \end{aligned}$$

Therefore, $\lim_{x \to c} c \cdot f(x) = c \cdot L$.

3. Let $\varepsilon > 0$. Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$. Then, there exist δ_1 and δ_2 such that $x \in (c - \delta_1, c) \cup (c, c + \delta_1)$ implies $|f(x) - L| < \dfrac{\varepsilon}{2 + 2|M|}$ and $x \in (c - \delta_2, c) \cup (c, c + \delta_2)$ implies $|g(x) - M| < \dfrac{\varepsilon}{2 + 2|L|}$. Take $\delta = \min\{\delta_1, \delta_2\}$. Moreover, we obtain $|g(x) - M| < 1$ whenever $x \in (c - \delta, c) \cup (c, c + \delta)$. Thus, $x \in (c - \delta, c) \cup (c, c + \delta)$ implies $|g(x)| \leq 1 + |M|$ by the Reverse Triangle Inequality. Then, $x \in (c - \delta, c) \cup (c, c + \delta)$ implies

$$\begin{aligned}
|f(x) \cdot g(x) - L \cdot M| &= |f(x) \cdot g(x) - L \cdot g(x) + L \cdot g(x) - L \cdot M| \\
&\leq |f(x) - L||g(x)| + |L||g(x) - M| \\
&\leq \frac{\varepsilon}{2 + 2|M|} \cdot (1 + |M|) + |L| \cdot \frac{\varepsilon}{2 + 2|L|} \\
&< \varepsilon.
\end{aligned}$$

Therefore, $\lim_{x \to c} f(x) \cdot g(x) = L \cdot M$.

4. We show if $f(x) \neq 0$ and $\lim_{x \to c} f(x) \neq 0$, then $\lim_{x \to c} \dfrac{1}{f(x)} = \dfrac{1}{L}$ first. Let $\varepsilon > 0$. Suppose $\lim_{x \to c} f(x) = L \neq 0$. Then, there exist $\delta > 0$ such that $x \in (c - \delta, c) \cup (c, c + \delta)$ implies $|f(x) - L| < \dfrac{L^2 \varepsilon}{2}$. In addition, if $x \in (c - \delta, c) \cup (c, c + \delta)$, then $|f(x) - L| < \dfrac{|L|}{2}$, which implies $\dfrac{|L|}{2} < |f(x)|$ by the Reverse Triangle Inequality. Then, $x \in (c - \delta, c) \cup (c, c + \delta)$ implies

$$\begin{aligned}
\left| \frac{1}{f(x)} - \frac{1}{L} \right| &= \left| \frac{L - f(x)}{f(x) \cdot L} \right| \\
&= \frac{|f(x) - L|}{|f(x) \cdot L|} \\
&= \frac{|f(x) - L|}{|L|} \cdot \frac{1}{|f(x)|} \\
&< \frac{\frac{L^2 \varepsilon}{2}}{|L|} \cdot \frac{2}{1 + |L|} \\
&< \varepsilon.
\end{aligned}$$

Thus, $\lim_{x \to c} \dfrac{1}{f(x)} = \dfrac{1}{L}$. Applying Theorem 6.20.3 completes the proof. $\square$

Remark 6.21.

We can also prove Theorem 6.20 by sequential limits.

Proposition 6.22.

Let $S \subseteq \mathbb{R}$, let c be a cluster point of S, and let $f : S \to \mathbb{R}$. Suppose $\lim\limits_{x \to c} f(x)$ exists, then $\lim\limits_{x \to c} |f(x)| = \left|\lim\limits_{x \to c} f(x)\right|$.

Proof. Let $\varepsilon > 0$. Suppose $\lim\limits_{x \to c} f(x) = L$. Then, there exists $\delta > 0$ such that $x \in (c - \delta, c) \cup (c, c + \delta) \cap S$ implies $|f(x) - L| < \varepsilon$. Then, by the Reverse Triangle Inequality, $x \in (c - \delta, c) \cup (c, c + \delta) \cap S$ implies

$$\begin{aligned} ||f(x)| - |L|| &\leq |f(x) - L| \\ &< \varepsilon. \end{aligned}$$

Therefore, $\lim\limits_{x \to c} |f(x)| = |L|$. □

Definition 6.23 (One-Sided Limit)**.**

Let $S \subseteq \mathbb{R}$, let c be a cluster point of $S \cap (-\infty, c)$, and let $f : S \to \mathbb{R}$. $f(x)$ converges to L as x increases in value approaching c if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in (c - \delta, c) \cap S$ implies $|f(x) - L| < \varepsilon$.

Definition 6.24 (One-Sided Limit)**.**

Let $S \subseteq \mathbb{R}$, let c be a cluster point of $S \cap (c, \infty)$, and let $f : S \to \mathbb{R}$. $f(x)$ converges to L as x decreases in value approaching c if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in (c, c + \delta) \cap S$ implies $|f(x) - L| < \varepsilon$.

Notation 6.25.

$f(x)$ converges to L as x increases in value approaching c denoted either $\lim\limits_{x \to c^-} f(x) = L$, $f(x) \overset{x \to c^-}{\longrightarrow} L$, or $f(x) \to L$ as $x \to c^-$. $f(x)$ converges to L as x decreases in value approaching c denoted either $\lim\limits_{x \to c^+} f(x) = L$, $f(x) \overset{x \to c^+}{\longrightarrow} L$, or $f(x) \to L$ as $x \to c^+$.

Example 6.26.

Let $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$. Then, $\lim\limits_{x \to 0^+} f(x) = 1$ and $\lim\limits_{x \to 0^-} f(x) = 0$.

Proof. This follows from the definitions. ■

Theorem 6.27.

Let $S \subseteq \mathbb{R}$, let c be a cluster point of S, and let $f : S \to \mathbb{R}$. Then, $\lim\limits_{x \to c^+} f(x) = \lim\limits_{x \to c^-} f(x) = L$ if and only if $\lim\limits_{x \to c} f(x) = L$.

Proof.

($\Longrightarrow$) Suppose $\lim\limits_{x \to c^+} f(x) = L$ and $\lim\limits_{x \to c^-} f(x) = L$. Then, there exist δ_1 and δ_2 such that $x \in (c, c + \delta_1)$ and $x \in S$ imply $|f(x) - L| < \varepsilon$ and $x \in (c - \delta_2, c) \cap S$ implies $|f(x) - L| < \varepsilon$. Take $\delta = \min\{\delta_1, \delta_2\}$, then $x \in (c - \delta, c) \cup (c, c + \delta) \cap S$ implies $|f(x) - L| < \varepsilon$. Hence, $\lim\limits_{x \to c} f(x) = L$.

($\Longleftarrow$) Suppose $\lim\limits_{x \to c} f(x) = L$. Then, there exists δ such that $x \in (c - \delta, c) \cup (c, c + \delta)$ and $x \in S$ imply $|f(x) - L| < \varepsilon$. Then, $x \in (c, c + \delta) \cap S$ implies $|f(x) - L| < \varepsilon$ and $x \in (c - \delta, c) \cap S$ implies $|f(x) - L| < \varepsilon$. Hence, $\lim\limits_{x \to c^+} f(x) = \lim\limits_{x \to c^-} f(x)$. $\square$

7 Continuity, Uniform Continuity, and Derivative

Definition 7.1 (Continuous)**.**

Let $S \subseteq \mathbb{R}$, let $c \in S$, and let $f : S \to \mathbb{R}$. f is <u>continuous</u> at c if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in (c-\delta, c+\delta) \cap S$ implies $|f(x) - f(c)| < \varepsilon$.

Definition 7.2 (Continuous)**.**

Let $S \subseteq \mathbb{R}$ and let $f : S \to \mathbb{R}$. f is <u>continuous</u> on U for $U \subseteq S$ if for all $c \in U$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in (c-\delta, c+\delta) \cap S$ implies $|f(x) - f(c)| < \varepsilon$.

Definition 7.3 (Continuous Function)**.**

Let $S \subseteq \mathbb{R}$ and let $f : S \to \mathbb{R}$. f is a <u>continuous function</u> if for all $c \in S$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in (c-\delta, c+\delta) \cap S$ implies $|f(x) - f(c)| < \varepsilon$.

Example 7.4.

$f(x) = ax + b$ is continuous on $\mathbb{R}$.

Proof. Let $\varepsilon > 0$. Let $c \in \mathbb{R}$. Take $\delta = \dfrac{\varepsilon}{1+|a|}$. Then, $|x - c| < \delta$ implies

$$\begin{aligned} |ax + b - ac - b| &= |a||x-c| \\ &< |a| \cdot \frac{\varepsilon}{1+|a|} \\ &< \varepsilon. \end{aligned}$$

Therefore, $f(x) = ax + b$ is continuous on $\mathbb{R}$. ■

Negation 7.5.

Let $S \subseteq \mathbb{R}$, let $c \in S$, and let $f : S \to \mathbb{R}$. f is <u>not continuous</u> at c if there exists $\varepsilon > 0$, for all $\delta > 0$ such that there exists $x \in (c-\delta, c+\delta) \cap S$ such that $|f(x) - f(c)| \geq \varepsilon$.

Example 7.6.

Let $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Then, $f(x)$ is not continuous at 0.

Proof. Take $\varepsilon = \frac{1}{2}$. Let $\delta > 0$. Then, $x = \frac{\delta}{2}$ satisfies $|x - 0| < \delta$ and $|1 - 0| \geq \frac{1}{2}$. Therefore, $f(x)$ is not continuous at 0. ■

Theorem 7.7.

Let $S \subseteq \mathbb{R}$, let $c \in \mathbb{S}$, and let $f : S \to \mathbb{R}$. Then,

1. if c is not a cluster point of f, then f is continuous at c;
2. if c is a cluster point of f, then f is continuous at c if and only if $\lim_{x \to c} f(x) = f(c)$;
3. f is continuous at c if and only if for every sequence $\{x_n\}$ of elements of S such that $\lim_{n \to \infty} x_n = c$, $\lim_{n \to \infty} f(x_n) = f(c)$.

Proof.

1. Suppose c is not a cluster point of f. Then, there exists $\delta^* > 0$ such that $(c - \delta^*, c + \delta^*) \cap S = \{c\}$. Thus, $x \in (c - \delta^*, c + \delta^*) \cap S$ implies $|f(x) - f(c)| = 0 < \varepsilon$. Hence, f is continuous at c.
2. Suppose c is a cluster point of f.

   ($\Longrightarrow$) Let $\varepsilon > 0$. Suppose f is continuous at c. Then, there exists $\delta > 0$ such that $x \in (c - \delta, c + \delta) \cap S$ implies $|f(x) - f(c)| < \varepsilon$. Thus, $x \in (c - \delta, c) \cup (c, c + \delta) \cap S$ implies $|f(x) - f(c)| < \varepsilon$. Hence, $\lim_{x \to c} f(x) = f(c)$.

   ($\Longleftarrow$) Let $\varepsilon > 0$. Suppose $\lim_{x \to c} f(x) = f(c)$. Then, there exists $\delta > 0$ such that $x \in (c - \delta, c) \cup (c, c + \delta) \cap S$ implies $|f(x) - f(c)| < \varepsilon$. Moreover, since $|f(c) - f(c)| = 0 < \varepsilon$, $x \in (c - \delta, c + \delta) \cap S$ implies $|f(x) - f(c)| < \varepsilon$. Hence, f is continuous at c.
3. ($\Longrightarrow$) Suppose f is continuous at c. Let $\varepsilon > 0$. Then, there exists $\delta > 0$ such that $x \in (c - \delta, c + \delta) \cap S$ implies $|f(x) - f(c)| < \varepsilon$. Thus, $x \in (c - \delta, c) \cup (c, c + \delta) \cap S$ implies $|f(x) - f(c)| < \varepsilon$. By Theorem 6.15, for every sequence $\{x_n\}$ of elements of S such that $\lim_{n \to \infty} x_n = c$, $\lim_{n \to \infty} f(x_n) = f(c)$.

   ($\Longleftarrow$) Suppose for every sequence $\{x_n\}$ of elements of S such that $\lim_{n \to \infty} x_n = c$, $\lim_{n \to \infty} f(x_n) = f(c)$. Then, by Theorem 6.15, $\lim_{x \to c} f(x) = f(c)$. Hence, f is continuous at c. □

Proposition 7.8.

$\sin x$ is a continuous function.

Proof. Let $c \in \mathbb{R}$. Let $\varepsilon > 0$. Take $\delta = \varepsilon$. Then, $|x - c| < \delta$ implies

$$\begin{aligned} |\sin x - \sin c| &= 2\left|\sin \frac{x-c}{2} \cos \frac{x+c}{2}\right| \\ &\leq 2\left|\sin \frac{x-c}{2}\right| \\ &\leq 2\left|\frac{x-c}{2}\right| \\ &< \varepsilon. \end{aligned}$$

Therefore, $\sin x$ is a continuous function. □

Proposition 7.9.

$\cos x$ is a continuous function.

Proof. Let $c \in \mathbb{R}$. Let $\{x_n\}$ be a sequence such that $\lim\limits_{n\to\infty} x_n = c$. Then, $\lim\limits_{n\to\infty} x_n + \frac{\pi}{2} = c + \frac{\pi}{2}$. Since $\sin x$ is continuous by Proposition 7.8,

$$\begin{aligned} \lim_{n\to\infty} \cos x_n &= \lim_{n\to\infty} \sin\left(x_n + \frac{\pi}{2}\right) \\ &= \sin\left(c + \frac{\pi}{2}\right) \\ &= \cos c. \end{aligned}$$

Therefore, $\cos x$ is a continuous function. □

Theorem 7.10.

Let $S \subseteq \mathbb{R}$, let $c \in \mathbb{S}$, let $f : S \to \mathbb{R}$ be continuous at c, and let $g : S \to \mathbb{R}$ be continuous at c. Then,

1. $f + g$ is continuous at c;
2. $f \cdot g$ is continuous at c;
3. if $g(x) \neq 0$ for all $x \in S$, then $\frac{f}{g}$ is continuous at c.

Proof.

1. Let $\varepsilon > 0$. Suppose $\lim\limits_{x\to c} f(x) = f(c)$ and $\lim\limits_{x\to c} g(x) = f(c)$. Then, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $x \in (c - \delta_1, c + \delta_1)$ implies $|f(x) - f(c)| < \frac{\varepsilon}{2}$ and $x \in (c - \delta_2, c + \delta_2)$ implies $|g(x) - g(c)| < \frac{\varepsilon}{2}$. Take

$\delta = \min\{\delta_1, \delta_2\}$, then $x \in (c-\delta, c) \cup (c, c+\delta)$ implies

$$\begin{aligned} |f(x)+g(x)-f(c)-g(c)| &\le |f(x)-f(c)| + |g(x)-g(c)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Therefore, $\lim\limits_{x\to c} f(x)+g(x) = f(c)+g(c)$, i.e., $f+g$.

2. Let $\varepsilon > 0$. Suppose $\lim\limits_{x\to c} f(x) = f(c)$ and $\lim\limits_{x\to c} g(x) = g(c)$. Then, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that $x \in (c-\delta_1, c+\delta_1)$ implies $|f(x)-f(c)| < \dfrac{\varepsilon}{2(1+|M|)}$ and $x \in (c-\delta_2, c+\delta_2)$ implies $|g(x)-g(c)| < \dfrac{\varepsilon}{2(|L|+1)}$. In addition, if $x \in (c-\delta_2, c+\delta_2)$, then $|g(x)-g(c)| < 1$, which implies $|g(x)| < 1+|g(c)|$ by the Reverse Triangle Inequality. Take $\delta = \min\{\delta_1, \delta_2\}$, then $x \in (c-\delta, c+\delta)$ implies

$$\begin{aligned} |f(x)\cdot g(x) - f(c)\cdot g(c)| &= |f(x)\cdot g(x) - f(c)\cdot g(x) + f(c)\cdot g(x) - f(c)\cdot g(c)| \\ &\le |g(x)||f(x)-f(c)| + |f(c)||g(x)-g(c)| \\ &< (1+|g(c)|)|f(x)-f(c)| + |f(c)||g(x)-g(c)| \\ &< (1+|g(c)|)\cdot \frac{\varepsilon}{2(1+|g(c)|)} + |f(c)|\cdot \frac{\varepsilon}{2(|f(c)|+1)} \\ &< \varepsilon. \end{aligned}$$

Therefore, $\lim\limits_{x\to c} f(x)\cdot g(x) = f(c)\cdot g(c)$, i.e., $f\cdot g$ is continuous at c.

3. We show if $f(x) \neq 0$ and $\lim\limits_{x\to c} f(x) \neq 0$, then $\lim\limits_{x\to c} \dfrac{1}{f(x)} = \dfrac{1}{f(c)}$ first. Let $\varepsilon > 0$. Suppose $\lim\limits_{x\to c} f(x) = f(c) \neq 0$. Then, there exist $\delta > 0$ such that $x \in (c-\delta, c+\delta)$ implies $|f(x)-f(c)| < \dfrac{(f(c))^2\varepsilon}{2}$. In addition, if $x \in (c-\delta, c+\delta)$, then $|f(x)-f(c)| < \dfrac{|f(c)|}{2}$, which implies $\dfrac{|f(c)|}{2} < |f(x)|$ by the Reverse Triangle Inequality. Then, $x \in (c-\delta, c+\delta)$ implies

$$\begin{aligned} \left|\frac{1}{f(x)} - \frac{1}{f(c)}\right| &= \left|\frac{f(c)-f(x)}{f(x)\cdot f(c)}\right| \\ &= \frac{|f(x)-f(c)|}{|f(x)\cdot L|} \\ &= \frac{|f(x)-f(c)|}{|f(c)|}\cdot\frac{1}{|f(x)|} \\ &< \frac{\frac{(f(c))^2\varepsilon}{2}}{|f(c)|}\cdot\frac{2}{1+|f(c)|} \\ &< \varepsilon. \end{aligned}$$

Thus, $\lim\limits_{x\to c} \dfrac{1}{f(x)} = \dfrac{1}{f(c)}$. Applying Theorem 7.10.2 completes the proof. $\square$

Corollary 7.11.

All polynomial functions are continuous on $\mathbb{R}$.

Proof. Since 1 and x are continuous, by Theorem 7.10, all polynomial functions are continuous. $\square$

Corollary 7.12.

All trigonometric functions are continuous functions.

Proof. Since $\sin x$ and $\cos x$ are continuous, by Theorem 7.10, all trigonometric functions are continuous. $\square$

Proposition 7.13.

Let $A \subseteq \mathbb{R}$, let $B \subseteq \mathbb{R}$, let $g : A \to B$, and let $f : B \to \mathbb{R}$. Then, if g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c.

Proof. Suppose g is continuous at c. Then, by Theorem 7.7.3, for any sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} x_n = c$, $\lim_{n\to\infty} g(x_n) = g(c)$. Suppose f is continuous at $g(c)$. Then, by Theorem 7.7.3, for any sequence $\{y_n\}$ in B such that $\lim_{n\to\infty} y_n = g(c)$, $\lim_{n\to\infty} f(y_n) = f(g(c))$. Hence, $\lim_{n\to\infty} f(g(x_n) = f(g(c))$. Therefore, by Theorem 6.15, $\lim_{x\to c} f(g(x)) = f(g(x))$, i.e., $f \circ g$ is continuous at c. $\square$

Example 7.14.

$\dfrac{1}{x^2}$ is continuous in $(0, \infty)$.

Proof. Since x is continuous in $(0, \infty)$, by Theorem 7.10.2 and Theorem 7.10.3, $\dfrac{1}{x^2}$ is continuous in $(0, \infty)$. $\blacksquare$

Example 7.15.

$\left(\cos \dfrac{1}{x^2}\right)^2$ is continuous in $(0, \infty)$.

Proof. Since $\dfrac{1}{x^2}$ is continuous in $(0, \infty)$ by Example 7.14 and $\cos x$ is continuous in $(0, \infty)$ by Proposition 7.9, by Proposition 7.13, $\left(\cos \dfrac{1}{x^2}\right)^2$ is continuous in $(0, \infty)$. $\blacksquare$

Proposition 7.16 (Dirichlet Function)**.**

$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ is not continuous on all of $\mathbb{R}$.

Proof. We separate cases for $c \in \mathbb{Q}$ and $c \notin \mathbb{Q}$.

1. Suppose $c \in \mathbb{Q}$. Then, for any $n \in \mathbb{N}$, there exists $x_n \notin \mathbb{Q}$ such that $c < x_n < c + \dfrac{1}{n}$. Thus, by the Squeeze Theorem, $\lim\limits_{n\to\infty} x_n = c$. However, $f(c) = 1$ and $\lim\limits_{n\to\infty} x_n = 0$, i.e., the function is not continuous at c.
2. Suppose $c \notin \mathbb{Q}$. Then, for any $n \in \mathbb{N}$, by Theorem 2.22, there exists $x_n \in \mathbb{Q}$ such that $c < x_n < c + \dfrac{1}{n}$. Thus, by the Squeeze Theorem, $\lim\limits_{n\to\infty} x_n = c$. However, $f(c) = 0$ and $\lim\limits_{n\to\infty} x_n = 1$, i.e., the function is not continuous at c.

Therefore, $f(x)$ is not continuous anywhere on $\mathbb{R}$. $\square$

Definition 7.17 (Bounded Function)**.**

A function $f : S \to \mathbb{R}$ is <u>bounded</u> if there exists $B \geq 0$ such that $|f(x)| \leq B$ for all $x \in S$.

Theorem 7.18.

If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded.

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is continuous. For the sake of contradiction, suppose f is unbounded. Then, for all $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $|f(x_n)| \geq n$. By the Bolzano-Weierstrass Theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges to some $x \in \mathbb{R}$. By Theorem 3.25, $a \leq x \leq b$. By assumption that f is continuous, we have $\lim\limits_{k\to\infty} f(x_{n_k}) = f(x)$. By Proposition 3.29, $\lim\limits_{k\to\infty} |f(x_{n_k})| = |f(x)|$. Thus, $\{|f(x_{n_k})|\}$ is convergent, thus is bounded. By assumption that f is unbounded, $|f(x_{n_k})| \geq n_k$. However, since $\{|f(x_{n_k})|\}$ is bounded, $\{n_k\}$ is bounded as well, which contradicts to the definition of subseqence that $n_k \geq k$ for all $k \in \mathbb{N}$. Therefore, f is bounded. $\square$

Definition 7.19 (Absolute Minimum)**.**

Let $f : S \to \mathbb{R}$. f achieves an <u>absolute minimum</u> at c if $f(x) \geq f(c)$ for all $x \in S$.

Definition 7.20 (Absolute Maximum)**.**

Let $f : S \to \mathbb{R}$. f achieves an absolute maximum at d if $f(x) \le f(d)$ for all $x \in S$.

Theorem 7.21 (Extreme Value Theorem)**.**

Let $f : [a, b] \to \mathbb{R}$. If f is continuous, then f achieves an absolute maximum and an absolute minimum.

Proof. Suppose f is continuous. By Theorem 7.18, f is bounded. Thus, the set $E = \{f(x) \mid x \in [a, b]\}$ is bounded above and bounded below. We do the absolute maximum part first. Let $L = \sup E$. By Theorem 4.3, there exists a sequence $\{a_n \in E\}$ such that $\lim_{n\to\infty} f(a_n) = L$. By the Bolzano-Weierstrass Theorem, there exists a subsequence $\{a_{n_k}\}$ and $d \in [a, b]$ such that $\lim_{k\to\infty} a_{n_k} = d$. Hence, by the assumption that f is continuous, we have

$$\begin{aligned}\lim_{k\to\infty} f(a_{n_k}) &= f(d)\\ &= \lim_{n\to\infty} f(a_n)\\ &= L.\end{aligned}$$

Thus, f achieves an absolute maximum at d. Now we do the absolute minimum part. Let $M = \inf E$. By Theorem 4.3, there exists a sequence $\{b_n \in E\}$ such that $\lim_{n\to\infty} f(b_n) = M$. By the Bolzano-Weierstrass Theorem, there exists a subsequence $\{b_{n_k}\}$ and $e \in [a, b]$ such that $\lim_{k\to\infty} b_{n_k} = e$. Hence, by the assumption that f is continuous, we have

$$\begin{aligned}\lim_{k\to\infty} f(b_{n_k}) &= f(e)\\ &= \lim_{n\to\infty} f(b_n)\\ &= M.\end{aligned}$$

Thus, f achieves an absolute minimum at e. □

Theorem 7.22 (Location of Roots Theorem)**.**

Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < 0$ and $f(b) > 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Proof. We prove this using the bisection method. Let $a_1 = a$ and $b_1 = b$. For all $n \in \mathbb{N}$, we define

- $(a_{n+1}, b_{n+1}) = \left(a_n, \dfrac{a_n + b_n}{2}\right)$ if $f\left(\dfrac{a_n + b_n}{2}\right) > 0$;

- $(a_{n+1}, b_{n+1}) = \left(\frac{a_n + b_n}{2}, b_n\right)$ if $f\left(\frac{a_n + b_n}{2}\right) < 0$.

Thus, we have

1. $a \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b$ for all $n \in \mathbb{N}$;
2. $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$ for all $n \in \mathbb{N}$;
3. $f(a_n) < 0$ and $f(b_n) > 0$ for all $n \in \mathbb{N}$.

By 1, we know that $\{a_n\}$ and $\{b_n\}$ are convergent by the Monotone Convergence Theorem. Thus, there exist c and d such that $\lim_{n\to\infty} a_n = c$ and $\lim_{n\to\infty} b_n = d$. By 2, we have

$$\begin{aligned} d - c &= \lim_{n\to\infty} (b_n - a_n) \\ &= \lim_{n\to\infty} \frac{b-a}{2^{n-1}} \\ &= 0. \end{aligned}$$

Thus, $d = c$. By 3, we have $f(c) = \lim_{n\to\infty} a_n \geq 0$ and $f(c) = \lim_{n\to\infty} b_n \leq 0$, therefore $f(c) = 0$. □

Example 7.23.

The function $f(x) = x^{2023} + 7x + 1$ has at least one real root.

Proof. Notice that $f(0) = 1$ and $f(-1) = -7$, by the Location of Roots Theorem, $f(x)$ has at least an root in $(-1, 0)$. ■

Theorem 7.24 (Intermediate Value Theorem)**.**

Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < f(b)$ and $y \in (f(a), f(b))$, then there exists $c \in (a, b)$ such that $f(c) = y$. If $f(b) < f(a)$ and $y \in (f(b), f(a))$, then there exists $c \in (a, b)$ such that $f(c) = y$.

Proof. Suppose $f(a) < f(b)$. Let $y \in (f(a), f(b))$. Define $g(x) = f(x) - y$, then $g : [a, b] \to \mathbb{R}$ is continuous. Moreover, $g(a) < 0$ and $g(b) > 0$. By the Location of Roots Theorem, there exists $c \in [a, b]$ such that $g(c) = 0$, i.e., $f(c) = y$. Now suppose $f(b) < f(a)$. Let $y \in (f(b), f(a))$. Define $g(x) = f(x) - y$, then $g : [a, b] \to \mathbb{R}$ is continuous. Moreover, $g(a) > 0$ and $g(b) < 0$. By the Location of Roots Theorem, there exists $c \in [a, b]$ such that $g(c) = 0$, i.e., $f(c) = y$. □

Remark 7.25.

The Location of Roots Theorem and the Intermediate Value Theorem are essentially the same.

Theorem 7.26.

Let $[a, b] \to \mathbb{R}$ be continuous. Let $c \in [a, b]$ be where f achieves an absolute minimum and $d \in [a, b]$ be where f achieves an absolute maximum. Then, $f([a, b]) = [f(c), f(d)]$.

Proof.

($\subseteq$) It is obvious that $f([a, b]) \subseteq [f(c), f(d)]$.

($\supseteq$) By the Intermediate Value Theorem, if $y \in [f(c), f(d)]$, then $y \in f([c, d])$. Hence, $[f(c), f(d)] \subseteq f([c, d])$. Since $c \in [a, b]$ and $d \in [a, b]$, $f([c, d]) \subseteq f([a, b])$. Thus, $[f(c), f(d)] \subseteq f([a, b])$.

Therefore, $f([a, b]) = [f(c), f(d)]$. $\square$

Example 7.27.

The function $f(x) = \dfrac{1}{x}$ is continuous in $(0, 1)$.

Proof. Let $\varepsilon > 0$. Let $c \in (0, 1)$. Take $\delta = \min\left\{\dfrac{c}{2}, \dfrac{c^2}{2}\varepsilon\right\}$. Notice that whenever $|x - c| < \dfrac{c}{2}$, we have

$$\begin{aligned} -\frac{c}{2} &< x - c < \frac{c}{2} \\ \frac{c}{2} &< x. \end{aligned}$$

Thus, $|x - c| < \delta$ implies

$$\begin{aligned} \left|\frac{1}{x} - \frac{1}{c}\right| &= \frac{|c - x|}{|x||c|} \\ &< \frac{\delta}{|x||c|} \\ &< \frac{\delta}{\frac{c^2}{2}} \\ &\leq \varepsilon. \end{aligned}$$

Therefore, $\dfrac{1}{x}$ is continuous in $(0, 1)$. $\blacksquare$

Definition 7.28 (Uniformly Continuous)**.**

Let $S \subseteq \mathbb{R}$ and let $f : S \to \mathbb{R}$. f is <u>uniformly continuous</u> on S if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in (c - \delta, c + \delta) \cap S$ implies $|f(x) - f(c)| < \varepsilon$ for all $c \in S$.

Example 7.29.

The function $f(x) = x^2$ is uniformly continuous on $[0, 1]$.

Proof. Let $\varepsilon > 0$. Take $\delta = \dfrac{\varepsilon}{2}$. Then, $x \in (c - \delta, c + \delta)$, $x \in [0, 1]$, and $c \in [0, 1]$ imply

$$
\begin{aligned}
|x^2 - c^2| &= |x + c||x - c| \\
&\leq 2|x - c| \\
&< 2\delta \\
&= \varepsilon.
\end{aligned}
$$

Therefore, $f(x) = x^2$ is uniformly continuous on $[0, 1]$. ■

Proposition 7.30.

Let $f : [a, b] \to \mathbb{R}$. If f is uniformly continuous, then f is continuous.

Proof. Suppose $f : [a, b] \to \mathbb{R}$ is uniformly continuous. Let $\varepsilon > 0$. Then, there exists $\delta_0 > 0$ such that for all $c \in [a, b]$, $x \in (c - \delta_0, c + \delta_0) \cap [a, b]$ implies $|f(x) - f(c)| < \varepsilon$. Take $\delta = \delta_0$. Then, for all $c \in [a, b]$, $x \in (c - \delta, c + \delta) \cap [a, b]$ implies $|f(x) - f(c)| < \varepsilon$. □

Negation 7.31 (Not Uniformly Continuous)**.**

Let $S \subseteq \mathbb{R}$ and let $f : S \to \mathbb{R}$. f is <u>not uniformly continuous</u> on S if there exists $\varepsilon > 0$, for all $\delta > 0$, there exists $c \in S$ such that there exists $x \in (c - \delta, c + \delta) \cap S$ such that $|f(x) - f(c)| \geq \varepsilon$.

Example 7.32.

The function $f(x) = \dfrac{1}{x}$ is not uniformly continuous in $(0, 1)$.

Proof. Take $\varepsilon = 1$. Let $\delta > 0$. Take $c = \min\{\delta, 1\}$. Take $x = \dfrac{c}{2}$. Then, $|x - c| < \delta$ and

$$\begin{aligned}\left|\frac{1}{x}-\frac{1}{c}\right| &= \left|\frac{2}{c}-\frac{1}{c}\right| \\ &= \frac{1}{c} \\ &\geq 1.\end{aligned}$$

Therefore, $f(x) = \dfrac{1}{x}$ is not uniformly continuous in $(0,1)$. ■

Theorem 7.33.

Let $f : [a,b] \to \mathbb{R}$. Then, f is continuous if and only if f is uniformly continuous.

Proof.

($\Longrightarrow$) Suppose f is continuous. For the sake of contradiction, suppose f is not uniformly continuous. Then, there exists $\varepsilon^* > 0$ such that for all $n \in \mathbb{N}$, there exist $x_n \in [a,b]$ and $c_n \in [a,b]$ such that $|x_n - c_n| < \dfrac{1}{n}$ and $|f(x_n) - f(c_n)| \geq \varepsilon^*$. By the Bolzano-Weierstrass Theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = x$ and there exists a subsequence $\{c_{n_k}\}$ of $\{c_n\}$ such that $\lim_{k\to\infty} c_{n_k} = c$. Then,

$$\begin{aligned}|x-c| &= \lim_{k\to\infty} |x_{n_k} - c_{n_k}| \\ &\leq \lim_{n\to\infty} \frac{1}{n} \\ \Longrightarrow \quad |x-c| &= 0.\end{aligned}$$

By assumption, f is continuous thus continuous at c. Hence, $\lim_{k\to\infty} |f(x_{n_k}) - f(c_{n_k})| = 0$. However, assumption states that $|f(x_n) - f(c_n)| \geq \varepsilon^*$ for all $n \in \mathbb{N}$, which is a contradiction.

($\Longleftarrow$) See Proposition 7.30. □

Definition 7.34 (Differentiable)**.**

Let I be an open interval, let $f : I \to \mathbb{R}$, and let $c \in I$. f is differentiable at c if $\lim_{x\to c} \dfrac{f(x)-f(c)}{x-c}$ exists.

Definition 7.35 (Derivative)**.**

Let I be an open interval, let $f : I \to \mathbb{R}$, and let $c \in I$. If f is differentiable at every $c \in I$, then the function $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) := \lim_{u\to x} \dfrac{f(u)-f(x)}{u-x}$ is called the derivative of f.

Example 7.36.

Let $f(x) = ax + b$. Then $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) = a$ for all $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$. Since $\lim_{u \to x} \dfrac{au + b - ax + b}{u - x} = a$, we obtain $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) = a$ for all $x \in \mathbb{R}$. ■

Proposition 7.37 (Power Rule)**.**

Let $f(x) = ax^n$, where $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then, $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) = nax^{n-1}$.

Proof. Let $x \in \mathbb{R}$. Since

$$\begin{aligned}
\lim_{u \to x} \frac{au^n - ax^n}{u - x} &= a \cdot \lim_{u \to x} \frac{u^n - x^n}{u - x} \\
&= a \cdot \lim_{u \to x} \frac{(u - x)(u^{n-1} + u^{n-2}x + \cdots + ux^{n-2} + x^{n-1})}{u - x} \\
&= a \cdot \lim_{u \to x} (u^{n-1} + u^{n-2}x + \cdots + ux^{n-2} + x^{n-1}) \\
&= a \cdot n \cdot x^{n-1},
\end{aligned}$$

we obtain $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) = nax^{n-1}$ for all $x \in \mathbb{R}$. □

Theorem 7.38.

If $f : I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c.

Proof. Suppose f is differentiable at c, then the limit $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists. Then,

$$\begin{aligned}
\lim_{x \to c} (f(x) - f(c)) &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} (x - c) \\
&= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot 0 \\
&= 0.
\end{aligned}$$

Thus, f is continuous at c. □

Example 7.39.

Let $f(x) = |x|$. Then, f is not differentiable at 0.

Proof. Since $\lim_{x \to 0^+} \dfrac{|x| - |0|}{x - 0} = 1$ and $\lim_{x \to 0^-} \dfrac{|x| - |0|}{x - 0} = -1$, $\lim_{x \to 0} \dfrac{|x| - |0|}{x - 0}$ does not exist by Theorem 6.27. ■

Lemma 7.40.

Let $c \in \mathbb{R}$. Then, for all $K \in \mathbb{N}$, there exists $y \in \left(c + \frac{\pi}{K}, c + \frac{\pi}{K}\right)$ such that $|\cos(Kc) - \cos(Ky)| \geq 1$.

Proof. The function $f(x) = \cos(Kx)$ is a $\frac{2\pi}{K}$-periodic function. If $\cos(Kc) \geq 0$, then we choose a y such that $\cos(Ky) = -1$; if $\cos(Kc) < 0$, then we choose a y such that $\cos(Ky) = 1$. $\square$

Lemma 7.41.

For all $a \in \mathbb{R}$, $b \in \mathbb{R}$, and $c \in \mathbb{R}$, $|a + b + c| \geq |a| - |b| - |c|$.

Proof. We apply the Triangle Inequality tiwce, having

$$\begin{aligned}
|a| &= |a + b + c + (-b) + (-c)| \\
&\leq |a + b + c| + |b + c| \\
&\leq |a + b + c| + |b| + |c| \\
\iff |a| - |b| - |c| &\leq |a + b + c|.
\end{aligned}$$

$\square$

Lemma 7.42.

The function $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{\cos(160^k x)}{4^k}$ is bounded and continuous.

Proof. Since

$$\begin{aligned}
f(x) &= \sum_{k=0}^{\infty} \frac{\cos(160^k x)}{4^k} \\
&\leq \sum_{k=0}^{\infty} \frac{1}{4^k} \\
&= \frac{4}{3},
\end{aligned}$$

f is bounded. Suppose $c \in \mathbb{R}$ and $\lim_{n\to\infty} x_n = c$. Since $\{f(x_n) - f(c)\}_n$ is bounded, $\lim_{n\to\infty} |f(x_n) - f(c)| = 0$ if and only if $\limsup_{n\to\infty} |f(x_n) - f(c)| = 0$. Let $\varepsilon > 0$. Choose $M \in \mathbb{N}$ such that $\displaystyle\sum_{k=M+1}^{\infty} \frac{1}{4^k} < \frac{\varepsilon}{2}$. Then,

$$
\begin{aligned}
\limsup_{n\to\infty} |f(x_n) - f(c)| &= \limsup_{n\to\infty} \left| \sum_{k=0}^{\infty} \frac{\cos(160^k x_n) - \cos(160^k c)}{4^k} \right| \\
&\le \limsup_{n\to\infty} \left| \sum_{k=0}^{M} \frac{\cos(160^k x_n) - \cos(160^k c)}{4^k} \right| + \limsup_{n\to\infty} \left| \sum_{k=M+1}^{\infty} \frac{\cos(160^k x_n) - \cos(160^k c)}{4^k} \right| \\
&\le \limsup_{n\to\infty} \left| \sum_{k=0}^{M} \frac{160^k x_n - 160^k c}{4^k} \right| + \varepsilon \\
&\le \limsup_{n\to\infty} \sum_{k=0}^{M} 40^k |x_n - c| + \varepsilon \\
&= \varepsilon,
\end{aligned}
$$

in which the last equation follows from the assumption $\lim_{n\to\infty} x_n = c$. Therefore, f is continuous. $\square$

Theorem 7.43.

The function $f(x) = \sum_{k=0}^{\infty} \frac{\cos(160^k x)}{4^k}$ is nowhere differentiable.

Proof. We want to construct a sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = c$ and $\left\{ \frac{f(x_n) - f(c)}{x_n - c} \right\}_n$ is unbounded. By Lemma 7.40, there exists x_n such that $\frac{\pi}{160^n} < x_n - c < \frac{3\pi}{160^n}$ and $|\cos(160^n x_n) - \cos(160^n c)| \ge 1$ for all $n \in \mathbb{N}$. Since $\frac{\pi}{160^n} < x_n - c < \frac{3\pi}{160^n}$, we know that $x_n \ne c$ and $\lim_{n\to\infty} x_n - c = 0$ by the Squeeze Theorem. Let $f_k(x) = \frac{\cos(160^k x)}{4^k}$, then $f(x) = \sum_{k=0}^{\infty} f_k(x)$. Let $n \in \mathbb{N}$. Let $a_n = f_n(x_n) - f_n(c)$, let $b_n = \sum_{k=0}^{n-1} (f_k(x_n) - f_k(c))$, and let $c_n = \sum_{k=n+1}^{\infty} (f_k(x_n) - f_k(c))$, then $|f(x_n) - f(c)| \ge |a_n| - |b_n| - |c_n|$ by Lemma 7.41. Since $|\cos(160^n x_n) - \cos(160^n c)| \ge 1$, we have $|a_n| \ge \frac{1}{4^n}$. Moreover,

$$
\begin{aligned}
|b_n| &= \sum_{k=0}^{n-1} \left| \frac{\cos(160^k x_n)}{4^k} - \frac{\cos(160^k c)}{4^k} \right| \\
&\le \sum_{k=0}^{n-1} \left| \frac{160^k x_n - 160^k c}{4^k} \right| \\
&= \sum_{k=0}^{n-1} 40^k |x_n - c| \\
&\le \sum_{k=0}^{n-1} 40^k \cdot \frac{3\pi}{160^n} \\
&= \frac{3\pi}{160^n} \cdot \frac{40^n - 1}{39} \\
&< \frac{3\pi}{160^n} \cdot \frac{40^n}{39} \\
&< \frac{4}{13} \cdot \frac{1}{4^n}.
\end{aligned}
$$

Finally, we have

$$\begin{aligned}
|c_n| &\leq \sum_{k=n+1}^{\infty} |f_k(x_n) - f_k(c)| \\
&= \sum_{k=n+1}^{\infty} \left| \frac{\cos(160^k x_n)}{4^k} - \frac{\cos(160^k c)}{4^k} \right| \\
&\leq \sum_{k=n+1}^{\infty} \left| \frac{\cos(160^k x_n)}{4^k} \right| + \left| \frac{\cos(160^k c)}{4^k} \right| \\
&\leq \sum_{k=n+1}^{\infty} 2 \cdot \frac{1}{4^k} \\
&= 2 \cdot \frac{1}{3 \cdot 4^{n+1}} \\
&< \frac{2}{3} \cdot \frac{1}{4^n}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|f(x_n) - f(c)| &\geq \left(1 - \frac{4}{13} - \frac{2}{3}\right) \frac{1}{4^n} \\
&= \frac{1}{39} \cdot \frac{1}{4^n}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{|f(x_n) - f(c)|}{|x_n - c|} &\geq \frac{\frac{1}{39} \cdot \frac{1}{4^n}}{\frac{3\pi}{160^n}} \\
&= \frac{1}{39} \cdot \frac{1}{4^n} \cdot \frac{160^n}{3\pi} \\
&= \frac{40^n}{117\pi},
\end{aligned}$$

which implies $\left\{ \frac{f(x_n) - f(c)}{x_n - c} \right\}_n$ is unbounded. $\square$

8 Differentiation Rules, Rolle's Theorem, and Mean Value Theorem

Theorem 8.1.

Let $f : I \to \mathbb{R}$ and let $g : I \to \mathbb{R}$ be two functions which are differentiable at $c \in I$. Then,

1. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\alpha f + g)(c) = \alpha \cdot \dfrac{\mathrm{d}f}{\mathrm{d}x}(c) + \dfrac{\mathrm{d}g}{\mathrm{d}x}(c)$;
2. $\dfrac{\mathrm{d}}{\mathrm{d}x}(f \cdot g)(c) = \dfrac{\mathrm{d}f}{\mathrm{d}x}(c) \cdot g(c) + f(c) \cdot \dfrac{\mathrm{d}g}{\mathrm{d}x}(c)$;
3. if $g(x) \neq 0$ for all $x \in I$, then $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{f}{g}\right)(c) = \dfrac{\dfrac{\mathrm{d}f}{\mathrm{d}x}(c) \cdot g(c) - f(c) \cdot \dfrac{\mathrm{d}g}{\mathrm{d}x}(c)}{(g(x))^2}$.

Proof.

1. We compute directly, having

$$\begin{aligned}
\lim_{x \to c} \frac{(\alpha f + g)(x) - (\alpha f + g)(c)}{x - c} &= \lim_{x \to c} \frac{\alpha f(x) + g(x) - (\alpha f(c) + g(c))}{x - c} \\
&= \lim_{x \to c} \frac{\alpha f(x) - \alpha f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\
&= \alpha \cdot \frac{\mathrm{d}f}{\mathrm{d}x}(c) + \frac{\mathrm{d}g}{\mathrm{d}x}(c).
\end{aligned}$$

2. We compute directly, having

$$\begin{aligned}
\lim_{x \to c} \frac{(f \cdot g)(x) - (f \cdot g)(c)}{x - c} &= \lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x - c} \\
&= \lim_{x \to c} \frac{f(x)g(x) + f(c)g(x) - f(c)g(x) - f(c)g(c)}{x - c} \\
&= \lim_{x \to c} \frac{f(x)g(x) - f(c)g(x)}{x - c} + \lim_{x \to c} \frac{f(c)g(x) - f(c)g(c)}{x - c} \\
&= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c} g(x) + f(c) \cdot \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\
&= \frac{\mathrm{d}f}{\mathrm{d}x}(c) \cdot g(c) + f(c) \cdot \frac{\mathrm{d}g}{\mathrm{d}x}(c).
\end{aligned}$$

3. Let $f(x) = g(x)h(x)$, where $h : I \to \mathbb{R}$ is differentiable at $c \in I$. Then, by Theorem 8.1.2

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}x}(c) &= \frac{\mathrm{d}g}{\mathrm{d}x}(c) \cdot h(c) + g(c) \cdot \frac{\mathrm{d}h}{\mathrm{d}x}(c) \\
&= \frac{\mathrm{d}g}{\mathrm{d}x}(c) \cdot \frac{f(c)}{g(c)} + g(c) \cdot \frac{\mathrm{d}h}{\mathrm{d}x}(c) \\
\iff \qquad \frac{\dfrac{\mathrm{d}f}{\mathrm{d}x}(c) - \dfrac{\mathrm{d}g}{\mathrm{d}x}(c) \cdot \dfrac{f(c)}{g(c)}}{g(c)} &= \frac{\mathrm{d}h}{\mathrm{d}x}(c) \\
\iff \quad \frac{\dfrac{\mathrm{d}f}{\mathrm{d}x}(c) \cdot g(c) - f(c) \cdot \dfrac{\mathrm{d}g}{\mathrm{d}x}(c)}{(g(c))^2} &= \frac{\mathrm{d}h}{\mathrm{d}x}(c).
\end{aligned}$$

□

Theorem 8.2 (Chain Rule)**.**

Let I_1 and I_2 be two intervals, let $g : I_1 \to I_2$ be differentiable at $c \in I_1$, and let $f : I_2 \to \mathbb{R}$ be differentiable at $g(c)$. Then, $f \circ g : I_1 \to \mathbb{R}$ is differnetiable at c and $\dfrac{\mathrm{d}}{\mathrm{d}x}(f \circ g)(c) = \dfrac{\mathrm{d}f}{\mathrm{d}x}(g(c)) \cdot \dfrac{\mathrm{d}g}{\mathrm{d}x}(c)$.

Proof. Let $h(x) = f(g(x))$ and let $d = g(c)$. We want to show $\dfrac{\mathrm{d}}{\mathrm{d}x}(f \circ g)(c) = \dfrac{\mathrm{d}f}{\mathrm{d}x}(d) \cdot \dfrac{\mathrm{d}g}{\mathrm{d}x}(c)$. Define $u(x) = \begin{cases} \dfrac{f(x) - f(d)}{x - d} & \text{if } x \neq d \\ \dfrac{\mathrm{d}f}{\mathrm{d}x}(d) & \text{if } x = d \end{cases}$ and $v(x) = \begin{cases} \dfrac{g(x) - g(c)}{x - c} & \text{if } x \neq c \\ \dfrac{\mathrm{d}g}{\mathrm{d}x}(c) & \text{if } x = c \end{cases}$. We can check by definition that $u(x)$ is continuous at d and $v(x)$ is continuous at c. If $x \in I_2 \setminus \{c\}$, then

$$
\begin{aligned}
h(x) - h(c) &= f(g(x)) - f(d) \\
&= u(g(x)) \cdot (g(x) - d) \\
&= u(g(x)) \cdot (g(x) - g(c)) \\
&= u(g(x)) \cdot v(c) \cdot (x - c).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\lim_{x \to c} \frac{h(x) - h(c)}{x - c} &= \lim_{x \to c} u(g(x)) \cdot v(c) \\
&= \frac{\mathrm{d}f}{\mathrm{d}x}(d) \cdot \frac{\mathrm{d}g}{\mathrm{d}x}(c) \\
&= \frac{\mathrm{d}f}{\mathrm{d}x}(f(c)) \cdot \frac{\mathrm{d}g}{\mathrm{d}x}(c).
\end{aligned}
$$

$\square$

Definition 8.3 (Relative Maximum)**.**

Let $S \subseteq \mathbb{R}$ and let $f : S \to \mathbb{R}$. Then, f has a <u>relative maximum</u> at $c \in S$ if there exists $\delta > 0$ such that $x \in (c - \delta, c + \delta) \cap S \implies f(x) \leq f(c)$.

Definition 8.4 (Relative Minimum)**.**

Let $S \subseteq \mathbb{R}$ and let $f : S \to \mathbb{R}$. Then, f has a <u>relative minimum</u> at $d \in S$ if there exists $\delta > 0$ such that $x \in (d - \delta, d + \delta) \cap S \implies f(d) \leq f(x)$.

Proposition 8.5.

If $f : [a,b] \to \mathbb{R}$ has a relative extremum at $c \in (a,b)$ and f is differentiable at c, then $\dfrac{\mathrm{d}f}{\mathrm{d}x}(c) = 0$.

Proof. Suppose $f : [a,b] \to \mathbb{R}$ is differnetiable at $c \in (a,b)$ and has a relative maximum at c. Then, there exists $\delta_1 > 0$ such that $x \in (c - \delta_1, c + \delta_1) \implies f(c) \geq f(x)$. Let $x_n = c - \dfrac{\delta_1}{2n} \in (c - \delta_1, c)$, then

$$\begin{aligned}\lim_{x \to c} \frac{f(x) - f(c)}{x - c} &= \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} \\ &\geq 0.\end{aligned}$$

Let $x_n = c + \dfrac{\delta_1}{2n} \in (c, c + \delta_1)$, then

$$\begin{aligned}\lim_{x \to c} \frac{f(x) - f(c)}{x - c} &= \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} \\ &\leq 0.\end{aligned}$$

Thus, $\displaystyle\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0$. Now suppose f is differnetiable at $d \in (a,b)$ and has a relative minimum at d. Then, there exists $\delta_2 > 0$ such that $x \in (d - \delta_2, d + \delta_2) \implies f(d) \leq f(x)$. Let $x_n = c - \dfrac{\delta_2}{2n} \in (d - \delta_2, d)$, then

$$\begin{aligned}\lim_{x \to d} \frac{f(x) - f(d)}{x - d} &= \lim_{n \to \infty} \frac{f(x_n) - f(d)}{x_n - d} \\ &\leq 0.\end{aligned}$$

Let $x_n = d + \dfrac{\delta_2}{2n} \in (d, d + \delta_2)$, then

$$\begin{aligned}\lim_{x \to d} \frac{f(x) - f(d)}{x - d} &= \lim_{n \to \infty} \frac{f(x_n) - f(d)}{x_n - d} \\ &\geq 0.\end{aligned}$$

Thus, $\displaystyle\lim_{x \to d} \frac{f(x) - f(d)}{x - d} = 0$. □

Theorem 8.6 (Rolle's Theorem)**.**

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b). If $f(a) = f(b)$, then there exists $c \in (a,b)$ such that $\dfrac{\mathrm{d}f}{\mathrm{d}x}(c) = 0$.

Proof. Let $K = f(a)$. By the Extreme Value Theorem, there exists $c \in [a,b]$ such that $f(c) \geq f(x)$ for all $x \in [a,b]$ and there exists $d \in [a,b]$ such that $f(d) \leq f(x)$ for all $x \in [a,b]$. If $f(c) > K$, then $c \in (a,b)$, which implies $\dfrac{\mathrm{d}f}{\mathrm{d}x}(c) = 0$ by Proposition 8.5. If $f(d) < K$, then $d \in (a,b)$, which implies $\dfrac{\mathrm{d}f}{\mathrm{d}x}(d) = 0$ by Proposition 8.5. If $f(c) \leq K \leq f(d)$, then $f(x) = K$ for all $x \in [a,b]$. Thus, $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) = 0$ for all $x \in [a,b]$. □

Theorem 8.7 (Mean Value Theorem)**.**

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in (a, b). Then, there exists $c \in (a, b)$ such that $\dfrac{\mathrm{d}f}{\mathrm{d}x}(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof. Define $g : [a, b] \to \mathbb{R}$ with $g(x) = f(x) - f(b) + \dfrac{f(b) - f(a)}{b - a} \cdot (b - x)$. Thus, $g(a) = g(b)$. By the Rolle's Theorem, there exists $c \in (a, b)$ such that $\dfrac{\mathrm{d}g}{\mathrm{d}x}(c) = 0$. Hence,

$$\begin{aligned}
\frac{\mathrm{d}g}{\mathrm{d}x}(c) &= \frac{\mathrm{d}f}{\mathrm{d}x}(c) + \frac{f(b) - f(a)}{b - a} \cdot (-1) \\
0 &= \frac{\mathrm{d}f}{\mathrm{d}x}(c) - \frac{f(b) - f(a)}{b - a} \\
\frac{\mathrm{d}f}{\mathrm{d}x}(c) &= \frac{f(b) - f(a)}{b - a}.
\end{aligned}$$

Therefore, there exists $c \in (a, b)$ such that $\dfrac{\mathrm{d}f}{\mathrm{d}x}(c) = \dfrac{f(b) - f(a)}{b - a}$. □

Proposition 8.8.

If $f : I \to \mathbb{R}$ is differentiable and $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) = 0$ for all $x \in I$, then f is a constant.

Proof. Let $a < b$, where $a \in I$ and $b \in I$. Then, since f is continuous on $[a, b]$ and differentiable in (a, b), by the Mean Value Theorem, there exists $c \in (a, b)$ such that $\dfrac{\mathrm{d}f}{\mathrm{d}x}(c) = \dfrac{f(b) - f(a)}{b - a} \iff 0 = f(b) - f(a)$. Therefore, $f(a) = f(b)$ for all $a \in I$ and $b \in I$ such that $a < b$. □

Theorem 8.9.

Let $f : I \to \mathbb{R}$ be differnetiable. Then,

1. f is increasing if and only if $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) \geq 0$ for all $x \in I$;
2. f is decreasing if and only if $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) \leq 0$ for all $x \in I$.

Proof.

1. $(\implies)$ Suppose f is increasing, then $x_1 > x_2$ implies $f(x_1) \geq f(x_2)$ for all $x_1 \in I$ and $x_2 \in I$. Let $c \in I$. On the one hand, there exists $\delta_1 > 0$ such that $x \in (c, c + \delta_1) \subseteq I$ implies $\dfrac{f(x) - f(c)}{x - c} \geq 0$. Thus, $\displaystyle\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \geq 0$. On the other hand, there exists $\delta_2 > 0$ such that $\dfrac{f(x) - f(c)}{x - c} \geq 0$ for all $x \in (c - \delta_2, c) \subseteq I$. Thus, $\displaystyle\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$.

($\Longleftarrow$) Suppose $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) \geq 0$ for all $x \in I$, then for $a \in I$ and $b \in I$ such that $a < b$, there exists $c \in I$ such that $\dfrac{f(b) - f(a)}{b - a} = \dfrac{\mathrm{d}f}{\mathrm{d}x}(c)$. Since $\dfrac{\mathrm{d}f}{\mathrm{d}x}(x) \geq 0$ for all $x \in I$ and $c \in I$, we obtain either $f(b) \geq f(a)$ if $b > a$ or $f(b) \leq f(a)$ if $b < a$.

2. Notice that f is increasing if and only if $-f$ is decreasing. Appling Theorem 8.9 to $-f$ completes the proof. □

Notation 8.10.

$\dfrac{\mathrm{d}^n f}{\mathrm{d}x^n}$ denotes the n-th derivative of f.

Definition 8.11 (n-Times Differentiable)**.**

A function $f : I \to \mathbb{R}$ is <u>n-times differentiable</u> on $J \subseteq I$ if all of $\dfrac{\mathrm{d}f}{\mathrm{d}x}$, $\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2}$, ..., and $\dfrac{\mathrm{d}^n f}{\mathrm{d}x^n}$ exist in J.

Remark 8.12.

The following Taylor's Theorem is essentially the Mean Value Theorem for higher order derivatives.

Definition 8.13 (Taylor Polynomial)**.**

In the Taylor's Theorem, $P_n(x) = \displaystyle\sum_{k=1}^{n} \frac{\frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x_0)}{k!}(x - x_0)^k$ is the n-th <u>Taylor polynomial</u> of f at x_0.

Definition 8.14 (Remainder Term)**.**

In the Taylor's Theorem, $R_n(x) = \dfrac{\frac{\mathrm{d}^{n+1} f}{\mathrm{d}x^{n+1}}(c)}{(n+1)!}(x - x_0)^{n+1}$ is the n-th <u>remainder term</u> of f at x_0.

Theorem 8.15 (Taylor's Theorem)**.**

Suppose $f : [a, b] \to \mathbb{R}$ is continuous and has n continuous derivatives on [a,b] such that $\dfrac{\mathrm{d}^{n+1} f}{\mathrm{d}x^{n+1}}$ exist in (a, b). Fix an arbitrary point $x_0 \in [a, b]$. Then, for all $x \in [a, b]$ such that $x > x_0$, there exists $c \in (x_0, x)$ such that $f(x) = \displaystyle\sum_{k=1}^{n} \frac{\frac{\mathrm{d}^k f}{\mathrm{d}x^k}(x_0)}{k!}(x - x_0)^k + \frac{\frac{\mathrm{d}^{n+1} f}{\mathrm{d}x^{n+1}}(c)}{(n+1)!}(x - x_0)^{n+1}$.

Proof. We apply the Mean Value Theorem $n+1$ times to prove this theorem. Suppose $x > x_0$ such that $x \in [a,b]$ and let $M(x,x_0) = \dfrac{f(x) - P_n(x)}{(x-x^0)^{n+1}}$, then $f(x) = P_n(x) + M(x,x_0)\cdot(x-x_0)^{n+1}$. For $k \in \{0,1,2,\dots,n\}$, we have $\dfrac{\mathrm{d}^k f}{\mathrm{d}x^k}(x_0) = \dfrac{\mathrm{d}^k P_n}{\mathrm{d}x^k}(x_0)$. Let $g(s) = f(s) - P_n(s) - M(x,x_0)\cdot(s-x_0)^{n+1}$, which is $n+1$-times differentiable. Then,

$$\begin{aligned} g(x_0) &= f(x_0) - P_n(x_n) - M(x,x_0)\cdot(x_0-x_0)^{n+1} \\ &= 0; \end{aligned} \tag{8.15.1}$$

$$\begin{aligned} \frac{\mathrm{d}g}{\mathrm{d}s}(x_0) &= \frac{\mathrm{d}f}{\mathrm{d}s}(x_0) - \frac{\mathrm{d}P_n}{\mathrm{d}s}(x_0) - M(x,x_0)\cdot(n+1)(x_0-x_0)^n \\ &= 0; \end{aligned} \tag{8.15.2}$$

$$\vdots$$

$$\begin{aligned} \frac{\mathrm{d}^n g}{\mathrm{d}s^n}(x_0) &= \frac{\mathrm{d}^n f}{\mathrm{d}s^n}(x_0) - \frac{\mathrm{d}^n P_n}{\mathrm{d}s^n}(x_0) - M(x,x_0)\cdot(n+1)!(x_0-x_0) \\ &= 0. \end{aligned} \tag{8.15.3}$$

Notice that $g(x) = 0$ and $g(x_0) = 0$ by equation 8.15.1. Thus, by the Mean Value Theorem, there exists $x_1 \in (x_0, x)$ such that $\dfrac{\mathrm{d}g}{\mathrm{d}s}(x_1) = 0$. Since $\dfrac{\mathrm{d}g}{\mathrm{d}s}(x_1) = 0$ and $\dfrac{\mathrm{d}g}{\mathrm{d}s}(x_0) = 0$ by equation 8.15.2, by the Mean Value Theorem, there exists $x_2 \in (x_0, x_1)$ such that $\dfrac{\mathrm{d}^2 g}{\mathrm{d}s^2}(x_2) = 0$. Continuing in this fashion, we can find that there exists $x_k \in (x_0, x_{k-1})$ such that $\dfrac{\mathrm{d}^k g}{\mathrm{d}s^k}(x_k) = 0$ for $k \in \{0,1,2,\dots,n+1\}$. Let $c = x_{n+1}$, then we obtain

$$\begin{aligned} \frac{\mathrm{d}^{n+1} g}{\mathrm{d}s^{n+1}}(c) &= \frac{\mathrm{d}^{n+1} f}{\mathrm{d}s^{n+1}}(c) - \frac{\mathrm{d}^{n+1} P_n}{\mathrm{d}s^{n+1}}(c) - M(x,x_0)\cdot(n+1)!(c-x_0)^0 \\ 0 &= \frac{\mathrm{d}^{n+1} f}{\mathrm{d}s^{n+1}}(c) - M(x,x_0)\cdot(n+1)! \\ \iff \quad \frac{\mathrm{d}^{n+1} f}{\mathrm{d}s^{n+1}}(c) &= M(x,x_0)\cdot(n+1)! \\ \iff \quad M(x,x_0) &= \frac{\frac{\mathrm{d}^{n+1} f}{\mathrm{d}s^{n+1}}(c)}{(n+1)!}, \end{aligned}$$

thus $f(x) = P_n(x) + \dfrac{\frac{\mathrm{d}^{n+1} f}{\mathrm{d}x^{n+1}}(c)}{(n+1)!}(x-x_0)^{n+1}$. □

Theorem 8.16 (Second Derivative Test)**.**

Suppose $f : (a, b) \to \mathbb{R}$ is twice differentiable and the two derivatives are continuous. If $x_0 \in (a, b)$ such that $\frac{\mathrm{d}f}{\mathrm{d}x}(x_0) = 0$ and $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x_0) > 0$, then f has a strict relative minimum at x_0.

Proof. Since $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ is continuous at x_0 and $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x_0) > 0$, we have $\lim_{x \to x_0} \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) > 0$. Then, there exists $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta)$ implies $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) > 0$. Let $x \in (x_0 - \delta, x_0 + \delta)$. By the Taylor's Theorem, there exists c between x and x_0 such that

$$\begin{aligned} f(x) &= f(x_0) + \frac{\frac{\mathrm{d}f}{\mathrm{d}x}(x_0)}{1!} \cdot (x - x_0) + \frac{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(c)}{2!} \cdot (x - x_0)^2 \\ &= f(x_0) + \frac{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(c)}{2!} \cdot (x - x_0)^2 \\ &\geq f(x_0). \end{aligned}$$

Therefore, if $x \neq x_0$, then $f(x) > f(x_0)$. □

Theorem 8.17 (Second Derivative Test)**.**

Suppose $f : (a, b) \to \mathbb{R}$ is twice differentiable and the two derivatives are continuous. If $x_0 \in (a, b)$ such that $\frac{\mathrm{d}f}{\mathrm{d}x}(x_0) = 0$ and $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x_0) < 0$, then f has a strict relative maximum at x_0.

Proof. Since $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$ is continuous at x_0 and $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x_0) < 0$, we have $\lim_{x \to x_0} \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) < 0$. Then, there exists $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta)$ implies $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x) < 0$. Let $x \in (x_0 - \delta, x_0 + \delta)$. By the Taylor's Theorem, there exists c between x and x_0 such that

$$\begin{aligned} f(x) &= f(x_0) + \frac{\frac{\mathrm{d}f}{\mathrm{d}x}(x_0)}{1!} \cdot (x - x_0) + \frac{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(c)}{2!} \cdot (x - x_0)^2 \\ &= f(x_0) + \frac{\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(c)}{2!} \cdot (x - x_0)^2 \\ &\leq f(x_0). \end{aligned}$$

Therefore, if $x \neq x_0$, then $f(x) < f(x_0)$. □

9 Riemann Sums, Integration, and Fundamental Theorem of Calculus

Definition 9.1.

We define the set $C([a,b]) \coloneqq \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$.

Definition 9.2 (Partition)**.**

A partition of $[a,b]$ is a finite set $\underline{x} = \{x_i \in [a,b] \mid a = x_0 < x_1 < x_2 < \cdots < x_n = b\}_{i=0}^{n}$.

Definition 9.3 (Norm)**.**

The norm of $\underline{x}$ is the number $\|\underline{x}\| = \max\{x_i - x_{i-1} \mid i \in \{1, 2, \ldots, n\}\}$.

Definition 9.4 (Tag)**.**

A tag of $\underline{x}$ is a finite set $\underline{\xi} = \{\xi_i \mid i \in \{1, 2, \ldots, n\}\}$ such that $x_{i-1} \leq \xi_i \leq x_i$ for $i \in \{1, 2, \ldots, n\}$. The pair containing two sets $(\underline{x}, \underline{\xi})$ is referred to as a tagged partition.

Definition 9.5 (Riemann Sum)**.**

The Riemann sum of f corresponding to $(\underline{x}, \underline{\xi})$ is the number $S_f(\underline{x}, \underline{\xi}) \coloneqq \displaystyle\sum_{k=1}^{n} f(\xi_k) \cdot (x_k - x_{k-1})$, where $n = |\underline{x}| - 1$, $|\underline{x}|$ is the cardinality of $\underline{x}$.

Definition 9.6 (Modulus of Continuity)**.**

The modulus of continuity of $f \in C([a,b])$ is defined by $w_f(\eta) = \sup\{|f(x) - f(y)| \mid |x - y| \leq \eta\}$, where $\eta > 0$.

Lemma 9.7.

For any $f \in C([a,b])$, we have $\lim\limits_{\eta \to 0} w_f(\eta) = 0$.

Proof. Since f is continuous on $[a,b]$, f is uniformly continuous on $[a,b]$ by Theorem 7.33. Thus, there exists $\delta > 0$ such that $|x_1 - x_2| < \delta \implies |f(x_1) - f(x_2)| < \varepsilon$. Let $0 < \eta < \delta$. If $|x_1 - x_2| < \eta$, then $|f(x_1) - f(x_2)| < \varepsilon$.

Thus, ε is an upper bound for $\{|f(x)-f(y)| \mid |x-y| \leq \eta\}$. Therefore, $w_f(\eta) < \varepsilon$, which is equivalent to $\lim_{\eta\to 0} w_f(\eta) = 0$. □

Definition 9.8 (Refinement)**.**

$\underline{x}'$ is a <u>refinement</u> of $\underline{x}$ if $(\underline{x}, \underline{\xi})$ and $(\underline{x}', \underline{\xi}')$ are tagged partitions of $[a,b]$ such that $\underline{x}' \subseteq \underline{x}$.

Remark 9.9.

Refinements of $\underline{x}$ are obtained by adding more partition points.

Lemma 9.10.

If $\underline{x}'$ is a refinement of $\underline{x}$ and $f \in C([a,b])$, then $\left|S_f(\underline{x},\underline{\xi}) - S_f(\underline{x}',\underline{\xi}')\right| \leq w_f(\|\underline{x}\|)\cdot(b-a)$.

Proof. Let $n = |\underline{x}| - 1$. For $k \in \{1,2,\ldots,n\}$, let $\underline{y}(k)$ be a partition of $[x_{k-1}, x_k]$ and let $\underline{\eta}(k)$ be a tag of $\underline{y}(k)$. More specifically, $\underline{y}(k) = \{x_i' \in [x_{k-1},x_k] \mid x_{k-1} = x_0' < x_1' < x_2' < \cdots < x_m' = x_k\}$ and $\underline{\eta}(k) = \{\xi_i' \mid i \in \{1,2,\ldots,n\}\}$ such that $x_{i-1}' \leq \xi_i' \leq x_i'$ for $i \in \{1,2,\ldots,m\}$. Since $\displaystyle\sum_{\ell=1}^{m}\left(x_\ell' - x_{\ell-1}'\right) = x_k - x_{k-1}$, we have

$$\begin{aligned}\left|f(\xi_k)\cdot(x_k-x_{k-1}) - S_f\left(\underline{y}(k),\underline{\eta}(k)\right)\right| &= \left|f(\xi_k)\cdot(x_k-x_{k-1}) - \sum_{\ell=1}^{m} f(\xi_\ell')\cdot(x_\ell'-x_{\ell-1}')\right| \\ &= \left|\sum_{\ell=1}^{m}\left(f(\xi_k)-f(\xi_\ell')\right)\cdot(x_\ell'-x_{\ell-1}')\right|.\end{aligned}$$

By Lemma 5.18,

$$\begin{aligned}\left|f(\xi_k)\cdot(x_k-x_{k-1}) - S_f\left(\underline{y}(k),\underline{\eta}(k)\right)\right| &\leq \left|\sum_{\ell=1}^{m}\left(f(\xi_k)-f(\xi_\ell')\right)\cdot(x_\ell'-x_{\ell-1}')\right| \\ &= \sum_{\ell=1}^{m}\left|\left(f(\xi_k)-f(\xi_\ell')\right)\cdot(x_\ell'-x_{\ell-1}')\right|.\end{aligned}$$

By Definition 9.6,

$$\begin{aligned}\left|f(\xi_k)\cdot(x_k-x_{k-1}) - S_f\left(\underline{y}(k),\underline{\eta}(k)\right)\right| &\leq \sum_{\ell=1}^{m}\left|\left(f(\xi_k)-f(\xi_\ell')\right)\cdot(x_\ell'-x_{\ell-1}')\right| \\ &= \sum_{\ell=1}^{m}\left|f(\xi_k)-f(\xi_\ell')\right|\cdot(x_\ell'-x_{\ell-1}') \\ &\leq \sum_{\ell=1}^{m} w_f(|x_k-x_{k-1}|)\cdot(x_\ell'-x_{\ell-1}') \\ &= w_f(|x_k-x_{k-1}|)\cdot\left(\sum_{\ell=1}^{m}(x_\ell'-x_{\ell-1}')\right).\end{aligned}$$

By Definition 9.3,

$$\begin{aligned}\left|f(\xi_k)\cdot(x_k-x_{k-1})-S_f\left(\underline{y}(k),\underline{\eta}(k)\right)\right| &\le w_f(|x_k-x_{k-1}|)\cdot\left(\sum_{\ell=1}^{m}(x'_\ell-x'_{\ell-1})\right)\\ &\le w_f(\|\underline{x}\|)\cdot\left(\sum_{\ell=1}^{m}(x'_\ell-x'_{\ell-1})\right)\\ &= w_f(\|\underline{x}\|)\cdot(x_k-x_{k-1})\,.\end{aligned}$$

Thus,

$$\begin{aligned}\left|S_f(\underline{x},\underline{\xi})-S_f(\underline{x}',\underline{\xi}')\right| &= \left|\sum_{k=1}^{n}\left(f(\xi_k)\cdot(x_k-x_{k-1})-S_f\left(\underline{y}(k),\underline{\eta}(k)\right)\right)\right|\\ &\le \left|\sum_{k=1}^{n}\left(w_f(\|\underline{x}\|)\cdot(x_k-x_{k-1})\right)\right|\\ &= w_f(\|\underline{x}\|)\cdot\left|\sum_{k=1}^{n}(x_k-x_{k-1})\right|\\ &= w_f(\|\underline{x}\|)\cdot(x_n-x_0)\\ &= w_f(\|\underline{x}\|)\cdot(b-a)\,.\end{aligned}$$

□

Lemma 9.11.

If $(\underline{x},\underline{\xi})$ and $(\underline{x}',\underline{\xi}')$ are any two tagged partitions of $[a,b]$ and $f\in C([a,b])$, then

$$\left|S_f(\underline{x},\underline{\xi})-S_f(\underline{x}',\underline{\xi}')\right|\le\left(w_f(\|\underline{x}\|)+w_f(\|\underline{x}'\|)\right)\cdot(b-a).$$

Proof. Let $\underline{x}''=\underline{x}\cup\underline{x}'$, i.e., a common refinement, and let $\underline{\xi}''$ be a tag of $\underline{x}''$. Then, by the Triangle Inequality and Lemma 9.10,

$$\begin{aligned}\left|S_f(\underline{x},\underline{\xi})-S_f(\underline{x}',\underline{\xi}')\right| &\le \left|S_f(\underline{x},\underline{\xi})-S_f(\underline{x}'',\underline{\xi}'')\right|+\left|S_f(\underline{x}',\underline{\xi}')-S_f(\underline{x}'',\underline{\xi}'')\right|\\ &\le w_f(\|\underline{x}\|)(b-a)+w_f(\|\underline{x}\|)(b-a).\end{aligned}$$

□

Theorem 9.12 (Riemann Integral)**.**

Let $f\in C([a,b])$. Then, there exists a unique number denoted $\displaystyle\int_a^b f(x)\,\mathrm{d}x\in\mathbb{R}$ such that for all sequences of tagged partitions $\{(\underline{x}_r,\underline{\xi}_r)\}_r$ such that $\lim\limits_{r\to\infty}\|\underline{x}_r\|=0$, we have $\lim\limits_{r\to\infty}S_f(\underline{x}_r,\underline{\xi}_r)=\displaystyle\int_a^b f(x)\,\mathrm{d}x$.

Proof.

(Existence) We have to show $\{S_f(\underline{x}_r,\underline{\xi}_r)\}_r$ converges for all sequences of partitions with $\lim\limits_{r\to\infty}\|\underline{x}_r\|=0$. Let $\{(\underline{x}_r,\underline{\xi}_r)\}_r$ be a sequence of tagged partitions with $\lim\limits_{r\to\infty}\|\underline{x}_r\|=0$. We want to show $\{S_f(\underline{x}_r,\underline{\xi}_r)\}_r$ is Cauchy, then it is convergent by Theorem 4.16. Let $\varepsilon > 0$. By Lemma 9.7, since $f \in C([a,b])$, we have $\lim_{\eta \to 0} w_f(\eta) = 0$. By Definition 6.9, there exists $\delta > 0$ such that $w_f(\eta) < \dfrac{\varepsilon}{2(b-a)}$ for all $\eta < \delta$. Moreover, since $\lim_{r \to \infty} \|\underline{x}_r\| = 0$, by Definition 6.11, there exists $N \in \mathbb{N}$ such that $n > N$ implies $\|\underline{x}_n\| < \delta$. For any two integers $r > N$ and $s > N$, by Lemma 9.11, we have

$$\begin{aligned}
\left| S_f(\underline{x}_r, \underline{\xi}_r) - S_f(\underline{x}_s, \underline{\xi}_s) \right| &\le (w_f \|\underline{x}_r\| + w_f \|\underline{x}_s\|) \cdot (b-a) \\
&< \left(\frac{\varepsilon}{2(b-a)} + \frac{\varepsilon}{2(b-a)} \right) \cdot (b-a) \\
&= \varepsilon.
\end{aligned}$$

Thus, $\{(\underline{x}_r, \underline{\xi}_r)\}_r$ is Cauchy and thus convergent by Theorem 4.16.

(Uniqueness) The uniqueness of $\lim_{r \to \infty} S_f(\underline{x}_r, \underline{\xi}_r)$ follows from Theorem 3.6 immediately. □

Theorem 9.13 (Linearity)**.**

If $f \in C([a,b])$, $g \in C([a,b])$, and $\alpha \in \mathbb{R}$, then $\displaystyle\int_a^b (\alpha f + g)(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$.

Proof. Let $\{(\underline{x}_r, \underline{\xi}_r)\}_r$ be a sequence of tagged partitions of $[a,b]$ such that $\lim_{r \to \infty} \|\underline{x}_r\| = 0$. Then,

$$\begin{aligned}
\int_a^b (\alpha f + g)(x)\,\mathrm{d}x &= \lim_{r \to \infty} S_{\alpha f + g}(\underline{x}_r, \underline{\xi}_r) \\
&= \lim_{r \to \infty} \Big(S_f(\underline{x}_r, \underline{\xi}_r) + S_g(\underline{x}_r, \underline{\xi}_r) \Big) \\
&= \alpha \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x.
\end{aligned}$$

□

Theorem 9.14 (Additivity)**.**

If $f \in C([a,b])$ and $a < c < b$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$.

Proof. Let $\{(\underline{x}_r^1, \underline{\xi}_r^1)\}_r$ be a sequence of tagged partitions of $[a,c]$ such that $\lim_{r \to \infty} \left\|\underline{x}_r^1\right\| = 0$ and let $\{(\underline{x}_r^2, \underline{\xi}_r^2)\}_r$ be a sequence of tagged partitions of $[c,b]$ such that $\lim_{r \to \infty} \left\|\underline{x}_r^2\right\| = 0$. Define $\underline{x}_r = \underline{x}_r^1 \cup \underline{x}_r^2$ and $\underline{\xi}_r = \underline{\xi}_r^1 \cup \underline{\xi}_r^2$, then $\{(\underline{x}_r, \underline{\xi}_r)\}_r$ a sequence of tagged partitions of $[a,b]$. Since $0 \le \|\underline{x}_r\| \le \left\|\underline{x}_r^1\right\| + \left\|\underline{x}_r^2\right\|$, $\lim_{r \to \infty} \left\|\underline{x}_r^1\right\| = 0$, and $\lim_{r \to \infty} \left\|\underline{x}_r^2\right\| = 0$, we have $\lim_{r \to \infty} \|\underline{x}_r\| = 0$ by the Squeeze Theorem. Thus,

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x &= \lim_{r \to \infty} S_f(\underline{x}_r, \underline{\xi}_r) \\
&= \lim_{r \to \infty} \Big(S_f(\underline{x}_r^1, \underline{\xi}_r^1) + S_f(\underline{x}_r^2, \underline{\xi}_r^2) \Big) \\
&= \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.
\end{aligned}$$

□

Proposition 9.15.

Let $f \in C([a,b])$, let $m_f = \min\{f(x) \mid x \in [a,b]\}$, and let $M_f = \max\{f(x) \mid x \in [a,b]\}$. Then,

$$m_f \cdot (b-a) \le \int_a^b f(x)\,\mathrm{d}x \le M_f \cdot (b-a).$$

Proof. Let $\{(\underline{x}_r, \underline{\xi}_r)\}_r$ be a sequence of tagged partitions of $[a,b]$ such that $\lim\limits_{r\to\infty} \|\underline{x}_r\| = 0$. Then,

$$\begin{aligned} S_f(\underline{x}_r, \underline{\xi}_r) &= \sum_{k=1}^{n} f(\xi_{k_r}) \cdot (\xi_{k_r} - \xi_{k-1_r}) \\ &\ge \sum_{k=1}^{n} m_f \cdot (\xi_{k_r} - \xi_{k-1_r}) \\ &= m_f \cdot (b-a) \end{aligned} \tag{9.15.1}$$

$$\begin{aligned} S_f(\underline{x}_r, \underline{\xi}_r) &= \sum_{k=1}^{n} f(\xi_{k_r}) \cdot (\xi_{k_r} - \xi_{k-1_r}) \\ &\le \sum_{k=1}^{n} M_f \cdot (\xi_{k_r} - \xi_{k-1_r}) \\ &= M_f \cdot (b-a) \end{aligned} \tag{9.15.2}$$

Thus, by equation 9.15.1 and equation 9.15.2, for all $r \in \mathbb{N}$, $m_f \cdot (b-a) \le S_f(\underline{x}_r, \underline{\xi}_r) \le M_f \cdot (b-a)$. Therefore, $m_f \cdot (b-a) \le \displaystyle\int_a^b f(x)\,\mathrm{d}x \le M_f \cdot (b-a)$. □

Proposition 9.16.

Let $f \in C([a,b])$ and let $g \in C([a,b])$. If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\displaystyle\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x$.

Proof. Let $\{(\underline{x}_r, \underline{\xi}_r)\}_r$ be a sequence of tagged partitions of $[a,b]$ such that $\lim\limits_{r\to\infty} \|\underline{x}_r\| = 0$. Then, for all $r \in \mathbb{N}$,

$$\begin{aligned} S_f(\underline{x}_r, \underline{\xi}_r) &= \sum_{k=1}^{n} f(\xi_{k_r}) \cdot (x_{k_r} - x_{k-1_r}) \\ &\le \sum_{k=1}^{n} g(\xi_{k_r}) \cdot (x_{k_r} - x_{k-1_r}) \\ &= S_g(\underline{x}_r, \underline{\xi}_r). \end{aligned}$$

Thus, $\displaystyle\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x$. □

Proposition 9.17 (Triangle Inequality for Integrals)**.**

Let $f \in C([a,b])$. If $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$.

Proof. Notice that $f(x) \leq |f(x)|$ and $-f(x) \leq |f(x)|$, thus by Proposition 9.16, $\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b |f(x)|\,\mathrm{d}x$ and $-\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b |f(x)|\,\mathrm{d}x$. Therefore, $-\int_a^b |f(x)|\,\mathrm{d}x \leq \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b |f(x)|\,\mathrm{d}x$, which is equivalent to $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leq \int_a^b |f(x)|\,\mathrm{d}x$. ☐

Theorem 9.18 (Fundamental Theorem of Calculus)**.**

Suppose $f \in C([a,b])$. Then,

1. if $F : [a,b] \to \mathbb{R}$ is differentiable and $\dfrac{\mathrm{d}F}{\mathrm{d}x} = f$, then $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$;
2. the function $G(x) \coloneqq \int_a^x f(t)\,\mathrm{d}t$ is differentiable in (a,b) and $\dfrac{\mathrm{d}G}{\mathrm{d}x} = f$.

Proof.

1. Let $\{\underline{x}_r\}_r$ be a sequence of partitions with $\lim_{r\to\infty} \|\underline{x}\| = 0$. By the Mean Value Theorem, for all $r \in \mathbb{N}$ and for all $j \in \{1, 2, \ldots, n\}$, there exists $\xi_{j_r} \in [x_{j-1_r}, x_{j_r}]$ such that $F(x_{j_r}) - F(x_{j-1_r}) = f(\xi_{j_r}) \cdot (x_{j_r} - x_{j-1_r})$. Thus,
$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \lim_{r\to\infty}\left(\sum_{j=1}^{n_r}\big(f(\xi_{j_r}) \cdot (x_{j_r} - x_{j-1_r})\big)\right)\\ &= \lim_{r\to\infty}\left(\sum_{j=1}^{n_r}\big(F(x_{j_r}) - F(x_{j-1_r})\big)\right)\\ &= \lim_{r\to\infty}(F(b) - F(a))\\ &= F(b) - F(a).\end{aligned}$$
2. We want to show $\lim_{x\to c} \dfrac{\int_a^x f(t)\,\mathrm{d}t - \int_a^c f(t)\,\mathrm{d}t}{x - c} = f(c)$ for all $c \in [a,b]$. Let $c \in [a,b]$. Let $\varepsilon > 0$. Since f is continuous at c, there exists $\delta > 0$ such that $t \in (c-\delta, c+\delta)$ implies $|f(t) - f(c)| < \varepsilon$. On the one hand,

suppose $x > c$ and $x - c < \delta$. If $t \in [c, x]$, then $|t - c| < \delta$. Thus,

$$\begin{aligned}
\left|\frac{1}{x-c}\cdot\left(\int_c^x f(t)\,\mathrm{d}t\right) - f(c)\right| &= \left|\frac{1}{x-c}\cdot\left(\int_c^x f(t)\,\mathrm{d}t\right) - \frac{f(c)\cdot(x-c)}{x-c}\right| \\
&= \left|\frac{1}{x-c}\cdot\left(\int_c^x f(t)\,\mathrm{d}t\right) - \frac{1}{x-c}\cdot\left(\int_c^x f(c)\,\mathrm{d}t\right)\right| \\
&= \frac{1}{x-c}\cdot\left|\int_c^x (f(t) - f(c))\,\mathrm{d}t\right| \\
&\leq \frac{1}{x-c}\cdot\int_c^x |f(t) - f(c)|\,\mathrm{d}t \\
&< \frac{1}{x-c}\cdot\int_c^x \varepsilon\,\mathrm{d}t \\
&= \varepsilon.
\end{aligned}$$

On the other hand, suppose $x < c$ and $x - c < \delta$. If $t \in [x, c]$, then $|t - c| < \delta$. Thus,

$$\begin{aligned}
\left|\frac{1}{x-c}\cdot\left(\int_c^x f(t)\,\mathrm{d}t\right) - f(c)\right| &= \left|\frac{1}{x-c}\cdot\left(\int_c^x f(t)\,\mathrm{d}t\right) - \frac{f(c)\cdot(x-c)}{x-c}\right| \\
&= \left|\frac{1}{x-c}\cdot\left(\int_c^x f(t)\,\mathrm{d}t\right) - \frac{1}{x-c}\cdot\left(\int_c^x f(c)\,\mathrm{d}t\right)\right| \\
&= \frac{1}{c-x}\cdot\left|\int_c^x (f(t) - f(c))\,\mathrm{d}t\right| \\
&\leq \frac{1}{x-c}\cdot\int_x^c |f(t) - f(c)|\,\mathrm{d}t \\
&< \frac{1}{x-c}\cdot\int_x^c \varepsilon\,\mathrm{d}t \\
&= \varepsilon.
\end{aligned}$$

Therefore, $\displaystyle\lim_{u\to x}\frac{\int_a^u f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t}{u-x} = f(x)$ for all $x \in [a, b]$. ☐

Theorem 9.19 (Integration by Parts)**.**

Suppose $f \in C([a, b])$, $g \in C([a, b])$, $\dfrac{\mathrm{d}f}{\mathrm{d}x} \in C([a, b])$, and $\dfrac{\mathrm{d}g}{\mathrm{d}x} \in C([a, b])$. Then,

$$\int_a^b \left(\frac{\mathrm{d}f}{\mathrm{d}x}\cdot g\right)(x)\,\mathrm{d}x = (f\cdot g)(x)|_a^b - \int_a^b \left(f\cdot\frac{\mathrm{d}g}{\mathrm{d}x}\right)(x)\,\mathrm{d}x.$$

Proof. Since we have $\dfrac{\mathrm{d}}{\mathrm{d}x}(f\cdot g)(c) = \dfrac{\mathrm{d}f}{\mathrm{d}x}(c)\cdot g(c) + f(c)\cdot\dfrac{\mathrm{d}g}{\mathrm{d}x}(c)$ by Theorem 8.1.2, by the Fundamental Theorem of Calculus, we have

$$\begin{aligned}
\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}(f\cdot g)(x)\,\mathrm{d}x &= \int_a^b \frac{\mathrm{d}f}{\mathrm{d}x}(x)\cdot g(x)\,\mathrm{d}x + \int_a^b f(x)\cdot\frac{\mathrm{d}g}{\mathrm{d}x}(x)\,\mathrm{d}x \\
\iff \qquad (f\cdot g)(x)|_a^b &= \int_a^b \left(\frac{\mathrm{d}f}{\mathrm{d}x}\cdot g\right)(x)\,\mathrm{d}x + \int_a^b \left(f\cdot\frac{\mathrm{d}g}{\mathrm{d}x}\right)(x)\,\mathrm{d}x.
\end{aligned}$$

☐

Lemma 9.20 (Riemann-Lebesgue Lemma)**.**

Suppose $f \in C([-\pi,\pi])$, $\dfrac{\mathrm{d}f}{\mathrm{d}x} \in C([-\pi,\pi])$, and f is 2π-periodic with $f(-\pi) = f(\pi)$. For $n \in \mathbb{N} \cup \{0\}$, let $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x$ and let $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x$. Then, $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = 0$.

Proof. By Integration by Parts, we have

$$\begin{aligned}
|a_n| &= \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x\right| \\
&= \left|\frac{1}{\pi}\left(f(x)\cdot\frac{1}{n}\cdot(-\cos(nx))\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{\mathrm{d}f}{\mathrm{d}x}(x)\cdot\frac{1}{n}\cdot(-\cos(nx))\right)\mathrm{d}x\right)\right|.
\end{aligned}$$

Notice that $\cos(n\pi) = \cos(n\cdot(-\pi))$ and $|\cos(nx)| \le 1$, hence

$$\begin{aligned}
|a_n| &= \left|\frac{1}{\pi}\left(f(x)\cdot\frac{1}{n}\cdot(-\cos(nx))\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{\mathrm{d}f}{\mathrm{d}x}(x)\cdot\frac{1}{n}\cdot(-\cos(nx))\right)\mathrm{d}x\right)\right| \\
&= \left|\frac{1}{\pi}\left(\left(f(\pi)\cdot\frac{1}{n}\cdot(-\cos(n\pi)) - f(-\pi)\cdot\frac{1}{n}\cdot(-\cos(-n\pi))\right) + \int_{-\pi}^{\pi}\left(\frac{\mathrm{d}f}{\mathrm{d}x}(x)\cdot\frac{1}{n}\cdot\cos(nx)\right)\mathrm{d}x\right)\right| \\
&\le \left|\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{\mathrm{d}f}{\mathrm{d}x}(x)\cdot\frac{1}{n}\cdot\cos(nx)\right)\mathrm{d}x\right| \\
&\le \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\frac{\mathrm{d}f}{\mathrm{d}x}(x)\cdot\frac{1}{n}\cdot\cos(nx)\right|\mathrm{d}x \\
&\le \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\frac{\mathrm{d}f}{\mathrm{d}x}(x)\cdot\frac{1}{n}\right|\mathrm{d}x \\
&\le \frac{1}{n\pi}\int_{-\pi}^{\pi}\left|\frac{\mathrm{d}f}{\mathrm{d}x}(x)\right|\mathrm{d}x.
\end{aligned}$$

Since $\lim\limits_{n\to\infty}\dfrac{1}{n\pi}\displaystyle\int_{-\pi}^{\pi}\left|\frac{\mathrm{d}f}{\mathrm{d}x}(x)\right|\mathrm{d}x = 0$, by the Squeeze Theorem, $\lim\limits_{n\to\infty}|a_n| = 0$ and thus $\lim\limits_{n\to\infty} a_n = 0$ by Proposition 3.23. By Integration by Parts again, we have

$$\begin{aligned}
|b_n| &= \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x\right| \\
&= \left|\frac{1}{\pi}\left(f(x)\cdot\frac{1}{n}\cdot\sin(nx)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\left(\frac{\mathrm{d}f}{\mathrm{d}x}(x)\cdot\frac{1}{n}\cdot\sin(nx)\right)\mathrm{d}x\right)\right|.
\end{aligned}$$

Notice that $\sin(k\pi) = 0$ for all $k \in \mathbb{Z}$ and $|\sin(nx)| \le 1$, hence

$$
\begin{aligned}
|b_n| &= \left| \frac{1}{\pi} \left(f(x) \cdot \frac{1}{n} \cdot \sin(nx) \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \left(\frac{\mathrm{d}f}{\mathrm{d}x}(x) \cdot \frac{1}{n} \cdot \sin(nx) \right) \mathrm{d}x \right) \right| \\
&= \left| \frac{1}{\pi} \left(\left(f(\pi) \cdot \frac{1}{n} \cdot \sin(n\pi) - f(-\pi) \cdot \frac{1}{n} \cdot \sin(-n\pi) \right) - \int_{-\pi}^{\pi} \left(\frac{\mathrm{d}f}{\mathrm{d}x}(x) \cdot \frac{1}{n} \cdot \sin(nx) \right) \mathrm{d}x \right) \right| \\
&\leq \left| \frac{1}{\pi} \cdot \left(- \int_{-\pi}^{\pi} \left(\frac{\mathrm{d}f}{\mathrm{d}x}(x) \cdot \frac{1}{n} \cdot \sin(nx) \right) \mathrm{d}x \right) \right| \\
&\leq \frac{1}{\pi} \int_{-\pi}^{\pi} \left| \frac{\mathrm{d}f}{\mathrm{d}x}(x) \cdot \frac{1}{n} \cdot \sin(nx) \right| \mathrm{d}x \\
&\leq \frac{1}{\pi} \int_{-\pi}^{\pi} \left| \frac{\mathrm{d}f}{\mathrm{d}x}(x) \cdot \frac{1}{n} \right| \mathrm{d}x \\
&\leq \frac{1}{n\pi} \int_{-\pi}^{\pi} \left| \frac{\mathrm{d}f}{\mathrm{d}x}(x) \right| \mathrm{d}x.
\end{aligned}
$$

Since $\lim_{n\to\infty} \frac{1}{n\pi} \int_{-\pi}^{\pi} \left| \frac{\mathrm{d}f}{\mathrm{d}x}(x) \right| \mathrm{d}x = 0$, by the Squeeze Theorem, $\lim_{n\to\infty} |b_n| = 0$ and thus $\lim_{n\to\infty} b_n = 0$ by Proposition 3.23. □

Definition 9.21 (Fourier Coefficients)**.**

The a_n and b_n defined in the Riemann-Lebesgue Lemma are referred to as the Fourier coefficients of f.

Theorem 9.22 (Change of Variables)**.**

Let $\varphi : [a,b] \to [c,d]$ be continuously differentiable, let $\varphi(a) = c$, and let $\varphi(b) = d$. Then, $\int_c^d f(u)\,\mathrm{d}u = \int_a^b f(\varphi(x)) \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x}\,\mathrm{d}x$.

Proof. Let $F : [c,d] \to \mathbb{R}$ such that $\frac{\mathrm{d}F}{\mathrm{d}x}(x) = f(x)$ for all $x \in [c,d]$. On the one hand, by the Fundamental Theorem of Calculus,

$$
\begin{aligned}
\text{LHS} &= \int_c^d f(u)\,\mathrm{d}u \\
&= \int_c^d \frac{\mathrm{d}F}{\mathrm{d}x}(u)\,\mathrm{d}u \\
&= F(d) - F(c).
\end{aligned}
$$

On the other hand, by the Fundamental Theorem of Calculus again,

$$
\begin{aligned}
\text{RHS} &= \int_a^b f(\varphi(x)) \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x}(x)\,\mathrm{d}x \\
&= \int_a^b \frac{\mathrm{d}F}{\mathrm{d}x}(\varphi(x))\,\mathrm{d}x \\
&= F(\varphi(b)) - F(\varphi(a)) \\
&= F(d) - F(c).
\end{aligned}
$$

Thus, $\int_c^d f(u)\,\mathrm{d}u = \int_a^b f(\varphi(x)) \cdot \frac{\mathrm{d}\varphi}{\mathrm{d}x}\,\mathrm{d}x$. $\square$