The Gambler's Ruin Problem for One-Dimensional Random Walks: Simple Symmetric Random Walk and Some Extensions

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Abstract

The gambler's ruin problem is a classical entry point into the random walk theory. For the simple symmetric random walk, an explicit expression for hitting probabilities can be derived. For more general step distributions, the analysis is challenging. This study explores the problem in three settings: the simple symmetric random walk, the spread-out (bounded-step) model, and the finite-variance case.

Definitions and the Gambler's Ruin Problem

Definition (*Random Walk*). A random walk is defined by $\mathcal{S} = (S_0, S_1, S_2, \ldots)$, where S_n is the sum of the starting point S_0 and independent, identically distributed steps X_i,

$$S_n = S_0 + X_1 + \cdots + X_n.$$

We assume that the walk is unbiased in this work, i.e., $\mathbb{E}(X_i) = 0$.

Definition (*Stopping Time*). Let $\mathcal{S}$ be a random walk. The stopping time η_r is the first n such that $\mathcal{S}$ exits $(0, r)$, i.e.,

$$\eta_r = \inf\{n \in \mathbb{N} : S_n \le 0 \text{ or } S_n \ge r\}.$$

The Gambler's Ruin Problem. A gambler starts to bet with $S_0 = x \in (0, r)$ dollars with outcomes X_i in each bet. They play a game until they either go broke (hit ≤ 0) or reach an intended wealth of r dollars (hit $\ge r$). What is the probability $\mathbb{P}^x\left(S_{\eta_r} \ge r\right)$ that the gambler wins?

Simple Symmetric Random Walk

A simple symmetric random walk has step probabilities

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}.$$

Proposition. For a simple symmetric random walk $\mathcal{S}$ starting at x, the probability of the gambler winning is

$$\mathbb{P}^x\left(S_{\eta_r} = r\right) = \frac{x}{r}.$$

This result can be derived by using the optional sampling theorem. The walk $\mathcal{S}$ is a martingale, and the stopped process $M_n \coloneqq S_{n \wedge \eta_r}$ is a bounded martingale. By the optional sampling theorem, $\mathbb{E}^x(M_0) = \mathbb{E}^x\left(M_{\eta_r}\right)$. Thus,

$$x = r \cdot \mathbb{P}^x\left(S_{\eta_r} = r\right) + 0 \cdot \mathbb{P}^x\left(S_{\eta_r} = 0\right).$$

We can also solve a second-order recurrence as in [1],

$$\mathbb{P}^x\left(S_{\eta_r} = r\right) = \mathbb{P}^{x+1}\left(S_{\eta_r} = r\right) \cdot \mathbb{P}(X_1 = 1) + \mathbb{P}^{x-1}\left(S_{\eta_r} = r\right) \cdot \mathbb{P}(X_1 = -1).$$

Spread-Out Model (Bounded Step Lengths)

The difficulty emerges here. We now assume X_i's are independent, identically distributed **continuous** random variables with $\mathbb{E}(X_i) = 0$.

The core challenge is *overshooting*. The walk does not stop exactly at 0 or at r: it overshoots the boundary.

Theorem. Let $\varepsilon > 0$ and let $K < \infty$. For a one-dimensional random walk, there exist constants c_1 and c_2 such that if $\mathbb{P}(|X_i| > K) = 0$ and $\mathbb{P}(X_i > 0) > 0$, then

$$c_1 \frac{x+1}{r} \le \mathbb{P}^x\left(S_{\eta_r} \ge r\right) \le c_2 \frac{x+1}{r}.$$

The theorem is proven by the same strategy as the simple case, but adapted to handle the overshoot. Notably, the bound can be tightened in the proof in [2], and it can be even tighter using the Markov Inequality.

The following lemma will be used to prove this theorem.

Lemma. Let $\mathcal{S}$ be a random walk with steps independent, identically distributed mean-zero almost-surely non-constant random variables. Let $r > 0$ and let $x \in (0, r)$. Then,

$$\mathbb{E}^x\left(S_{\eta_r}\right) = x.$$

For this lemma, the optional sampling theorem still gives

$$x = \mathbb{E}^x\left(S_{\eta_r} \cdot \mathbf{1}_{\{S_{\eta_r} \ge r\}}\right) + \mathbb{E}^x\left(S_{\eta_r} \cdot \mathbf{1}_{\{S_{\eta_r} \le 0\}}\right);$$

however, we can no longer replace the first term with $r \cdot \mathbb{P}^x\left(S_{\eta_r} = r\right)$.

The previous theorem is almost ideal for this study. However, the choice of constants depends on the maximum step size K. We may wonder, is it possible to choose constants regardless of maximum step size, for a more general walk?

Finite-Variance Case

Theorem (*Gambler's Ruin Estimate*). Let $\delta, K \in \mathbb{R}^+$ and let $b, \rho \in (0, 1)$. Let $\mathcal{S}$ be a random walk with

$$\mathbb{E}(X_i^2) \le K^2, \mathbb{P}(X_i \ge 1) \ge \delta, \inf_{n \in \mathbb{N}} \mathbb{P}\left(\bigcap_{i=1}^{n^2} S_i > -n\right) \ge b, \text{and } \rho \le \inf_{n \in \mathbb{N}} \mathbb{P}(S_{n^2} \le -n).$$

Then, there exist constants c_1 and c_2 such that

$$c_1 \frac{x+1}{r} \le \mathbb{P}^x\left(S_{\eta_r} \ge r\right) \le c_2 \frac{x+1}{r}.$$

This theorem is a generalization of the previous theorem. For any mean zero, finite non-zero variance random walk $\mathcal{S}$, we can find a $t > 0$ and some K, δ, b, ρ such that the estimates above hold for tS_n. Hence, the scaling is uniform.

The constants c_1, c_2 uniformly depend only on the **family parameters** (K, δ, b, ρ), not on the particular walk.

Comparison

The linear x/r scaling for the gambler's ruin probability is universal for one-dimensional mean-zero walks. The primary difference is the nature of the result and the difficulty of the proof.

	Simple Symmetric	**Finite-Variance**
Goal	$\mathbb{P}^x\left(S_{\eta_r} = r\right)$	$\mathbb{P}^x\left(S_{\eta_r} \ge r\right)$
Result	$\frac{x}{r}$	$\asymp \frac{x+1}{r}$
Boundary	$\{0, r\}$	The walk overshoots 0 or r

Conclusion

The gambler's ruin problem shows how a simple, exact result from the discrete simple symmetric case can become an estimate in the general case. The core difficulty moves from a simple algebraic solve to controlling the "overshoot" of the walk, which requires a deeper analysis of the walk's behavior.

Finally, with the general gambler's ruin estimate, we know that the hitting probability is robust, and constants for the estimate can be chosen uniformly over a class of distributions instead of depending on each specific case.

Bibliography

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[2] G. F. Lawler and V. Limic, *Random Walk: A Modern Introduction*. Cambridge University Press, 2010. [Online]. Available: https://www.math.uchicago.edu/~lawler/srwbook.pdf