Honor Calculus Problem Solving Session (I) Final Report

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Abstract

This paper aims to answer questions in given suggested problem set in the Honor Calculus Problem Solving Section (I) in 2022 Fall. This is sorted and formatted by Chang, Yung-Hsuan.

*He has charge of all answers except Problem 5.4 on Nov. 18 2022 and Problem 7.1 on Dec. 2 2022.

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‡He has charge of Problem 5.4 on Nov. 18 2022.

1 Sep. 23 2022

1.1 Problem 2.1

By $\lim\limits_{x\to\infty} F_1(x) = 0$, we can have $\lim\limits_{\substack{k\to\infty \\ k\in\mathbb{R}}} F_1(x+kT) = 0$ for all x. This implies that $\lim\limits_{\substack{k\to\infty \\ k\in\mathbb{R}}} F_1(x) = 0$ for all x by $F_1(x+T) = F_1(x)$ for all x. Thus, we can have $F_1(x) = 0$ for all x, which ends the proof.

1.2 Problem 2.2

By $\lim\limits_{x\to\infty} F_2(x) = A$, we can have $\lim\limits_{\substack{k\to\infty \\ k\in\mathbb{N}}} F_2(x\cdot 2^k) = A$ for $x > 0$. This implies that $\lim\limits_{\substack{k\to\infty \\ k\in\mathbb{N}}} F_2(x) = A$ for $x > 0$ by $F_2(2x) = F_2(x)$ for $x > 0$. Thus, we can have $F_2(x) = A$ for $x > 0$, which implies that $F_2(x)$ is a constant function for $x > 0$, which ends the proof.

2 Oct. 7 2022

2.1 Problem 3.1

2.1.1 Part 3.1.3

By observation, we guess $\inf f(x) = 0$. If we want to find the greatest lower bound of f for $x \geq 0$ is 0, then we have to make sure that 0 is the lower bound of f and other lower bounds are not greater than 0.

1. (Show that 0 is a lower bound of f.)

   (a) Since $x^2 \geq 0$ for $x \geq 0$, $x^2 + 1 > 0$ for $x \geq 0$.

   (b) Since $x^4 - x^2 + 1 = (x^2 - \frac{1}{2})^2 + \frac{3}{4}$ and $(x^2 - \frac{1}{2})^2 \geq 0$ for all x, $x^4 - x^2 + 1 > 0$ for $x \geq 0$.

   Since the numerator of f is greater than 0 and the denominator of f is greater than 0 by (a) and (b), $f > 0$ for $x \geq 0$. Hence, 0 is a lower bound of f.

2. (Show that lower bounds of f are not greater than 0.)

   For the sake of contradiction, let A be a lower bound of f and $A > 0$, i.e., $f(x) > A$ for $x \geq 0$.

   If $f(x) > A$, then

$$
\begin{aligned}
1 + x^2 &> A(1 - x^2 + x^4) \\
1 + x^2 &> A - Ax^2 + Ax^4 \\
0 &> (A-1) - (A+1)x^2 + Ax^4 \\
0 &> x^4 - \frac{A+1}{A}x^2 + \frac{A-1}{A} \\
0 &> \left(x^2 + \frac{A+1}{2A}\right)^2 + \frac{4A^3 - 9A^2 + 2A - 1}{4A^2} \geq \frac{4A^3 - 9A^2 + 2A - 1}{4A^2} \\
0 &> 4A^3 - 9A^2 + 2A - 1,
\end{aligned}
$$

   which yields contradiction with the assumption that $A > 0$. Hence, there is no lower bound of f which is greater then 0.

Therefore, $\inf f(x) = 0$.

2.1.2 Part 3.1.1

We want to show that $f(x)$ is bounded on $[0, \infty)$ for $x \geq 0$. We achieve this goal by showing $f(x) \geq 0$ and showing that f has an upper bound.

1. (Show that $f(x) \geq 0$.)

   By 3.1.3, $\inf f(x) = 0$. Thus, $f(x) \geq 0$ for $x \geq 0$.

2. (Show that f has an upper bound.)

   We consider the function $g(x) = 3 - f(x)$. Since $g(x) = \dfrac{3x^4 - 4x^2 + 2}{x^4 - x^2 + 1}$, and both numerator and denominator of $g(x)$ is greater than 0, $g(x) > 0$ for $x \geq 0$. Thus, 3 is an upper bound of $f(x)$.

By 1. and 2., $f(x)$ is bounded on $[0, 3]$, hence bounded on $[0, \infty)$, which ends the proof.

2.1.3 Part 3.1.2

$f(x) = \dfrac{1 + x^2}{1 - x^2 + x^4}$, then $f'(x) = \dfrac{-2x^5 - 4x^3 + 4x}{(1 - x^2 + x^4)^2}$. We look for the maximum via checking $f'(x) = 0$. Since the denominator of $f(x)$ is always positive, we just need to look for solutions to $-2x^5 - 4x^3 + 4x = 0$. Since $-2x^5 - 4x^3 + 4x = -x(x^4 + 2x^2 - 2)$, the solution to $-2x^5 - 4x^3 + 4x = 0$ is 0, $\sqrt{\sqrt{3} - 1}$, or $-\sqrt{\sqrt{3} - 1}$. Since $f(\sqrt{\sqrt{3} - 1}) > f(0)$, we conclude that $x_0 = \sqrt{\sqrt{3} - 1}$.

2.2 Problem 3.3

By definition, for all $M \in \mathbb{R}$, there exists $N \in \mathbb{R}$ such that $f^2(x) - M > 0$ for $x > N$, which yields $f(x) > \sqrt{M}$ or $f(x) < -\sqrt{M}$ for $x > N$.

Assume that $f(x)$ is not always positive or always negative for $x > N$, then there exists $a > N$ and $b > N$ such that $f(a) > 0$ and $f(b) < 0$.

Since f is continuous, $f(a) > 0$, and $f(b) < 0$, there exists c in (a, b) such that $f(c) = 0$, which yields contradiction with $f(x) > \sqrt{M}$ for $x > N$.

Thus, $f(x)$ must be always positive or always negative on (N, ∞). Hence, $f(x) > \sqrt{M}$ or $f(x) < -\sqrt{M}$, which ends the proof.

2.3 Problem 3.4

2.3.1 Part 3.4.1

We observe that

$$
\begin{aligned}
\left| x_n - \frac{f'(0)}{n^2} \sum_{k=1}^{n} k \right| &= \left| \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) - \frac{k}{n^2} f'(0) \right| \\
&\le \sum_{k=1}^{n} \left| f\left(\frac{k}{n^2}\right) - \frac{k}{n^2} f'(0) \right| \\
&= \sum_{k=1}^{n} \frac{k}{n^2} \left| \frac{f\left(\frac{k}{n^2}\right)}{\frac{k}{n^2}} - f'(0) \right|.
\end{aligned}
$$

By definition, for $\varepsilon > 0$, we can find $\delta > 0$ such that $\left| \frac{f(h)}{h} - f'(0) \right| < \varepsilon$, where $0 < |h| < \delta$. Take $N = \left[\frac{1}{\delta}\right] + 1$, then, when $n \ge N$, we have

$$
0 < \frac{k}{n^2} \le \frac{1}{n} \le \frac{1}{N} < \delta.
$$

Let $h = \frac{k}{n^2}$, then we get

$$
\left| x_n - \frac{f'(0)}{n^2} \sum_{k=1}^{n} k \right| < \varepsilon \cdot \sum_{k=1}^{n} \frac{k}{n^2} < \varepsilon.
$$

Hence, we have

$$
\begin{aligned}
\lim_{n\to\infty} x_n &= \lim_{n\to\infty} \frac{f'(0)}{n^2} \sum_{k=1}^{n} k \\
&= \frac{f'(0)}{2} \\
&= \frac{1}{2}.
\end{aligned}
$$

3 Oct. 21 2022

3.1 Problem 4.2

3.1.1 Part 4.2.1

By Rolle's theorem, since $Q_2(0) = Q_2(1) = 0$, there exists $\alpha_{1_1} \in (0,1)$ such that ${Q_2}'(\alpha_{1_1}) = 0$.

Once again by Rolle's theorem, since ${Q_2}'(0) = {Q_2}'(\alpha_{1_1}) = {Q_2}'(1) = 0$, there exist $\alpha_{2_1} \in (0, \alpha_{1_1})$ and $\alpha_{2_2} \in (\alpha_{1_1}, 1)$ such that ${Q_2}''(\alpha_{2_1}) = 0$ and ${Q_2}''(\alpha_{2_2}) = 0$, which ends the proof.

3.1.2 Part 4.2.2

We prove this by induction.

1. (Show that the statement holds when $n = 1$.)

   By Rolle's theorem, since $Q_n(0) = Q_n(1) = 0$, there exists $\alpha_{1_1} \in (0,1)$ such that ${Q_n}'(\alpha_{1_1}) = 0$.

2. (Assume that the statement holds when $n = k$, examine the statement when $n = k + 1$.)

   By Rolle's theorem, since ${Q_n}^{(n)}(\alpha_{n_1}) = {Q_n}^{(n)}(\alpha_{n_2}) = \cdots = {Q_n}^{(n)}(\alpha_{n_{n-1}}) = {Q_n}^{(n)}(\alpha_{n_n}) = 0$, there exist $\alpha_{n+1_1} \in (0, \alpha_{n_1})$, $\alpha_{n+1_2} \in (\alpha_{n_1}, \alpha_{n_2})$, ..., $\alpha_{n+1_n} \in (\alpha_{n_{n-1}}, \alpha_{n_n})$, and $\alpha_{n+1_{n+1}} \in (\alpha_{n_n}, 1)$ such that ${Q_n}^{(n+1)}(\alpha_{n+1_1}) = {Q_n}^{(n+1)}(\alpha_{n+1_2}) = \cdots = {Q_n}^{(n+1)}(\alpha_{n+1_n}) = {Q_n}^{(n+1)}(\alpha_{n+1_{n+1}}) = 0$, which proves that the statement holds when $n = k + 1$.

Therefore, the statement holds for $n \in \mathbb{N}$.

4 Nov. 4 2022

4.1 Problem 4.3

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, we want to find a solution to $f(x) = 0$.

We first consider the definite integral of $f(x)$ on $(0, 1)$:

$$\int_0^1 f(x)dx = \frac{a_n}{n+1} + \frac{a_{n-1}}{n} + \cdots + \frac{a_1}{2} + a_0.$$

By given information, we know that

$$\int_0^1 f(x)dx = 0.$$

If $f(x) > 0$ for $0 < x < 1$, then there is contradiction.

If $f(x) < 0$ for $0 < x < 1$, then there is contradiction as well.

Let $a \in (0, 1)$, then we have the 2 cases:

1. $f(a) = 0$

   a is a solution to $f(x) = 0$.

2. $f(a) \neq 0$

   Since there exists $b \in (0, 1)$ such that $f(a)f(b) < 0$ (the function changes sign at least once on $(0, 1)$),

   then by I.V.T., there exists $c \in (0, 1)$ such that $f(c) = 0$, then c is a solution to $f(x) = 0$.

Since the 2 cases have the same result, i.e., there exists at least 1 solution, the statement is proved.

5 Nov. 18 2022

5.1 Problem 5.4

Let $u = 1 + x^{\frac{1}{3}}$, then $du = \frac{1}{3}x^{-\frac{2}{3}}dx$.

Thus, we have

$$\begin{aligned}\int \frac{1}{\sqrt{x}(1+x^{\frac{1}{3}})}dx &= \int \frac{1}{(u-1)^{\frac{3}{2}}(u)} \cdot 3(u-1)^2 du \\ &= 3\int \frac{\sqrt{u-1}}{u}du.\end{aligned}$$

We need to substitute once again to complete the integration.

Let $u = \sec^2\theta$, then $du = 2\tan\theta\sec^2\theta d\theta$.

Thus, we have

$$\begin{aligned}3\int \frac{\sqrt{u-1}}{u}du &= 3\int \frac{\tan\theta}{\sec^2\theta} \cdot 2\tan\theta\sec^2\theta d\theta \\ &= 6\int \tan^2\theta d\theta \\ &= 6(\tan\theta - \theta) + C \\ &= 6x^{\frac{1}{6}} - 6\,\mathrm{arcsec}\sqrt{1+x^{\frac{1}{3}}} + C.\end{aligned}$$

Thus, we have $\int \frac{dx}{\sqrt{x}(1+x^{\frac{1}{3}})} = 6x^{\frac{1}{6}} - 6\,\mathrm{arcsec}\sqrt{1+x^{\frac{1}{3}}} + C.$[1]

[1] Cheng, Hong-Yu has charge of this answer.

6 Dec. 2 2022

6.1 Problem 7.1

We separate the two sides of the inequality to see which is greater.

1. $\displaystyle\int_0^1 \frac{\cos x}{\sqrt{1-x^2}}dx$

   Let $x = \sin t$, then $dx = \cos t dt$.

$$\begin{aligned}\int_0^1 \frac{\cos x}{\sqrt{1-x^2}}dx &= \int_0^{\frac{\pi}{2}} \frac{\cos(\sin t)}{\cos t}(\cos t)dt \\ &= \int_0^{\frac{\pi}{2}} \cos(\sin t)dt.\end{aligned}$$

2. $\displaystyle\int_0^1 \frac{\sin x}{\sqrt{1-x^2}}dx$

   Let $x = \cos t$, then $dx = -\sin t dt$.

$$\begin{aligned}\int_0^1 \frac{\cos x}{\sqrt{1-x^2}}dx &= \int_{\frac{\pi}{2}}^{0} \frac{\sin(\cos t)}{\sin t}(-\sin t)dt \\ &= \int_0^{\frac{\pi}{2}} \sin(\cos t)dt.\end{aligned}$$

Since $x > \sin x$ for $x > 0$, we have $\cos t > \sin(\cos t)$ for $t \in (0, \frac{\pi}{2})$.

Since $\cos t$ is decreasing on $(0, \frac{\pi}{2})$, we have $\cos t < \cos(\sin t)$.

Thus,

$$\sin(\cos t) < \cos(\sin t)$$

$$\Leftrightarrow \cos(\sin t) > \sin(\cos t).$$

Then we have

$$\int_0^{\frac{\pi}{2}} \sin(\cos t)dt > \int_0^{\frac{\pi}{2}} \cos(\sin t)dt,$$

which ends the proof.[2]

[2] Hsieh, Hsuan-Yu has charge of this answer.

6.2 Problem 7.4

Consider $x > \sin x$ for $x > 0$, we have

$$\begin{aligned} \int_0^x t dt &> \int_0^x \sin t dt \\ \Rightarrow \frac{x^2}{2} &> 1 - \cos x. \end{aligned}$$

We further integrate both sides of the inequality, getting

$$\begin{aligned} \int \frac{x^2}{2} dx &> \int 1 - \cos x \\ \Rightarrow \frac{x^3}{6} &> x - \sin x \\ \Leftrightarrow \sin x &> x - \frac{x^3}{6}. \end{aligned}$$

Once again, we integrate both sides of the inequality on $(0, x)$, getting

$$\begin{aligned} \int_0^x \sin x &> \int_0^x x - \frac{x^3}{6} \\ \Rightarrow 1 - \cos x &> \frac{x^2}{2} - \frac{x^4}{24} \\ \Leftrightarrow \cos x &< 1 - \frac{x^2}{2} + \frac{x^4}{24}. \end{aligned}$$

Again, we integrate both sides of the inequality, getting

$$\begin{aligned} \int \cos x &< \int 1 - \frac{x^2}{2} + \frac{x^4}{24} \\ \Rightarrow \sin x &< x - \frac{x^3}{6} + \frac{x^5}{120}. \end{aligned}$$

By above, we have proved that

$$x - \frac{x^3}{6} < \sin x < x - \frac{x^3}{6} + \frac{x^5}{120}.$$

7 Dec. 16 2022

7.1 Problem 7.1

7.1.1 Statement A

Take $f(x) = -e^{-x}$, then $\dfrac{d}{dx}(-e^{-x}) = e^{-x} > 0$. However, $\lim\limits_{x\to\infty} f(x) = 0$, which disproves this statement.

7.1.2 Statement B

For $x > 0$, there exists $a \in (0, x)$ such that $f(x) = f(0) + xf'(a)$ by Lagrange theorem.

Since $f''(x) > 0$ for $x > 0$ and $a > 0$, we have $f'(a) > f'(0)$. Thus, $f(x) > f(0) + xf'(0)$ for $x > 0$. Therefore, we have $\lim\limits_{x\to\infty} f(x) = \infty$, which proves this statement.

7.1.3 Statement C

From Statement A, $\dfrac{d^2}{dx^2}(-e^{-x}) = -e^{-x} < 0$. However, $\lim\limits_{x\to\infty} f(x) = 0$, which disproves this statement.

7.2 Problem 7.2

7.2.1 Part 7.2.1

We prove this by induction.

1. (Show that the statement holds when $n = 1$.)

$$\begin{aligned} f'(x) &= 2x^{-3}e^{-\frac{1}{x^2}} \\ &= 2\left(\frac{1}{x}\right)^3 e^{-\frac{1}{x^2}}. \end{aligned}$$

2. (Assume that the statement holds when $n = k$, examine the statement when $n = k + 1$.)

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}(P_k\left(\frac{1}{x}\right)e^{-\frac{1}{x^2}}) \\ &= (\frac{d}{dx}P_k\left(\frac{1}{x}\right))e^{-\frac{1}{x^2}} + P_k\left(\frac{1}{x}\right)\frac{d}{dx}(e^{-\frac{1}{x^2}}) \\ &= ((\frac{d}{dx}P_k\left(\frac{1}{x}\right)) + 2\left(\frac{1}{x}\right)^3 P_k\left(\frac{1}{x}\right)) \cdot e^{-\frac{1}{x^2}}. \end{aligned}$$

Since $(\dfrac{d}{dx}P_k\left(\dfrac{1}{x}\right)) + 2\left(\dfrac{1}{x}\right)^3 P_k\left(\dfrac{1}{x}\right)$ is a polynomial with degree less than $3(k+1)$, the statement holds when $n = k+1$.

Therefore, the statement holds for $n \in \mathbb{N}$.

7.3 Problem 7.3

We first use substitution to change the form of the integral.

Let $\alpha u = x$, then

$$\begin{aligned}\int_0^\alpha f(x)dx &= \int_0^1 f(\alpha u)\alpha du \\ &= \alpha \int_0^1 f(\alpha u)du.\end{aligned}$$

If we can show $\displaystyle\int_0^1 f(\alpha u)du > \int_0^1 f(u)du$, then the statement is proved.

By using Riemann sum to represent the two definite integral, we have

$$\int_0^1 f(\alpha u)du = \lim_{n\to\infty}\sum_{i=0}^{n} f(\frac{\alpha i}{n})\cdot\frac{1}{n} \quad \text{and} \quad \int_0^1 f(u)du = \lim_{n\to\infty}\sum_{i=0}^{n} f(\frac{i}{n})\cdot\frac{1}{n}.$$

Since f is monotone decreasing on $[0,1]$, $f(\dfrac{\alpha i}{n}) > f(\dfrac{i}{n})$ for $\alpha \in (0,1)$. Hence, we have

$$\lim_{n\to\infty}\sum_{i=0}^{n} f(\frac{\alpha i}{n})\cdot\frac{1}{n} \geq \lim_{n\to\infty}\sum_{i=0}^{n} f(\frac{i}{n})\cdot\frac{1}{n},$$

which ends the proof.