National Yang Ming Chiao Tung University

Calculus A (I) Report 2

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Question 1

(1) Compute the integral $\displaystyle\int_0^\infty \frac{\arctan ax - \arctan bx}{x}dx$, where $a < b$.

Solution:

This is an improper integral. Let $I = \displaystyle\int_0^\infty \frac{\arctan ax - \arctan bx}{x}dx$, then we have

$$\begin{aligned} I &= \int_0^1 \frac{\arctan ax - \arctan bx}{x}dx + \int_1^\infty \frac{\arctan ax - \arctan bx}{x}dx \\ &= \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{\arctan ax - \arctan bx}{x}dx + \lim_{R\to\infty}\int_1^R \frac{\arctan ax - \arctan bx}{x}dx. \end{aligned}$$

We discuss for five cases below:

(a) $0 < a < b$

In this case, we have

$$\begin{aligned} I &= \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{\arctan ax - \arctan bx}{x}dx + \lim_{R\to\infty}\int_1^R \frac{\arctan ax - \arctan bx}{x}dx \\ &= \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{\arctan ax}{x}dx - \lim_{\varepsilon\to 0^+}\int_\varepsilon^1 \frac{\arctan bx}{x}dx \\ &\quad + \lim_{R\to\infty}\int_1^R \frac{\arctan ax}{x}dx - \lim_{R\to\infty}\int_1^R \frac{\arctan bx}{x}dx. \end{aligned}$$

Let $u = ax$ in the first and third term; let $u = bx$ in the second and fourth term, then we have

$$\begin{aligned} I &= \lim_{\varepsilon\to 0^+}\int_{a\varepsilon}^a \frac{\arctan u}{\frac{u}{a}}\frac{1}{a}du - \lim_{\varepsilon\to 0^+}\int_{b\varepsilon}^b \frac{\arctan u}{\frac{u}{b}}\frac{1}{b}du \\ &\quad + \lim_{R\to\infty}\int_a^{aR} \frac{\arctan u}{\frac{u}{a}}\frac{1}{a}du - \lim_{R\to\infty}\int_b^{bR} \frac{\arctan u}{\frac{u}{b}}\frac{1}{b}du \\ &= \lim_{\varepsilon\to 0^+}\int_{a\varepsilon}^a \frac{\arctan u}{u}du - \lim_{\varepsilon\to 0^+}\int_{b\varepsilon}^b \frac{\arctan u}{u}du \\ &\quad + \lim_{R\to\infty}\int_a^{aR} \frac{\arctan u}{u}du - \lim_{R\to\infty}\int_b^{bR} \frac{\arctan u}{u}du \\ &= \lim_{\varepsilon\to 0^+}\int_{a\varepsilon}^{b\varepsilon} \frac{\arctan u}{u}du - \lim_{\varepsilon\to 0^+}\int_a^b \frac{\arctan u}{u}du \\ &\quad + \lim_{R\to\infty}\int_{aR}^{bR} \frac{\arctan u}{u}du - \lim_{R\to\infty}\int_b^a \frac{\arctan u}{u}du \end{aligned}$$

Since the second and fourth term cancel out, we have

$$I = \lim_{\varepsilon \to 0^+} \int_{a\varepsilon}^{b\varepsilon} \frac{\arctan u}{u} du + \lim_{R \to \infty} \int_{aR}^{bR} \frac{\arctan u}{u} du.$$

By M.V.T., there exists $cr \in (ar, br)$ for r in $\mathbb{R}^+$ such that

$$\int_{ar}^{br} \frac{\arctan u}{u} du = \arctan cr \int_{ar}^{br} \frac{1}{u} du.$$

Thus, we have

$$\begin{aligned} I &= \lim_{\varepsilon \to 0^+} \arctan c_1 \varepsilon \int_{a\varepsilon}^{b\varepsilon} \frac{1}{u} du + \lim_{R \to \infty} \arctan c_2 R \int_{aR}^{bR} \frac{1}{u} du \\ &= \lim_{\varepsilon \to 0^+} \arctan c_1 \varepsilon \ln \frac{b\varepsilon}{a\varepsilon} + \lim_{R \to \infty} \arctan c_2 R \ln \frac{aR}{bR} \\ &= \ln \frac{b}{a} \lim_{\varepsilon \to 0^+} \arctan c_1 \varepsilon + \ln \frac{a}{b} \lim_{R \to \infty} \arctan c_2 R. \end{aligned}$$

Since $c_1 > 0$ and $c_2 > 0$,

$$\begin{aligned} I &= \ln \frac{b}{a} \cdot 0 + \ln \frac{a}{b} \cdot \frac{\pi}{2} \\ &= \frac{\pi}{2} \ln \frac{a}{b}. \end{aligned}$$

Therefore, $I = \dfrac{\pi}{2} \ln \dfrac{a}{b}$ whenever $0 < a < b$.

(b) $0 = a < b$

In this case, we have

$$I = -\Big(\lim_{\varepsilon \to 0^+} \int_{\varepsilon}^{1} \frac{\arctan bx}{x} dx + \lim_{R \to \infty} \int_{1}^{R} \frac{\arctan bx}{x} dx \Big).$$

We observe that since $\dfrac{\arctan x}{x} \geq \dfrac{\arctan 1}{x}$ for $x \geq 1$, $\dfrac{\arctan bx}{x} \geq \dfrac{\arctan b}{x}$ for $b > 0$ and $x \geq 1$.

Thus, we have

$$\lim_{R \to \infty} \int_{1}^{R} \frac{\arctan bx}{x} dx \geq \lim_{R \to \infty} \int_{1}^{R} \frac{\arctan b}{x} dx.$$

Since the limit $\lim\limits_{R \to \infty} \Big((\arctan b) \displaystyle\int_{1}^{R} \frac{1}{x} dx \Big)$ diverges, $\lim\limits_{R \to \infty} \displaystyle\int_{1}^{R} \frac{\arctan bx}{x} dx$ diverges as well by the comparison test for improper integrals.

Therefore, I diverges whenever $0 = a < b$.

(c) $a < 0 < b$

Let $a = -\alpha$. Since arctangent is an odd function, we have

$$I = -\Big(\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{\arctan\alpha x + \arctan bx}{x}dx + \lim_{R\to\infty}\int_{1}^{R}\frac{\arctan\alpha x + \arctan bx}{x}dx\Big).$$

We look at the second term:

$$\lim_{R\to\infty}\int_{1}^{R}\frac{\arctan\alpha x + \arctan bx}{x}dx = \lim_{R\to\infty}\int_{1}^{R}\frac{\arctan\alpha x}{x}dx + \lim_{R\to\infty}\int_{1}^{R}\frac{\arctan bx}{x}dx.$$

Since $\lim_{R\to\infty}\int_{1}^{R}\frac{\arctan\alpha x}{x}dx$ and $\lim_{R\to\infty}\int_{1}^{R}\frac{\arctan bx}{x}dx$ both diverge (we can prove these by using the same method with regard to the comparison test for improper integrals in Question 1 (1) (b)), $\lim_{R\to\infty}\int_{1}^{R}\frac{\arctan\alpha x + \arctan bx}{x}dx$ diverges; thus, I diverges.

Therefore, I diverges whenever $a < 0 < b$.

(d) $a < b = 0$

Let $a = -\alpha$. Since arctangent is an odd function, we have

$$I = -\Big(\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{\arctan\alpha x}{x}dx + \lim_{R\to\infty}\int_{1}^{R}\frac{\arctan\alpha x}{x}dx\Big).$$

Since $\lim_{R\to\infty}\int_{1}^{R}\frac{\arctan\alpha x}{x}dx$ diverges (we can prove this by using the same method with regard to the comparison test for improper integrals in Question 1 (1) (b)), I diverges.

Therefore, I diverges whenever $a < b = 0$.

(e) $a < b < 0$

Let $a = -\alpha$ and $b = -\beta$, then $\beta < \alpha$. Since arctangent is an odd function, we have

$$\begin{aligned} & I = -\Big(\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{\arctan\alpha x - \arctan\beta x}{x}dx + \lim_{R\to\infty}\int_{1}^{R}\frac{\arctan\alpha x - \arctan\beta x}{x}dx\Big) \\ \Leftrightarrow\quad & I = \lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{\arctan\beta x - \arctan\alpha x}{x}dx + \lim_{R\to\infty}\int_{1}^{R}\frac{\arctan\beta x - \arctan\alpha x}{x}dx \end{aligned}$$

We find the answer just by algebraically replacing β to the position of a and replacing α to the position of b in the result from Question 1 (1) (a), having

$$\lim_{\varepsilon\to 0^+}\int_{\varepsilon}^{1}\frac{\arctan\beta x - \arctan\alpha x}{x}dx + \lim_{R\to\infty}\int_{1}^{R}\frac{\arctan\beta x - \arctan\alpha x}{x}dx = \frac{\pi}{2}\ln\frac{\beta}{\alpha}.$$

Therefore, $I = \frac{\pi}{2}\ln\frac{b}{a}$ whenever $a < b < 0$.

(2) Compute the limit $\lim_{n\to\infty}\sum_{k=1}^{2n}(-1)^k\left(\frac{k}{2n}\right)^{111}$.

Solution:

We first change the form of the summation, having

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{2n}(-1)^k\left(\frac{k}{2n}\right)^{111} &= \lim_{n\to\infty}\Bigg(\quad (-1)\left(\frac{1}{2n}\right)^{111}+(-1)^3\left(\frac{3}{2n}\right)^{111}+\cdots+(-1)^{2n-1}\left(\frac{2n-1}{2n}\right)^{111} \\ &\qquad + (-1)^2\left(\frac{2}{2n}\right)^{111}+(-1)^4\left(\frac{4}{2n}\right)^{111}+\cdots+(-1)^{2n}\left(\frac{2n}{2n}\right)^{111}\Bigg) \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}(-1)\left(\frac{2i-1}{2n}\right)^{111}+\left(\frac{2i}{2n}\right)^{111} \\ &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{2n}\right)^{111}-\left(\frac{2i-1}{2n}\right)^{111}.\end{aligned}$$

Consider $f(x)=x^{111}$, by M.V.T., there exists c_i in $\left(\frac{2i-1}{2n},\frac{2i}{2n}\right)$ such that

$$\begin{aligned}\left(\frac{d}{dx}f(x)\right)\Big|_{x=c_i} &= \frac{f\left(\frac{2i}{2n}\right)-f\left(\frac{2i-1}{2n}\right)}{\frac{2i}{2n}-\frac{2i-1}{2n}} \\ \Leftrightarrow \quad \frac{1}{2n}\cdot\left(\left(\frac{d}{dx}f(x)\right)\Big|_{x=c_i}\right) &= f\left(\frac{2i}{2n}\right)-f\left(\frac{2i-1}{2n}\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i}{2n}\right)^{111}-\left(\frac{2i-2}{2n}\right)^{111} &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2n}\cdot\left(\left(\frac{d}{dx}f(x)\right)\Big|_{x=c_i}\right) \\ &= \frac{1}{2}\lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(\frac{d}{dx}f(x)\right)\Big|_{x=c_i}\right)\frac{1}{n} \\ &= \frac{1}{2}\int_0^1\frac{d}{dx}f(t)dt \\ &= \frac{1}{2}\left(f(x)\Big|_0^1\right) \\ &= \frac{1}{2}(1^{111}-0^{111}) \\ &= \frac{1}{2}.\end{aligned}$$

Therefore, $\lim_{n\to\infty}\sum_{k=1}^{2n}(-1)^k\left(\frac{k}{2n}\right)^{111}=\frac{1}{2}$.

Question 2

We consider the function $f_n(x) = \dfrac{1}{1+x^2} - \sum_{k=1}^{n} (-1)^{k-1} x^{2(k-1)}$.

(1) Show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$.

Solution:

We directly substitute $f_n(x) = \dfrac{1}{1+x^2} - \sum_{k=1}^{n} (-1)^{k-1} x^{2(k-1)}$, having

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 f_n(x)dx &= \lim_{n\to\infty} \int_0^1 \frac{1}{1+x^2} - \sum_{k=1}^{n} (-1)^{k-1} x^{2(k-1)} dx \\
&= \lim_{n\to\infty} \int_0^1 \frac{1}{1+x^2} - \sum_{k=1}^{n} (-x^2)^{k-1} dx \\
&= \lim_{n\to\infty} \int_0^1 \frac{1}{1+x^2} - 1 \cdot \frac{1-(-x^2)^n}{1-(-x^2)} dx \\
&= \lim_{n\to\infty} \int_0^1 \frac{(-x^2)^n}{1+x^2} dx.
\end{aligned}$$

Let $I_n = \int_0^1 \dfrac{(-x^2)^n}{1+x^2} dx$. Thus, we have

$$\begin{aligned}
|I_n| &\le \int_0^1 \frac{|(-x^2)^n|}{1+x^2} dx \\
&= \int_0^1 \frac{x^{2n}}{1+x^2} dx \\
&\le \int_0^1 \frac{x^{2n}}{1} dx \\
&= \int_0^1 x^{2n} dx \\
&= \frac{1}{2n+1} x^{2n+1} \Big|_0^1 \\
&= \frac{1}{2n+1} \\
\Rightarrow \quad \lim_{n\to\infty} |I_n| &\le \lim_{n\to\infty} \frac{1}{2n+1} \\
&= 0 \\
\Rightarrow \quad \lim_{n\to\infty} \; I_n \; &= 0.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$.

(2) Using (1), show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$.

Solution:

By $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$, we have

$$\begin{aligned}
0 &= \lim_{n\to\infty} \int_0^1 \frac{1}{1+x^2} - \sum_{k=1}^{n} (-1)^{k-1} x^{2(k-1)} dx \\
&= \int_0^1 \frac{1}{1+x^2} dx - \lim_{n\to\infty} \int_0^1 \sum_{k=1}^{n} (-1)^{k-1} x^{2(k-1)} dx \\
&= \arctan x \Big|_0^1 - \lim_{n\to\infty} \int_0^1 \sum_{k=1}^{n} (-1)^{k-1} x^{2(k-1)} dx \\
&= (\arctan 1 - \arctan 0) - \lim_{n\to\infty} \int_0^1 \sum_{k=1}^{n} (-1)^{k-1} x^{2(k-1)} dx \\
&= \frac{\pi}{4} - \lim_{n\to\infty} \int_0^1 \sum_{k=1}^{n} (-1)^{k-1} x^{2(k-1)} dx \\
\Leftrightarrow \quad \frac{\pi}{4} &= \lim_{n\to\infty} \int_0^1 \sum_{k=1}^{n} (-1)^{k-1} x^{2(k-1)} dx.
\end{aligned}$$

Since the integration is with respect to x, we have

$$\begin{aligned}
\frac{\pi}{4} &= \lim_{n\to\infty} \int_0^1 \sum_{k=1}^{n} (-1)^{k-1} x^{2(k-1)} dx \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} (-1)^{k-1} \int_0^1 x^{2(k-1)} dx \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} (-1)^{k-1} \int_0^1 x^{2k-2} dx \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{2k-1} x^{2k-1} \Big|_{x=0}^{x=1} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{2k-1} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}.
\end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$.

Question 3

Describe your thought as clearly as possible about what is Calculus within 400 words.

Answer:

Calculus is the fundamental of many essential applications including physics, finance, economics, etc. Since I major in Applied Mathematics, I know that I must master Calculus A (I) and (II) to get better understanding in Introduction to Analysis, which is the most important steppingstone to the field of analysis.

Calculus can be simply separated into 2 parts, differentiation and integration. Differentiation is the rate of change or can be viewed as the slope of the tangent line; integration is highly related to the area under curves. The most impressive to me is the Fundamental Theorem of Calculus, which connects the seemly unrelated 2 concept together. I think this is the beauty of mathematics: Things look unrelated on the surface, but there is an exquisite relationship between them.

I have always admired Newton that he had the state-of-the-art physics concept and dared to represent in language of mathematics. In addition, the effort paid by Cauchy and Bolzano, the pioneers in analysis nowadays, are most pivotal for me; each time I understand their concepts or proofs, my ability of logical thinking becomes better. People say that reading proofs and mathematics articles is not merely knowing how to prove the theorems but learning the logic way of thinking as well. I have strong resonating with this passage; every time I encounter proving a theorem, I have a clearer path to follow.

Some experts in fields of application (or average Joe and Jane) say that there is no need for overly strict definition, such as $\varepsilon - \delta$ definition, and I think they are incorrect. In applications, we indeed just need to know how to apply the correct model corresponding to the phenomena; In mathematics, we care about the rigidness and logic. However, the application relies on well-defined mathematics, or there will be contradictions to actual phenomena. Hence, the existence of strict definitions is needed.

I look forward to the coming lectures related to calculus, Calculus A (II), Introduction to Analysis, Real Analysis, Complex Analysis, etc. Only through paying effort can I chase for my dream and then achieve my goal, and thus I will go great length to try to consume more decent knowledge regarding calculus.

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References

[1] Stewart, J., Clegg, D. K., & Watson, S. (2020). *Calculus*(9th ed.). Cengage.

[2] Shuan Mu Zhi Yue Tong. (2020, November 18). *1-1/3+1/5-1/7+1/9-......=?*. Zhihu. `https://zhuanlan.zhihu.com/p/299011986`