A NOTE ON GOLDBACH'S CONJECTURE

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Abstract. Goldbach's conjecture is one of unsolved problem in number theory. It states: Every even integer greater than 4 is the sum of two odd primes. Bui Minh Phong and Li Dongdong proved that Goldbach's conjecture is equivalent to $S(k) \geq p_{k+1} + 3$ for all $k \in \mathbb{N}$ where $S(k) = min\{2n > p_k \,|\, 2n - p_i$ is composite for $i = 1, \cdots, k\}$ and p_i is the i-th prime [1]. In this paper, we prove if $S(k) > p_{k+1}$ for all $k \in \mathbb{N}$ then every even integer greater than 4 is the sum of two odd primes. This means if Goldbach's conjecture is true and we use the proof by contradiction to prove Goldbach's conjecture, we can remove one case $S(r) = p_{r+1} + 1$ when assume that $S(r) < p_{r+1} + 3$ for some $r \in \mathbb{N}$.

1. Introduction

Goldbach wrote a letter to Euler in 1742 suggesting that every integer $n > 5$ is the sum of three primes. Euler replied that this is equivalent to the following statement:

Every even integer $2n > 4$ is the sum of two odd primes.

This is now known as Goldbach's conjecture. It has been proven that every odd integer greater than 5 is the sum of three primes [2]. Numerous attempts have been made to prove Goldbach's conjecture by solving the equivalence statements. Bui Minh Phong and Li Dongdong showed that the following statement is equivalent to Goldbach's conjecture:

$$\forall k \in \mathbb{N},\ S(k) \geq p_{k+1} + 3$$

where $S(k) = min\{2n > p_k \,|\, 2n - p_i$ is composite for $i = 1, \cdots, k\}$ and p_i is the i-th prime [1]. Proving the equivalent statement is also difficult since there is so little information about the sequence $S(k)$.

We shall prove if $S(k) > p_{k+1}$ for all $k \in \mathbb{N}$ then every even integer greater than 4 is the sum of two odd primes. The purpose of this proving is to remove unnecessary case when we use the proof by contradiction to prove the equivalent statement assuming that $S(r) < p_{r+1} + 3$ for some $r \in \mathbb{N}$. Strictly speaking, this approach to remove unnecessary case is no useful when if Goldbach's conjecture is false. The reason will be discussed in detail later. We start by dealing with the definitions required for proof.

Key words and phrases. Goldbach's conjecture, equivalent, remove one case.

Definition 1. For an positive integer k, we define $S(k)$ as follows:

$$S(k) = min\{2n > p_k \,|\, 2n - p_i \text{ is composite for } i = 1, \cdots, k\}$$

where p_i is the i-th prime. In this paper, a composite number is a positive integer that can be formed by multiplying two smaller positive integers.

Example 1.1. $S(1) = 6$

Proof. We know that $4 - 2 = 2$ is not a composite and $6 - 2 = 4$ is a composite. 6 is the smallest even number that satisfies the condition. □

Example 1.2. $S(2) = 12$

Proof. We know that $6-3 = 3$, $8-3 = 5$ and $10-3 = 7$ are not a composite and $12 - 3 = 9$ is a composite. Also, We know $12 - 2 = 10$ is a composite. 12 is the smallest even number that satisfies the condition. □

We don't need to check if $4 - 3$ is composite or not because we already know $4 - 2$ is not composite.

Definition 2. For an positive interger m, we define $C(m)$ as follows:

$$C(m) = \{2n \in \mathbb{N} \,|\, 2n - p \text{ is composite for all } p < m\}$$

where p is a prime.

Example 1.3. $C(3) = \{6, 8, 10, 12, 14, \cdots\}$

Proof. The prime less than 3 is 2. We know $2 - 2 = 0$, $4 - 2 = 2$ are not composite. $2n - 2$ is a composite for all even integer 2n greater than 4. □

Example 1.4. $C(4) = \{12, 18, 24, 28, 30, 36, 38, 42, \cdots\}$

Proof. $C(4) = \{2n \in \mathbb{N} \,|\, 2n - p \text{ is composite for all } p < 4\}$. Let $C_2 = \{2n \in \mathbb{N} \,|\, 2n - 2 \text{ is composite}\} = \{6, 8, 10, 12, \cdots\}$, $C_3 = \{2n \in \mathbb{N} \,|\, 2n - 3 \text{ is composite}\} = \{12, 18, 24, 28, 30, 36, 38, 42, \cdots\}$, respectively. Then we get $C(4) = \bigcap_{p<4} C_p = C_2 \cap C_3 = \{12, 18, 24, 28, 30, 36, 38, 42, \cdots\}$. □

We can check that $S(\cdot)$ and $C(\cdot)$ have the following association.

Lemma 1.5. Let $p_k < 2n \leq p_{k+1}$. Then $S(k) = min\, C(2n)$.

Since the above result is just by definition, we can easily see that this is correct. The basic preparation for proof is now complete.

2. Sufficient conditions for Goldbach's conjecture

We shall prove the theorem below.

Theorem 2.1. The following statement is a sufficient condition for Goldbach's conjecture:

$$\forall k \in \mathbb{N},\ S(k) > p_{k+1}$$

Proof. Suppose that $S(k) > p_{k+1}$ for all $k \in \mathbb{N}$. We know that for all even integer $2n$ greater than 4 there exists the r-th prime p_r such that $p_r < 2n < p_{r+1}$. We also know that $S(r) = min\, C(2n)$. This implies $min\, C(2n) > 2n$ for all even integer $2n$ greater than 4. This means $2n$ is sum of two odd primes since $2n \notin C(2n)$. Since we supposed that $2n$ is a arbitrary even integer greater than 4, this implies Goldbach's conjecture. □

Strictly speaking, it is a equivalent statement to Goldbach's conjecture. Remember that the following statement is equivalent to Goldbach's conjecture:

$$\forall k \in \mathbb{N},\ S(k) \geq p_{k+1} + 3.$$

This implies that if Goldbach's conjecture holds, then $S(k) \geq p_{k+1} + 3$ for all $k \in \mathbb{N}$. It means that the following statements is a neccessary condition for Goldbach's conjecture:

$$\forall k \in \mathbb{N},\ S(k) > p_{k+1}.$$

Goldbach's conjecture	$\forall\ k \in \mathbb{N},\ S(k) > p_{k+1}$	$\forall\ k \in \mathbb{N},\ S(k) \geq p_{k+1} + 3$
T	T	T
F	F	F

FIGURE 1. Truth table of the three statements.

If Goldbach's conjecture is true and we use the proof by contradiction to prove Goldbach's conjecture, we can choose to assume that $S(r) \leq p_{r+1}$ for some $r \in \mathbb{N}$ instead of assuming that $S(r) < p_{r+1} + 3$ for some $r \in \mathbb{N}$. This means that we can remove one case $S(r) = p_{r+1} + 1$ when we assume that $S(r) < p_{r+1} + 3$ for some $r \in \mathbb{N}$.

Unfortunately, if Goldbach's conjecture is false, we can't say anything about removed cases. It is not guaranteed $r_1 = r_2$ in the two statements: (i) $\exists\, r_1 \in \mathbb{N}$ s.t $S(r_1) \leq p_{r_1+1}$ and (ii) $\exists\, r_2 \in \mathbb{N}$ s.t $S(r_2) < p_{r_2+1} + 3$.

REFERENCES

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