Simulation

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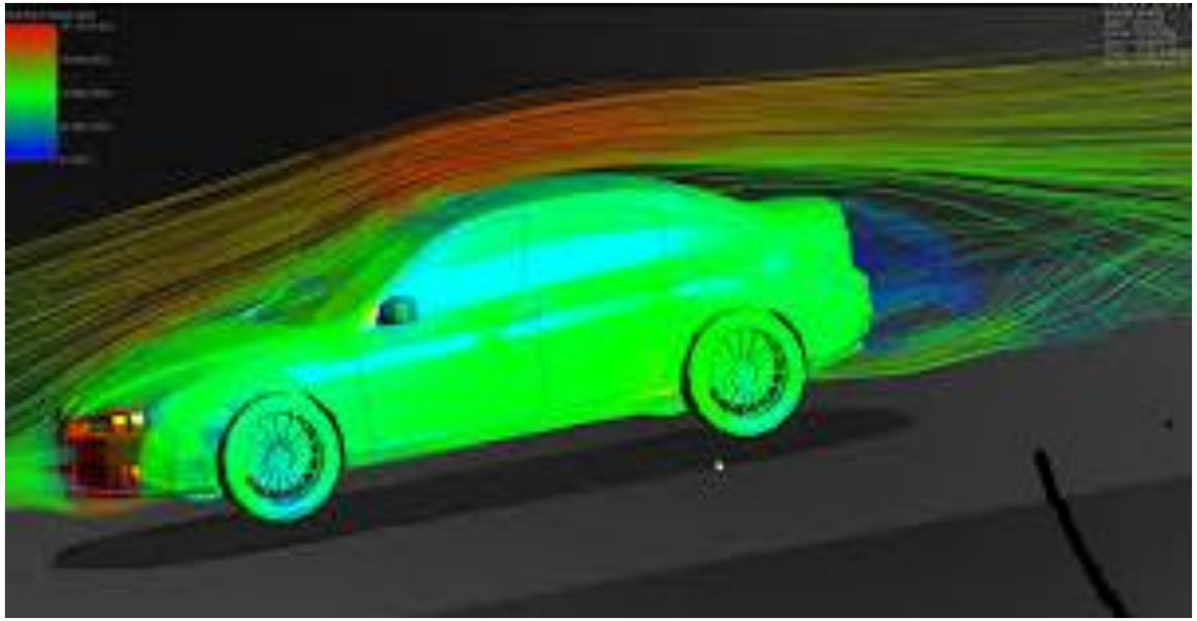
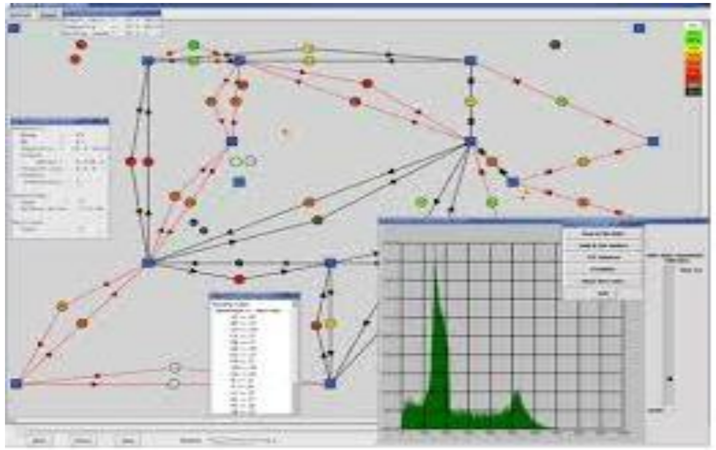
- Monte Carlo simulation
- Random walk

Simulation

Simulation

Examples: circuit design, network design, economic model, financial model, wind tunnel, etc





Simulation

- Imitation of a situation or process
- A numerical technique for conducting experiments on a digital computer, which involves certain types of mathematics or logical models
- Use to solve problems in a data-driven manner

Simulation

Each *experiment* is carried out in two steps:

1. Drawing a random outcome
2. Applying an estimation function to the drawn data

Analytical solution vs. simulation

Die rolling

- Consider rolling a die once, how probable is the output 2?
- Consider rolling a die three times, how probable is the output 222?

Die rolling

- Consider rolling a die once, how probable is the output 2?

  Actual probability =

  Estimated probability =

- Consider rolling a die three times, how probable is the output 222?

  Actual probability =

  Estimated probability =

Die rolling

Why simulation did not give me the exact answer?

Die rolling

- Why simulation did not give me the exact answer?
  - It takes a lot of repetition to get a good estimate of the frequency of occurrence of an event, more so if it is a rare event
  - One should not confuse the estimated probability with the actual probability
- There was really no need to do this experiment of die rolling by simulation
- However there are many examples where simulations are often useful

Analytical solution vs. simulation

Same birthday

- What is the probability of at least two people in a group having the same birthday?
    - If there are 367 people in the group?
    - What about a smaller group, say a group of 15?

Same birthday

- If we assume that each birthdate is equally likely, then for a group of n people with n < 366

P(at least 2 in a group of n people having the same birthday)

$$= 1 - \frac{366!}{366^n \times (366 - n)!}$$

- Simulation?

Same birthday

How about the probability of at least 3 people sharing the same birthday?

Analytical solution: why 3 is much harder?

- For 2 the complementary problem is “all birthdays distinct”
- For 3 people, the complementary problem is a complicated disjunction
  - All birthdays distinct or
  - One pair and rest distinct or
  - Two pairs and rest distinct or
  - ...
- But changing the simulation is easy

Same birthday

- We assume each birthdate is equally likely
- Are they?

Another advantage of simulation

- Adjusting analytical model is a pain
- Adjusting simulation model is easy

Simulation models

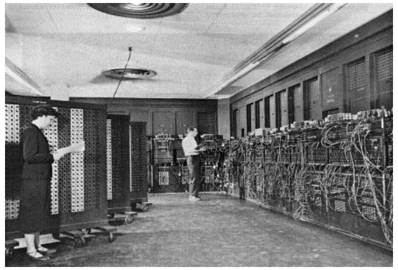
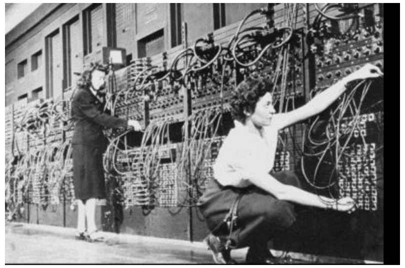
- A description of computations that provide useful information about the possible behaviors of the system being modeled
- Descriptive, not prescriptive
- Only an approximation to reality
- “All models are wrong, but some are useful.” – George Box

Monte Carlo Simulation

A little history



- Stanisław Ulam was playing a lot of solitaire while recovering from an illness
- Tried to figure out probability of winning……….and failed
- Thought about playing lots of hands and counting the number of wins, but decided it would take ages
- Asked John von Neumann if he could build an algorithm to simulate many hands on ENIAC

A little history

- Ulam, von Neuman, together with Nicholas Metropolis developed algorithms for computer implementations
- Their work enable transformation of non-random problems into random forms that can be solve via statistical sampling
- Metropolis named the new method after the casinos of Monte Carlo

Monte Carlo simulation

- A method of estimating the value of an unknown quantity using the principles of inferential statistics

Monte Carlo Simulation

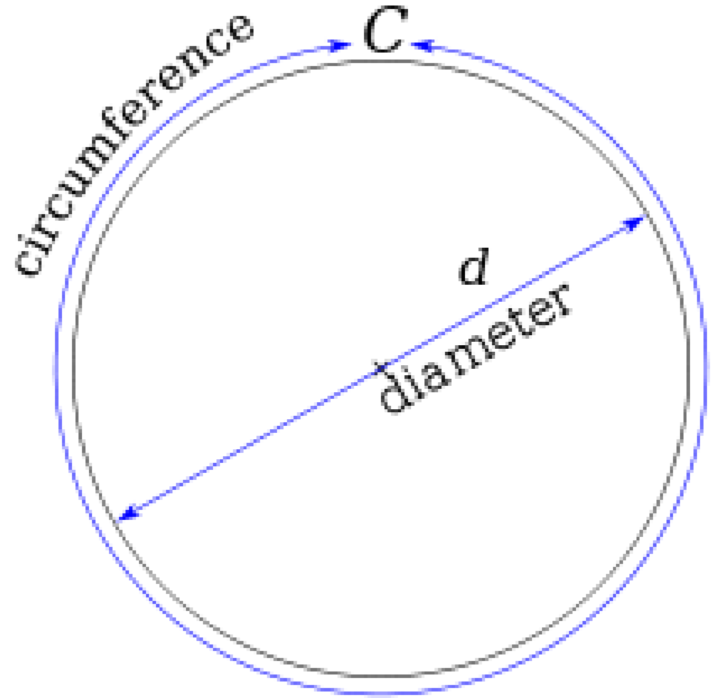
- Assume a known universe and assume something about the statistical distribution of certain event
- Numerically draw many realisations from this distribution, calculate the outcome and analyse the result

$$\pi$$

The value of π

- π = 3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825342117067982148086513282306647093844609550582231725359408 1. . .

π



$$\pi = \frac{circumference}{diameter}$$

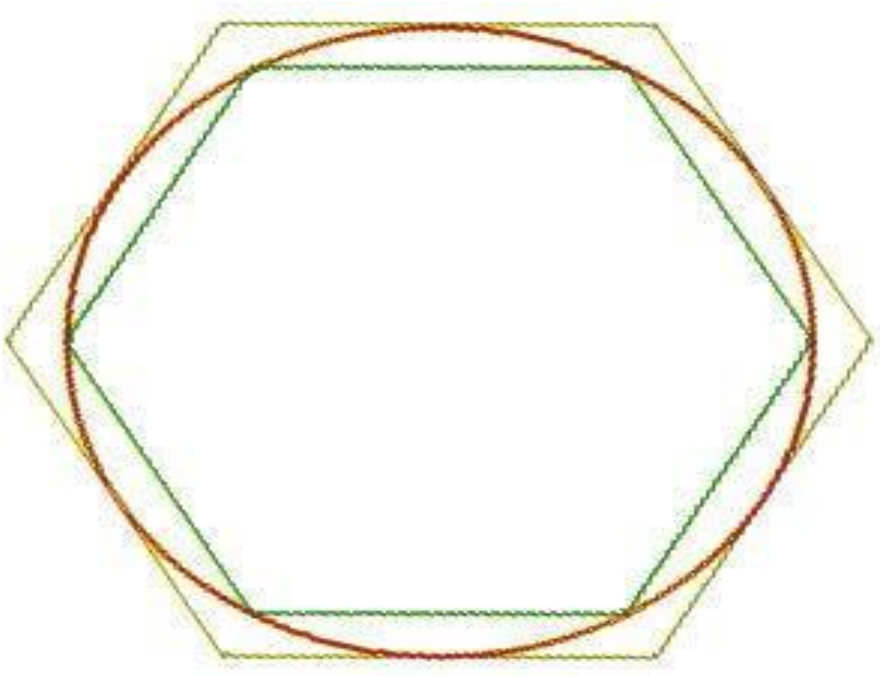
2000 BCE

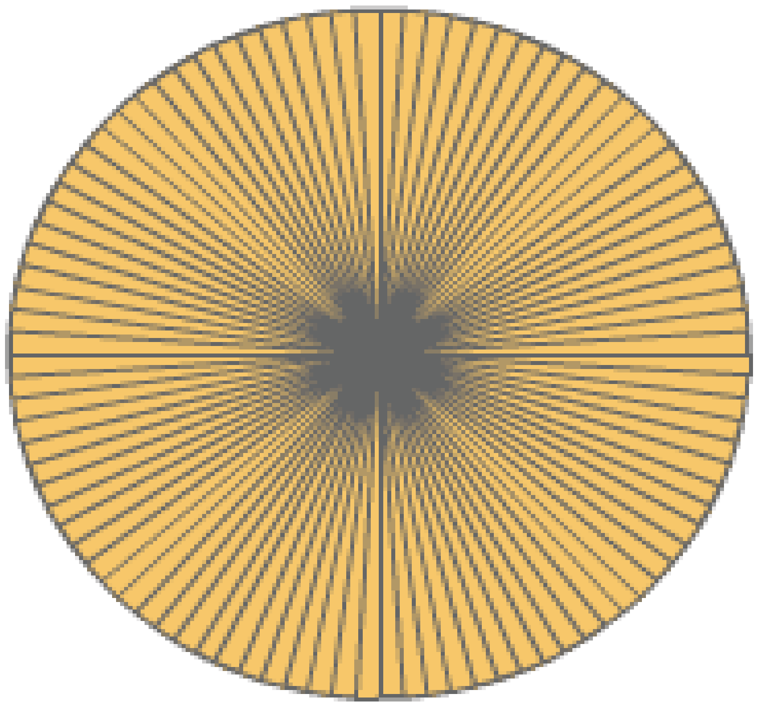
- Rhind Papyrus, approximated the circle by an octagon, $\pi = 4 \times \left(\frac{8}{9}\right)^2$

- Ancient Babylonians, $\pi = 3 + \frac{1}{8}$
- Shatapatha Brahmana (6th century BCE) used fractional approximation

$$\pi = \frac{339}{108}$$

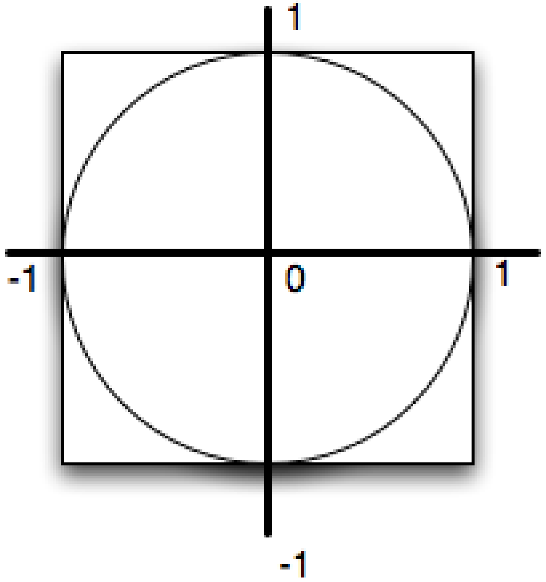
Archimedes (3rd century BCE)





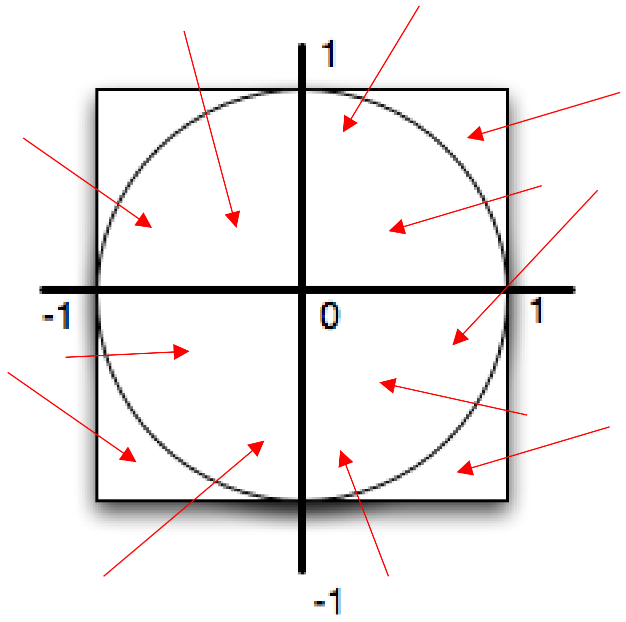
Approximating the circle by an 96-gon, $\frac{223}{71} < \pi < \frac{22}{7}$

The value of π



$$\frac{area\ of\ circle}{area\ of\ square}$$

The value of π



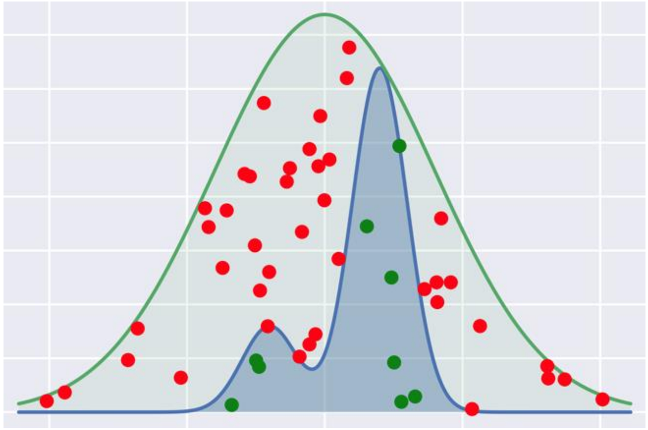
$$\frac{area\ of\ circle}{area\ of\ square} = \frac{points\ in\ circle}{points\ in\ square}$$

The value of π

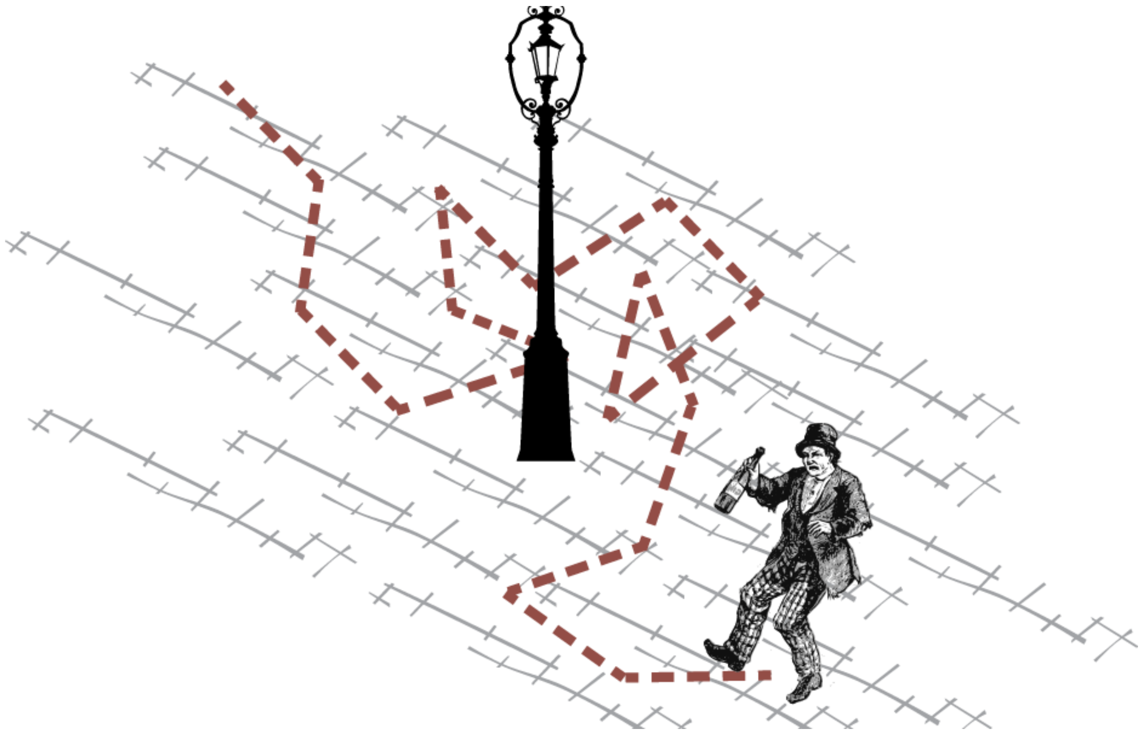
- Monte Carlo simulation

Monte Carlo simulation

- Generally useful techniques to estimate the area of some region, R
  - Pick an enclosed region, E, that is easy to calculate and R lies entirely in E
  - Pick a set of random points that lie within E
  - Let F be the fraction of the points that fall within R
  - Multiply the area of E by F
- A way to estimate integrals

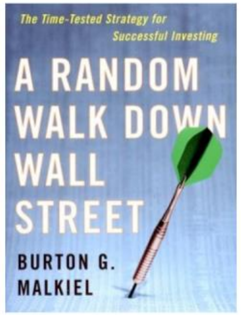


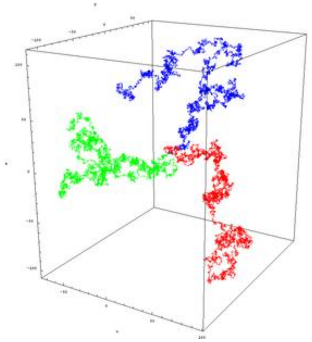
Random Walk



Random walk

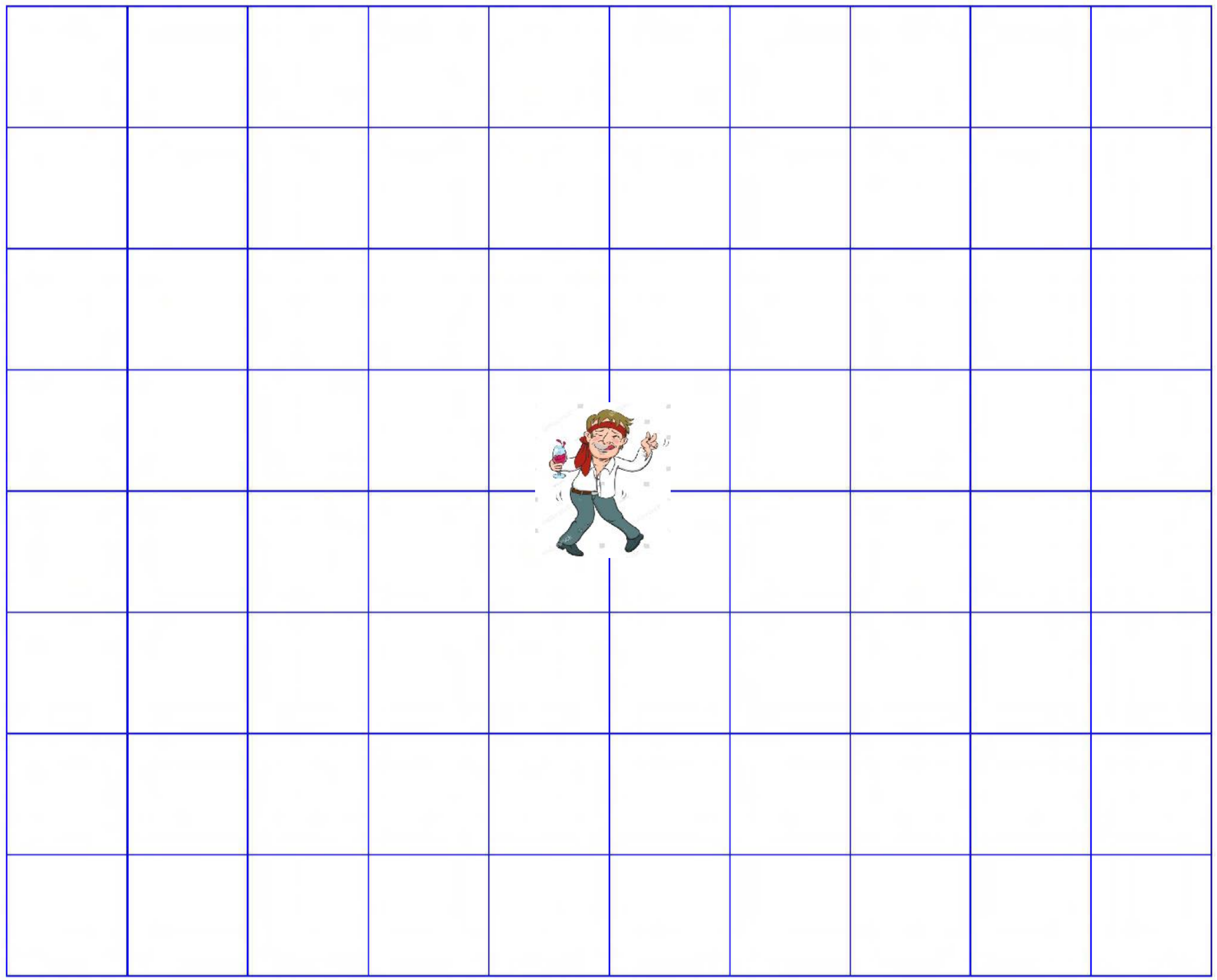
- Random walks are important in many domains
  - Understanding the stock market
  - Modelling diffusion processes
  - Etc.



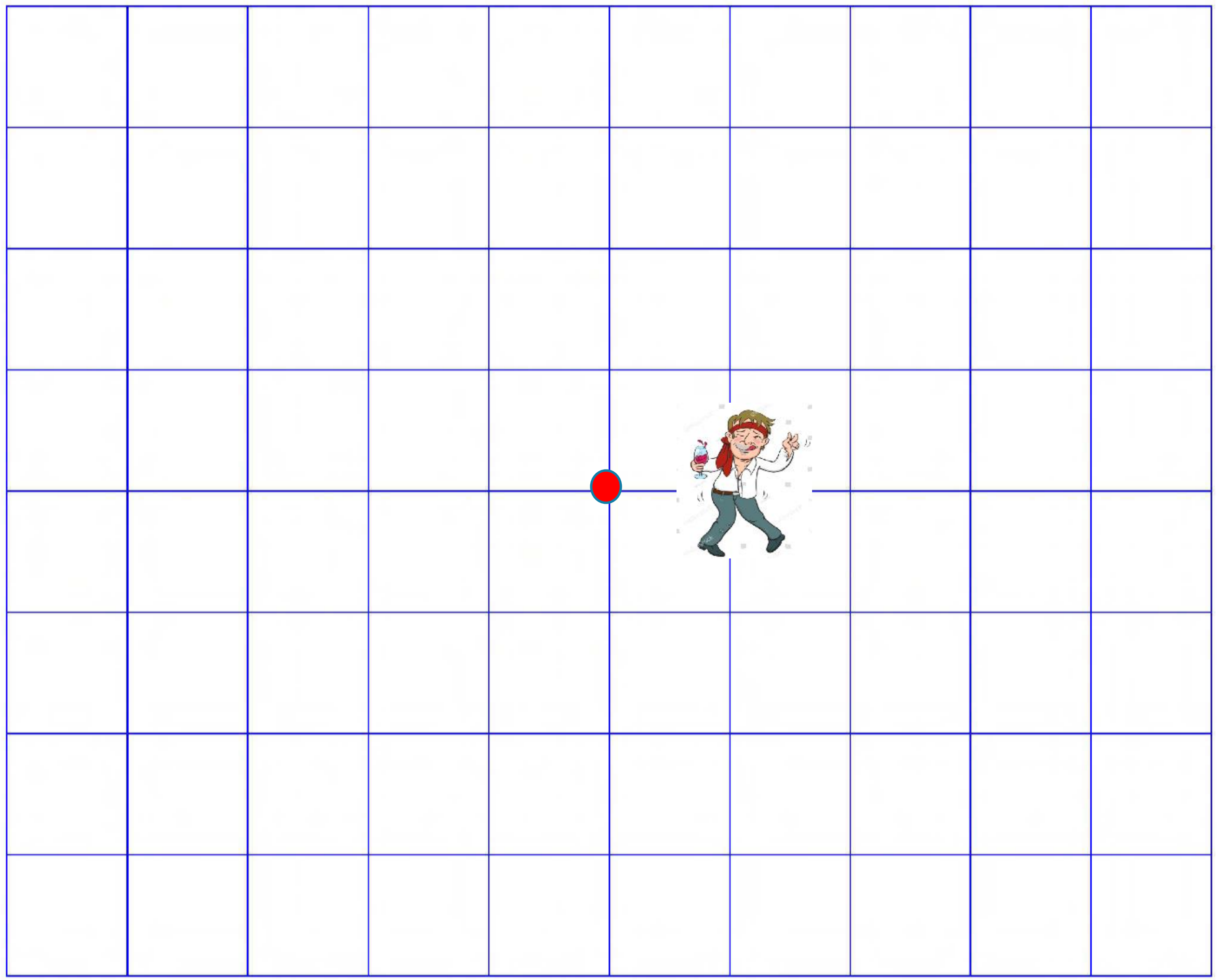


- Good illustration of how to use simulations to understand things

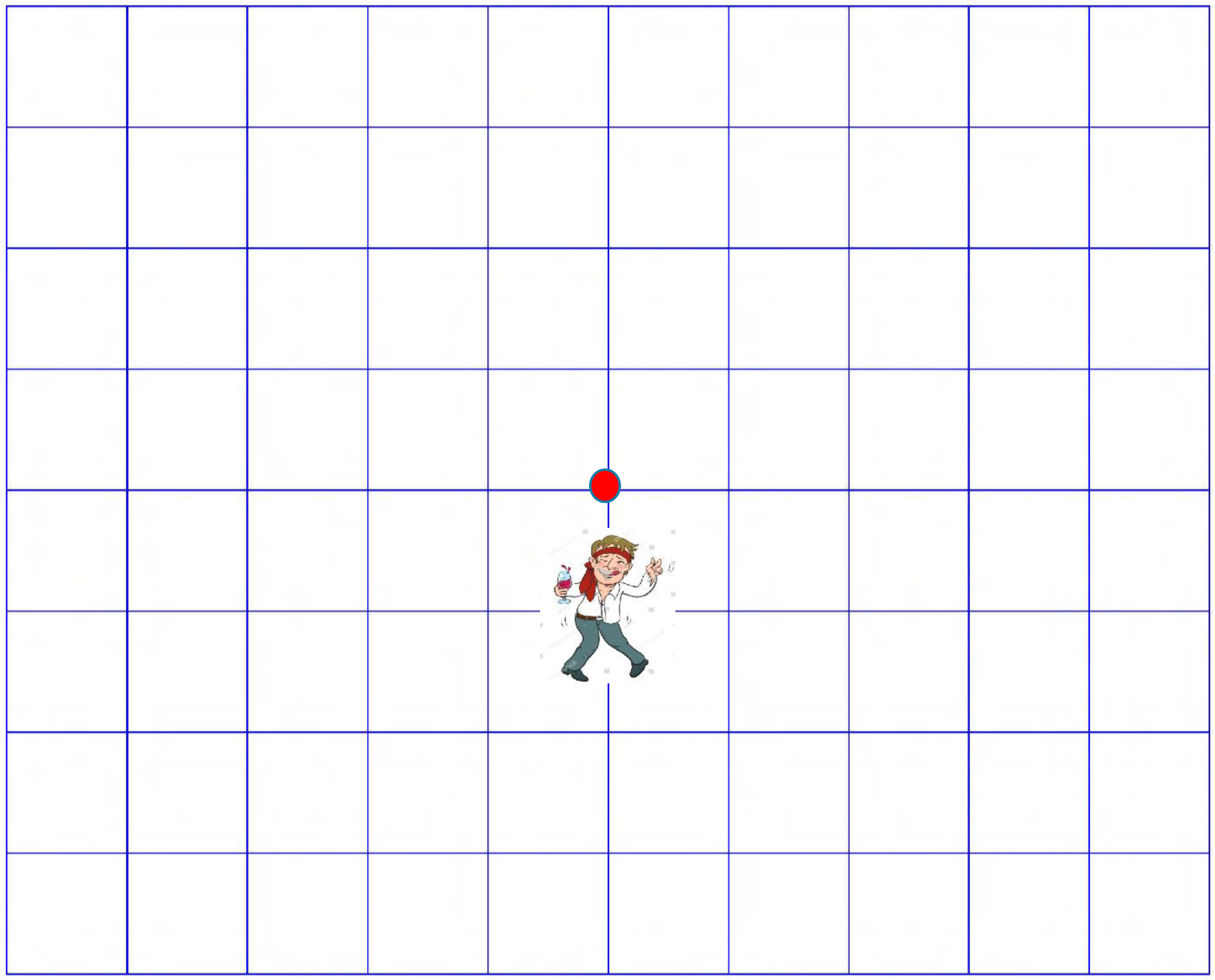
Drunkard’s walk



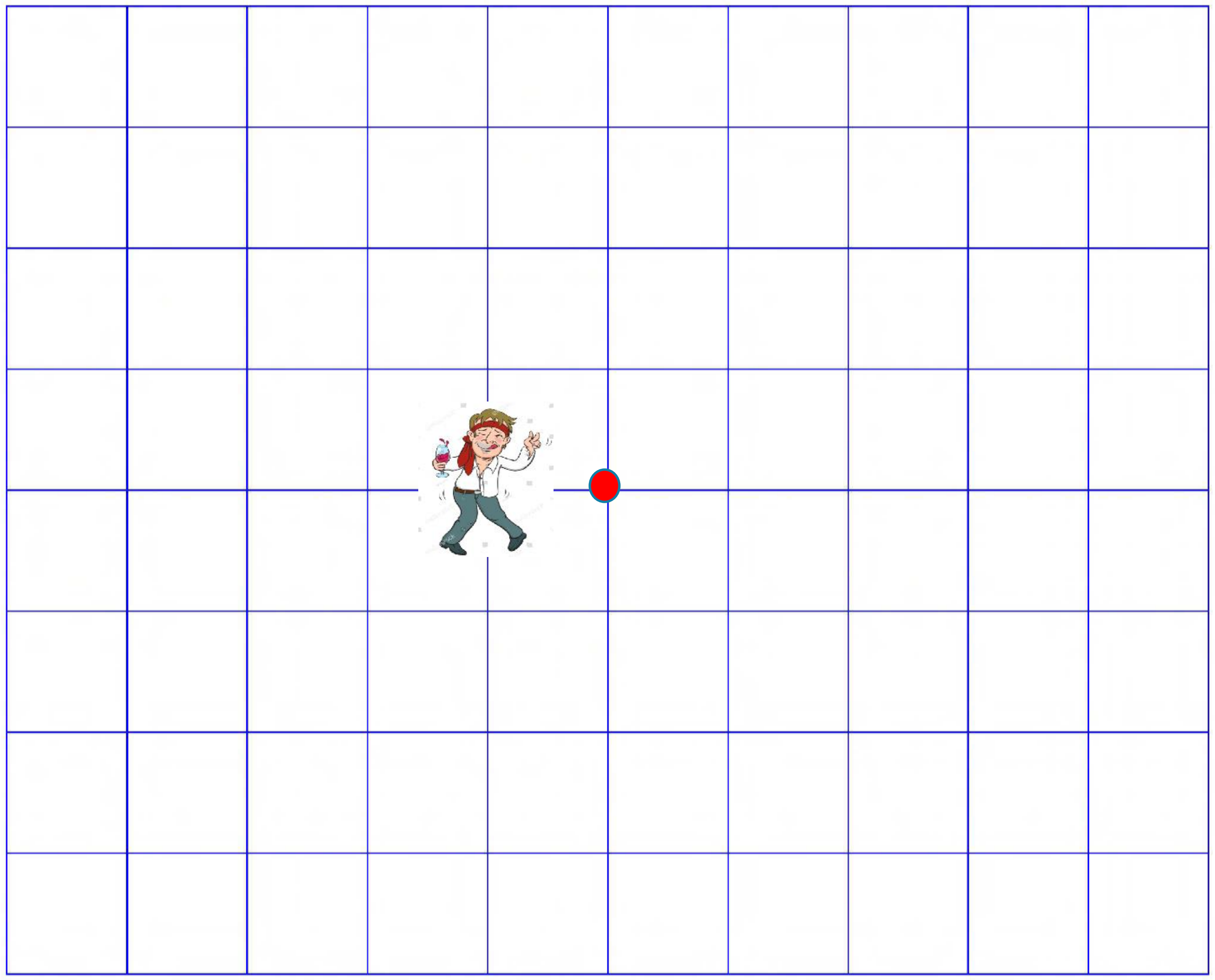
One possible first step



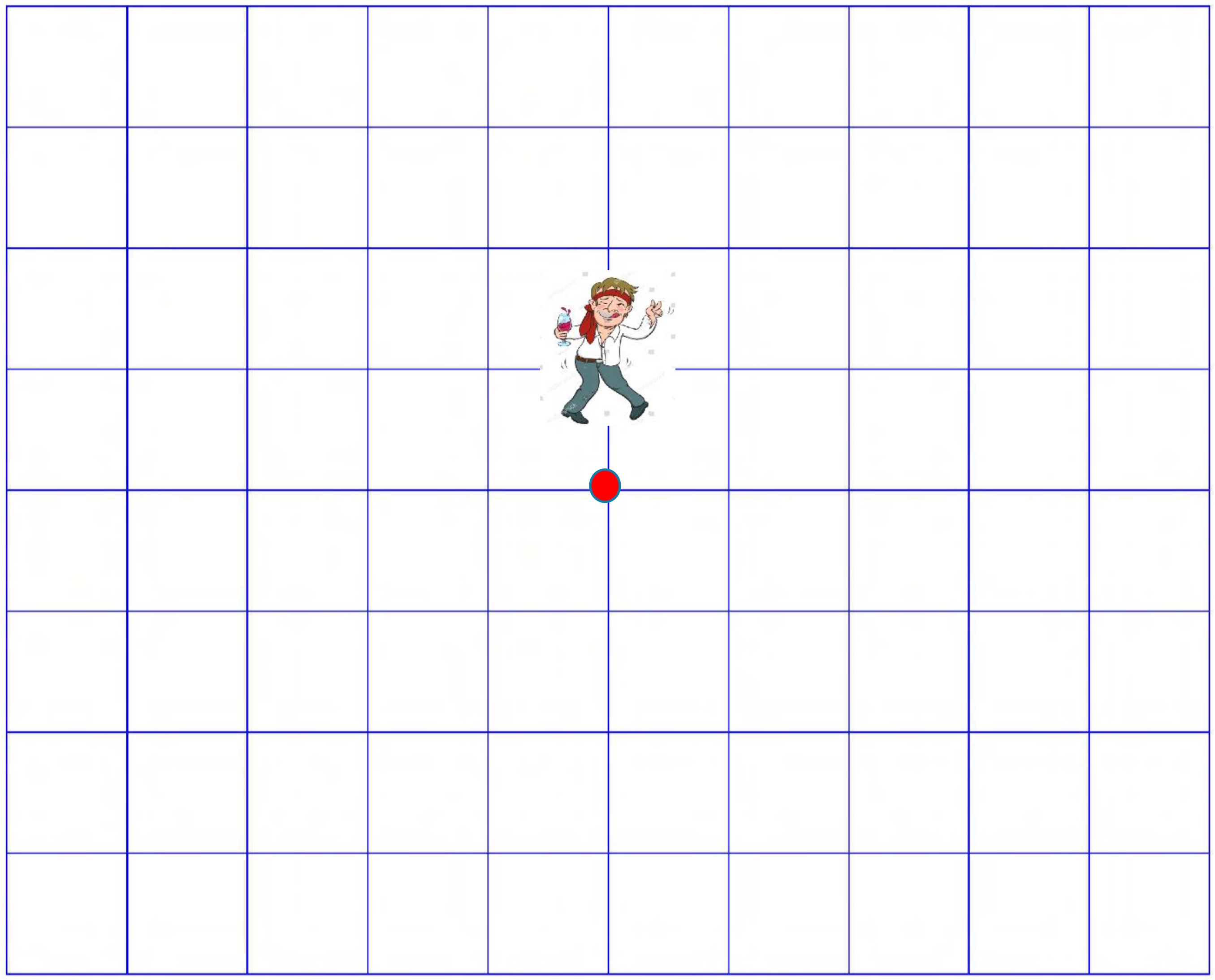
Another possible first step



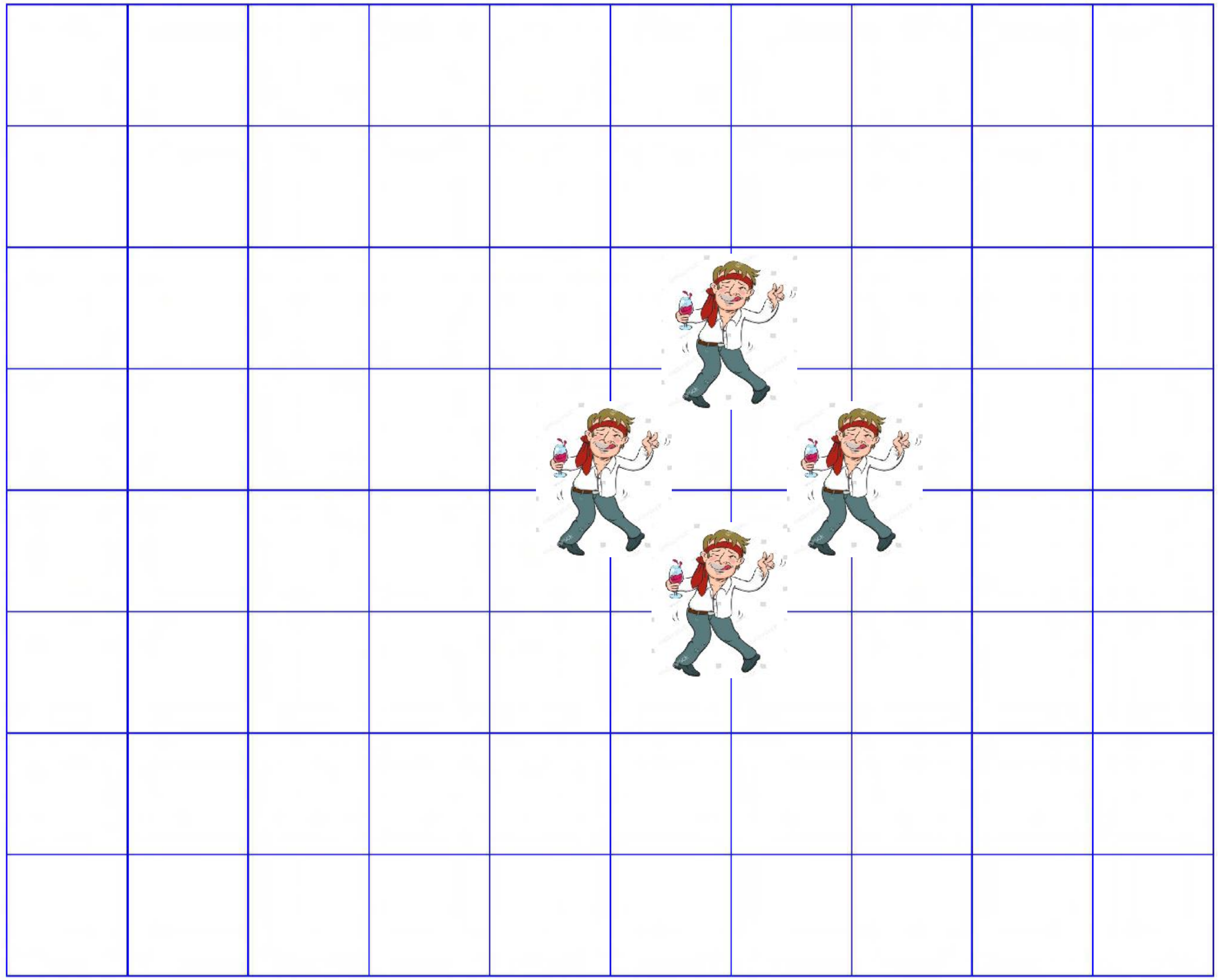
Yet another possible first step



Last possible first step



Some possible locations after two steps



Drunkard's walk

- Expected distance from the origin after 1 step?
- Expected distance from the origin after 2 steps?
- Expected distance from the origin after 10,000 steps?

Drunkard’s walk

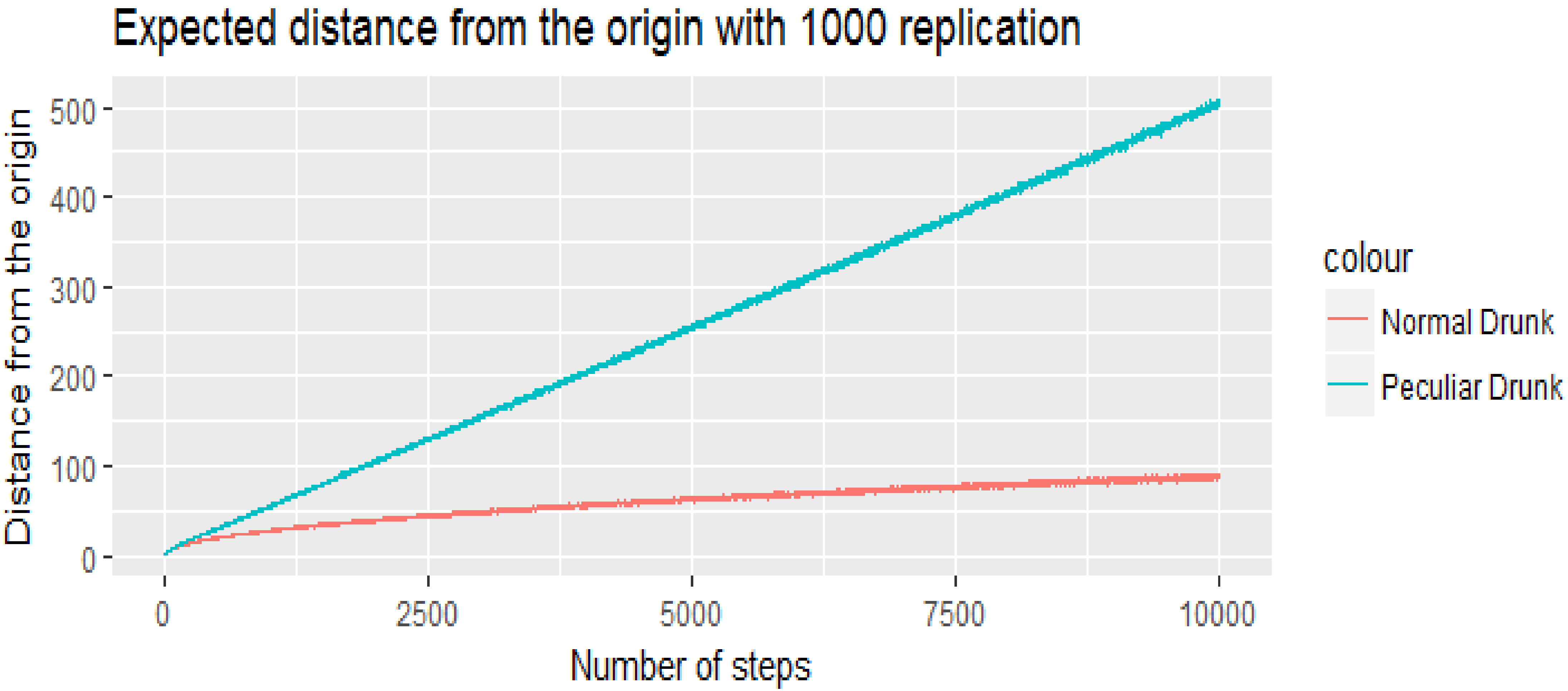
- Monte Carlo simulation
- Sanity check

Drunkard’s walk

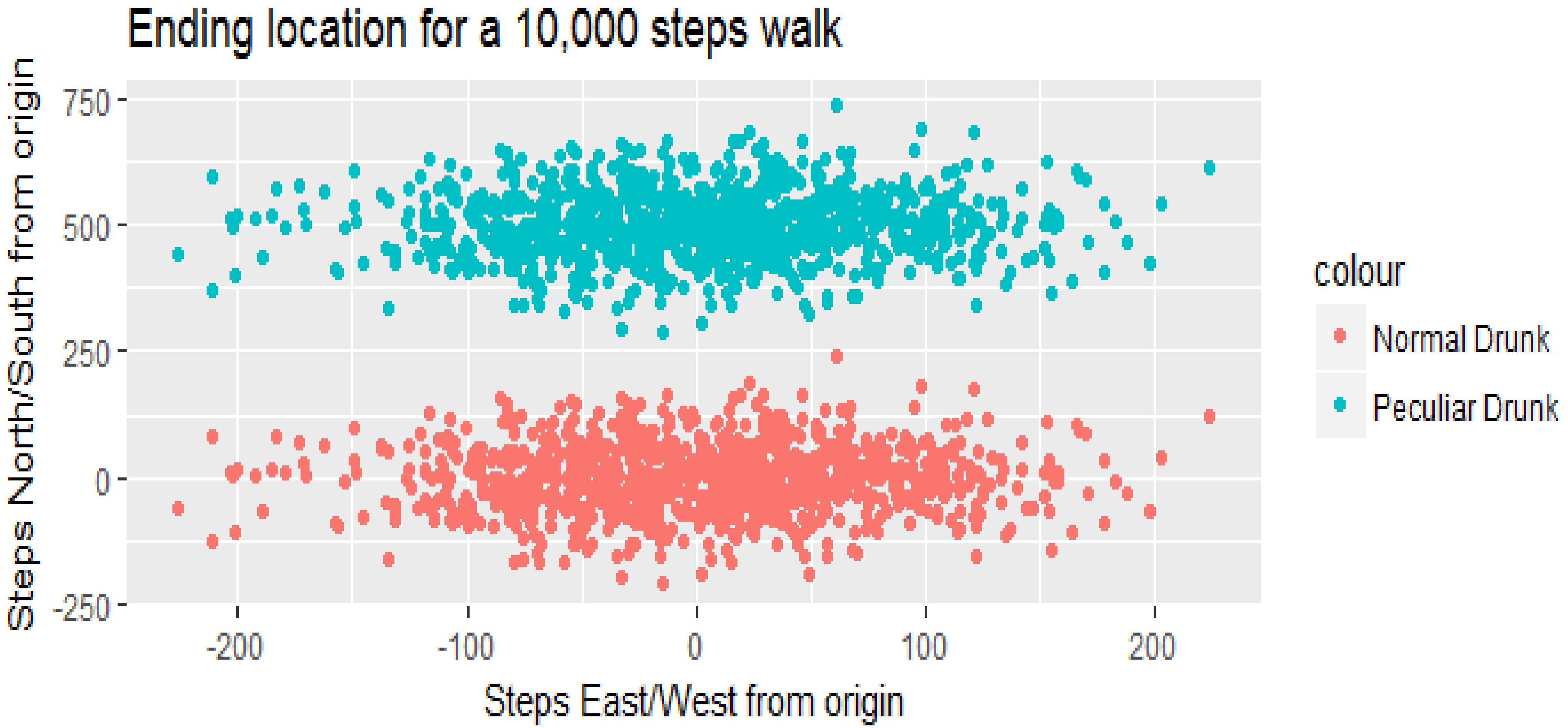
What if the drunkard’s walk is strange?

- A peculiar drunk who walk north with larger step

Drunkard’s walk



Drunkard’s walk (1000 replication)



Summary

- Examples showed how to built stochastic simulation
- Get simple version working first
- Built functions corresponding to
  - One trial, multiple trials, result reporting
- Do a sanity check!
- Make series of incremental changes to the simulation so that we could investigate different questions
- Use plots to get insights

Why simulation

- Expensive to obtain data
- Deal with a complex system
- Analytic techniques are inadequate
- Impossible/costly to perform validation
- Enable replication of experiment