Opportunity Hunters*

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Abstract

We propose a novel framework for analyzing competition for opportunities. We introduce a new type of games, called "opportunity-hunting games," in which two players compete to discover an uncertain event ("opportunity") that occurs at an unobserved and random point in time. Players can inspect whether the event has already occurred again and again, but each inspection is costly. Varying the parameters of the model spans the range from games where competition between the players to be the first to identify the opportunity is the dominant force, to games in which free-riding on the other player's effort is the dominant force. The game has a unique symmetric Markov Perfect Equilibrium. Depending on the parameters, the equilibrium takes one of two forms: the first involves frequent synchronized inspections, and the second exhibits slow diffusion in which players inspect randomly at different times.

Keywords: opportunities, sequential inspection, opportunity-hunting games.

JEL Classification Codes: C72, C73, D83.

1 Introduction

According to the Oxford Dictionary, an *opportunity* is a set of circumstances that makes it possible to do something. Identifying an opportunity, however, is oftentimes difficult. Indeed, opportunities do not make appointments – the exact moment in time in which they appear is uncertain, identifying them is costly, and the window of time to act upon them is narrow. When multiple players are interested in the same opportunity, a competition to find it emerges.

There are many economic situations in which players compete to identify an opportunity and seize it quickly. For example, consider two firms vying to capture a certain market. The firms can capture the market only if it is ripe, and checking for this condition incurs an unrecoverable cost (say, advertising costs), which is paid by the firm who moves first. If the inspection reveals that the market is not ripe, then firms may inspect again, and again, at later dates.

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The dynamics of such competition are influenced by the characteristics of the market in question. In some markets, being the first mover enhances the chances of successfully capturing the ripe market. This advantage is evident in industries like technology and fashion, where the ability to introduce innovative products or set new trends can lead to significant market share. On the other hand, there are markets where the second mover has the upper hand. In these cases, the second firm can learn from the experiences and strategies of the first mover, enabling them to improve upon the initial attempt. A notable example is the retail sector, where Walmart, as a follower, managed to offer comparable products at lower prices, ultimately becoming a dominant force in the industry.

The advantage of the first- or second-mover, as well as the cost associated with inspecting the market's ripeness, play crucial roles in shaping the nature of the competition between the two firms. Understanding these dynamics is essential for devising optimal strategies in scenarios where firms compete to seize opportunities swiftly while considering the market conditions and associated costs.

Other natural contexts in which a competition for opportunities emerges include innovation, patenting, launching of new products, asset sales, headhunting and dating. What these examples all have in common is that the optimal timing of action depends on identifying an unobservable event while considering the strategy of the other player. Additionally, players may act more than once in their attempt to identify this event. These two features distinguish the hunting for opportunities and add an important new dimension to what has been learned from the literature on strategic timing games such as "preemption games" and "war of attrition games." What characterizes the competition for opportunities? How do its equilibria look like? What are the lessons for the design of the search for opportunities?

To address these questions we introduce the following *opportunity-hunting* game. An event (an opportunity) occurs at some unobserved and random point in time. There are two players who wish to discover the event (seize the opportunity) once it occurs. At each point in time each player can inspect whether the event has already occurred at a cost of $c > 0$ per inspection. The inspections are publicly observable. If a player discovers that the event has already occurred, she obtains the payoff v_1, the other player obtains the payoff v_2, and the game ends; otherwise, the players can inspect again, and again, later in time. The player who inspects at t is referred to as the "leader" at t; the other player is referred to as the "follower" at t. If the two players inspect simultaneously, then they are randomly assigned to be leader and follower. By varying the parameters of the model, it is possible to span the range from games in which the competition between the players to identify the opportunity is the dominant force, to games in which free-riding on the other player's effort is the dominant force.

We characterize the symmetric Markov Perfect Equilibrium (MPE) of the game and show that it is unique. We show that depending on one parameter (the follower prize v_2), equilibrium takes one of two forms. When the follower prize v_2 is smaller than some threshold value, the equilibrium involves frequent synchronized inspections by the players. Intuitively, when the benefit to the follower is small, both players compete over being the first to discover the prize. This competition pushes them to inspect as early as possible, but not so early that the probability that the prize has already appeared is too small. Consequently, the two players synchronize their inspections at the earliest mutually beneficial time. We show that the players' discount rate does not affect the rate of inspection in this type of equilibrium.

By contrast, when the benefit to the follower is large, the incentive of players to "free-ride" on each other's inspections is more intense. In this case, the players cannot synchronize their inspections in equilibrium, because each player always prefers to slightly delay her inspection so she benefits from the other player's inspection without incurring the associated cost. It follows that in this case the unique equilibrium exhibits slow diffusion in which the players inspect randomly at different times. Notably, in this equilibrium, changing the value of v_2 has no effect on the players' equilibrium payoffs. This implies that from a designer's perspective, it is better to concentrate efforts on decreasing the cost of inspection c rather than increasing the reward v_2.

The direct effect of v_2 on the equilibrium renders our model valuable for applied work in two complementary ways. Firstly, the theory provides clear and testable implications. If v_2 can be inferred within certain markets, one can examine whether the theoretical predictions align with the observed market data. Secondly, the model offers flexibility by accommodating two modes of competition that correspond to the two types of equilibria mentioned earlier. Applied economists can utilize our framework to construct a generalized model tailored to a specific market or industry, without preconceived commitments to a particular mode of competition in that market. In such cases, relevant parameters can be estimated from the available data, allowing for the inference of the dominant form of competition based on market observations.

From a methodological perspective, in addition to the introduction of opportunity hunting games and the characterization of its MPEs, our work also offers two more specialized contributions. First, the characterization of equilibrium hinges on a condition that requires players to be indifferent among all the inspection times in the support of the equilibrium distribution, which is expressed as a functional equation. While this in itself is not new, here the functional equation that characterizes the equilibrium also depends on the equilibrium payoff itself. Our solution method demonstrates how to find the equilibrium despite the difficulty implied by this endogeneity. Second, we prove that the equilibrium is unique. While for the case where v_2 is small the proof is direct, the case in which v_2 is large is more challenging. In this case, our proof utilizes a result from the Theory of Distributions (Hörmann and Steinbauer, 2009), which implies that if our functional equation has a continuous solution, then this solution is unique.

Related Literature

The game we consider falls into the general category of stochastic games with a partially observable state. See, e.g., Davis and Varaiya (1973) and Hansen, Bernstein and Zilberstein (2004). Such games are usually intractable, and studied only in the case of two-players, often with zero-sum payoffs. Papers in this literature are mostly concerned with proving the existence of a value or equilibrium for the game and not in the qualitative properties of equilibrium or in comparative statics, which is our focus here.

Numerous studies have examined war of attrition games and preemption games as simple timing (or stopping) games, where players must choose the optimal moment to act.[1,2] The "classical" war of attrition game studied by Hendricks, Weiss and Wilson (1988) involves two players engaged in a cost-intensive struggle, with the last player standing declared the winner. The symmetric Nash equilibrium in this game entails a mixture of quitting times. Fudenberg and Tirole (1985) analyze a preemption game where firms compete to adopt a new technology. The first adopter benefits, but early adoption comes at a higher cost. Fudenberg and Tirole (1985) characterize the equilibria of the game, demonstrating that firms either adopt at different deterministic times or simultaneously at one of many possible times in some interval, depending on the parameters. More recent papers in this literature include Argenziano and Schmidt-Dengler (2014), Anderson, Smith and Park (2017), and Smirnov and Wait (2022).

Subsequent works in this literature have focused on the incorporation of private information into one-shot timing games, in diverse contexts. Notable examples include Hopenhayn and Squintani (2011), Murto and Valimaki (2011), Awaya and Krishna (2021), Bobtcheff, Levy and Mariotti (2021), Shahanaghi (2022), and Cetemen and Margaria (2023). In contrast to this line of research, our extension of the basic model introduces the possibility of repeated actions.

Our work is also related to the substantial body of literature on experimentation. Papers in this literature study games in which players who face uncertainty about the value of different choices engage in strategic learning over time to reduce this uncertainty. In some of these papers the focus is on the rate of experimentation. See, e.g., Bonatti and Hörner (2011) and Décamps and Mariotti (2004). In other papers, experimentation is modeled as a multiple-armed bandit problem. See, e.g., Keller, Rady and Cripps (2005), Rosenberg, Solan and Vieille (2007), Rosenberg, Salomon and Vieille (2013), and Hörner, Klein and Rady (2022). Unlike in our paper, where the focus is on *when* to next check whether the event has already occurred, in this literature, the focus is either on how much to experiment or on which experiment (arm) players should conduct next. Accordingly, papers in this literature have studied the interplay between private and public learning, and the implications of the possibility of free-riding on learning by others.

Our model is also related to the literature on contests and the design of prize schemes. See, Moldovanu and Sela (2001), Che and Gale (2003), and Moldovanu, Sela and Shi (2007). However, this literature has a different focus and the games studied in this literature typically have a one-dimensional strategy space.

Finally, Ball and Knopfle (2023) study optimal inspection policy in a dynamic Principal-Agent model with partial observability, and derive conditions for the optimality of deterministic vs. random inspections.

[1] When the number of opportunities for players to act is bounded, a timing game can be viewed as a sequence of standard timing games, where players can act only once. See, e.g., Laraki and Solan (2005). In our case, the number of possible actions for each player is potentially infinite.

[2] In a different vein, Liu and Wong (forthcoming) investigate a model of competition where two players decide *where* to search for a prize, rather than *when*.

The paper is organized as follows. In Section 2 we present the model. In Section 3 we analyze the one-player problem, which serves as a benchmark for the rest of the analysis. The two-player game is presented and analyzed in Section 4, and its symmetric Markov Perfect Equilibria are presented in Section 5. In Section 6 we discuss alternative modelling assumptions and equilibrium concepts. All proofs are relegated to the Appendix.

2 Model

We consider a continuous-time two-player game, which we call an *opportunity-hunting* game. A prize appears at a random time according to an exponential distribution with parameter λ. That is, for every time $t \geq 0$, the probability that the prize has appeared by time t is $1 - e^{-\lambda t}$. After the prize appears, it remains hidden from the players until they actively search for it. The first player to discover the prize obtains a payoff $v_1 > 0$; the other player obtains a payoff $v_2 \geq 0$.[3] Both players discount future payoffs at the rate r (i.e., a unit prize at t generates a current payoff e^{-rt}). If the prize is never found, then the payoff to both players is zero.

At each point in time each player can check whether the prize has already appeared. A player who inspects at time t is referred to as the *leader* at t. The other player is referred to as the *follower* at t. The players' inspection times are commonly observed.[4] The cost of checking is $c > 0$ per inspection. We assume that the cost of inspection is smaller than the leader's prize, i.e., $c < v_1$.[5] If, upon inspection, a player discovers the prize, then the game ends, and the two players' payoffs are as described above. Otherwise, the game continues and players can inspect again and again. If the two players inspect at the exact same time, then they are randomly assigned to be leader (who incurs the cost of inspection and obtains v_1 if she discovers the prize) or follower (who does not incur the cost of inspection, and obtains the payoff v_2 if the prize is discovered).[6]

In the subsequent analysis, our primary focus will be on symmetric Markov perfect equilibria of the game, which we formally define in Section 2.2. In these equilibria, both players employ Markov strategies, which possess a simple structure, as explained below. However, it is important to note that in our model, Markov perfect equilibria are robust against deviations to *any* strategy, not just Markov strategies. To establish this result, we must present a comprehensive definition of a strategy in the game.

Because time is continuous, a strategy cannot simply be a function from time and history into actions. The problem lies in the difficulty of mapping such strategies into distributions of game plays.[7] Therefore,

[3] The assumption that v_2 is non-negative ensures that equilibrium payoffs are non-negative, which is important for some of the results presented below. There are, of course, interesting applications where $v_2 < 0$. The analysis of such cases is different and falls outside the scope of this paper.

[4] On the one hand, this assumption simplifies the analysis because it implies that the players are symmetrically informed. On the other hand, it complicates the analysis because it expands the strategy space. This assumption implies that we consider "closed-loop" equilibria (see, e.g., Fudenberg and Tirole (1991), p. 130). In such equilibria, each player can observe, and therefore also respond to, deviations from equilibrium of other players. By contrast, an "open-loop" equilibrium is an equilibrium where deviations by other players are unobserved.

[5] The case in which $c \geq v_1$ is trivial because the players would never inspect.

[6] In Section 6.2 we discuss the implications of the alternative assumption that when both players inspect simultaneously, they both incur the cost of inspection.

[7] A standard example that illustrates this difficulty is a strategy in which in each point in time, a player inspects with probability

we adopt the following approach: a strategy instructs a player when to inspect next, as a function of the times in which past inspections took place and the identity of the player who inspected at each time. In principle, the decision of when to inspect next also depends on the outcomes of past inspections. However, since the game ends once the prize is discovered, we can assume that all prior inspections have failed to find the prize. Additionally, a player may choose not to inspect at all, which we denote as an inspection at time $t = \infty$. In order to correctly define a strategy in this way, we need to address a few technical subtleties, which we take care of in the next section. Readers who wish to skip the technical details can proceed directly to Section 2.2 without any disruption to overall coherence.

2.1 On the Definition of Strategies

In this section, we formally define the notions of history, strategy, and play. The challenge is to define these notions in a way that is, on the one hand, both general and natural, and on the other hand, ensures that a pair of strategies induces a unique distribution over the set of plays. This ensures that the expected payoffs associated with any pair of strategies are well defined.

In principle, a player may inspect as often as she likes. For example, a player may inspect at times $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, 1\frac{1}{2}, 1\frac{2}{3}, 1\frac{3}{4}, 1\frac{4}{5}, \ldots$. Such an inspection policy is likely to be suboptimal, but a definition of a strategy must accommodate it.

We begin by defining the notion of history.

Definition (**History**). *A* history *is a 3-tuple* $\langle \alpha^*, \{t_\alpha\}_{\alpha \le \alpha^*}, \iota \rangle$ *where: (i)* α^* *is a countable ordinal; (ii)* $\{t_\alpha\}_{\alpha \le \alpha^*}$ *is an increasing sequence of non-negative real numbers, such that* $t_\alpha = \lim_{\beta < \alpha} t_\beta$ *for every limit ordinal* $\alpha \le \alpha^*$*; and (iii)* ι *is a function that assigns a nonempty set of players to every successor ordinal* $\alpha \le \alpha^*$.

The interpretation of a history is as follows. The players' inspection times before t_{α^*} are given by

$$\left\{ t_\alpha \colon \alpha \le \alpha^*, \alpha \text{ is a successor ordinal} \right\}.$$

For each time t_α in this set, the players in $\iota(\alpha)$ are the players who search at that time.[8,9]

Example. To illustrate the definition above, consider a history in which a player inspects at times $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$, and then at times $1\frac{1}{2}, 1\frac{2}{3}, 1\frac{3}{4}, \ldots$. To describe this history, we set $\alpha^* = 2\omega$.[10] Then, we have that $t_0 = 0$, $t_1 = \frac{1}{2}$,

one-half, independently of past play. It is impossible to embed such a sequence in a well defined probability space. Another example is the following strategy (see, Kamada and Rao (2023)): Inspect at time t if no player inspected until t *and* $t = 1/n$ for some natural number n; otherwise, do not inspect. It can be shown that there is no play of the game that is consistent with both players employing this strategy.

[8] By definition, no player inspects at times t_α for *limit* ordinals α. It is possible to also consider a more general set of strategies where players may also inspect at times that are associated with limit ordinals. Doing so would not affect our results.

[9] The requirement that α^* is countable is without loss of generality, because between each t_α and $t_{\alpha+1}$ there is at least one rational number, and the set of rational numbers is countable.

[10] Ordinal numbers are defined as linearly ordered labels that include the natural numbers and have the property that every set of ordinals has a least element. This facilitates the definition of an ordinal number ω that is greater than every natural number, along with ordinal numbers $\omega + 1, \omega + 2, \ldots$, which are greater than ω. Similarly, the ordinal number 2ω is greater than ω plus every natural number, and so forth.

$t_2 = \frac{2}{3}$, $t_3 = \frac{3}{4}, \ldots$, $t_\omega = 1$, $t_{\omega+1} = 1\frac{1}{2}$, $t_{\omega+2} = 1\frac{2}{3}$, $t_{\omega+3} = 1\frac{3}{4}, \ldots$, and $t_{2\omega} = 2$. In this description, $0, \omega$ and 2ω are limit ordinals. All other ordinals, namely $1, 2, 3, \ldots, \omega+1, \omega+2, \omega+3, \ldots$, are successor ordinals, each associated with a specific inspection time. In this example, since only one player inspects during this history, ι assigns the set containing this player to every successor ordinal. ■

Next, we define the notion of strategy. A strategy is a measurable function from histories to distributions over future inspection times. For measurability to be well defined, we endow both of these spaces with appropriate sigma-algebras. The sigma-algebra over the space of distributions on $[0,\infty]$ is the weak-$*$ topology. The sigma-algebra over the space of histories is the one inherited from $\cup_\alpha \mathbb{R}^\alpha$, where the union is over all countable ordinals. Formally,

Definition (**Strategy**). *A* strategy σ_i *of player* i *is a measurable function that assigns to every history* $h = \langle \alpha^*, \{t_\alpha\}_{\alpha \le \alpha^*}, \iota \rangle$ *a probability distribution over* $(t_{\alpha^*}, \infty]$. *An atom of* $\sigma_i(h)$ *at* ∞ *is interpreted as if the player assigns positive probability to never inspecting.*

A *play* is an object that indicates how the game was played from beginning to end. In our model, the formal definition of play coincides with that of history, with one exception: when $\langle \alpha^*, \{t_\alpha\}_{\alpha \le \alpha^*}, \iota \rangle$ is a history, $\{t_\alpha\}_{\alpha \le \alpha^*}$ is a collection of non-negative real numbers. In a play, we allow t_{α^*} to be equal to ∞. This is because along the play the players may continue inspecting ad infinitum and never discover the prize. We endow the space of plays with the sigma-algebra inherited from $\cup_\alpha (\mathbb{R} \cup \{\infty\})^\alpha$, where the union is over all countable ordinals.

The next result ensures that every pair of strategies induces a well defined distribution over plays. This implies that the players' expected payoffs from any pair of strategies are well defined.

Lemma 1. *Every pair of strategies* (σ_1, σ_2) *induces a unique probability distribution over plays.*

2.2 Markov Perfect Equilibrium

Suppose that the last inspection has occurred at time t. For every time $t' > t$, the probability that the players assign to the event that the prize has already appeared by time t' is determined by Bayes' rule and is given by $1 - e^{-\lambda(t'-t)}$. Because of our assumption that inspection times are observable, the players are symmetrically informed. Thus, they hold the same beliefs after every possible history of the game.

Since the prize appears according to an exponential distribution, which is memoryless, and since payoffs are exponentially discounted, if an inspection at time t fails to discover the prize, then the subgame that starts at time $t+$ (i.e., immediately after t) is equivalent to the whole game.[11] In such environments, it

[11]This is not exactly accurate: in the whole game a player can inspect at $t = 0$, while in the subgame that starts at time $t+$ a player cannot inspect at time t. Yet since in the whole game inspecting at time 0 will never be done in equilibrium, we ignore this difference.

is natural for players to employ *Markov strategies*. Generally, Markov strategies are strategies that are measurable with respect to payoff-relevant histories (see, e.g., Fudenberg and Tirole (1991)). In our model, this implies that Markov strategies may only depend on the time of the last (failed) inspection.

Definition (**Markov Strategy**)**.** *A Markov strategy is a cumulative probability distribution function (CDF) F over $(0,\infty]$, with $F(0)=0$.*

The interpretation of a Markov strategy is that a player selects the first inspection time according to a distribution F; and for each time $t>0$ at which an inspection fails to discover the prize, the process of inspection "restarts" at t, and the player selects again a new inspection time according to the distribution F shifted to $(t,\infty]$. The requirement that $F(0)=0$ reflects the fact that a player cannot inspect at the exact same time as the previous inspection. The difference $1-\lim_{t\to\infty}F(t)$ is interpreted as the probability of never inspecting.

In terms of the general definition of strategy given in Section 2.1, a Markov strategy is a strategy σ that assigns for every history of past inspections h (formally defined in Section 2.1) the same cumulative probability distribution over the time of the next inspection by the player. Formally,

$$\sigma(h)(t_{\alpha^*}+t)=\begin{cases} F(t), & \forall t\geq 0,\\ 0, & \forall t<0. \end{cases}$$

where t_{α^*} is the last inspection time in the history h.[12]

We say that a pair of Markov strategies constitutes a Markov Perfect Equilibrium (MPE) if they are best responses to each other at the beginning of the game, and after every history. Note that if a pair of Markov strategies is an MPE, then they are best responses to each other after any t, because if a player has a profitable deviation at some time t, then the player also has a profitable deviation immediately after any past inspection. We restrict attention to symmetric Markov Perfect Equilibria of the game (for discussion of alternative notions of equilibrium see Section 6.1).

Two remarks are in order. First, a necessary condition for a Markov strategy F to be part of an MPE is that the player is indifferent between all the inspection times in the support of F. We refer to this property as the *Indifference Principle*.

Second, to verify that a given pair of Markov strategies is an MPE, it is sufficient to verify that a player cannot benefit by deviating to another *Markov* strategy. While this property is well known in discrete time models, in our model because time is continuous, the argument is more subtle. The next result provides the first step in this argument.

Lemma 2. *Every Markov strategy has an ε-best response, which is itself a Markov strategy.*

Let (σ_1,σ_2) be a pair of Markov strategies. Lemma 2 implies that if Player 1 has a profitable deviation to some strategy, then she has a profitable deviation to a Markov strategy. Thus, to verify that a pair of strategies

[12]More precisely, as explained in Section 2.1, the time t_{α^*} may be an accumulation point of previous inspection times in the history $h=\langle\alpha^*,\{t_\alpha\}_{\alpha\leq\alpha^*},\iota\rangle$.

is an MPE, it is sufficient to verify that no player has a profitable deviation to a Markov strategy. We record this observation in the following corollary.

Corollary 1. *A pair of Markov strategies* (σ_1, σ_2) *is an MPE if and only if no player has a profitable deviation to another Markov strategy.*

We emphasize that an MPE is immune against *all* deviations, including deviations to non-Markov strategies. We will be interested in the symmetric MPE of the game. Note, however, that the game may also have MPEs that are not symmetric and equilibria that are not MPE. We discuss these other equilibria in Section 6.1 below.

3 Benchmark: The One-Player Problem

In this section, we examine a simpler problem involving a single player searching for a prize. This serves as a benchmark for analyzing the two-player game and provides us with useful results. We maintain all the model's assumptions, except for the presence of only one player, who receives a payoff of v_1 upon discovering the prize.

For every $t > 0$, let $w(t)$ be the (single-) player's expected payoff under a pure Markov strategy in which the player inspects at time t (and because the strategy is Markov, inspects again t units of time after every failed inspection):

$$w(t) = e^{-rt}\left(-c + (1 - e^{-\lambda t})v_1 + e^{-\lambda t} w(t)\right), \quad \forall t > 0. \tag{1}$$

The three terms in parentheses on the right-hand side of Eq. (1) capture the expected payoff from inspection at time t: c is the cost of inspection; $(1 - e^{-\lambda t})v_1$ is the expected payoff upon discovering the prize; and $e^{-\lambda t} w(t)$ is the expected payoff upon failing to discover the prize and "restarting" the process of inspection.

Simplifying Eq. (1) yields:

$$w(t) = \frac{(1 - e^{-\lambda t})v_1 - c}{e^{rt} - e^{-\lambda t}}. \tag{2}$$

The next lemma establishes two basic properties of the player's payoff function $w(\cdot)$.

Lemma 3. *The function* $w(\cdot)$ *is strictly quasi-concave and has a unique maximizer on the interval* $(0, \infty)$.

We denote the unique maximizer of $w(\cdot)$ by $\tau^* \in (0, \infty)$. Intuitively, when deciding when to inspect, the player balances two opposite effects of slightly postponing (or advancing) the time of inspection. On the one hand, postponement increases the probability that the prize has already appeared. On the other hand, discounting lowers the benefit from finding the prize. Overall, the sum of the two effects is the total gain from postponement. This gain from postponement is given by the derivative of the function $w(t)$ according to t. At the optimal inspection time τ^* this derivative is equal to zero.

Inspection every τ^* units of time is optimal among all Markov strategies. Our next result records this observation and shows that this Markov strategy is in fact optimal among all strategies.

Proposition 1. *The player has an optimal inspection strategy, which is Markov. This strategy instructs the player to inspect at time τ^*, and τ^* units of time after every failed inspection, until the prize is discovered. The player's expected payoff from this strategy is given by $w(\tau^*)$.*

Proposition 1 asserts the existence of an optimal strategy, the fact that it is Markov, and relates it to the maximizer of the function $w(\cdot)$. Intuitively, existence can be deduced from Lemma 2 (invoked for the case where the other player's Markov strategy is to never inspect), which implies that if an optimal strategy exists, its payoff can be approximated by a Markov strategy. Because the strategy outlined in the proposition is optimal among all Markov strategies, it follows that it is optimal among all strategies as well. However, in the proof, we adopt a more direct approach, leveraging recent findings in the literature by Jasso-Fuentes, Menaldi and Prieto-Rumeau (2020) and Stachurski and Zhang (2021).

Intuitively, when the value of the prize v_1 increases, the optimal inspection rate increases as well. This is because the effective cost of postponement is greater. Similarly, when the inspection cost c increases, the optimal inspection rate decreases. We record these two observations in the next lemma.

Lemma 4. *The optimal inspection rate τ^* is decreasing in v_1 and increasing in c.*

4 The Two-Player Game

We now shift the focus back to the two-player game. The addition of another player introduces a strategic element into the players' considerations. On the one hand, the presence of the other player creates competition for the prize. Consequently, each player is inclined to inspect more frequently to increase her chances of being the first to discover the prize. On the other hand, the existence of another player also introduces the possibility of "free-riding." This means that a player can delay her inspection in the hope that the other player will conduct the inspection instead. By doing so, she can save on inspection costs and potentially secure a payoff of v_2 if the other player happens to discover the prize. The parameters of the model determine the relative strength of the competition versus the free-riding effects, as well as the qualitative characteristics of the equilibrium.

4.1 The Follower Gain and the Delay Gain

Fix a symmetric MPE (not necessarily continuous), and denote the players' symmetric payoff in this MPE by v_0. What does Player 1 gain by slightly postponing her inspection at some time t? The answer depends on whether or not Player 2 inspects in the meantime.

Suppose that Player 1 slightly postpones her inspection at some time t (while keeping the times of future inspections unchanged), and Player 2 inspects in the meantime. We call Player 1's gain from this postponement the *follower gain* at t, and define it formally as follows.

Definition (**Follower Gain**). *The follower gain at t is the quantity*

$$(1-e^{-\lambda t})(v_2-v_1)+c. \tag{3}$$

The discounted follower gain at t is the follower gain multiplied by the discount factor e^{-rt}.

Intuitively, the follower gain at t is a player's gain from a postponement of inspection at time t, which turns her into the follower rather than the leader. Accordingly, the first term in the sum in Eq. (3) is the difference between the second and first prizes, multiplied by the probability that the prize has appeared by time t. The second term is the cost, which is saved for sure if the player is not the leader. Notably, if $v_2 \geq v_1 - c$, then the follower gain is positive for all $t > 0$. Indeed, in this case, it is always better to be the follower rather than the leader: if the prize has already appeared at t, then the benefit from switching from leader to follower is $v_2 - (v_1 - c) \geq 0$, and if the prize has not appeared yet, then the benefit from switching from leader to follower is $c > 0$. On the other hand, if $v_2 < v_1 - c$ then the follower gain changes sign: for a sufficiently small time t it is positive, whereas it is negative for all large enough times t.

Alternatively, suppose that Player 1 slightly postpones her inspection at time t (while keeping the times of future inspections unchanged), but Player 2 does not inspect in the meantime. Player 1's gain from being the leader at t is given by

$$L(t) \equiv e^{-rt}\left(-c+(1-e^{-\lambda t})v_1+e^{-\lambda t}v_0\right). \tag{4}$$

The expression in the right-hand side of Eq. (4) is similar to the right-hand side of Eq. (1), except that $w(t)$ is replaced by the equilibrium payoff v_0. It is noteworthy that v_0, and so also $L(t)$, are determined endogenously in equilibrium. This key feature distinguishes our model of repeated inspections from previous simple stopping games that were analyzed in the literature, where the payoff from taking an action is exogenously fixed.

Thus, the discounted gain from postponing the inspection from t to $t+\varepsilon$, conditional on the other player not inspecting during $[t, t+\varepsilon]$ and keeping the times of future inspections unchanged, is given by the difference $L(t+\varepsilon) - L(t) \approx \varepsilon L'(t)$. We define the *discounted delay gain* at t to be the derivative of $L(t)$ according to t. The *delay gain* at t is the discounted delay gain divided by e^{-rt}. Consequently, we have that,

Definition (**Delay Gain**). *The delay gain at t is the quantity*

$$L'(t)/e^{-rt} = rc - rv_1 + (\lambda+r)e^{-\lambda t}v_1 - (\lambda+r)e^{-\lambda t}v_0. \tag{5}$$

The discounted delay gain is the delay gain multiplied by e^{-rt}.

Note that if $v_2 < v_1$, then the follower gain is decreasing in t, and if $v_0 < v_1$, then the delay gain is decreasing in t. It is also noteworthy that for sufficiently large t, the delay gain is negative. Intuitively, when t is large, the prize has almost surely appeared already and additional delay is costly.

The next lemma, which will be useful in the analysis that follows, relates the delay gain and the optimal inspection time in the one-player problem.

Lemma 5. *If* $v_0 = w(\tau^*)$ *then* τ^* *is the unique time at which the delay gain is equal to zero.*

4.2 Qualitative Properties of Equilibrium

In this section we derive qualitative properties of the equilibria of the game. The next observation follows immediately from our definition of the follower gain and the delay gain.

Observation 1. *Let* F *be a symmetric MPE strategy. If* t *belongs to the support of* F*, then the delay gain and the follower gain cannot both be positive at* t *and cannot both be negative at* t*.*

Intuitively, if both the delay gain and follower gain are positive at a particular time t in the support of a symmetric MPE strategy, then the continuity of these gains suggests the existence of a small interval around t where both gains remain positive. Consequently, it is beneficial for a player to postpone all inspections conducted within this interval to its upper end. This deviation benefits the player regardless of whether the other player conducts inspections during the same interval.

Likewise, when both the delay gain and follower gain are negative at a specific time t, similar reasoning suggests that it is more favorable for a player to slightly advance all inspections around t. However, this contradicts the optimality of the strategies employed.

In principle, the players may employ strategies that refrain from inspection with a positive probability. As shown by the next result, such strategies cannot be part of a symmetric MPE.

Proposition 2. *Let* F *be a symmetric MPE strategy. Then,* $\lim_{t\to\infty} F(t) = 1$.

The intuition behind this result is as follows. Suppose there exists a symmetric MPE strategy F such that $\lim_{t\to\infty} F(t) < 1$. According to the Indifference Principle introduced earlier, it is optimal for Player 1 to never inspect. Since $\lim_{t\to\infty} F(t) < 1$, there exists a sufficiently large time t where: (i) The probability of Player 2 not inspecting before time t is positive. (ii) Given that Player 2 has not inspected before time t, the conditional probability of her inspecting at or after time t is extremely small. (iii) The probability of the prize appearing before time t is arbitrarily close to 1. Thus, given that time t has been reached, the payoff to Player 1 (evaluated at time t) from not inspecting at all is negligibly small. On the other hand, the expected payoff from inspecting at time t (evaluated at t) is close to $v_1 - c > 0$. This leads to a contradiction.

Proposition 2 implies that the support of a symmetric MPE strategy is a subset of the open interval $(0,\infty)$. The next result describes the atoms of symmetric MPE strategies, if any.

Proposition 3. *Let* F *be a symmetric MPE strategy. If* F *has an atom at* t*, then*

$$t = t^\circ \equiv -\frac{1}{\lambda} \ln\left(\frac{v_1 - v_2 - c}{v_1 - v_2}\right), \tag{6}$$

where t° *is the time at which the follower gain, as given by Eq. (3), is equal to zero (if such a time exists).*

To see the intuition for this result, suppose that F has an atom of size 1 at t (in the proof, we extend this argument to an atom of any positive mass). In this case, the probability that Player 1 is the leader at

t is one-half. By slightly postponing the inspection, Player 1 becomes the follower with probability 1, and by slightly advancing the inspection, Player 1 becomes the leader with probability 1. In an equilibrium, a player cannot profit from postponing or advancing the inspection, which implies that the follower gain, as given by Eq. (3), must be equal to zero, which pins down t. In other words, t° is the time in which a player is indifferent between being the leader and follower, if such a time exists.

Inspection of Eq. (6) reveals that when $v_2 < v_1 - c$ the value of t° is positive. Then, the follower gain is equal to zero at t°. It is straightforward to verify that the follower gain is positive at earlier times, and negative at later times.

When $v_2 > v_1 - c$ being the follower is always better than being the leader. Indeed, in this case the follower gain is always strictly positive and t° is either negative or undefined. The next result follows immediately.

Corollary 2. *If $v_2 > v_1 - c$, then a symmetric MPE strategy has no atoms.*

Fix a symmetric MPE strategy. Denote the equilibrium payoff under this MPE by v_0, and the lower and upper bounds of its support by $\underline{t}$ and $\overline{t}$, respectively. The next result establishes a link between v_0 and $\underline{t}$.

Proposition 4. *Let F be a symmetric MPE strategy whose support has a minimum of $\underline{t}$. Then, the players' equilibrium payoff is given by $v_0 = w(\underline{t})$, where $w(\cdot)$ is the single player payoff function given by Eq. (2).*

The proof of this result is the following. Suppose first that there is no atom at $\underline{t}$. The Indifference Principle implies that a player has to be indifferent between all the inspection times in the support of her strategy. In particular, a player has to be willing to inspect at $\underline{t}$, knowing that the other player has not inspected until that time. In fact, inspecting at $\underline{t}, 2\underline{t}, \ldots$ ensures that the other player will never get to inspect. This makes the player's payoff identical to her payoff from inspecting every $\underline{t}$ units of time in the one-player game, which is given by $w(\underline{t})$. If there is an atom at $\underline{t}$, then Proposition 3 implies that $\underline{t} = t^{\circ}$. Hence, the player is indifferent between being the leader or the follower at $\underline{t}$, and the conclusion of Proposition 4 follows.

The next result follows immediately.

Corollary 3. *Let F be a symmetric MPE strategy with an equilibrium payoff v_0. Then, v_0 cannot exceed the leader's prize minus the cost, i.e., $v_0 < v_1 - c$.*

Indeed, by Proposition 4, the equilibrium payoff is equal to the payoff of a single player who inspects at $\underline{t}$. This payoff is (weakly) smaller than the optimal payoff of a single player, which is smaller than $v_1 - c$ because of the potential costs associated with wasted inspections and the time required for the prize to appear.

The next result characterizes the upper bound, $\overline{t}$, of the support of a symmetric MPE strategy.

Proposition 5. *Let F be a symmetric MPE strategy. Then, the supremum of the support of F is either $\overline{t} = \infty$ or $\overline{t} = t^{\circ} > 0$ (defined in Eq. (6)).*

To see the intuition for this result, suppose that $\overline{t}$ is finite and smaller than t°, and that the game has reached time $\overline{t} - \varepsilon$ for some small $\varepsilon > 0$. If a player adheres to the original strategy, symmetry of the MPE implies that she would end up becoming the leader and follower with probability one-half each. However, by postponing all inspections until slightly after $\overline{t}$, the player would become the follower with probability one. Because $\overline{t} < t^\circ$ the follower gain is positive and therefore this deviation is profitable. An analogous argument applies for the case in which $\overline{t}$ is finite and larger than t°.

4.3 The Indifference Condition

Let F be a symmetric MPE strategy (not necessarily continuous). By the Indifference Principle, a player must be indifferent among all inspection times in the support of the strategy F. Thus, the following indifference condition must hold for every t in the support of F:

$$v_0 = (1-F(t))e^{-rt}\left(-c + (1-e^{-\lambda t})v_1 + e^{-\lambda t}v_0\right) + \int_{s=0}^{t} e^{-rs}\left((1-e^{-\lambda s})v_2 + e^{-\lambda s}v_0\right)\mathrm{d}F(s). \tag{IND}$$

To understand the indifference condition (IND), note that the right-hand side of (IND) is a player's expected payoff from inspecting at t, provided that (i) the other player employs F, and (ii) the players would keep employing F in the future. Indeed, if Player 2 does not inspect before or at t, which occurs with probability $1-F(t)$, then Player 1's expected payoff is given by $-c+(1-e^{-\lambda t})v_1 + e^{-\lambda t}v_0$ (see the explanation that follows Eq. (1)). And, if Player 2 inspects at time $s < t$, then Player 1's payoff is given by $(1-e^{-\lambda s})v_2 + e^{-\lambda s}v_0$. The first term describes Player 1's expected payoff if Player 2 discovers the prize at s, and the second term describes her expected payoff if Player 2 does not discover the prize, and the game "restarts" at s.

The indifference condition (IND) is a necessary condition that has to be satisfied for all the times t in the support of F. Our objective is to solve for F and to show that it is indeed a symmetric MPE strategy. We face two challenges. First, the support of F can be a complicated set, and (IND) says nothing about times that lie outside this support. Second, a natural approach for solving (IND) is to differentiate both sides of the equation according to t. But, it is not a-priori clear that (IND) is differentiable, because it is not a-priori clear that F is differentiable. Finally, we have to show that a function F that solves (IND) is indeed a symmetric MPE strategy. The next proposition addresses the first challenge.

Proposition 6. *The support of a symmetric MPE strategy F is convex.*

The proof of this "no holes in the support" property hinges on Corollary 3, which ensures that $v_1 > v_0$. This implies that the delay gain, as given by Eq. (5), is decreasing in t *in equilibrium*, which is necessary in order to prove the proposition.[13]

[13]Thus, if t_1 and t_2 are in the support of F, but the open interval (t_1, t_2) is not in the support of F, then it is beneficial for a player to either postpone inspection from t_1 to $t_1+\varepsilon$ or to advance inspection from t_2 to $t_2-\varepsilon$ for some small $\varepsilon > 0$, which is a contradiction.

4.4 Properties of Continuous Equilibria

Eq. (IND) defines a necessary condition for a symmetric MPE. If the MPE strategy is continuous, then it is possible to derive from Eq. (IND) tighter conditions that facilitate the characterization of the MPE.

Suppose that F is a continuous symmetric MPE strategy that is supported on an interval $[\underline{t}, \overline{t}]$ (and if $\overline{t} = \infty$, then F is supported on $[\underline{t}, \overline{t})$), and denote its density by f. Differentiate both sides of the indifference condition (IND) according to t to obtain the following differential equation:

$$0 = f(t)\left(c - (1-e^{-\lambda t})v_1 + (1-e^{-\lambda t})v_2\right) + (1-F(t))\left(rc - rv_1 + (\lambda+r)e^{-\lambda t}v_1 - (\lambda+r)e^{-\lambda t}v_0\right), \tag{7}$$

which holds for all $t \in (\underline{t}, \overline{t})$.

For the analysis below, it is useful to use the following notation: $x(t) = e^{-\lambda t}$, and

$$A = c - v_1 + v_2, \qquad B = v_1 - v_2, \qquad D = r(c - v_1), \qquad E(v_0) = (\lambda + r)(v_1 - v_0). \tag{8}$$

Note that the quantities A, B, and D depend only on the parameters of the problem. The term $E(v_0)$ depends also on the endogenous equilibrium payoff. Corollary 3 implies that $E(v_0) > 0$. Rearrangement of Eq. (7) yields,

$$f(t)\cdot\left(A + Bx(t)\right) = (1-F(t))\cdot\left(-D - E(v_0)x(t)\right) \tag{9}$$

for all $t \in (\underline{t}, \overline{t})$. Eq. (9) is a differential equation whose solution is a candidate for a continuous symmetric MPE, which is parametrized by v_0.

It is noteworthy that (i) the term $A + Bx(t)$ is the follower gain at t, given by Eq. (3), and (ii) the term $D + E(v_0)\,x(t)$ is the delay gain at t, given by Eq. (5). Thus, by Observation 1, the expressions $A + Bx(t)$ and $D + E(v_0)\,x(t)$ must have different signs in every time t in the support of a continuous MPE.

Eq. (9) is a first order linear differential equation. The next lemma characterizes its solution F under the initial condition that $F(\underline{t}) = 0$.

Lemma 6. *If $v_2 = v_1 - c$ (so that $A = 0$), then the solution of Eq.* (9) *under the initial condition $F(\underline{t}) = 0$ is given by*

$$F(t) = 1 - e^{J(t) - J(\underline{t})}, \tag{10}$$

where $J(t) = \frac{D}{B\lambda}e^{\lambda t} + \frac{E(w(\underline{t}))}{B}t$.

If $v_2 \neq v_1 - c$ (so that $A \neq 0$) and $A + Be^{-\lambda t} > 0$ for all $t \in [\underline{t}, \overline{t})$, then the solution of Eq. (9) under the initial condition $F(\underline{t}) = 0$ is given by:

$$F(t) = 1 - e^{G(t) - G(\underline{t})}, \quad \forall t \in [\underline{t}, \overline{t}], \tag{11}$$

where $G(t) = \frac{D}{A}t + \frac{1}{\lambda}\left(\frac{D}{A} - \frac{E(w(\underline{t}))}{B}\right)\ln\left(A + Be^{-\lambda t}\right)$.

Given a support $[\underline{t}, \overline{t}]$, Eqs. (10) and (11) describe a *candidate* for a continuous symmetric MPE. To show that this candidate is indeed a continuous symmetric MPE, we need to show that it is a CDF, and that there are no profitable deviations outside the support $[\underline{t}, \overline{t}]$.

Remark 1. *To better understand the economic interpretation of Eq. (9), suppose that Player 2 inspects according to the distribution F. What is the gain for Player 1 from delaying her inspection at time t by Δt? Conditional on the event that Player 2 did not inspect until t, Player 1 switches from being the leader to being the follower with probability $\Delta t \cdot f(t)/(1-F(t))$, and she gains $A + Bx(t)$. With the complementary probability, Player 1 inspects at time $t+\Delta t$, and gains $\Delta t \cdot (D + E(v_0)x(t))$. Taken together, this implies that the expected gain from a small delay at time t is given by*

$$H(t,\Delta t) = \frac{\Delta t \cdot f(t)}{1-F(t)} \cdot (A + Bx(t)) + \left(1 - \frac{\Delta t \cdot f(t)}{1-F(t)}\right) \cdot \Delta t \cdot (D + E(v_0)\,x(t)).$$

A necessary condition for the strategy F to be a symmetric MPE strategy is that $\lim_{\Delta t \to 0} (H(t,\Delta t)/\Delta t) = 0$. *This yields Eq. (9).*

5 Markov Perfect Equilibria in the Two-Player Game

In this section we characterize the symmetric MPE of the two-player game. We show that the MPE may take one of two forms, depending on the parameters of the game. The first type of MPE involves frequent synchronized inspections, and the second exhibits slow diffusion in which the players inspect randomly at different times.

Denote $x^* = x(\tau^*) = e^{-\lambda\tau^*}$, where τ^* is the optimal inspection time in the one-player game (see Proposition 1). The following threshold value plays a crucial role in the characterization of equilibrium:

$$\widetilde{v}_2 \equiv v_1 - \frac{c}{1-x^*}. \tag{12}$$

Simple algebraic manipulation shows that $v_2 \geq \widetilde{v}_2$ if and only if $t^\circ \geq \tau^*$, with equality when $v_2 = \widetilde{v}_2$.[14] When $v_2 \leq \widetilde{v}_2$ we say that the follower prize is *small*, and when $v_2 > \widetilde{v}_2$ we say that the follower prize is *large*.

5.1 The Case of a Small Follower Prize: Synchronized Inspection

Suppose that the follower prize is small. Namely, $v_2 \leq \widetilde{v}_2$. Recall that the two-player game exhibits two strategic considerations. On the one hand, the players compete to discover the prize. On the other hand, the presence of another player introduces the possibility to "free-ride." The analysis in this section shows that when the follower prize is small, the competition effect dominates, and each player wants to be the first to discover the prize.

Formally, if $v_2 \leq \widetilde{v}_2$, then the unique symmetric MPE strategy has an atom of probability one at t°. It thus exhibits perfectly synchronized inspections.

Theorem 1. *Suppose that $v_2 \leq \widetilde{v}_2$. The unique symmetric MPE strategy of the two-player game is given by*

$$F^S(t) = \begin{cases} 0, & t < t^\circ, \\ 1, & t \geq t^\circ, \end{cases} \tag{13}$$

[14] To see this, see Step 2 of the proof of Theorem 2.

where t° is defined in Eq. (6).

In the equilibrium described in Theorem 1, both players synchronize their inspections and inspect at times $t^\circ, 2t^\circ, \ldots$. Namely, the players inspect simultaneously at the exact times in which they are indifferent between being leader and follower.

To see why this is an equilibrium, recall that by Proposition 4, the players' payoff under this strategy is $w(t^\circ)$. A player cannot benefit from inspecting earlier because $v_2 \leq \tilde{v}_2$ implies that $t^\circ \leq \tau^*$. This means that by advancing the inspection to $t < t^\circ$, the player's payoff would be equal to the payoff of the player in the one-player game who inspects at the same times. Lemma 3 implies that this payoff is quasiconcave in the inspection time, and so such a deviation is not profitable.[15] And, deviating and inspecting after time t° would turn the player into a follower at t°, which would not affect her payoff.

The intuition for why F^S is the *unique* symmetric MPE strategy hinges on the observation that when $v_2 \leq \tilde{v}_2$ there is no time t in which the follower gain and the delay gain have opposite signs. Therefore the support of the strategy F must be concentrated at t° where the follower gain is zero. While the delay gain at t° is positive, the players cannot profit by delaying their inspections because this would turn them into followers with probability one.[16]

The following corollary is an immediate implication of Theorem 1.

Corollary 4. *If $v_2 \leq \tilde{v}_2$, then the unique symmetric MPE strategy does not exhibit "diffusion": the two players synchronize their inspections. Moreover, they inspect more frequently compared to the one-player benchmark.*

The values of v_1, v_2, and c, affect the frequency of inspection and the equilibrium payoff. As intuitively expected, a larger value of v_1 induces a lower value of t° and hence a higher frequency of inspection. The effect on the equilibrium payoff is more subtle: on the one hand, a higher value of v_1 increases the players' payoff, but on the other hand, the faster inspection decreases it. It can be shown that the first effect dominates, and that the player's equilibrium payoff increases in the value of v_1. The effect of the cost c is exactly the opposite. Increasing it decreases the frequency of inspection as well as the equilibrium payoff.

To see the effect of v_2, note that a larger value of v_2 makes being the leader relatively less attractive and so decreases the frequency of inspection. By Proposition 4, this also increases the players' equilibrium payoff because it delays inspections.[17] While this second result holds true for small values of v_2, this is not the case for large values of v_2, as shown in the next section.

Finally, it is worth highlighting that the rate of inspection in the MPE described in Theorem 1 is unaffected by the players' discount rate r, as long as v_2 is smaller than the cutoff $\tilde{v}_2$ (notice that the discounting rate does affect the value of the cutoff $\tilde{v}_2$).

[15] This argument makes use of the observation that if a player prefers to deviate to an earlier inspection time once, then the player would also want to deviate every time she finds herself in the same situation.

[16] Recall that by definition, a player enjoys the delay gain only if the other player does not inspect in the meantime.

[17] It is noteworthy that unlike v_1 and c, the value of v_2 does not affect the function $w(\cdot)$. Thus, it affects the players' equilibrium payoff only through its effect on the timing of inspection.

5.2 The Case of a Large Follower Prize: Diffused Inspection

When the follower's prize v_2 is large, both players benefit from free-riding on each other. In this section, we show that this has a significant effect on the symmetric equilibrium of the game.

Suppose that $v_2 > \tilde{v}_2$. As we show below, the upper bound of the distribution of the MPE strategy F is given by:

$$\tau^\circ = \begin{cases} t^\circ, & \text{if } v_2 < v_1 - c, \\ \infty, & \text{if } v_2 \geq v_1 - c, \end{cases} \tag{14}$$

where t° is defined in Eq. (6).

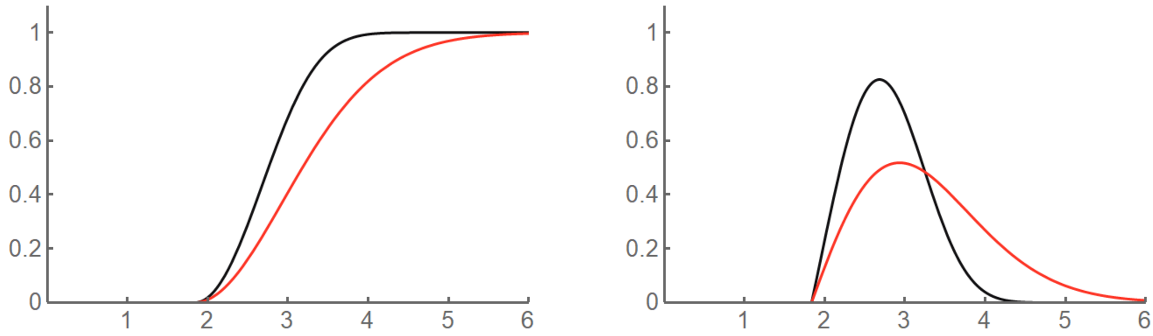
The next proposition describes the unique symmetric MPE of the two-player game in the case where $v_2 > \tilde{v}_2$. Notice that this MPE exhibits diffused inspections.

Theorem 2. *Suppose that $v_2 > \tilde{v}_2$. Then, there is a unique symmetric MPE strategy. The players' strategy in this equilibrium is characterized by a continuous CDF that is given by Eq. (10) (if $v_2 = v_1 - c$) or Eq. (11) (if $v_2 \neq v_1 - c$), that is supported on the interval $\left[\underline{t}, \overline{t}\right] = [\tau^*, \tau^\circ]$.*

To prove Theorem 2 we start by showing that the continuous CDF that is given by Eq. (10) or (11), and is supported on the interval $[\tau^*, \tau^\circ]$, is indeed an MPE. To show this, we first prove that the follower gain $A + Be^{-\lambda t}$ is positive for all t in the support of F, which is a necessary condition for Lemma 6. Next, we show that the delay gain is negative for all t in the support of F. It follows that the derivative f that solves Eq. (9) is positive. Therefore F is monotone increasing, and hence describes a CDF. Finally, we show that inspections outside the support of F are not profitable deviations.

To prove that F is the *unique* symmetric MPE, we first show that the lower bound of the support of any symmetric MPE must be equal to τ^*, and the upper bound is at most t°. This pins down the equilibrium payoff $v_0 = w(\tau^*)$. We then employ a general result in the Theory of Distributions. The right-hand side of (IND) is of the form $(1 - F(t))g(t) + \int_{r=0}^{t} h(r)\mathrm{d}F(r)$, where g and h are two smooth functions, and the left-hand side is a constant. Simple calculations show that the functions g and h can be equal only a finite number of times on the interval $[\underline{t}, \overline{t}]$. According to Theorem 2.24 in Hörmann and Steinbauer (2009), on any interval in which g and h are not equal, if a continuous solution to (IND) exists, then this solution is unique among all possible solutions, continuous or not. Because the distribution F in Eq. (10) or (11) is differentiable, it follows that (IND) does not admit any non-differentiable solutions.

Figure 1 depicts a symmetric MPE strategy and its density for different parameter values. It is noteworthy that the density of the distribution of inspection times in an MPE is first increasing and then decreasing in t. As can be seen in the figure, a higher value of the follower prize v_2 stretches the support of the MPE strategy to the right (towards larger values). By contrast, the lower bound of the support of the MPE CDF is always equal to τ^*, for any value of $v_2 > \tilde{v}_2$. This last fact, together with Proposition 4, implies the following result.



(a) CDFs of symmetric MPEs

(b) PDFs of symmetric MPEs

Figure 1: CDFs and PDFs of symmetric MPE strategies for different parameter values. The black curves correspond to the parameters $v_1 = 10$, $v_2 = 7.5$, $c = 2$, $r = 0.75$, $\lambda = 0.25$, and $\tilde{v}_2 = 4.566$. The red curves correspond to the same parameters, with the only difference that $v_2 = 9.5$.

Corollary 5. *Suppose that $v_2 > \tilde{v}_2$. Then, the players' symmetric MPE payoff v_0 is equal to the payoff in the one-player problem.*

It is also noteworthy that by Corollary 5, as long as $v_2 > \tilde{v}_2$ the players' equilibrium payoff is *independent of the value of* v_2. Intuitively, this independence is due to the fact that as v_2 increases, the players intensify free-riding on each other by delaying their inspections. This is illustrated in Figure 1 that shows that the symmetric MPE strategy with a larger v_2 first-order stochastically dominates the symmetric equilibrium strategy with a smaller v_2.

5.3 Social Welfare

Suppose that the players' utilities are quasilinear, so that utility is transferable. Thus, the socially optimal inspection policy is given by the solution to the one-player problem, where the prize is given by $v_1 + v_2$.

Throughout the analysis the value of v_2 played a crucial role in the characterization of equilibrium. The next proposition demonstrates the effect of the value of v_2 on social welfare.

Proposition 7. *There exists a unique threshold $\hat{v}_2 = \hat{v}_2(v_1, c, \lambda, r) \leq \tilde{v}_2$ such that:*

(i) when $v_2 < \hat{v}_2$, the players inspect "too frequently" relative to the socially optimal policy;

(ii) when $v_2 > \hat{v}_2$, the players inspect "too rarely" relative to the socially optimal policy;

(iii) when $v_2 = \hat{v}_2$, the players inspect according to the socially optimal policy.

The key idea is that when v_2 is small, the competition between the two players is fierce. The players inspect too frequently in an attempt to be the first player to find the prize, which erodes their joint welfare. By contrast, when v_2 is high the players try to free-ride on each other, and therefore they inspect too infrequently. There is a unique value of v_2 for which the two effects exactly offset each other.

An implication of the fact that $\hat{v}_2 \leq \tilde{v}_2$ is that when an economist lacks direct access to the model's parameters but can observe the equilibrium dynamics in play, Proposition 7 indicates that the observation of diffused inspections suggests that players inspect too rarely compared to the social optimum.

Furthermore, our analysis can also potentially inform the design of regulatory policy. For example, consider the problem faced by a social planner (or, a regulator) who needs to divide a resource amount $v > c$ between a leader's prize (v_1) and a follower's prize (v_2) such that $v_1 + v_2 = v$. One conclusion that follows from Corollary 5 is that having a follower prize v_2 greater than $\widetilde{v}_2$ is Pareto dominated. Moreover, when players' utilities are quasi-linear, it is straightforward to verify that the socially optimal inspection policy can be obtained by solving the one-player problem with a prize of size v.

Denote the socially optimal (recurrent) inspection time by $\tau^*(v)$ (see Proposition 1). Can the designer choose a prize scheme (v_1, v_2) that satisfies $v_1 + v_2 = v$ *and* induces socially optimal inspection level *in equilibrium*? The answer is positive. Intuitively, if the prize scheme is $(v_1 = v, v_2 = 0)$, the competition force governs the players' equilibrium behavior, causing them to inspect too frequently compared to the socially optimal inspection level. Conversely, if the prize scheme is $(v_1 = c + \varepsilon, v_2 = v - c - \varepsilon)$ for some small $\varepsilon > 0$, then the free-riding force governs the players' equilibrium behavior, causing them to inspect too little compared to the socially optimal inspection level. Continuity of the quantities τ^*, t°, and $\widetilde{v}_2$ in v_1 and in v_2 (see Eqs. (1), (6), and (12)) implies the existence of a value of the follower prize $\overline{v}_2$ for which the equilibrium inspection frequency coincides with the socially optimal level.[18]

6 Discussion

6.1 Other Equilibria

Thus far, our analysis has focused on the symmetric Markov perfect equilibria (MPEs) of the game. Symmetric MPEs possess two appealing characteristics. Firstly, they consist of Markov strategies, which, intuitively, require less sophistication from the players as they allow the players to only condition their behavior on payoff-relevant variables. Secondly, since the game and the players are ex-ante symmetric, studying symmetric equilibria is a natural choice. Consequently, symmetric MPEs serve as a natural starting point for analyzing the game.

However, the game also has asymmetric equilibria. For example, consider the case in which $v_2 > v_1$. Then, the following pair of strategies is an equilibrium. Player 1 inspects at the optimal frequency in the single-player problem, i.e., every τ^* units of time; Player 2 never inspects. To see that this is an equilibrium of the game, notice that given that Player 2 never inspects, Player 1 essentially faces a single-player problem and so responds optimally. And, because $v_2 > v_1$, Player 2 cannot do better than free-riding on Player 1's inspection.

The game also exhibits symmetric equilibria that are not Markov strategies. An example of such an equilibrium can be illustrated as follows: Assume that $v_1 = 1$, $v_2 = 0$, $c = 0.05$, $r = \lambda = 0.5$. Computation shows that in this case, $\tau^* \approx 0.646$. Consider the following symmetric strategies: Both players inspect every

[18] Specifically, $\overline{v}_2$ has to satisfy the equation $t^\circ(v_1 = v - \overline{v}_2, v_2 = \overline{v}_2) = \tau^*(v)$, and thus can be computed numerically.

τ^* units of time until the prize is discovered or until one of the players deviates. If a player deviates, then the players revert to playing the MPE strategies characterized in Theorem 1, where they inspect every $t^\circ \approx 0.1$ units of time. Notice that because $v_2 = 0$, the players' continuation payoff after reverting to the strategies characterized by Theorem 1 is zero.

To understand why this is an equilibrium, observe that the most profitable deviation for a player is to inspect at $\tau^* - \varepsilon$ for some small $\varepsilon > 0$. However, calculations reveal that the gain from such a deviation is not significant enough to warrant the deviation, taking into account the fact that in the continuation game the players will inspect "too frequently," every $t^\circ \approx 0.1$ units of time. These strategies are not Markov strategies because the players' actions depend on components of history (specifically, whether a player has deviated before or not), which are not payoff relevant.

6.2 Modelling Assumptions

In this section we discuss two of our modelling assumptions and explore two natural alternatives to these assumptions and their implications.

Observable vs. Unobservable Actions

Throughout our analysis, we have assumed that the players' inspection times are observable. This assumption implies that both players have symmetric information at all times. While this assumption holds in certain environments, there are certainly other scenarios where it is less plausible.

In the parameter range in which the follower prize is small (i.e., $v_2 < \tilde{v}_2$) our analysis and the characterization of the symmetric MPE are unchanged under the assumption that inspection times are unobservable. The reason is that, as stated in Theorem 1, when the follower prize is small, the players inspect simultaneously at t°, which is the point in time in which a player is indifferent between being the leader and the follower. This also implies that a player would still inspect at the same time as long as the strategy of the other player is to inspect at t°, even if the inspection time itself is not observable.

This reasoning no longer holds in the parameter range in which the follower prize is large because in this case the fact that the symmetric MPE involves mixed strategies implies that the players would be asymmetrically informed about past inspection times, which affects their strategic considerations. The characterization of MPE in this environment remains an open question.

Payoffs upon Simultaneous Inspection

In our model, we made the assumption that if the players inspect simultaneously, they are randomly assigned the roles of leader (incurring the inspection cost c and obtaining the payoff v_1 if the prize has already appeared) and follower (not incurring the inspection cost and obtaining the payoff v_2 if the prize has already appeared). An alternative and plausible assumption is that if both players perform their inspections simultaneously, then *both* players incur the inspection cost c, while the allocation of the leader and follower prizes remains random.

In the parameter range where the follower prize is large ($v_2 > \widetilde{v}_2$), the MPE characterized in Theorem 2 remains valid. This is due to the fact that this MPE involves continuous strategies, and therefore the probability of the event in which both players inspect simultaneously is zero.

However, in the parameter range where the follower prize is small ($v_2 < \widetilde{v}_2$), the MPE characterized in Theorem 1 does not hold under this alternative assumption. This is because a player can gain a payoff that is bounded away from zero by slightly advancing their inspection time. Intuitively, this reasoning suggests that a symmetric MPE strategy cannot include an atom. Therefore, in the absence of any other special adjustments to the model (such as the use of a correlation device), no symmetric MPE exists in this case.[19]

[19]The relationship between the type of tie-breaking assumption employed and the existence or non-existence of equilibrium is well-established. For further discussion on this matter and the potential adaptations introduced by the literature, see Fudenberg and Tirole (1991).

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Appendix: Proofs

Proof of Lemma 1

We will construct the unique probability distribution over plays induced by (σ_1, σ_2) by transfinite induction. The distribution of t_0 is determined by $\sigma_1(\emptyset)$ and $\sigma_2(\emptyset)$.

Let α_* be a limit ordinal, and suppose that the distribution over histories with $\alpha < \alpha_*$ is uniquely defined. Then $t_{\alpha_*} = \sup_{\alpha<\alpha_*} t_\alpha$.

Let $\alpha_* + 1$ be a successor ordinal, and suppose that the distribution over histories with $\alpha \le \alpha_*$ is uniquely defined. Denote by h the random variable of the history up to t_{α_*}. The conditional distribution of t_{α_*+1} given h is determined by $\sigma_1(h)$ and $\sigma_2(h)$.

Proof of Lemma 2

To prove the proposition we need some notation. For every pair of strategies (σ_1, σ_2) and every history $h = \langle \alpha^*, \{t_\alpha\}_{\alpha \le \alpha^*}, \iota \rangle$, denote by $U(\sigma_1, \sigma_2; h)$ the expected payoff of Player 1 under (σ_1, σ_2) in the subgame that starts at h. The quantities $(U(\sigma_1, \sigma_2; h))_h$ are related through the following recursive equation:

$$\begin{aligned} U(\sigma_1,\sigma_2;h) \;\; = \;\; & \int_{s_1,s_2\in(t_{\alpha^*},\infty]} \Bigg(\left(1_{s_1<s_2}+\frac{1}{2}\cdot 1_{s_1=s_2}\right)e^{-r(s_1-t_{\alpha^*})}\left(-c+(1-e^{-\lambda(s_1-t_{\alpha^*})})v_1+e^{-\lambda(s_1-t_{\alpha^*})}U(\sigma_1,\sigma_2;h')\right) \\ & +\left(1_{s_1>s_2}+\frac{1}{2}\cdot 1_{s_1=s_2}\right)e^{-r(s_1-t_{\alpha^*})}\left((1-e^{-\lambda(s_2-t_{\alpha^*})})v_2+e^{-\lambda(s_2-t_{\alpha^*})}U(\sigma_1,\sigma_2;h')\right)\Bigg)\sigma_1(h)(\mathrm{d}s_1)\sigma_2(h)(\mathrm{d}s_2), \end{aligned} \tag{15}$$

where h' is the history after the current one, so it is created by adding to h either $(t_{\alpha^*+1} = s_1, \iota_{\alpha^*+1} = \{1\})$ (if $s_1 < s_2$), or $(t_{\alpha^*+1} = s_2, \iota_{\alpha^*+1} = \{2\})$ (if $s_1 > s_2$), or $(t_{\alpha^*+1} = s_1, \iota_{\alpha^*+1} = \{1,2\})$ (if $s_1 = s_2 < \infty$).

Denote

$$\begin{aligned} \widetilde{U}(\sigma_1,\sigma_2,h) \;\; := \;\; & \int_{s_1,s_2\in(t_{\alpha^*},\infty]} \Bigg(\left(1_{s_1<s_2}+\frac{1}{2}\cdot 1_{s_1=s_2}\right)e^{-r(s_1-t_{\alpha^*})}\left(-c+(1-e^{-\lambda(s_1-t_{\alpha^*})})v_1\right) \\ & +\left(1_{s_1>s_2}+\frac{1}{2}\cdot 1_{s_1=s_2}\right)e^{-r(s_1-t_{\alpha^*})}\left((1-e^{-\lambda(s_2-t_{\alpha^*})})v_2\right)\Bigg)\sigma_1(h)(\mathrm{d}s_1)\sigma_2(h)(\mathrm{d}s_2), \end{aligned}$$

the unconditional expected payoff of Player 1 due to the first inspection done after t_{α^*}, and by

$$\widetilde{P}(\sigma_1,\sigma_2,h) := \int_{s_1,s_2\in(t_{\alpha^*},\infty]} e^{-r(\min\{s_1,s_2\}-t_{\alpha^*})}\sigma_1(h)(\mathrm{d}s_1)\sigma_2(h)(\mathrm{d}s_2),$$

the expected discounted time (normalized to time t_{α^*}) until the next inspection. Finally, define

$$\Lambda(\sigma_1,\sigma_2,h) := \widetilde{U}(\sigma_1,\sigma_2,h)/\widetilde{P}(\sigma_1,\sigma_2,h),$$

to be the normalized expected payoff to Player 1 until the next inspection. When σ_1 and σ_2 are Markov, $\Lambda(\sigma_1, \sigma_2, h)$ is independent of h, and coincides with Player 1's expected payoff under (σ_1, σ_2).

Player 1's expected payoff under (σ_1, σ_2) is a convex combination of $(\Lambda(\sigma_1, \sigma_2, h))_h$, where the weight of $\Lambda(\sigma_1, \sigma_2, h)$ is given by the unconditional probability that the first inspection done after history h is successful.

Fix now a Markov strategy σ_2 of Player 2, and let σ_1 be an $(\varepsilon/2)$-best response of Player 1 to σ_2. Denote

$u := \sup_h \Lambda(\sigma_1, \sigma_2, h)$, where the supremum is over all histories h. Since the expected payoff under (σ_1, σ_2) is a convex combination of $(\Lambda(\sigma_1, \sigma_2, h))_h$, it follows that this expected payoff is at most u.

Let h_0 be a history such that

$$\Lambda(\sigma_1, \sigma_2, h_0) \geq u - \frac{\varepsilon}{2}.$$

Finally, let σ_1' be the Markov strategy that is defined by $\sigma_1(h_0)$: after each inspection, the distribution of the next inspection time of Player 1 is according to $\sigma_1(h_0)$, shifted to time of the last inspection.

Since σ_1' and σ_2 are Markov strategies, the expected payoff under (σ_1', σ_2) is $\Lambda(\sigma_1', \sigma_2, h)$ (and this quantity is independent of h). By the choice of h_0, this expected payoff is at least $u - \varepsilon/2$, and hence at least the expected payoff under (σ_1, σ_2) plus ε. The proposition follows.

Proof of Lemma 3

Substituting $x = e^{-\lambda t}$ in $w(\cdot)$, and differentiating $w(\cdot)$ with respect to x, yields:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{-c + (1-x)\, v_1}{x^{-(r/\lambda)} - x}\right) = \frac{(v_1 - c)\, x^{(r/\lambda)+1} + \left(-v_1 \cdot \frac{\lambda + r}{\lambda}\right) x + (v_1 - c)\,(r/\lambda)}{x^{(r/\lambda)+1}\left(x^{-(r/\lambda)} - x\right)^2}. \tag{16}$$

The denominator on the right-hand side of Eq. (16) is positive. Hence, the sign of the derivative is determined solely by the sign of the numerator on the right-hand side. It is easy to verify that this sign is positive at $x = 0$ and negative at $x = 1$. And, because the derivative of the numerator with respect to x is given by $-\frac{1}{\lambda}(\lambda + r)\left(\left(1 - x^{\frac{r}{\lambda}}\right) v_1 + c x^{\frac{1}{\lambda} r}\right)$, which is negative for all $x \in [0,1]$, the numerator is decreasing in x. Let x^* denote the (unique) value for which the numerator is equal to zero. Thus, $w(x)$ is increasing for all $x < x^*$ and decreasing for all $x > x^*$. The result follows from the fact that $x = e^{-\lambda t}$ is decreasing in t.

Proof of Proposition 1

We first prove that the player has an optimal strategy which is Markov. We then show that it is given by inspecting at times $k\tau^*$, $k \in \mathbb{N}$, until the prize is found.

To prove that the player has an optimal strategy which is Markov, we use a result due to Jasso-Fuentes, Menaldi and Prieto-Rumeau (2020) or Stachurski and Zhang (2021), which provide conditions under which a Markov decision problem with general state space and state/action-dependent discount factor admits the dynamic programming principle.

Our first goal is, then, to provide an alternative representation of the decision problem, where time is discrete. Suppose the set of states is $\mathbb{N} \times [0, \infty]$: the first coordinate counts the number of inspections that were already made by the players, and the second coordinate corresponds to time. The initial state is $(0,0)$. The set of actions of the player at state (k, t) for $t < \infty$ is $[t, \infty]$; the interpretation of action $a \in (t, \infty)$ is that the player inspects at time a, the action $a = \infty$ corresponds to no future inspection, and the action $a = t$ is interpreted as doing another inspection at time t. The payoff that corresponds to action a

is $-c+(1-e^{-\lambda(a-t)})v_1$, with probability $1-e^{-\lambda(a-t)}$ the prize is found and the game terminates, and with probability $e^{-\lambda(a-t)}$ the prize is not found, and the game continues to state $(k+1,a)$, with a state/action-dependent discount factor $e^{-r(a-t)}$. The states (k,∞) for $k\in\mathbb{N}$ are absorbing; the player has no available actions, and the payoff is 0.

The payoff function is continuous, the action sets are compact, and the transitions are continuous, hence by Jasso-Fuentes, Menaldi and Prieto-Rumeau (2020) or Stachurski and Zhang (2021) the dynamic-programming principle applies, and the player has an optimal strategy σ, which is pure and Markov. The reader can verify that σ is also optimal in the original problem.

The definition of τ^* implies that the optimal pure Markov strategy is to inspect at times $k\tau^*$ for $k\in\mathbb{N}$, which completes the proof.

Proof of Lemma 4

As in the proof of Lemma 3, substitute $x=e^{-\lambda t}$ in $w(\cdot)$ and recall that the optimal sampling time τ^* is determined by the value x^* that solves $\frac{\mathrm{d}w(\cdot)}{\mathrm{d}x}(x)=0$. By Eq. (16),

$$\frac{\lambda\cdot(x^*)^{\frac{r+\lambda}{\lambda}}+r}{x^*}=\frac{v_1}{v_1-c}(\lambda+r)\,. \tag{17}$$

The right-hand side is decreasing in v_1 and increasing in c, and since $x^*\in(0,1]$, the left-hand side is decreasing in x^*. It follows that x^* is increasing in v_1, and decreasing in c, as claimed.

Proof of Lemma 5

By Proposition 1, in the single player problem, the optimal strategy is to inspect every τ^* units of time. By the Dynamic Programming Principle of Optimality (which holds because of the argument presented in the proof of Proposition 1), it follows that in the one-player problem, if the continuation payoff following a failed inspection is fixed at $w(\tau^*)$, then the optimal inspection time is τ^* (possibly not unique).

Therefore, the derivative according to t of the expression $e^{-rt}\left(-c+(1-e^{\lambda t})v_1+e^{\lambda t}w(\tau^*)\right)$, which is similar to the expression on the right-hand side of Eq. (1) where $w(t)$ is replaced by $w(\tau^*)$, is equal to zero at $t=\tau^*$. It follows that if $w(\tau^*)=v_0$, then $L'(\tau^*)=0$ and so also the delay gain $L'(\tau^*)/e^{-r\tau^*}=0$.

Uniqueness follows from the fact that $v_0=w(\tau^*)<v_1$, and therefore the expression in Eq. (5) is monotonic in t and can be equal to zero for at most one value of t.

Proof of Proposition 2

Assume to the contrary that $F(\infty):=\lim_{t\to\infty}F(t)<1$. Let $\varepsilon>0$ be sufficiently small such that $v_1>c+\varepsilon$. Let $s>0$ be sufficiently large such that $\frac{F(\infty)-F(s)}{1-F(s)}<\varepsilon$. In words, the conditional probability that a player will inspect after time s, provided she did not inspect up to time s, is smaller than ε. Such s exists since $F(\infty)<1$. Assume, without loss of generality, that s is sufficiently large so that $(1-e^{-\lambda s})v_1>\varepsilon(v_1+v_2)$.

The probability that no player inspects before time s is $(1-F(s))^2 > 0$. All subsequent calculations assume that no player inspected before time s and are discounted to time s.

Since the probability that a player inspects after time s is $\frac{F(\infty)-F(s)}{1-F(s)} < \varepsilon$, the expected payoff of a player who follows F is smaller than $\varepsilon \cdot (v_1 + v_2)$. Since $v_2 > 0$, the payoff of a player who inspects at s is at least $(1-e^{-\lambda s})v_1$.[20] Yet, by our choice of s we have $(1-e^{-\lambda s})v_1 > \varepsilon(v_1+v_2)$, which contradicts the assumption that F is an equilibrium.

Proof of Proposition 3

The idea of the proof is as follows: when both players inspect at t, the identity of the leader and the follower is determined randomly. If there is an atom of size $\widehat{q}$ at t, then the probability that at least one of the players inspects at t is $1-(1-\widehat{q})^2$, and hence the probability that a specific player will be the leader is half this quantity, which is $\widehat{q} - \frac{\widehat{q}^2}{2} = \widehat{q}(1-\frac{\widehat{q}}{2})$, which is smaller than $\widehat{q}$ and larger than $\widehat{q}(1-\widehat{q})$. By slightly delaying the inspection to $t+\varepsilon$ (and ignoring the probability of an inspection between t and $t+\varepsilon$, which goes to 0 with ε), the player lowers the probability to be the leader because of this atom to $\widehat{q}(1-\widehat{q})$ while by slightly advancing the inspection to $t-\varepsilon$ (again, ignoring the probability of an inspection between $t-\varepsilon$ and t), the player increases the probability to be the leader because of this atom to $\widehat{q}$. In an equilibrium, the player cannot profit by delaying the inspection at t or by advancing it in time, which implies that both eventualities yield her the same payoff. It turns out that this condition pins down t.

We turn this idea into a formal proof. All calculation we do next are discounted to t and conditional that no player inspected before t. The payoff of a player who inspects alone at t is

$$\gamma_1 := -c + (1-e^{-\lambda t})v_1 + e^{-\lambda t}v_0, \tag{18}$$

and the payoff of the other player is

$$\gamma_2 := (1-e^{-\lambda t})v_2 + e^{-\lambda t}v_0. \tag{19}$$

where v_0 in both equations denotes the equilibrium payoff. The payoff of the players if they both inspect at t is

$$\gamma_{12} := \frac{1}{2}\gamma_1 + \frac{1}{2}\gamma_2. \tag{20}$$

Denote by γ_0 the payoff of each player if no-one inspects at t.

Denote the size of the atom at t by

$$\widehat{q} := F(t) - F(t_-),$$

and the conditional probability of inspecting at t if no inspection was done by that player before t by

$$q := \frac{\widehat{q}}{1-F(t_-)}.$$

[20] With probability $1-e^{-\lambda s}$, the player discovers the prize at s, and obtains the payoff v_1; with the complimentary probability, she obtains the equilibrium payoff v_0, which is positive because $v_2 \geq 0$.

Discounted to t, a player's payoff under F, conditional that no player inspected before t, is

$$g_1 = q^2\gamma_{12} + q(1-q)\gamma_1 + q(1-q)\gamma_2 + (1-q)^2\gamma_0. \tag{21}$$

For every $\varepsilon > 0$ denote by F_-^ε the strategy that moves the atom at t to $t-\varepsilon$, and by F_+^ε the strategy that moves the atom at t to $t+\varepsilon$. Denote by g_-^ε the payoff of a player who adopts the strategy F_-^ε until the first inspection and afterwards follows F, while her opponent follows F (discounted to $t-\varepsilon$, conditional that the prize was not inspected before $t-\varepsilon$). Then

$$g_- := \lim_{\varepsilon\to 0} g_-^\varepsilon = q\gamma_1 + q(1-q)\gamma_2 + (1-q)^2\gamma_0. \tag{22}$$

For every $\varepsilon > 0$ denote by g_+^ε the payoff of a player who adopts the strategy F_+^ε until the first inspection and afterwards follows F, while her opponent follows F (discounted to $t+\varepsilon$, conditional that the prize was not inspected before t). Then

$$g_+ := \lim_{\varepsilon\to 0} g_+^\varepsilon = q\gamma_2 + q(1-q)\gamma_1 + (1-q)^2\gamma_0. \tag{23}$$

In particular, $g_1 = \frac{1}{2}g_- + \frac{1}{2}g_+$. Since F is an equilibrium, $g_-, g_+ \le g_1$. Hence,

$$g_1 = \frac{1}{2}g_- + \frac{1}{2}g_+ \le \frac{1}{2}g_1 + \frac{1}{2}g_1 = g_1.$$

It follows that $g_+ = g_- = g_1$, and hence $\gamma_1 = \gamma_2$, or, equivalently, $e^{-\lambda t} = \frac{v_1 - v_2 - c}{v_1 - v_2}$. The result follows. The equality $\gamma_1 = \gamma_2$ further implies that the player is indifferent between being the leader or the follower at t.

Proof of Proposition 4

The proof appears in the text.

Proof of Proposition 5

By Proposition 2, $F(\bar{t}) = 1$. Suppose by way of contradiction that $\bar{t}$ is finite and different from t°. Thus, by Proposition 3 the follower gain at $\bar{t}$ is different from zero. We will derive a contradiction, whether the follower gain is positive or negative.

If the follower gain at $\bar{t}$ is positive, then fixing the strategy of Player 2, it is beneficial for Player 1 to postpone all inspections in the interval $[\bar{t}-\varepsilon, \bar{t}]$ to $\bar{t}+\delta$, for any small $\delta > 0$ (i.e., immediately after $\bar{t}$). By doing so, Player 1 ensures that she becomes the follower conditional on having reached $\bar{t}-\varepsilon$. This improves her payoff because the follower gain is bounded away from zero, while the cost of discounting that is induced by the additional delay is negligible.[21]

Analogously, if the follower gain at $\bar{t}$ is negative, then it is beneficial to advance all inspections in the interval $[\bar{t}-\varepsilon, \bar{t}]$ to $\bar{t}-\varepsilon$. This way, conditional on reaching $\bar{t}-\varepsilon$, Player 1 increases the probability of being the leader from 1/2 to 1 while the probability to find the prize is only negligibly affected.

[21]This cost arises due to the delay in future inspections.

Proof of Proposition 6

Recall that the support of F is a closed set, so its complement is an open set. Suppose by contradiction that the support of F is not convex. Then there is an interval $(\widehat{t}_0, \widehat{t}_1)$ that is disjoint from the support of F, yet $\widehat{t}_0$ and $\widehat{t}_1$ lie in the support of F. It follows that the integral on the right-hand side of (IND) is the same for every $t \in [\widehat{t}_0, \widehat{t}_1)$, because $\mathrm{d}F(r)$ vanishes inside this interval.

We distinguish between two cases, where $\widehat{t}_1 \neq t^\circ$ and where $\widehat{t}_0 \neq t^\circ$. Since $\widehat{t}_1 \neq \widehat{t}_0$, at least one of these cases holds.

Case 1: Suppose that $\widehat{t}_1 \neq t^\circ$. Since there is no atom at $\widehat{t}_1$, both $1 - F(t)$ and the integral on the right-hand side of (IND) are constant on $[\widehat{t}_0, \widehat{t}_1]$ (and not only on $[\widehat{t}_0, \widehat{t}_1)$). Denote the coefficient of $(1 - F(t))$ in (IND) by

$$h(t) := e^{-rt}\left(-c + (1 - e^{-\lambda t})v_1 + e^{-\lambda t} v_0\right), \quad \forall s \in [\widehat{t}_0, \widehat{t}_1]. \tag{24}$$

The function h is differentiable. Since F is a symmetric MPE strategy, for every $t \in (\widehat{t}_0, \widehat{t}_1)$ the left-hand side of (IND) is larger than or equal to the right-hand side of (IND). It follows that

$$h(t) \leq h(\widehat{t}_0) = h(\widehat{t}_1), \quad \forall t \in [\widehat{t}_0, \widehat{t}_1].$$

Hence, $h'(\widehat{t}_0) \leq 0$ and $h'(\widehat{t}_1) \geq 0$. The derivative of h is

$$h'(t) = e^{-rt}\left((r+\lambda)e^{-\lambda t}(v_1 - v_0) - r(v_1 - c)\right). \tag{25}$$

The term e^{-rt} does not change the sign of the product in Eq. (25), and the term in the parenthesis is decreasing with t (since $v_1 > v_0$ by Corollary 3). This contradicts the fact that $h'(\widehat{t}_0) \leq 0$ and $h'(\widehat{t}_1) \geq 0$.

Case 2: Suppose that $\widehat{t}_0 \neq t^\circ$. For every $t \leq \widehat{t}_1$ denote

$$f(t) := e^{-\lambda t}v_1 + (1 - e^{-\lambda t})v_0, \quad g(t) := e^{-r(\widehat{t}_1 - t)}\left(e^{-\lambda \widehat{t}_1}v_1 + (1 - e^{-\lambda \widehat{t}_1})v_0\right). \tag{26}$$

Consider the situation faced by a player at $\widehat{t}_0$, conditional on the other player not inspecting before, and discount payoffs to $\widehat{t}_0$. Since there is no atom at $\widehat{t}_0$, if the player inspects at $\widehat{t}_0$, her payoff is $f(\widehat{t}_0)$. Since $(\widehat{t}_0, \widehat{t}_1)$ is disjoint of the support of F, the probability that the other player inspects in the interval $[\widehat{t}_0, \widehat{t}_1)$ is zero. Hence, if the player inspects at $\widehat{t}_1$, her payoff is $g(\widehat{t}_0)$.[22]

By the indifference principle, the player is indifferent between inspection at times $\widehat{t}_0$ and $\widehat{t}_1$, and therefore $f(\widehat{t}_0) = g(\widehat{t}_0)$. But this is impossible when $\widehat{t}_0 \neq \widehat{t}_1$. To see this, notice that by Eq. (26) we have that $f(\widehat{t}_1) = g(\widehat{t}_1)$,

$$f'(t) = -\lambda e^{-\lambda t}(v_1 - v_0) < 0, \quad \forall t \leq \widehat{t}_1,$$

and

$$g'(t) = re^{-r(\widehat{t}_1 - t)}\left(e^{-\lambda \widehat{t}_1}v_1 + (1 - e^{-\lambda \widehat{t}_1})v_0\right) > 0, \quad \forall t \leq \widehat{t}_1.$$

Therefore there does not exist a time $t < \widehat{t}_1$ such that $f(t) = g(t)$.

[22] If $\widehat{t}_1 = t^\circ$, the player is indifferent between being the leader and follower, and this equation still holds.

Proof of Lemma 6

When $A = 0$, we necessarily have $B = c$, and Eq. (9) reduces to

$$f(t)\cdot cx(t) = (1-F(t))\cdot\left(-D - E(v_0)x(t)\right).$$

This equation can be written as

$$-\frac{f(t)}{1-F(t)} = \frac{-D - E(v_0)e^{\lambda t}}{ce^{\lambda t}},$$

whose solution is $F(t) = 1 - e^{J(t)-J(\underline{t})}$, where $J(t) = \frac{D}{c\lambda}e^{\lambda t} + \frac{E(w(\underline{t}))}{c}t$.

When $A \neq 0$, Eq. (9) can be equivalently presented as

$$-\frac{f(t)}{1-F(t)} = \frac{D + E(\underline{t})\cdot e^{-\lambda t}}{A + Be^{-\lambda t}} = \frac{E(\underline{t})}{B} + \left(D - \frac{AE(\underline{t})}{B}\right)\frac{1}{A + Be^{-\lambda t}}, \quad \forall t \in (\underline{t}, \overline{t}),$$

and therefore in the interval $[\underline{t}, \overline{t}]$ we have

$$\int\left(-\frac{f(t)}{1-F(t)}\right)\mathrm{d}t = \int\left(\frac{E(\underline{t})}{B} + \left(D - \frac{AE(\underline{t})}{B}\right)\frac{1}{A + Be^{-\lambda t}}\right)\mathrm{d}t.$$

Using the facts that $\int\frac{1}{A+Be^{-\lambda t}}\mathrm{d}t = \left(\frac{t}{A} + \frac{1}{A\lambda}\ln\left(A + Be^{-\lambda t}\right)\right) + K$ (when $A, \lambda \neq 0$ and $A + Be^{-\lambda t} > 0$ in the interval of integration) and $\int -\frac{f(t)}{1-F(t)}\mathrm{d}t = \ln(1-F(t)) + K$, we obtain:

$$\begin{aligned}\ln(1-F(t)) &= \frac{E(\underline{t})}{B}t + \left(D - \frac{AE(\underline{t})}{B}\right)\left(\frac{t}{A} + \frac{1}{A\lambda}\ln\left(A + Be^{-\lambda t}\right)\right) + K\\ &= \frac{D}{A}t + \frac{1}{\lambda}\left(\frac{D}{A} - \frac{E(\underline{t})}{B}\right)\ln\left(A + Be^{-\lambda t}\right) + K, \quad \forall t \in [\underline{t}, \overline{t}].\end{aligned}$$

Denoting $G(t) = \frac{D}{A}t + \frac{1}{\lambda}\left(\frac{D}{A} - \frac{E(\underline{t})}{B}\right)\ln\left(A + Be^{-\lambda t}\right)$, we obtain Eq. (11). Thus, when $A \neq 0$, the only continuous solution of Eq. (9) is given by Eq. (11).

Proof of Theorem 1

The proof consists of four steps. We first argue that F^S is a symmetric MPE.

Step 1: F^S defined in Eq. (13) is a symmetric MPE strategy.

By Proposition 4, the payoff under F^S is $w(t^\circ)$. As mentioned before, the condition $v_2 \leq \tilde{v}_2$ implies that $t^\circ \leq \tau^*$, and because by Lemma 3 $w(\cdot)$ is quasiconcave and attains its maximum at τ^*, it follows that no player can profit by deviating and inspecting at some time $t < t^\circ$ (a deviation that yields $w(t)$). And, deviating and inspecting after time t° only increases the probability that the player is the follower at t°. This is not profitable because, by Proposition 3, the follower gain at t° is zero.

To prove that F^S is the unique symmetric MPE, we fix a symmetric MPE strategy $F \neq F^S$, and show that (i) the support of F is included in $\left[\min\left\{t^\circ, t^D\right\}, \max\left\{t^\circ, t^D\right\}\right]$, where t^D is the value that sets the delay gain equal to zero (that is, such that $D + E(v_0)\cdot x\left(t^D\right) = 0$), (ii) $t^D \leq t^\circ$, and (iii) $t^D \geq t^\circ$. It follows that the probability mass in F must be concentrated at t°, which implies that it coincides with F^S. A contradiction.

Let $\underline{t}$ and $\overline{t}$ denote the lower and upper bounds, respectively, of the support of an MPE strategy F, and let $v_0 = w(\underline{t})$ denote the equilibrium payoff (see Proposition 4). Recall that t° is such that the follower gain $A + Bx(t^\circ) = 0$.

Step 2: The support of F is included in the interval $\left[\min\left\{t^\circ, t^D\right\}, \max\left\{t^\circ, t^D\right\}\right]$.

By Corollary 3, the delay gain is decreasing in t. Since $v_2 \le \tilde{v}_2 < v_1$, the follower gain is decreasing in t as well. Therefore, for every $t < \min\left\{t^\circ, t^D\right\}$ both $A + Bx(t)$ and $D + E(v_0) \cdot x(t)$ are positive, and for every $t > \max\left\{t^\circ, t^D\right\}$ both $A + Bx(t)$ and $D + E(v_0)) \cdot x(t)$ are negative. By Observation 1, these two quantities must have different signs at any t in the support of F, and the result follows.

Step 3: $t^D \le t^\circ$.

Suppose by way of contradiction that $t^D > t^\circ$. Since $F \neq F^S$, and since F assigns probability 0 to the interval $[0, t^\circ)$, we have $\overline{t} > t^\circ$. By Lemma 3, there is no atom at $\overline{t}$. It follows that there is some time t that slightly precedes $\overline{t}$, in which the players inspect under F (formally, $F(\overline{t}) > F(\overline{t} - \varepsilon)$ for every $\varepsilon > 0$).

Fix $\varepsilon > 0$ sufficiently small, such that $\overline{t} - \varepsilon > t^\circ$. Since the follower gain is decreasing in t, it is negative on $(\overline{t} - \varepsilon, \overline{t}]$, and by Observation 1, the delay gain is positive on this interval. Suppose a player considers advancing all inspections from the interval $(\overline{t} - \varepsilon, \overline{t}]$ to $\overline{t} - \varepsilon$, *conditional on no player inspecting before* $\overline{t} - \varepsilon$. This has two effects on the player's payoff. The first effect is the delay gain. Because the length of the interval $(\overline{t} - \varepsilon, \overline{t}]$ is ε, this gain is in the order of magnitude of ε. The second effect is the follower gain. By advancing the inspection to $\overline{t} - \varepsilon$, the player ensures that she becomes the leader rather than the follower. Because the follower gain is positive on $(\overline{t} - \varepsilon, \overline{t}]$, switching from being the follower on $(\overline{t} - \varepsilon, \overline{t}]$ to being the leader at $\overline{t} - \varepsilon$, yields a positive payoff that does not vanish as ε becomes small. Consequently, for ε sufficiently small, the overall effect of advancing the inspection is dominated by the follower gain, which is positive. Thus, the deviation considered above is profitable, contradicting the assumption that F is an MPE.

Step 4: $t^D \ge t^\circ$.

Suppose by way of contradiction that $t^D < t^\circ$. Recall that by definition of t^D we have $x\left(t^D\right) = \frac{-D}{E(v_0)} = \frac{r}{\lambda + r} \frac{v_1 - c}{v_1 - v_0}$. By Lemma 5, $x(\tau^*) = \frac{r}{\lambda + r} \frac{v_1 - c}{v_1 - w(\tau^*)}$. As before, the fact that $v_2 \le \tilde{v}_2$, implies that $\tau^* \ge t^\circ > t^D$. We therefore have $x(\tau^*) = e^{-\lambda \tau^*} < e^{-\lambda t^D} = x\left(t^D\right)$, or, equivalently, that

$$\frac{r}{\lambda + r} \frac{v_1 - c}{v_1 - w(\tau^*)} < \frac{r}{\lambda + r} \frac{v_1 - c}{v_1 - v_0}.$$

It follows that $w(\tau^*) < v_0 = w(\underline{t})$. This is a contradiction because by definition τ^* is the maximizer of $w(\cdot)$.

Proof of Theorem 2

The proof consists of six steps. In the first three steps we show that $\tau^* < t^\circ$ and that the strategy F defined in Eq. (11) and Eq. (10) with $[\underline{t}, \overline{t}] = [\tau^*, t^\circ]$ is a symmetric MPE. In the last three steps, we show that this is the unique symmetric MPE.

Step 1: If $t^\circ < \infty$ then $A < 0$.

Since t° is finite, and since by Proposition 3 we have $e^{-\lambda t^\circ} = \frac{v_1-(v_2+c)}{v_1-v_2}$, it follows that the ratio $\frac{v_1-(v_2+c)}{v_1-v_2}$ is in $(0,1)$. If $v_1 < v_2$ this ratio is larger than 1, if $v_1 = v_2$ this ratio is undefined, and if $v_1 \in (v_2, v_2+c]$ this ratio is nonpositive. Therefore $v_1 > v_2 + c$, or, equivalently, $A < 0$.

Step 2: $\tau^* < t^\circ$.

If $t^\circ = \infty$ the claim holds trivially. If $t^\circ < \infty$, then by Step 1 we have $A < 0$. Because $v_2 > \tilde{v}_2 = v_1 - \frac{c}{1-x^*}$, it follows that

$$x^* > 1 - \frac{c}{v_1 - v_2} = -\frac{A}{B}.$$

Thus, $\tau^* = -\frac{1}{\lambda}\ln(x^*) < -\frac{1}{\lambda}\ln(-B/A) = t^\circ$.

Step 3: The strategy F defined by Eq. (10) (if $v_2 = v_1 - c$) or by Eq. (11) (if $v_2 \neq v_1 - c$) with $[\underline{t}, \overline{t}] = [\tau^*, t^\circ]$ is a symmetric MPE.

If $v_2 > v_1 - c$, then $A + Be^{-\lambda t} = (v_2 - v_1 + c)(1 - e^{-\lambda t}) + ce^{-\lambda t}$ is positive for all t. If $v_2 < v_1 - c$, then, since $v_2 > \tilde{v}_2$, the quantity $A + Be^{-\lambda t}$ is positive for all t in the support $[\tau^*, t^\circ)$. This is because the follower gain is decreasing and is equal to zero at $t = t^\circ$. Therefore, the conditions of Lemma 6 are satisfied, and the function F given by Eq. (10) or Eq. (11), depending on the value of v_2, is a solution of (IND).

To show that this solution is a CDF, we need to show that f is non-negative on the support $[\underline{t}, \overline{t}]$ and that $F(\overline{t}) = 1$. By Lemma 5 and Corollary 3, the delay gain, $D + E(v_0)e^{-\lambda t}$, is negative for all t in the support. Therefore, Eq. (9) implies that the density f is non-negative on the support. Next, inspection of Eqs. (10) and (11) reveals that when $\overline{t} = t^\circ$, we have that $F(\overline{t}) = F(t^\circ) = 1$.[23]

It remains to show that no player wants to deviate and inspect at a time outside the support of F. To see this, recall that by Proposition 4, the payoff under F is equal to $w(\underline{t})$. Since $w(\cdot)$ is maximized at τ^*, no player can profit from inspecting before $\underline{t} = \tau^*$. And, because (i) the equilibrium payoff is positive (see footnote 3), and (ii) $F(t^\circ) = 1$, no player can profit by inspecting after time t°.

We now turn to show that the equilibrium described above is unique.

Step 4: When $v_2 > \tilde{v}_2$, the lower bound $\underline{t}$ of the support of any symmetric MPE is equal to τ^*.

Suppose that the lower bound $\underline{t}$ of the support of a symmetric MPE is smaller than τ^*. Because $\underline{t} < \tau^* < t^\circ$, it follows that the follower gain at $\underline{t}$ is positive. We show that, in this case, the delay gain is also positive at $\underline{t}$, which implies a contradiction to Observation 1. To see why the delay gain is positive, recall that by Lemma 5, if the continuation payoff upon a failed inspection was $w(\tau^*)$, then the delay gain would have been zero at τ^* and positive at $\underline{t}$. However, the continuation payoff following a failed inspection is $v_0 = w(\underline{t}) < w(\tau^*)$, which implies that a small delay in inspection (conditional on the other player not inspecting

[23]If $A = 0$ then $t^\circ = \infty$, and so $F(t^\circ) = 1$ by Eq. (10); if $A > 0$ then $t^\circ = \infty$, and so $F(t^\circ) = 1$ by Eq. (11); and, if $A < 0$ then t° is determined such that $A + Be^{\lambda t^\circ} = 0$, and so $F(t^\circ) = 1$ by Eq. (11).

in the meantime) is even *more* profitable than in the case in which the continuation payoff is $w(\tau^*)$. Thus, the delay gain is also positive at $\underline{t}$ when the continuation payoff is $v_0 = w(\underline{t})$.

We have therefore established that $\underline{t} \geq \tau^*$. Suppose by way of contradiction that $\underline{t} > \tau^*$. Proposition 4 implies that the payoff under the symmetric MPE is equal to $w(\underline{t})$. However, by deviating to a Markov strategy that inspects at τ^*, a player can guarantee $w(\tau^*) > w(\underline{t})$. A contradiction.

Step 5: When $v_2 > \tilde{v}_2$, the upper bound $\underline{t}$ of the support of any symmetric MPE is no greater than t°.

At $t = t^\circ$ the follower gain is zero. By Step 4 and Lemma 5, at $t = \tau^*$ the delay gain is zero. By Step 2 we have $\tau^* < t^\circ$. Since both follower gain and delay gain are decreasing, the upper bound of the support of the MPE strategy cannot be greater than t°.

Step 6: The strategy F defined in Eq. (10) (if $A = 0$) or Eq. (11) (if $A \neq 0$) with $[\underline{t}, \bar{t}] = [\tau^*, t^\circ]$ is the *unique* symmetric MPE.

Eq. (IND) holds for every t in the support of F. By Proposition 6 the support of F is convex, hence Eq. (IND) holds for every $t \in [\underline{t}, \bar{t}] = [\tau^*, \bar{t}]$.

By Proposition 4, the equilibrium payoff under F is $v_0 = w(\underline{t}) = w(\tau^*)$. The function F defined in Eq. (10) (if $A = 0$) or in Eq. (11) (if $A \neq 0$) is a continuous solution of Eq. (IND). The Theory of Distributions states that if the integral equation (IND) has a solution that is a continuous distribution, then it is the unique solution (see, e.g., Theorem 2.24 in Hörmann and Steinbauer, 2009). Therefore, F is the only solution of (IND) for $\bar{t}$. Inspecting the formula of F, we see that if $\bar{t} < t^\circ$, then $F(\bar{t}) < 1$, which is a contradiction. This implies that $\bar{t}$ must coincide with t°, and the proof is complete.

Proof of Proposition 7

By Theorem 2, for values $v_2 \geq \tilde{v}_2$ the players start sampling at τ^*, where τ^* is the optimal sampling time for a single player when the prize is v_1. Lemma 4 (and the fact that $v_2 > 0$) implies that the optimal inspection time for one player with a prize that is equal to $v_1 + v_2$ is smaller than τ^*. Thus, when $v_2 \geq \tilde{v}_2$ the players inspect too infrequently relative to the social optimum.

When $v_2 = 0$, the socially optimal inspection time is τ^* because the social gain from discovering the prize is v_1. Theorem 1 implies that in this case the players inspect at t°, which is smaller than τ^*. Thus, in equilibrium, the players inspect too frequently relative to the social optimum.

The MPE inspection time t° increases continuously in v_2 (see Eq. 6), and the optimal inspection time for one player with a prize that is equal to $v_1 + v_2$ decreases continuously in v_2 (see Lemma 4). Therefore, by the discussion above, these two curves (of the MPE inspection time t°, and of the optimal inspection time for one player) intersect for some value $\hat{v}_2 < \tilde{v}_2$. It follows that when $v_2 = \hat{v}_2$ the two players inspect synchronously at the socially optimal frequency.