Quantum Entanglement and the CHSH Inequality: Circuit Simulations and Graph Analysis

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Abstract

The CHSH inequality tests the limits of local realism versus quantum mechanics. We create and simulate quantum circuits for the CHSH game, and analyze findings to observe and understand quantum phenomena and their impact on classical bounds. Violations of the classical bound reveal the influence of entanglement in quantum correlations.

1 Introduction

In Classical Computing, information is stored as bits, or what one could represent as 1's and 0's. When we migrate to Quantum Computing, we see the change of these being represented as qubits, which can be represented as 1's, 0's, or any proportion of 0 and 1, which exists in a superposition state. This is the key component towards why Quantum Algorithms can process information faster than different Classical Systems. Now lets dive deeper, and introduce the CHSH inequality.

The CHSH Game is a well-known problem in quantum information theory, designed to illustrate the power of quantum entanglement. In the Quantum Variation, two players, Alice and Bob, receive binary inputs x and y, respectively, and produce outputs a and b. They win if,

$$x \cdot y = a \oplus b,$$

This means that:

For $x = y = 1$, they win if $a \neq b$.
For other input combinations, they win if $a = b$.

Classical Strategy:
With a classical strategy where Alice and Bob independently choose their outputs a and b (e.g., both always output 0), the best they can achieve is a winning probability of 75%. This is optimal among all deterministic strategies.

Quantum Strategy:
Using shared entanglement, Alice and Bob can achieve a higher winning probability. They share an entangled pair of qubits in the state $|\phi_{AB}\rangle$ and use specific measurement bases depending on their inputs:

- Alice uses the computational basis for $x = 0$ and the Hadamard basis for $x = 1$.

- Bob uses the computational basis rotated by $\pi/8$ for $y = 0$ and rotated by $-\pi/8$ for $y = 1$.

Measurement Probabilities:

- When $x = y = 1$, the angle between Alice's and Bob's measurement bases is $3\pi/8$.

- For other input combinations, the angle is $\pi/8$.

Probability of Matching Outputs:

$$P(a = b) = \cos^2(\theta),$$

where θ is the angle between the measurement bases.

- For $x = y = 1$, the winning probability is $\sin^2(3\pi/8)$.

- For other cases, it is $\cos^2(\pi/8)$.

Overall Winning Probability:

$$P_{\text{win}} = \frac{3}{4} \cdot \cos^2(\pi/8) + \frac{1}{4} \cdot \sin^2(3\pi/8).$$

Evaluating this, we get approximately 0.853.

2 Using Qiskit to Provide Circuit Outcomes

2.1 Creating circuits with respect to θ Values:

The θ values referred to in this chapter are used to prepare quantum states by rotating qubits and to measure outcomes in various bases, helping to compute CHSH expectation values.

The function generates a list of quantum circuits for testing the CHSH inequality. Each circuit is configured for different combinations of measurement bases and rotation angles θ . The circuits prepare entangled qubits using Hadamard and CNOT gates, rotate qubits around the y-axis, and apply additional Hadamard gates based on the measurement basis before measuring both qubits.

2.2 Computing the Expectation Values:

The function calculates CHSH expectation values from measurement outcomes. It processes counts from measurements in various bases, computes the total number of shots, and applies the CHSH formula to obtain two values, witness 1 and witness 2, which are normalized by the total number shots.

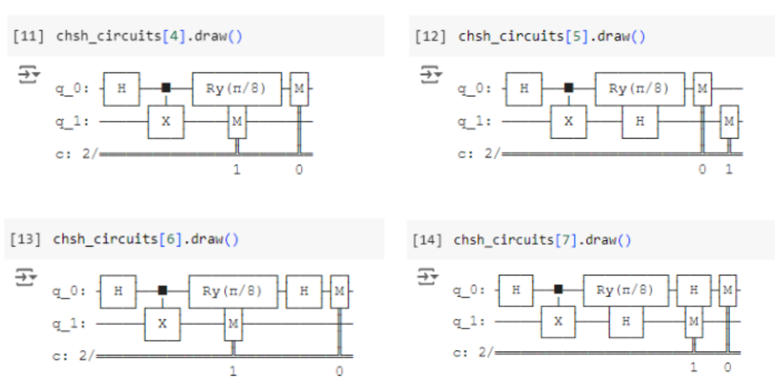

Figure 1: Quantum Circuits

Within this code, the number of θ values defines the number of rotational angles used, creating an array of equally spaced θ values from 0 to 2π. These values are typically used to define the rotation angles for the quantum circuits or to compute and plot various quantum measurements or calculations over this range of angles.

Below, we test specific values of θ rather than spacing them evenly as shown above. Previously, we implemented a code that returned an output for 17 values evenly spaced between 0 and 2π. Each θ value represented a distinct rotation angle used in preparing and measuring quantum states in the CHSH experiment. In contrast, this approach specifies a particular angle θ, such as $\frac{\pi}{2}$, and generates the corresponding quantum circuit in Qiskit. Modifying the function allows for a single θ value to be returned, rather than a list of values.

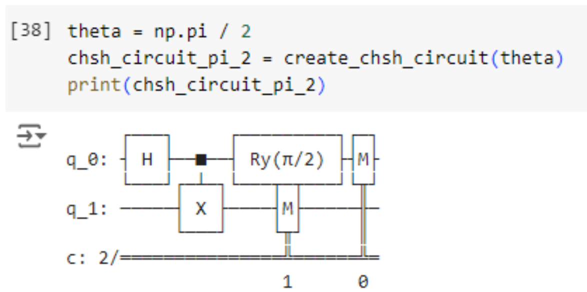

Figure 2: Specified Quantum Circuit

3 Influence of Angles on the CHSH Inequality

3.1 Classical Bound

In this context of this chapter, θ values are used to compute correlation functions and check whether the quantum correlations exceed the classical bound of the CHSH inequality. The angles chosen for this computation can be specific values that might be set to test particular aspects of the inequality.

Definition:
The classical bound of the CHSH inequality is derived from local hidden variable theories. According to these theories, the CHSH inequality is constrained to values between -2 and 2. This bound is a consequence of local realism and assumes that measurement outcomes are independent of the spatial separation between the particles.

Significance:
Local hidden variable theories propose that measurement outcomes are determined by pre-existing hidden variables, which fully account for the results of experiments. The classical bound of 2 ensures that no correlations beyond this limit can be explained by classical physics.

Inequality Formulation:
For any local hidden variable model, the CHSH inequality is expressed as:

$$S = E(A_1, B_1) + E(A_1, B_2) + E(A_2, B_1) - E(A_2, B_2), \tag{1}$$

where S must satisfy:

$$|S| \leq 2. \tag{2}$$

3.2 Quantum Bound

Definition:
In quantum mechanics, the CHSH value can exceed the classical bound of 2. For entangled quantum states, particularly the maximally entangled Bell states, the CHSH value can reach up to $2\sqrt{2}$, approximately 2.828.

Significance:
The quantum bound of $2\sqrt{2}$ highlights the non-local nature of quantum mechanics. It arises from the quantum correlations between entangled particles, which cannot be explained by local hidden variables.

Maximal Violation:
The value $2\sqrt{2}$ is achieved with the maximally entangled Bell state and specific choices of measurement settings. This violation of the classical bound provides a clear demonstration of the unique and non-classical properties of quantum entanglement.

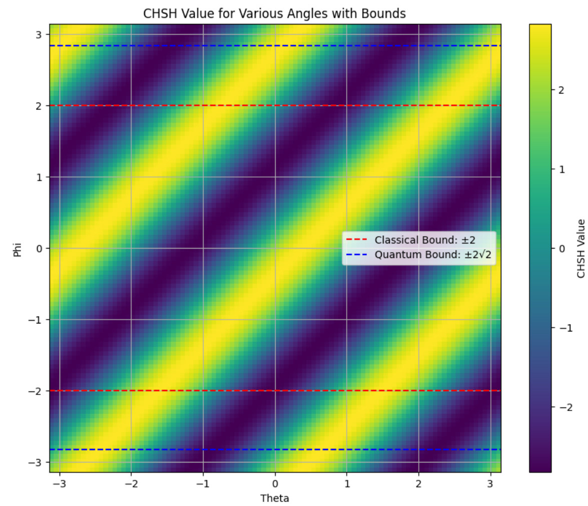

Figure 3: CHSH Value for Various Angles with Bounds

The figure above shows how the CHSH parameter S changes with different angles, with color indicating the magnitude of S. This visualization helps in analyzing how different angle settings influence the CHSH value and its bounds. By examining these variations, one can better understand the implications of measurement angles on the CHSH inequality and the quantum correlations that it reveals.

Understanding the Diagonal Pattern:
The diagonal pattern observed in a CHSH heatmap is a result of the trigonometric relationships between the measurement angles θ and ϕ. The CHSH value S is influenced by these angles through correlation functions that depend on phase differences such as $\theta - \phi$. This dependence generates diagonal patterns in the heatmap, which visually represent how quantum mechanical interference leads to correlations that can surpass classical limits. Consequently, these diagonal bands indicate significant angles where quantum effects are particularly strong, highlighting the regions where quantum correlations become most apparent.

High CHSH Values:
Regions of the graph with high values of S (close to or exceeding 2) indicate settings of measurement angles where the CHSH inequality is maximally violated. This highlights the non-classical, entangled nature of the quantum state being measured.

Low CHSH Values:
Regions where S is close to or less than 2 suggest measurement settings where the CHSH inequality is satisfied within classical bounds. This might be due to specific choices of angles that do not leverage the full quantum mechanical potential for violation.

Optimal Angles:
The graph helps identify specific angles where the CHSH value reaches its maximum quantum violation. These are the angles where the measurement settings are optimized to demonstrate the most significant deviation from classical expectations.

4 Analysis of CHSH Inequality Violations, Quantum Bound Achievements, and Non-Locality in Simulated Bell Test Experiments

4.1 CHSH Expectation Value Graph outcomes

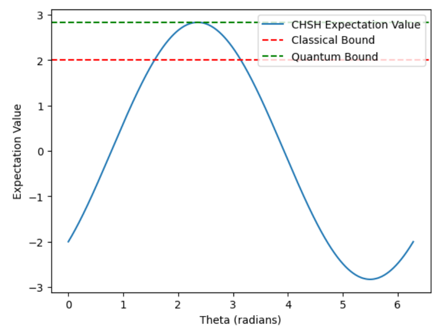

Figure 4: CHSH Expectation Values simulated classically

The experiment demonstrates that quantum mechanics allows for correlations between entangled particles that violate the classical bound of 2, predicted by local hidden variable theories. Instead, quantum mechanics predicts a maximum CHSH value of $2\sqrt{2} = 2.828$. The plot produced by the code simulation compares the computed CHSH values against these classical and quantum mechanical bounds. Observing a value that exceeds 2 and approaches 2.828 provides empirical evidence of quantum entanglement and non-locality, confirming the predictions of quantum theory and showcasing the limitations of classical interpretations.

4.2 CHSH inequality Violation & Winning Probability Graph outcomes

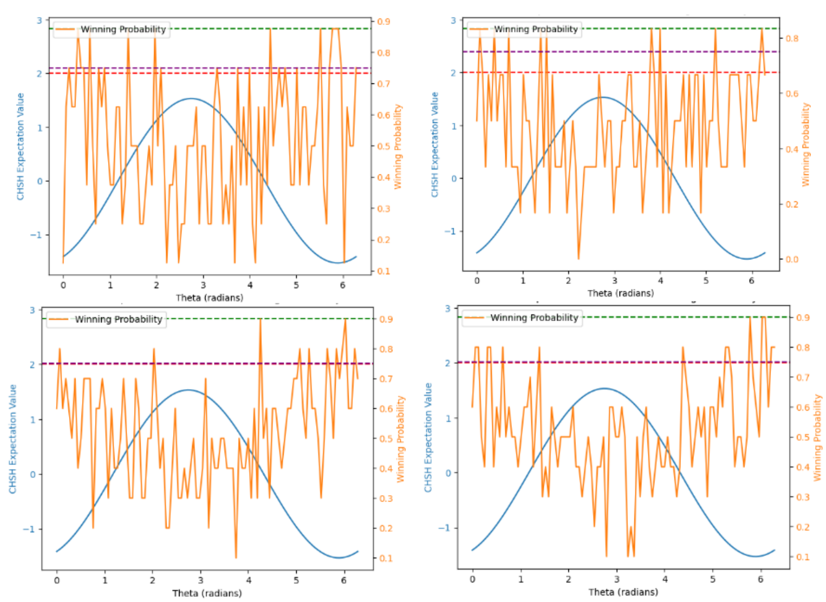

Figure 5: Quantum Bound Reached

Winning Probability:
The fraction of times the outcome of the Bell test experiment matches the predicted outcome based on the chosen measurement settings. In a Bell test, this typically refers to the probability that the observed correlations align with the theoretical predictions given by quantum mechanics.

Quantum Bound (Green Dashed Line):
The green dashed line represents the maximum CHSH value predicted by quantum mechanics, which is $2\sqrt{2}$. This is a theoretical upper limit that can be achieved under optimal conditions according to quantum theory.

Implications of Winning Probability Touching the Green Line:

- **Violation of Classical Bound:** If the winning probability reaches or exceeds the quantum bound (represented by the green dashed line), it indicates that the CHSH value is achieving the maximum quantum mechanical prediction, which violates the classical bound of 2 (red dashed line). This violation demonstrates the non-classical, entangled nature of the quantum states being tested.
- **Optimal Settings for Quantum Violation:** Touching the quantum bound suggests that the angles used in the measurements are optimized for demonstrating quantum mechanical effects. In practice, this would mean that the chosen angles lead to the maximal violation of the CHSH inequality allowed by quantum mechanics.
- **Verification of Quantum Predictions:** If the winning probability aligns with the quantum bound, it verifies that the experimental setup is effectively capturing quantum mechanical correlations. This is a key result in experiments designed to test the validity of quantum mechanics versus classical theories of local realism.