Homework 4

1. (a) $u(t) = V_L + V_R + V_C$ $\qquad y(t) = V_C$.

$= L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt.$ $\qquad = \frac{1}{C}\int i\,dt.$

$U(s) = \mathcal{L}(u(t)) = sL I(s) + R I(s) + \frac{1}{Cs} I(s)$ $\quad$ Transfer function $G(s) = \frac{Y(s)}{U(s)} = \frac{\frac{1}{Cs}I(s)}{sL I(s) + R I(s) + \frac{1}{Cs} I(s)}$

$Y(s) = \mathcal{L}(y(t)) = \frac{1}{Cs} I(s).$ $\qquad = \frac{1}{s^2 CL + CsR + 1}$

$U(s) = s^2 CL\, Y(s) + CsR\, Y(s) + Y(s)$

$u = \mathcal{L}^{-1}(U(s))$

$u = CL\ddot{y} + CR\dot{y} + y$ differential equation.

define $y = x_1$. $\dot{x}_1 = x_2 = \dot{y}$ $\quad \ddot{y} = \dot{x}_2 = \frac{u - x_1 - CRx_2}{CL}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{CL} \end{bmatrix} u \quad \text{state equation.}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) $u = V_1 + V_2.$

$V_1 = R_1 i$

$V_2 = \frac{1}{C_1}\int i_1\,dt = R_2 i_2 + \frac{1}{C_2}\int i_2\,dt$

$i = i_1 + i_2.$ $\quad y = \frac{1}{C_2}\int i_2\,dt$

$\Rightarrow$

$U(s) = V_1(s) + V_2(s)$

$V_1(s) = R_1 I(s)$

$V_2 = \frac{1}{C_1 s} I_1(s) = (R_2 + \frac{1}{C_2 s}) I_2(s)$

$I(s) = I_1(s) + I_2(s).$

$Y(s) = \frac{1}{C_2 s} I_2(s)$

$\Rightarrow$

$\frac{1}{C_1 s} I_1(s) = (R_2 + \frac{1}{C_2 s}) I_2(s)$

$I_1(s) = (C_1 R_2 s + \frac{C_1}{C_2}) I_2(s)$

$= \frac{C_1 C_2 R_2 s + C_1}{C_1 C_2 R_2 s + C_1 + C_2} I(s).$

$I_2(s) = \frac{C_2}{C_1 C_2 R_2 s + C_1 + C_2} I(s).$

$V_2 = \frac{1}{C_1 s} I_1(s)$

$= \frac{C_2 R_2 s + 1}{C_1 C_2 R_2 s^2 + C_1 s + C_2 s} I(s)$

$Y(s) = \frac{1}{C_2 s} \frac{C_2}{C_1 C_2 R_2 s + C_1 + C_2} I(s)$

$= \frac{1}{C_1 C_2 R_2 s^2 + (C_1 + C_2) s} I(s)$

$U(s) = V_1(s) + V_2(s)$

$= R_1 I(s) + \frac{C_2 R_2 s + 1}{C_1 C_2 R_2 s^2 + (C_1 + C_2) s} I(s)$

$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{C_1 C_2 R_1 R_2 s^2 + (C_1 + C_2) R_1 s + C_2 R_2 s + 1}$

$= \frac{C_1 C_2 R_1 R_2 s^2 + (C_1 + C_2) R_1 s + C_2 R_2 s + 1}{C_1 C_2 R_2 s^2 + (C_1 + C_2) s} I(s)$

$U(s) = C_1 C_2 R_1 R_2 s^2 Y(s) + (C_1 R_1 + C_2 R_1 + C_2 R_2) s\, Y(s) + Y(s)$

$u = C_1 C_2 R_1 R_2 \ddot{y} + (C_1 R_1 + C_2 R_1 + C_2 R_2)\dot{y} + y$

define $y = x_1$, $\dot{x}_1 = x_2 = \dot{y}$, $\ddot{y} = \dot{x}_2 = \frac{u - (C_1 R_1 + C_2 R_1 + C_2 R_2) x_2 - x_1}{C_1 C_2 R_1 R_2}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{C_1 C_2 R_1 R_2} & -\frac{C_1 R_1 + C_2 R_1 + C_2 R_2}{C_1 C_2 R_1 R_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C_1 C_2 R_1 R_2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1.(c)

$$u = u_1 + u_2. \qquad\qquad U(s) = U_1(s) + U_2(s)$$
$$u_1 = R_1 i \qquad\qquad U_1(s) = R_1 I(s)$$
$$u_2 = L\frac{di_1}{dt} = R_2 i_2 + \frac{1}{C_2}\int i_2\,dt \quad \Rightarrow \quad U_2(s) = sL I_1(s) = R_2 I_2(s) + \frac{1}{C_2 s} I_2(s)$$
$$i = i_1 + i_2. \qquad\qquad I(s) = I_1(s) + I_2(s)$$
$$y = \frac{1}{C_2}\int i_2\,dt \qquad\qquad Y(s) = \frac{1}{C_2 s} I_2(s)$$

$$sL I_1(s) = R_2 I_2(s) + \frac{1}{C_2 s} I_2(s)$$
$$I_1(s) = \frac{R_2C_2s+1}{C_2Ls^2} I_2(s) \qquad\qquad I_2 = \frac{C_2Ls^2}{R_2C_2s+C_2Ls^2+1} I(s)$$
$$= \frac{R_2C_2s+1}{R_2C_2s+C_2Ls^2+1} I(s)$$

$$U(s) = U_1(s) + U_2(s) \qquad\qquad Y(s) = \frac{1}{C_2 s} I_2(s)$$
$$= R_1 I(s) + sL I_1(s) \qquad\qquad = \frac{1}{C_2 s}\,\frac{C_2Ls^2}{R_2C_2s+C_2Ls^2+1} I(s)$$
$$= R_1 I(s) + \frac{sL(R_2C_2s+1)}{R_2C_2s+C_2Ls^2+1} I(s) \qquad\qquad = \frac{sL}{R_2C_2s+C_2Ls^2+1} I(s)$$
$$= \frac{R_1R_2C_2s + R_1C_2Ls^2 + R_1 + R_2C_2Ls^2 + sL}{R_2C_2s+C_2Ls^2+1} I(s)$$

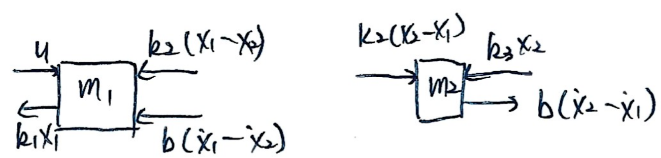
$$G(s) = \frac{Y(s)}{U(s)} = \frac{sL}{R_1R_2C_2s + R_1C_2Ls^2 + R_1 + R_2C_2Ls^2 + sL} = \frac{s}{(R_1+R_2)C_2s^2 + (\frac{R_1R_2C_2}{L}+1)s + R_1/L}$$

define $y = x_1$,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{R_1}{L(R_1+R_2)C_2} & -\frac{R_1R_2C_2+L}{L(R_1+R_2)C_2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

2.



$$u(t) = m_1\ddot{x}_1 + b(\dot{x}_1-\dot{x}_2) + k_1x_1 + k_2(x_1-x_2) \Rightarrow U(s) = (m_1s^2+bs+k_1+k_2)X_1(s) - (bs+k_2)X_2(s)$$
$$0 = m_2\ddot{x}_2 + b(\dot{x}_2-\dot{x}_1) + k_3x_2 + k_2(x_2-x_1) \qquad 0 = (m_2s^2+bs+k_2+k_3)X_2(s) - (bs+k_2)X_1(s)$$
$$\Downarrow$$
$$U(s) = \frac{(m_1s^2+bs+k_1+k_2)(m_2s^2+bs+k_2+k_3)-(bs+k_2)^2}{m_2s^2+bs+k_2+k_3}X_1(s) \quad \Leftarrow \quad X_1(s) = \frac{m_2s^2+bs+k_2+k_3}{bs+k_2}X_2(s)$$
$$X_2(s) = \frac{bs+k_2}{m_2s^2+bs+k_2+k_3}X_1(s)$$

$$\frac{X_1(s)}{U(s)} = \frac{m_2s^2+bs+k_2+k_3}{(m_1s^2+bs+k_1+k_2)(m_2s^2+bs+k_2+k_3)-(bs+k_2)^2}$$

$$U(s) = \frac{(m_1s^2+bs+k_1+k_2)(m_2s^2+bs+k_2+k_3)-(bs+k_2)^2}{bs+k_2}X_2(s)$$

$$\frac{X_2(s)}{U(s)} = \frac{bs+k_2}{(m_1s^2+bs+k_1+k_2)(m_2s^2+bs+k_2+k_3)-(bs+k_2)^2}$$

3. $A = \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $C = [1, 2]$

(a) $G(s) = C(sI - A)^{-1}B + D$

$$= [1,2]\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -5 & -1 \\ 3 & -1 \end{bmatrix}\right)^{-1}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= [1,2]\begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= \frac{1}{(s+5)(s+1)+3}[1,2]\begin{bmatrix} s+1 & -1 \\ 3 & s+5 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$= \frac{12s+59}{s^2+6s+8}$$

(b) $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

$$e^{At} = L^{-1}[sI - A]^{-1}$$

$$= L^{-1}\begin{bmatrix} s+5 & 1 \\ -3 & s+1 \end{bmatrix}^{-1}$$

$$= L^{-1}\begin{bmatrix} \frac{s+1}{s^2+6s+8} & -\frac{1}{s^2+6s+8} \\ \frac{3}{s^2+6s+8} & \frac{s+5}{s^2+6s+8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2}e^{-4t} - \frac{1}{2}e^{-2t} & \frac{1}{2}e^{-4t} - \frac{1}{2}e^{-2t} \\ \frac{3}{2}e^{-2t} - \frac{3}{2}e^{-4t} & \frac{3}{2}e^{-2t} - \frac{1}{2}e^{-4t} \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \frac{3}{2}e^{-4t} - \frac{1}{2}e^{-2t}, & \frac{1}{2}e^{-4t} - \frac{1}{2}e^{-2t} \\ \frac{3}{2}e^{-2t} - \frac{3}{2}e^{-4t}, & \frac{3}{2}e^{-2t} - \frac{1}{2}e^{-4t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$= \begin{bmatrix} 2e^{-4t} - e^{-2t} \\ 3e^{-2t} - 2e^{-4t} \end{bmatrix} + e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau$$

$$= \begin{bmatrix} 2e^{-4t} - e^{-2t} \\ 3e^{-2t} - 2e^{-4t} \end{bmatrix} + e^{At}\int_0^t e^{-A\tau}\begin{bmatrix} 2 \\ 5 \end{bmatrix}d\tau$$

$$= \begin{bmatrix} 2e^{-4t} - e^{-2t} \\ 3e^{-2t} - 2e^{-4t} \end{bmatrix} + e^{At}\begin{bmatrix} \frac{11}{8}e^{4t} - \frac{7}{4}e^{2t} + \frac{3}{8} \\ \frac{21}{4}e^{2t} - \frac{11}{8}e^{4t} - \frac{31}{8} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{8}e^{-4t} + \frac{3}{4}e^{-2t} - \frac{3}{8} \\ -\frac{5}{8}e^{-4t} - \frac{9}{4}e^{-2t} + \frac{31}{8} \end{bmatrix}$$

$y(t) = [1 \;\; 2]\, x(t)$

$$= +\frac{5}{8}e^{-4t} + \frac{3}{4}e^{-2t} - \frac{3}{8} - \frac{10}{8}e^{-4t} - \frac{18}{4}e^{-2t} + \frac{62}{8}$$

$$= -\frac{5}{8}e^{-4t} - \frac{15}{4}e^{-2t} + \frac{59}{8}$$

(c) $\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -5+\delta & -1 \\ 3 & -1 \end{bmatrix}$

$$= \begin{bmatrix} \lambda+5-\delta & 1 \\ -3 & \lambda+1 \end{bmatrix}$$

$(\lambda+5-\delta)(\lambda+1) - (-3)$

$= \lambda^2 + (6-\delta)\lambda + 8 - \delta$

$6 - \delta > 0$ $\quad$ $8 - \delta > 0$

$\delta < 6$ $\quad$ $\delta < 8$

$\therefore \delta < 6$

4. $G(s) = \frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{25}{s^2+4s+25}$

$Y(s) = G(s)U(s)$ unit step $u(t)=1 \Rightarrow U(s)=\frac{1}{s}$

$= \frac{25}{(s^2+4s+25)s} \underset{L^{-1}}{\Rightarrow} y(t) = 1-e^{-2t}\left(\cos(\sqrt{21}t)+\frac{2\sqrt{21}\sin(\sqrt{21}t)}{21}\right)$

5.(a) $f_3=0 \quad n_3=0.$
$\omega_0=0 \quad \dot{\omega}_0=0. \quad {}^0\dot{v}_0 = g\hat{Y}_0 = \begin{bmatrix}0\\g\\0\end{bmatrix} \quad {}^1P_{c_1} = \begin{bmatrix}l_1\\0\\0\end{bmatrix} \quad {}^2P_{c_2} = \begin{bmatrix}l_2\\0\\0\end{bmatrix}$

$${}^1\omega_1 = \dot\theta_1 {}^1\hat{Z}_1 = \begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix} \quad {}^1\dot\omega_1 = \ddot\theta_1 {}^1\hat{Z}_1 = \begin{bmatrix}0\\0\\\ddot\theta_1\end{bmatrix} \quad {}^1\dot{v}_1 = \begin{bmatrix}c_1 & s_1 & 0\\-s_1 & c_1 & 0\\0&0&1\end{bmatrix}\begin{bmatrix}0\\g\\0\end{bmatrix} = \begin{bmatrix}gs_1\\gc_1\\0\end{bmatrix}$$

$${}^1\dot{v}_{c_1} = \begin{bmatrix}0\\0\\\ddot\theta_1\end{bmatrix}\times\begin{bmatrix}l_1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix}\times\left(\begin{bmatrix}0\\0\\\dot\theta_1\end{bmatrix}\times\begin{bmatrix}l_1\\0\\0\end{bmatrix}\right) + \begin{bmatrix}gs_1\\gc_1\\0\end{bmatrix} = \begin{bmatrix}-l_1\dot\theta_1^2+gs_1\\l_1\ddot\theta_1+gc_1\\0\end{bmatrix}$$

$${}^1F_1 = m_1 {}^1\dot{v}_{c_1} = \begin{bmatrix}-m_1l_1\dot\theta_1^2+m_1gs_1\\m_1l_1\ddot\theta_1+m_1gc_1\\0\end{bmatrix} \quad {}^1N_1 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$${}^2\omega_2 = \begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_2\end{bmatrix} \quad {}^2\dot\omega_2 = \begin{bmatrix}0\\0\\\ddot\theta_1+\ddot\theta_2\end{bmatrix} \quad {}^2\dot{v}_2 = \begin{bmatrix}c_2 & s_2 & 0\\-s_2 & c_2 & 0\\0&0&1\end{bmatrix}\begin{bmatrix}-l_1\dot\theta_1^2+gs_1\\l_1\ddot\theta_1+gc_1\\0\end{bmatrix} = \begin{bmatrix}l_1\ddot\theta_1s_2-l_1\dot\theta_1^2c_2+gs_{12}\\l_1\ddot\theta_1c_2+l_1\dot\theta_1^2s_2+gc_{12}\\0\end{bmatrix}$$

$${}^2\dot{v}_{c_2} = \begin{bmatrix}0\\0\\\ddot\theta_1+\ddot\theta_2\end{bmatrix}\times\begin{bmatrix}l_2\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_2\end{bmatrix}\times\left(\begin{bmatrix}0\\0\\\dot\theta_1+\dot\theta_2\end{bmatrix}\times\begin{bmatrix}l_2\\0\\0\end{bmatrix}\right) + \begin{bmatrix}l_1\ddot\theta_1s_2-l_1\dot\theta_1^2c_2+gs_{12}\\l_1\ddot\theta_1c_2+l_1\dot\theta_1^2s_2+gc_{12}\\0\end{bmatrix}$$
$$= \begin{bmatrix}-l_2(\dot\theta_1+\dot\theta_2)^2+l_1\ddot\theta_1s_2-l_1\dot\theta_1^2c_2+gs_{12}\\l_2(\ddot\theta_1+\ddot\theta_2)+l_1\ddot\theta_1c_2+l_1\dot\theta_1^2s_2+gc_{12}\\0\end{bmatrix} \quad {}^2F_2 = m_2 {}^2\dot{v}_{c_2} \quad {}^2N_2 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$${}^2f_2 = {}^2F_2, \quad {}^2n_2 = \begin{bmatrix}0\\0\\m_2l_1l_2c_2\ddot\theta_1+m_2l_1l_2s_2\dot\theta_1^2+m_2l_2gc_{12}+m_2l_2^2(\ddot\theta_1+\ddot\theta_2)\end{bmatrix}$$

$${}^1f_1 = \begin{bmatrix}c_2 & -s_2 & 0\\s_2 & c_2 & 0\\0&0&1\end{bmatrix}\begin{bmatrix}m_2l_1s_2\ddot\theta_1-m_2l_1c_2\dot\theta_1^2+m_2gs_{12}-m_2l_2(\dot\theta_1+\dot\theta_2)^2\\m_2l_1c_2\ddot\theta_1+m_2l_1s_2\dot\theta_1^2+m_2gc_{12}+m_2l_2(\ddot\theta_1+\ddot\theta_2)\\0\end{bmatrix} + \begin{bmatrix}-m_1l_1\dot\theta_1^2+m_1gs_1\\m_1l_1\ddot\theta_1+m_1gc_1\\0\end{bmatrix}$$

$${}^1n_1 = \begin{bmatrix}0\\0\\m_2l_1l_2c_2\ddot\theta_1+m_2l_1l_2s_2\dot\theta_1^2+m_2l_2gc_{12}+m_2l_2^2(\ddot\theta_1+\ddot\theta_2)\end{bmatrix} + \begin{bmatrix}0\\0\\m_1l_1^2\ddot\theta_1+m_1l_1gc_1\end{bmatrix} + \begin{bmatrix}0\\0\\m_2l_1^2\ddot\theta_1-m_2l_1l_2s_2(\dot\theta_1+\dot\theta_2)^2+\\m_2l_1gs_2s_{12}+m_2l_1l_2c_2(\ddot\theta_1+\ddot\theta_2)\\+m_2l_1gc_2c_{12}\end{bmatrix}$$

$$\tau_1 = m_2l_2^2(\ddot\theta_1+\ddot\theta_2)+m_2l_1l_2c_2(2\ddot\theta_1+\ddot\theta_2)+(m_1+m_2)l_1^2\ddot\theta_1-m_2l_1l_2s_2\dot\theta_2^2$$
$$-2m_2l_1l_2s_2\dot\theta_1\dot\theta_2+m_2l_2gc_{12}+(m_1+m_2)l_1gc_1$$

$$\tau_2 = m_2l_1l_2c_2\ddot\theta_1+m_2l_1l_2s_2\dot\theta_1^2+m_2l_2gc_{12}+m_2l_2^2(\ddot\theta_1+\ddot\theta_2)$$

$$\begin{bmatrix}\tau_1\\\tau_2\end{bmatrix} = \begin{bmatrix}l_2^2m_2+2l_1l_2m_2c_2+l_1^2(m_1+m_2) & l_2^2m_2+l_1l_2m_2c_2\\l_2^2m_2+l_1l_2m_2c_2 & l_2^2m_2\end{bmatrix}\begin{bmatrix}\ddot\theta_1\\\ddot\theta_2\end{bmatrix} + \begin{bmatrix}-m_2l_1l_2s_2\dot\theta_2^2-2m_2l_1l_2s_2\dot\theta_1\dot\theta_2\\m_2l_1l_2s_2\dot\theta_1^2\end{bmatrix}$$
$$+\begin{bmatrix}m_2l_2gc_{12}+(m_1+m_2)l_1gc_1\\m_2l_2gc_{12}\end{bmatrix}$$

$$\tau = M(\theta)\ddot\theta + V(\theta,\dot\theta) + G(\theta)$$

6. $${}^0_1T = \begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1_2T = \begin{bmatrix} C_2 & -S_2 & 0 & L_1 \\ 0 & 0 & -1 & 0 \\ S_2 & C_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^1P_2 = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \quad {}^2P_3 = \begin{bmatrix} L_2 \\ 0 \\ 0 \end{bmatrix} \quad {}^1P_{C_1} = \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} \quad {}^0\dot{V}_0 = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \quad W_0 = 0. \ \dot{W}_0 = 0. \quad f_3 = 0 \quad n_3 = 0.$$

$${}^1w_1 = \dot\theta_1 {}^1\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} \quad {}^1\dot{w}_1 = \ddot\theta_1 {}^1\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot\theta_1 \end{bmatrix} \quad {}^1\dot{V}_1 = \begin{bmatrix} C_1 & S_1 & 0 \\ -S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

$${}^1\dot{V}_{C_1} = {}^1\dot{w}_1 \times {}^1P_{C_1} + {}^1W_1 \times ({}^1W_1 \times {}^1P_{C_1}) + {}^1\dot{V}_1 = \begin{bmatrix} -L_1\dot\theta_1^2 \\ L_1\ddot\theta_1 \\ g \end{bmatrix} \quad {}^1F_1 = \begin{bmatrix} -m_1L_1\dot\theta_1^2 \\ m_1L_1\ddot\theta_1 \\ mg \end{bmatrix} \quad {}^1N_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^2W_2 = {}^2_1R\,{}^1w_1 + \dot\theta_2\,{}^2\hat{Z}_2 = \begin{bmatrix} C_2 & 0 & S_2 \\ -S_2 & 0 & C_2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_2 \end{bmatrix} = \begin{bmatrix} S_2\dot\theta_1 \\ C_2\dot\theta_1 \\ \dot\theta_2 \end{bmatrix} \quad {}^2\dot{W}_2 = \begin{bmatrix} S_2\ddot\theta_1 + C_2\dot\theta_1\dot\theta_2 \\ C_2\ddot\theta_1 - S_2\dot\theta_1\dot\theta_2 \\ \ddot\theta_2 \end{bmatrix}$$

$${}^2\dot{V}_2 = {}^2_1R\left[{}^1\dot{w}_1 \times {}^1P_2 + {}^1W_1 \times ({}^1W_1 \times {}^1P_2) + {}^1\dot{V}_1\right] = \begin{bmatrix} -L_1C_2\dot\theta_1^2 + gS_2 \\ L_1S_2\dot\theta_1^2 + gC_2 \\ -L_1\ddot\theta_2 \end{bmatrix} \quad {}^2F_2 = m_2\,{}^2\dot{V}_{C_2} \quad {}^2N_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^2\dot{V}_{C_2} = {}^2\dot{w}_2 \times {}^2P_{C_2} + {}^2W_2 \times ({}^2W_2 \times {}^2P_{C_2}) + {}^2\dot{V}_2 = \begin{bmatrix} -(L_1C_2 + L_2C_2^2)\dot\theta_1^2 - L_2\dot\theta_2^2 + gS_2 \\ (L_1S_2 + L_2S_2C_2)\dot\theta_1^2 + L_2\ddot\theta_2 + gC_2 \\ 2L_2S_2\dot\theta_1\dot\theta_2 - L_1\ddot\theta_1 - L_2C_2\ddot\theta_1 \end{bmatrix}$$

$${}^2n_2 = {}^2N_2 + {}^2_3R\,{}^3n_3 + {}^2P_{C_2} \times {}^2F_2 + {}^2P_3 \times {}^2_3R\,{}^3F_3 \qquad {}^1n_1 = {}^1N_1 + {}^1_2R\,{}^2n_2 + {}^1P_{C_1} \times {}^1F_1 + {}^1P_2 \times {}^1_2R\,{}^2F_2$$

$$= \begin{bmatrix} 0 \\ m_2L_1L_2\ddot\theta_1 - 2m_2L_2^2S_2\dot\theta_1\dot\theta_2 + m_2L_2^2C_2\ddot\theta_1 \\ m_2L_2(L_1 + L_2C_2)S_2\dot\theta_1^2 - m_2gL_2C_2 + m_2L_2^2\ddot\theta_2 \end{bmatrix} \qquad = \begin{bmatrix} \cdots \\ \cdots \\ (m_1L_1^2 + m_2L_1^2 + m_2L_2^2C_2 + 2m_2L_1L_2C_2)\ddot\theta_1 - 2(L_1 + L_2C_2)m_2L_2S_2\dot\theta_1\dot\theta_2 \end{bmatrix}$$

$$\tau_1 = \left[m_1L_1^2 + m_2(L_1 + L_2C_2)^2\right]\ddot\theta_1 - 2(L_1 + L_2C_2)m_2L_2S_2\dot\theta_1\dot\theta_2 + V_1\dot\theta_1$$

$$\tau_2 = m_2L_2^2\ddot\theta_2 + (L_1 + L_2C_2)m_2L_2S_2\dot\theta_1^2 + m_2gL_2C_2 + V_2\dot\theta_2$$

7.(2) $1+k\dfrac{s+3}{s(s+1)(s^2+4s+16)}=0.$

$s^4+5s^3+20s^2+(16+k)s+3k=0.$

Routh-Hurwitz

$$\begin{array}{lll} 1 & 20 & 3k \\ 5 & 16+k & \\ \frac{84-k}{5} & 3k & \\ \frac{\frac{(84-k)}{5}(16+k)-5\times 3k}{\frac{84-k}{5}} & & \end{array}$$

C_1

$\Rightarrow \dfrac{(84-k)(16+k)}{5}-15k \geqslant 0.$

$1344+68k-k^2-75k \geqslant 0$

$k^2+7k-1344 \leqslant 0.$

$k \leq 33.$

Matlab codes and figures in the folder.