ME 5250 Final Exam

Eileen Wang

1. (a) $u = H + M\ddot{y}$

Position of pendulum $p = (y + l\sin\theta)\hat{x} + l\cos\theta\,\hat{y}$

$a_p = (\ddot{y} + \ddot{\theta}l\cos\theta - \dot{\theta}^2 l\sin\theta)\hat{x} + (-l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta)\hat{y}$

$H = m(\ddot{y} + \ddot{\theta}l\cos\theta - \dot{\theta}^2 l\sin\theta)$

$V = m(-l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta) + mg$

$\therefore u = (M+m)\ddot{y} + m\ddot{\theta}l\cos\theta - m\dot{\theta}^2 l\sin\theta$

$(H\cos\theta - V\sin\theta + mg\sin\theta)l = m\ddot{y}l\cos\theta + m\ddot{\theta}l^2$

$Hl\cos\theta - Vl\sin\theta = 0$ (Assume $I = 0$.)

$\therefore mgl\sin\theta = m\ddot{y}l\cos\theta + m\ddot{\theta}l^2$

The dynamical equations of this system: $u = (M+m)\ddot{y} + m\ddot{\theta}l\cos\theta - m\dot{\theta}^2 l\sin\theta$

$mgl\sin\theta = m\ddot{y}l\cos\theta + m\ddot{\theta}l^2$

(b) when θ is small, $\sin\theta \doteq \theta$. $\cos\theta \doteq 1$. $\dot{\theta}^2 \doteq 0$

then $u = (M+m)\ddot{y} + m\ddot{\theta}l$

$m\ddot{y}l + m\ddot{\theta}l^2 = mgl\theta$

(c) Laplace transform:

$(M+m)Y(s)s^2 + ml\theta(s)s^2 = U(s)$

$ml^2\theta(s)s^2 + mlY(s)s^2 = mgl\theta(s) \Rightarrow Y(s) = \frac{mgl - ml^2s^2}{mls^2}\theta(s) = \frac{g - ls^2}{s^2}\theta(s)$

$(M+m)s^2\frac{g-ls^2}{s^2}\theta(s) + mls^2\theta(s) = U(s)$

$[(M+m)g - Mls^2]\theta(s) = U(s)$

$\frac{\theta(s)}{U(s)} = \frac{1}{(M+m)g - Mls^2}$ $\qquad$ $\frac{Y(s)}{U(s)} = \frac{g-ls^2}{s^2}\frac{1}{(M+m)g - MLs^2}$

$(M+m)g - Mls^2 = 0$ $\qquad$ $s^2[(M+m)g - MLs^2] = 0.$

$s = \pm\sqrt{\frac{(M+m)g}{ML}}$ $\qquad$ $S_1 = 0,\ S_{2,3} = \pm\sqrt{\frac{(M+m)g}{ML}}$

There's real pole in the right half plane for both of these two transfer function, so the system is unstable.

(d) $X_1 = \theta,\ X_2 = \dot{\theta},\ X_3 = y,\ X_4 = \dot{y}$

$u = (M+m)\ddot{y} + ml\ddot{\theta}$

$\ddot{y} = \frac{u - ml\ddot{\theta}}{(M+m)}$

$ml\frac{u - ml\ddot{\theta}}{M+m} + ml^2\ddot{\theta} = mgl\theta$

$\ddot{\theta} = \frac{(M+m)g}{ML}\theta - \frac{1}{ML}u$

$\ddot{y} = -\frac{mg}{M}\theta + \frac{1}{M}u$

$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \theta \\ x \end{bmatrix} = \begin{bmatrix} X_1 \\ X_3 \end{bmatrix}$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(M+m)g}{ML} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{mg}{M} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{ML} \\ 0 \\ \frac{1}{M} \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

(e) Matlab Code: Eileen Wang

```
M=2;
m=0.1;
L=0.5;
g=10;
P1=-1;
P2=-2;
P3=-3;
P4=-4;
P=[P1,P2,P3,P4];
A=[0 1 0 0;(M+m)*g/(M*L) 0 0 0;0 0 0 1;-m*g/M 0 0 0];
B=[0;-1/(M*L);0;1/M];
k=place(A,B,P);
disp(k)
```

The result of running this code is -57.2 -12.5 -2.4 -5.0

Due to the state feedback control law, k= 57.2 12.5 2.4 5.0

Eileen Wang

2. (a) $J=\begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$, $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J(\theta) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$

Case 1. $l_1=2$ $l_2=3$ $\theta_1=60^\circ$ $\theta_2=30^\circ$ $\dot{\theta}_1=1$ $\dot{\theta}_2=3$.

$s_1=\sin 60^\circ=\frac{\sqrt{3}}{2}$
$c_1=\cos 60^\circ=\frac{1}{2}$
$s_{12}=\sin 90^\circ=1$
$c_{12}=\cos 90^\circ=0$

$$J=\begin{bmatrix} -2\times\frac{\sqrt{3}}{2}-3\times 1 & -3\times 1 \\ 2\times\frac{1}{2}+3\times 0 & 3\times 0 \end{bmatrix} = \begin{bmatrix} -(3+\sqrt{3}) & -3 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -(3+\sqrt{3}) & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -12-\sqrt{3} \\ 1 \end{bmatrix} = \begin{bmatrix} -13.73 \\ 1 \end{bmatrix}$$

Case 2. $l_1=2$ $l_2=3$ $\theta_1=167^\circ$ $\theta_2=-156^\circ$ $\dot{\theta}_1=1$ $\dot{\theta}_2=3$

$s_1=\sin 167^\circ=0.225$
$c_1=\cos 167^\circ=-0.974$
$s_{12}=\sin 11^\circ=0.191$
$c_{12}=\cos 11^\circ=0.982$

$$J=\begin{bmatrix} -2\times 0.225-3\times 0.191 & -3\times 0.191 \\ 2\times(-0.974)+3\times 0.982 & 3\times 0.982 \end{bmatrix} = \begin{bmatrix} -1.023 & -0.573 \\ 0.998 & 2.946 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -1.023 & -0.573 \\ 0.998 & 2.946 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2.742 \\ 9.836 \end{bmatrix}$$

(b) $\tau = J^T F$

Case 1. $\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -4.732 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 30 \\ 20 \end{bmatrix} = \begin{bmatrix} -121.96 \\ -90 \end{bmatrix}$

Case 2. $\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -1.023 & 0.998 \\ -0.573 & 2.946 \end{bmatrix} \begin{bmatrix} 30 \\ -20 \end{bmatrix} = \begin{bmatrix} -50.65 \\ -76.11 \end{bmatrix}$

(c) $\tau = M(\theta)\ddot{\theta} + V(\theta,\dot{\theta}) + G(\theta)$

$\tau = \alpha\tau' + \beta$.

$\alpha = M(\theta)$ $\beta = V(\theta,\dot{\theta}) + G(\theta)$

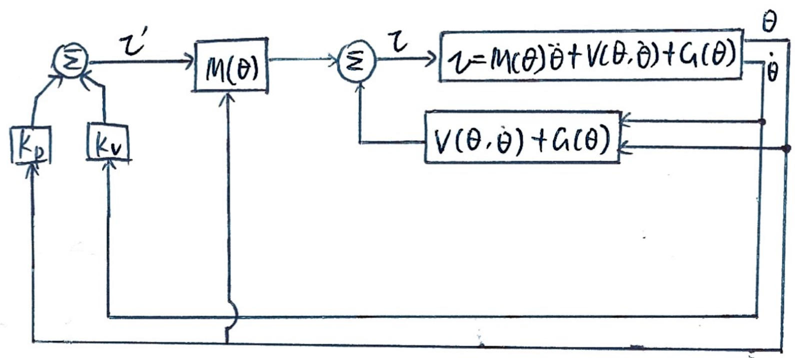
$\ddot{\theta} = \tau'$, $\tau' = -k_v\dot{\theta} - k_p\theta$

$\ddot{\theta} + k_v\dot{\theta} + k_p\theta = 0$

$\ddot{\theta}_1 + k_{v1}\dot{\theta}_1 + k_{p1}\theta_1 = 0$, $k_{v1} = 2\sqrt{k_{p1}}$

$\ddot{\theta}_2 + k_{v2}\dot{\theta}_2 + k_{p2}\theta_2 = 0$, $k_{v2} = 2\sqrt{k_{p2}}$

$K_p = \begin{bmatrix} k_{p1} & 0 \\ 0 & k_{p2} \end{bmatrix}$ $K_v = \begin{bmatrix} k_{v1} & 0 \\ 0 & k_{v2} \end{bmatrix}$



Eileen Wang

3. (a) $\tau = ml^2\ddot{\theta} + f\dot{\theta} + mgl\cos\theta$

$m=1,\ l=1,\ f=7,\ g=10.$

$\tau = \ddot{\theta} + 7\dot{\theta} + 10\cos\theta.$

$\alpha = ml^2 = 1$

$\beta = f\dot{\theta} + mgl\cos\theta = 7\dot{\theta} + 10\cos\theta.$

$\ddot{\theta} = \tau',\quad \tau' = -k_v\dot{\theta} - k_p\theta$

$\ddot{\theta} + k_v\dot{\theta} + k_p\theta = 0.$

$s^2 + k_v s + k_p = s^2 + 2\xi\omega_n s + \omega_n^2. \quad \omega_n = 10.$

$k_p = \omega_n^2 = 100. \quad k_v = 2\sqrt{k_p} = 20.$

(b) $\theta_d = A\sin(2\pi t/T)$, $A = 0.1$ rad $T = 2s$.

$\theta_d = 0.1\sin(\pi t)$

$\tau = ml^2\ddot{\theta} + f\dot{\theta} + mgl\cos\theta$

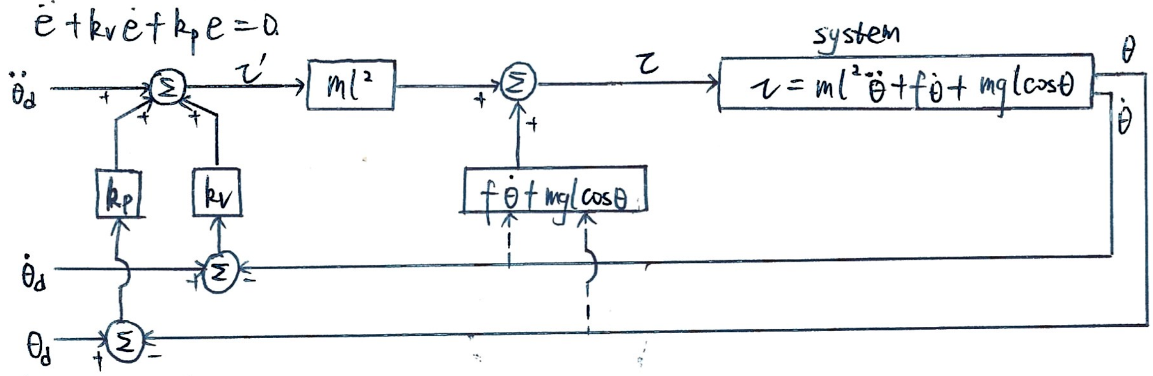
$\tau = \alpha\tau' + \beta. \quad \ddot{\theta} = \tau'$

$\alpha = ml^2$

$\beta = f\dot{\theta} + mgl\cos\theta$

$\tau' = \ddot{\theta}_d + k_v\dot{e} + k_p e \qquad e = \theta_d - \theta$

$\ddot{e} + k_v\dot{e} + k_p e = 0$



(c) $ml^2\ddot{\theta} + f\dot{\theta} + mgl\cos\theta = \tau$

$x_1 = \theta$

$x_2 = \dot{x}_1 = \dot{\theta}$

$\dot{x}_2 = \ddot{\theta} = \dfrac{\tau - f\dot{\theta} - mgl\cos\theta}{ml^2}$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{1}{ml^2}fx_2 - \frac{g}{l}\cos x_1 + \frac{1}{ml^2}\tau\end{bmatrix} = f(x_1, x_2, \tau)$$

$x_0 = \begin{bmatrix}\frac{\pi}{2}\\ 0\end{bmatrix} \quad \tau_0 = 0$

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\sin x_1 & -\frac{1}{ml^2}f\end{bmatrix}$$

use the parameters from (a),

$$A = \begin{bmatrix} 0 & 1 \\ 10 & -7\end{bmatrix}. \quad B = \frac{\partial f}{\partial \tau} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 10 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\tau$$

$$\det(\lambda I - A) = \begin{bmatrix}\lambda & \lambda-1 \\ \lambda-10 & \lambda+7\end{bmatrix}$$

$= \lambda(\lambda+7) - (\lambda-1)(\lambda-10)$

$= 18\lambda - 10 = 0 \qquad \lambda = \frac{5}{9}$

The pole is in the right half plane, the system is unstable.