ICCM

Hey Pentti, We Did It Again!:
Differentiable vector-symbolic types that prove polynomial termination

Eilene Tomkins-Flanagan*, Connor Hanley*, Mary Kelly*
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Outline

Demise of the Power Law

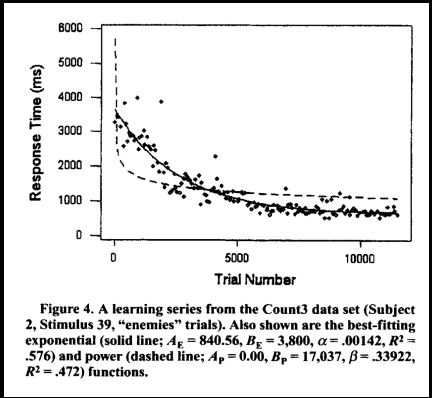
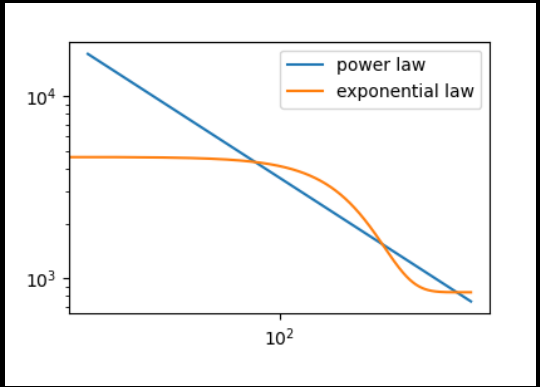
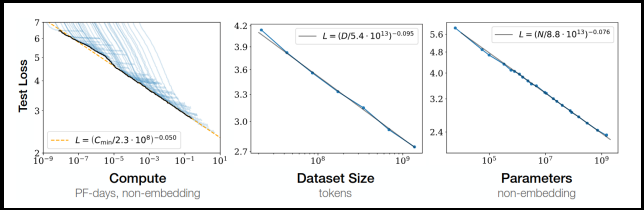


Figure 4. A learning series from the Count3 data set (Subject 2, Stimulus 39, "enemies" trials). Also shown are the best-fitting exponential (solid line; $A_E = 840.56$, $B_E = 3{,}800$, $\alpha = .00142$, $R^2 = .576$) and power (dashed line; $A_P = 0.00$, $B_P = 17{,}037$, $\beta = .33922$, $R^2 = .472$) functions.



Reproduced from Heathcote et al. (2000)

Are Transformers So Great After All?



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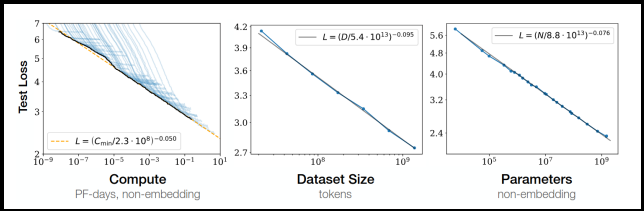
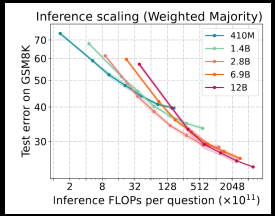
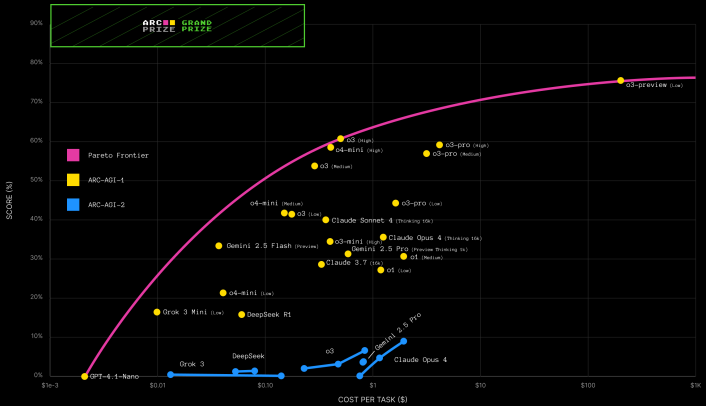




Reproduced from Kaplan et al. (2020) and Wu et al. (2025)

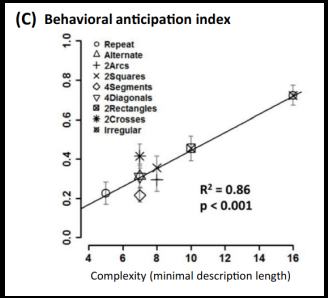
The Wages of GPS



Reproduced from ARC Prize Foundation (2025)

There is an alternative to the Empiricist idea that all learning consists of a kind of statistical inference, realized by adjusting parameters; it's the Rationalist idea that some learning is a kind of theory construction, effected by framing hypotheses and evaluating them against evidence. We seem to remember having been through this argument before. (Fodor & Pylyshyn, 1988)

Is Rationalist Learning Possible?



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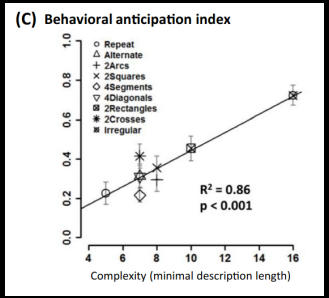

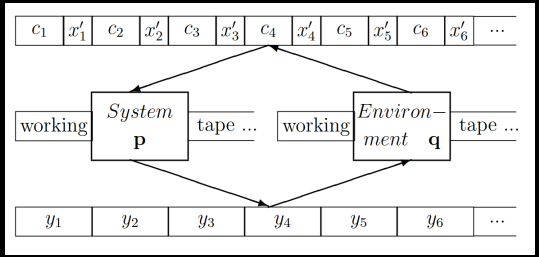

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Goals

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2. We need constraints on program search!

Cutting Edge Technology: Lisp 1.5

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  - To good approximation, the λ-calculus is a subset of Lisp 1.5 (McCarthy, 1960)

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Inessential to our VSA definition, but used frequently (and here), are permutations of symbols $\mathbf{P}(v)$

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We denote $\mathsf{cons}(x, \mathsf{cons}(y, \ldots))$ as $(x\ y\ \ldots)$, omitting the final nil, as is convention.

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$$((\texttt{LAMBDA}\ ()\ e)) \mapsto e \tag{1}$$

$$((\texttt{LAMBDA}\ (x\ y\ \ldots)\ e)\ a) \mapsto (\texttt{LAMBDA}\ (y\ \ldots)\ e[x := a]) \tag{2}$$

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\mathsf{nil}(n, t) &:= \mathbf{Nil} + (\mathbf{level} \otimes n) + (\mathbf{type} \otimes t) \\
\mathsf{dollar}(n) &:= \mathbf{Dollar} + (\mathbf{level} \otimes n) \\
\mathsf{app}(t, s) &:= \mathbf{App} + (\mathbf{rator} \otimes t) + (\mathbf{rand} \otimes s) \\
\mathsf{lambda}(x, \tau, t) &:= \mathbf{Lambda} + (\mathbf{var} \otimes x) + (\mathbf{type} \otimes \tau) \\
\mathsf{box}(n, x, s, t) &:= \mathbf{Box} + (\mathbf{let} \otimes x) + (\mathbf{this} \otimes s) + (\mathbf{that} \otimes t) + (\mathbf{level} \otimes n)
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Doug (encoding LLFPL) uses the special *dollar* and *box* terms with *levels* to constrain recursion to polynomial runtime

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- To achieve these properties, we use an encoding based on Holographic Declarative Memory (HDM; Kelly et al., 2020)

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- As a result, types will be similar if their set encodings share any subsets (or different subsets with similar encodings)

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- Designing neural networks that efficiently learn constraints over the type space

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References I

- ARC Prize Foundation. (2025). Arcpize.org [Accessed: 2025-07-27].
- Dehaene, S., Al Roumi, F., Lakretz, Y., Planton, S., & Sablé-Meyer, M. (2022).Symbols and mental programs: A hypothesis about human singularity. *Trends in Cognitive Sciences*, *26*(9), 751–766. https://doi.org/10.1016/j.tics.2022.06.010
- Fodor, J. A., & Pylyshyn, Z. W. (1988).Connectionism and cognitive architecture: A critical analysis. *Cognition*, *28*(1), 3–71. https://doi.org/10.1016/0010-0277(88)90031-5
- Gayler, R. (2003).Vector symbolic architectures answer jackendoff's challenges for cognitive neuroscience. *Proceedings of the ICCS 2003*.

References II

- Heathcote, A., Brown, S., & Mewhort, D. J. K. (2000).The power law repealed: The case for an exponential law of practice. *Psychonomic Bulletin & Review*, *7*(2), 185–207. https://doi.org/10.3758/BF03212979
- Hutter, M. (2000). Towards a universal theory of artificial intelligence based on algorithmic probability and sequential decision theory.
- Hutter, M. (2005). *Universal artificial intelligence: Sequential decisions based on algorithmic probability*. Springer.
- Jackendoff, R. (2002). *Foundations of language : Brain, meaning, grammar, evolution*. Oxford University Press.
- Kaplan, J., McCandlish, S., Henighan, T., Brown, T. B., Chess, B., Child, R., Gray, S., Radford, A., Wu, J., & Amodei, D. (2020). Scaling laws for neural language models. https://arxiv.org/abs/2001.08361

References III

- Kelly, M. A., Arora, N., West, R. L., & Reitter, D. (2020). Holographic declarative memory: Distributional semantics as the architecture of memory. *Cognitive Science*, *44*(11), e12904. https://doi.org/10.1111/cogs.12904
- Lawvere, F. W. (1969). Diagonal arguments and cartesian closed categories. *Category Theory, Homology Theory and their Applications II*, 134–145.
- McCarthy, J. (1960). Recursive functions of symbolic expressions and their computation by machine, part i. *Commun. ACM*, *3*(4), 184–195. https://doi.org/10.1145/367177.367199
- Schimanski, S. (2009). *Polynomial time calculi* [Doctoral dissertation, Ludwig Maximilian University of Munich].

References IV

- Tomkins-Flanagan, E., & Kelly, M. A. (2024).Hey Pentti, We Did It!: A Fully Vector-Symbolic Lisp (C. Sibert, Ed.). *Proceedings of the 22nd International Conference on Cognitive Modeling*, 65–72. https://github.com/eilene-ftf/holis
- Wu, Y., Sun, Z., Li, S., Welleck, S., & Yang, Y. (2025). Inference scaling laws: An empirical analysis of compute-optimal inference for problem-solving with language models. https://arxiv.org/abs/2408.00724

https://carleton.ca/animus


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