ECON2125/6012

Fedor Iskhakov

Aug 09, 2023

CONTENTS

Preliminary schedule

Week	Date	Topic	Notes
1	July 27	*Introduction*	Recorded lecture
2	Aug 3	*Univariate and bivariate optimization*	Tutorials start
3	Aug 10	*Elements of set theory and analysis*	
4	Aug 17	*Elements of linear algebra*	
Test		15%	Submit by Aug 23
5	Aug 24	*Elements of Probability*	
6	Aug 31	*Fundamentals of optimization*	
Test		15%	Submit by Sept 3
Break			2 weeks
7	Sept 21	*Unconstrained optimization*	
8	Sept 28	*Constrained optimization*	
Test		15%	Submit by Oct 4
9	Oct 5	*Practical session/invited speaker*	TBA
10	Oct 12	*Envelope and maximum theorems*	
11	Oct 19	*Dynamic optimization*	
12	Oct 26	*Revision*	
Exam		55%	During exam period

ANU course pages

Course Wattle page Schedule, announcements, teaching team contacts, recordings, assignement, grades

Course overview Class summary General course description in ANU Programs and Courses

CHAPTER ONE

WELCOME

Course title: **"Optimization for Economics and Financial Economics"**

- Elective second year course in the *Bachelor of Economics* program ECON2125
- Compulsory second math course in the *Master of Economics* program ECON6012

The two courses are identical in content and assessment, but final grades may be adjusted depending on your program.

1.1 Plan for this lecture

1. Organization
2. Administrative topics
3. Course content
4. Self-learning materials

1.2 Instructor

Fedor Iskhakov Professor of Economics at RSE

- Office: 1021 HW Arndt Building
- Email: fedor.iskhakov@anu.edu.au
- Web: fedor.iskh.me
- Contact hours: Thursday 9:30-11:30

1.3 Timetable

Face-to-face:

- Lectures: Thursday 15:30 — 17:30
- Location: **DNF Dunbar Lecture Theatre, Physics Bldg 39A**

Online:

- Echo-360 recordings on Wattle
- All notes and materials on **optim.iskh.me**

Face-to-face is strictly preferred

1.4 Course web pages

- Wattle Schedule, announcements, teaching team contacts, recordings, assignment, grades
- Online notes Lecture notes, slides, assignment tasks
- Lecture slides should appear online the previous day before the lecture
- Details on assessment including the exam instructions will appear on Wattle

1.5 Tutorials

- Enrollments open on *Wattle*

Tutorial questions

- posted on the course website
- not assessed, help you learn and prepare

Tutorials start on week 2

1.6 Tutors

Wending Liu

- Email: Wending.Liu@anu.edu.au
- Room: Room 2084, Copland Bld (24)
- Office hours: **Friday 1pm-3pm**

Chien Yeh

- Email: Chien.Yeh@anu.edu.au
- Room: Room 2106, Copland Bld (24)
- Office hours: **Monday 2pm-4pm**

1.7 Prerequisites

See Course overview and Class summary

What you actually need to know:

- basic algebra
- basic calculus
- some idea of what a matrix is, etc.

≈ content of EMET1001/EMET7001 math course

1.8 Focus?

Q: Is this optimization or a general math-econ course?

A: A general course on mathematical modeling for economics and financial economics. Optimization will be an important and recurring theme.

1.9 Assessment

- 3 timed open book tests (15% each)
- Final exam (55%)

The three tests spread out through the semester will check the knowledge of the immediately preceding material. The final closed book in-person exam will cover the entire course.

1.10 Questions

1. Administrative questions: RSE admin

- **Bronwyn Cammack** Senior School Administrator
- Email: enquiries.rse@anu.edu.au
- "I can not register for the tutorial group"

2. Content related questions: please, refer to the tutors

- "I don't understand why this function is convex"

3. Other questions: to Fedor

- "I'm working hard but still can not keep up"
- "Can I please have extra assignment for more practice"

1.11 Attendance

- Please, **do not** use email for *instructional* questions\Instead make use of the office hours
- Attendance of tutorials is *very highly* recommended
  You will make your life much easier this way
- Attendance of lectures is *highly* recommended
  But not mandatory

1.12 Comments for lectures notes/slides

- Cover exactly what you are required to know
- Code inserts are the exception, they are not assessable

In particular, you need to know:

- The definitions from the notes
- The facts from the notes
- How to apply facts and definitions

If a concept in not in the lecture notes, it is not assessable

1.13 Definitions and facts

The lectures notes/slides are full of definitions and facts.

Definition

Functions $f : \mathbb{R} \to \mathbb{R}$ is called *continuous at* x if, for any sequence $\{x_n\}$ converging to x, we have $f(x_n) \to f(x)$.

Possible exam question: "Show that if functions f and g are continuous at x, so is $f + g$."

You should start the answer with the definition of continuity:

"Let $\{x_n\}$ be any sequence converging to x. We need to show that $f(x_n) + g(x_n) \to f(x) + g(x)$. To see this, note that $\ldots$"

1.14 Facts

In the lecture notes/slides you will often see

Fact

The only N-dimensional subset of $\mathbb{R}^N$ is $\mathbb{R}^N$.

This means either:

- theorem
- proposition
- lemma
- true statement

All well known results. You need to remember them, have some intuition for, and be able to apply.

1.15 Note on Assessments

Assessable = definitions and facts + last year level math + a few simple steps of logic

Exams and tests will award:

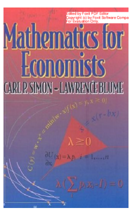
- Hard work
- Deeper understanding of the concepts

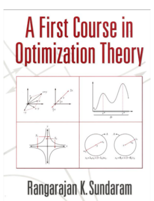
In each question there will be a *easy* path to the solution

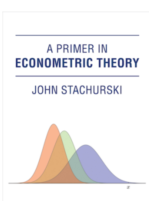
1.16 Reading materials

Primary reference: lecture slides

Books:







- "Mathematics for Economists" (1994) by Simon, C. and L. Blume
- "A First Course in Optimization" (1996) Theory by Rangarajan Sundaram
- "A Primer in Econometric Theory" (2016) by John Stachurski

Readings are supplementary but will provide a more detailed explanation with additional examples.

- Each lecture will reference book chapters

1.17 Key points for the administrative part

- Tutorials start next week, **please register before the next lecture**
- Course content = what's in lecture notes/slides
- Lecture slides are available online and will be updated throughout the semester
- Optimization is a recurring theme but not the only topic

1.18 What you will learn in the course

- The lecture plan is on the course website optim.iskh.me and Class summary
- See the list of topics on the left

Essentially:

1. **Mathematical foundations**

- elements of analysis
- elements of linear algebra
- elements of probability

2. **Optimization theory**

- when solution exists
- unconstrained optimization
- optimization with equality constraints
- optimization with inequality constraints

3. **Further topics**

- Parameterized optimization problems
- Optimization in dynamics

1.19 Further material and self-learning

- Each lecture will suggest some material for further reading and learning
- Today: **The Wason Selection Task** logical problem
- Mathematics relies on rules of logic
- Yet, for human brain applying mathematical logic may be difficult, and dependent on the domain

Please, watch the video and try to solve the puzzle yourself youtu.be/iR97LBgpsl8

CHAPTER

TWO

UNIVARIATE AND BIVARIATE OPTIMIZATION

ECON2125/6012 Lecture 2 Fedor Iskhakov

2.1 Announcements & Reminders

- **Tutorials start tomorrow (Aug 4)**
- Register for tutorials on Wattle if you have not done so already
- Office hours of the tutors are updated:
  - **Wending Liu**
    - Email: Wending.Liu@anu.edu.au
    - Room: Room 2084, Copland Bld (24) (*updated!*)
    - Office hours: **Friday 1pm-3pm**
  - **Chien Yeh**
    - Email: Chien.Yeh@anu.edu.au
    - Room: Room 2106, Copland Bld (24)
    - Office hours: **Monday 2pm-4pm**
- Reminder on how to ask questions:
  1. Administrative: RSE admin
  2. Content/understanding: tutors
  3. Other: to Fedor

2.2 Plan for this lecture

1. Motivation (math vs. computing)
2. Univariate optimization
3. Working with bivariate functions
4. Bivariate optimization

Supplementary reading:

- Simon & Blume: part 1 (revision)

- Sundaram: sections 1.1, 1.4, chapter 2, chapter 4

2.3 Computing

The *classic* way we do mathematics is pencil and paper

> In 1944, Hans Bethe solved following problem *by hand*:
> Will detonating an atom bomb ignite the atmosphere and thereby destroy life on earth?
> source

These days we rarely calculate with actual numbers

Almost all calculations are done on computers

Example: numerical integration

$$\frac{1}{\sqrt{2\pi}} \int_{-2}^{2} \exp\left\{-\frac{x^2}{2}\right\} dx$$

```
from scipy.stats import norm
from scipy.integrate import quad
phi = norm()
value, error = quad(phi.pdf, -2, 2)
print('Integral value =',value)
```

```
Integral value = 0.9544997361036417
```

Example: Numerical optimization

$$f(x) = -\exp\left\{-\frac{(x-5.0)^4}{1.5}\right\} \to \min$$

```
from scipy.optimize import fminbound
import numpy as np
f = lambda x: -np.exp(-(x - 5.0)**4 / 1.5)
res = fminbound(f, -10, 10)  # find approx solution
print('Minimum value is attained approximately at', res)
```

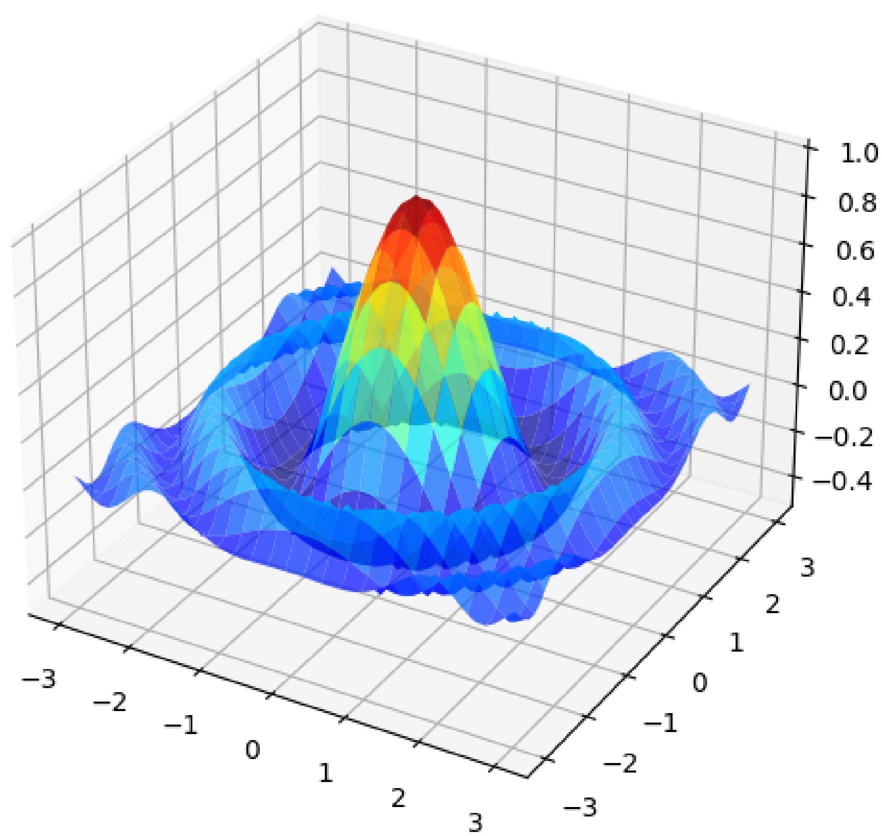
```
Minimum value is attained approximately at 4.999941901210501
```

Example: Visualization

What does this function look like?

$$f(x,y) = \frac{\cos(x^2+y^2)}{1+x^2+y^2}$$

```python
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d.axes3d import Axes3D
import numpy as np
from matplotlib import cm
f = lambda x, y: np.cos(x**2 + y**2) / (1 + x**2 + y**2)
xgrid = np.linspace(-3, 3, 50)
ygrid = xgrid
x, y = np.meshgrid(xgrid, ygrid)
fig = plt.figure(figsize=(8, 6))
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(x,
                y,
                f(x, y),
                rstride=2, cstride=2,
                cmap=cm.jet,
                alpha=0.7,
                linewidth=0.25)
ax.set_zlim(-0.5, 1.0)
plt.show()
```



Example: Symbolic calculations

Differentiate $f(x) = (1 + 2x)^5$.

Forgotten how? No problems, just ask a computer for *symbolic* derivative

```python
import sympy as sp
x = sp.Symbol('x')
fx = (1 + 2 * x)**5
print("Derivative of",fx,"is",fx.diff(x))
```

```
Derivative of (2*x + 1)**5 is 10*(2*x + 1)**4
```

So if computers can do our maths for us, why learn maths?

The difficulty is

- giving them the right inputs and instructions
- interpreting what comes out

The skills we need are

- Understanding of fundamental concepts
- Sound deductive reasoning

These are the focus of the course

2.3.1 Computer Code in the Lectures

While computation is not a formal part of the course
there will be little bits of code in the lectures to illustrate the kinds of things we can do.

- All the code will be written in the Python programming language
- It is not assessable

You might find value in actually running the code shown in lectures
If you want to do so please refer to **linked GitHub repository** in optim.iskh.me

2.4 Univariate Optimization

Let $f\colon [a, b] \to \mathbb{R}$ be a differentiable (smooth) function

- $[a, b]$ is all x with $a \leq x \leq b$
- $\mathbb{R}$ is "all numbers"
- f takes $x \in [a, b]$ and returns number $f(x)$
- derivative $f'(x)$ exists for all x with $a < x < b$

Definition

A point $x^* \in [a, b]$ is called a

- ***maximizer*** of f on $[a, b]$ if $f(x^*) \geq f(x)$ for all $x \in [a, b]$
- ***minimizer*** of f on $[a, b]$ if $f(x^*) \leq f(x)$ for all $x \in [a, b]$

Example

Let

- $f(x) = -(x-4)^2 + 10$
- $a = 2$ and $b = 8$

Then

- $x^* = 4$ is a maximizer of f on $[2, 8]$
- $x^{**} = 8$ is a minimizer of f on $[2, 8]$

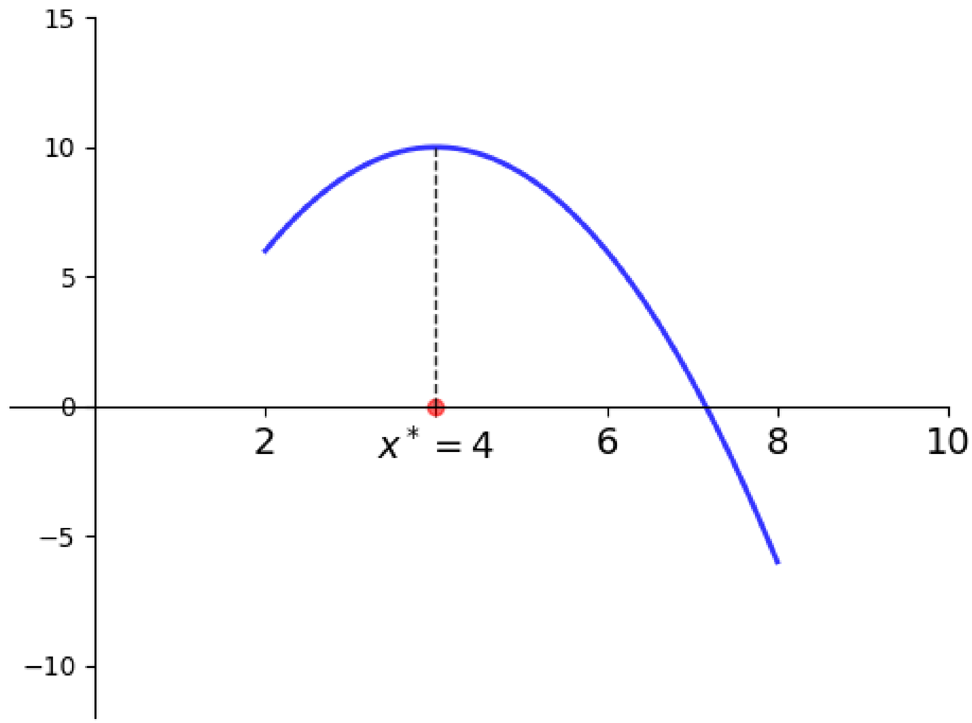


Fig. 2.1: Maximizer on $[a, b] = [2, 8]$ is $x^* = 4$

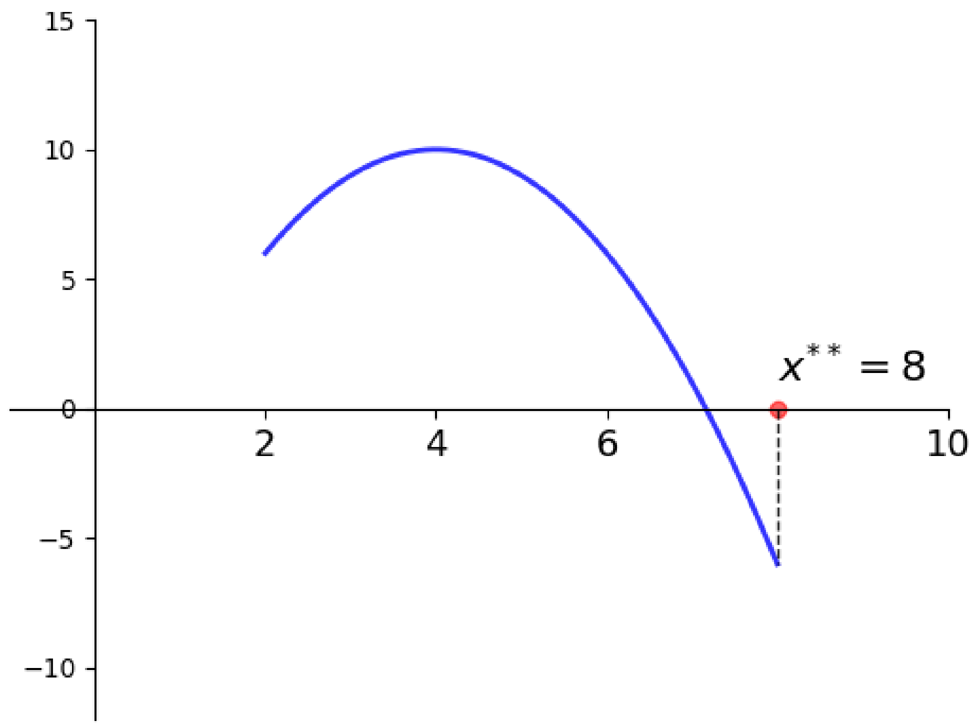


Fig. 2.2: Minimizer on $[a, b] = [2, 8]$ is $x^{**} = 8$

The set of maximizers/minimizers can be

- empty
- a singleton (contains one element)
- infinite (contains infinitely many elements)

Example: infinite maximizers

$f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 1$
has infinitely many maximizers and minimizers on $[0, 1]$

Example: no maximizers

The following function has no maximizers on $[0, 2]$

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 1/2 & \text{otherwise} \end{cases}$$

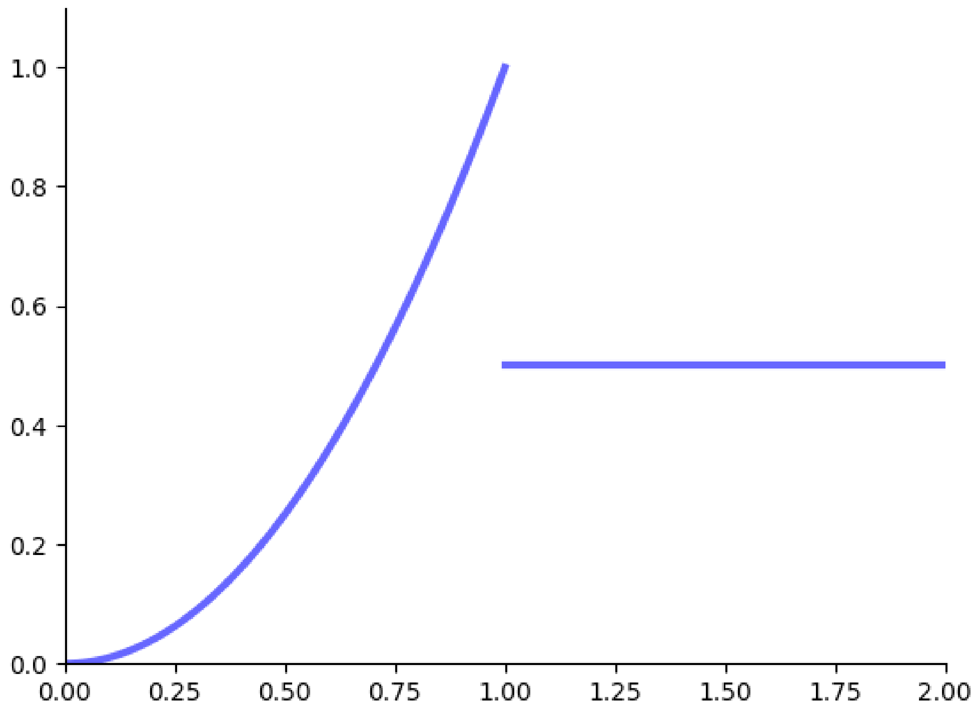


Fig. 2.3: No maximizer on $[0, 2]$

Definition

Point x is called ***interior*** to $[a, b]$ if $a < x < b$

The set of all interior points is written (a, b)

We refer to $x^* \in [a, b]$ as

- ***interior maximizer*** if both a maximizer and interior
- ***interior minimizer*** if both a minimizer and interior

2.5 Finding optima

Definition

A ***stationary point*** of f on $[a, b]$ is an interior point x with $f'(x) = 0$

Fact

If f is differentiable and x^* is either an interior minimizer or an interior maximizer of f on $[a, b]$, then x^* is stationary

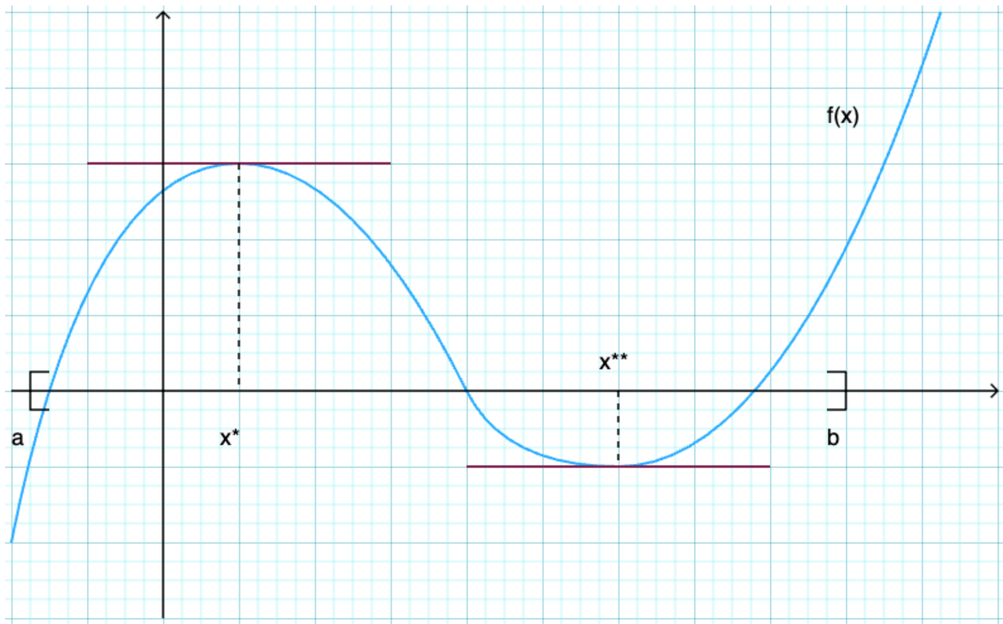


Fig. 2.4: Both x^* and x^{**} are stationary

Sketch of proof, for maximizers:

$$f'(x^*) = \lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h} \qquad \text{(by def.)}$$

$$\Rightarrow f(x^* + h) \approx f(x^*) + f'(x^*)h \qquad \text{for small } h$$

If $f'(x^*) \neq 0$ then exists small h such that $f(x^* + h) > f(x^*)$

Hence interior maximizers must be stationary — otherwise we can do better

$\Rightarrow$ any interior maximizer stationary
$\Rightarrow$ set of interior maximizers $\subset$ set of stationary points
$\Rightarrow$ maximizers $\subset$ stationary points $\cup \{a\} \cup \{b\}$

Usage:

1. Locate stationary points
2. Evaluate $y = f(x)$ for each stationary x and for a, b
3. Pick point giving largest y value

Minimization: same idea

Example

Let's solve

$$\max_{-2 \leq x \leq 5} f(x) \quad \text{where} \quad f(x) = x^3 - 6x^2 + 4x + 8$$

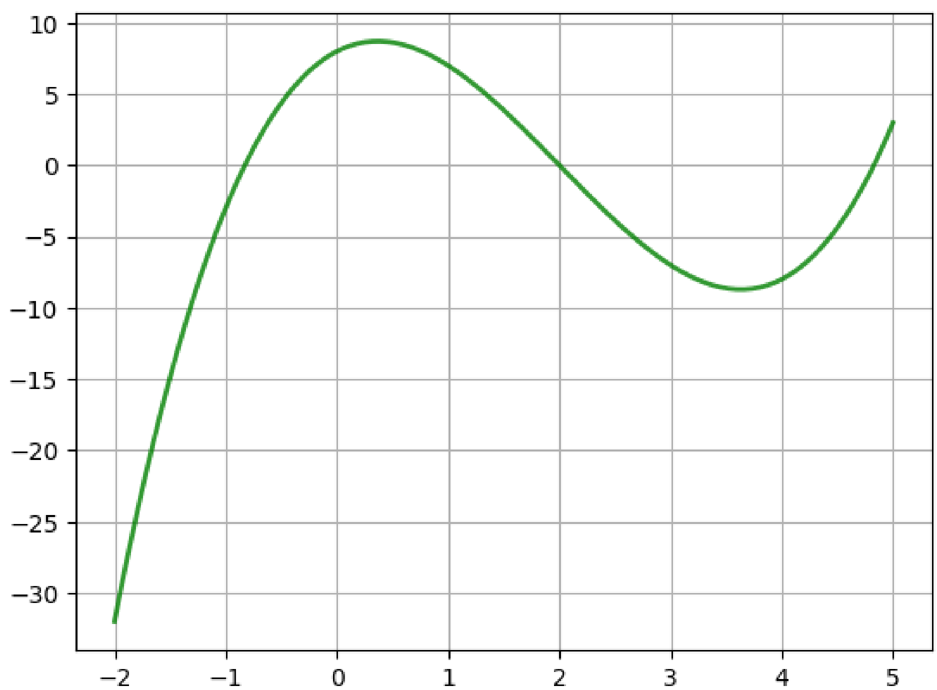
Steps

- Differentiate to get $f'(x) = 3x^2 - 12x + 4$
- Solve $3x^2 - 12x + 4 = 0$ to get stationary x

- Discard any stationary points outside $[-2, 5]$
- Eval f at remaining points plus end points -2 and 5
- Pick point giving largest value

```
from sympy import *
x = Symbol('x')
points = [-2, 5]
f = x**3 - 6*x**2 + 4*x + 8
fp = diff(f, x)
spoints = solve(fp, x)
points.extend(spoints)
v = [f.subs(x, c).evalf() for c in points]
maximizer = points[v.index(max(v))]
print("Maximizer =", str(maximizer),'=',maximizer.evalf())
```

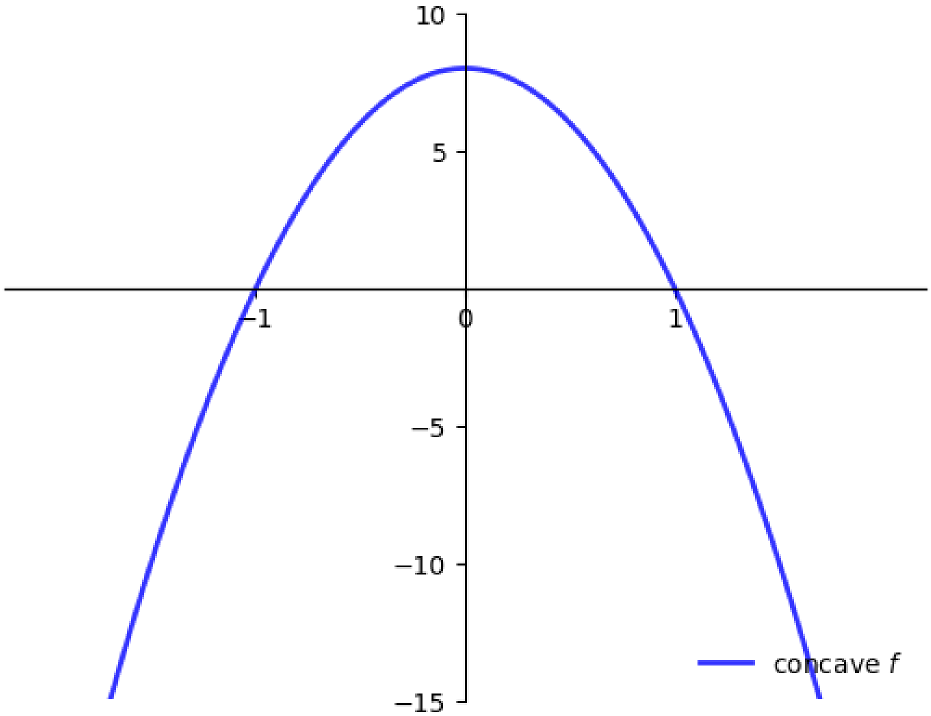
```
Maximizer = 2 - 2*sqrt(6)/3 = 0.367006838144548
```



2.6 Shape Conditions and Sufficiency

When is $f'(x^*) = 0$ sufficient for x^* to be a maximizer?

One answer: When f is concave



(Full definition deferred)

Sufficient conditions for *concavity* in one dimension

Let $f: [a, b] \to \mathbb{R}$

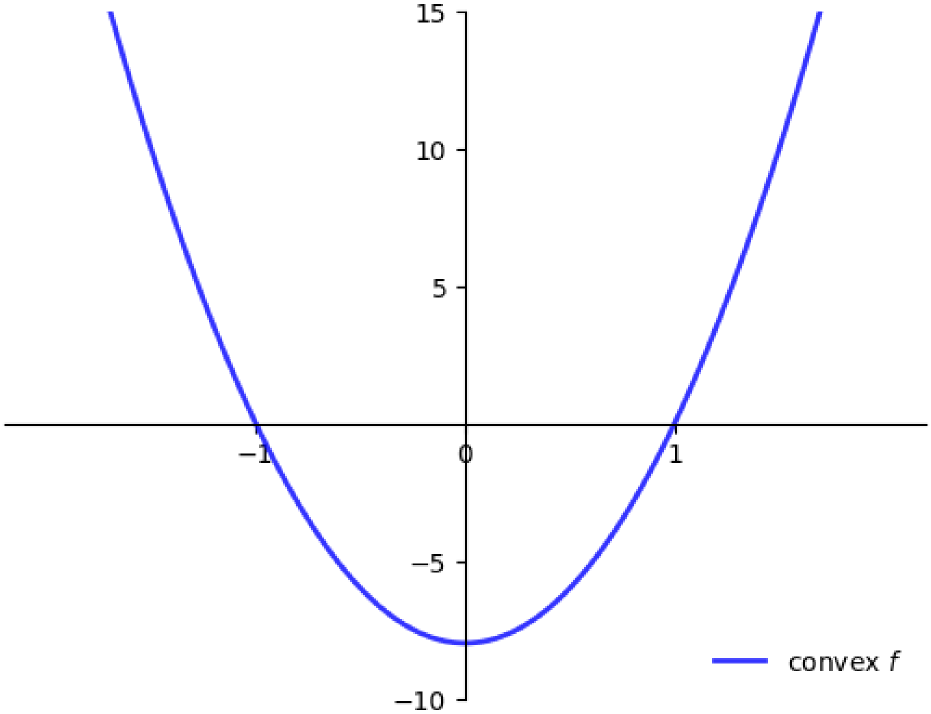
- If $f''(x) \leq 0$ for all $x \in (a, b)$ then f is concave on (a, b)
- If $f''(x) < 0$ for all $x \in (a, b)$ then f is **strictly** concave on (a, b)

Example

- $f(x) = a + bx$ is concave on $\mathbb{R}$ but not strictly
- $f(x) = \log(x)$ is strictly concave on $(0, \infty)$

When is $f'(x^*) = 0$ sufficient for x^* to be a minimizer?

One answer: When f is convex



(Full definition deferred)

Sufficient conditions for *convexity* in one dimension

Let $f \colon [a, b] \to \mathbb{R}$

- If $f''(x) \geq 0$ for all $x \in (a, b)$ then f is convex on (a, b)
- If $f''(x) > 0$ for all $x \in (a, b)$ then f is **strictly** convex on (a, b)

Example

- $f(x) = a + bx$ is convex on $\mathbb{R}$ but not strictly
- $f(x) = x^2$ is strictly convex on $\mathbb{R}$

2.6.1 Sufficiency and uniqueness with shape conditions

Fact

For maximizers:

- If $f \colon [a, b] \to \mathbb{R}$ is concave and $x^* \in (a, b)$ is stationary then x^* is a maximizer
- If, in addition, f is strictly concave, then x^* is the unique maximizer

Fact

For minimizers:

- If $f: [a, b] \to \mathbb{R}$ is convex and $x^* \in (a, b)$ is stationary then x^* is a minimizer
- If, in addition, f is strictly convex, then x^* is the unique minimizer

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Example

A price taking firm faces output price $p > 0$, input price $w > 0$

Maximize profits with respect to input ℓ

$$\max_{\ell \ge 0} \pi(\ell) = pf(\ell) - w\ell,$$

where the production technology is given by

$$f(\ell) = \ell^{\alpha}, 0 < \alpha < 1.$$

Evidently

$$\pi'(\ell) = \alpha p \ell^{\alpha - 1} - w,$$

so unique stationary point is

$$\ell^* = (\alpha p / w)^{1/(1-\alpha)}$$

Moreover,

$$\pi''(\ell) = \alpha(\alpha - 1) p \ell^{\alpha - 2} < 0$$

for all $\ell \ge 0$ so ℓ^* is unique maximizer.

2.7 Functions of two variables

Let's have a look at some functions of two variables

- How to visualize them
- Slope, contours, etc.

Example: Cobb-Douglas production function

Consider production function

$$f(k, \ell) = k^{\alpha} \ell^{\beta}$$
$$\alpha \ge 0, \, \beta \ge 0, \, \alpha + \beta < 1$$

Let's graph it in two dimensions.

Like many 3D plots it's hard to get a good understanding

Let's try again with contours plus heat map

In this context the contour lines are called ***isoquants***

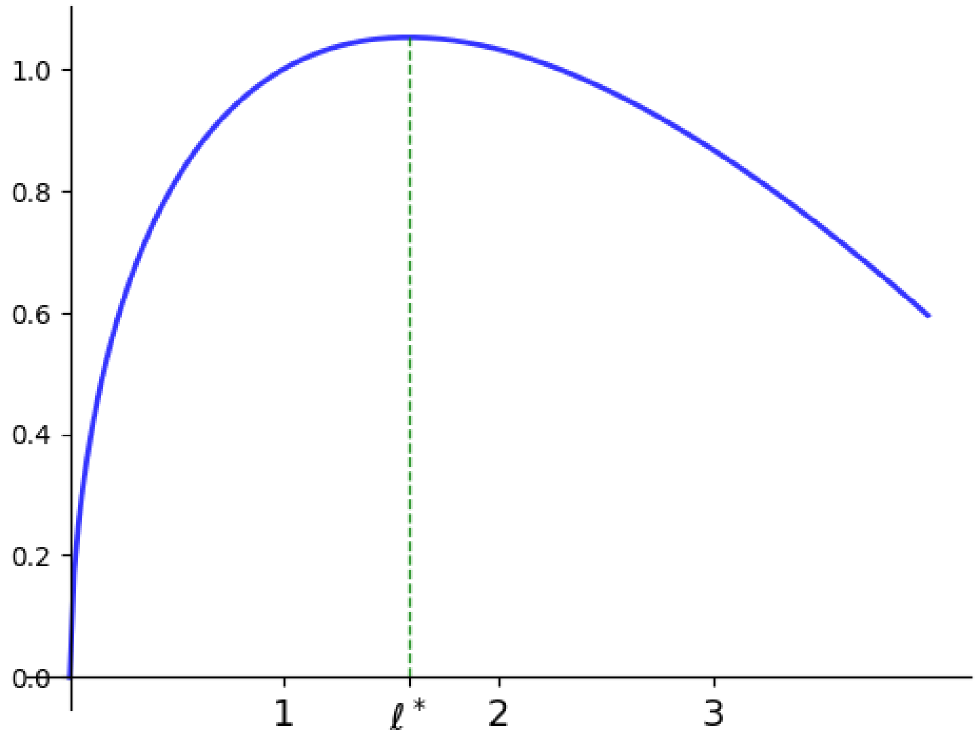


Fig. 2.5: Profit maximization with $p = 2$, $w = 1$, $\alpha = 0.6$, ℓ^* =1.5774

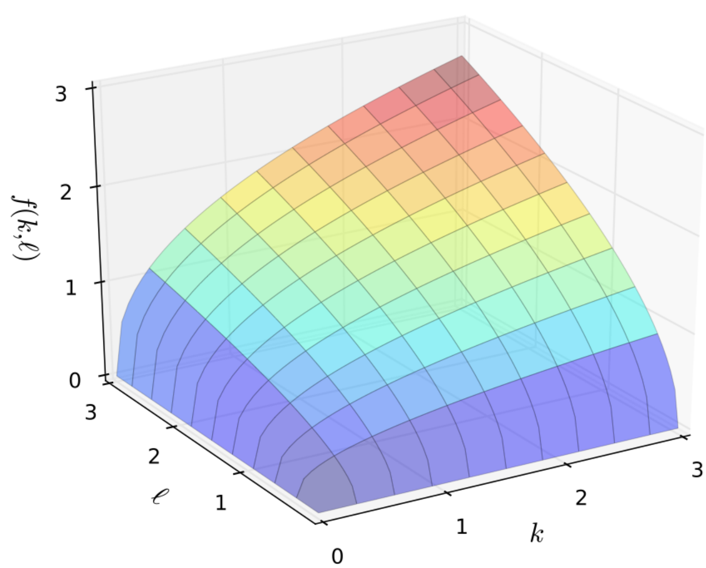


Fig. 2.6: Production function with $\alpha = 0.4$, $\beta = 0.5$ (a)

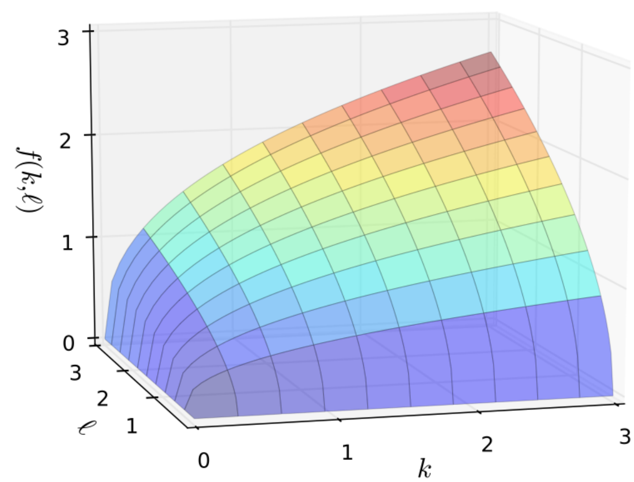


Fig. 2.7: Production function with $\alpha = 0.4$, $\beta = 0.5$ (b)

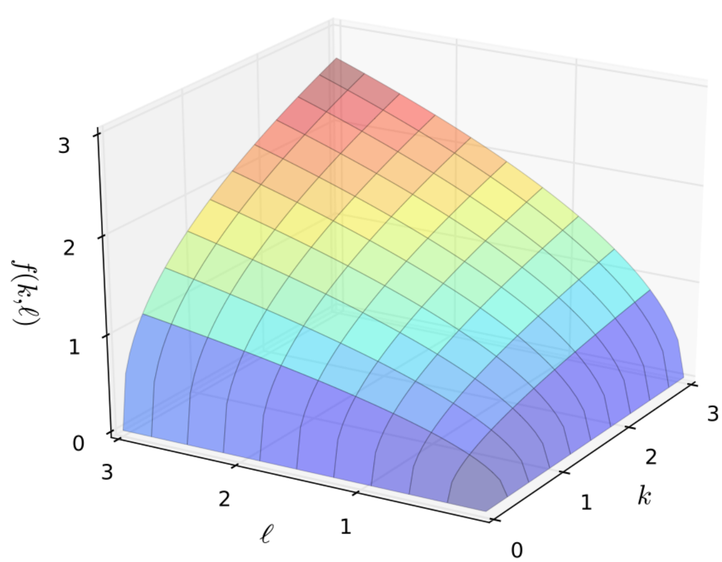


Fig. 2.8: Production function with $\alpha = 0.4$, $\beta = 0.5$ (c)

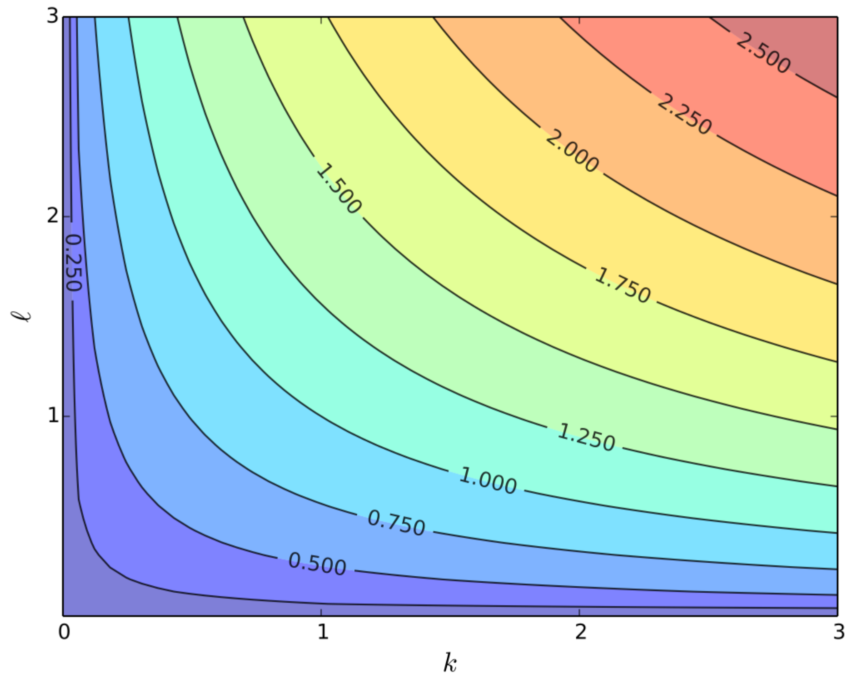


Fig. 2.9: Production function with $\alpha = 0.4$, $\beta = 0.5$, contours

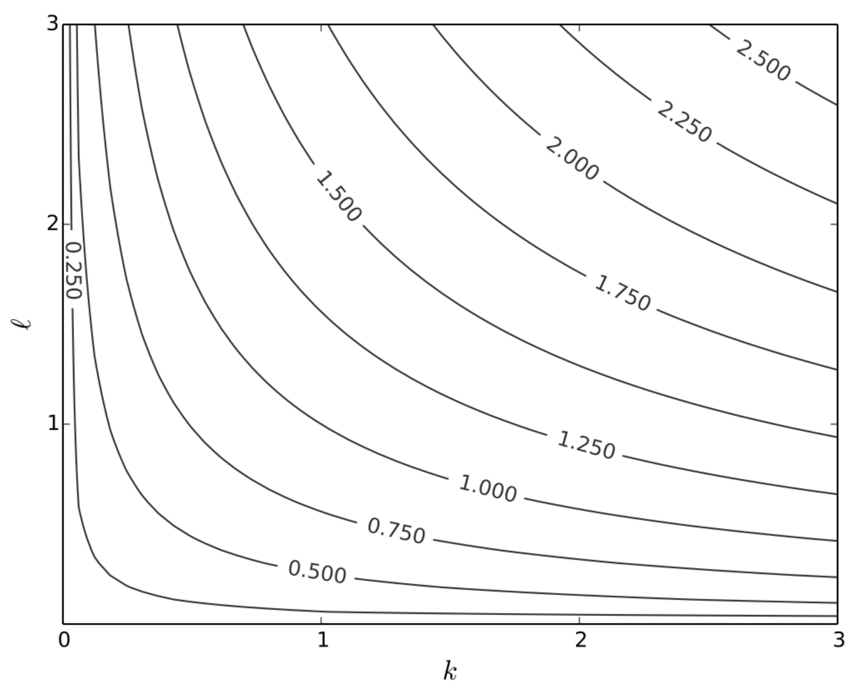


Fig. 2.10: Production function with $\alpha = 0.4$, $\beta = 0.5$

Can you see how $\alpha < \beta$ shows up in the slope of the contours?

We can drop the colours to see the numbers more clearly

Example: log-utility

Let $u(x_1, x_2)$ be "utility" gained from x_1 units of good 1 and x_2 units of good 2

We take

$$u(x_1, x_2) = \alpha \log(x_1) + \beta \log(x_2)$$

where

- α and β are parameters
- we assume $\alpha > 0$, $\beta > 0$
- The log functions mean "diminishing returns" in each good

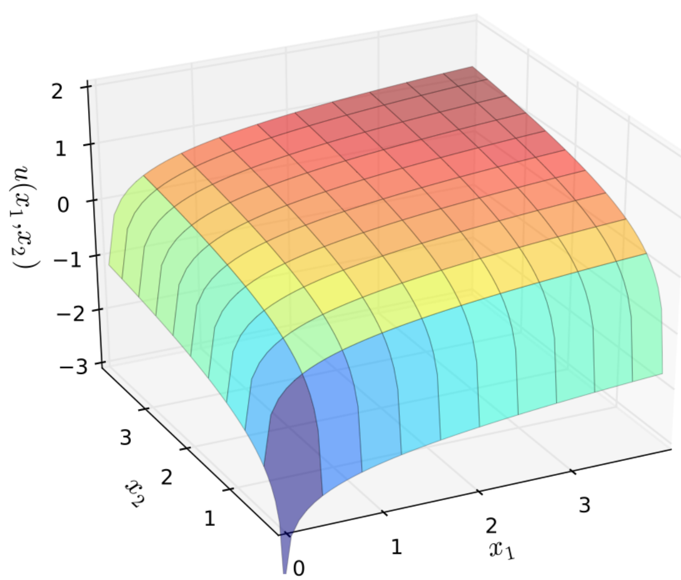


Fig. 2.11: Log utility with $\alpha = 0.4$, $\beta = 0.5$

Let's look at the contour lines

For utility functions, contour lines called ***indifference curves***

Example: quasi-linear utility

$$u(x_1, x_2) = x_1 + \log(x_2)$$

- Called quasi-linear because linear in good 1

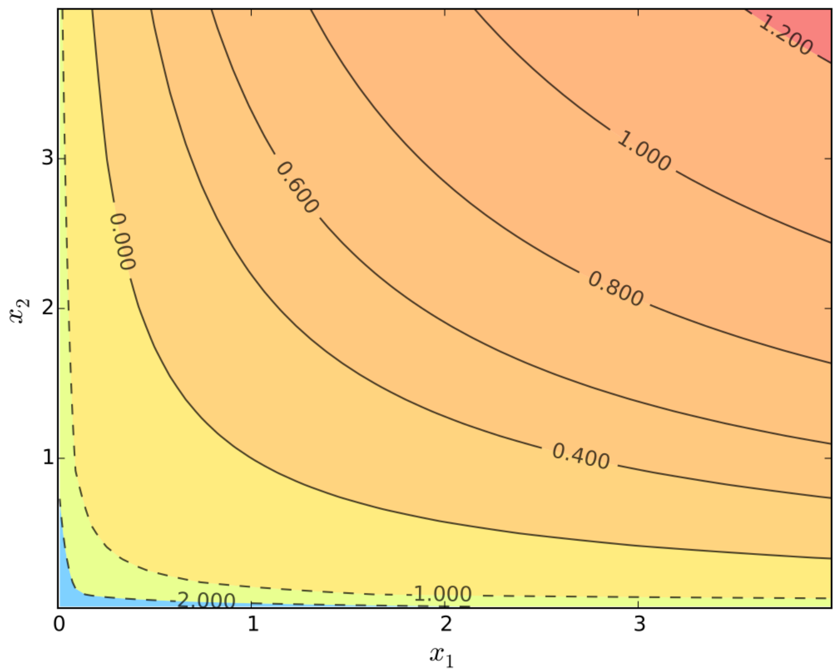


Fig. 2.12: Indifference curves of log utility with $\alpha = 0.4$, $\beta = 0.5$

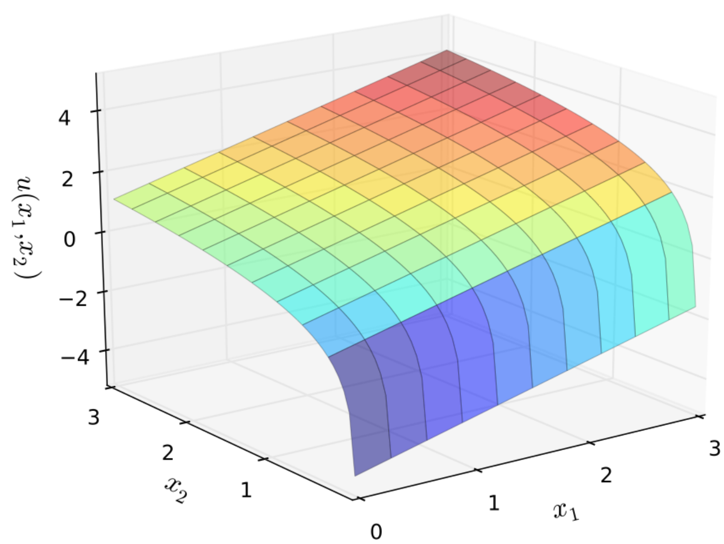


Fig. 2.13: Quasi-linear utility

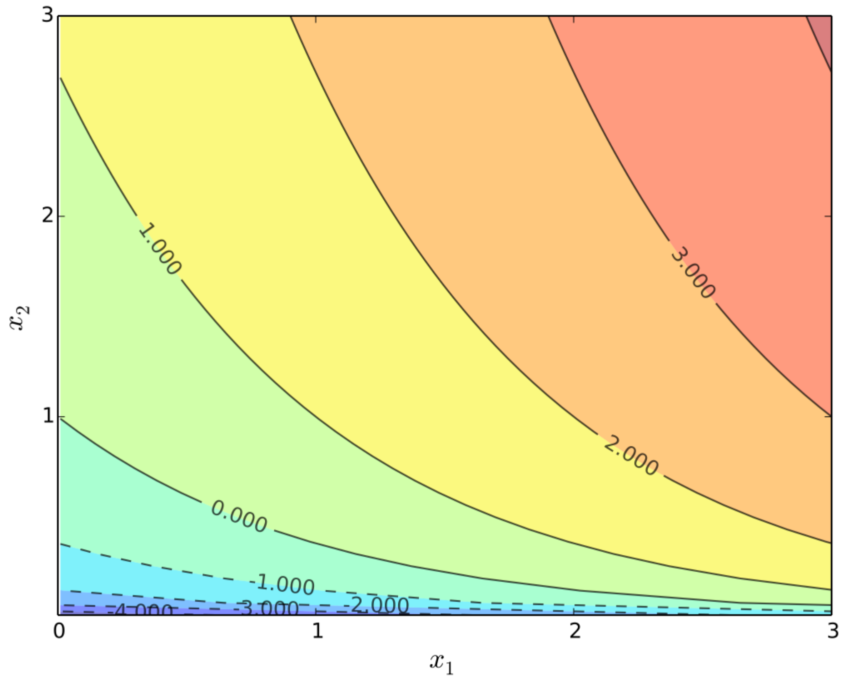


Fig. 2.14: Indifference curves of quasi-linear utility

Example: quadratic utility

$$u(x_1, x_2) = -(x_1 - b_1)^2 - (x_2 - b_2)^2$$

Here

- b_1 is a "satiation" or "bliss" point for x_1
- b_2 is a "satiation" or "bliss" point for x_2

Dissatisfaction increases with deviations from the bliss points

2.8 Bivariate Optimization

Consider $f: I \to \mathbb{R}$ where $I \subset \mathbb{R}^2$

The set $\mathbb{R}^2$ is all (x_1, x_2) pairs

Definition

A point $(x_1^*, x_2^*) \in I$ is called a ***maximizer*** of f on I if

$$f(x_1^*, x_2^*) \geq f(x_1, x_2) \quad \text{for all} \quad (x_1, x_2) \in I$$

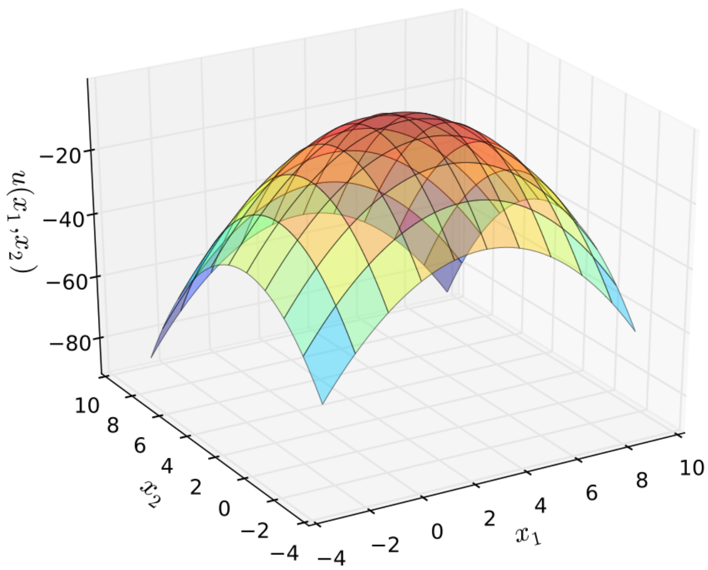


Fig. 2.15: Quadratic utility with $b_1 = 3$ and $b_2 = 2$

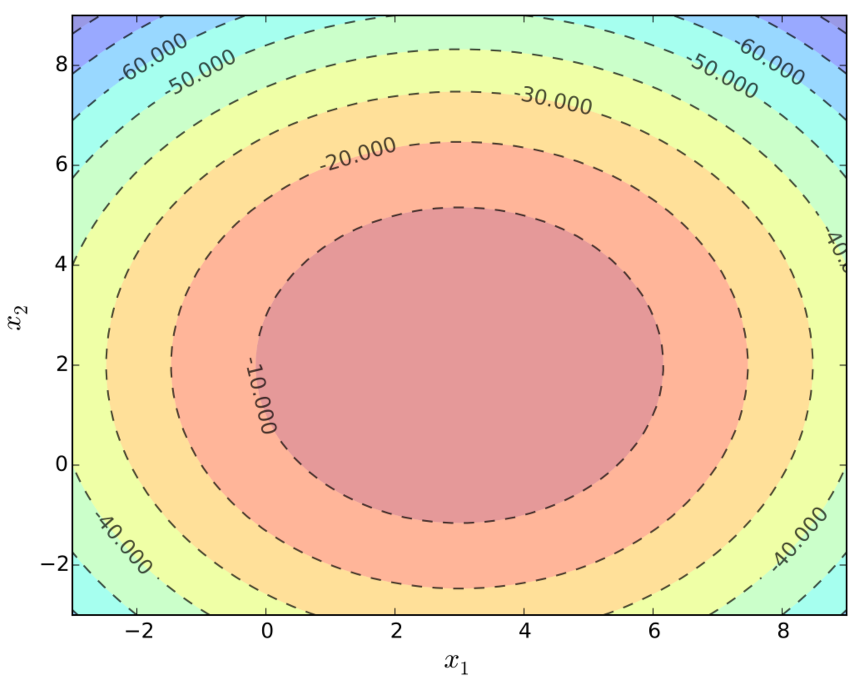


Fig. 2.16: Indifference curves quadratic utility with $b_1 = 3$ and $b_2 = 2$

Definition

A point $(x_1^*, x_2^*) \in I$ is called a ***minimizer*** of f on I if

$$f(x_1^*, x_2^*) \leq f(x_1, x_2) \quad \text{for all} \quad (x_1, x_2) \in I$$

When they exist, the partial derivatives at $(x_1, x_2) \in I$ are

$$f_1(x_1, x_2) = \frac{\partial}{\partial x_1} f(x_1, x_2)$$
$$f_2(x_1, x_2) = \frac{\partial}{\partial x_2} f(x_1, x_2)$$

Example

When $f(k, \ell) = k^\alpha \ell^\beta$,

$$f_1(k, \ell) = \frac{\partial}{\partial k} f(k, \ell) = \frac{\partial}{\partial k} k^\alpha \ell^\beta = \alpha k^{\alpha - 1} \ell^\beta$$

Definition

An interior point $(x_1, x_2) \in I$ is called ***stationary*** for f if

$$f_1(x_1, x_2) = f_2(x_1, x_2) = 0$$

Fact

Let $f \colon I \to \mathbb{R}$ be a continuously differentiable function. If (x_1^*, x_2^*) is either

- an interior maximizer of f on I, or
- an interior minimizer of f on I,

then (x_1^*, x_2^*) is a stationary point of f

Usage, for maximization:

1. Compute partials
2. Set partials to zero to find $S =$ all stationary points
3. Evaluate candidates in S and boundary of I
4. Select point (x_1^*, x_2^*) yielding highest value

Example

$$f(x_1, x_2) = x_1^2 + 4x_2^2 \to \min \quad \text{s.t.} \quad x_1 + x_2 \leq 1$$

Setting

$$f_1(x_1, x_2) = 2x_1 = 0 \quad \text{and} \quad f_2(x_1, x_2) = 8x_2 = 0$$

gives the unique stationary point $(0, 0)$, at which $f(0, 0) = 0$

On the boundary we have $x_1 + x_2 = 1$, so

$$f(x_1, x_2) = f(x_1, 1 - x_1) = x_1^2 + 4(1 - x_1)^2$$

Exercise: Show right hand side > 0 for any x_1

Hence minimizer is $(x_1^*, x_2^*) = (0, 0)$

2.8.1 Nasty secrets

Solving for (x_1, x_2) such that $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$ can be hard

- System of nonlinear equations
- Might have no analytical solution
- Set of solutions can be a continuum

Example

(Don't) try to find all stationary points of

$$f(x_1, x_2) = \frac{\cos(x_1^2 + x_2^2) + x_1^2 + x_1}{2 + p(-x_1^2) + \sin^2(x_2)}$$

Also:

- Boundary is often a continuum, not just two points
- Things get even harder in higher dimensions

On the other hand:

- Most classroom examples are chosen to avoid these problems
- Life is still pretty easy if we have concavity / convexity
- Clever tricks have been found for certain kinds of problems

2.9 Second Order Partials

Let $f \colon I \to \mathbb{R}$ and, when they exist, denote

$$f_{11}(x_1, x_2) = \frac{\partial^2}{\partial x_1^2} f(x_1, x_2)$$

$$f_{12}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} f(x_1, x_2)$$

$$f_{21}(x_1, x_2) = \frac{\partial^2}{\partial x_2 \partial x_1} f(x_1, x_2)$$

$$f_{22}(x_1, x_2) = \frac{\partial^2}{\partial x_2^2} f(x_1, x_2)$$

Example: Cobb-Douglas technology with linear costs

If $\pi(k, \ell) = pk^{\alpha}\ell^{\beta} - w\ell - rk$ then

$$\pi_{11}(k, \ell) = p\alpha(\alpha - 1)k^{\alpha-2}\ell^{\beta}$$

$$\pi_{12}(k, \ell) = p\alpha\beta k^{\alpha-1}\ell^{\beta-1}$$

$$\pi_{21}(k, \ell) = p\alpha\beta k^{\alpha-1}\ell^{\beta-1}$$

$$\pi_{22}(k, \ell) = p\beta(\beta - 1)k^{\alpha}\ell^{\beta-2}$$

Fact

If $f \colon I \to \mathbb{R}$ is twice continuously differentiable at (x_1, x_2), then

$$f_{12}(x_1, x_2) = f_{21}(x_1, x_2)$$

Exercise: Confirm the results in the exercise above.

2.10 Shape conditions in 2D

Let I be an "open" set (only interior points – formalities next week)

Let $f \colon I \to \mathbb{R}$ be twice continuously differentiable

The function f is strictly **concave** on I if, for any $(x_1, x_2) \in I$

1. $f_{11}(x_1, x_2) < 0$
2. $f_{11}(x_1, x_2)\, f_{22}(x_1, x_2) > f_{12}(x_1, x_2)^2$

The function f is strictly **convex** on I if, for any $(x_1, x_2) \in I$

1. $f_{11}(x_1, x_2) > 0$
2. $f_{11}(x_1, x_2)\, f_{22}(x_1, x_2) > f_{12}(x_1, x_2)^2$

When is stationarity sufficient?

Fact

If f is differentiable and strictly concave on I, then any stationary point of f is also a unique maximizer of f on I

Fact

If f is differentiable and strictly convex on I, then any stationary point of f is also a unique minimizer of f on I

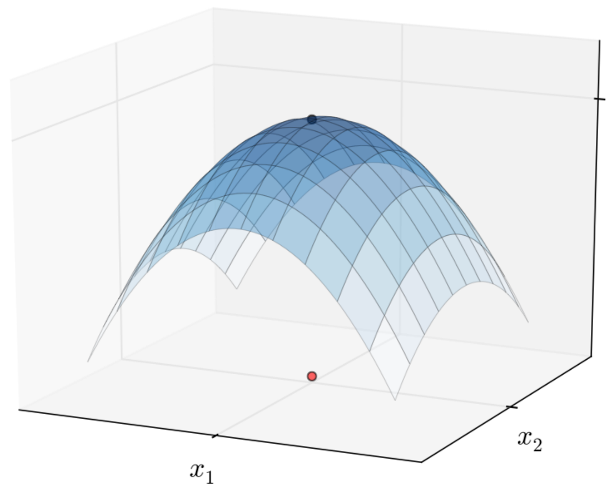


Fig. 2.17: Maximizer of a concave function

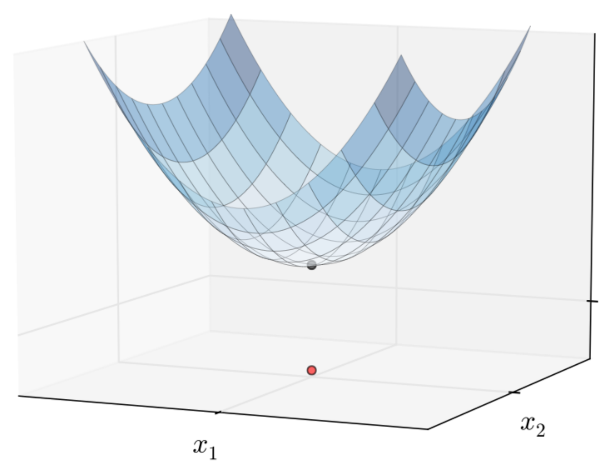


Fig. 2.18: Minimizer of a convex function

Example: unconstrained maximization of quadratic utility

$$u(x_1, x_2) = -(x_1 - b_1)^2 - (x_2 - b_2)^2 \to \max_{x_1, x_2}$$

Intuitively the solution is $x_1^* = b_1$ and $x_2^* = b_2$

Analysis above leads to the same conclusion

First let's check first order conditions (*F.O.C.*)

$$\frac{\partial}{\partial x_1} u(x_1, x_2) = -2(x_1 - b_1) = 0 \quad \implies \quad x_1 = b_1$$
$$\frac{\partial}{\partial x_2} u(x_1, x_2) = -2(x_2 - b_2) = 0 \quad \implies \quad x_2 = b_2$$

How about (strict) concavity?

Sufficient condition is

1. $u_{11}(x_1, x_2) < 0$
2. $u_{11}(x_1, x_2) u_{22}(x_1, x_2) > u_{12}(x_1, x_2)^2$

We have

- $u_{11}(x_1, x_2) = -2$
- $u_{11}(x_1, x_2) u_{22}(x_1, x_2) = 4 > 0 = u_{12}(x_1, x_2)^2$

Example: Profit maximization with two inputs

$$\pi(k, \ell) = p k^\alpha \ell^\beta - w\ell - rk \to \max_{k, \ell}$$

where α, β, p, w are all positive and $\alpha + \beta < 1$

Derivatives:

- $\pi_1(k, \ell) = p\alpha k^{\alpha-1} \ell^\beta - r$
- $\pi_2(k, \ell) = p\beta k^\alpha \ell^{\beta-1} - w$
- $\pi_{11}(k, \ell) = p\alpha(\alpha - 1) k^{\alpha-2} \ell^\beta$
- $\pi_{22}(k, \ell) = p\beta(\beta - 1) k^\alpha \ell^{\beta-2}$
- $\pi_{12}(k, \ell) = p\alpha\beta k^{\alpha-1} \ell^{\beta-1}$

First order conditions: set

$$\pi_1(k, \ell) = 0$$
$$\pi_2(k, \ell) = 0$$

and solve simultaneously for k, ℓ to get

$$k^* = \left[p(\alpha/r)^{1-\beta}(\beta/w)^\beta\right]^{1/(1-\alpha-\beta)}$$
$$\ell^* = \left[p(\beta/w)^{1-\alpha}(\alpha/r)^\alpha\right]^{1/(1-\alpha-\beta)}$$

Exercise: Verify

Now we check second order conditions, hoping for strict concavity

What we need: for any $k, \ell > 0$

1. $\pi_{11}(k, \ell) < 0$
2. $\pi_{11}(k, \ell)\, \pi_{22}(k, \ell) > \pi_{12}(k, \ell)^2$

Exercise: Show both inequalities satisfied when $\alpha + \beta < 1$

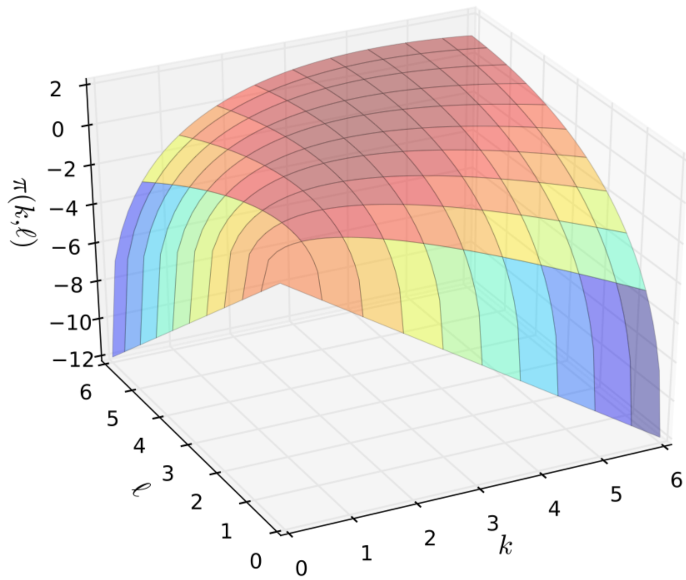


Fig. 2.19: Profit function when $p = 5$, $r = w = 2$, $\alpha = 0.4$, $\beta = 0.5$

2.11 Exercises and materials for self study

Exercise set A

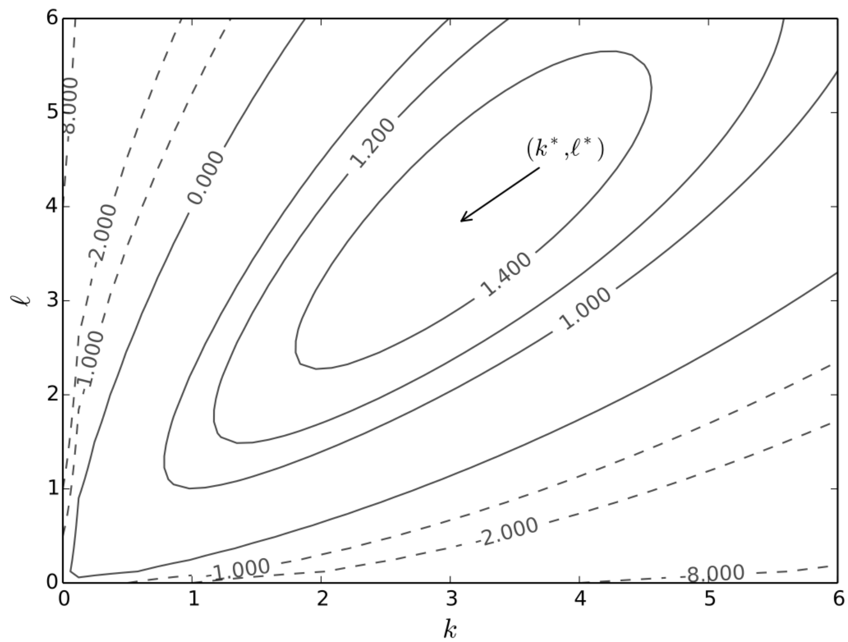


Fig. 2.20: Optimal choice, $p = 5$, $r = w = 2$, $\alpha = 0.4$, $\beta = 0.5$

CHAPTER

THREE

ELEMENTS OF SET THEORY

ECON2125/6012 Lecture 3
Fedor Iskhakov

3.1 Announcements & Reminders

3.2 Plan for this lecture

We now turn to more formal / foundational ideas

1. Logic and proofs
2. Sets, operations with sets
3. Sequences, limits, operations with limits
4. Functions, properties of functions
5. Differentiation, Taylor series
   +
6. Analysis in $\mathbb{R}^n$

Mainly review of key ideas

Supplementary reading:

- Simon & Blume:
- Sundaram:

Common symbols

- $P \implies Q$ means "P implies Q"
- $P \iff Q$ means "$P \implies Q$ and $Q \implies P$"
- $\exists$ means "there exists"
- $\forall$ means "for all"
- s.t. means "such that"
- $\because$ means "because" (not used very often)
- $\therefore$ means "therefore" (not used very often)

- $a := 1$ means "a is defined to be equal to 1" (alternatively $a \equiv 1$ or $a \stackrel{def.}{=} 1$)
- $\mathbb{R}$ means all real numbers
- $\mathbb{N}$ means the natural numbers $\{1, 2, \ldots\}$
- $\mathbb{Z}$ means integers $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{Q}$ means the rational numbers (ratios of two integers)

3.3 Logic

Let P and Q be statements, such as

- x is a negative integer
- x is an odd number
- the area of any circle in the plane is $-2\pi R$

Law of the excluded middle: Every mathematical statement is either `true` or `false`

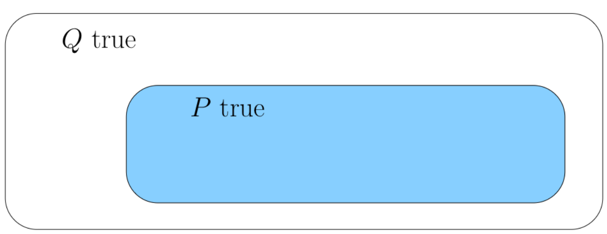
Statement "$P \implies Q$" means "P implies Q"

Example

k is even $\implies k = 2n$ for some integer n

Equivalent forms of $P \implies Q$:

1. If P is true then Q is true
2. P is a *sufficient condition* for Q
3. Q is a necessary condition for P
4. If Q fails then P fails

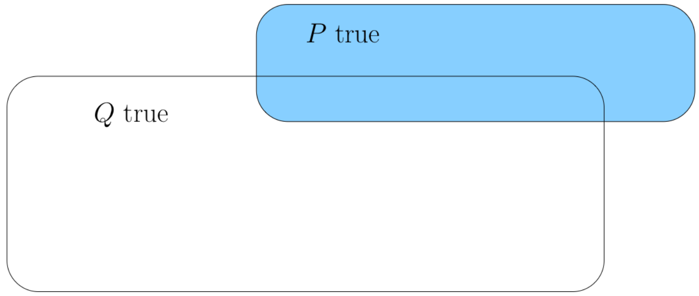


Equivalent ways of saying $P \not\Rightarrow Q$ (*does not imply*):

1. P does not imply Q
2. P is not sufficient for Q
3. Q is not necessary for P
4. Even if Q fails, P can still hold

Example

Let



- $P :=$ "$n \in \mathbb{N}$ and even"
- $Q :=$ "n even"

Then

1. $P \implies Q$
2. P is sufficient for Q
3. Q is necessary for P
4. If Q fails then P fails

Example

Let

- $P :=$ "R is a rectangle"
- $Q :=$ "R is a square"

Then

1. $P \not\Rightarrow Q$
2. P is not sufficient for Q
3. Q is not necessary for P
4. Just because Q fails does not mean that P fails

3.4 Proof by contradiction

Suppose we wish to prove a statement such as $P \implies Q$

1. A proof by contradiction starts by **assuming the opposite**: P holds and yet Q fails.
2. We then show that this scenario leads to a contradiction

Examples of contradictions

- $1 < 0$
- 10 is an odd number

We then conclude that $P \implies Q$ is valid after all.

Example: proof by contradiction

Suppose that island X is populated only by pirates and knights:

- pirates always lie
- knights always tell the truth

Claim to prove: If person Y says `"I'm a pirate"` then person Y is ***not*** a native of island X

Strategy for the **Proof:**

1. Suppose person Y is a native of the island
2. Show that this leads to a contradiction
3. Conclude that Y is not a native of island X, as claimed

Proof: Suppose to the contrary that person Y ***is*** a native of island X

- then Y is either a pirate or a knight

1. Suppose first that Y is knight

- Y is a knight who claims to be a pirate
- This is impossible, since knights always tell the truth

2. Suppose next that Y is pirate

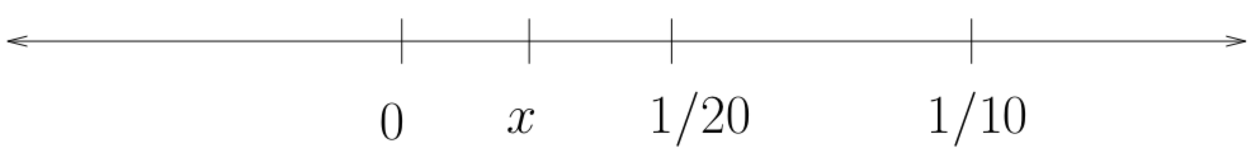
- Y is a pirate who claims to be a pirate
- Since pirates always lie, they would not make such a statement

Either way we get a contradiction $\implies$ Y is not a native of the island!

Example

There is ***no*** $x \in \mathbb{R}$ such that $0 < x < 1/n$, $\forall n \in \mathbb{N}$.

Proof: Suppose to the contrary that such an x exists



Since $x > 0$ the number $1/x$ is finite

Let k be the smallest integer such that $k \geq 1/x$

- if $x = 0.3$ then $1/x = 3.333 \cdots$, so set $k = 4 \in \mathbb{N}$
- if $x = 0.02$ then $1/x = 50 \cdots$, so set $k = 50 \in \mathbb{N}$

Since $k \geq 1/x$ we also have $1/k \leq x$

On the other hand, since $k \in \mathbb{N}$, we have $x < 1/k$

But then $1/k \leq x < 1/k$, and in particular $1/k < 1/k$, which is impossible — a contradiction!

Example

Let $n \in \mathbb{N}$. Show that n^2 odd $\implies$ n odd

Proof: Suppose to the contrary that is:

1. $n \in \mathbb{N}$ and n^2 is odd
2. but n is even

Then $n = 2k$ for some $k \in \mathbb{N}$

Hence $n^2 = (2k)^2$

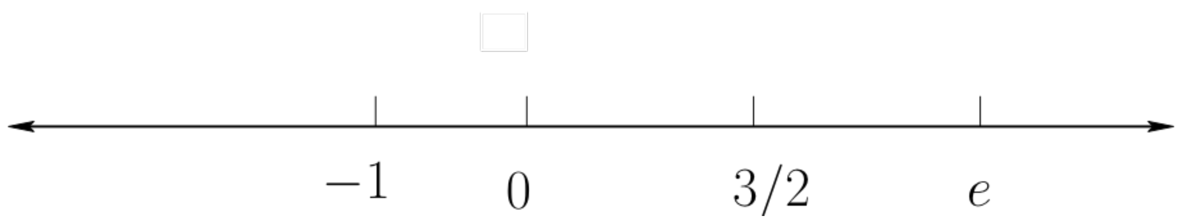
But then $n^2 = 2m$ for $m := 2k^2 \in \mathbb{N}$, and thus n^2 is even!

Contradiction

3.5 Sets

Will often refer to the ***real numbers***, $\mathbb{R}$

Understand it to contain "all of the numbers" on the "real line"



Contains both the rational and the irrational numbers

$\mathbb{R}$ is an example of a ***set***

A set is a collection of objects viewed as a whole

(In case of $\mathbb{R}$, the objects are numbers)

Other examples of sets:

- set of all rectangles in the plane
- set of all prime numbers
- set of students in the class

Notation:

- Sets: A, B, C
- Elements: x, y, z

Important sets:

- $\mathbb{N} := \{1, 2, 3, ...\}$
- $\mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$
- $\mathbb{Q} := \{p/q : p, q \in \mathbb{Z},\ q \neq 0\}$
- $\mathbb{R} := \mathbb{Q} \cup \{$ irrationals $\}$

Definition of a set

A set A can be defined by either

- direct enumeration of its elements
- defining a formula for infinite number of elements
- as a *subset* of already defined set B and known function $\psi(x)$

$$A = \{\psi(x), x \in B \colon \text{condition on x}\}$$

3.6 Intervals of $\mathbb{R}$

Common notation:

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}$$

$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$$

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$

$$[a, \infty) := \{x \in \mathbb{R} : a \leq x\}$$

$$(-\infty, b) := \{x \in \mathbb{R} : x < b\}$$

Etc.

3.7 Operations with sets

Let A and B be any sets

Statement $x \in A$ means that x is an element of A

$A \subset B$ means that any element of A is also an element of B

Example

- $\mathbb{N} \subset \mathbb{Z}$
- irrational numbers are a subset of $\mathbb{R}$

Equality of A and B

Let S be any set and A and B be subsets of S

$A = B$ means that A and B contain the same elements

Equivalently, $A = B \iff A \subset B$ and $B \subset A$

Union of A and B

$$A \cup B := \{x \in S : x \in A \text{ or } x \in B\}$$

Intersection of A and B

$$A \cap B := \{x \in S : x \in A \text{ and } x \in B\}$$

Set theoretic difference of A and B

$$A \quad B := \{x \in S : x \in A \text{ and } x \notin B\}$$

In other words, all points in A that are not points in B

Example

- $\mathbb{Z} \quad \mathbb{N} = \{\dots, -2, -1, 0\}$
- $\mathbb{R} \quad \mathbb{Q} =$ the set of irrational numbers
- $\mathbb{R} \quad [0, \infty) = (-\infty, 0)$
- $\mathbb{R} \quad (a, b) = (-\infty, a] \cup [b, \infty)$

Complement of A

All elements of S that are not in A:

$$A^c := S \quad A =: \{x \in S : x \notin A\}$$

Remarks:

- Need to know what S is before we can determine A^c
- If not clear better write $S \quad A$

Example

$(a, \infty)^c$ generally understood to be $(-\infty, a]$

3.8 Set operations properties

If A and B subsets of S, then

1. $A \cup B = B \cup A$ and $A \cap B = B \cap A$

- $(A \cup B)^c = B^c \cap A^c$ and $(A \cap B)^c = B^c \cup A^c$
- $A \quad B = A \cap B^c$

9. $(A^c)^c = A$

The ***empty set*** $\emptyset$ is the set containing no elements

If $A \cap B = \emptyset$, then A and B said to be ***disjoint***

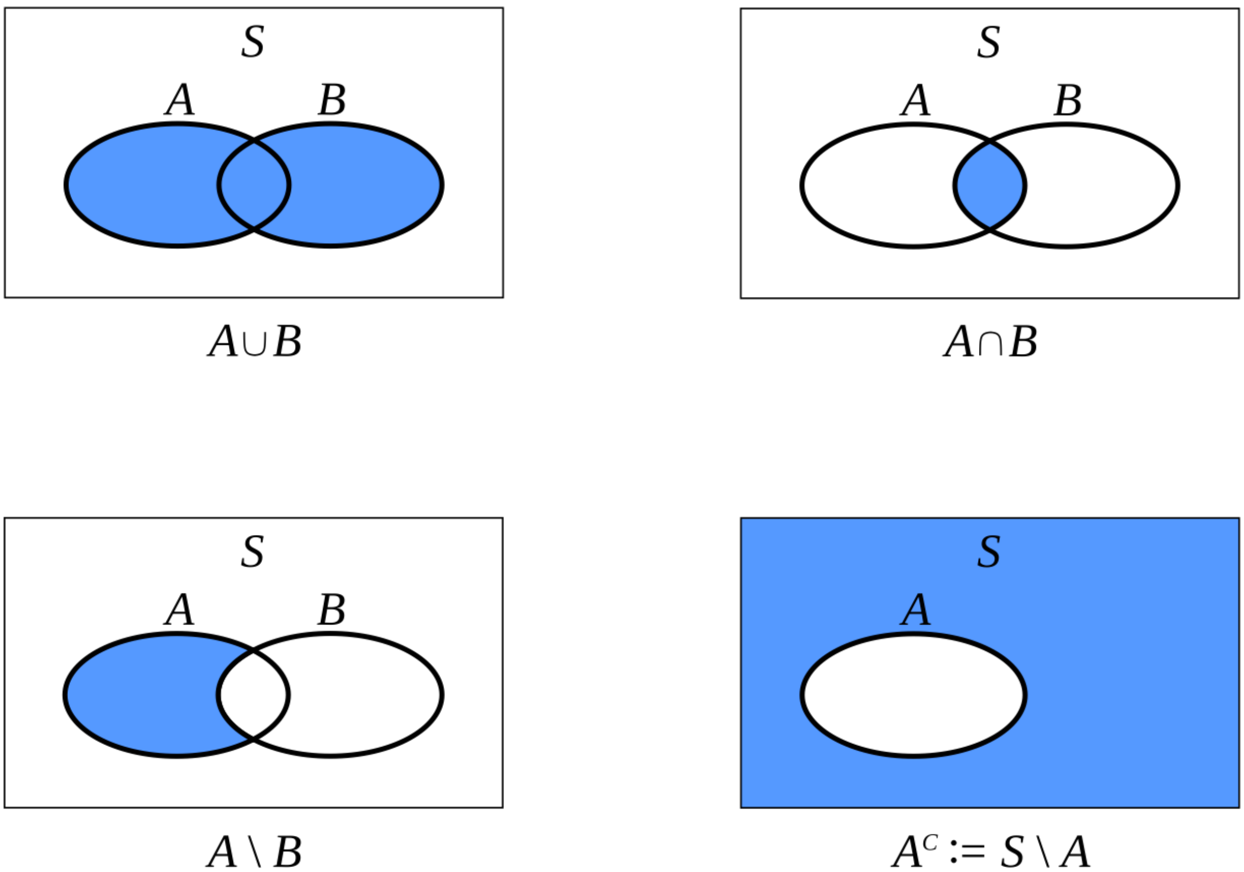


Fig. 3.1: \label{f:allsets} Unions, intersections and complements

3.9 Infinite Unions and Intersections

Given a family of sets $K_\lambda \subset S$ with $\lambda \in \Lambda$,

$$\bigcap_{\lambda \in \Lambda} K_\lambda := \{x \in S : x \in K_\lambda \text{ for all } \lambda \in \Lambda\}$$

$$\bigcup_{\lambda \in \Lambda} K_\lambda := \{x \in S \colon \text{there exists an } \lambda \in \Lambda \text{ such that } x \in K_\lambda\}$$

- "there exists" means "there exists ***at least*** one"

Example

Let $A := \cap_{n \in \mathbb{N}} (0, 1/n)$

Claim: $A = \emptyset$

Proof: We need to show that A contains no elements

Suppose to the contrary that $x \in A = \cap_{n \in \mathbb{N}} (0, 1/n)$

Then x is a number satisfying $0 < x < 1/n$ for all $n \in \mathbb{N}$

No such x exists as we showed above. Contradiction.

Example

For any $a < b$ we have $\cup_{\epsilon > 0} \, [a + \epsilon, b) = (a, b)$

Proof: To show equality of the sets, we show that RHS $\subset$ LHS and LHS $\subset$ RHS

Pick any $a < b$

Suppose first that $x \in \cup_{\epsilon > 0} \, [a + \epsilon, b)$

This means there exists $\epsilon > 0$ such that $a + \epsilon \leq x < b$

Clearly $a < x < b$, and hence $x \in (a, b)$

Conversely, if $a < x < b$, then $\exists \, \epsilon > 0$ s.t. $a + \epsilon \leq x < b$

Hence $x \in \cup_{\epsilon > 0} \, [a + \epsilon, b)$

Fact: de Morgan's laws

Let S be any set and let $K_\lambda \subset S$ for all $\lambda \in \Lambda$. Then

$$\left[\bigcup_{\lambda \in \Lambda} K_\lambda \right]^c = \bigcap_{\lambda \in \Lambda} K_\lambda^c \quad \text{and} \quad \left[\bigcap_{\lambda \in \Lambda} K_\lambda \right]^c = \bigcup_{\lambda \in \Lambda} K_\lambda^c$$

Let's prove that $A := (\cup_{\lambda \in \Lambda} K_\lambda)^c = \cap_{\lambda \in \Lambda} K_\lambda^c =: B$

Suffices to show that $A \subset B$ and $B \subset A$

Let's just do $A \subset B$

Must show that every $x \in A$ is also in B

Fix $x \in A$

Since $x \in A$, it must be that x is not in $\cup_{\lambda \in \Lambda} K_\lambda$

$$\text{therefore } \ x \text{ is not in any } K_\lambda$$

$$\text{therefore } x \in K_\lambda^c \text{ for each } \lambda \in \Lambda$$

$$\text{therefore } x \in \cap_{\lambda \in \Lambda} K_\lambda^c =: B$$

3.10 Tuples

We often organize collections with natural order into "tuples"

Definition

A ***tuple*** is

- a finite ordered sequence of terms
- denoted using notation such as (a_1, a_2) or (x_1, x_2, x_3)

Example

Flip a coin 10 times and let

- 0 represent tails and 1 represent heads

Typical outcome $(1, 1, 0, 0, 0, 0, 1, 0, 1, 1)$

Generic outcome $(b_1, b_2, \dots, b_{10})$ for $b_n \in \{0, 1\}$

3.11 Cartesian Products

We make collections of tuples using Cartesian products

Definition

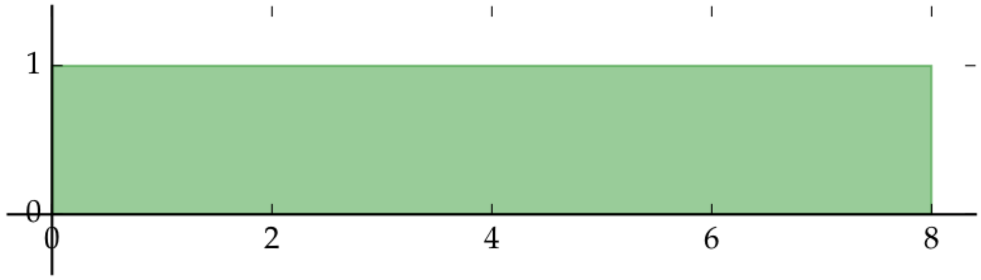
The ***Cartesian product*** of $A_1, \dots, A_N$ is the set

$$A_1 \times \cdots \times A_N := \{(a_1, \dots, a_N) : a_n \in A_n \text{ for } n = 1, \dots, N\}$$

Example

$$\text{```}[0, 8] \times [0, 1] = \{(x_1, x_2) : 0 \leq x_1 \leq 8, \, 0 \leq x_2 \leq 1\}$$

Example



Set of all outcomes from flip experiment is

$$B := \left\{(b_1, \dots, b_{10}) : b_n \in \{0, 1\} \text{ for } n = 1, \dots, 10\right\}$$

$$= \{0, 1\} \times \cdots \times \{0, 1\} \quad (10 \text{ products})$$

Example

The ***vector space*** $\mathbb{R}^N$ is the Cartesian product

$$\mathbb{R}^N = \mathbb{R} \times \cdots \times \mathbb{R} \quad (N \text{ times})$$

$$= \{ \text{all tuples } (x_1, \dots, x_N) \text{ with } x_n \in \mathbb{R}\}$$

3.12 Counting Finite Sequences

Counting methods answer common questions such as

- How many arrangements of a sequence?
- How many subsets of a set?

They also address deeper problems such as

- How "large" is a given set?
- Can we compare size of sets even when they are infinite?

The key rule is: multiply possibilities

Example

Can travel from Sydney to Tokyo in 3 ways and Tokyo to NYC in 8 ways $\implies$ can travel from Sydney to NYC in 24 ways

Example

Number of 10 letter passwords from the lowercase letters `a, b, ..., z` is $\$26^{10} = 141,167,095,653,376\$$

Example

Number of possible distinct outcomes (i, j) from 2 rolls of a dice is $\$6 \times 6 = 36\$$

3.13 Counting Cartesian Products

Fact

If A_n are finite for $n = 1, \ldots, N$, then

$$\#(A_1 \times \cdots \times A_N) = (\# A_1) \times \cdots \times (\# A_N)$$

That is, number of possible tuples = product of the number of possibilities for each element

Example

Number of binary sequences of length 10 is $\$\#[\{0,1\} \times \cdots \times \{0,1\}] = 2 \times \cdots \times 2 = 2^{10}\$$

Infinite Cartesian Products

If $\{A_n\}$ is a collection of sets, one for each $n \in \mathbb{N}$, then

$$A_1 \times A_2 \times \cdots := \{(a_1, a_2, \ldots) : a_n \in A_n \text{ for each } n \in \mathbb{N}\}$$

Sometimes denoted $\times_{n=1}^{\infty} A_n$

If $A_n = A$ for all n, then $\times_{n=1}^{\infty} A$ also written as $A^{\mathbb{N}}$

Example

The set of all binary sequences $\{0,1\}^{\mathbb{N}}$

3.14 Functions

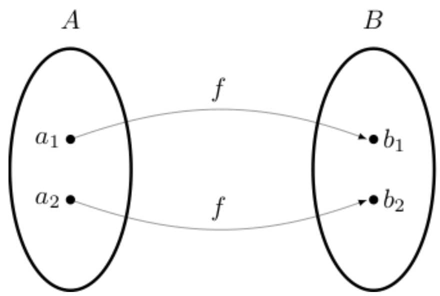
Definition

A ***function*** $f : A \to B$ from set A to set B is a rule that associates to each element of A a uniquely determined element of B

- $f : A \to B$ means that f is a function from A to B

A is called the ***domain*** of f and B is called the ***codomain***

Example



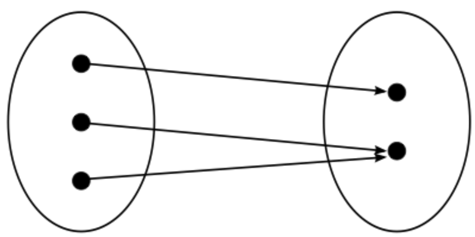
f defined by

$$f(x) = \exp(-x^2)$$

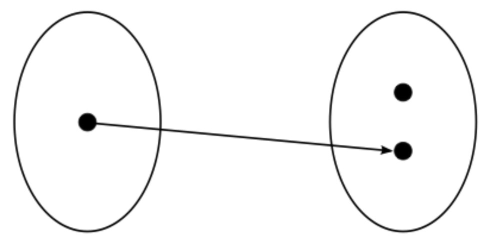
is a function from $\mathbb{R}$ to $\mathbb{R}$

Sometimes we write the whole thing like this

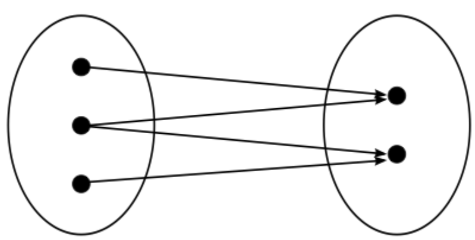
$$\begin{aligned} f \colon \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \exp(-x^2), \text{ or} \\ f \colon \mathbb{R} \ni x &\mapsto \exp(-x^2) \in \mathbb{R} \end{aligned}$$



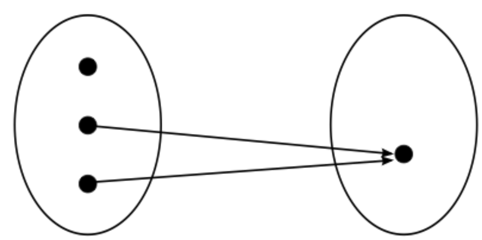
A function



A function

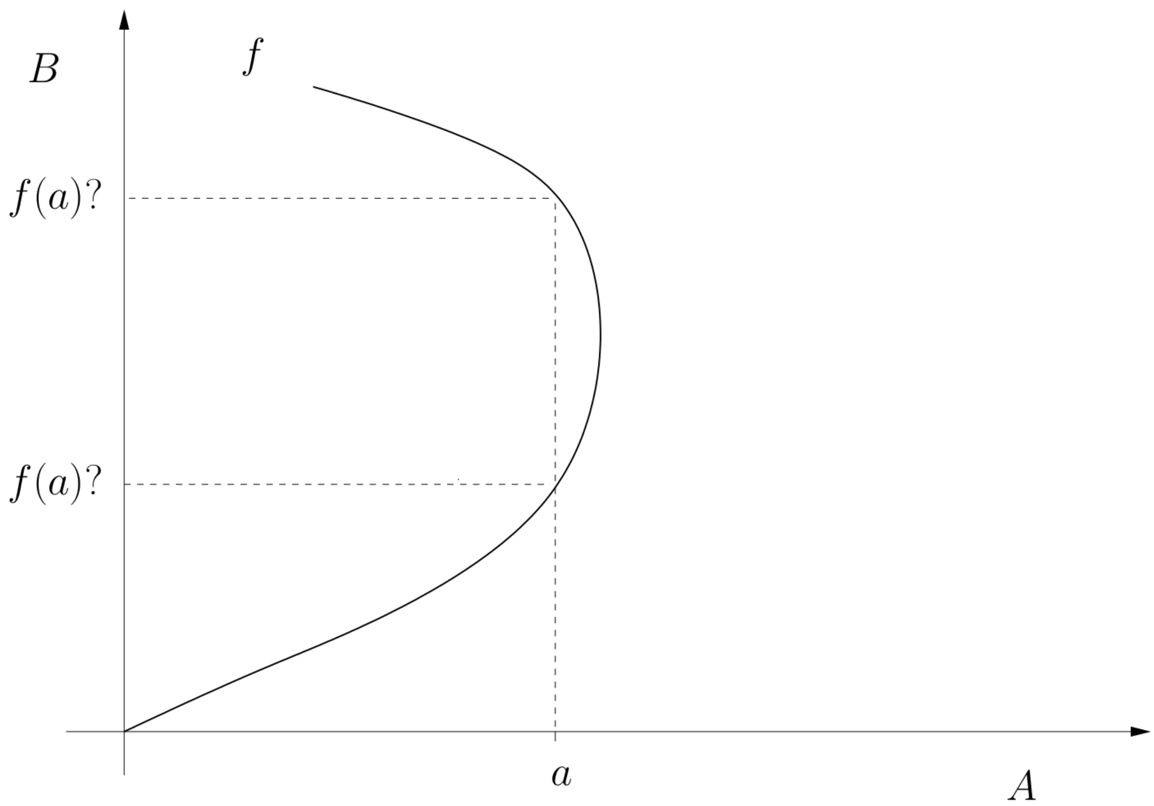


Not a function



Not a function

Example: not a function



For each $a \in A$, $f(a) \in B$ is called the ***image of*** a under f

If $f(a) = b$ then a is called a ***preimage of*** b under f

A point in B can have one, many or zero preimages

The codomain of a function is not uniquely pinned down

Example

Consider the mapping defined by \$$f(x) = \exp(-x^2)$\$

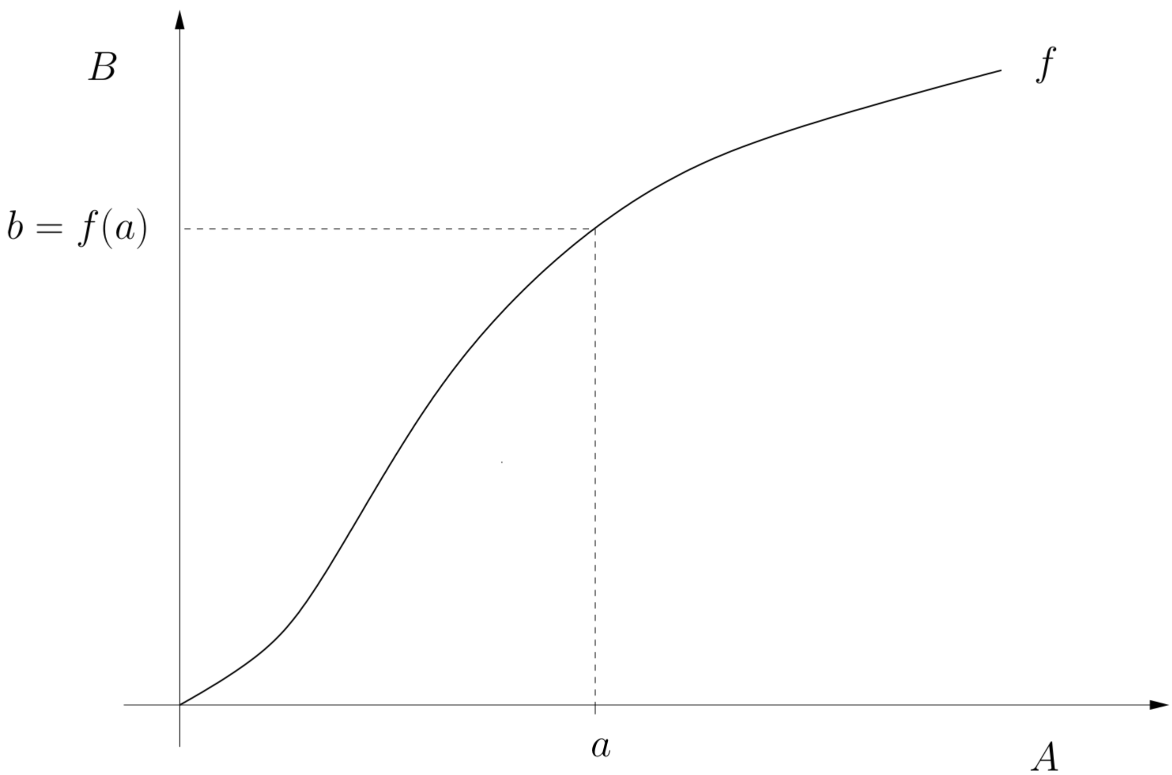
Both of these statements are valid:

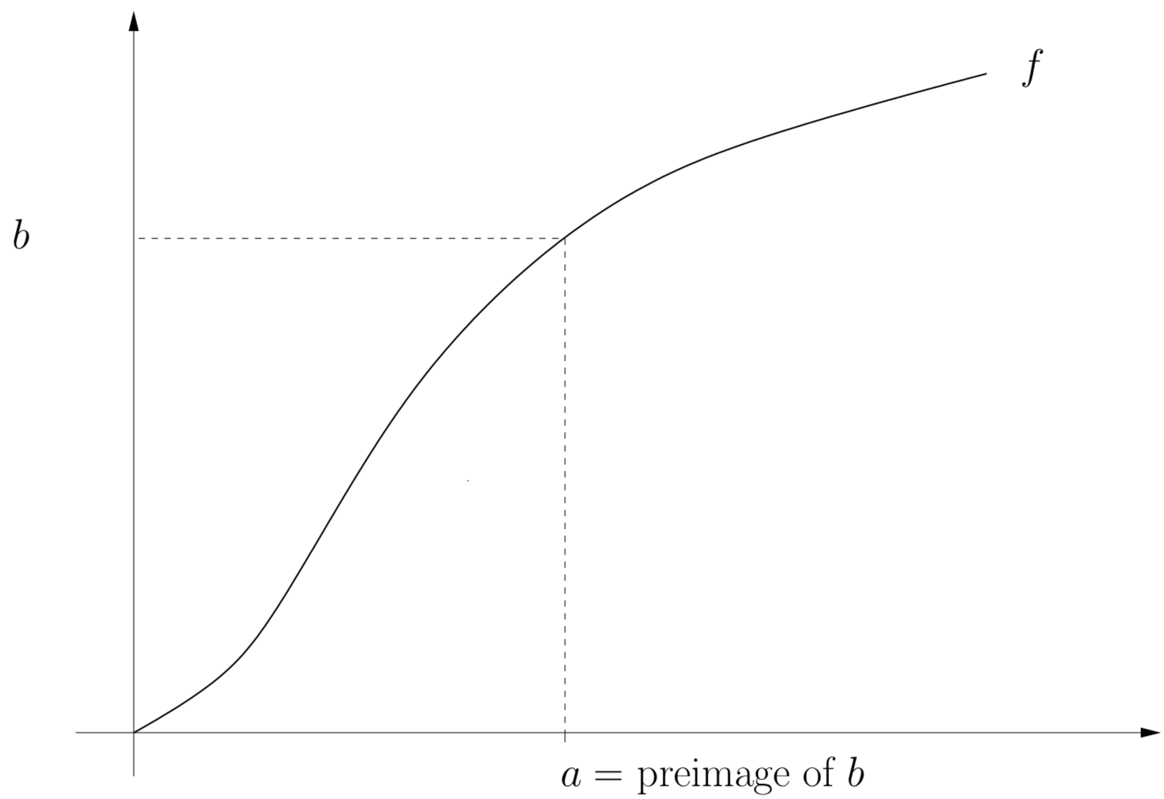
- f a function from $\mathbb{R}$ to $\mathbb{R}$
- f a function from $\mathbb{R}$ to $(0, \infty)$

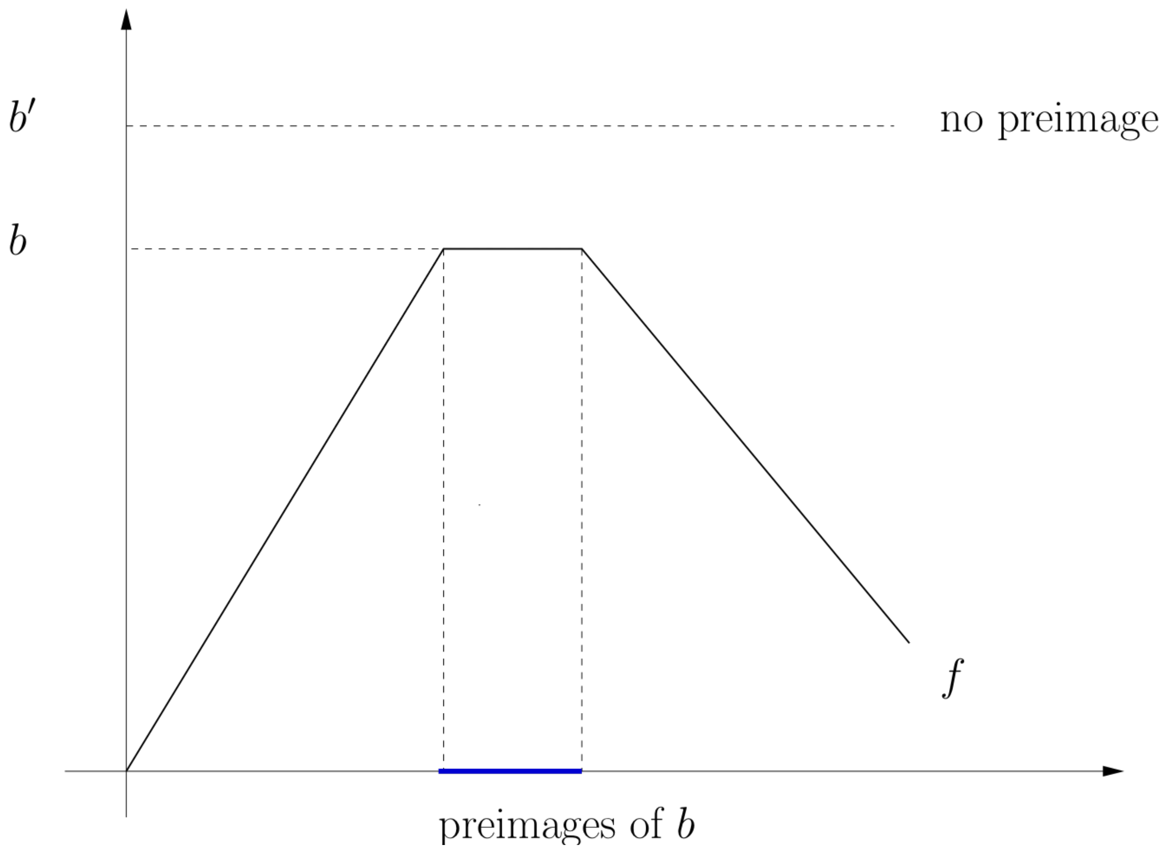
The smallest possible codomain is called the ***range*** of $f \colon A \to B$:

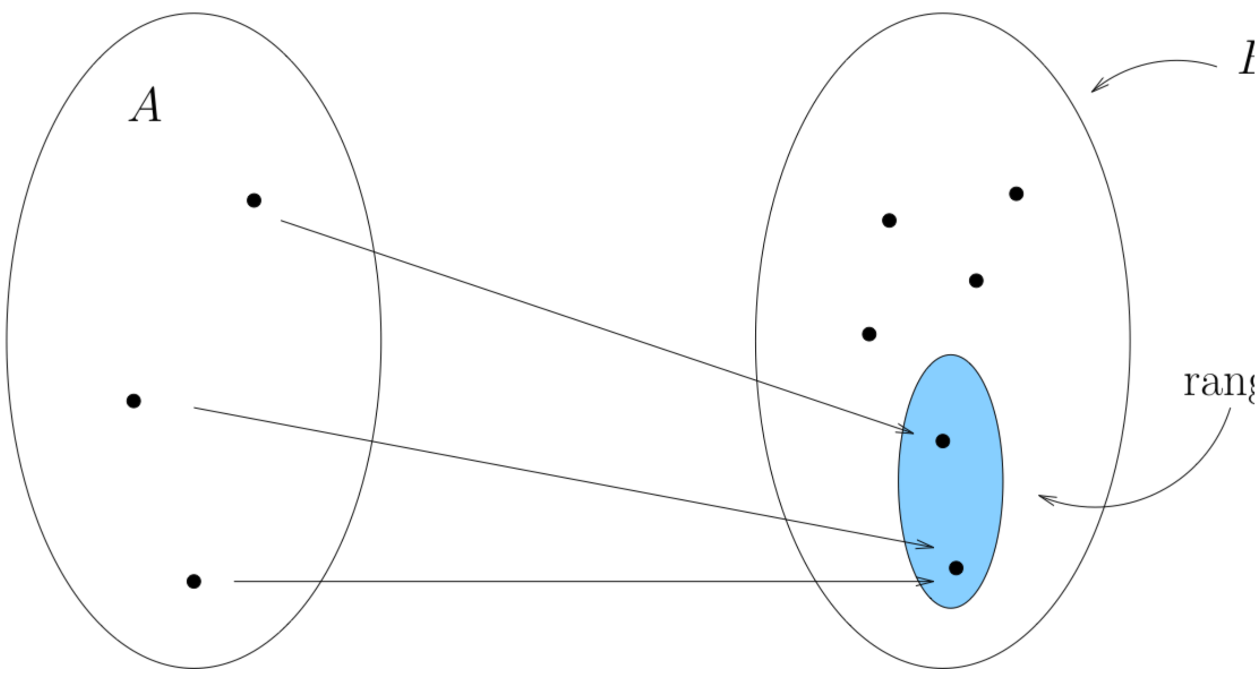
$$\text{rng}(f) := \{b \in B : f(a) = b \text{ for some } a \in A\}$$

Example

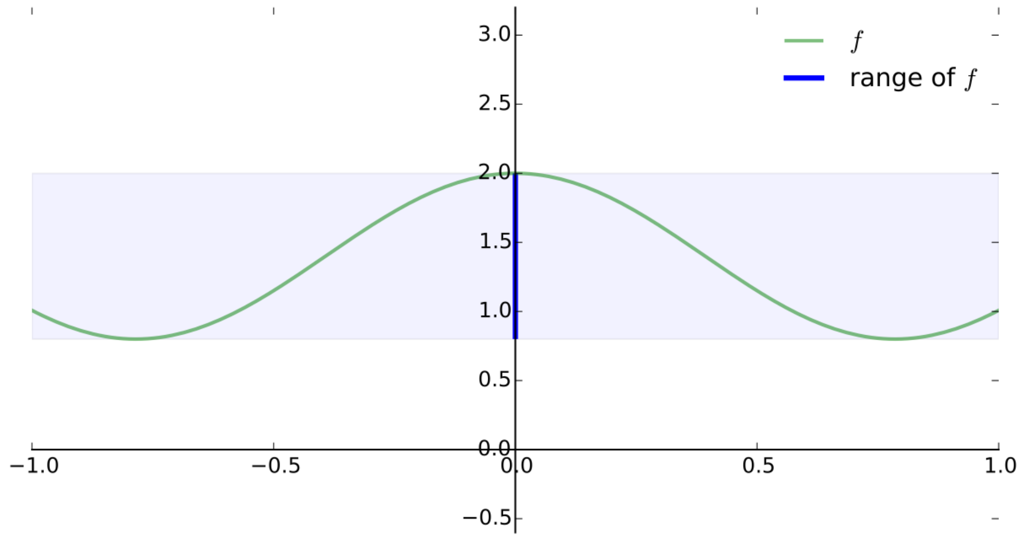

B
f
$b = f(a)$
a
A


f
b
$a =$ preimage of b


b'
no preimage
b
f
preimages of b


A

Let $f \colon [-1, 1] \to \mathbb{R}$ be defined by \$$f(x) = 0.6\cos(4x) + 1.4 Then$\mathrm{rng}(f) = [0.8, 2.0]\$



Example

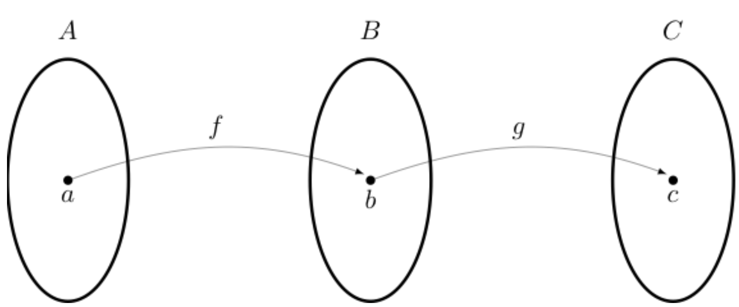
If $f \colon [0, 1] \to \mathbb{R}$ is defined by \$$f(x) = 2x then$\mathrm{rng}(f) = [0, 2]\$

Example

If $f \colon \mathbb{R} \to \mathbb{R}$ is defined by \$$f(x) = \exp(x) then$\mathrm{rng}(f) = (0, \infty)\$

The ***composition*** of $f \colon A \to B$ and $g \colon B \to C$ is the function $g \circ f$ from A to C defined by

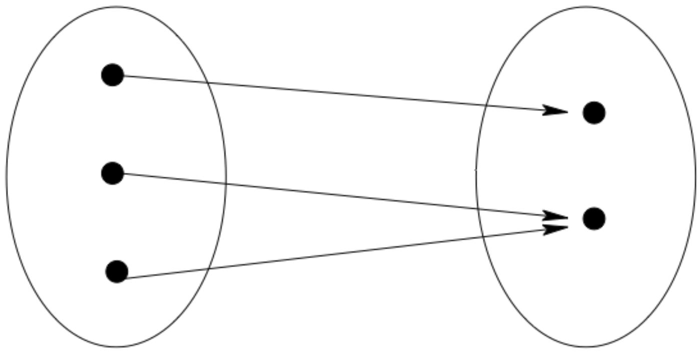
$$(g \circ f)(a) = g(f(a)) \quad (a \in A)$$



3.15 Onto Functions

A function $f\colon A \to B$ is called ***onto*** if every element of B is the image under f of at least one point in A.

Equivalently, $\mathrm{rng}(f) = B$



Fact

$f\colon A \to B$ is onto if and only if each element of B has at least one preimage under f

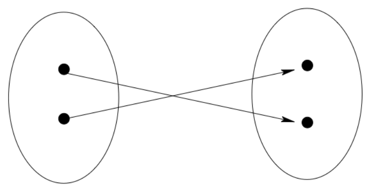


Fig. 3.2: Onto

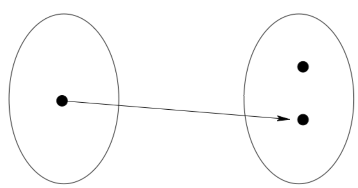


Fig. 3.3: Not *onto*!

Example

The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = ax^3 + bx^2 + cx + d$$

is onto whenever $a \neq 0$

To see this pick any $y \in \mathbb{R}$

We claim $\exists\, x \in \mathbb{R}$ such that $f(x) = y$

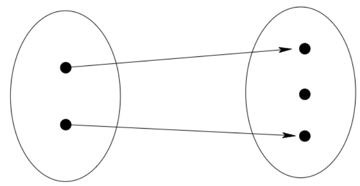


Fig. 3.4: Not *onto*!

Equivalent:

$$\exists\, x \in \mathbb{R} \text{ s.t. } ax^3 + bx^2 + cx + d - y = 0$$

Fact

Every cubic equation with $a \neq 0$ has at least one real root

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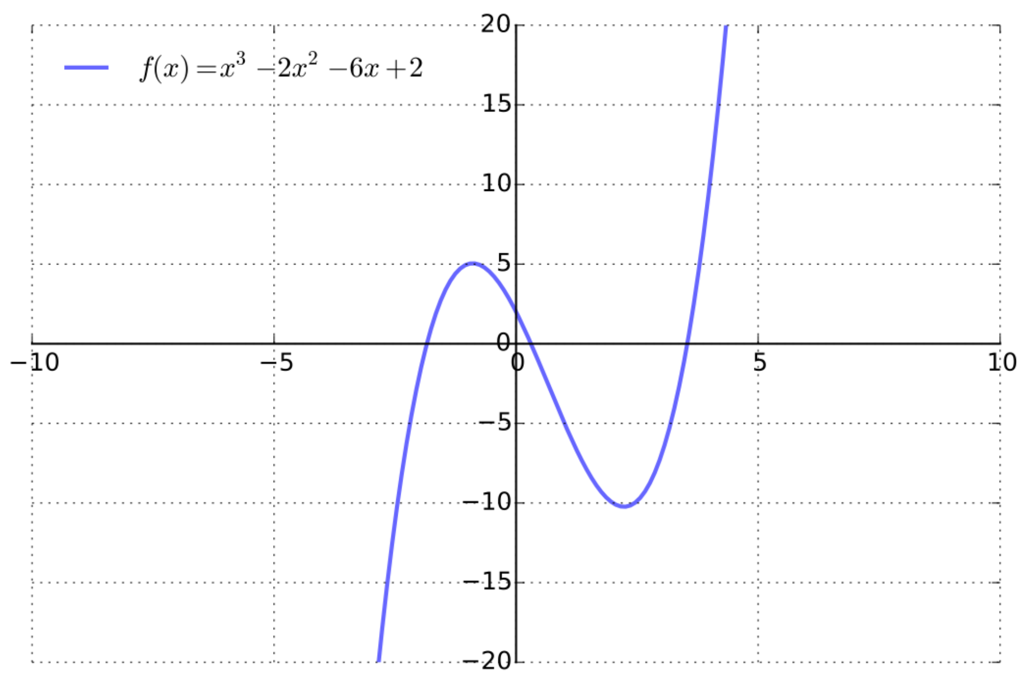


Fig. 3.5: Cubic functions from $\mathbb{R}$ to $\mathbb{R}$ are always onto

3.16 One-to-One Functions

A function $f\colon A \to B$ is called ***one-to-one*** if distinct elements of A are always mapped into distinct elements of B. That is, f is one-to-one if

$$a \in A,\ a' \in A \text{ and } a \neq a' \implies f(a) \neq f(a')$$

Equivalently,

$$f(a) = f(a') \implies a = a'$$

Fact

$f\colon A \to B$ is one-to-one if and only if each element of B has at most one preimage under f

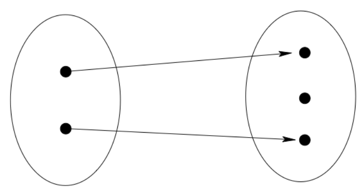


Fig. 3.6: One-to-one

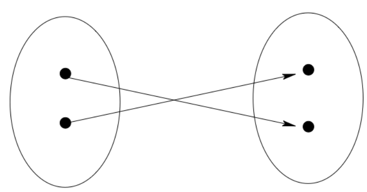


Fig. 3.7: One-to-one

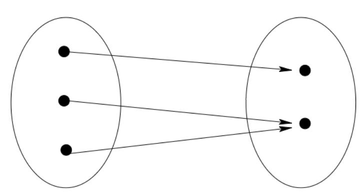
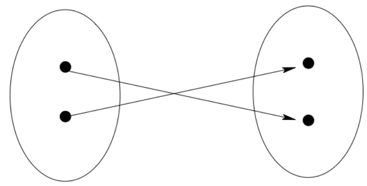


Fig. 3.8: Not one-to-one

3.17 Bijections

A function that is

1. one-to-one and
2. onto



is called a ***bijection*** or ***one-to-one correspondence***

Fact

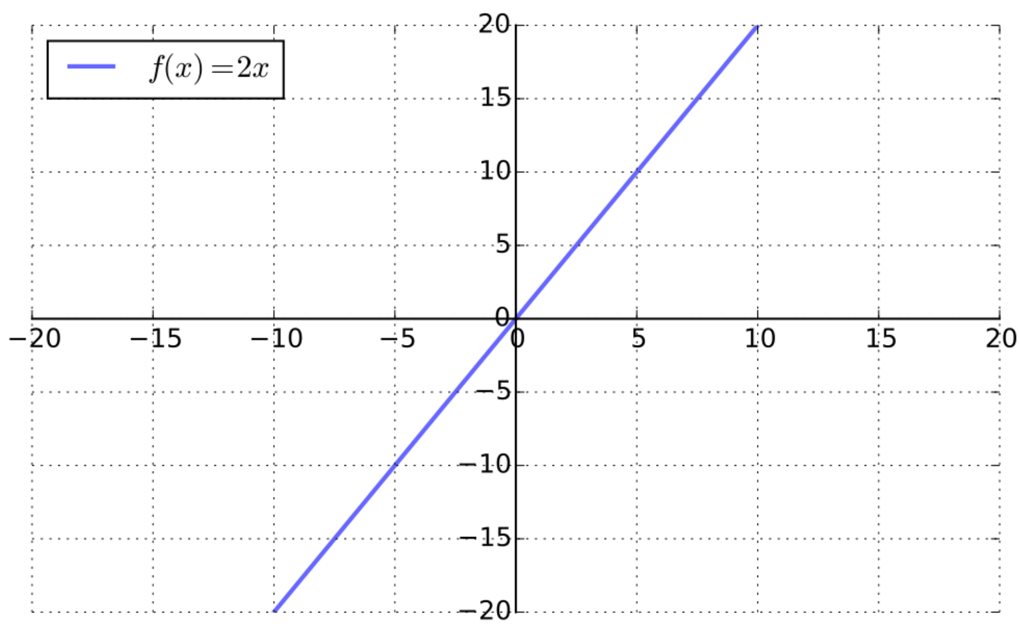
$f\colon A \to B$ is a bijection if and only if each $b \in B$ has

---

one and only one preimage in A

Example

$x \mapsto 2x$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$

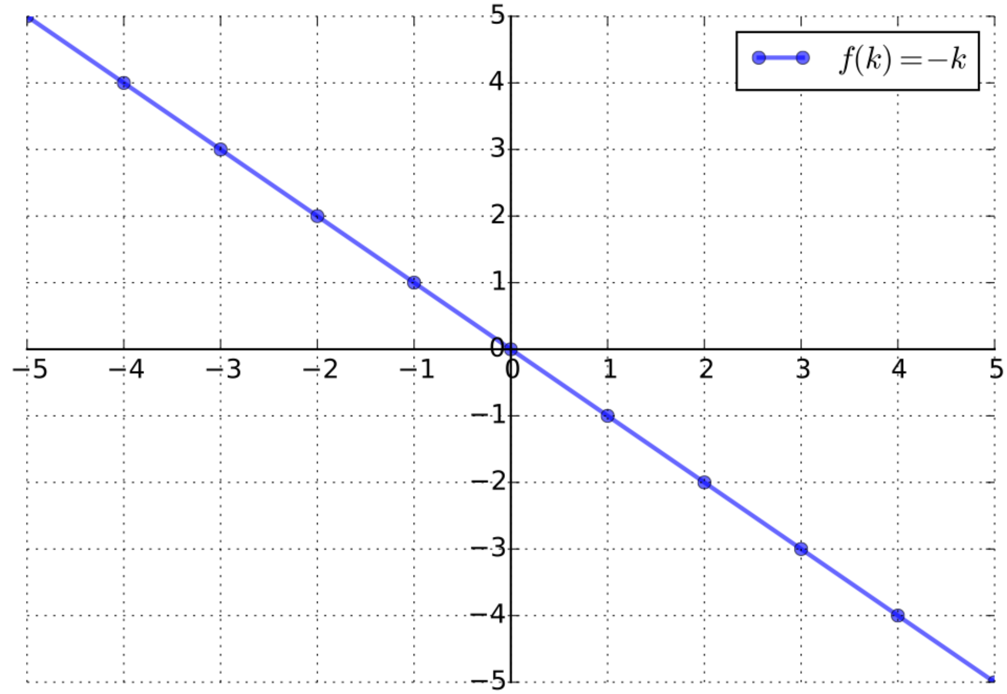


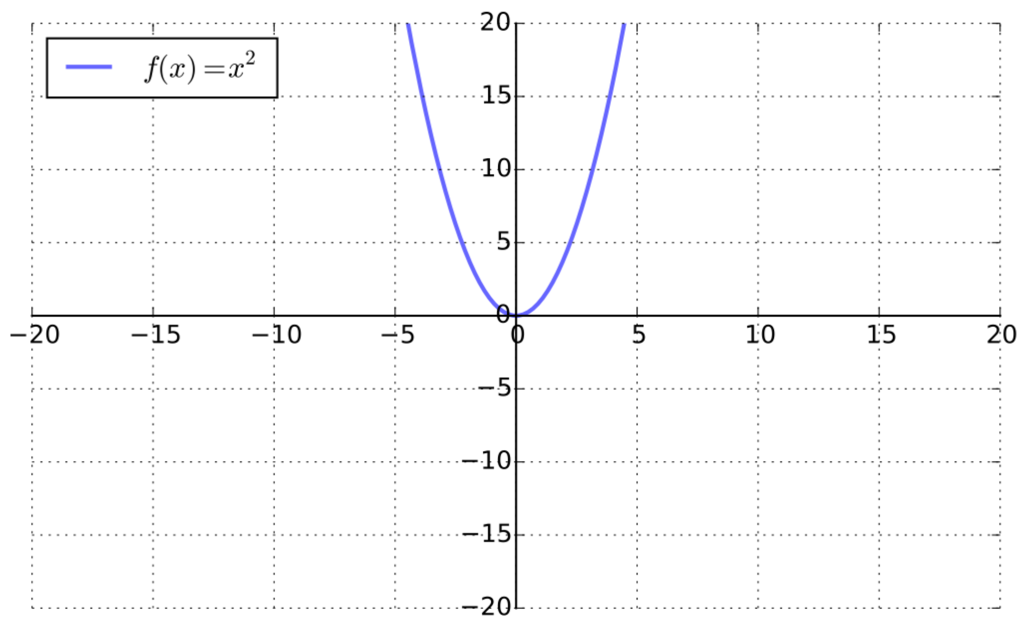
Example

$k \mapsto -k$ is a bijection from $\mathbb{Z}$ to $\mathbb{Z}$

Example

$x \mapsto x^2$ is ***not*** a bijection from $\mathbb{R}$ to $\mathbb{R}$


$f(k) = -k$
−5 −4 −3 −2 −1 0 1 2 3 4 5
5 4 3 2 1 0 −1 −2 −3 −4 −5

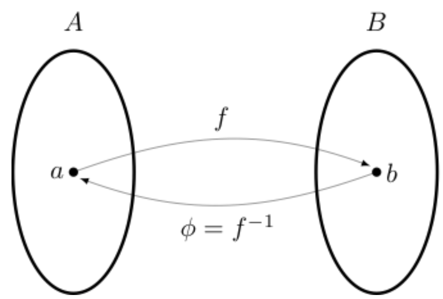

$f(x) = x^2$
−20 −15 −10 −5 0 5 10 15 20
20 15 10 5 0 −5 −10 −15 −20

Fact

If $f: A \to B$ a bijection, then there exists a unique function $\phi: B \to A$ such that

$$\phi(f(a)) = a, \quad \forall\, a \in A$$

That function ϕ is called the ***inverse*** of f and written f^{-1}



Example

Let

- $f: \mathbb{R} \to (0, \infty)$ be defined by $f(x) = \exp(x) := e^x$
- $\phi: (0, \infty) \to \mathbb{R}$ be defined by $\phi(x) = \log(x)$

Then $\phi = f^{-1}$ because, for any $a \in \mathbb{R}$,

$$\phi(f(a)) = \log(\exp(a)) = a$$

Fact

If $f: A \to B$ is one-to-one, then $f: A \to \operatorname{rng}(f)$ is a bijection

Fact

Let $f: A \to B$ and $g: B \to C$ be bijections

1. f^{-1} is a bijection and its inverse is f

- $f^{-1}(f(a)) = a$ for all $a \in A$
- $f(f^{-1}(b)) = b$ for all $b \in B$
- $g \circ f$ is a bijection from A to C and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ \end{enumerate}

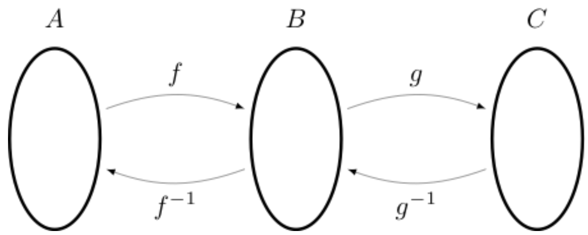


Fig. 3.9: Illustration of $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

3.18 Cardinality

If a bijection exists between sets A and B they are said to have the ***same cardinality***, and we write $|A| = |B|$

Fact

If $|A| = |B|$ and A and B are finite then A and B have the same number of elements (same cardinality).

Exercise: Convince yourself this is true

Hence "same cardinality" is analogous to "same size"

- But cardinality applies to infinite sets as well!

Fact

If $|A| = |B|$ and $|B| = |C|$ then $|A| = |C|$

Proof:

- Since $|A| = |B|$, there exists a bijection $f \colon A \to B$
- Since $|B| = |C|$, there exists a bijection $g \colon B \to C$

Let $h := g \circ f$

Then h is a bijection from A to C

Hence $|A| = |C|$

A nonempty set A is called ***finite*** if \$$|A| = |\{1, 2, \ldots, n\}|$ for some $n \in \mathbb{N}$\$

Otherwise called ***infinite***

Sets that either

1. are finite, or
2. have the same cardinality as $\mathbb{N}$

are called ***countable***, denoted $|A| = \aleph_0$

Example

$-\mathbb{N} := \{\ldots, -4, -3, -2, -1\}$ is countable

$$\begin{array}{ccc}
-1 & \leftrightarrow & 1 \\
-2 & \leftrightarrow & 2 \\
-3 & \leftrightarrow & 3 \\
 & \vdots & \\
-n & \leftrightarrow & n \\
 & \vdots &
\end{array}$$

Formally: $f(k) = -k$ is a bijection from $-\mathbb{N}$ to $\mathbb{N}$

Example

$E := \{2, 4, ...\}$ is countable

$$\begin{array}{ccc}
2 & \leftrightarrow & 1 \\
4 & \leftrightarrow & 2 \\
6 & \leftrightarrow & 3 \\
 & \vdots & \\
2n & \leftrightarrow & n \\
 & \vdots &
\end{array}$$

Formally: $f(k) = k/2$ is a bijection from E to $\mathbb{N}$

Example

$\{100, 200, 300, ...\}$ is countable

$$\begin{array}{ccc}
100 & \leftrightarrow & 1 \\
200 & \leftrightarrow & 2 \\
300 & \leftrightarrow & 3 \\
 & \vdots & \\
100n & \leftrightarrow & n \\
 & \vdots &
\end{array}$$

Fact

Nonempty subsets of countable sets are countable

Fact

Finite unions of countable sets are countable

Sketch of proof, for

- A and B countable $\implies$ $A \cup B$ countable
- A and B are disjoint and infinite

By assumption, can write $A = \{a_1, a_2, ...\}$ and $B = \{b_1, b_2, ...\}$

Now count it like so:

$$\begin{array}{ccccccccc} a_1 & & a_2 & & a_3 & & a_4 & \cdots \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow \\ b_1 & & b_2 & & b_3 & & b_4 & \cdots \end{array}$$

Example

$\mathbb{Z} = \{... , -2, -1\} \cup \{0\} \cup \{1, 2, ...\}$ is countable

---

Fact

Finite Cartesian products of countable sets is countable

Sketch of proof, for

- A and B countable $\implies A \times B$ countable
- A and B are disjoint and infinite

Now count like so:

$$\begin{array}{ccccccc} (a_1, b_1) & \to & (a_1, b_2) & & (a_1, b_3) & \to \cdots \\ & \swarrow & & \nearrow & & \\ (a_2, b_1) & & (a_2, b_2) & & \cdots & \\ \downarrow & \nearrow & & & & \\ (a_3, b_1) & & \vdots & & & \\ \vdots & & & & & \end{array}$$

Example

$\mathbb{Z} \times \mathbb{Z} = \{(p, q) : p \in \mathbb{Z}, q \in \mathbb{Z}\}$ is countable

---

Fact

$\mathbb{Q}$ is countable

Proof: By definition

$$\mathbb{Q} := \left\{ \text{all } \frac{p}{q} \text{ where } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \right\}$$

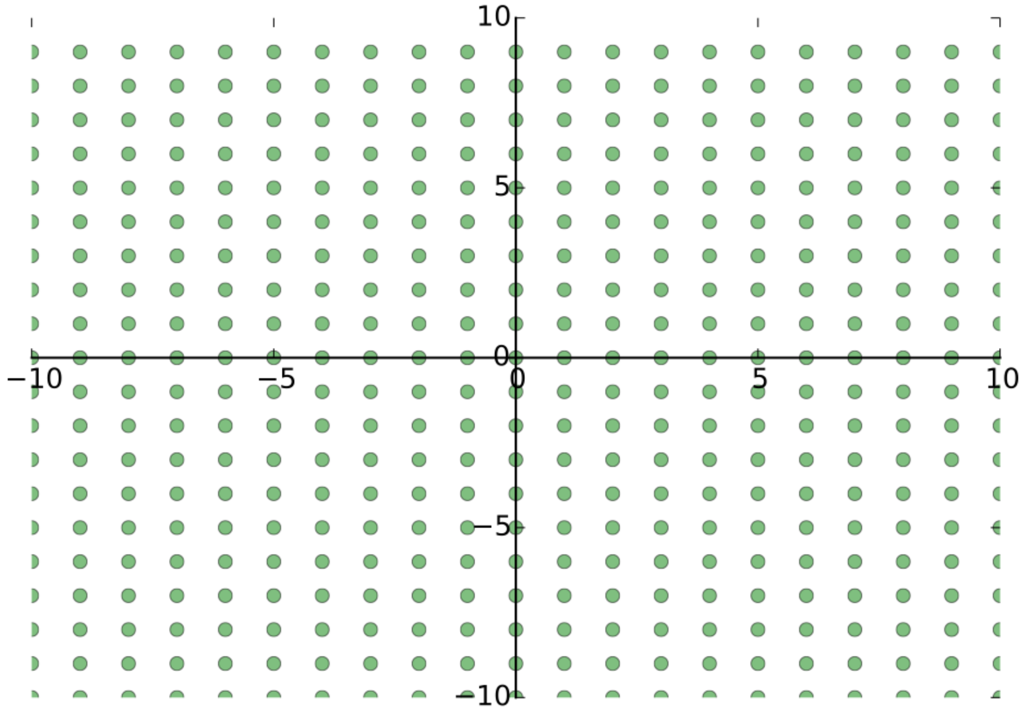
Consider the function ϕ defined by $\phi(p/q) = (p, q)$

- A one-to-one function from $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{N}$
- A bijection from $\mathbb{Q}$ to $\text{rng}(\phi)$

Since $\mathbb{Z} \times \mathbb{N}$ is countable, so is $\text{rng}(\phi) \subset \mathbb{Z} \times \mathbb{N}$

Hence $\mathbb{Q}$ is also countable

Example



An example of an ***uncountable*** set is all binary sequences $\{0, 1\}^{\mathbb{N}} := \{(b_1, b_2, ...) : b_n \in \{0, 1\} \text{ for each } n\}$

Sketch of proof: If this set were countable then it could be listed as follows:

$$\begin{array}{ccc}
1 & \leftrightarrow & \mathbf{a_1},\ a_2,\ a_3,\ a_4, ... \\
2 & \leftrightarrow & b_1,\ \mathbf{b_2},\ b_3,\ b_4, ... \\
3 & \leftrightarrow & c_1,\ c_2,\ \mathbf{c_3},\ c_4, ... \\
4 & \leftrightarrow & d_1,\ d_2,\ d_3,\ \mathbf{d_4}, ... \\
\vdots & & \vdots
\end{array}$$

Such a list is never complete: Cantor's diagonalization argument

Cardinality of $\{0, 1\}^{\mathbb{N}}$ called the ***power of the continuum***

Other sets with the power of the continuum

- $\mathbb{R}$
- (a, b) for any $a < b$
- $[a, b]$ for any $a < b$
- $\mathbb{R}^N$ for any finite $N \in \mathbb{N}$

Continuum hypothesis

Every nonempty subset of $\mathbb{R}$ is either countable or has the power of the continuum

---

- Not a homework exercise!

Example

$\mathbb{R}$ and $(-1, 1)$ have the same cardinality because $x \mapsto 2\arctan(x)/\pi$ is a bijection from $\mathbb{R}$ to $(-1, 1)$

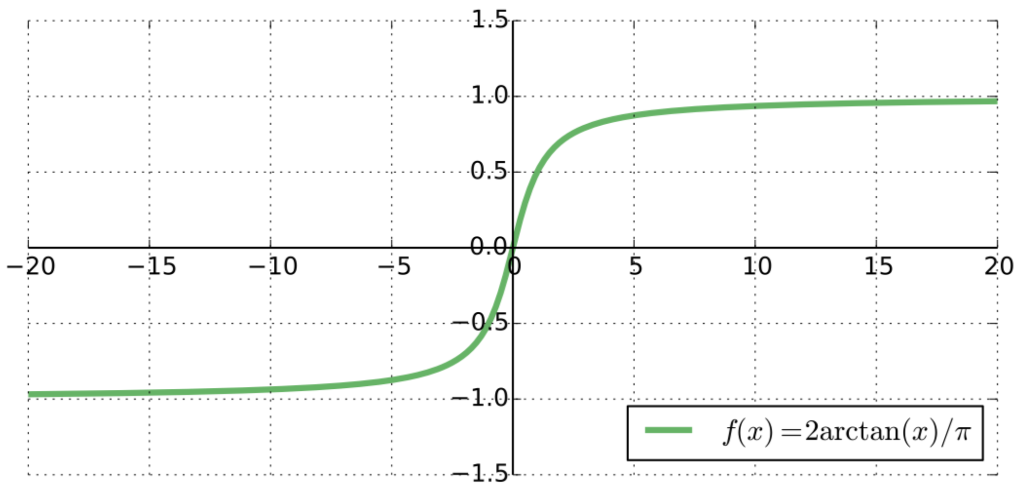


Fig. 3.10: Same cardinality

CHAPTER

FOUR

ELEMENTS OF LINEAR ALGEBRA

COMING SOON

CHAPTER

FIVE

ELEMENTS OF PROBABILITY

COMING SOON

CHAPTER

SIX

FUNDAMENTALS OF OPTIMIZATION

COMING SOON

CHAPTER

SEVEN

UNCONSTRAINED OPTIMIZATION

COMING SOON

CHAPTER

EIGHT

CONSTRAINED OPTIMIZATION

COMING SOON

CHAPTER

NINE

PRACTICAL SESSION

COMING SOON

CHAPTER

TEN

ENVELOPE AND MAXIMUM THEOREMS

COMING SOON

CHAPTER

ELEVEN

DYNAMIC OPTIMIZATION

COMING SOON

CHAPTER

TWELVE

REVISION

COMING SOON

CHAPTER THIRTEEN

EXERCISE SET A

General comment on the tutorial exercises

- these questions are not directly assessable and solutions are provided in a separate document
- the aim is to help you better understand the material we have covered so far and start to prepare for actual assessments
- if you are having no particular problems with this course then please carry on to the questions below; if, on the other hand, you are having difficulty with the material, then please read on
- this is an upper level course on mathematics for economists that pushes you beyond the boundaries of the kind of things we do in high school or first year university
- many people find this material hard at first, however, the experience is that anyone who works diligently and consistently can and will do well

Here are few tips on getting through and doing well:

1. It's hard to "wing" this course, even if you did well at maths in high school. It's also very hard to follow everything just from the lectures. It takes practice to do well, just like playing guitar or learning a language. The first resource is the lecture slides, and the more often you read them the more the definitions will stick and the material will gel in your head.
2. Each time you are having difficulty with a new concept try googling it. Have a look at Wikipedia, find a video on YouTube, or some of the other online resources. They might phrase or explain the concept in a way that fits better with your brain.
3. Work consistently throughout the semester. Concepts become clearer and more familiar the more times that you go over them—with at least one sleep in between to allow your brain to organize neurons and synapses to store and categorize this new information.
4. Make use of your tutors. They are very knowledgable and are willing to put in time to help anyone who is genuinely trying (although much less inclined to help those who aren't).
5. Send me feedback if you think it's something I can help with (e.g., more practice questions on a certain topic) or drop in during my office hours to discuss.
6. Above all, remember that the course material is nontrivial for a reason. Doing straightforward calculations applying well known rules or memorizing "cookbooks" of facts are not particularly useful, mainly because computers are far, far better than humans at these kinds of activities. What is still very useful—probably more than ever—is understanding concepts and how they relate to each other, and building up your ability to digest technical material and think in a logical way. If you complete this course successfully you will have significantly upgraded your mathematical skills.

Finally, here are some tips on doing proofs:

1. In many instances there will be an easy way to do things, if you can spot it. A question that seems to require a long calculation will likely have an easy answer if you know the relevant fact.

2. If you feel stuck, remember that the hardest step is getting started, and for proofs the best place to start is always the relevant definitions. If you are asked to show that the range of a given function is a linear subspace, start by writing down those two definitions. They will tell you more specifically what you need to show. If you're still stuck, review any facts from the lecture slides related to those definitions. Is there a different way to describe the range of this function? Is there some fact related to linear subspaces that might be helpful?
3. If you're still stuck, try flipping the problem around. In the previous example, suppose that the range of the function is not a linear subspace. What would that imply? Can you show that such an outcome is impossible?
4. Be patient and don't rush. You'll get quicker naturally, with practice.

Question A.1

Let $f\colon [-1, 1] \to \mathbb{R}$ be defined by $f(x) = 1 - |x|$, where $|x|$ is the absolute value of x.

- Is the point $x = 0$ a maximizer of f on $[-1, 1]$?
- Is it a unique maximizer?
- Is it an interior maximizer?
- Is it stationary?

Question A.2

Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin(x)$.

- Write down the set of stationary points of this function.
- Which of these, if any, are maximizers, and which are minimizers?

Tip: When we discussed these kinds of problems it was for functions of the form $f\colon [a, b] \to \mathbb{R}$. Now the domain is all of $\mathbb{R}$. However you can apply the same definitions and use similar reasoning. Also, feel free to look up and use any helpful facts on trigonometric functions.

13.1 Question A.3: Profit maximization with Cobb-Douglas production and linear costs

A firm uses capital and labor to produce output. When it employs k units of capital and ℓ units of labor, its output is $Ak^\alpha \ell^\beta$ units, where A is a positive number, and $\alpha + \beta < 1$.

The unit price of capital is r, and the unit price of labor is w; both are non-negative. The firm would like to maximize the profits taking the price p of the output as given.

The firm's chief economist *Bob* presented the following formulation of the firm's optimization problem to the CEO *Alice*:

$$\text{Choose } k, \ell, w \text{ and } r \text{ to maximize } pk^\alpha \ell^\beta - w\ell - rk \quad \text{s.t.} \quad \alpha + \beta < 1$$

Questions:

1. Is this formulation of the firm's optimization problem correct?

- What part reflects the revenue?
- What part reflects the costs?
- What are the choice variables?
- Are there any constraints to be taken into account?

2. Right down the problem after *Alice* have updated the formulation.
3. Approach the problem as unconstrained maximization, and follow the steps in the lecture to find find all stationary points (solve the FOCs).
4. Write down second order partial derivatives and verify the shape conditions for the profit function.
5. What is the optimal strategy for the firm? Is the maximizer unique? Why?

13.2 Question A.4*

Note: Exercises marked with an asterisk **(*)** are optional and more difficult.

1. Find all stationary points of the function $f(x, y) = \frac{\cos(x^2+y^2)}{1+x^2+y^2}$.
2. Find all maximizers and minimizers of this function on $\mathbb{R}^2$.

Hint: Is there a convenient change of variable to convert the problem to a univariate one.

Solutions

Question A.1

The point $x = 0$ is indeed a maximizer, since $f(x) = 1 - |x| \leq 1 = f(0)$ for any $x \in [-1, 1]$ ($|x| = 0$ if and only if $x = 0$). It is also a unique maximizer, since no other point is a maximizer (because $1 - |x| < 1$ for any other x). It is an interior maximizer since 0 is not an end point of $[-1, 1]$. It is not stationary because f is not differentiable at this point (sketch the graph if you like) and hence cannot satisfy $f'(x) = 0$.

Question A.2

The set S of stationary points of f are the points $x \in \mathbb{R}$ such that $f'(x) = \cos(x) = 0$. By the definition of the cosine function this is the set

$$S := \{x \in \mathbb{R} : x = \pi/2 + k\pi \text{ for } k \in \mathbb{Z}\}$$

Every point in the domain $\mathbb{R}$ is interior (i.e, not an end point) and the function f is differentiable, so the set of maximizers will be contained in the set of stationary points. The same is true of the set of minimizers. From the definition of the sine function, we have

$$f(\pi/2 + k\pi) = \begin{cases} 1 & \text{if } k \text{ is even} \\ -1 & \text{if } k \text{ is odd} \end{cases}$$

Hence the set of maximizers is

$$M^* := \{x \in \mathbb{R} : x = \pi/2 + k\pi \text{ for } k \text{ an even integer}\}$$

The set of minimizers is

$$M_* := \{x \in \mathbb{R} : x = \pi/2 + k\pi \text{ for } k \text{ an odd integer}\}$$

Question A.3

1. The formulation is not correct. The revenue (after reincerting constant A) is $pAk^\alpha \ell^\beta$, the costs are $w\ell + rk$, and the choice variables are k and ℓ (w and r are not chosen by the firm).
   The constraint $\alpha + \beta < 1$ is irrelevant for the optimization problem, instead it is a constraint on the parameters for the problem to be well posed. Relevant constraints on the optimization problem are $k > 0$ and $\ell > 0$, they can be first ignored and checked after we solve the unconstrained version of the problem.
2. The correct formulation is ($A, p, \alpha, \beta, w, r$ are parameters and should be fixed/found out before the firm solves the optimization problem)
$$pAk^\alpha \ell^\beta - w\ell - rk \to \max_{k,\ell}$$
$$\text{s.t.} \quad k \geq 0, \ell \geq 0$$
3. See lecture notes
4. See lecture notes
5. Optimal strategy k^*, ℓ^* are given in the lecture notes. The maximizer is unique because the objective function is strictly concave when $\alpha + \beta < 1$.

Proof:

We check second order conditions for strict concavity.

What we need: for any $k, \ell > 0$

1. $\pi_{11}(k, \ell) < 0$
2. $\pi_{11}(k, \ell)\, \pi_{22}(k, \ell) > \pi_{12}(k, \ell)^2$

The second order derivatives are

$$\begin{aligned}
\pi_{11}(k, \ell) &= (\alpha - 1)\alpha pAk^{\alpha-2}\ell^\beta \\
\pi_{22}(k, \ell) &= (\beta - 1)\beta pAk^{\alpha}\ell^{\beta-2} \\
\pi_{12}(k, \ell) &= \alpha\beta pAk^{\alpha-1}\ell^{\beta-1}.
\end{aligned}$$

Since $\alpha + \beta < 1$ and $\alpha, \beta \geq 0$, we have $\alpha - 1 < 0$, which implies $\pi_{11}(k, \ell) < 0$ for all $k, \ell > 0$.
Moreover, the second order differentials imply

$$\begin{aligned}
\pi_{11}(k, \ell)\, \pi_{22}(k, \ell) &= (\alpha - 1)(\beta - 1)\alpha\beta p^2 A^2 k^{2\alpha-2}\ell^{2\beta-2} \\
(\pi_{22}(k, \ell))^2 &= \alpha^2\beta^2 p^2 A^2 k^{2\alpha-2}\ell^{2\beta-2}.
\end{aligned}$$

Assuming that all parameters and variables are positive. Then, we obtain $\pi_{11}(k, \ell)\, \pi_{22}(k, \ell) > \pi_{12}(k, \ell)^2$ if and only if $(\alpha - 1)(\beta - 1) > \alpha\beta$ if and only if $1 > \alpha + \beta$.

Question A.4

The graph of $f(x, y)$ is

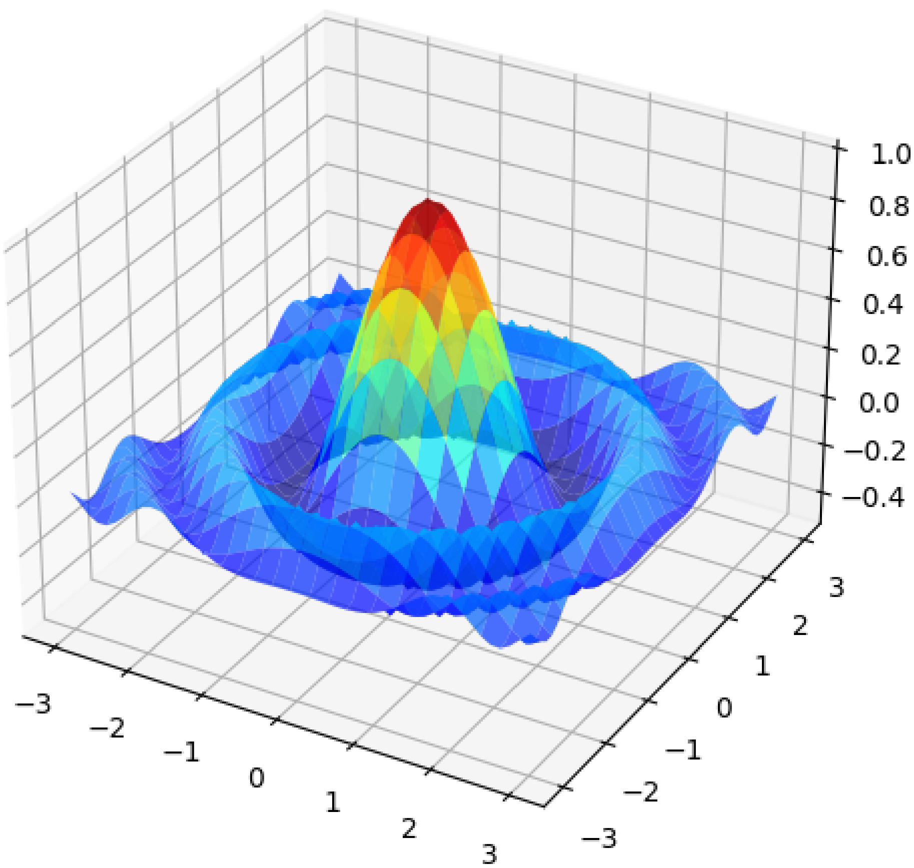
Let $t = x^2 + y^2 \geq 0$. The function becomes

$$f(x, y) = \frac{\cos(x^2 + y^2)}{1 + x^2 + y^2} = \frac{\cos(t)}{1 + t} =: g(t) \quad (t \geq 0).$$

First note that since $t \geq 0$ and $\cos(t) \leq 1$, we have $g(t) \leq 1$ and $g(0) = 1$. Hence, $t = 0$ is a maximizer for g, or $(x, y) = (0, 0)$ is the maximizer for f. It is a unique maximizer, since if $g(t) < 1$ for $t > 0$.

Next, we find the stationary points of f by finding the stationary points of g. The FOC is

$$g'(t) = \frac{-\sin(t)(1 + t) - \cos(t)}{(1 + t)^2} = 0.$$


1.0
0.8
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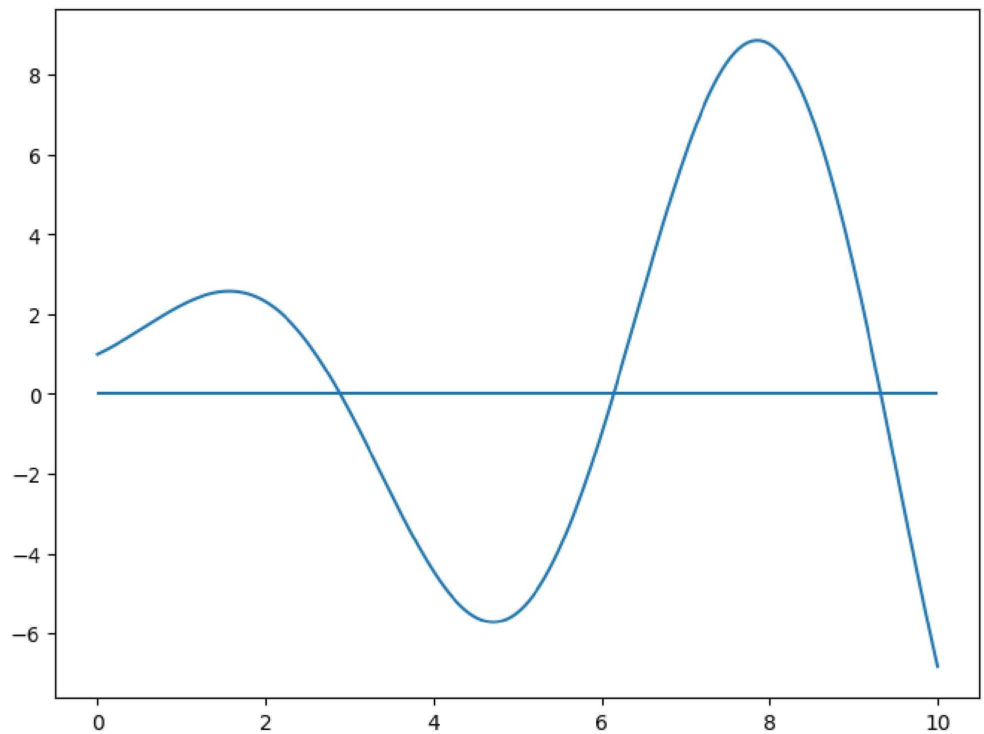

Since $(1+t)^2 > 0$, it must be

$$-\sin(t)(1+t) - \cos(t) = 0 \Leftrightarrow t\sin(t) + \sin(t) + \cos(t) = 0.$$

The numerical solutions for the smallest stationary point t_m such that $\cos(t_m) < 0$ are

'The smallest stationary point is tm=2.889969697678371'

'The minimum is -0.24897613487740497'

The minimizers are $\{(x, y) \in \mathbb{R} : x^2 + y^2 = t_m\}$. To verify that t_m is the unique minimizer for g, since $\cos^2(t) + \sin^2(t) = 1$, we rewrite FOC to get

$$\begin{aligned}
t\sin(t) + \sin(t) &= \pm\sqrt{1 - \sin^2(t)} \\
\Leftrightarrow \sin^2(t) &= \frac{1}{2 + 2t + t^2} \\
\Leftrightarrow \cos^2(t) = 1 - \frac{1}{2 + 2t + t^2} &= \frac{(1+t)^2}{2 + 2t + t^2} \\
\Rightarrow g(t) = \frac{\cos(t)}{1+t} = \pm\frac{1}{\sqrt{2 + 2t + t^2}} & \qquad (t \text{ is stationary point}).
\end{aligned}$$

Therefore, the smallest stationary point such that $\cos(t) < 0$ will be the unique minimizer for g.

CHAPTER

FOURTEEN

EXERCISE SET B

Please, see the ***general comment on the tutorial exercises***

14.1 Question B.1

Each of the definitions below is an attempt to define a set. Determine whether a set is indeed defined in each case, and if not explain why.

1. $\{1, e, -2, -\pi\}$
2. $\{15, \{1, 2, 3\}, \text{ANU}, \text{Europe}, \text{USA}\}$
3. $\{1, 2, \dots, 99\}$
4. $\{1, 4, 7, 91, \dots\}$
5. $\{x \in \mathbb{R} \colon x^2 \leq 5\}$
6. $\{(x, y) \in \mathbb{R}^2 \colon 5x^2 + y^2 \leq 10\}$
7. $\{f \colon [0, 1] \to \mathbb{R} \colon f \text{ is one-to-one}\}$
8. $\{f_n(x) \colon [0, 1] \to \mathbb{R} \colon f_n(x) = x^n\}$
9. $\{A \subset S : x_0 \in A\}$ for given S and $x_0 \in S$

14.2 Question B.2

Let A,B and C be any three sets. Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Hint: Hint, if you need it: One way to show that $E = F$ is show that a arbitrary element of E must also be in F and vice versa.

14.3 Question B.3

Let A, B, C and D be some set such that $A \subset C$ and $B \subset C$. Let $f \colon D \to C$ be a function.

Show that $f^{-1}(A \quad B) = f^{-1}(A) \quad f^{-1}(B)$.

14.4 Question B.4

Find the composition $g \circ f$ of two functions f and g, if it exists:

1. $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$ and $g \colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \frac{x}{1+x^2}$
2. $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 1 - x^4$ and $g \colon (1, \infty) \to \mathbb{R}$ defined by $g(x) = \log(x - 1)$
3. $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \cos(x)$ and $g \colon \mathbb{R} \quad \{1\} \to \mathbb{R}$ defined by $g(x) = \frac{x}{1-x}$

Hint: Is there a composition in each case?

14.5 Question B.5

Let f and g be any two functions from $\mathbb{R}$ to $\mathbb{R}$. Is it true that $g \circ f = f \circ g$ always holds?

Hint: There are two things implicit in this question. First, there is an implicit final sentence here, which is: If yes, prove it. If no, give a counterexample. Second, an equality sign between two functions means that they are the same function. Hence to show equality you need to show that they agree everywhere on the domain. To show inequality, you need to give just one point in the domain where the function values differ.

14.6 Question B.6

Fact: the sufficient conditions for concavity/convexity in 2D

Let $z = f(x, y)$ be a twice continuously differentiable function defined for all $(x, y) \in R^2$.

Then it holds:

- f is convex $\iff f''_{1,1} \geq 0,\; f''_{2,2} \geq 0,$ and $f''_{1,1}f''_{2,2} - (f''_{1,2})^2 \geq 0$
- f is concave $\iff f''_{1,1} \leq 0,\; f''_{2,2} \leq 0,$ and $f''_{1,1}f''_{2,2} - (f''_{1,2})^2 \geq 0$
- $f''_{1,1} > 0$ and $f''_{1,1}f''_{2,2} \implies f$ is strictly convex
- $f''_{1,1} < 0$ and $f''_{1,1}f''_{2,2} \implies f$ is strictly concave

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1. Find the largest domain S on which $f(x, y) = x^2 - y^2 - xy - x^3$ is concave.
2. How about strictly concave?