ECON2125/6012

Fedor Iskhakov

Aug 03, 2023

CONTENTS

Preliminary schedule

Week	Date	Topic	Notes
1	July 27	*Introduction*	Recorded lecture
2	Aug 3	*Univariate and bivariate optimization*	Tutorials start
3	Aug 10	*Elements of set theory and analysis*	
4	Aug 17	*Elements of linear algebra*	
Test		15%	Submit by Aug 23
5	Aug 24	*Elements of Probability*	
6	Aug 31	*Fundamentals of optimization*	
Test		15%	Submit by Sept 3
Break			2 weeks
7	Sept 21	*Unconstrained optimization*	
8	Sept 28	*Constrained optimization*	
Test		15%	Submit by Oct 4
9	Oct 5	*Practical session/invited speaker*	TBA
10	Oct 12	*Envelope and maximum theorems*	
11	Oct 19	*Dynamic optimization*	
12	Oct 26	*Revision*	
Exam		55%	During exam period

ANU course pages

Course Wattle page Schedule, announcements, teaching team contacts, recordings, assignement, grades

Course overview Class summary General course description in ANU Programs and Courses

CHAPTER

ONE

WELCOME

Course title: **"Optimization for Economics and Financial Economics"**

- Elective second year course in the *Bachelor of Economics* program ECON2125
- Compulsory second math course in the *Master of Economics* program ECON6012

The two courses are identical in content and assessment, but final grades may be adjusted depending on your program.

1.1 Plan for this lecture

1. Organization
2. Administrative topics
3. Course content
4. Self-learning materials

1.2 Instructor

Fedor Iskhakov Professor of Economics at RSE

- Office: 1021 HW Arndt Building
- Email: fedor.iskhakov@anu.edu.au
- Web: fedor.iskh.me
- Contact hours: Thursday 9:30-11:30

1.3 Timetable

Face-to-face:

- Lectures: Thursday 15:30 — 17:30
- Location: **DNF Dunbar Lecture Theatre, Physics Bldg 39A**

Online:

- Echo-360 recordings on Wattle
- All notes and materials on **optim.iskh.me**

Face-to-face is strictly preferred

1.4 Course web pages

- Wattle Schedule, announcements, teaching team contacts, recordings, assignment, grades
- Online notes Lecture notes, slides, assignment tasks
- Lecture slides should appear online the previous day before the lecture
- Details on assessment including the exam instructions will appear on Wattle

1.5 Tutorials

- Enrollments open on *Wattle*

Tutorial questions

- posted on the course website
- not assessed, help you learn and prepare

Tutorials start on week 2

1.6 Tutors

Wending Liu

- Email: Wending.Liu@anu.edu.au
- Room: 1018 HW Arndt Building
- Office hours: **Friday 1pm-3pm**

Chien Yeh

- Email: Chien.Yeh@anu.edu.au
- Room: Room 2106, Copland Bld (24)
- Office hours: **Monday 2pm-4pm**

1.7 Prerequisites

See Course overview and Class summary

What you actually need to know:

- basic algebra
- basic calculus
- some idea of what a matrix is, etc.

≈ content of EMET1001/EMET7001 math course

1.8 Focus?

Q: Is this optimization or a general math-econ course?

A: A general course on mathematical modeling for economics and financial economics. Optimization will be an important and recurring theme.

1.9 Assessment

- 3 timed open book tests (15% each)
- Final exam (55%)

The three tests spread out through the semester will check the knowledge of the immediately preceding material. The final closed book in-person exam will cover the entire course.

1.10 Questions

1. Administrative questions: RSE admin

- **Bronwyn Cammack** Senior School Administrator
- Email: enquiries.rse@anu.edu.au
- "I can not register for the tutorial group"

2. Content related questions: please, refer to the tutors

- "I don't understand why this function is convex"

3. Other questions: to Fedor

- "I'm working hard but still can not keep up"
- "Can I please have extra assignment for more practice"

1.11 Attendance

- Please, **do not** use email for *instructional* questions\Instead make use of the office hours
- Attendance of tutorials is *very highly* recommended
  You will make your life much easier this way
- Attendance of lectures is *highly* recommended
  But not mandatory

1.12 Comments for lectures notes/slides

- Cover exactly what you are required to know
- Code inserts are the exception, they are not assessable

In particular, you need to know:

- The definitions from the notes
- The facts from the notes
- How to apply facts and definitions

If a concept in not in the lecture notes, it is not assessable

1.13 Definitions and facts

The lectures notes/slides are full of definitions and facts.

Definition

Functions $f : \mathbb{R} \to \mathbb{R}$ is called *continuous at* x if, for any sequence $\{x_n\}$ converging to x, we have $f(x_n) \to f(x)$.

Possible exam question: "Show that if functions f and g are continuous at x, so is $f + g$."

You should start the answer with the definition of continuity:

"Let $\{x_n\}$ be any sequence converging to x. We need to show that $f(x_n) + g(x_n) \to f(x) + g(x)$. To see this, note that $\dots$"

1.14 Facts

In the lecture notes/slides you will often see

Fact

The only N-dimensional subset of $\mathbb{R}^N$ is $\mathbb{R}^N$.

This means either:

- theorem
- proposition
- lemma
- true statement

All well known results. You need to remember them, have some intuition for, and be able to apply.

1.15 Note on Assessments

Assessable = definitions and facts + last year level math + a few simple steps of logic

Exams and tests will award:

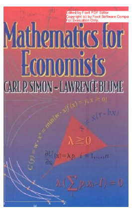
- Hard work
- Deeper understanding of the concepts

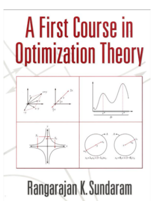
In each question there will be a *easy* path to the solution

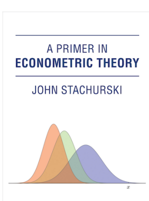
1.16 Reading materials

Primary reference: lecture slides

Books:







- “Mathematics for Economists” (1994) by Simon, C. and L. Blume
- “A First Course in Optimization” (1996) Theory by Rangarajan Sundaram
- “A Primer in Econometric Theory” (2016) by John Stachurski

Readings are supplementary but will provide a more detailed explanation with additional examples.

- Each lecture will reference book chapters

1.17 Key points for the administrative part

- Tutorials start next week, **please register before the next lecture**
- Course content = what's in lecture notes/slides
- Lecture slides are available online and will be updated throughout the semester
- Optimization is a recurring theme but not the only topic

1.18 What you will learn in the course

- The lecture plan is on the course website optim.iskh.me and Class summary
- See the list of topics on the left

Essentially:

1. **Mathematical foundations**

- elements of analysis
- elements of linear algebra
- elements of probability

2. **Optimization theory**

- when solution exists
- unconstrained optimization
- optimization with equality constraints
- optimization with inequality constraints

3. **Further topics**

- Parameterized optimization problems
- Optimization in dynamics

1.19 Further material and self-learning

- Each lecture will suggest some material for further reading and learning
- Today: **The Wason Selection Task** logical problem
- Mathematics relies on rules of logic
- Yet, for human brain applying mathematical logic may be difficult, and dependent on the domain

Please, watch the video and try to solve the puzzle yourself youtu.be/iR97LBgpsl8

CHAPTER

TWO

UNIVARIATE AND BIVARIATE OPTIMIZATION

ECON2125/6012 Lecture 2 Fedor Iskhakov

2.1 Announcements & Reminders

- **Tutorials start tomorrow (Aug 4)**
- Register for tutorials on Wattle if you have not done so already
- Office hours of the tutors are updated:
  - **Wending Liu**
    - Email: Wending.Liu@anu.edu.au
    - Room: 1018 HW Arndt Building
    - Office hours: **Friday 1pm-3pm**
  - **Chien Yeh**
    - Email: Chien.Yeh@anu.edu.au
    - Room: Room 2106, Copland Bld (24)
    - Office hours: **Monday 2pm-4pm**
- Reminder on how to ask questions:
  1. Administrative: RSE admin
  2. Content/understanding: tutors
  3. Other: to Fedor

2.2 Plan for this lecture

1. Motivation (math vs. computing)
2. Univariate optimization
3. Working with bivariate functions
4. Bivariate optimization

Supplementary reading:

- Simon & Blume: part 1 (revision)

- Sundaram: sections 1.1, 1.4, chapter 2, chapter 4

2.3 Computing

The *classic* way we do mathematics is pencil and paper

> In 1944, Hans Bethe solved following problem *by hand*:
> Will detonating an atom bomb ignite the atmosphere and thereby destroy life on earth?
> source

These days we rarely calculate with actual numbers

Almost all calculations are done on computers

Example: numerical integration

$$\frac{1}{\sqrt{2\pi}} \int_{-2}^{2} \exp\left\{-\frac{x^2}{2}\right\} dx$$

```
from scipy.stats import norm
from scipy.integrate import quad
phi = norm()
value, error = quad(phi.pdf, -2, 2)
print('Integral value =',value)
```

```
Integral value = 0.9544997361036417
```

Example: Numerical optimization

$$f(x) = -\exp\left\{-\frac{(x-5.0)^4}{1.5}\right\} \to \min$$

```
from scipy.optimize import fminbound
import numpy as np
f = lambda x: -np.exp(-(x - 5.0)**4 / 1.5)
res = fminbound(f, -10, 10)   # find approx solution
print('Minimum value is attained approximately at', res)
```

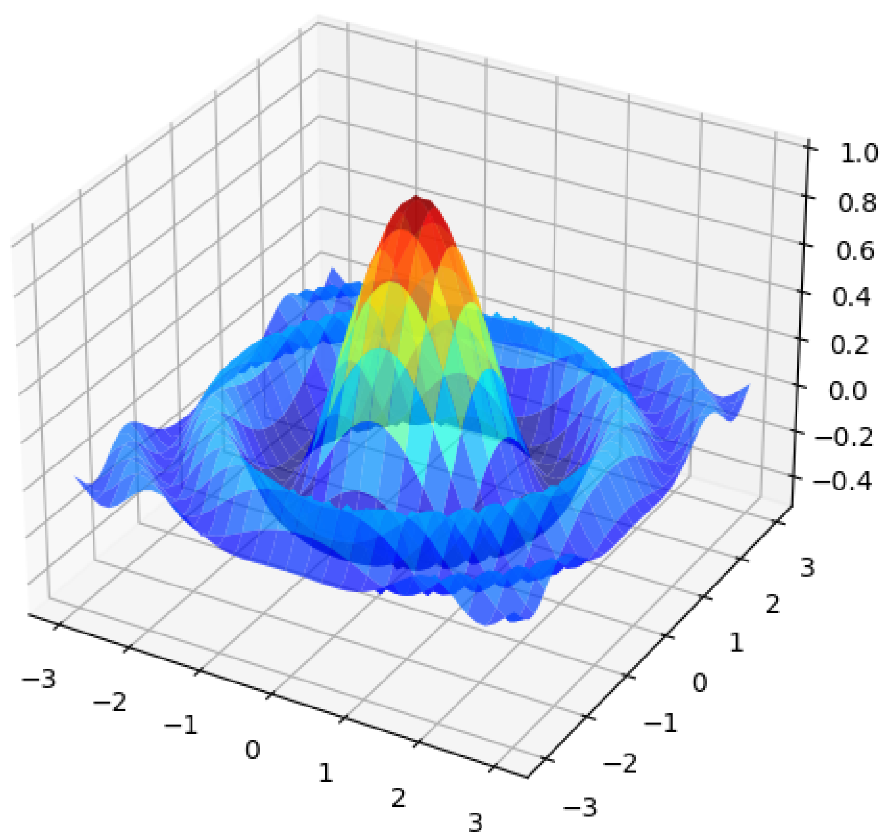
```
Minimum value is attained approximately at 4.999941901210501
```

Example: Visualization

What does this function look like?

$$f(x, y) = \frac{\cos(x^2 + y^2)}{1 + x^2 + y^2}$$

```
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d.axes3d import Axes3D
import numpy as np
from matplotlib import cm
f = lambda x, y: np.cos(x**2 + y**2) / (1 + x**2 + y**2)
xgrid = np.linspace(-3, 3, 50)
ygrid = xgrid
x, y = np.meshgrid(xgrid, ygrid)
fig = plt.figure(figsize=(8, 6))
ax = fig.add_subplot(111, projection='3d')
ax.plot_surface(x,
                y,
                f(x, y),
                rstride=2, cstride=2,
                cmap=cm.jet,
                alpha=0.7,
                linewidth=0.25)
ax.set_zlim(-0.5, 1.0)
plt.show()
```



Example: Symbolic calculations

Differentiate $f(x) = (1 + 2x)^5$.

Forgotten how? No problems, just ask a computer for *symbolic* derivative

```
import sympy as sp
x = sp.Symbol('x')
fx = (1 + 2 * x)**5
print("Derivative of",fx,"is",fx.diff(x))
```

```
Derivative of (2*x + 1)**5 is 10*(2*x + 1)**4
```

So if computers can do our maths for us, why learn maths?

The difficulty is

- giving them the right inputs and instructions
- interpreting what comes out

The skills we need are

- Understanding of fundamental concepts
- Sound deductive reasoning

These are the focus of the course

2.3.1 Computer Code in the Lectures

While computation is not a formal part of the course
there will be little bits of code in the lectures to illustrate the kinds of things we can do.

- All the code will be written in the Python programming language
- It is not assessable

You might find value in actually running the code shown in lectures
If you want to do so please refer to **linked GitHub repository** in optim.iskh.me

2.4 Univariate Optimization

Let $f: [a, b] \to \mathbb{R}$ be a differentiable (smooth) function

- $[a, b]$ is all x with $a \leq x \leq b$
- $\mathbb{R}$ is "all numbers"
- f takes $x \in [a, b]$ and returns number $f(x)$
- derivative $f'(x)$ exists for all x with $a < x < b$

Definition

A point $x^* \in [a, b]$ is called a

- ***maximizer*** of f on $[a, b]$ if $f(x^*) \geq f(x)$ for all $x \in [a, b]$
- ***minimizer*** of f on $[a, b]$ if $f(x^*) \leq f(x)$ for all $x \in [a, b]$

Example

Let

- $f(x) = -(x-4)^2 + 10$
- $a = 2$ and $b = 8$

Then

- $x^* = 4$ is a maximizer of f on $[2, 8]$
- $x^{**} = 8$ is a minimizer of f on $[2, 8]$

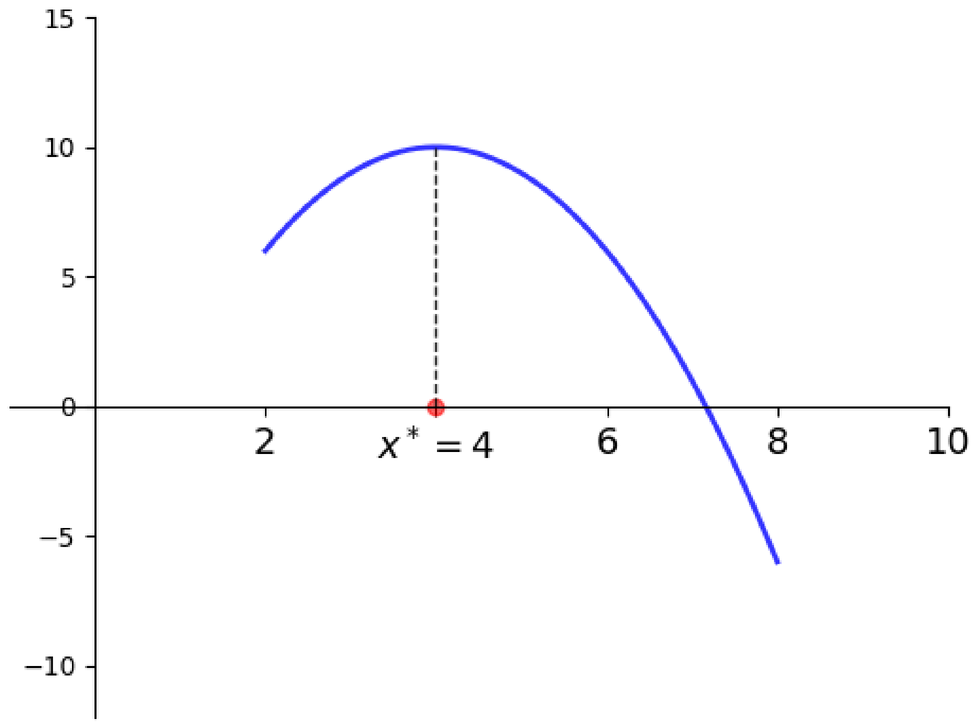


Fig. 2.1: Maximizer on $[a, b] = [2, 8]$ is $x^* = 4$

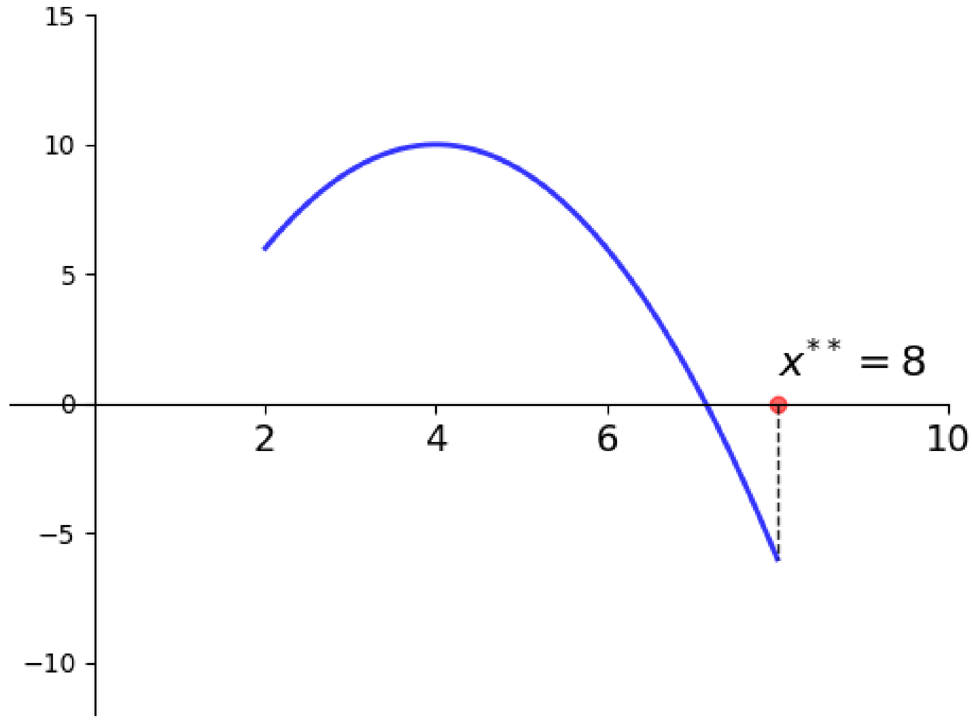


Fig. 2.2: Minimizer on $[a, b] = [2, 8]$ is $x^{**} = 8$

The set of maximizers/minimizers can be

- empty
- a singleton (contains one element)
- infinite (contains infinitely many elements)

Example: infinite maximizers

$f: [0, 1] \to \mathbb{R}$ defined by $f(x) = 1$
has infinitely many maximizers and minimizers on $[0, 1]$

Example: no maximizers

The following function has no maximizers on $[0, 2]$

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 1/2 & \text{otherwise} \end{cases}$$

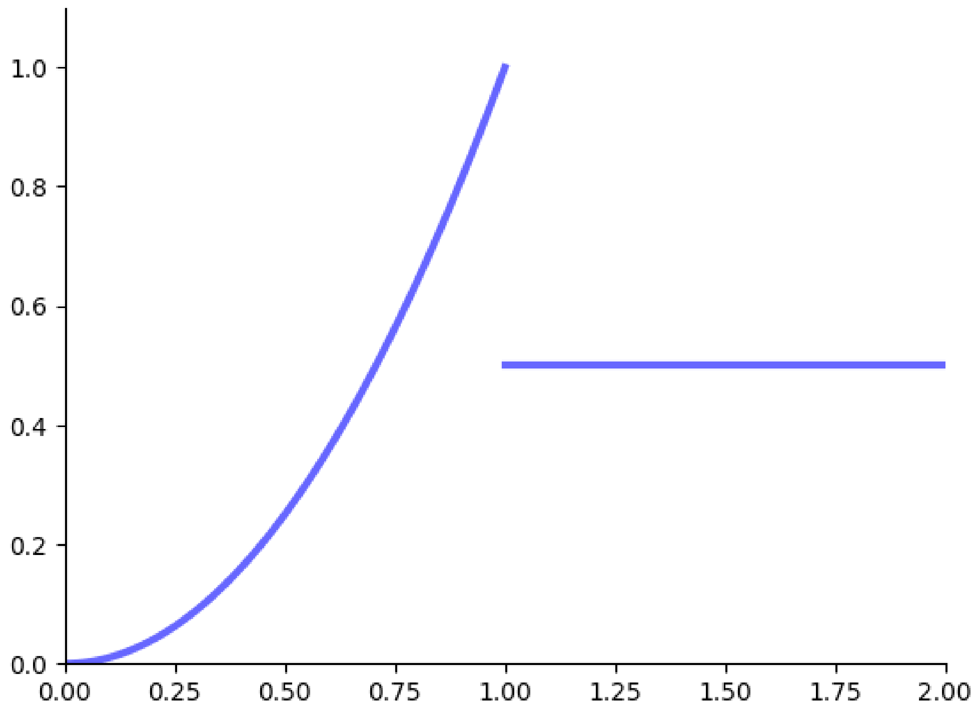


Fig. 2.3: No maximizer on $[0, 2]$

Definition

Point x is called ***interior*** to $[a, b]$ if $a < x < b$

The set of all interior points is written (a, b)

We refer to $x^* \in [a, b]$ as

- ***interior maximizer*** if both a maximizer and interior
- ***interior minimizer*** if both a minimizer and interior

2.5 Finding optima

Definition

A ***stationary point*** of f on $[a, b]$ is an interior point x with $f'(x) = 0$

Fact

If f is differentiable and x^* is either an interior minimizer or an interior maximizer of f on $[a, b]$, then x^* is stationary

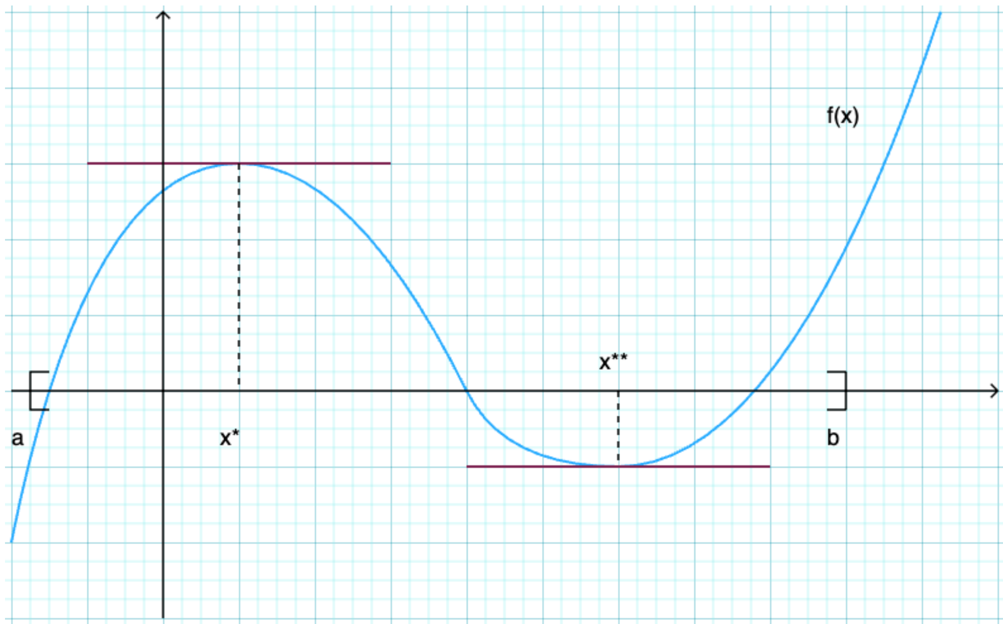


Fig. 2.4: Both x^* and x^{**} are stationary

Sketch of proof, for maximizers: \$$f'(x^*) = \lim_{h \to 0} \frac{f(x^*+h)-f(x^*)}{h}$ (by def.)$\Rightarrow$ $f(x^* + h) \approx f(x^*) + f'(x^*)h$ for small h\$

If $f'(x^*) \neq 0$ then exists small h such that $f(x^* + h) > f(x^*)$

Hence interior maximizers must be stationary — otherwise we can do better

$\Rightarrow$ any interior maximizer stationary
$\Rightarrow$ set of interior maximizers $\subset$ set of stationary points
$\Rightarrow$ maximizers $\subset$ stationary points $\cup \{a\} \cup \{b\}$

Usage:

1. Locate stationary points
2. Evaluate $y = f(x)$ for each stationary x and for a, b
3. Pick point giving largest y value

Minimization: same idea

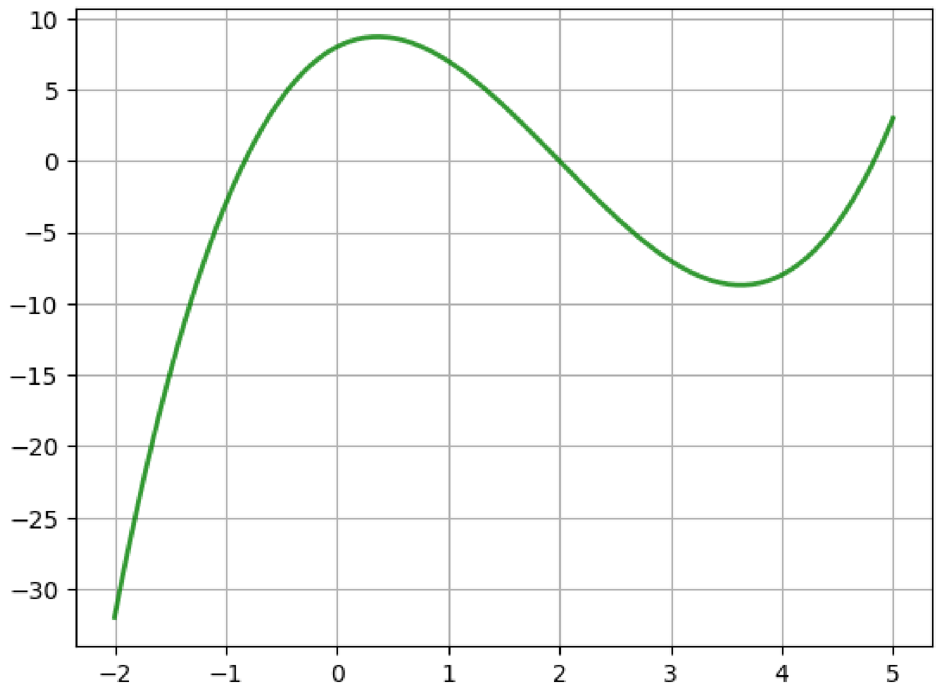
Example

Let's solve \$$\max_{-2 \leq x \leq 5} f(x) \quad \text{where} \quad f(x) = x^3 - 6x^2 + 4x + 8$\$ Steps

- Differentiate to get $f'(x) = 3x^2 - 12x + 4$
- Solve $3x^2 - 12x + 4 = 0$ to get stationary x
- Discard any stationary points outside $[-2, 5]$
- Eval f at remaining points plus end points -2 and 5
- Pick point giving largest value

```
from sympy import *
x = Symbol('x')
points = [-2, 5]
f = x**3 - 6*x**2 + 4*x + 8
fp = diff(f, x)
spoints = solve(fp, x)
points.extend(spoints)
v = [f.subs(x, c).evalf() for c in points]
maximizer = points[v.index(max(v))]
print("Maximizer =", str(maximizer),'=',maximizer.evalf())
```

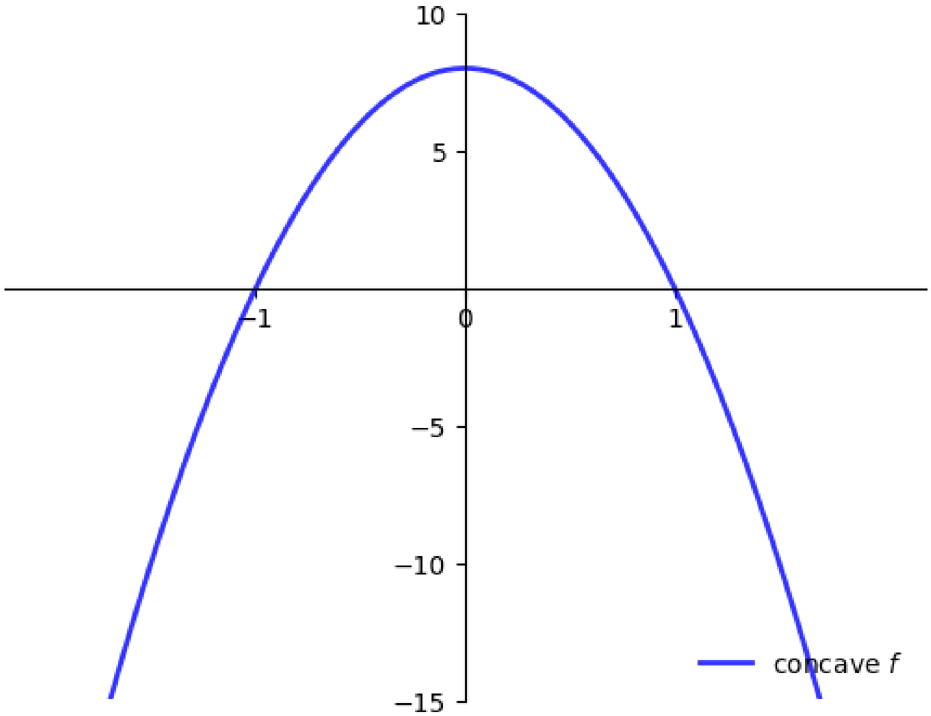
```
Maximizer = 2 - 2*sqrt(6)/3 = 0.367006838144548
```



2.6 Shape Conditions and Sufficiency

When is $f'(x^*) = 0$ sufficient for x^* to be a maximizer?

One answer: When f is concave



(Full definition deferred)

Sufficient conditions for *concavity* in one dimension

Let $f \colon [a, b] \to \mathbb{R}$

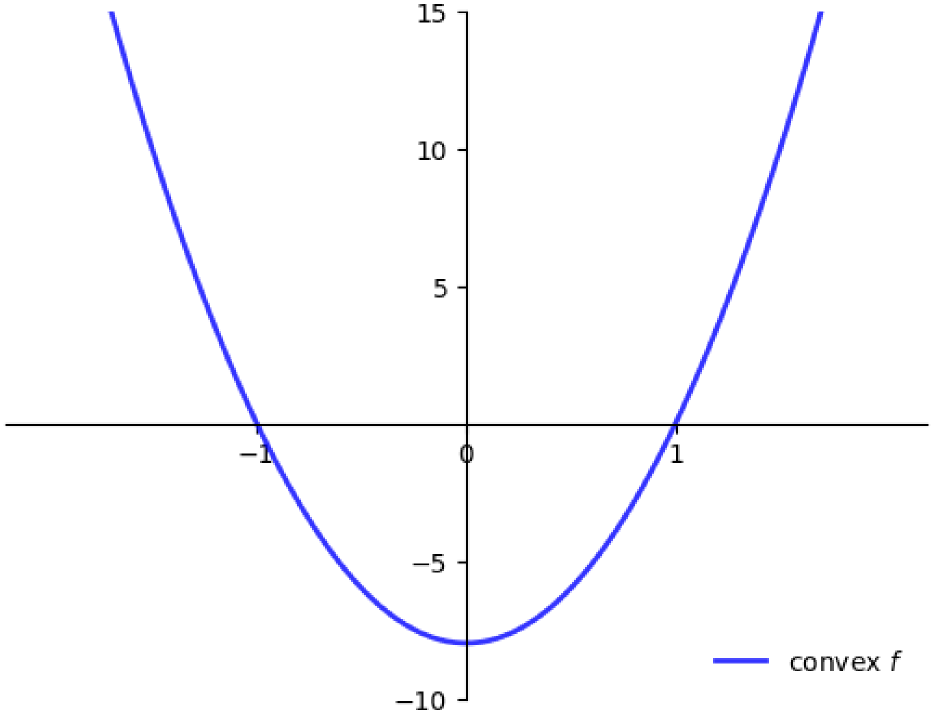
- If $f''(x) \leq 0$ for all $x \in (a, b)$ then f is concave on (a, b)
- If $f''(x) < 0$ for all $x \in (a, b)$ then f is **strictly** concave on (a, b)

Example

- $f(x) = a + bx$ is concave on $\mathbb{R}$ but not strictly
- $f(x) = \log(x)$ is strictly concave on $(0, \infty)$

When is $f'(x^*) = 0$ sufficient for x^* to be a minimizer?

One answer: When f is convex



(Full definition deferred)

Sufficient conditions for *convexity* in one dimension

Let $f : [a, b] \to \mathbb{R}$

- If $f''(x) \geq 0$ for all $x \in (a, b)$ then f is convex on (a, b)
- If $f''(x) > 0$ for all $x \in (a, b)$ then f is **strictly** convex on (a, b)

Example

- $f(x) = a + bx$ is convex on $\mathbb{R}$ but not strictly
- $f(x) = x^2$ is strictly convex on $\mathbb{R}$

2.6.1 Sufficiency and uniqueness with shape conditions

Fact

For maximizers:

- If $f : [a, b] \to \mathbb{R}$ is concave and $x^* \in (a, b)$ is stationary then x^* is a maximizer
- If, in addition, f is strictly concave, then x^* is the unique maximizer

Fact

For minimizers:

- If $f: [a,b] \to \mathbb{R}$ is convex and $x^* \in (a,b)$ is stationary then x^* is a minimizer
- If, in addition, f is strictly convex, then x^* is the unique minimizer

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Example

A price taking firm faces output price $p > 0$, input price $w > 0$

Maximize profits with respect to input ℓ

$$\max_{\ell \geq 0} \pi(\ell) = pf(\ell) - w\ell,$$

where the production technology is given by

$$f(\ell) = \ell^{\alpha}, 0 < \alpha < 1.$$

Evidently

$$\pi'(\ell) = \alpha p \ell^{\alpha - 1} - w,$$

so unique stationary point is

$$\ell^* = (\alpha p / w)^{1/(1-\alpha)}$$

Moreover,

$$\pi''(\ell) = \alpha(\alpha - 1) p \ell^{\alpha - 2} < 0$$

for all $\ell \geq 0$ so ℓ^* is unique maximizer.

2.7 Functions of two variables

Let's have a look at some functions of two variables

- How to visualize them
- Slope, contours, etc.

Example: Cobb-Douglas production function

Consider production function

$$f(k, \ell) = k^{\alpha} \ell^{\beta}$$
$$\alpha \geq 0, \; \beta \geq 0, \; \alpha + \beta < 1$$

Let's graph it in two dimensions.

Like many 3D plots it's hard to get a good understanding

Let's try again with contours plus heat map

In this context the contour lines are called ***isoquants***

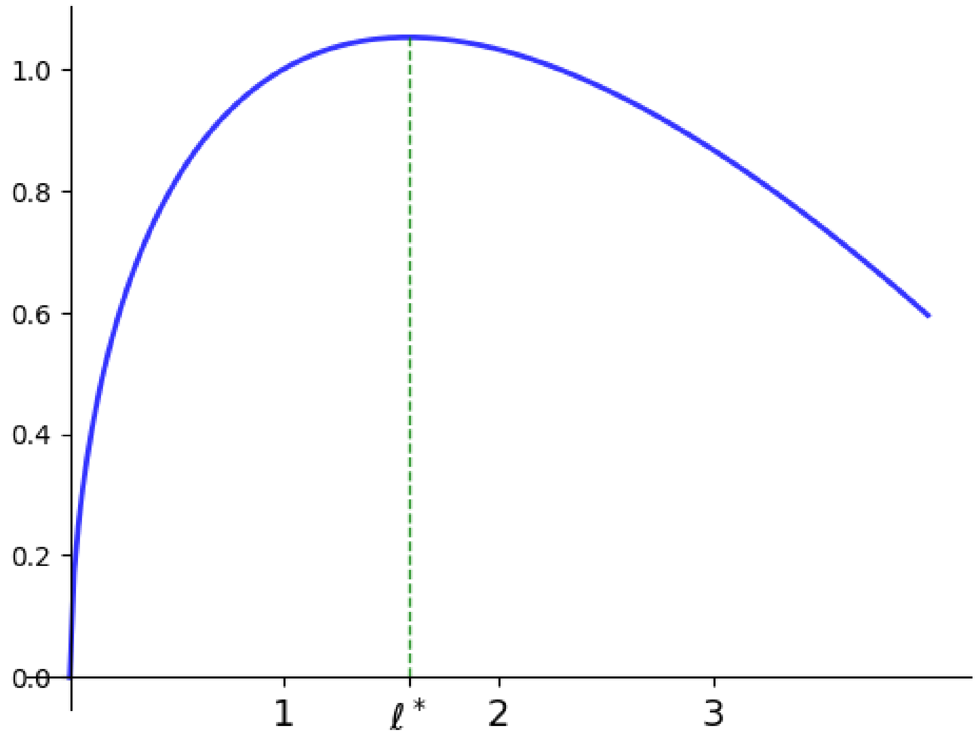

Fig. 2.5: Profit maximization with $p = 2$, $w = 1$, $\alpha = 0.6$, ℓ^* =1.5774

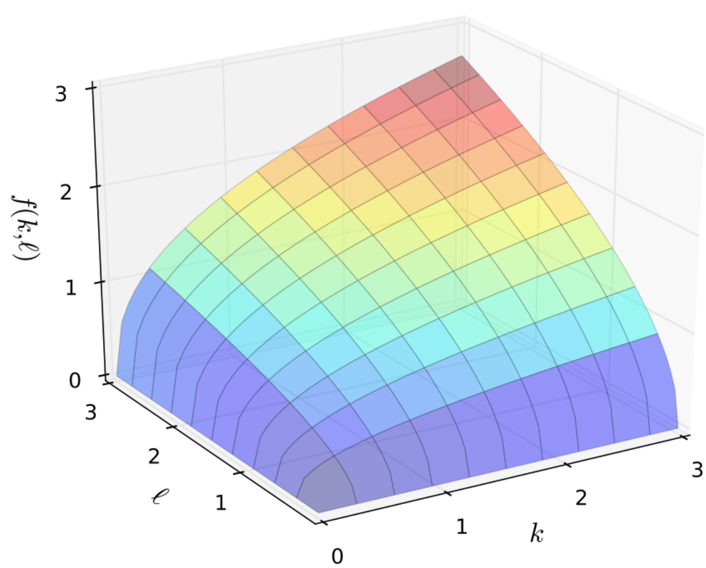

Fig. 2.6: Production function with $\alpha = 0.4$, $\beta = 0.5$ (a)

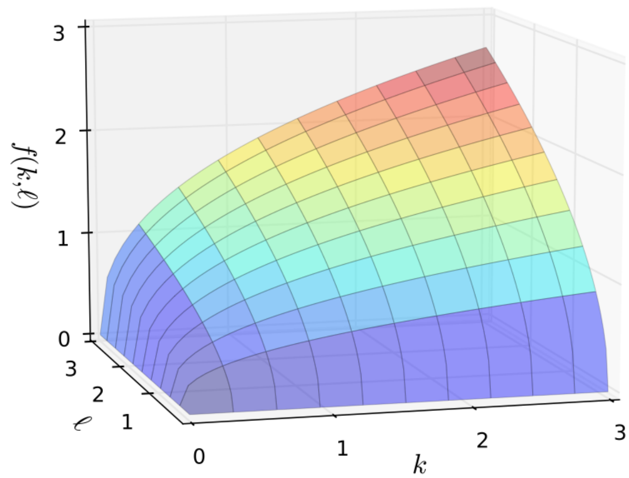

Fig. 2.7: Production function with $\alpha = 0.4$, $\beta = 0.5$ (b)

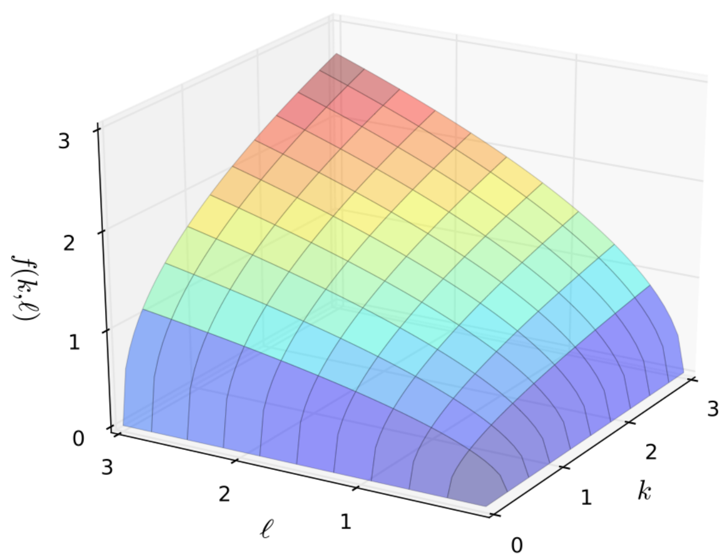

Fig. 2.8: Production function with $\alpha = 0.4$, $\beta = 0.5$ (c)

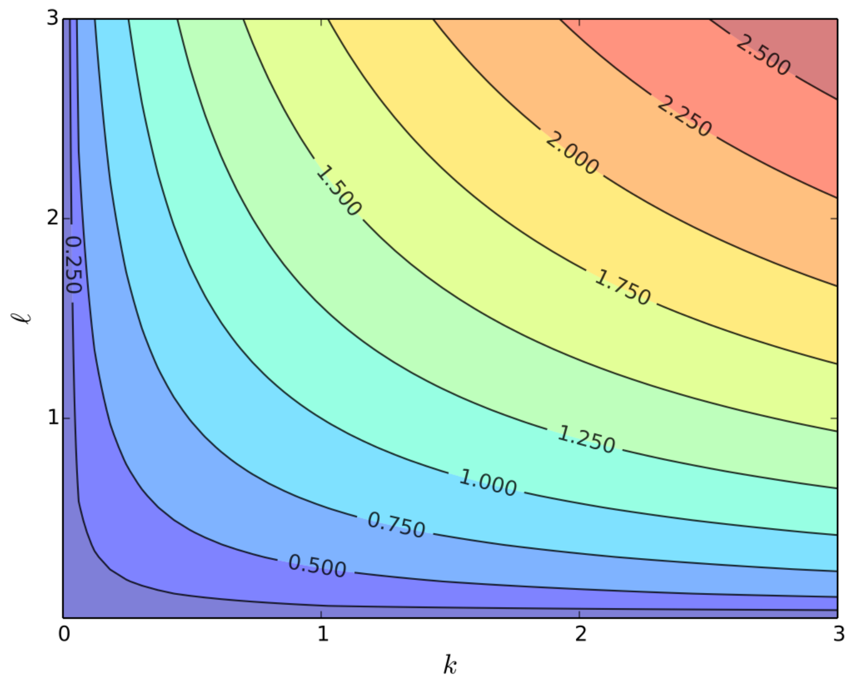

Fig. 2.9: Production function with $\alpha = 0.4$, $\beta = 0.5$, contours

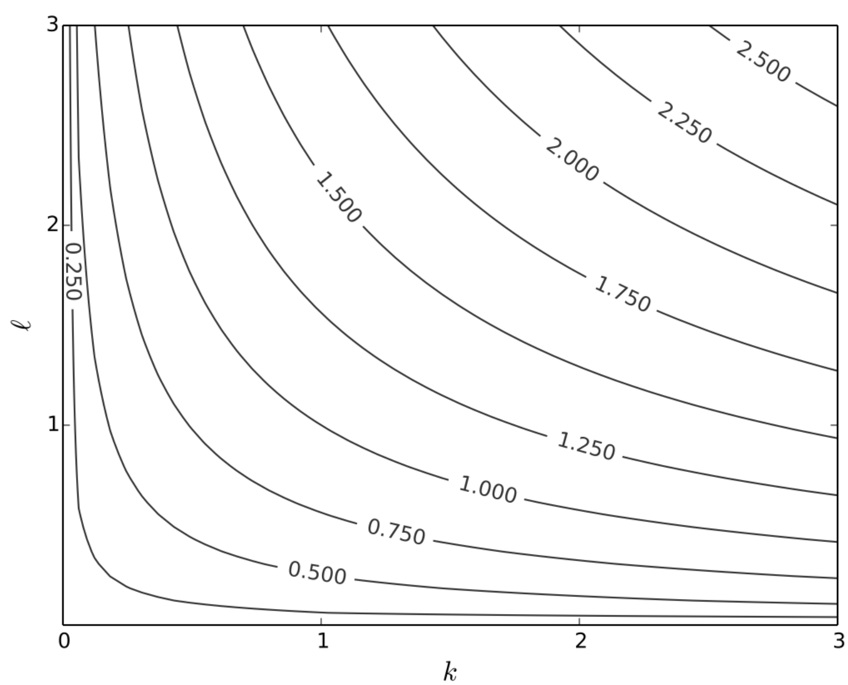

Fig. 2.10: Production function with $\alpha = 0.4$, $\beta = 0.5$

Can you see how $\alpha < \beta$ shows up in the slope of the contours?

We can drop the colours to see the numbers more clearly

Example: log-utility

Let $u(x_1, x_2)$ be "utility" gained from x_1 units of good 1 and x_2 units of good 2

We take

$$u(x_1, x_2) = \alpha \log(x_1) + \beta \log(x_2)$$

where

- α and β are parameters
- we assume $\alpha > 0, \ \beta > 0$
- The log functions mean "diminishing returns" in each good

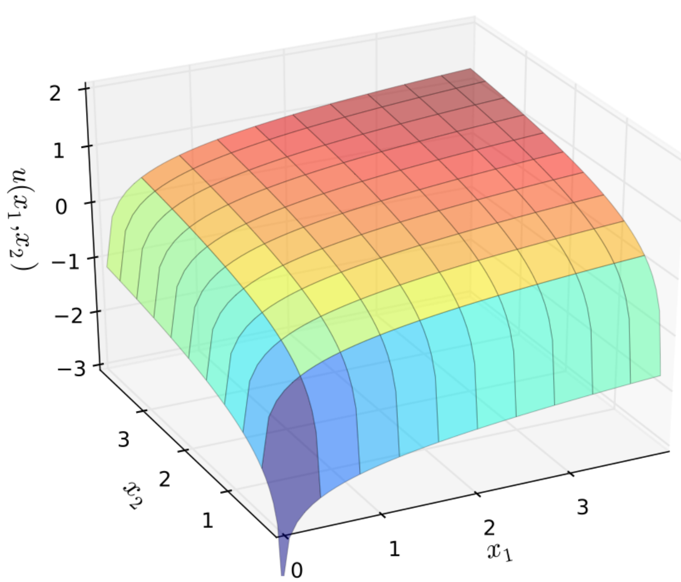

Fig. 2.11: Log utility with $\alpha = 0.4$, $\beta = 0.5$

Let's look at the contour lines

For utility functions, contour lines called ***indifference curves***

Example: quasi-linear utility

$$u(x_1, x_2) = x_1 + \log(x_2)$$

- Called quasi-linear because linear in good 1

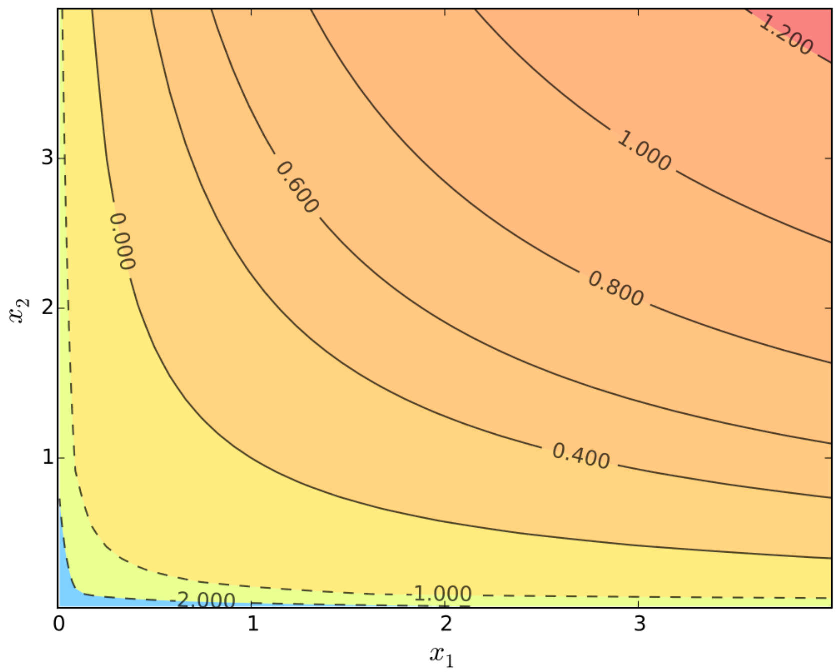

Fig. 2.12: Indifference curves of log utility with $\alpha = 0.4$, $\beta = 0.5$

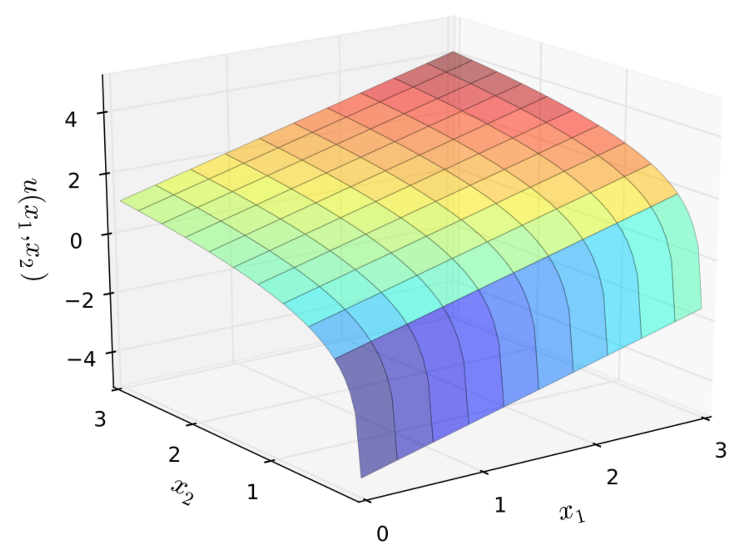

Fig. 2.13: Quasi-linear utility

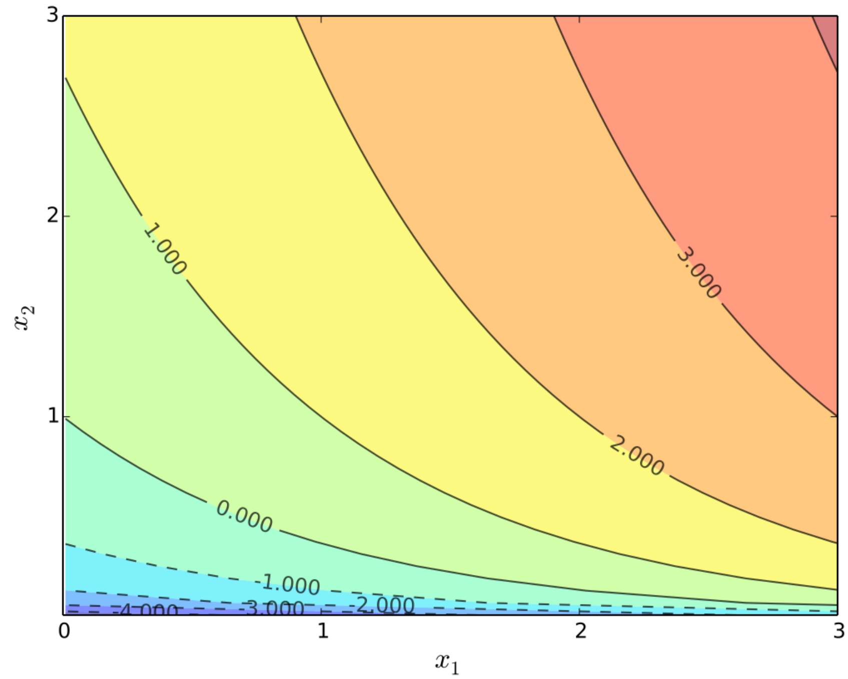

Fig. 2.14: Indifference curves of quasi-linear utility

Example: quadratic utility

$$u(x_1, x_2) = -(x_1 - b_1)^2 - (x_2 - b_2)^2$$

Here

- b_1 is a "satiation" or "bliss" point for x_1
- b_2 is a "satiation" or "bliss" point for x_2

Dissatisfaction increases with deviations from the bliss points

2.8 Bivariate Optimization

Consider $f: I \to \mathbb{R}$ where $I \subset \mathbb{R}^2$

The set $\mathbb{R}^2$ is all (x_1, x_2) pairs

Definition

A point $(x_1^*, x_2^*) \in I$ is called a ***maximizer*** of f on I if

$$f(x_1^*, x_2^*) \geq f(x_1, x_2) \quad \text{for all} \quad (x_1, x_2) \in I$$

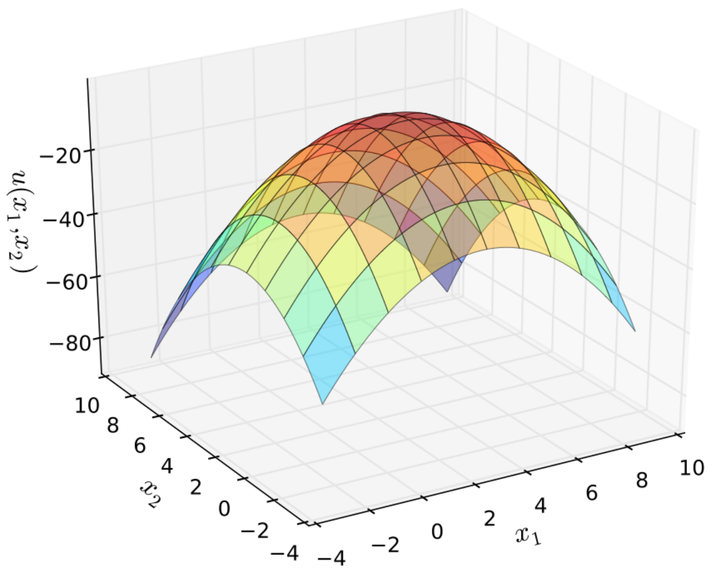

Fig. 2.15: Quadratic utility with $b_1 = 3$ and $b_2 = 2$

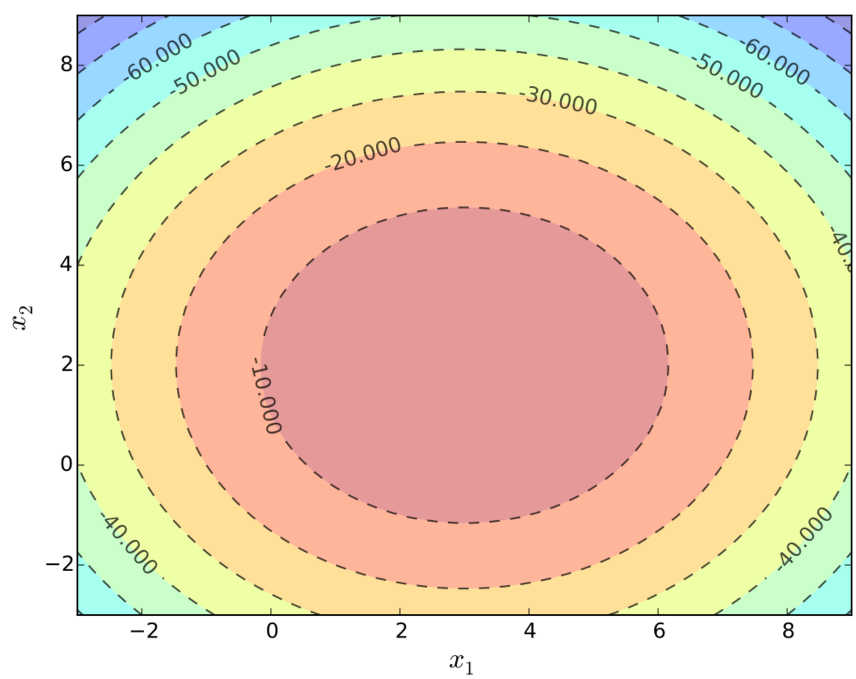

Fig. 2.16: Indifference curves quadratic utility with $b_1 = 3$ and $b_2 = 2$

Definition

A point $(x_1^*, x_2^*) \in I$ is called a ***minimizer*** of f on I if $f(x_1^*, x_2^*) \leq f(x_1, x_2) \quad$ for all $\quad (x_1, x_2) \in I$

When they exist, the partial derivatives at $(x_1, x_2) \in I$ are

$$f_1(x_1, x_2) = \frac{\partial}{\partial x_1} f(x_1, x_2)$$
$$f_2(x_1, x_2) = \frac{\partial}{\partial x_2} f(x_1, x_2)$$

Example

When $f(k, \ell) = k^\alpha \ell^\beta$,

$$f_1(k, \ell) = \frac{\partial}{\partial k} f(k, \ell) = \frac{\partial}{\partial k} k^\alpha \ell^\beta = \alpha k^{\alpha - 1} \ell^\beta$$

Definition

An interior point $(x_1, x_2) \in I$ is called ***stationary*** for f if

$$f_1(x_1, x_2) = f_2(x_1, x_2) = 0$$

Fact

Let $f \colon I \to \mathbb{R}$ be a continuously differentiable function. If (x_1^*, x_2^*) is either

- an interior maximizer of f on I, or
- an interior minimizer of f on I,

then (x_1^*, x_2^*) is a stationary point of f

Usage, for maximization:

1. Compute partials
2. Set partials to zero to find $S =$ all stationary points
3. Evaluate candidates in S and boundary of I
4. Select point (x_1^*, x_2^*) yielding highest value

Example

$$f(x_1, x_2) = x_1^2 + 4x_2^2 \to \min \quad \text{s.t.} \quad x_1 + x_2 \leq 1$$

Setting

$$f_1(x_1, x_2) = 2x_1 = 0 \quad \text{and} \quad f_2(x_1, x_2) = 8x_2 = 0$$

gives the unique stationary point $(0, 0)$, at which $f(0, 0) = 0$

On the boundary we have $x_1 + x_2 = 1$, so

$$f(x_1, x_2) = f(x_1, 1 - x_1) = x_1^2 + 4(1 - x_1)^2$$

Exercise: Show right hand side > 0 for any x_1

Hence minimizer is $(x_1^*, x_2^*) = (0, 0)$

2.8.1 Nasty secrets

Solving for (x_1, x_2) such that $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$ can be hard

- System of nonlinear equations
- Might have no analytical solution
- Set of solutions can be a continuum

Example

(Don't) try to find all stationary points of

$$f(x_1, x_2) = \frac{\cos(x_1^2 + x_2^2) + x_1^2 + x_1}{2 + p(-x_1^2) + \sin^2(x_2)}$$

Also:

- Boundary is often a continuum, not just two points
- Things get even harder in higher dimensions

On the other hand:

- Most classroom examples are chosen to avoid these problems
- Life is still pretty easy if we have concavity / convexity
- Clever tricks have been found for certain kinds of problems

2.9 Second Order Partials

Let $f : I \to \mathbb{R}$ and, when they exist, denote

$$f_{11}(x_1, x_2) = \frac{\partial^2}{\partial x_1^2} f(x_1, x_2)$$

$$f_{12}(x_1, x_2) = \frac{\partial^2}{\partial x_1 \partial x_2} f(x_1, x_2)$$

$$f_{21}(x_1, x_2) = \frac{\partial^2}{\partial x_2 \partial x_1} f(x_1, x_2)$$

$$f_{22}(x_1, x_2) = \frac{\partial^2}{\partial x_2^2} f(x_1, x_2)$$

Example: Cobb-Douglas technology with linear costs

If $\pi(k, \ell) = pk^{\alpha}\ell^{\beta} - w\ell - rk$ then

$$\pi_{11}(k, \ell) = p\alpha(\alpha - 1)k^{\alpha-2}\ell^{\beta}$$

$$\pi_{12}(k, \ell) = p\alpha\beta k^{\alpha-1}\ell^{\beta-1}$$

$$\pi_{21}(k, \ell) = p\alpha\beta k^{\alpha-1}\ell^{\beta-1}$$

$$\pi_{22}(k, \ell) = p\beta(\beta - 1)k^{\alpha}\ell^{\beta-2}$$

Fact

If $f: I \to \mathbb{R}$ is twice continuously differentiable at (x_1, x_2), then $f_{12}(x_1, x_2) = f_{21}(x_1, x_2)$

Exercise: Confirm the results in the exercise above.

2.10 Shape conditions in 2D

Let I be an "open" set (only interior points – formalities next week)

Let $f: I \to \mathbb{R}$ be twice continuously differentiable

The function f is strictly **concave** on I if, for any $(x_1, x_2) \in I$

1. $f_{11}(x_1, x_2) < 0$
2. $f_{11}(x_1, x_2)\, f_{22}(x_1, x_2) > f_{12}(x_1, x_2)^2$

The function f is strictly **convex** on I if, for any $(x_1, x_2) \in I$

1. $f_{11}(x_1, x_2) > 0$
2. $f_{11}(x_1, x_2)\, f_{22}(x_1, x_2) > f_{12}(x_1, x_2)^2$

When is stationarity sufficient?

Fact

If f is differentiable and strictly concave on I, then any stationary point of f is also a unique maximizer of f on I

Fact

If f is differentiable and strictly convex on I, then any stationary point of f is also a unique minimizer of f on I

Example: unconstrained maximization of quadratic utility

$$u(x_1, x_2) = -(x_1 - b_1)^2 - (x_2 - b_2)^2 \to \max_{x_1, x_2}$$

Intuitively the solution is $x_1^* = b_1$ and $x_2^* = b_2$

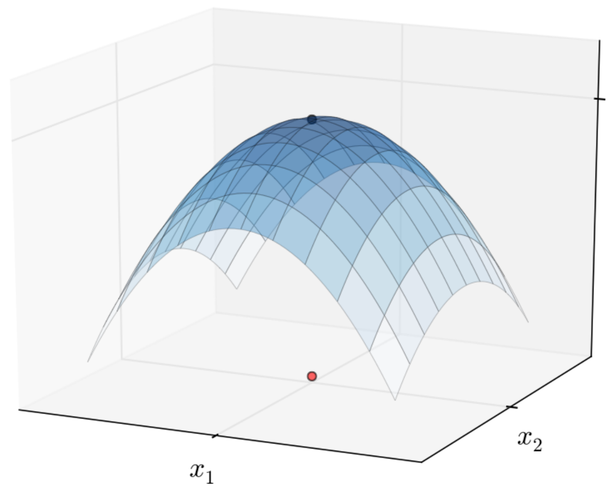

Fig. 2.17: Maximizer of a concave function

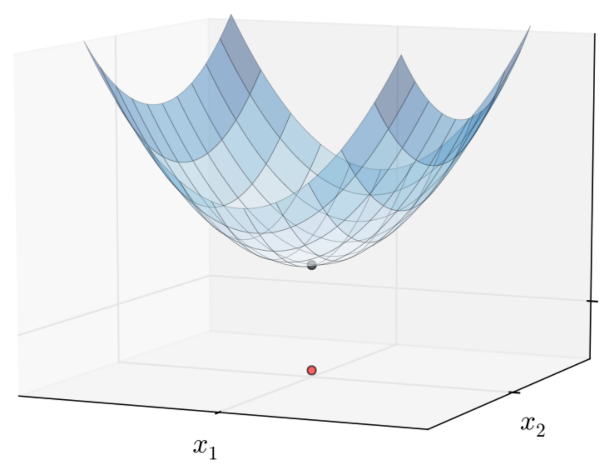

Fig. 2.18: Minimizer of a convex function

Analysis above leads to the same conclusion

First let's check first order conditions (*F.O.C.*)

$$\frac{\partial}{\partial x_1} u(x_1, x_2) = -2(x_1 - b_1) = 0 \quad \Longrightarrow \quad x_1 = b_1$$

$$\frac{\partial}{\partial x_2} u(x_1, x_2) = -2(x_2 - b_2) = 0 \quad \Longrightarrow \quad x_2 = b_2$$

How about (strict) concavity?

Sufficient condition is

1. $u_{11}(x_1, x_2) < 0$
2. $u_{11}(x_1, x_2) u_{22}(x_1, x_2) > u_{12}(x_1, x_2)^2$

We have

- $u_{11}(x_1, x_2) = -2$
- $u_{11}(x_1, x_2) u_{22}(x_1, x_2) = 4 > 0 = u_{12}(x_1, x_2)^2$

Example: Profit maximization with two inputs

$$\pi(k, \ell) = p k^\alpha \ell^\beta - w\ell - rk \to \max_{k, \ell}$$

where α, β, p, w are all positive and $\alpha + \beta < 1$

Derivatives:

- $\pi_1(k, \ell) = p\alpha k^{\alpha-1} \ell^\beta - r$
- $\pi_2(k, \ell) = p\beta k^\alpha \ell^{\beta-1} - w$
- $\pi_{11}(k, \ell) = p\alpha(\alpha - 1) k^{\alpha-2} \ell^\beta$
- $\pi_{22}(k, \ell) = p\beta(\beta - 1) k^\alpha \ell^{\beta-2}$
- $\pi_{12}(k, \ell) = p\alpha\beta k^{\alpha-1} \ell^{\beta-1}$

First order conditions: set

$$\pi_1(k, \ell) = 0$$
$$\pi_2(k, \ell) = 0$$

and solve simultaneously for k, ℓ to get

$$k^* = \left[p(\alpha/r)^{1-\beta}(\beta/w)^\beta\right]^{1/(1-\alpha-\beta)}$$
$$\ell^* = \left[p(\beta/w)^{1-\alpha}(\alpha/r)^\alpha\right]^{1/(1-\alpha-\beta)}$$

Exercise: Verify

Now we check second order conditions, hoping for strict concavity

What we need: for any $k, \ell > 0$

1. $\pi_{11}(k, \ell) < 0$
2. $\pi_{11}(k, \ell)\, \pi_{22}(k, \ell) > \pi_{12}(k, \ell)^2$

Exercise: Show both inequalities satisfied when $\alpha + \beta < 1$

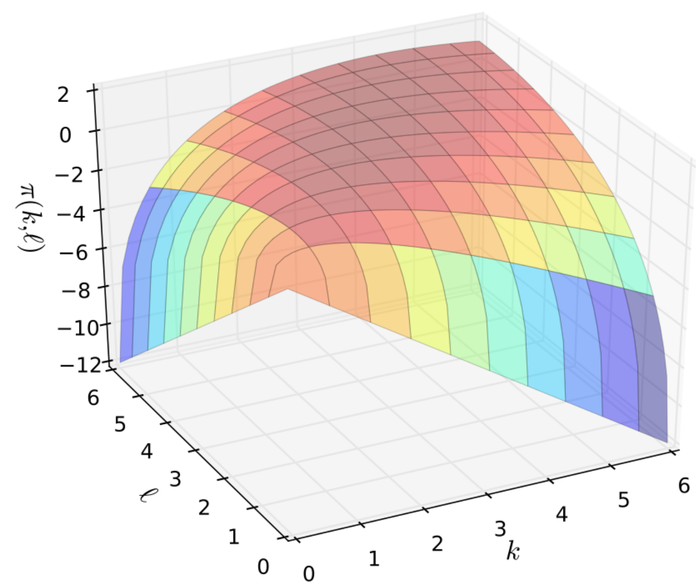

Fig. 2.19: Profit function when $p = 5$, $r = w = 2$, $\alpha = 0.4$, $\beta = 0.5$

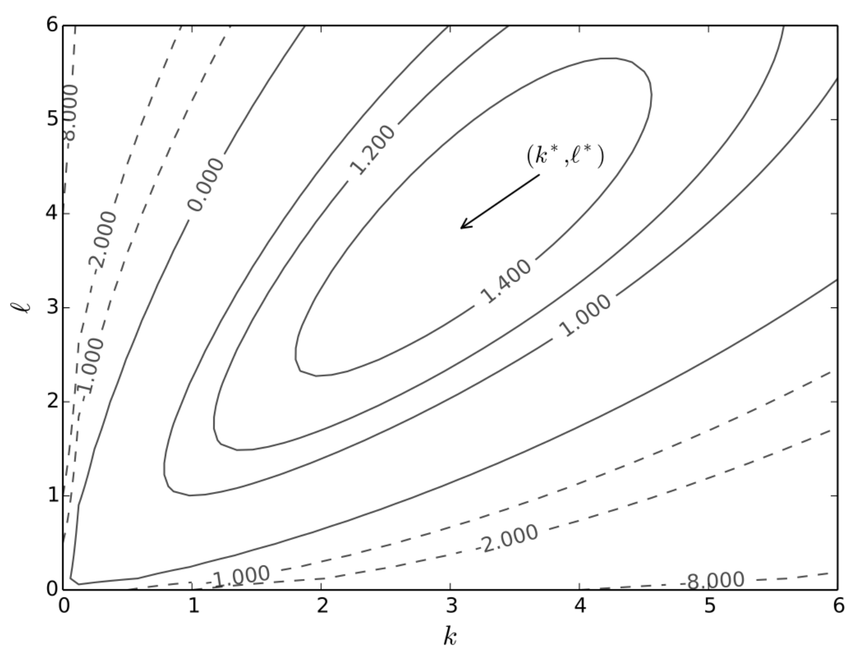

Fig. 2.20: Optimal choice, $p = 5$, $r = w = 2$, $\alpha = 0.4$, $\beta = 0.5$

CHAPTER

THREE

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COMING SOON

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