Heat equation weak formulation

1 Time dependent heat equation

$$\rho c_p \frac{\partial T}{\partial t} = \nabla \cdot (\kappa(T)\nabla T) + \sigma(T)(\nabla\varphi)^2 \tag{1.1}$$

$$\vec{n} \cdot (\kappa(T)\nabla T) = Q(T) \text{ on } \Omega \tag{1.2}$$

2 Euler implicit

Assuming that conductivities have only previous time step values (otherwise the problem would be non-linear) and $\gamma = \rho c_p/\Delta t$:

$$\gamma(T_{n+1} - T_n) = \nabla \cdot (\kappa(T_n)\nabla T_{n+1}) + \sigma(T_n)(\nabla\varphi_{n+1})^2 \tag{2.1}$$

Weak formulation (multiply by test function ϕ_i and integrate over the domain)

$$\gamma \int \phi_i(T_{n+1} - T_n) = \int \phi_i \nabla \cdot (\kappa(T_n)\nabla T_{n+1}) + \int \phi_i \sigma(T_n)(\nabla\varphi_{n+1})^2$$

$$\gamma \int \phi_i(T_{n+1} - T_n) = \oint \phi_i \vec{n} \cdot (\kappa(T_n)\nabla T_{n+1}) - \int \nabla\phi_i \cdot (\kappa(T_n)\nabla T_{n+1}) + \int \phi_i \sigma(T_n)(\nabla\varphi_{n+1})^2$$

$$\gamma \int \phi_i T_{n+1} + \int \nabla\phi_i \cdot (\kappa(T_n)\nabla T_{n+1}) = \gamma \int \phi_i T_n + \oint \phi_i Q(T_{n+1}) + \int \phi_i \sigma(T_n)(\nabla\varphi_{n+1})^2$$

The heat boundary condition $Q(T_{n+1})$ should be set to $Q(T_n)$, otherwise the problem still remains non-linear.

3 Crank Nicolson

Assuming that conductivities have only previous time step values (otherwise the problem would be non-linear) and $\gamma = \rho c_p/\Delta t$:

$$\gamma(T_{n+1} - T_n) = \frac{1}{2}(\nabla \cdot (\kappa(T_n)\nabla T_{n+1}) + \sigma(T_n)(\nabla\varphi_{n+1})^2 + \nabla \cdot (\kappa(T_n)\nabla T_n) + \ldots$$
$$\ldots + \sigma(T_n)(\nabla\varphi_n)^2) \tag{3.1}$$

$$2\gamma(T_{n+1} - T_n) = \nabla \cdot (\kappa(T_n)\nabla T_{n+1} + \kappa(T_n)\nabla T_n) + \sigma(T_n)((\nabla\varphi_{n+1})^2 + (\nabla\varphi_n)^2) \tag{3.2}$$

Weak formulation (multiply by test function ϕ_i and integrate over the domain)

$$2\gamma \int \phi_i (T_{n+1} - T_n) = \int \phi_i \nabla \cdot (\kappa(T_n) \nabla T_{n+1} + \kappa(T_n) \nabla T_n) + \dots$$
$$\dots + \int \phi_i \sigma(T_n)((\nabla \varphi_{n+1})^2 (\nabla \varphi_n)^2) \quad (3.3)$$

$$2\gamma \int \phi_i (T_{n+1} - T_n) = \oint \phi_i \vec{n} \cdot (\kappa(T_n) \nabla T_{n+1} + \kappa(T_n) \nabla T_n) - \dots$$
$$\dots - \int \nabla \phi_i \cdot (\kappa(T_n) \nabla T_{n+1} + \kappa(T_n) \nabla T_n) + \int \phi_i \sigma(T_n)((\nabla \varphi_{n+1})^2 + (\nabla \varphi_n)^2) \quad (3.4)$$

$$2\gamma \int \phi_i T_{n+1} + \int \nabla \phi_i \cdot (\kappa(T_n) \nabla T_{n+1}) = 2\gamma \int \phi_i T_n + \oint \phi_i (Q(T_n) + Q(T_{n+1})) - \dots$$
$$\dots - \int \nabla \phi_i \cdot (\kappa(T_n) \nabla T_n) + \int \phi_i \sigma(T_n)((\nabla \varphi_{n+1})^2 + (\nabla \varphi_n)^2) \quad (3.5)$$

The heat boundary condition $Q(T_{n+1})$ should be set to $Q(T_n)$, otherwise the problem still remains non-linear.