Behavioural Equivalences

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Process Algebras and Concurrent Systems

Behavioural Equivalence

Implementation

$$CM \stackrel{\text{def}}{=} coin.\overline{coffee}.CM$$

$$PR \stackrel{\text{def}}{=} \overline{hello}.\overline{coin}.coffee.\overline{drink}.PR$$

$$UNI \stackrel{\text{def}}{=} (CM \,|\, PR) \setminus \{coin, coffee\}$$

Behavioural Equivalence

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Specification

$$Spec \stackrel{\text{def}}{=} \overline{hello}.\tau.\tau.\overline{drink}.Spec$$

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$$Spec \stackrel{\text{def}}{=} \overline{hello}.\tau.\tau.\overline{drink}.Spec$$

Question

Are the processes *Uni* and *Spec* "behaviourally equivalent"?

$$Uni \equiv Spec$$

Goals

What should a reasonable behavioural equivalence satisfy?

- Abstract from states (consider only the behaviour – actions)
- Abstract from nondeterminism
- Abstract from internal behaviour

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- Abstract from nondeterminism
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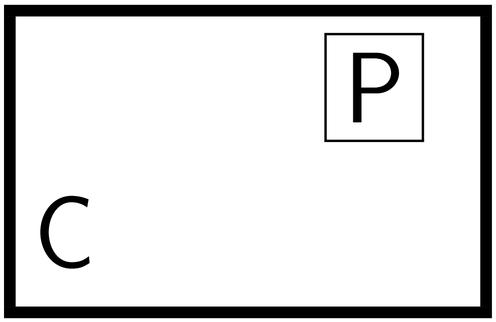
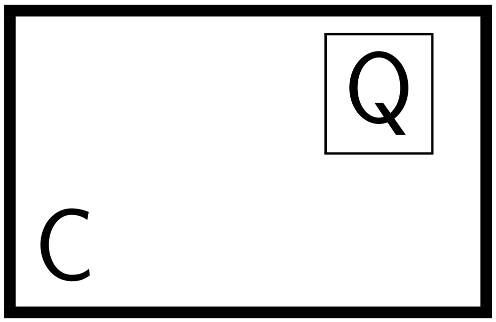
What else should a reasonable behavioural equivalence satisfy?

- Reflexivity: $P \equiv P$ for each process P
- Transitivity: $Spec_0 \equiv Spec_1 \equiv Spec_2 \equiv \cdots \equiv Impl$ gives that
$$Spec_0 \equiv Impl$$
- Symmetry: $P \equiv Q$ iff $Q \equiv P$

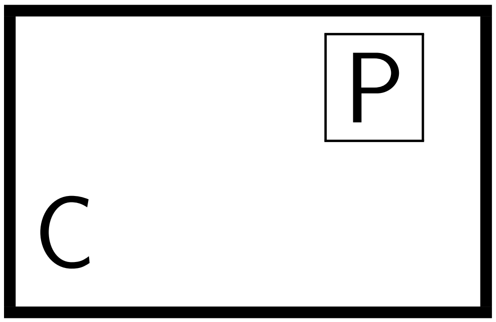


$C(P)$



$C(Q)$

Congruence



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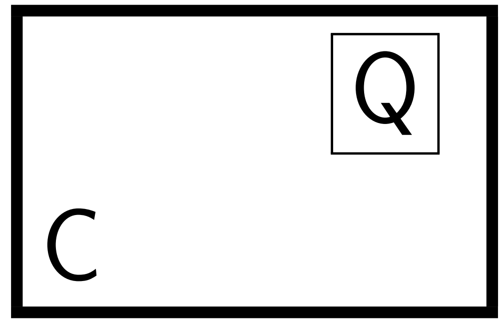
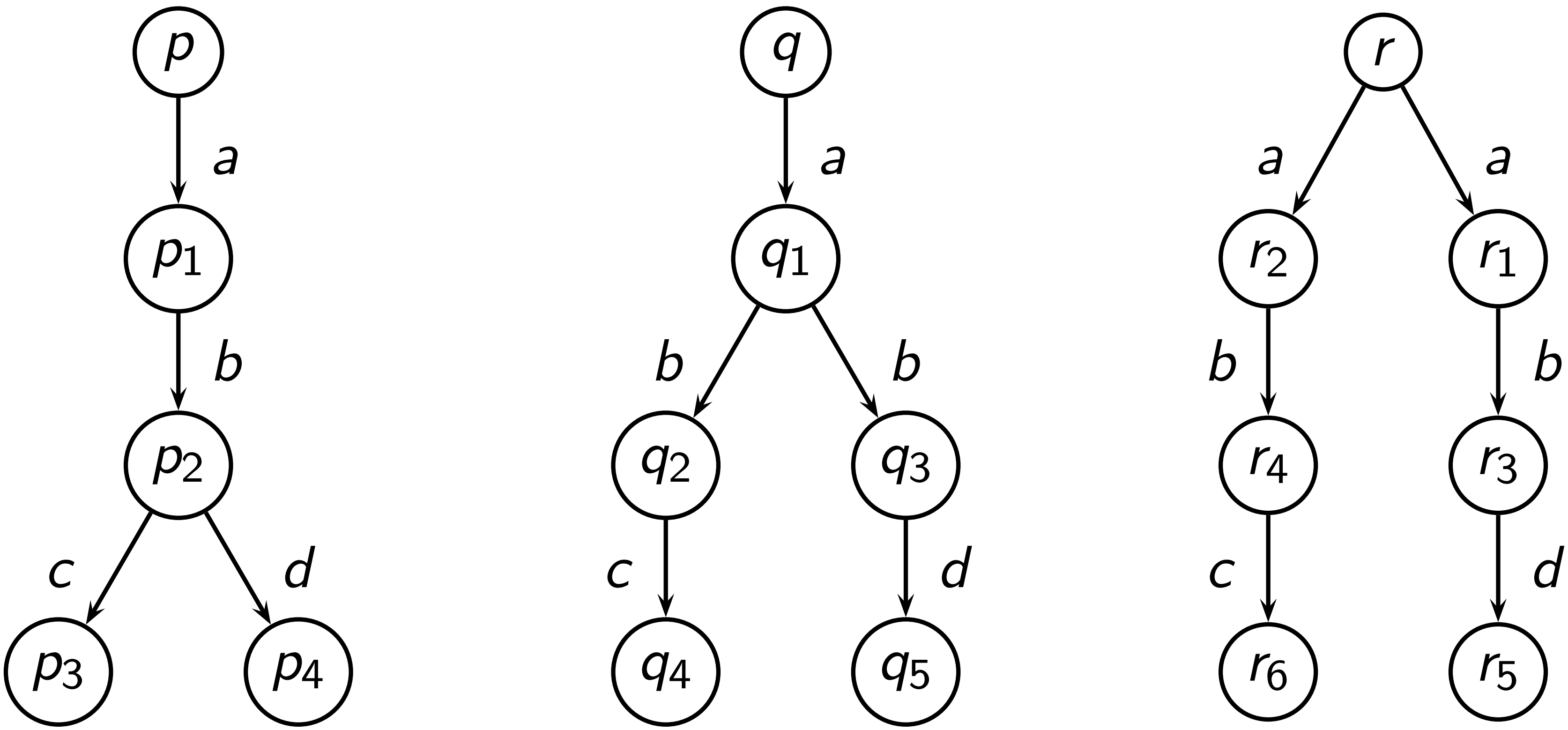


$C(Q)$

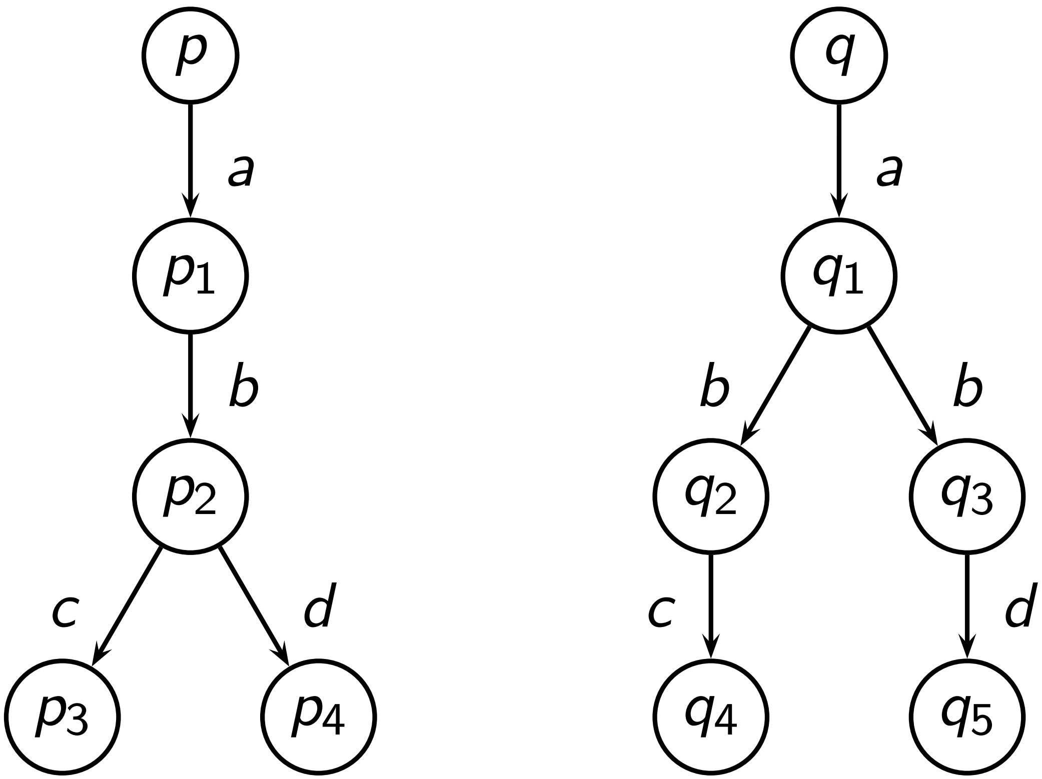
Congruence Property

$P \equiv Q$ implies that $C(P) \equiv C(Q)$

Behavioural Equivalences



Behavioural Equivalences



Problem: Are these three systems equivalent?

Traces/Language Equivalence

Let $\langle Q, A, \longrightarrow \rangle$ be an LTS, with $q \in Q$ and $s \in A^*$.

Traces

1. s is a *trace* of q if there exists $q' \in Q$ s.t. $q \xrightarrow{s} q'$.
2. $T(q)$ represents the set of all traces of q

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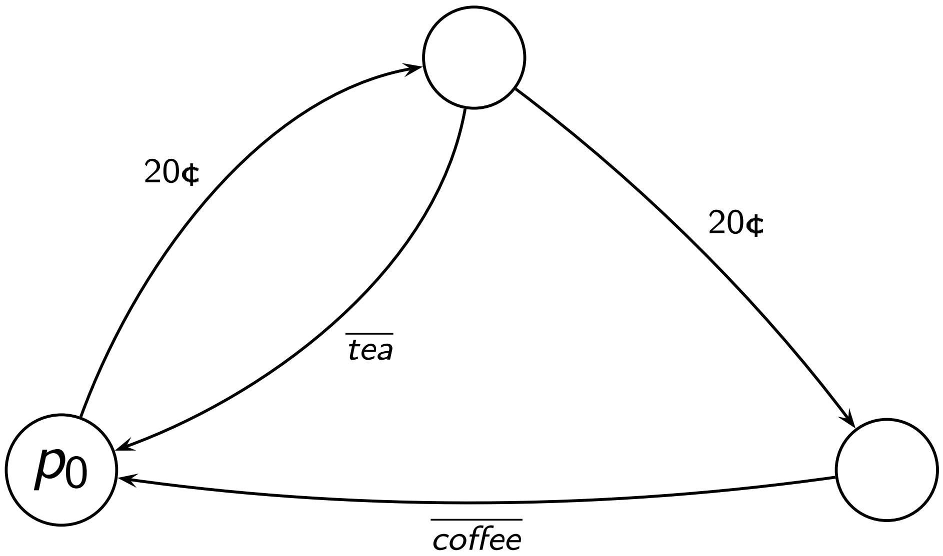
Traces Equivalence

Two states p e q are *trace equivalent*, written $p =_T q$, if $T(p) = T(p)$.

Two Traces Equivalent Systems



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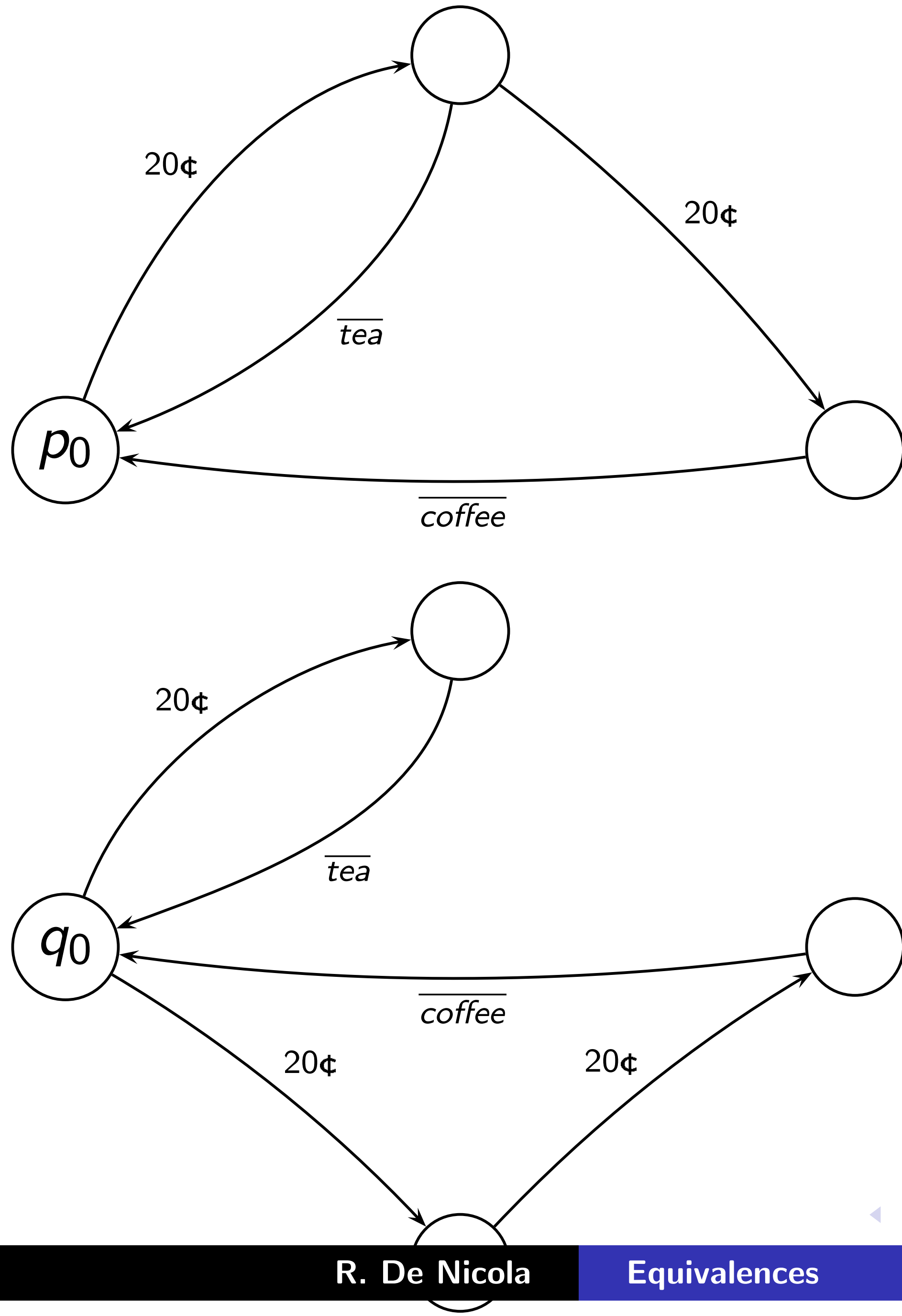

Bisimulation Relation

Strong Bisimulation

A relation $R \subseteq Q \times Q$ is *strong bisimulation* if, for any pair of states p e q such that $\langle p, q \rangle \in R$, the following holds:

1. for all $a \in A$ e $p' \in Q$, if $p \xrightarrow{a} p'$ then $q \xrightarrow{a} q'$ for some $q' \in Q$ such that $\langle p', q' \rangle \in R$;
2. for all $a \in A$ e $q' \in Q$, if $q \xrightarrow{a} q'$ then $p \xrightarrow{a} p'$ for some $p' \in Q$ such that $\langle p', q' \rangle \in R$.

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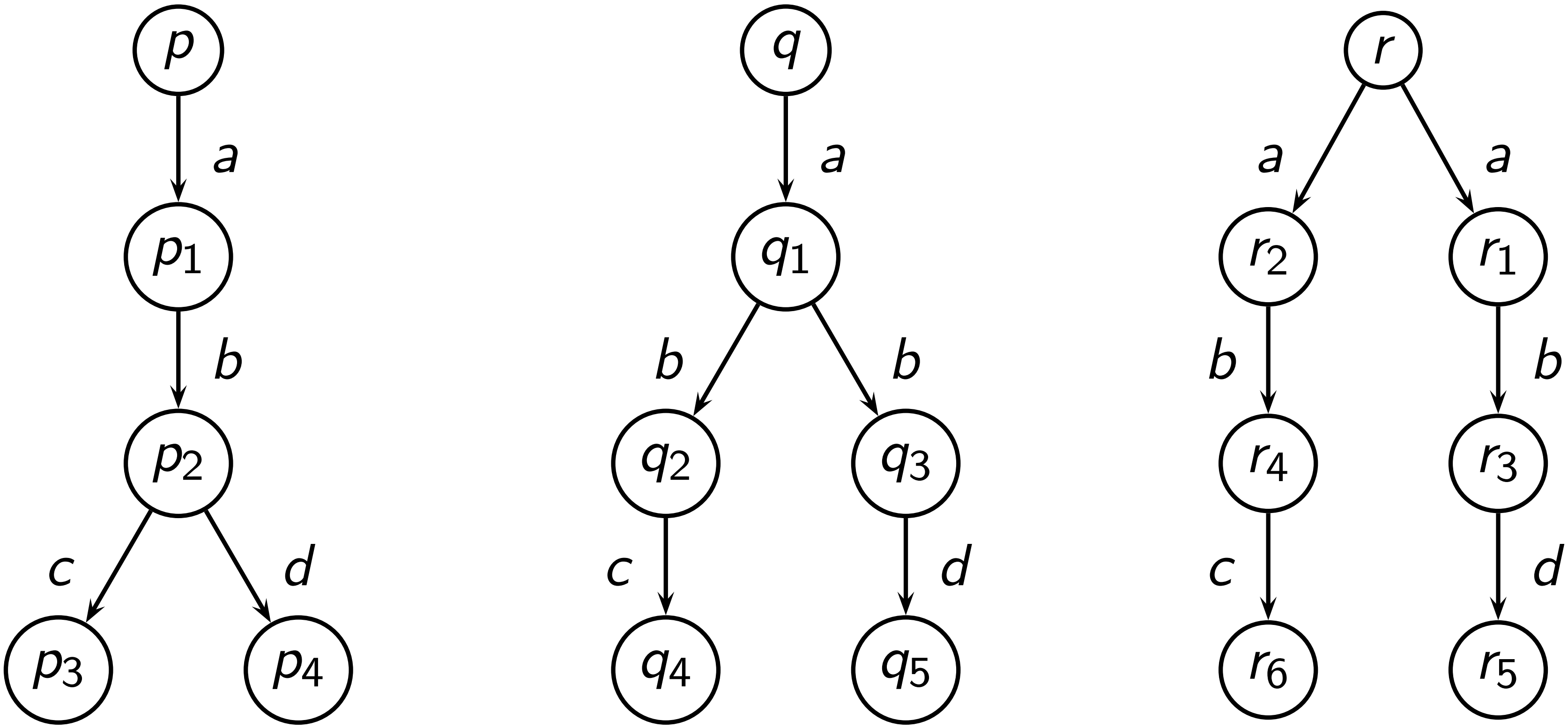
Bisimilarity

Two states $p, q \in Q$ are strongly *bisimilar*, written $p \sim q$, if there exists a strong bisimulation R such that $\langle p, q \rangle \in R$.

$$\sim \; = \bigcup \{R \mid R \text{ is a strong bisimulation}\}$$


p
a
p_1
b
p_2
c
d
p_3
p_4
q
a
q_1
b
b
q_2
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c
d
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c
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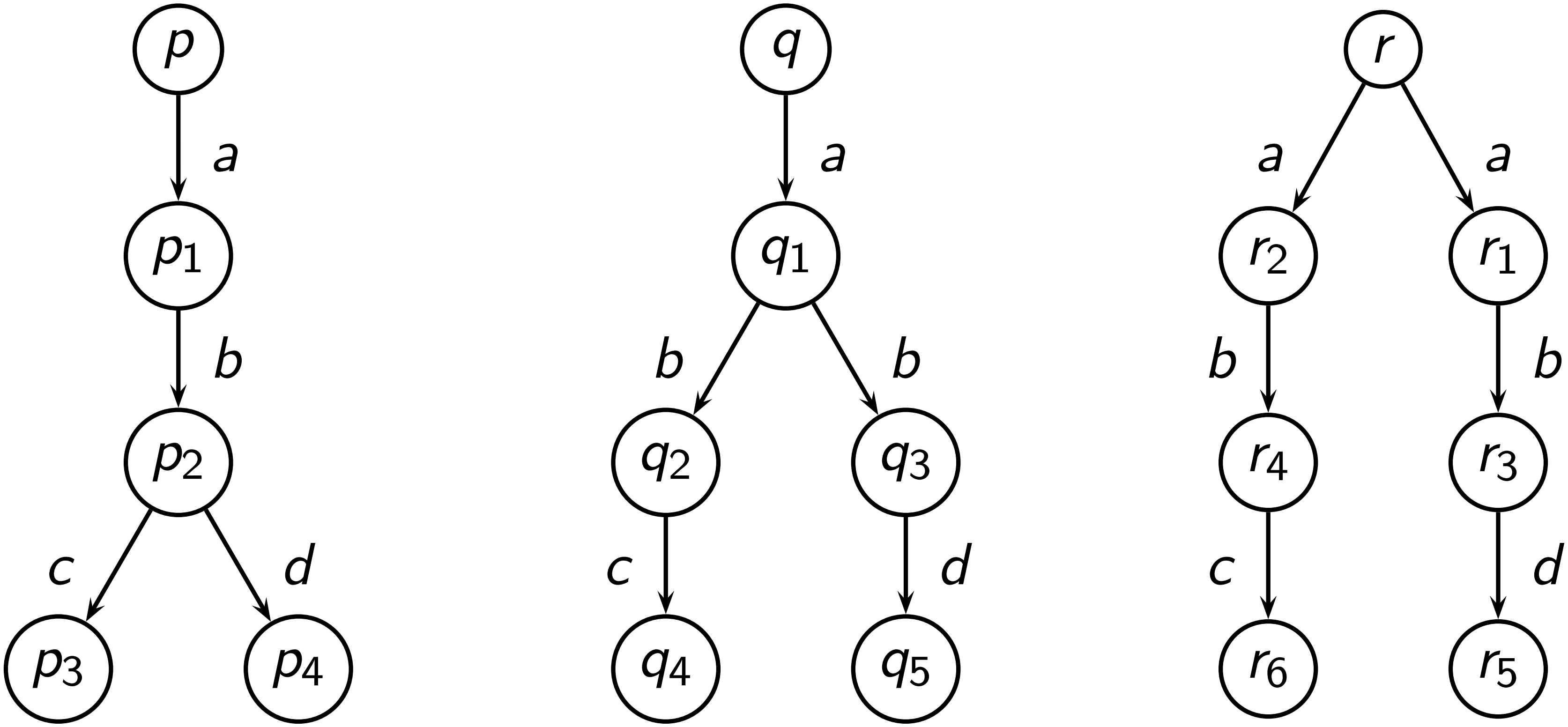
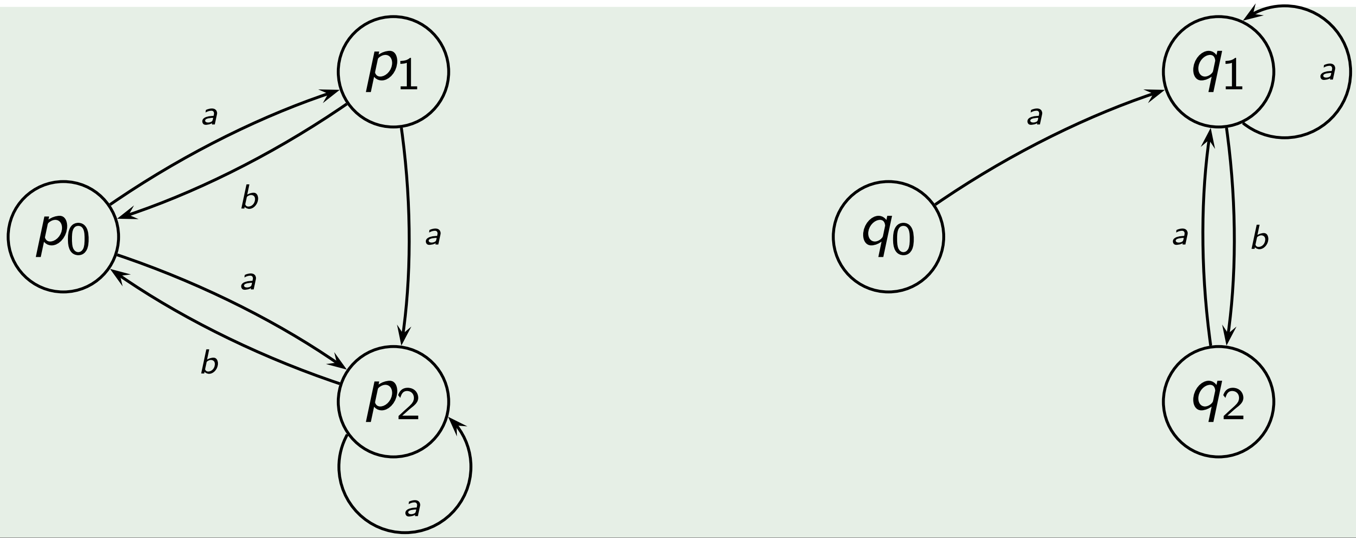
r_6
r_5



These three systems are not bisimulation equivalent

Two bisimilar Systems



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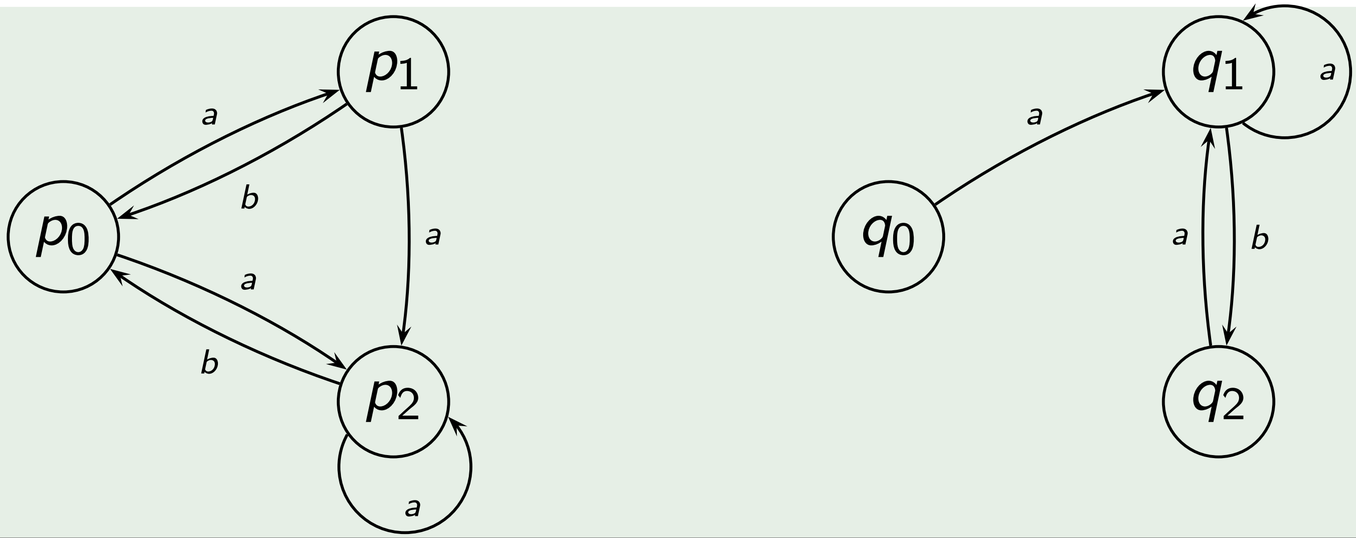
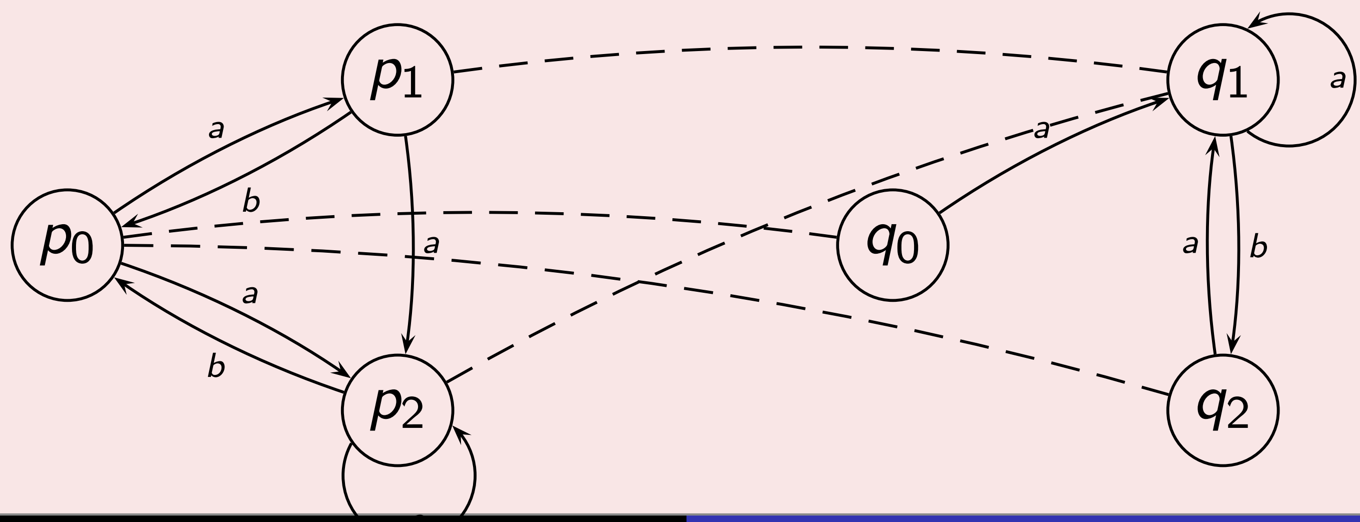


$R \triangleq \{\langle p_0, q_0\rangle, \langle p_0, q_2\rangle, \langle p_1, q_1\rangle, \langle p_2, q_1\rangle\}$ is a strong bisimulation



Basic Properties of Strong Bisimilarity

Theorem

$\sim$ *is an equivalence relation (reflexive, symmetric and transitive)*

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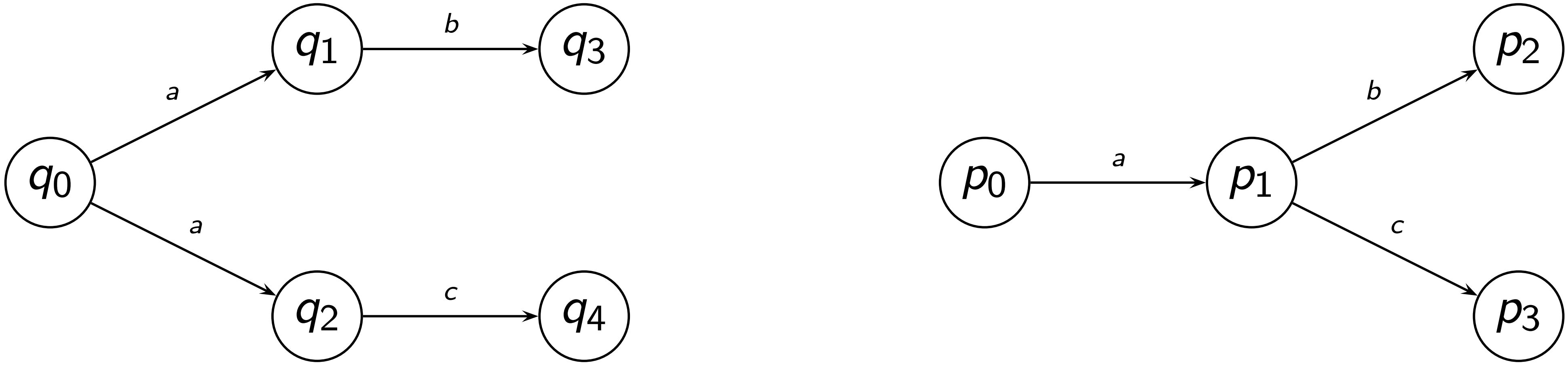
Theorem

$s \sim t$ *if and only if for each* $a \in Act$*:*

- *if* $s \xrightarrow{a} s'$ *then* $t \xrightarrow{a} t'$ *for some* t' *such that* $s' \sim t'$
- *if* $t \xrightarrow{a} t'$ *then* $s \xrightarrow{a} s'$ *for some* s' *such that* $s' \sim t'$.

Two Systems that are not bisimilar



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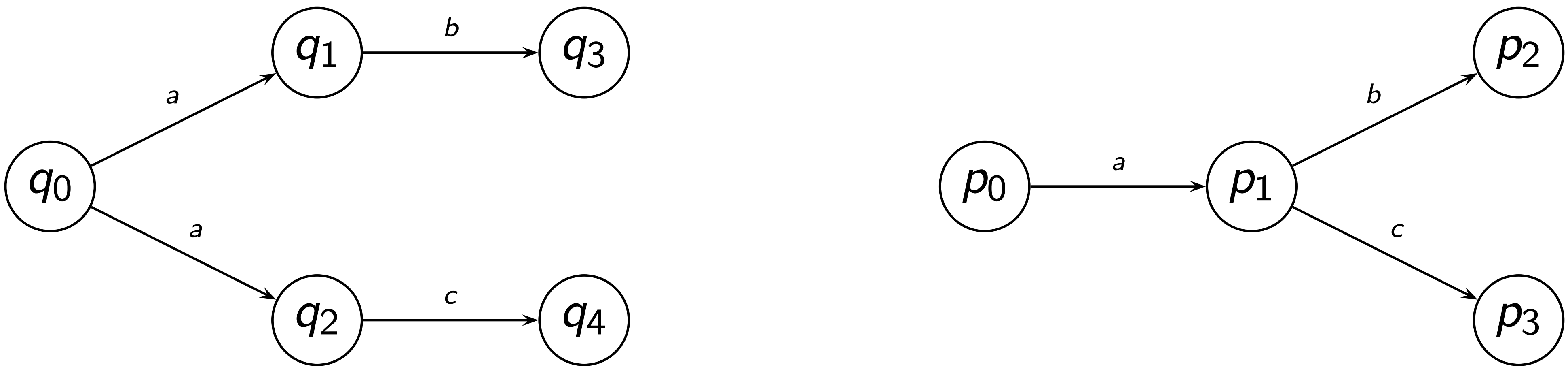

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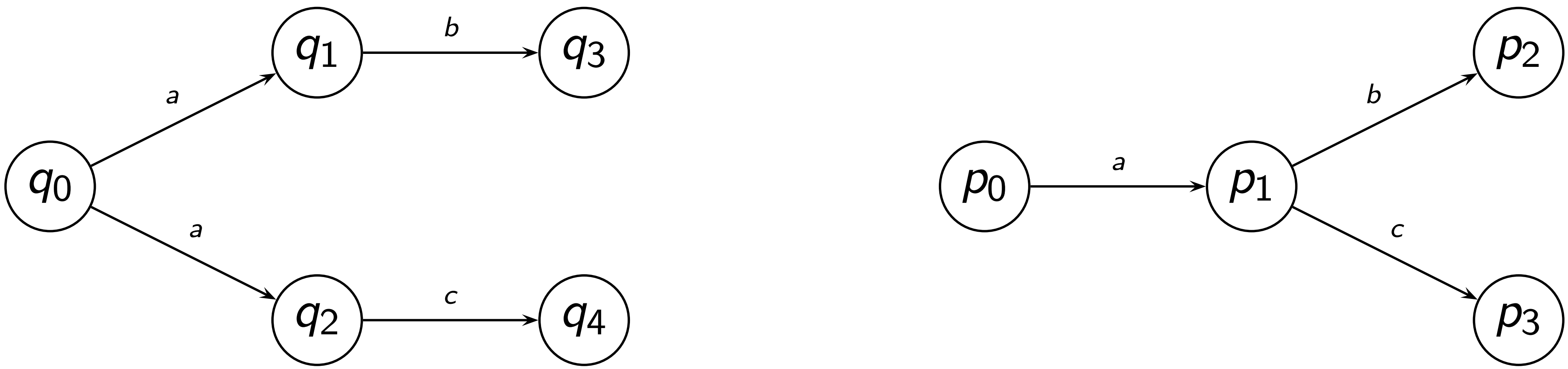

- States p_0 and q_0 are not strongly bisimilar.
- If they were equivalent, also states p_1 e q_1, had to be so.

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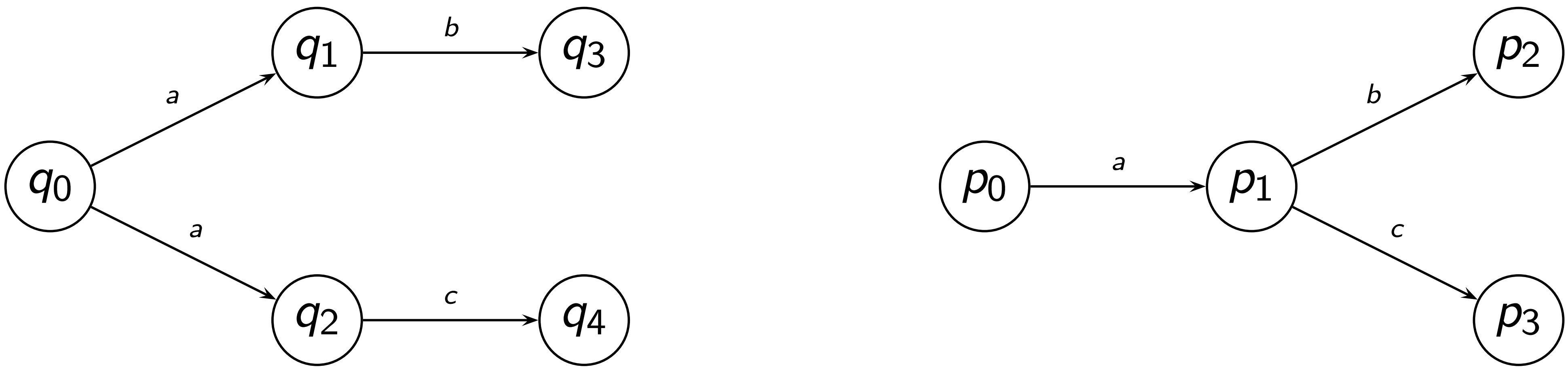
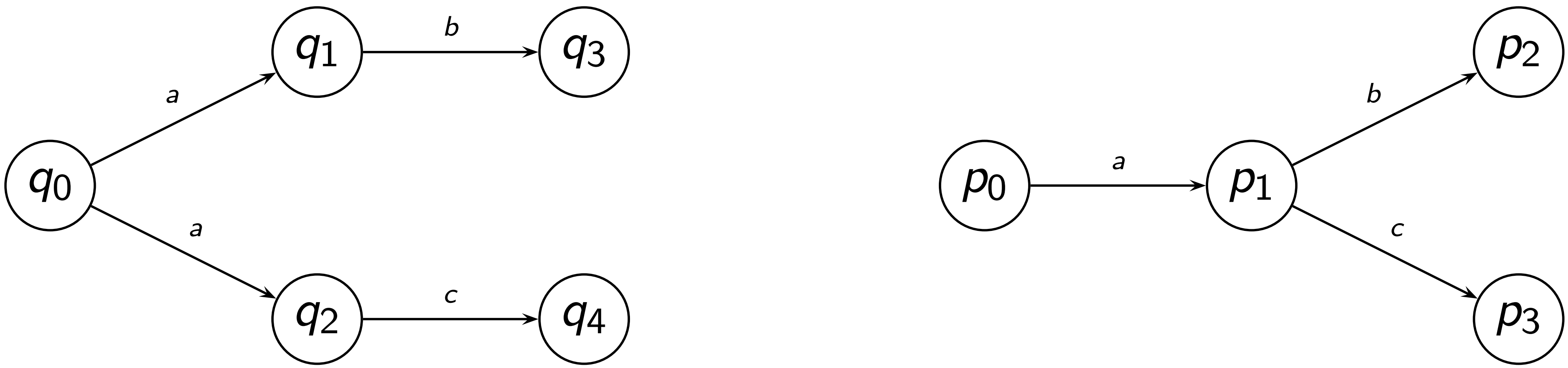


- States p_0 and q_0 are not strongly bisimilar.
- If they were equivalent, also states p_1 e q_1, had to be so.
- There is no strong bisimulation R that contains $\langle p_1, q_1 \rangle$.
- The c-transition from p_1 cannot be simulated by q_1 .

How to Show Nonbisimilarity?

Given:



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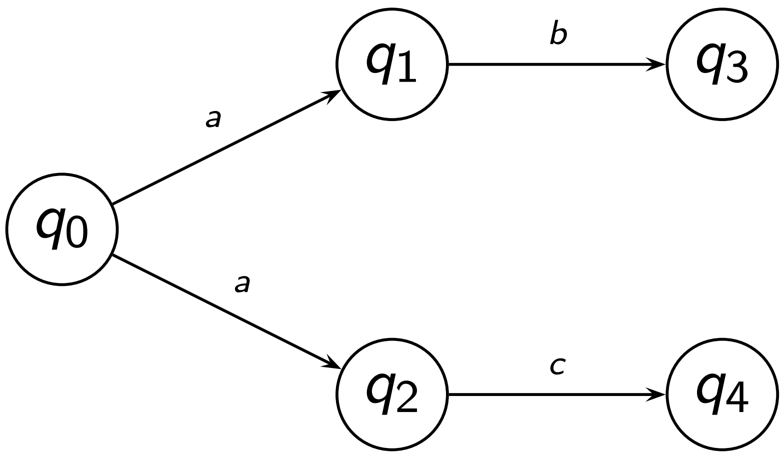

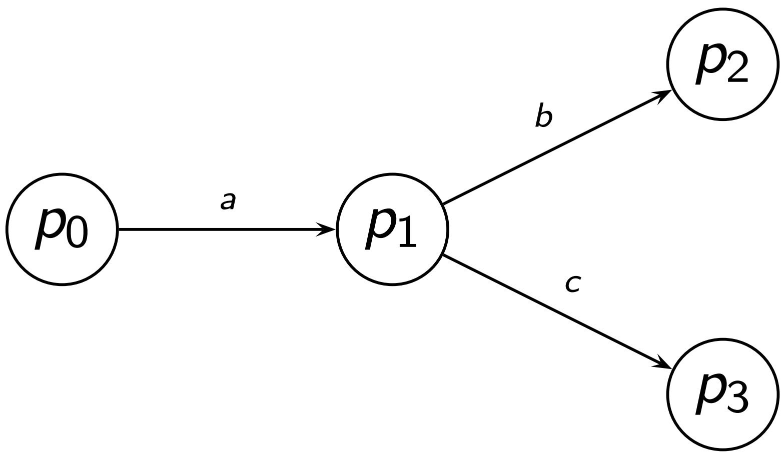

How to prove that $p_0 \not\sim q_0$:

- Enumerate all binary relations and show that none of them at the same time contains (s, t) and is a strong bisimulation. (Expensive: $2^{|Proc|^2}$ relations.)

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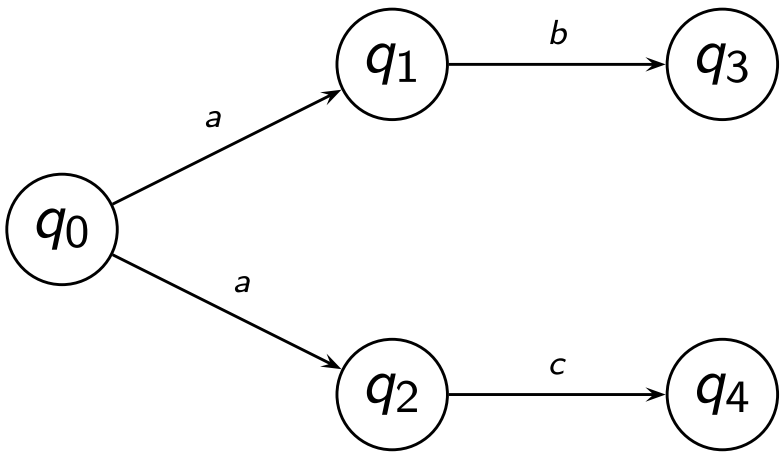

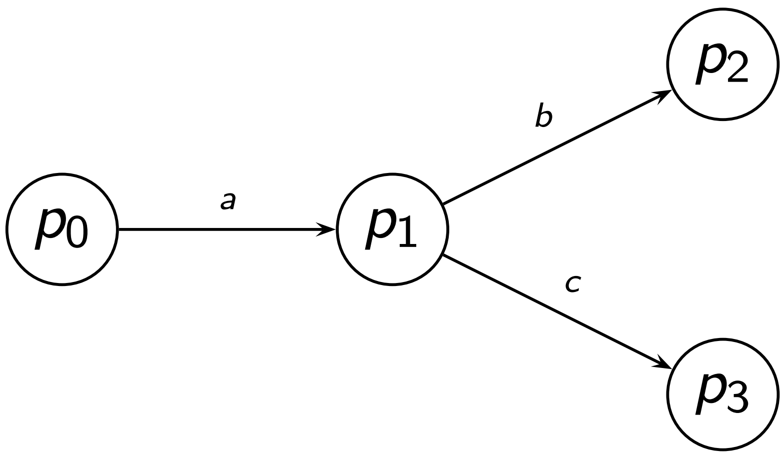

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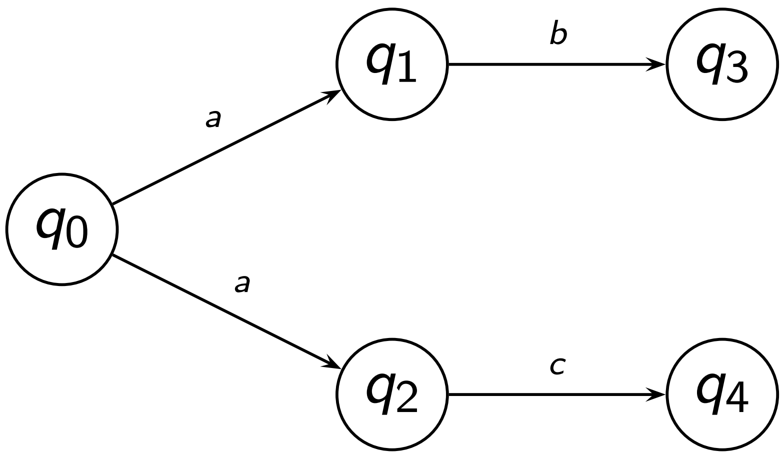

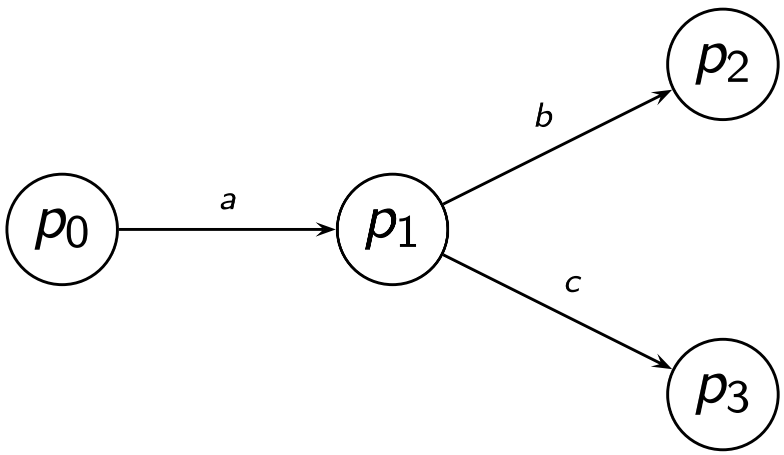

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- Make certain observations which enable us to disqualify many bisimulation candidates in one step.
- Use the game characterization of strong bisimilarity.

Strong Bisimulation Game

Let $(Proc, Act, \{\xrightarrow{a} \mid a \in Act\})$ be an LTS and $s, t \in Proc$.

We define a two-player game of an 'attacker' and a 'defender' starting from s and t.

- The game is played in rounds, and configurations of the game are pairs of states from $Proc \times Proc$.
- In every round exactly one configuration is called current. Initially the configuration (s, t) is the current one.

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Intuition

The defender wants to show that s and t are strongly bisimilar while the attacker aims at proving the opposite.

Rules of the Bisimulation Games

Game Rules

In each round the players change the current configuration as follows:

1. the attacker chooses one of the processes in the current configuration and makes an $\xrightarrow{a}$-move for some $a \in Act$, and
2. the defender must respond by making an $\xrightarrow{a}$-move in the other process under the same action a.

The newly reached pair of processes becomes the current configuration. The game then continues by another round.

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Result of the Game

- If one player cannot move, the other player wins.
- If the game is infinite, the defender wins.

Game Characterization of Strong Bisimilarity

Theorem

- States s and t are strongly bisimilar if and only if the defender has a universal winning strategy starting from the configuration (s, t).
- States s and t are not strongly bisimilar if and only if the attacker has a universal winning strategy starting from the configuration (s, t).

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Remark

The bisimulation game can be used to prove both bisimilarity and nonbisimilarity of two processes. It very often provides elegant arguments for the negative case.

Simulation Relation

Strong Simulation

A relation $R \subseteq Q \times Q$ is *strong simulation* if, for any pair of states p e q such that $\langle p, q \rangle \in R$, the following holds:

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Similarity

Two states $p, q \in Q$ are strongly *similar*, written $p \sqsubseteq q$, if there exists a strong simulation R such that $\langle p, q \rangle \in R$.

$$\sqsubseteq \; = \bigcup \{R \mid R \text{ is a strong simulation}\}$$

Double Similarity

Two states $p, q \in Q$ are *doubly similar*, written $p \simeq q$, if we have $p \sqsubseteq q$ and $q \sqsubseteq^{-1} p$ (i.e., $\simeq \triangleq \sqsubseteq \cap \sqsubseteq^{-1}$)

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2. To determine whether a state q satisfies an observer o the set $OBS(q, o)$ of all *computations* from $\langle q, o \rangle$ is considered
3. A process may satisfy an observer always or sometimes.

Observations

Given two LTS $\langle Q, A, \longrightarrow \rangle$ and $\langle O, A_w, \longrightarrow \rangle$, and two states $q \in Q$ e $o \in O$, an *observation* c from $\langle q, o \rangle$ is a sequence of pairs $\langle q_i, o_i \rangle$,such that

1. $\langle q_0, o_0 \rangle = \langle q, o \rangle$;
2. the transition $\langle q_i, o_i \rangle \xrightarrow{a} \langle q_{i+1}, o_{i+1} \rangle$ can be proved using:
$$\frac{E \xrightarrow{a} E' \qquad F \xrightarrow{a} F'}{\langle E, F \rangle \xrightarrow{a} \langle E', F' \rangle} \; a \in A$$
3. the last element of the sequence, say $\langle q_k, o_k \rangle$, is such that for no configuration $\langle q', o' \rangle$, with $q' \in Q$ e $o' \in O$, there exists $a \in A$ such that $\langle q_k, o_k \rangle \xrightarrow{a} \langle q', o' \rangle$ via the above rule.

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$OBS(q, o)$ is the set of all observations from the initial configuration $\langle q, o \rangle$.

Experimentations

Successful Experiments

An observation $c \in OBS(q, o)$ is *successful* if there exists a configuration $\langle q_n, o_n \rangle \in c$, with $n \geq 0$, such that $o_n \xrightarrow{w}$.

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Satisfaction of Observers

1. q MAY SATISFY o if there exists an observation $c \in OBS(q, o)$ that is successful;
2. q MUST SATISFY o if all observations $c \in OBS(q, o)$ are successful.

May, Must and Testing Equivalences

May Equivalence

p is *may* equivalent to q, $p \simeq_m q$, if for all observers $o \in \mathcal{O}$ we have p MAY SATISFY o if and only if q MAY SATISFY o;

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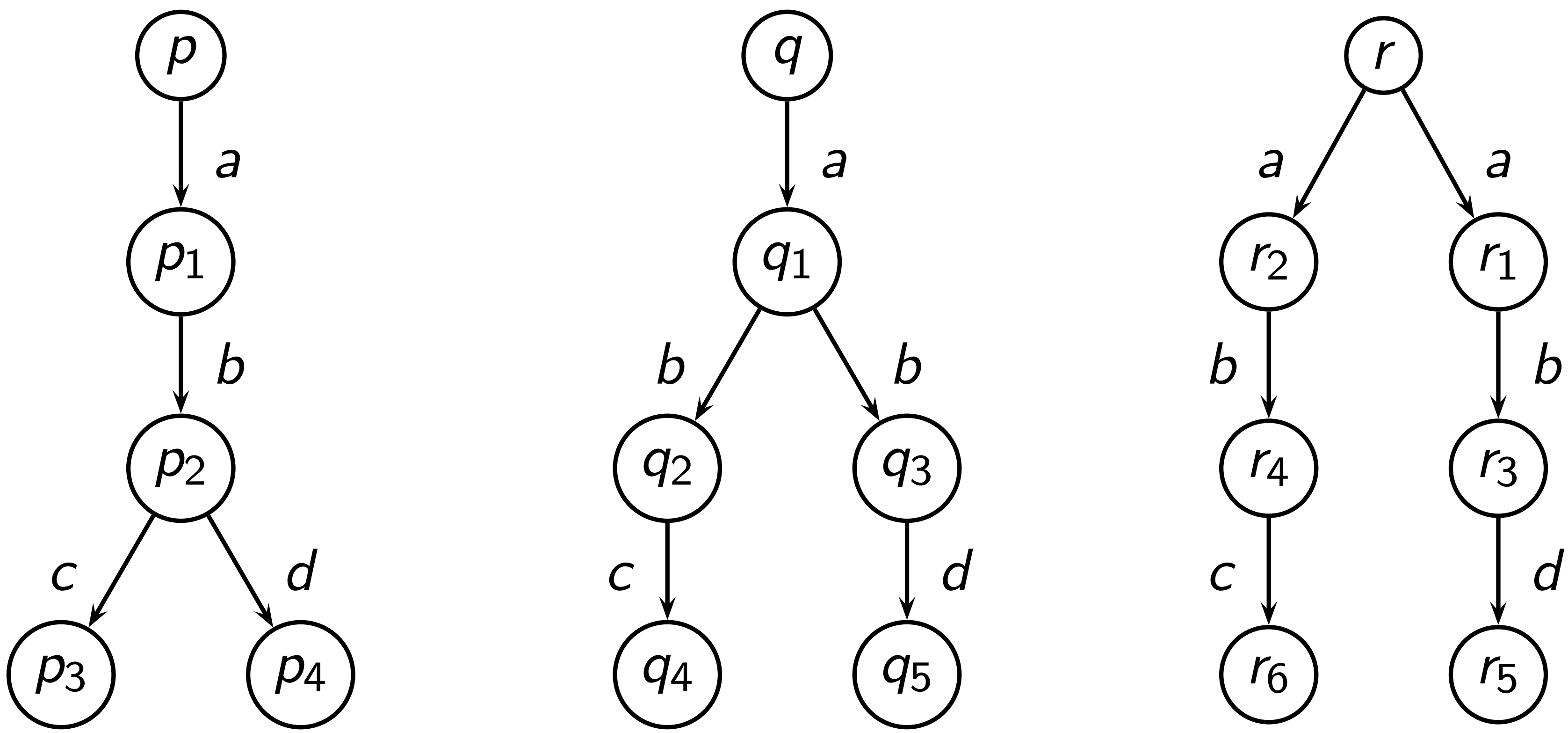
Testing Equivalence

p is *testing* equivalent to q, $p \simeq_{test} q$, if $p \simeq_m q$ and $p \simeq_M q$.

Examples for may, must and testing



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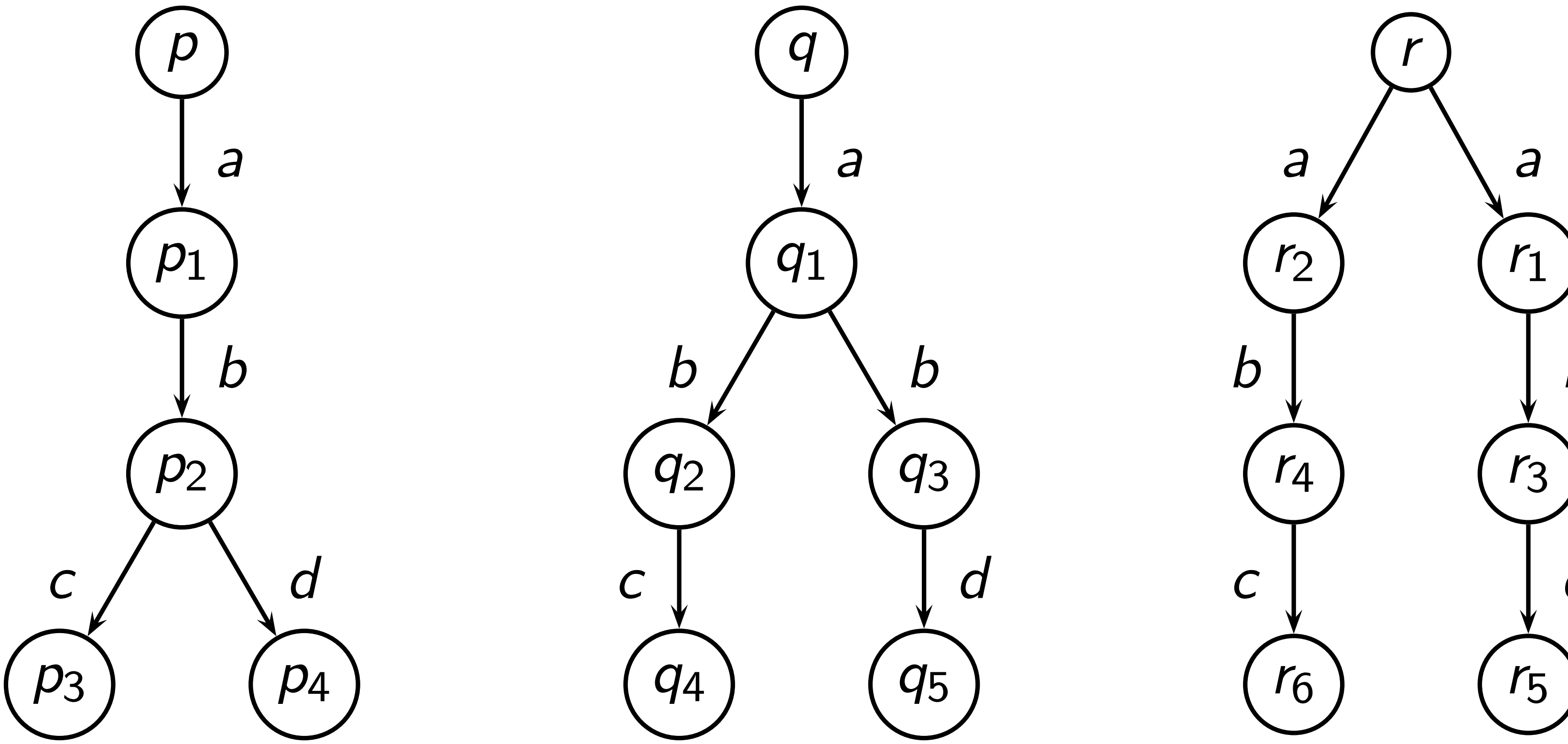
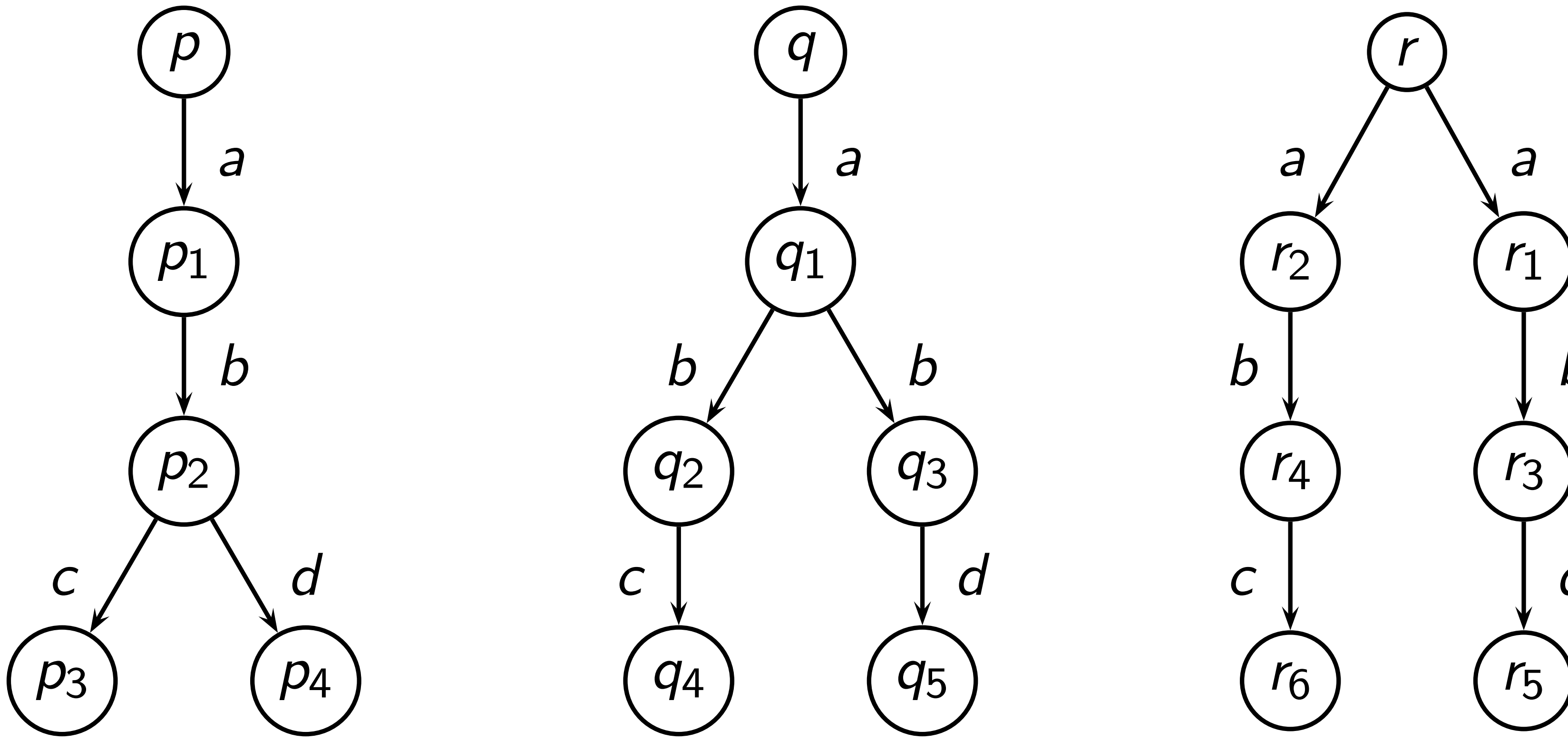


Examples for may, must and testing



- $p \simeq_m q$

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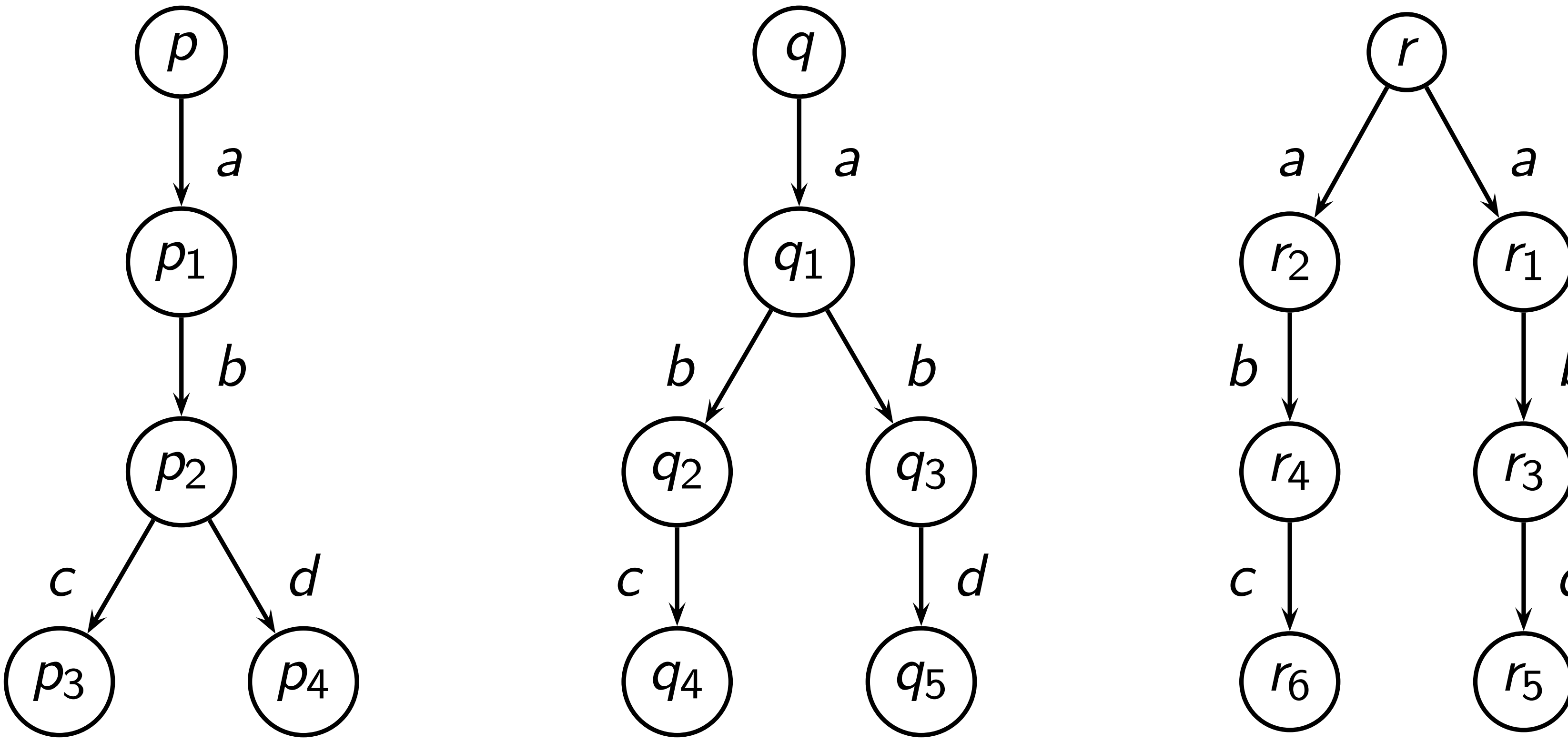

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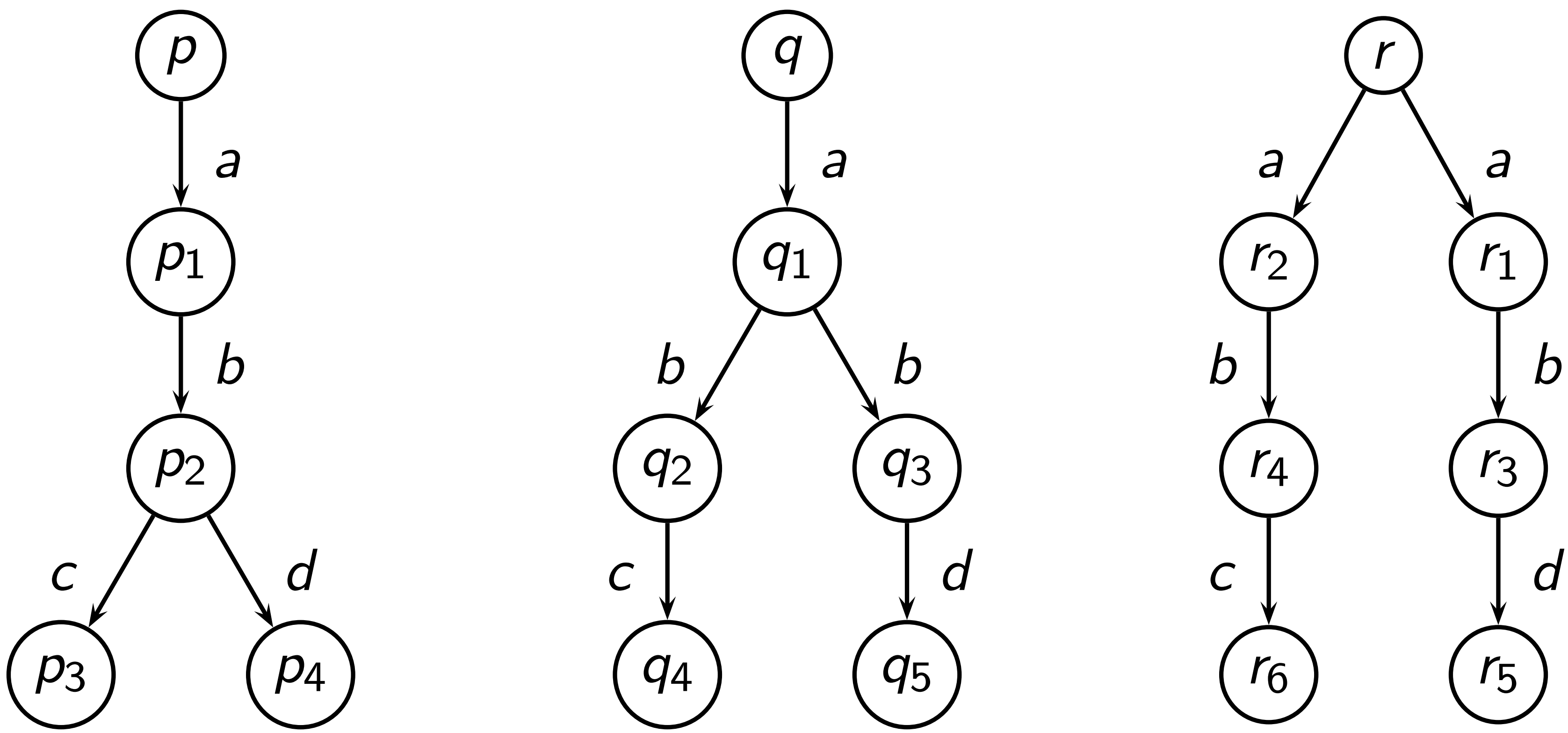

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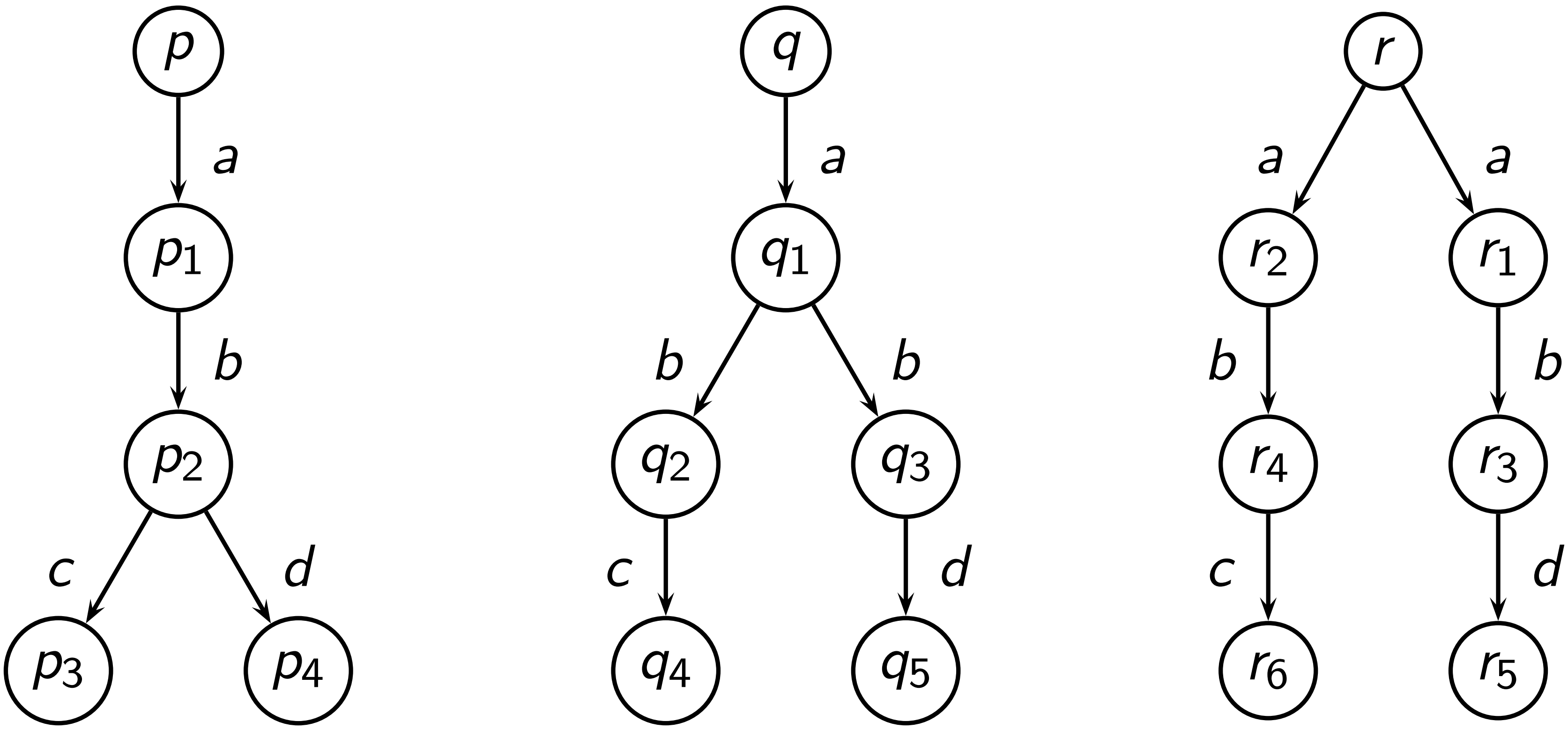
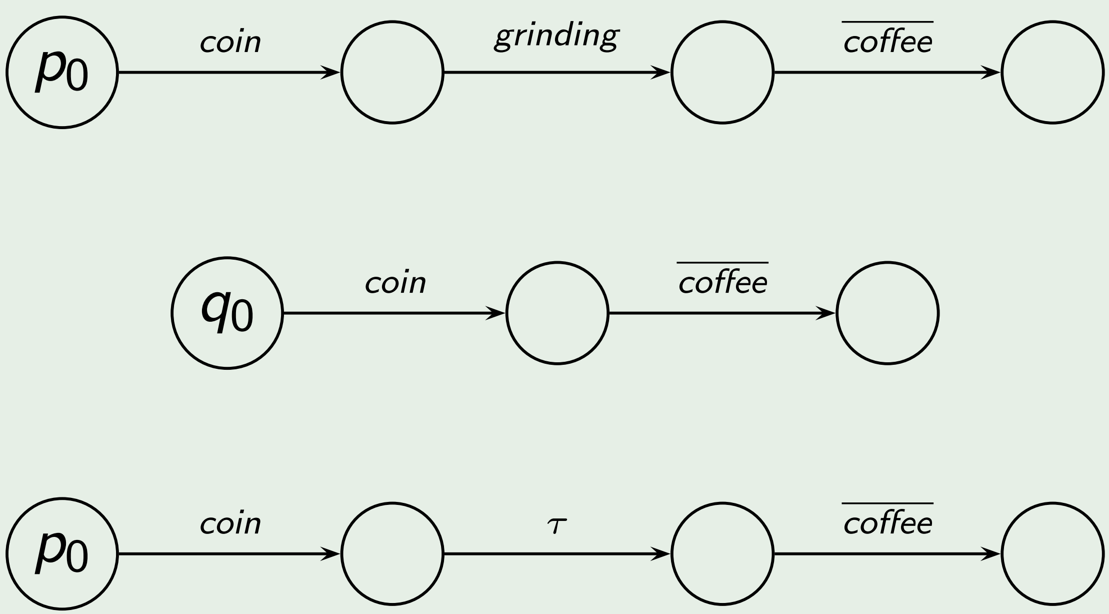


- $p \simeq_m q$
- NOT $p \simeq_M q$
- $q \simeq_M r$
- $q \simeq_{test} r$

Weak Equivalences

Is it right to consider different from a user point of view the three machine below, if

- grinding is an internal action?
- τ is an invisible action?



Weak Traces Equivalence

Let $\langle Q, A, \longrightarrow \rangle$ be an LTS, with $q \in Q$ and $s \in A^*$
and
Let $q \stackrel{s}{\Longrightarrow} q'$ denote that q reduces to q' by performing the sequence s of visible actions each of which can be preceded or followed by internal actions τ.

Traces

1. s is a *weak trace* of q if there exists $q' \in Q$ s.t. $q \stackrel{s}{\Longrightarrow} q'$.
2. $L(q)$ represents the set of all weak traces of q

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Traces Equivalence

Two states p e q are *trace equivalent*, written $p \approx_L q$, if $L(p) = L(p)$.

Weak Observations

To define the weak variants of may, must and testing equivalences it suffices to change experiments so that processes and observers can freely perform silent actions

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Given two LTS $\langle Q, A, \longrightarrow \rangle$ and $\langle O, A_w, \longrightarrow \rangle$, and two states $q \in Q$ e $o \in O$, a weak *experiment* c from $\langle q, o \rangle$ is a sequence of pairs $\langle q_i, o_i \rangle$,s.t.

1. $\langle q_0, o_0 \rangle = \langle q, o \rangle$;
2. the transition $\langle q_i, o_i \rangle \xrightarrow{a} \langle q_{i+1}, o_{i+1} \rangle$ can be proved using:

$$\frac{E \xrightarrow{\tau} E'}{\langle E, F \rangle \xrightarrow{\tau} \langle E', F \rangle} \qquad \frac{F \xrightarrow{\tau} F'}{\langle E, F \rangle \xrightarrow{\tau} \langle E, F' \rangle} \qquad \frac{E \xrightarrow{a} E' \quad F \xrightarrow{a} F'}{\langle E, F \rangle \xrightarrow{\tau} \langle E', F' \rangle}\ a \in A$$

3. the last element of the sequence, say $\langle q_k, o_k \rangle$, is such that for no configuration $\langle q', o' \rangle$, with $q' \in Q$ e $o' \in O$, there exists $a \in A$ such that $\langle q_k, o_k \rangle \xrightarrow{a} \langle q', o' \rangle$ via the above rule.

Weak Bisimulation Relation: An immediate generalization

Weak Bisimulation

A relation $R \subseteq Q \times Q$ is *weak bisimulation* if, for any pair of states p e q such that $\langle p, q \rangle \in R$, for any $s \in A^*$, the following holds:

1. for all $a \in A$ e $p' \in Q$, if $p \stackrel{s}{\Rightarrow} p'$ then $q \stackrel{s}{\Rightarrow} q'$ for some $q' \in Q$ such that $\langle p', q' \rangle \in R$;
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Weak Bisimilarity

Two states $p, q \in Q$ of an LTS $\langle Q, A_\tau, \longrightarrow \rangle$ are *weakly bisimilar*, written $p \approx q$, if there exists a weak bisimulation R such that $\langle p, q \rangle \in R$.

Weak Bisimulation Relation: A simpler definition

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where

$$\hat{\mu} = \begin{cases} \epsilon & \text{se } \mu = \tau \\ \mu & \text{se } \mu \neq \tau \end{cases}$$

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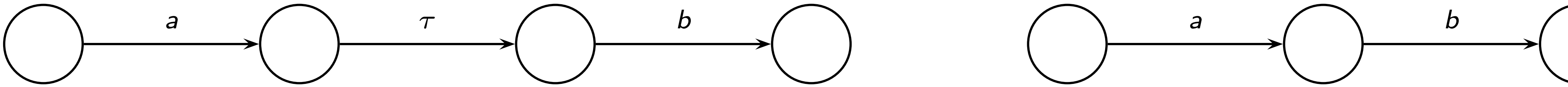
Two Pairs of Weakly Bisimilar Systems

Ignoring Tau's



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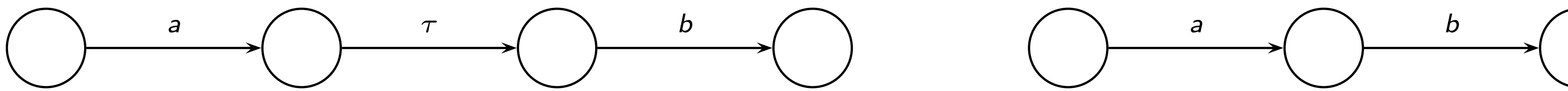
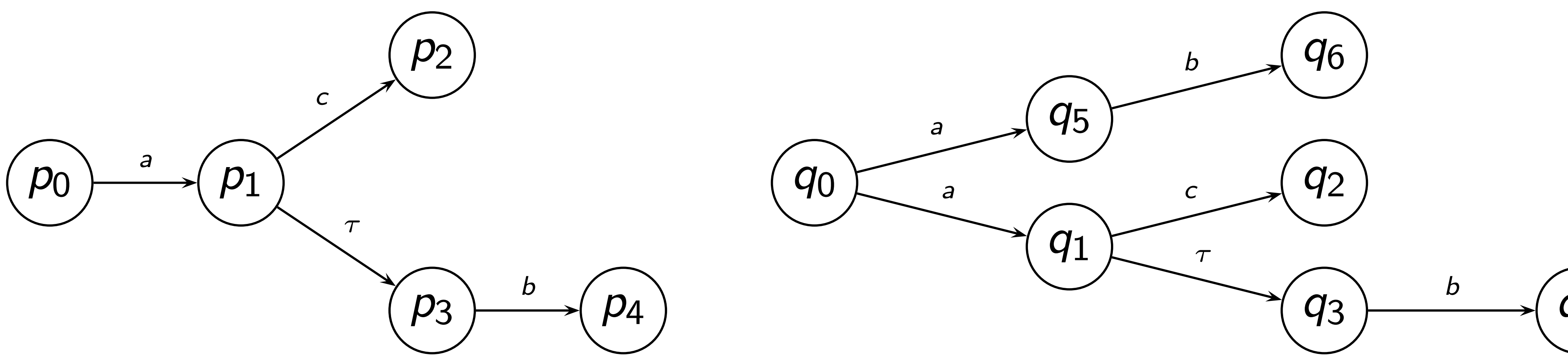


Ignoring Tau's and Branching



An Alternative to Weak Bisimulation

Branching Bisimulation

A symmetric relation $R \subseteq Q \times Q$ is *weak bisimulation* if, for any pair of states p e q such that $\langle p, q \rangle \in R$, if $p \xrightarrow{\mu} p'$, with $\mu \in A_\tau$ and $p' \in Q$, at least one of the following conditions holds:

1. $\mu = \tau$ e $\langle p', q \rangle \in R$
2. $q \Rightarrow q'' \xrightarrow{\mu} q'$ per qualche $q', q'' \in Q$ tali che $\langle p, q'' \rangle \in R$ e $\langle p', q' \rangle \in R$.

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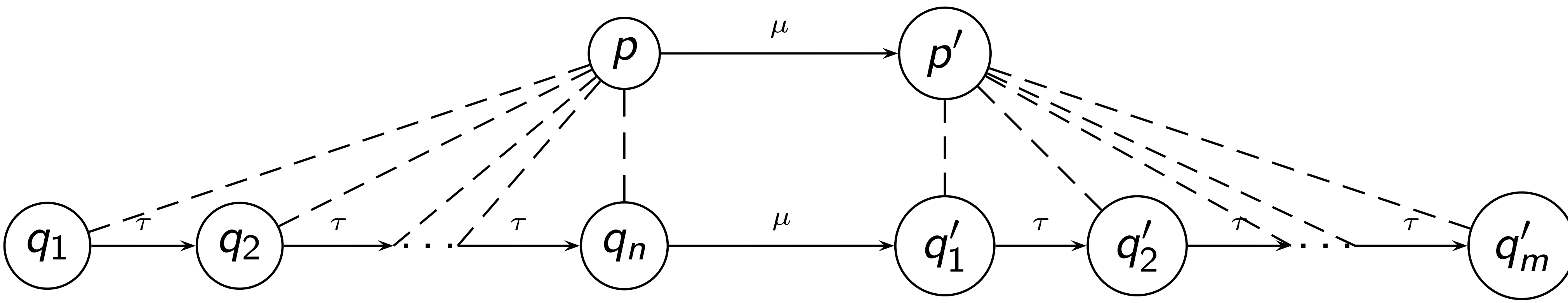
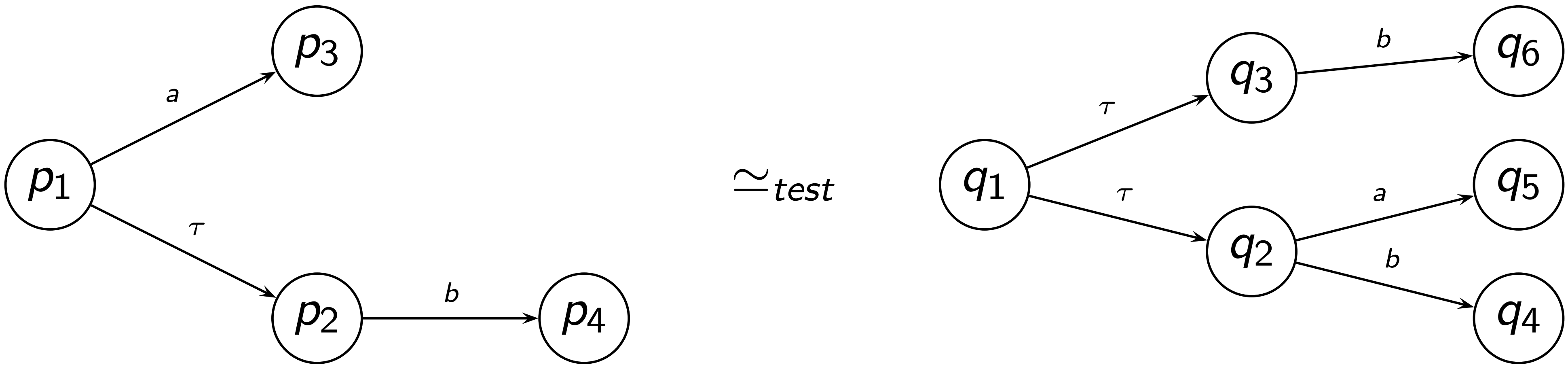
Branching Bisimilarity

Two states $p, q \in Q$ of an LTS $\langle Q, A_\tau, \longrightarrow \rangle$ are *Branching bisimilar*, written $p \approx_b q$, if there exists a branching bisimulation R such that $\langle p, q \rangle \in R$.

Branching Bisimulation, ... pictorially

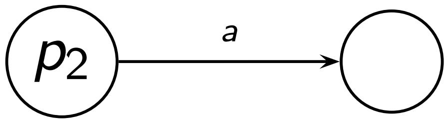


Testing vs Bisimulation - 1



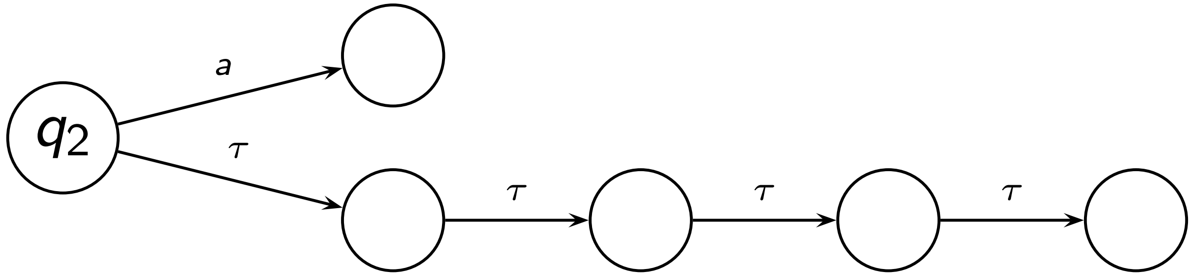
The systems above are weakly testing equivalent but NOT weakly (nor branching) bisimilar

Testing vs Bisimulation - 2



$\sqsupseteq_{test}$

$\not\sqsubseteq_{test}$



The systems above are NOT testing equivalent but are weakly (and branching) bisimilar

Equivalences Hierarchies

For strongly convergent systems we have:

$$
\begin{tikzcd}
 & \sim & \\
=_{CT} \arrow[ur] & & \simeq \arrow[ul] \\
 & =_{T} \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\approx_b \\
\approx \arrow[u] \\
\simeq_F \equiv \simeq_M \arrow[u] \\
\approx_L \equiv \simeq_m \arrow[u]
\end{tikzcd}
$$