



## Functional Geometry

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**Abstract.** An algebra of pictures is described that is sufficiently powerful to denote the structure of a well-known Escher woodcut, Square Limit. A decomposition of the picture that is reasonably faithful to Escher's original design is given. This illustrates how a suitably chosen algebraic specification can be both a clear description and a practical implementation method. It also allows us to address some of the criteria that make a good algebraic description.

**Keywords:** Functional programming, graphics, geometry, algebraic style, architecture, specification.

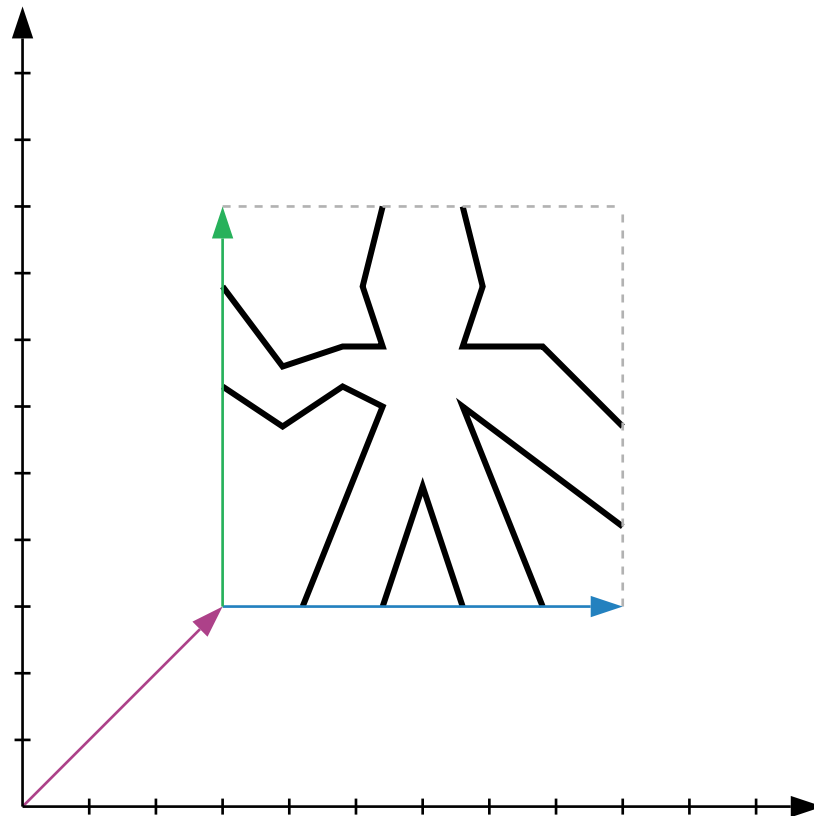
A **picture** is an example  
of a **complex object** that  
can be described in terms  
of its **parts**.

Let us define a picture as a **function** which takes three arguments, each being two-space **vectors** and returns **a set of graphical objects** to be rendered on the output device.

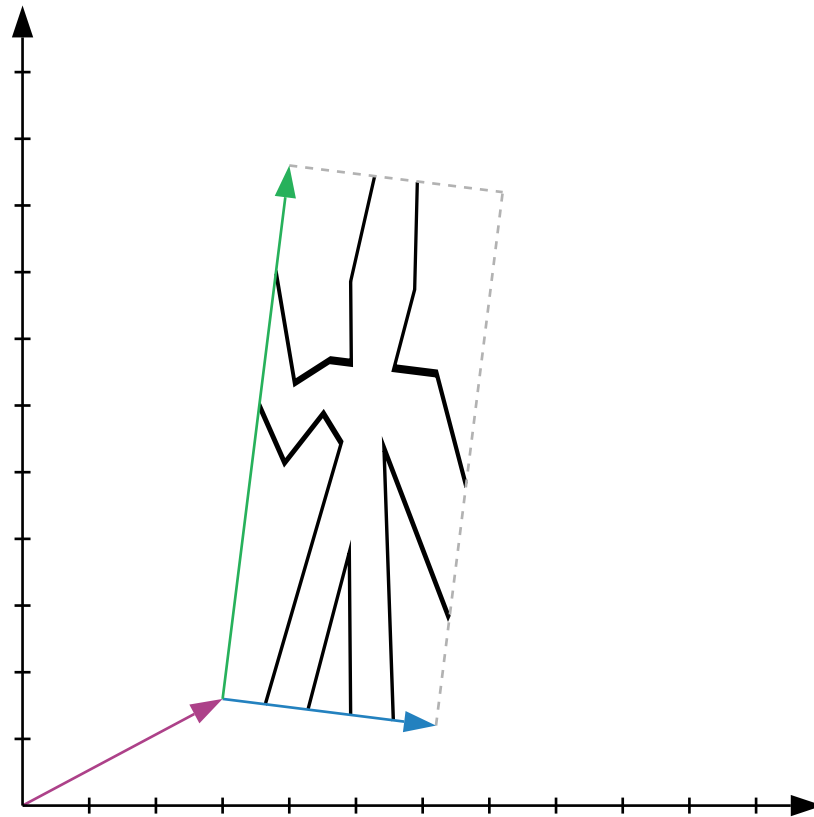
```
type Box = { a : Vector  
             b : Vector  
             c : Vector }
```

```
type Picture = Box -> Rendering
```

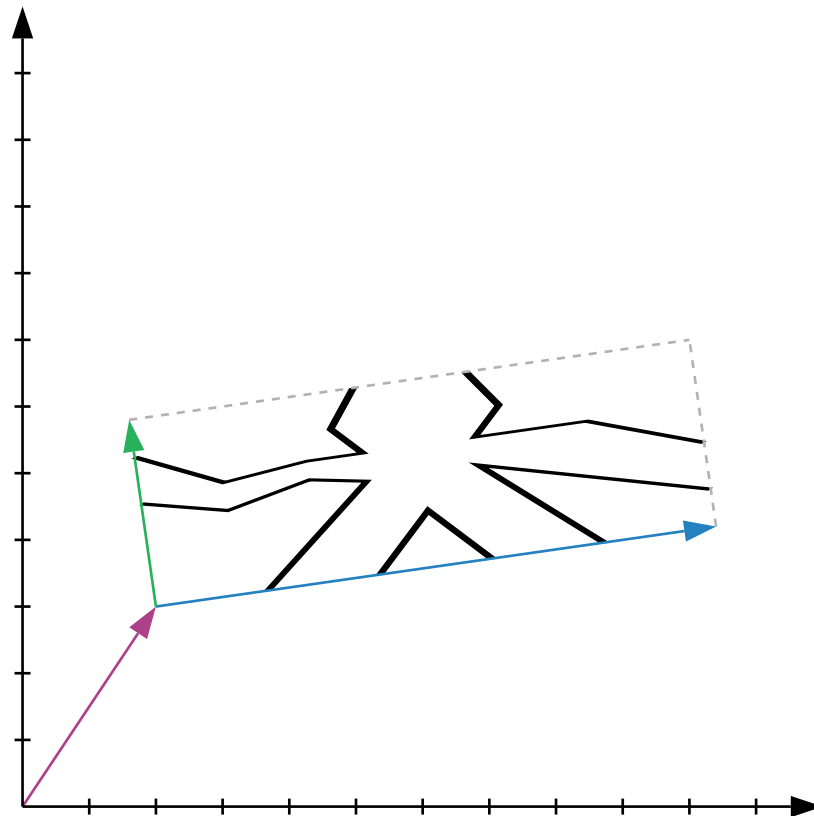
george



also george



still george





turn

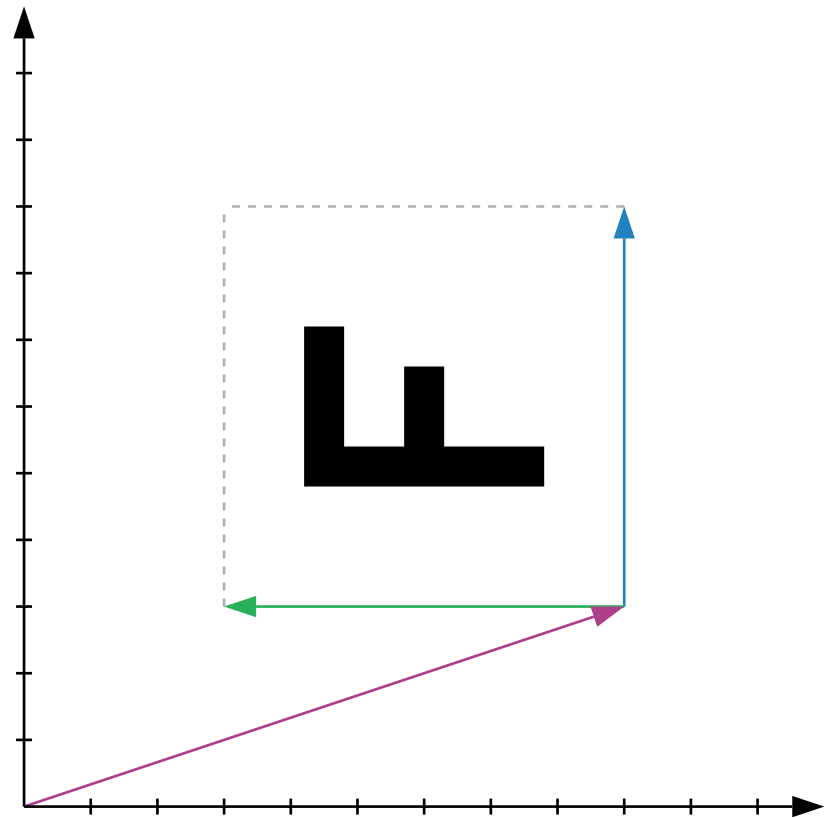


=>

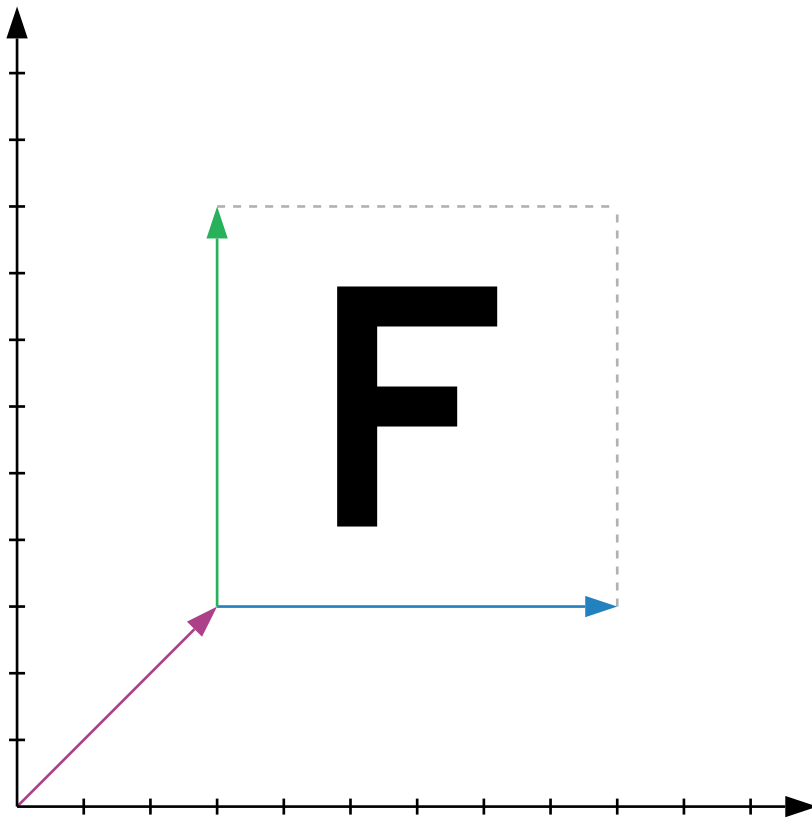


```
turnBox : Box -> Box
turnBox { a, b, c } = { a = add a b
                        , b = c
                        , c = neg b }
```

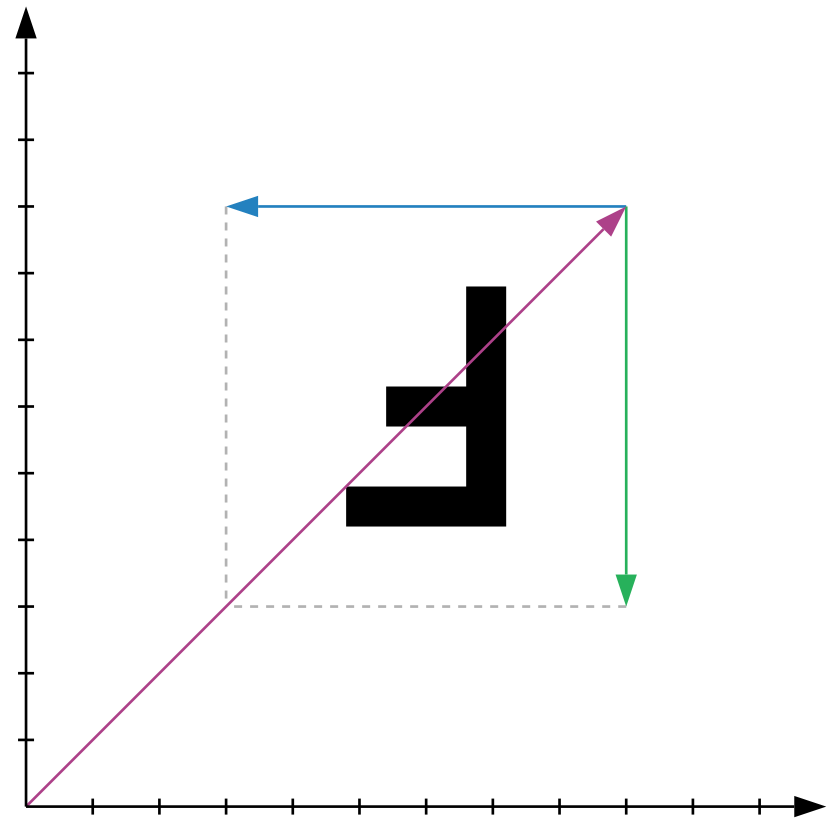
```
turn : Picture -> Picture
turn p = turnBox >> p
```

$$=>$$


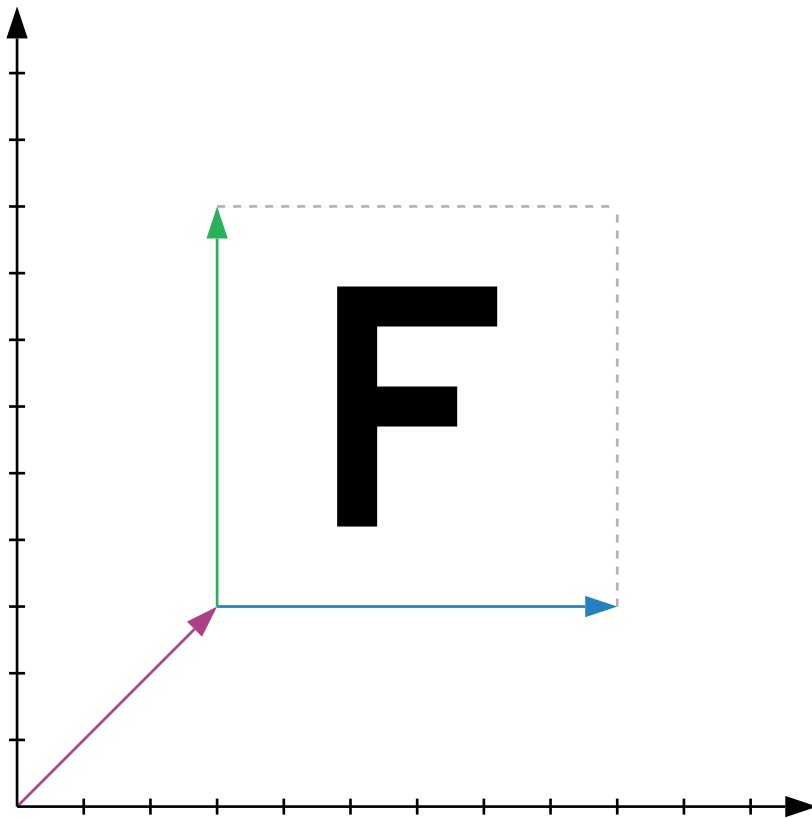
turn >> turn



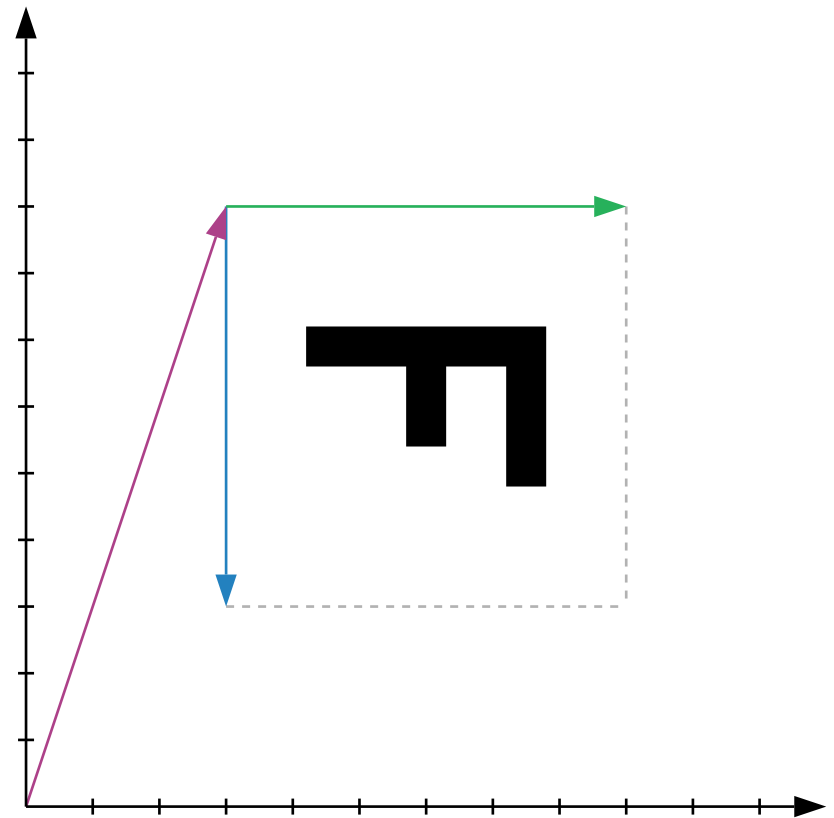
=>



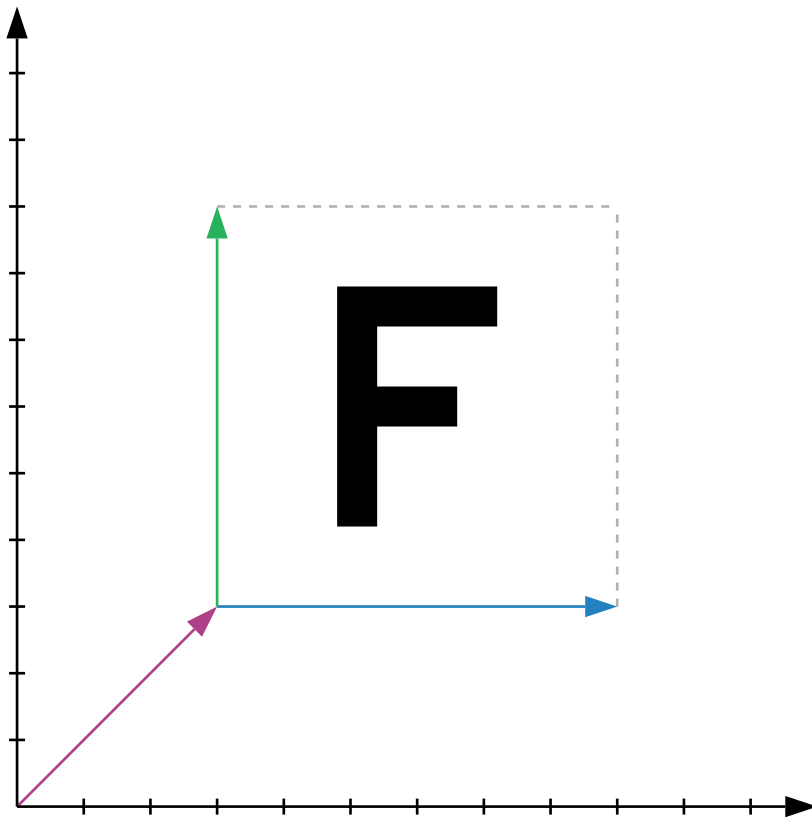
turn >> turn >> turn



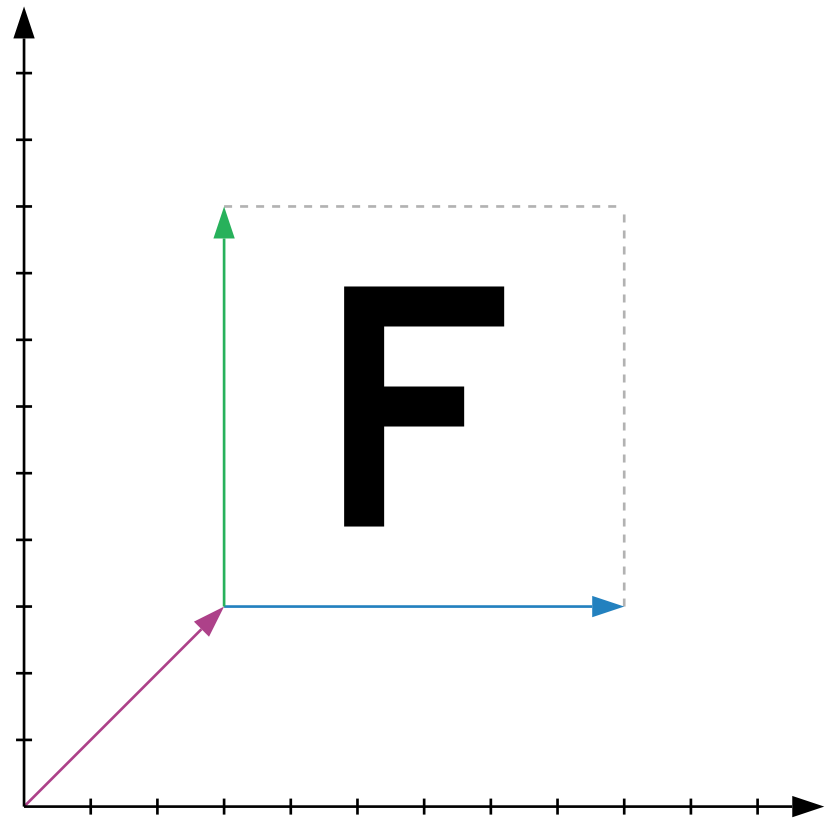
=>



turn >> turn >> turn >> turn



=>



flip

F

=>

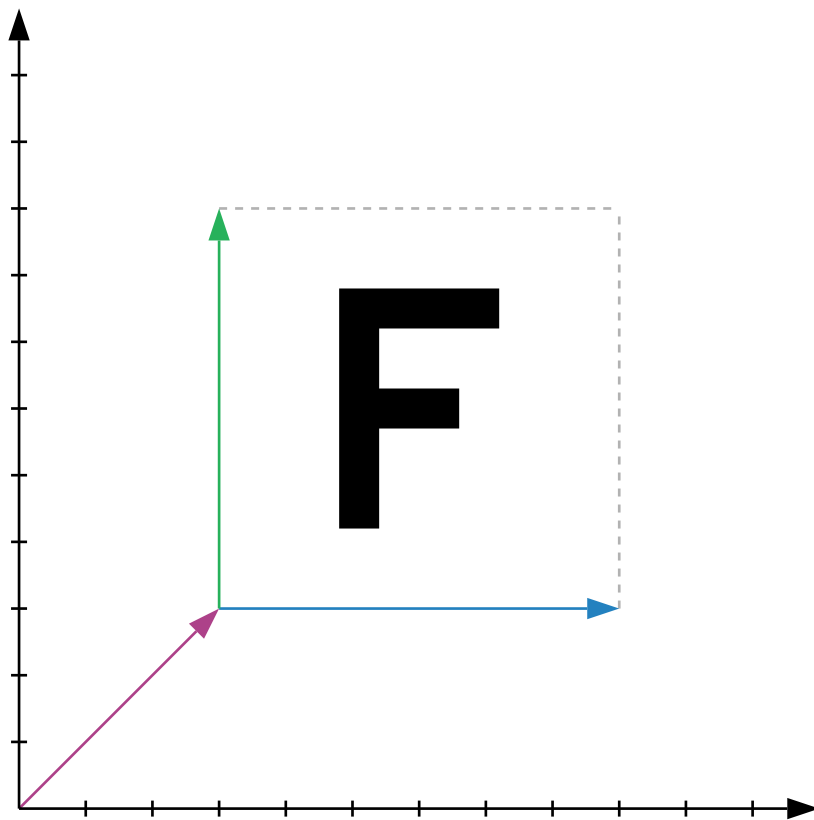
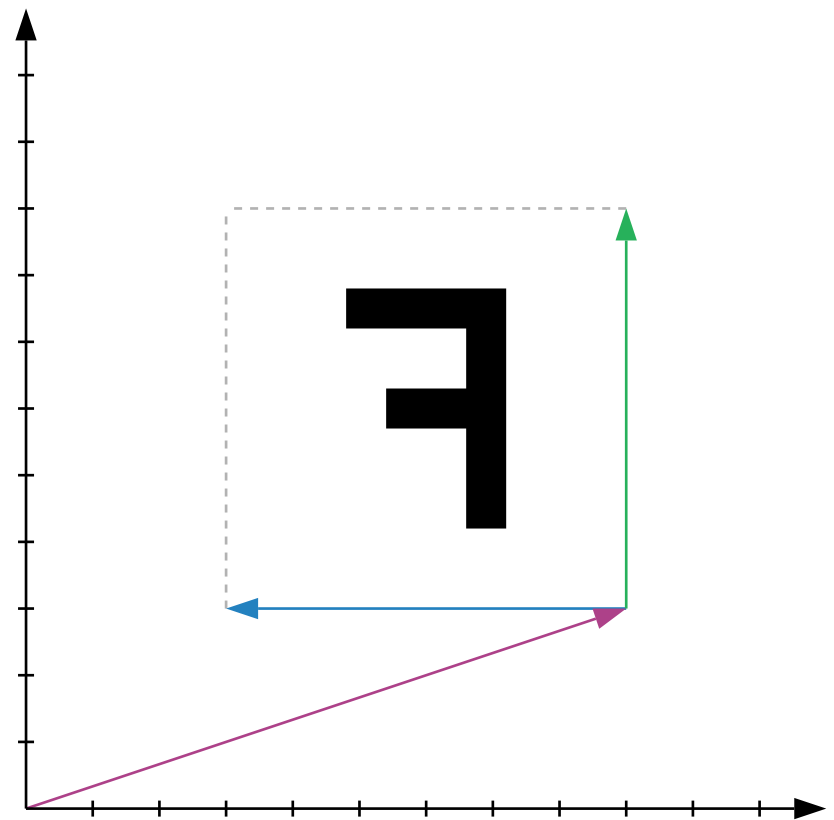
Ǝ

```
flipBox : Box -> Box
flipBox { a, b, c } = { a = add a b
                        , b = neg b
                        , c = neg c }
```

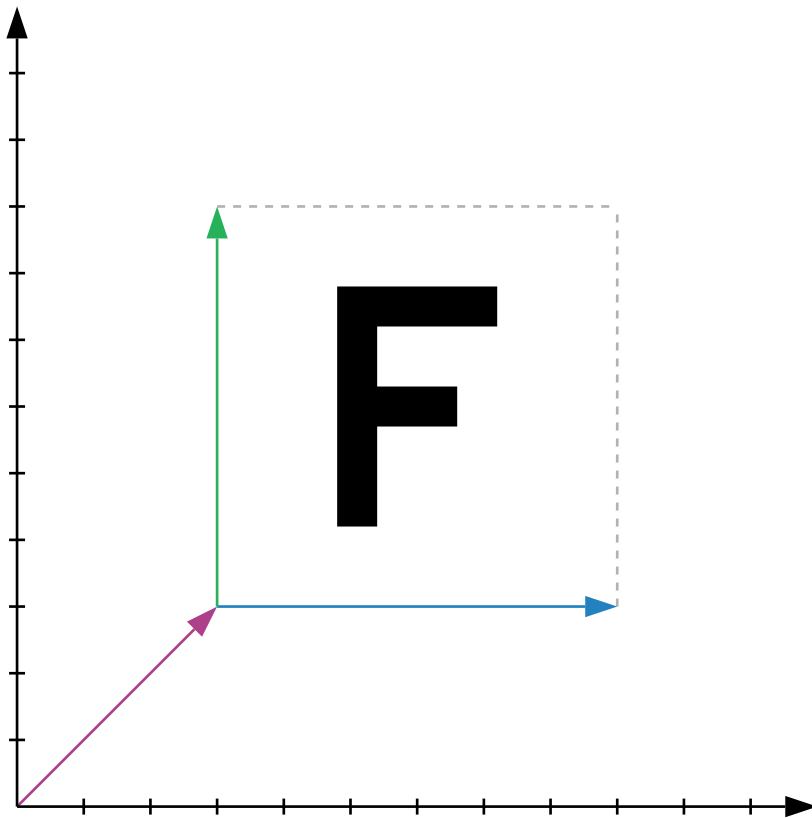
```
flip : Picture -> Picture
flip p = flipBox >> p
```



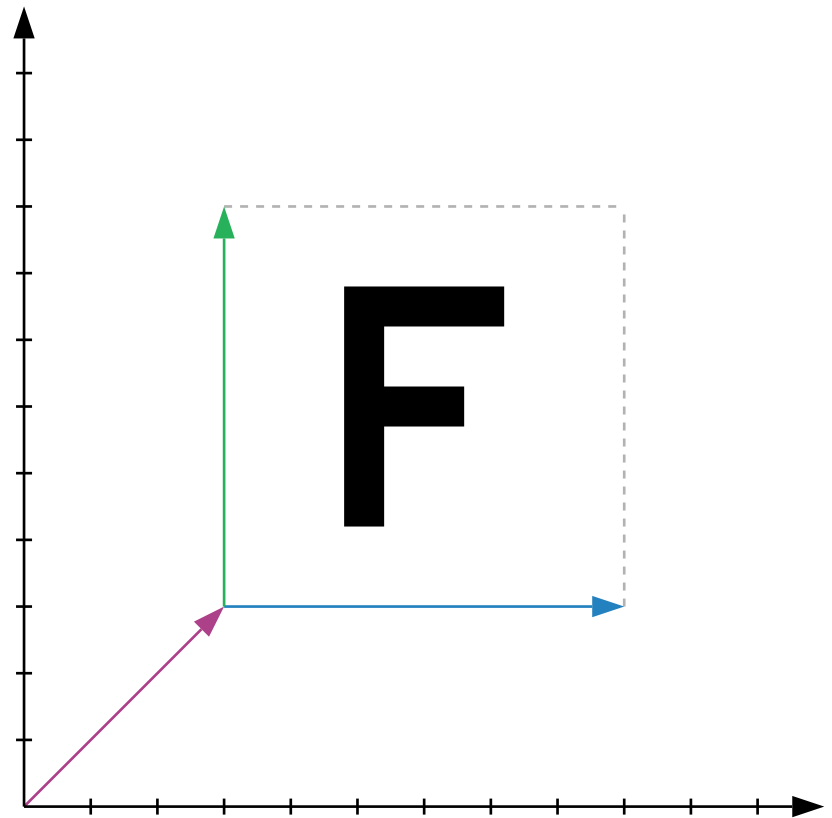
flip

 $\Rightarrow$ 

flip >> flip



=>



toss

F

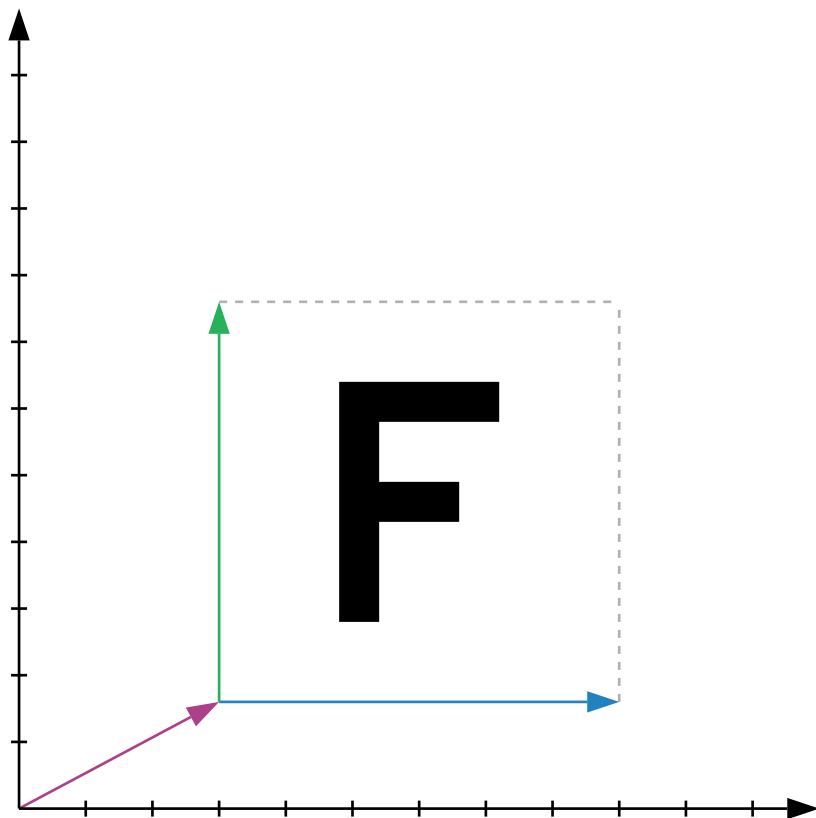
=>

F

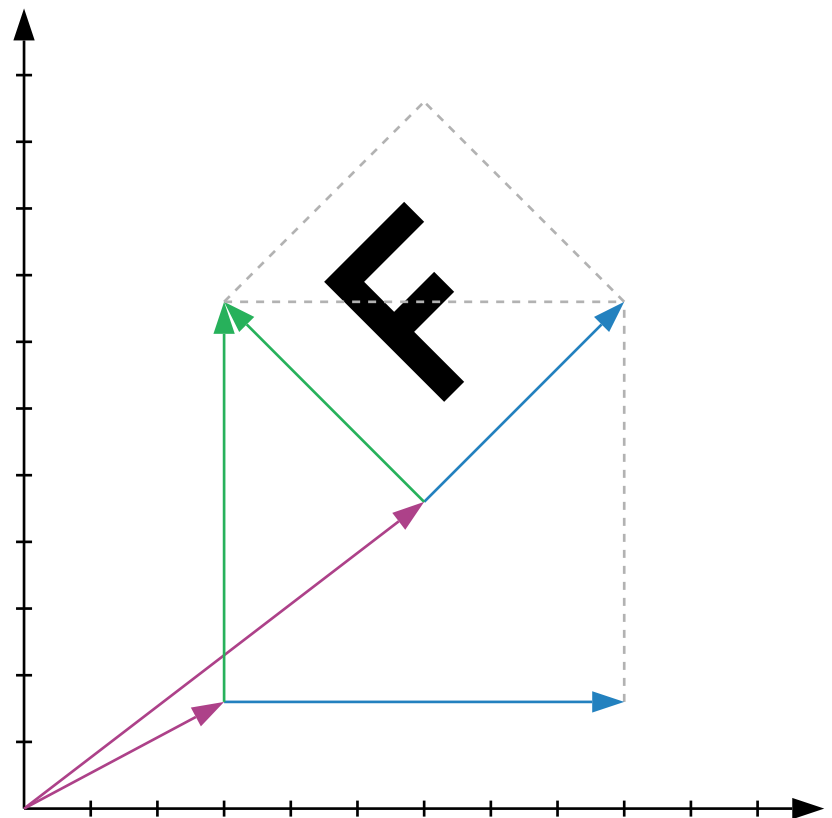
```
tossBox : Box -> Box
tossBox { a, b, c } =
  { a = add a (scale 0.5 (add b c))
    , b = scale 0.5 (add b c)
    , c = scale 0.5 (sub c b) }
```

```
toss : Picture -> Picture
toss p = tossBox >> p
```

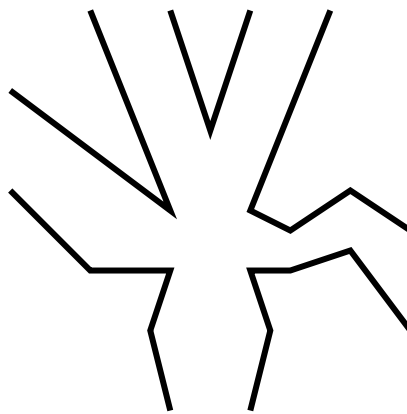
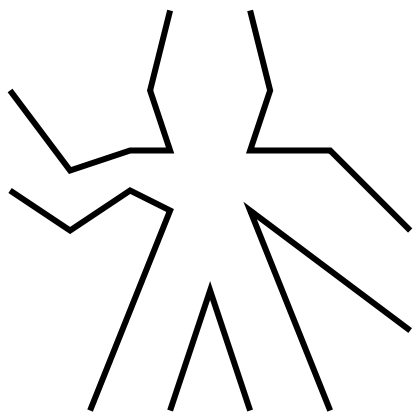
toss



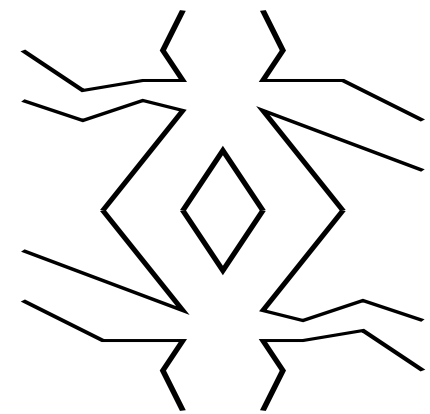
$\Rightarrow$



above george ((turn >> turn) george)



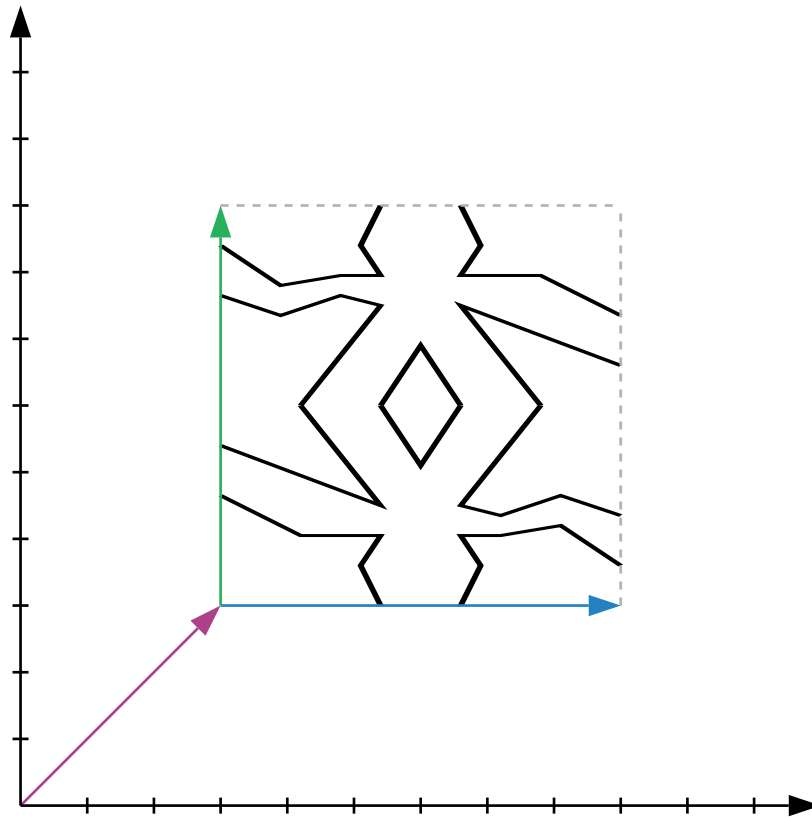
=>



```
aboveRatio : Int -> Int -> Pic -> Pic -> Pic
aboveRatio m n p1 p2 =
    \box ->
        let
            f = m / (m + n)
            (b1, b2) = splitVertically f box
        in
            (p1 b1) ++ (p2 b2)

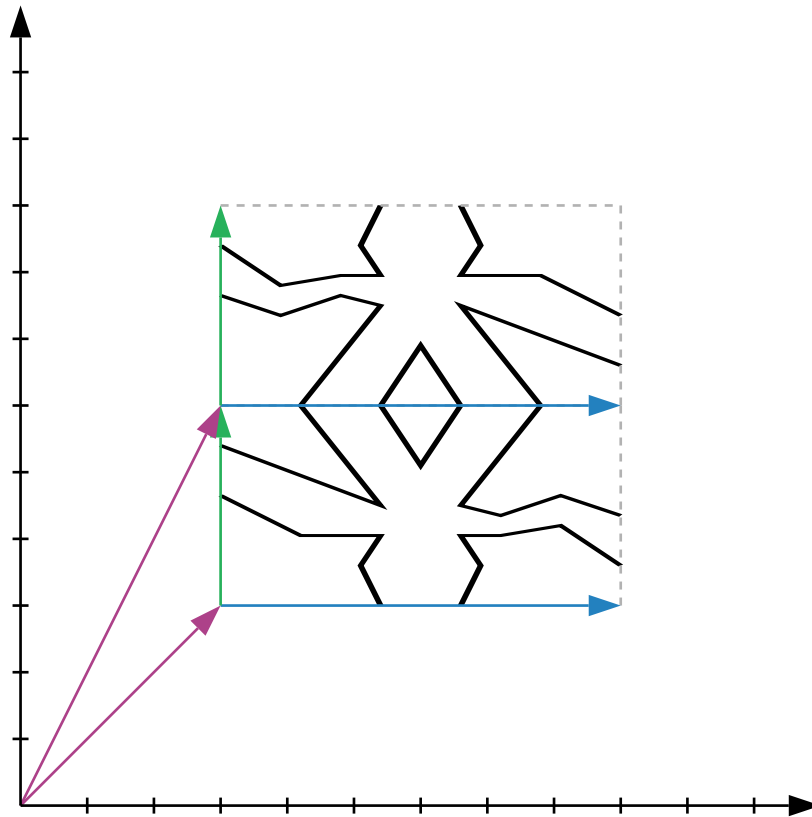
above : Pic -> Pic -> Pic
above p1 p2 = aboveRatio 1 1
```

above george ((turn >> turn) george)

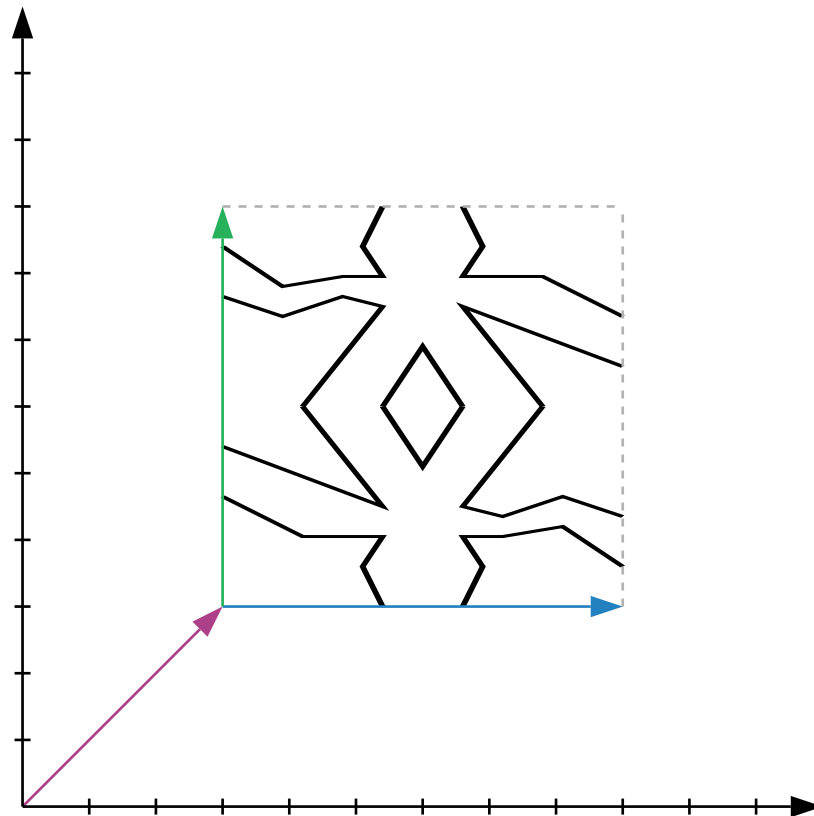




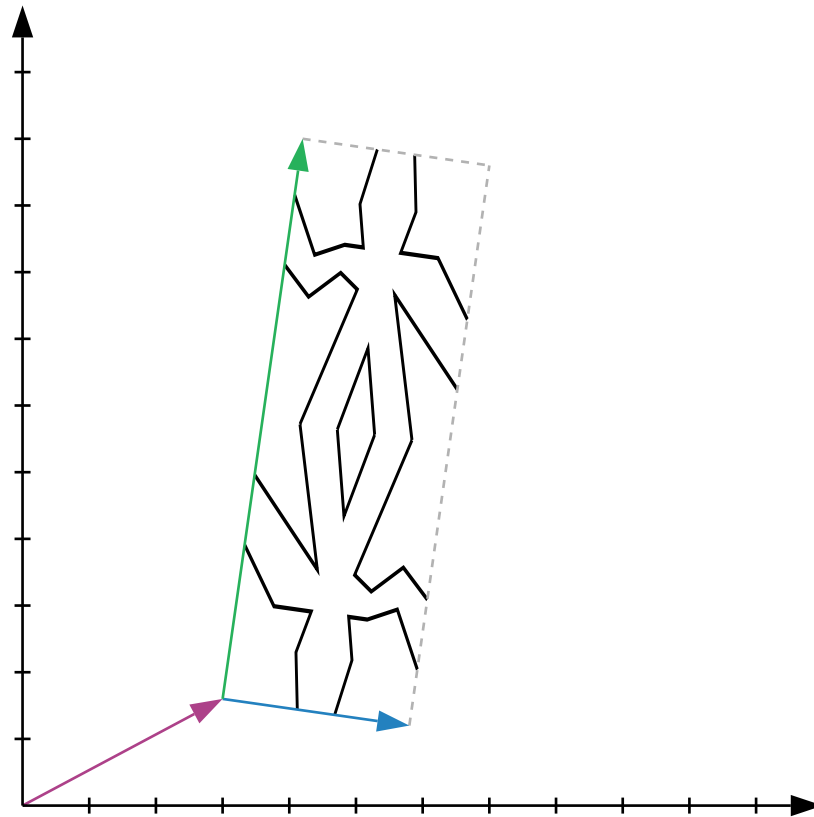
above george ((turn >> turn) george)



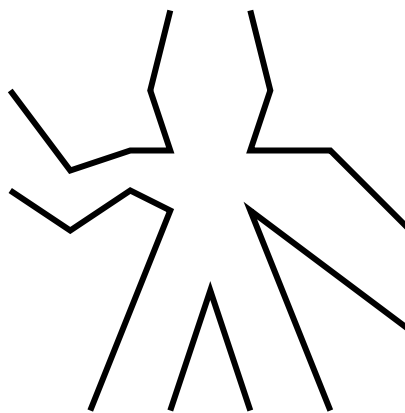
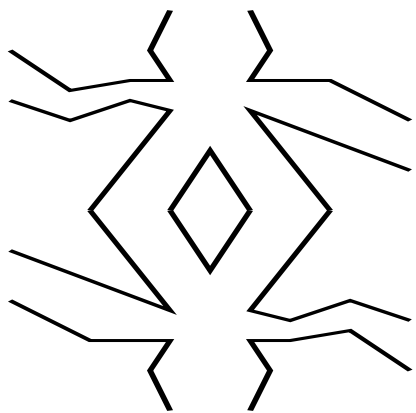
mirrorgeorge



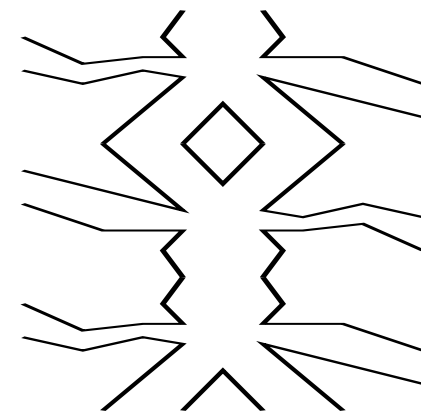
mirrorgeorge



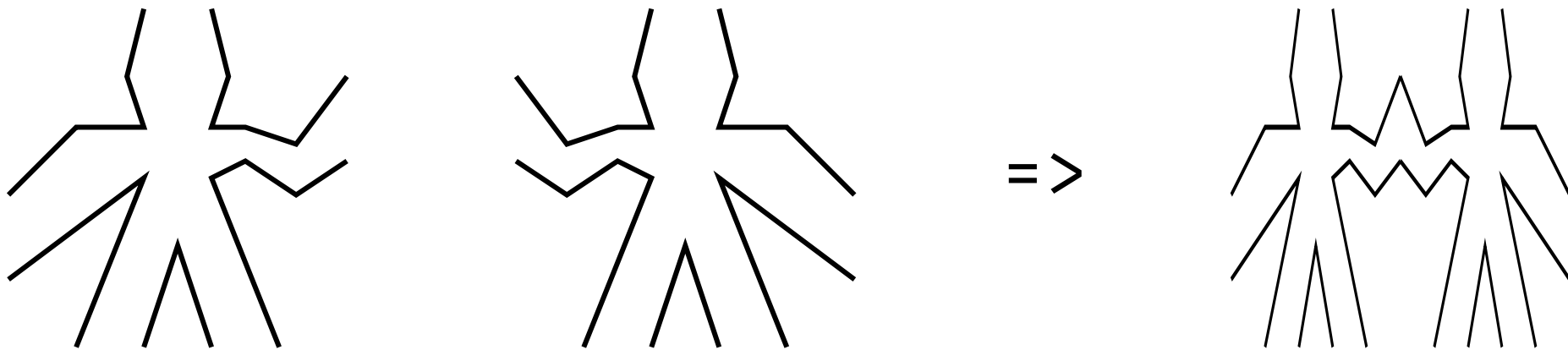
aboveRatio 2 1 mirrorgeorge george



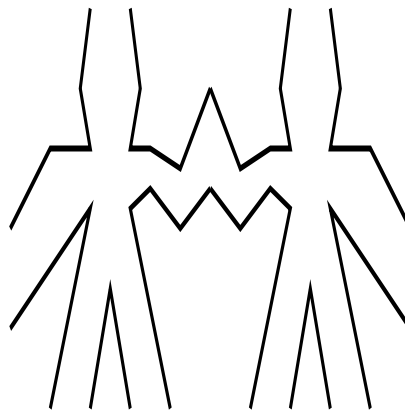
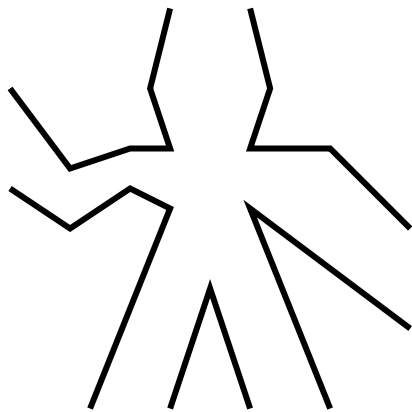
=>



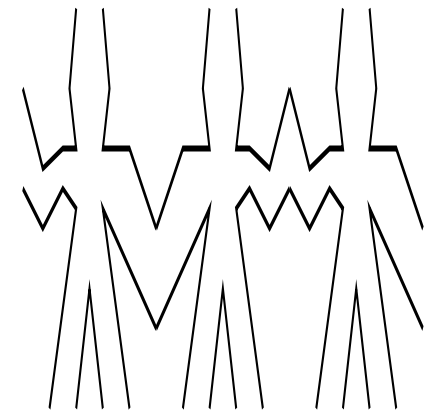
beside (flip george) george



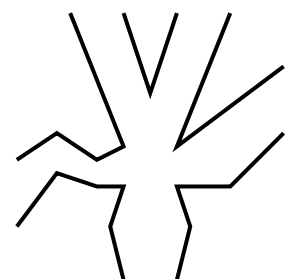
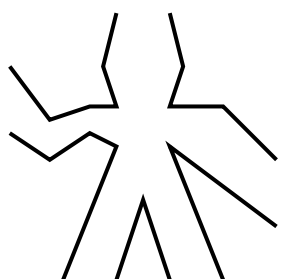
besideRatio 1 2 george twingeorge



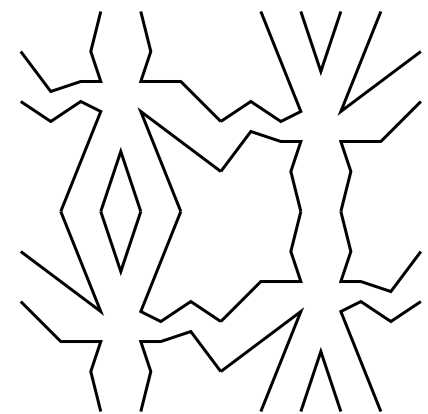
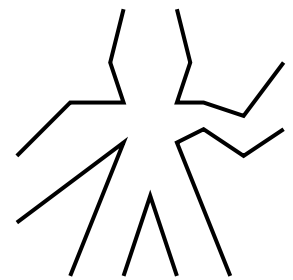
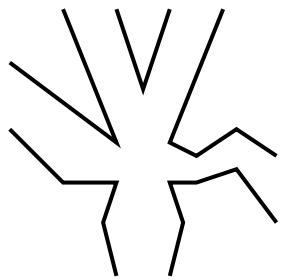
=>



quartet g1 g2 g3 g4



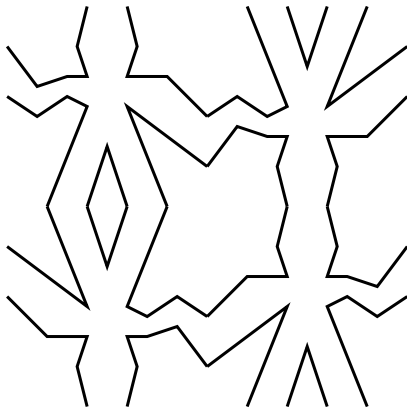
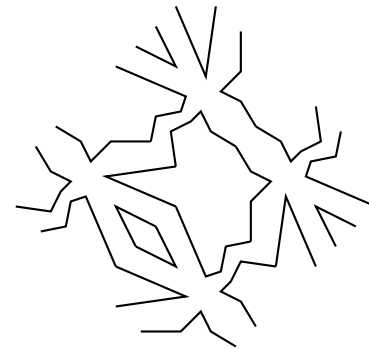
=>



```
quartet : P -> P -> P -> P -> P
quartet nw ne sw se =
    above (beside nw ne)
          (beside sw se)
```



toss

 $\Rightarrow$ 

nonet h e n d e r s o n

H E N

D E R

S O N

=>

H E N

D E R

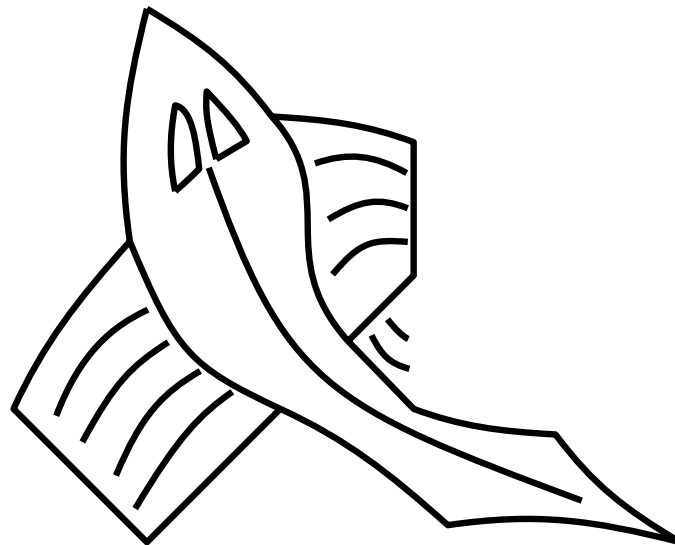
S O N

```
nonet : P -> P -> P -> P -> P -> P -> P -> P -> P
  let
    row w m e = besideRatio 1 2 w (beside m e)
    col n m s = aboveRatio 1 2 n (above m s)
  in
    col (row nw nm ne)
        (row nw nm ne)
        (row nw nm ne)
```

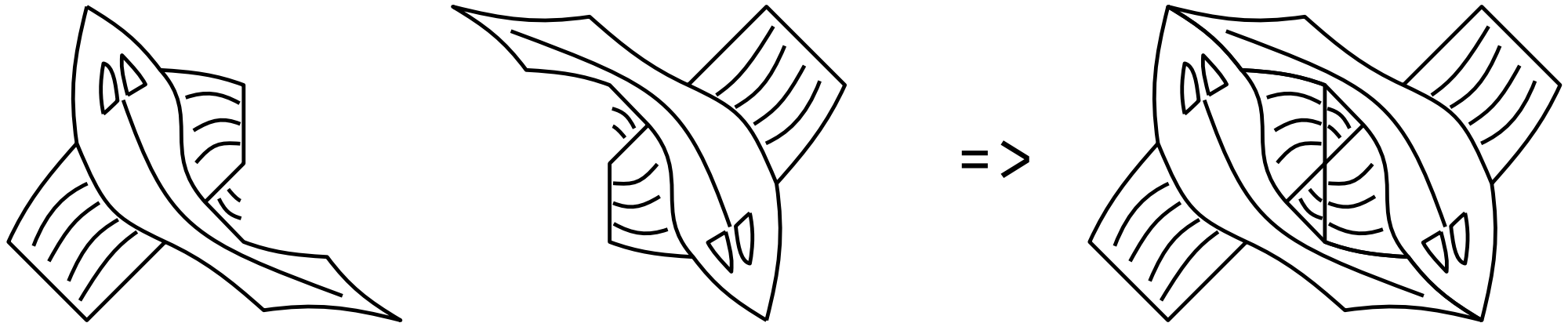
nonets are just pictures



a fish picture

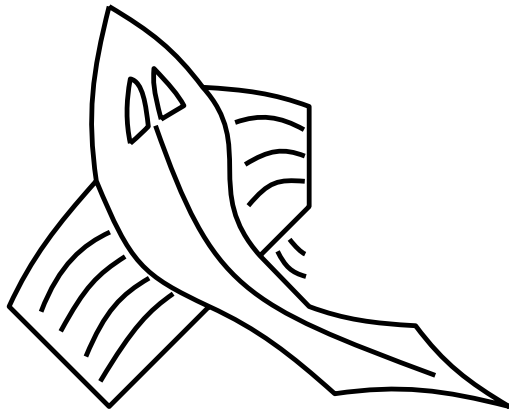
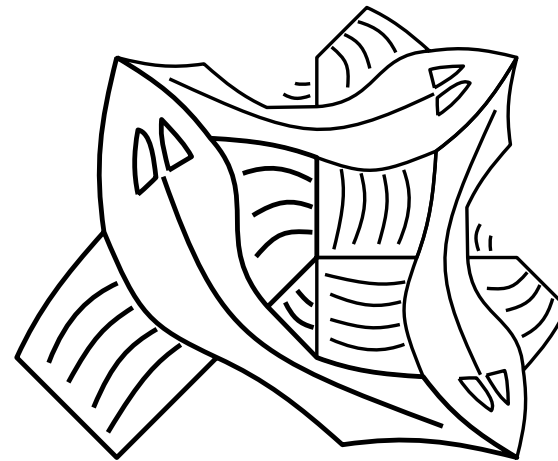


```
over fish ((turn >> turn) fish)
```



```
over : Pic -> Pic -> Pic
over p1 p2
    \box -> p1 box ++ p2 box
```

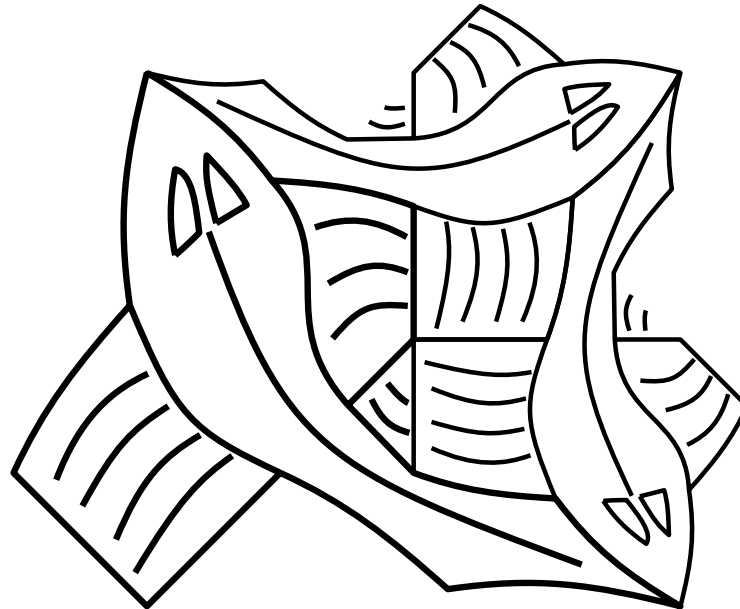
ttitle

 $\Rightarrow$ 

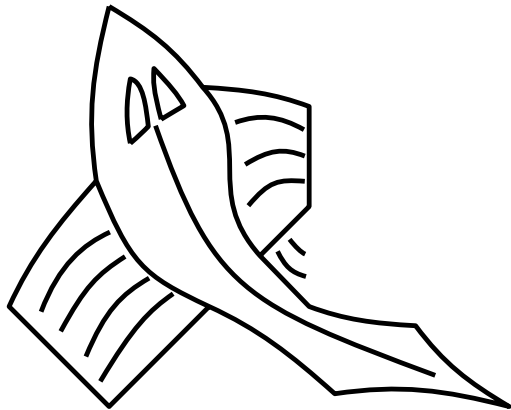


```
ttile : Picture -> Picture
ttile p =
    let
        pn = (toss >> flip) p
        pe = (turn >> turn >> turn) p
    in
        over p (over pn pe)
```

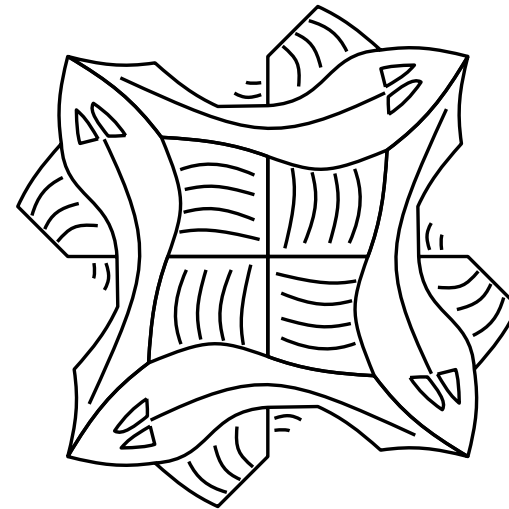
ttitle



utile

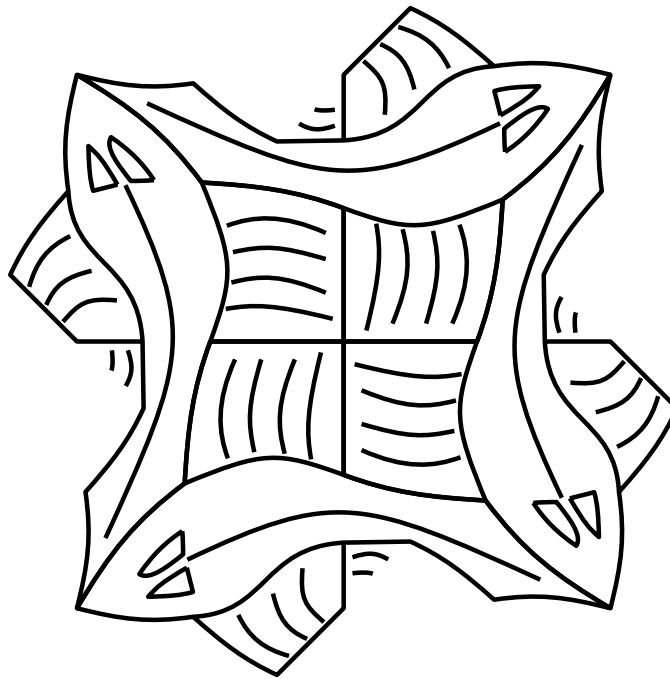


=>

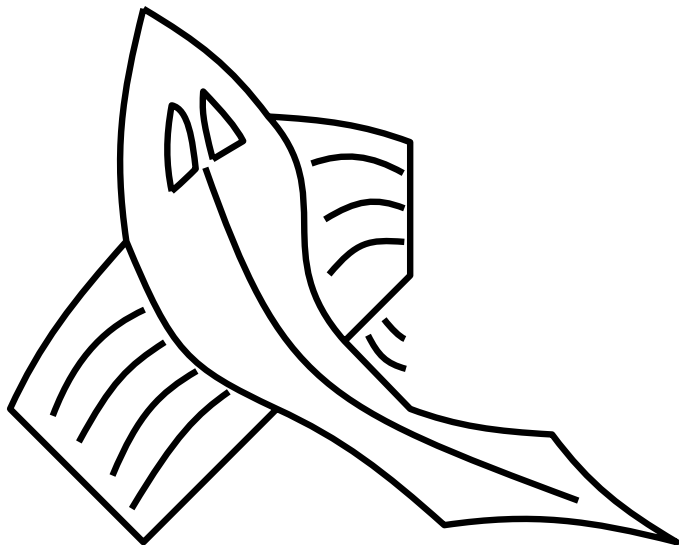


```
utile : Picture -> Picture
utile p =
  let
    pn = (toss >> flip) p
    pw = turn pn
    ps = turn pw
    pe = turn ps
  in
    over pn (over pw (over ps pe))
```

utile

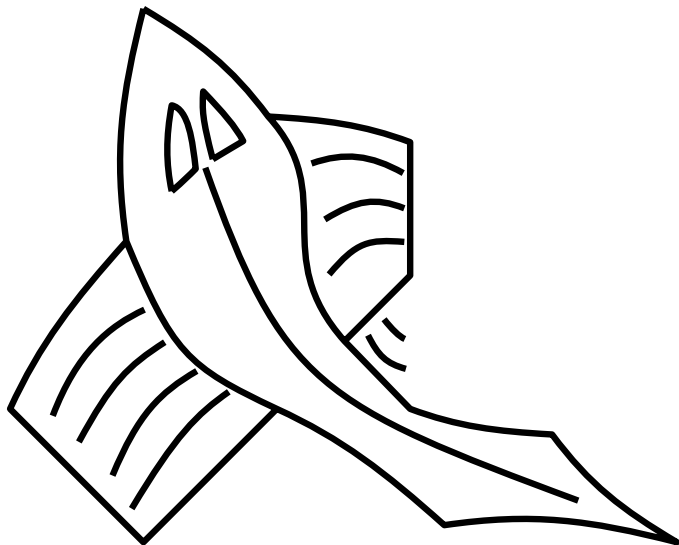
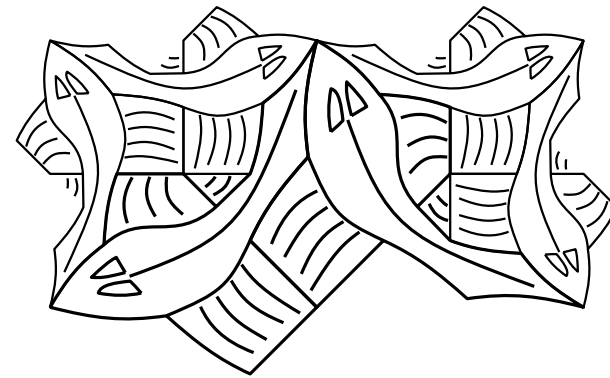


side 0

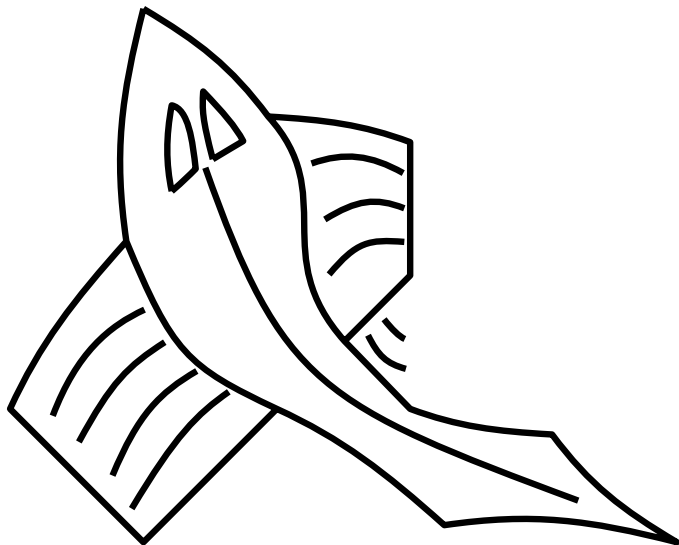


=>

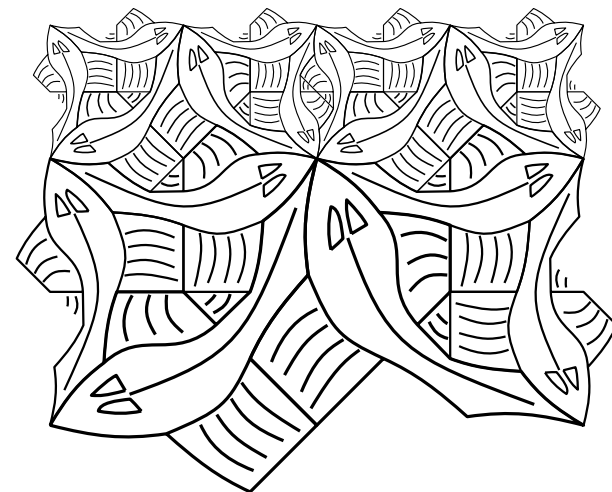
side 1

 $\Rightarrow$ 

side 2

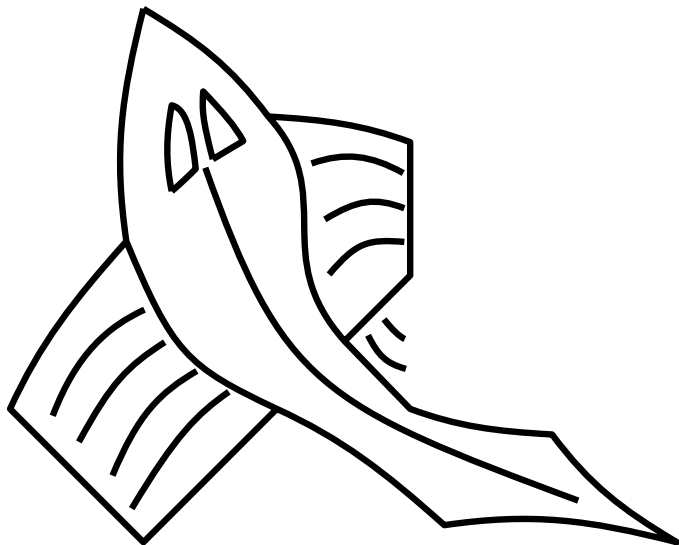
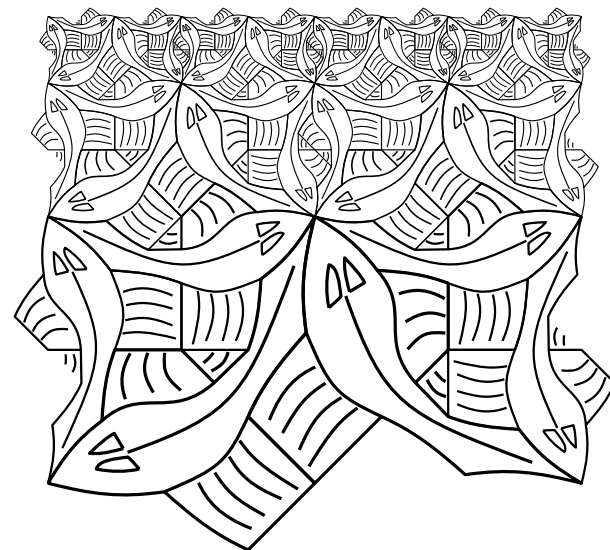


=>



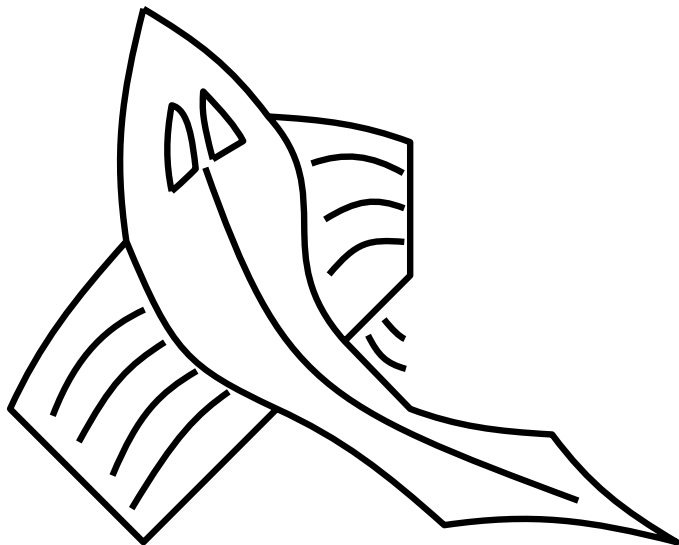


side 3

 $\Rightarrow$ 

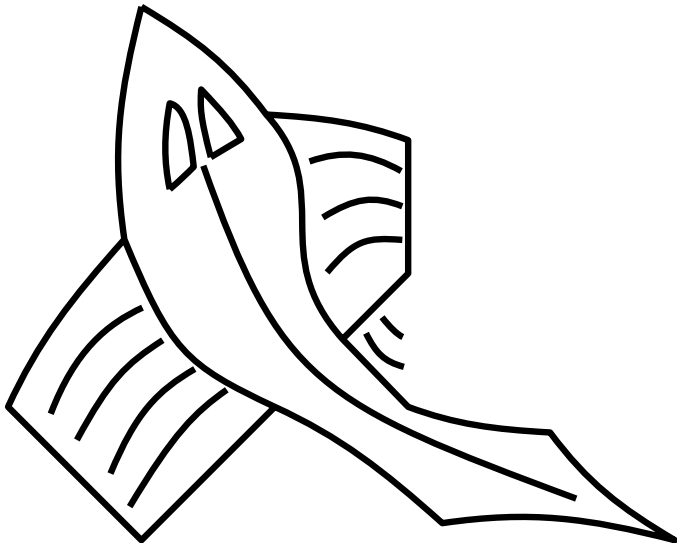
```
side : Int -> Picture -> Picture
side n p =
    if n <= 0 then blank
    else
        let
            s = side (n - 1) p
            t = ttile p
        in
            quartet s s (turn t) t
```

corner 0

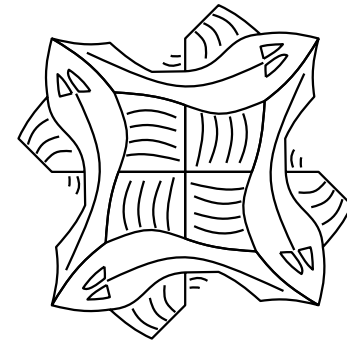


=>

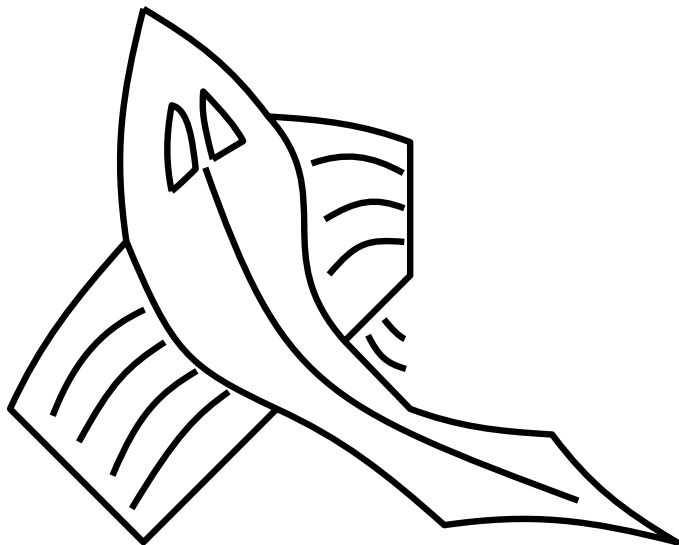
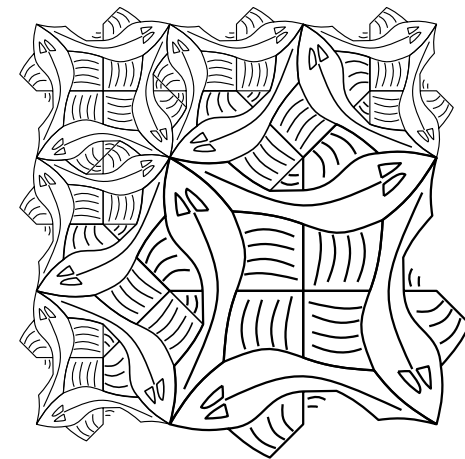
corner 1



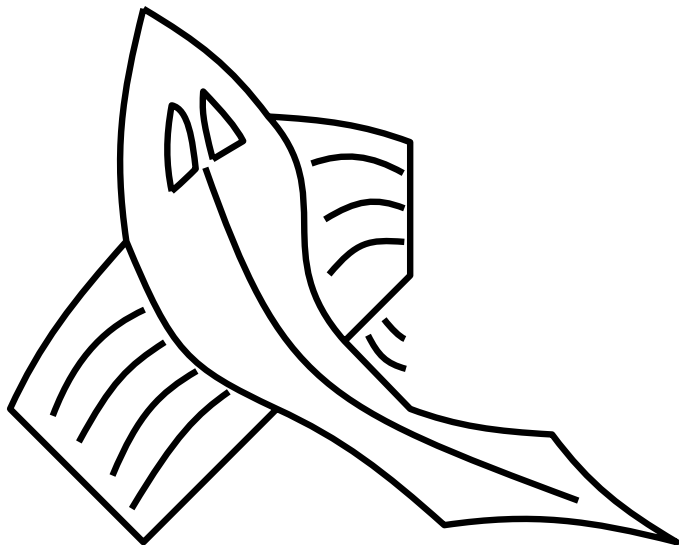
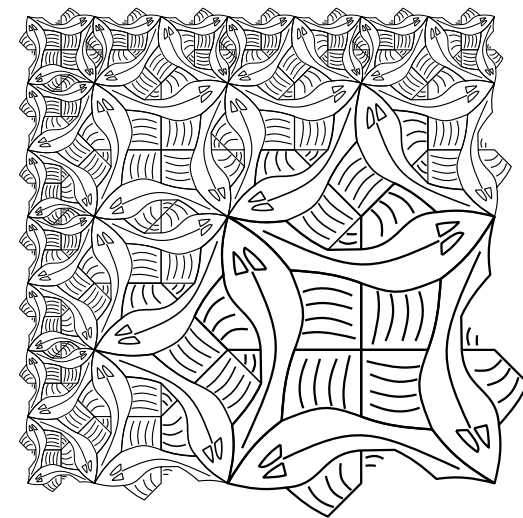
=>



corner 2

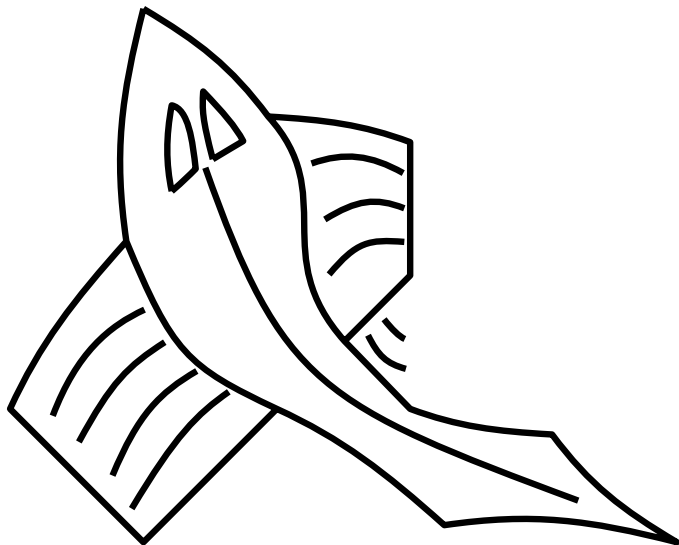
 $\Rightarrow$ 

corner 3

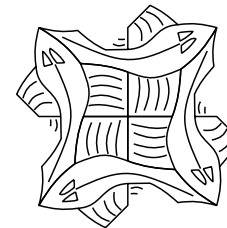
 $\Rightarrow$ 

```
corner : Int -> Picture -> Picture
corner n p =
  if n <= 0 then blank
  else
    let
      c = corner (n - 1) p
      s = side (n - 1) p
    in
      quartet c s (turn s) (utile p)
```

square-limit 0

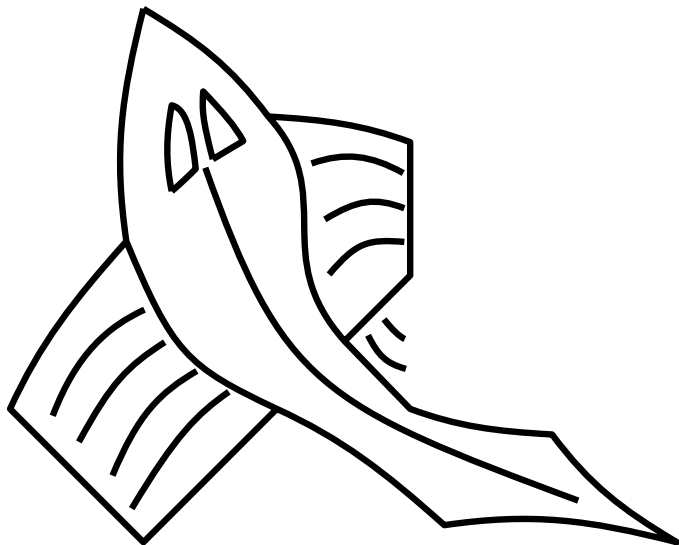


=>

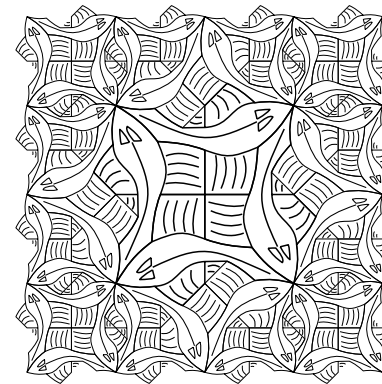




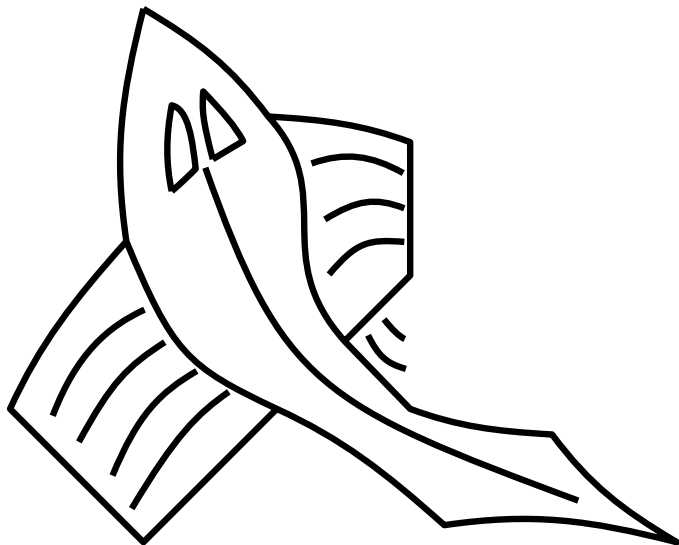
square-limit 1



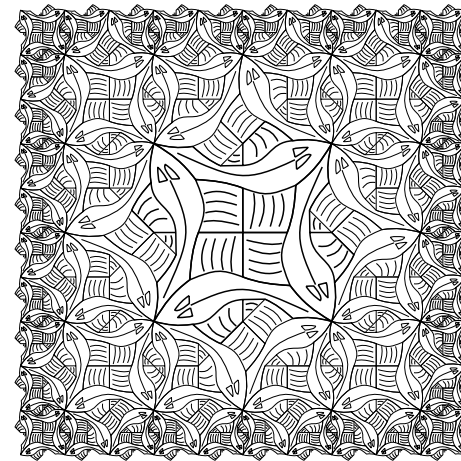
=>



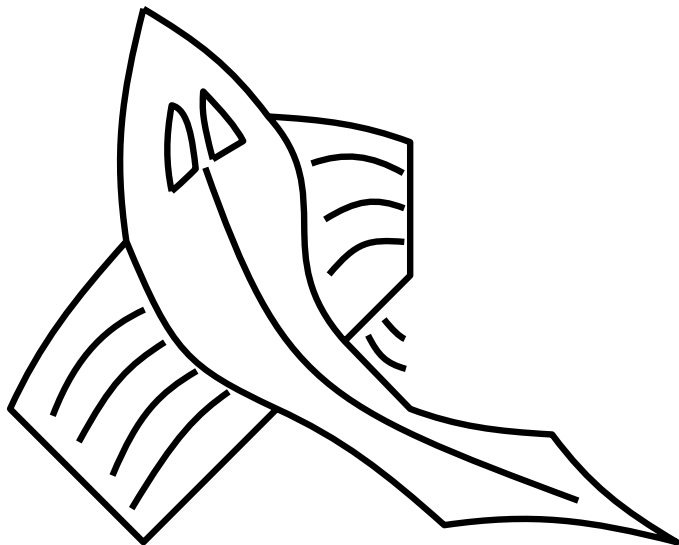
square-limit 2



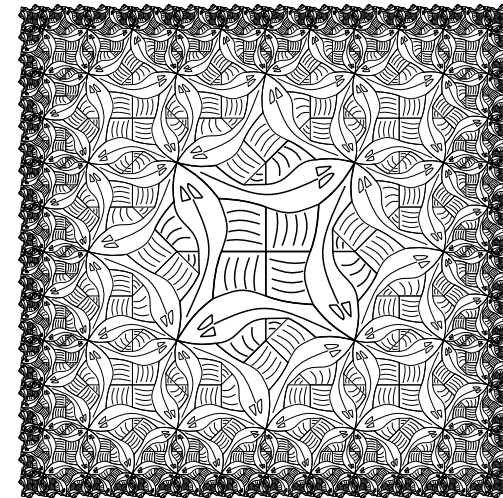
=>



square-limit 3



=>



```
squareLimit : Int -> Picture -> Picture
squareLimit n p =
  let
    mm = utile p
    nw = corner n p
    sw = turn nw
    se = turn sw
    ne = turn se
    nm = side n p
    mw = turn nm
    sm = turn mw
    me = turn sm
  in
    nonet nw nm ne mw mm me sw sm se
```

# Henderson's square limit

