



## Functional Geometry

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**Abstract.** An algebra of pictures is described that is sufficiently powerful to denote the structure of a well-known Escher woodcut, Square Limit. A decomposition of the picture that is reasonably faithful to Escher's original design is given. This illustrates how a suitably chosen algebraic specification can be both a clear description and a practical implementation method. It also allows us to address some of the criteria that make a good algebraic description.

**Keywords:** Functional programming, graphics, geometry, algebraic style, architecture, specification.

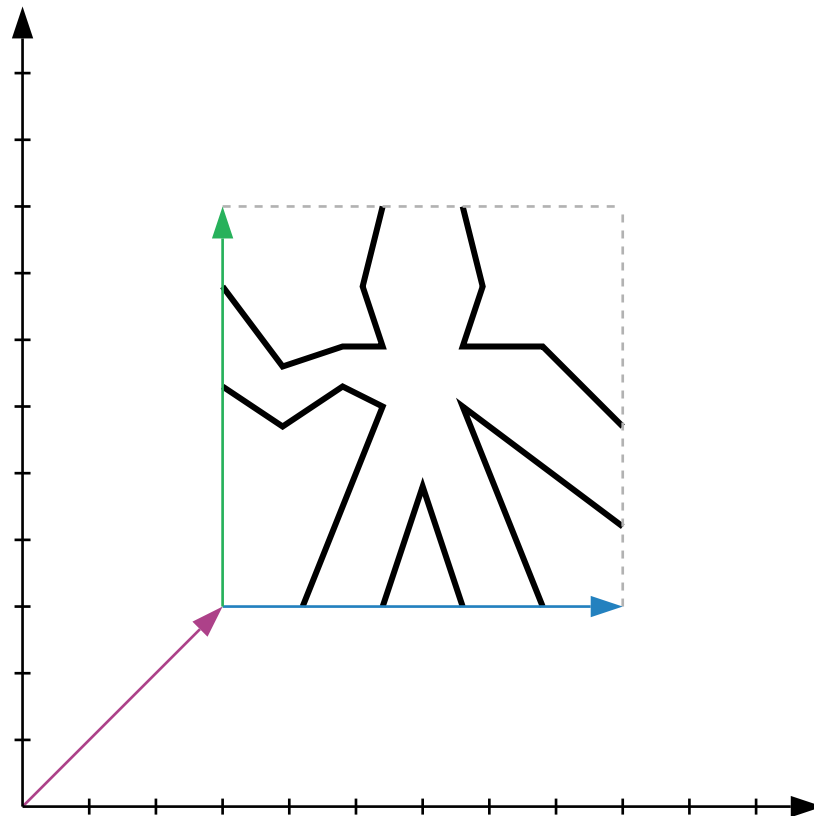
A **picture** is an example  
of a **complex object** that  
can be described in terms  
of its **parts**.

Let us define a picture as a **function** which takes three arguments, each being two-space **vectors** and returns **a set of graphical objects** to be rendered on the output device.

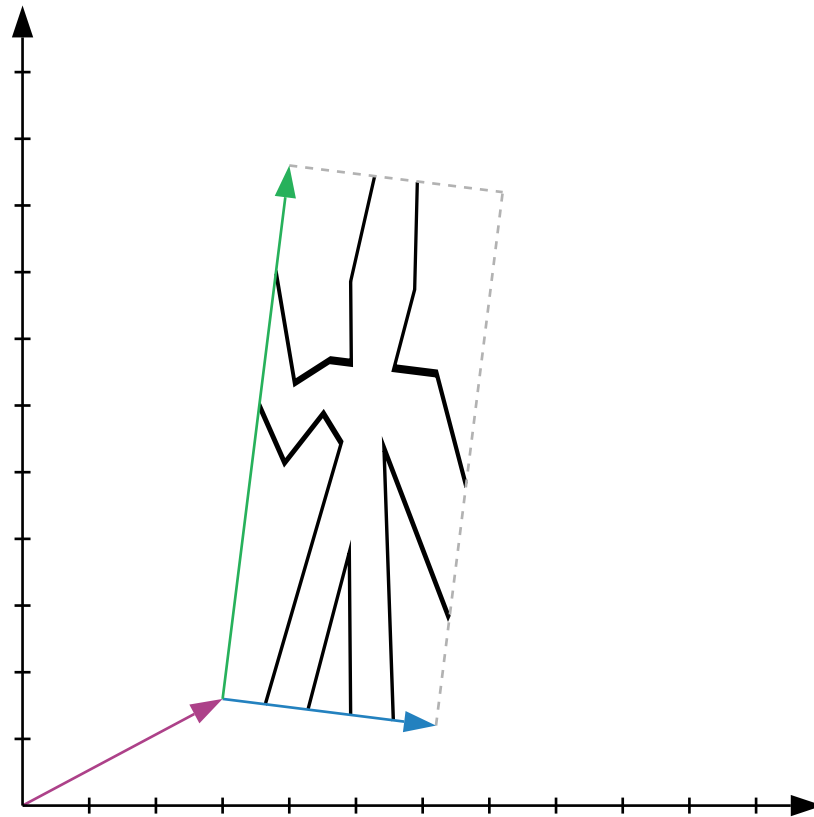
```
type Box = { a : Vector  
             b : Vector  
             c : Vector }
```

```
type Picture = Box -> Rendering
```

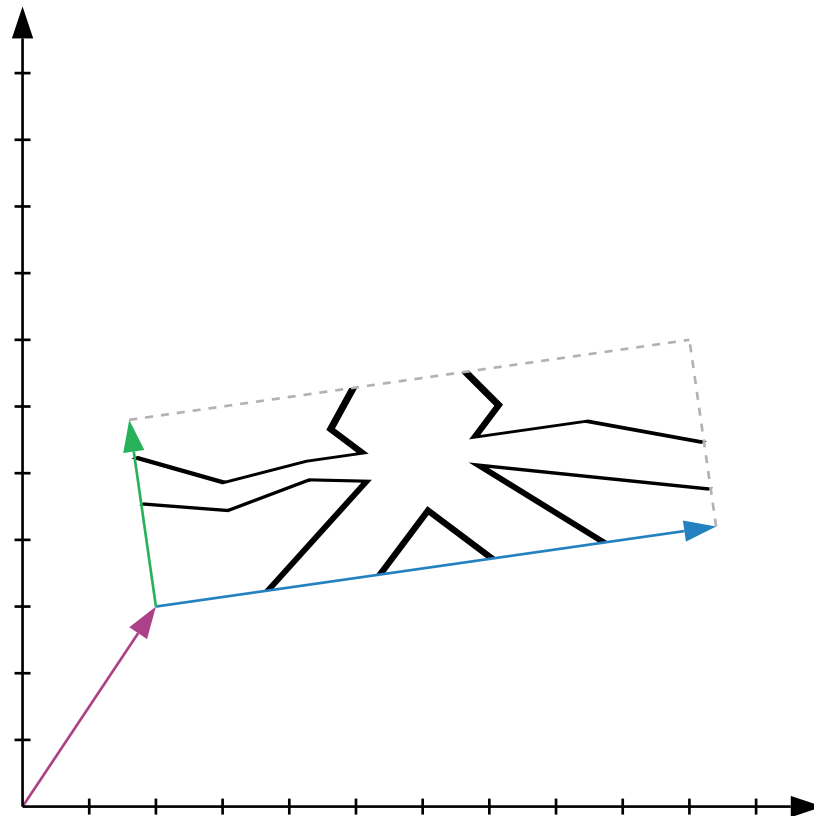
george



also george



still george





turn



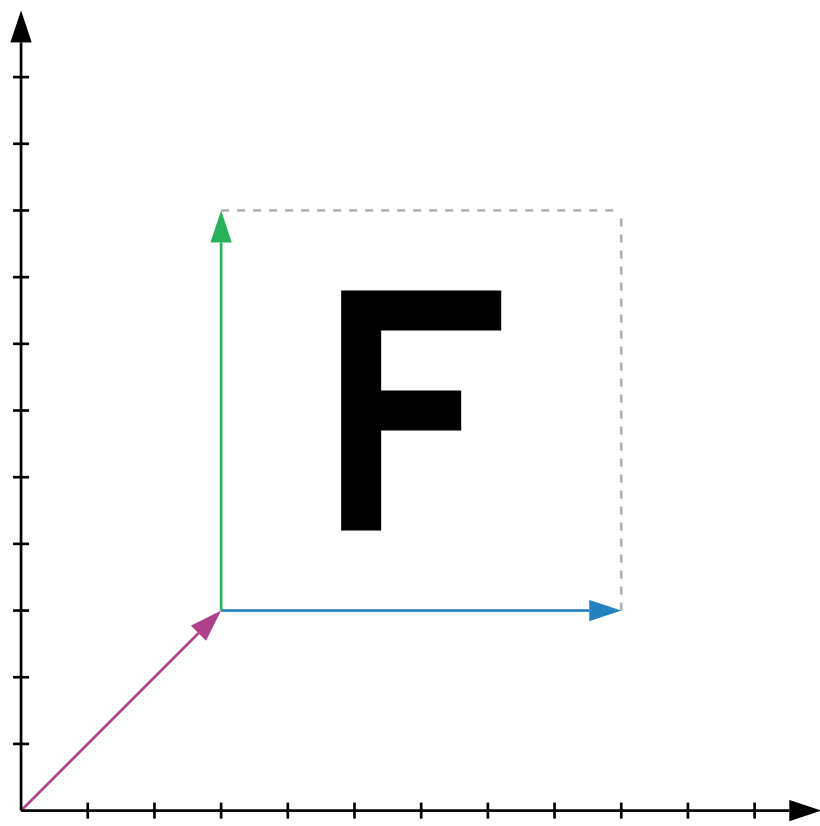
=>



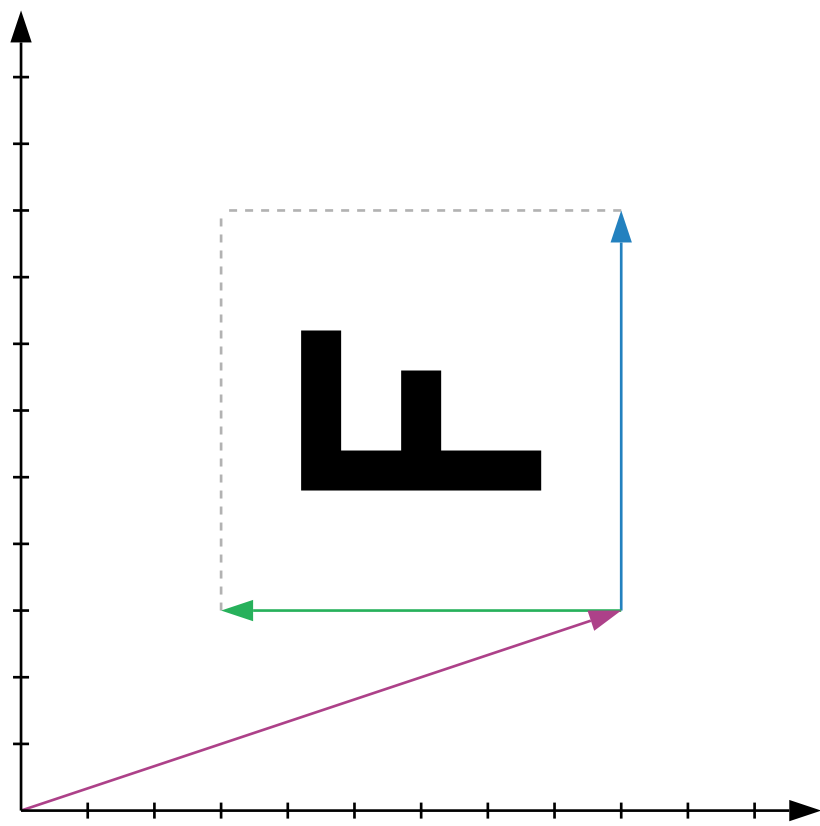
```
turnBox : Box -> Box
turnBox { a, b, c } = { a = add a b
                        , b = c
                        , c = neg b }
```

```
turn : Picture -> Picture
turn p = turnBox >> p
```

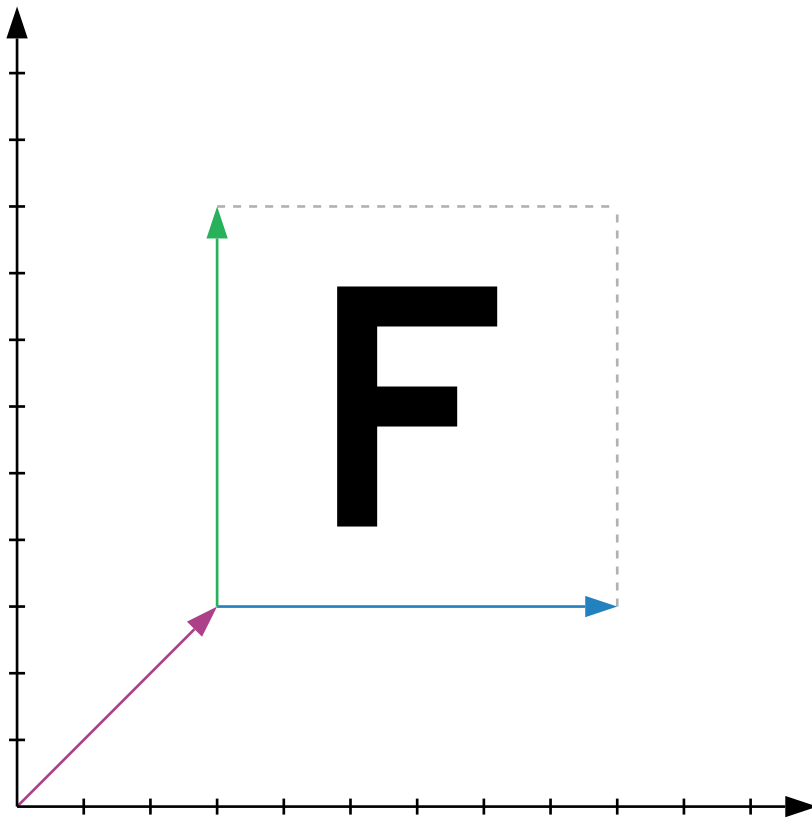
turn



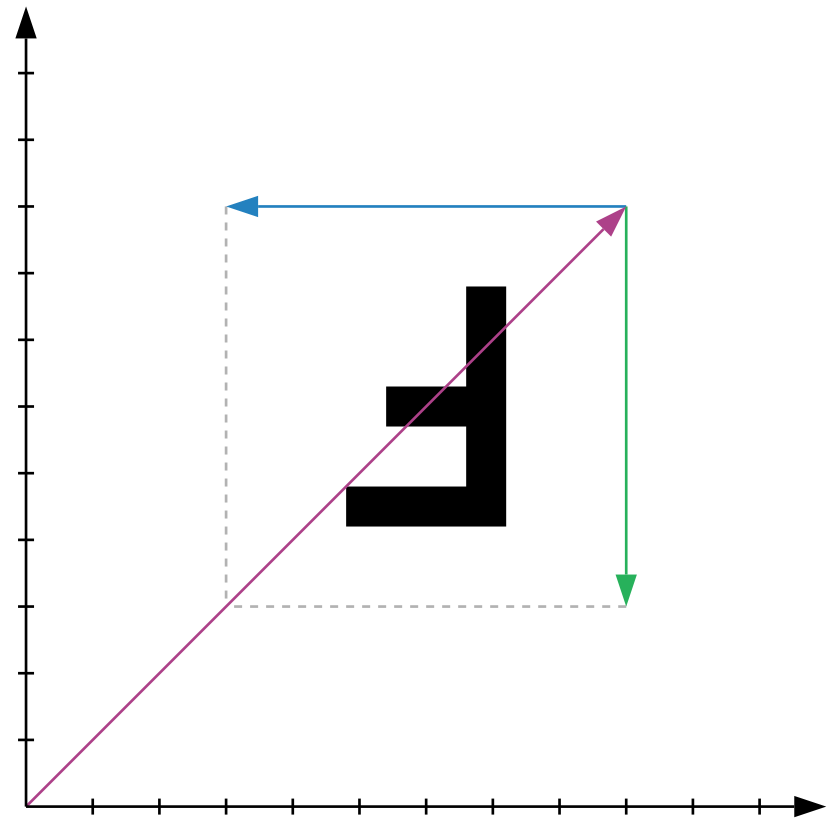
=>



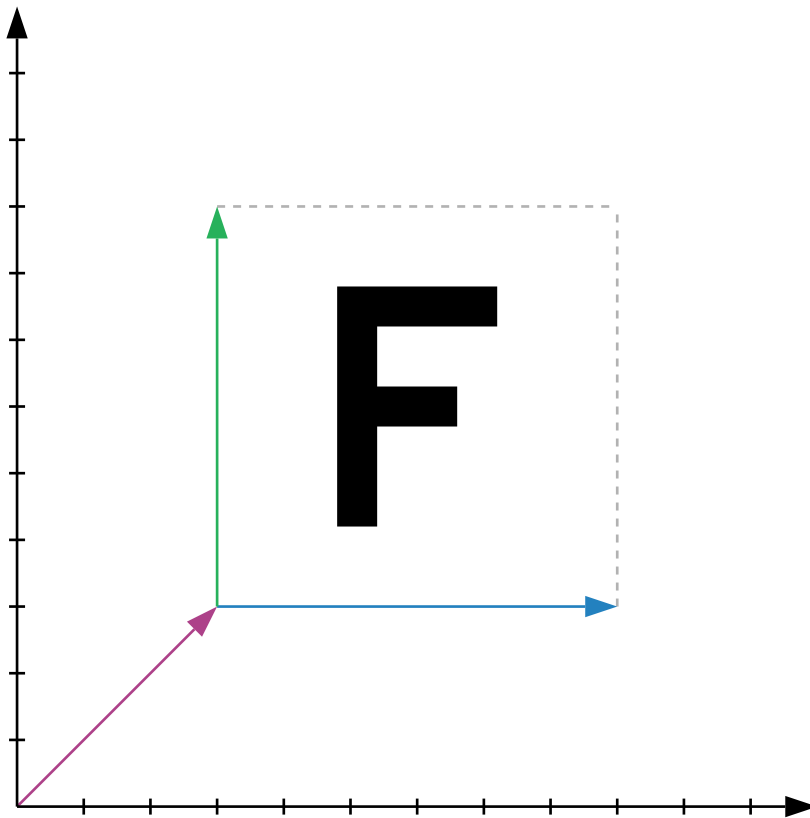
turn >> turn



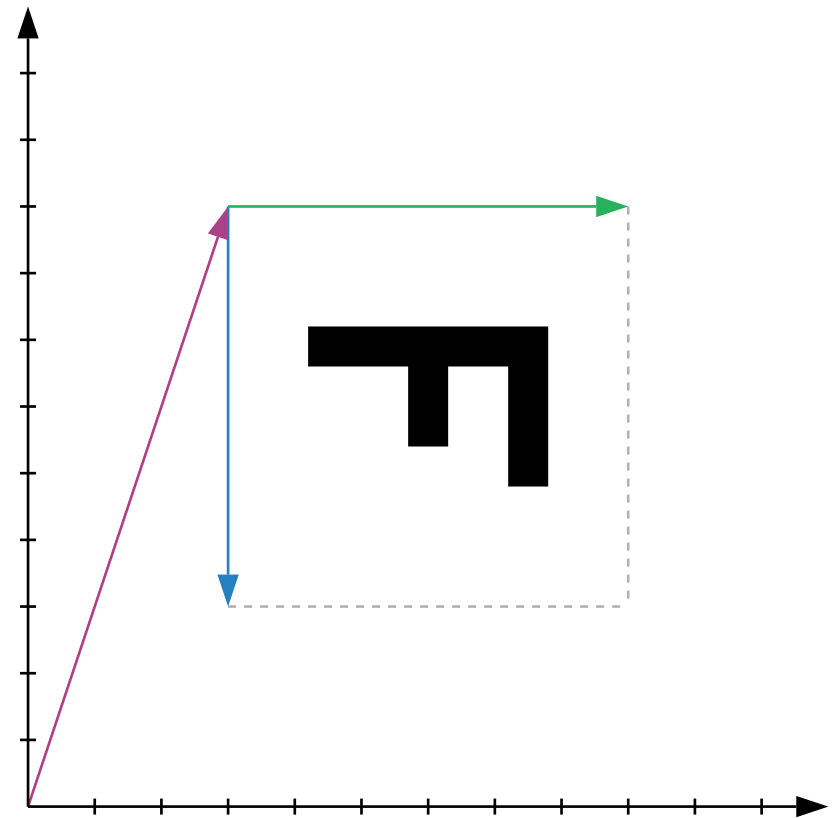
=>



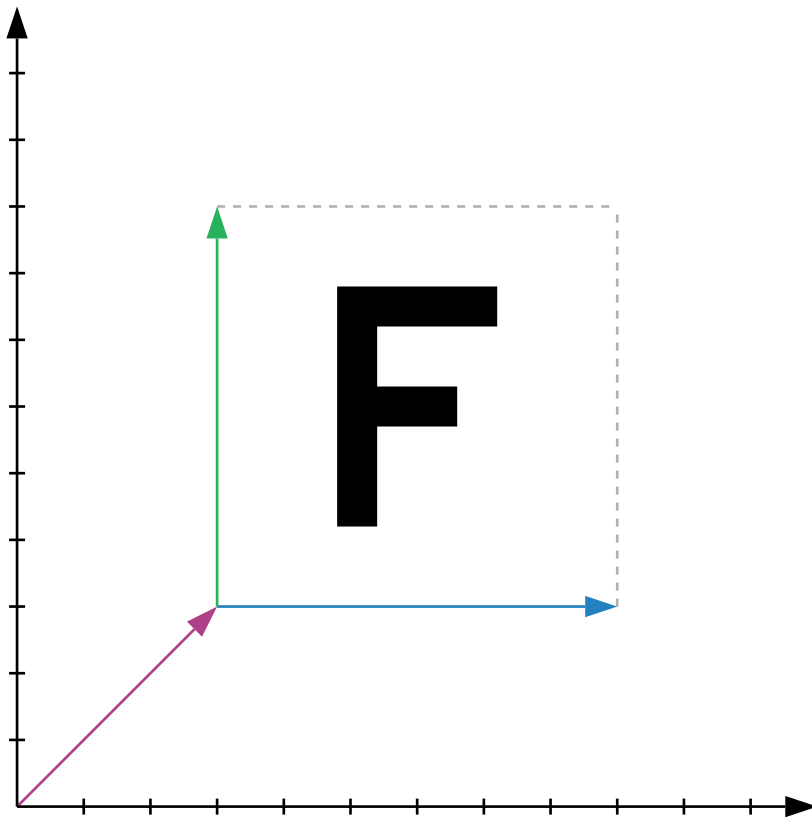
turn >> turn >> turn



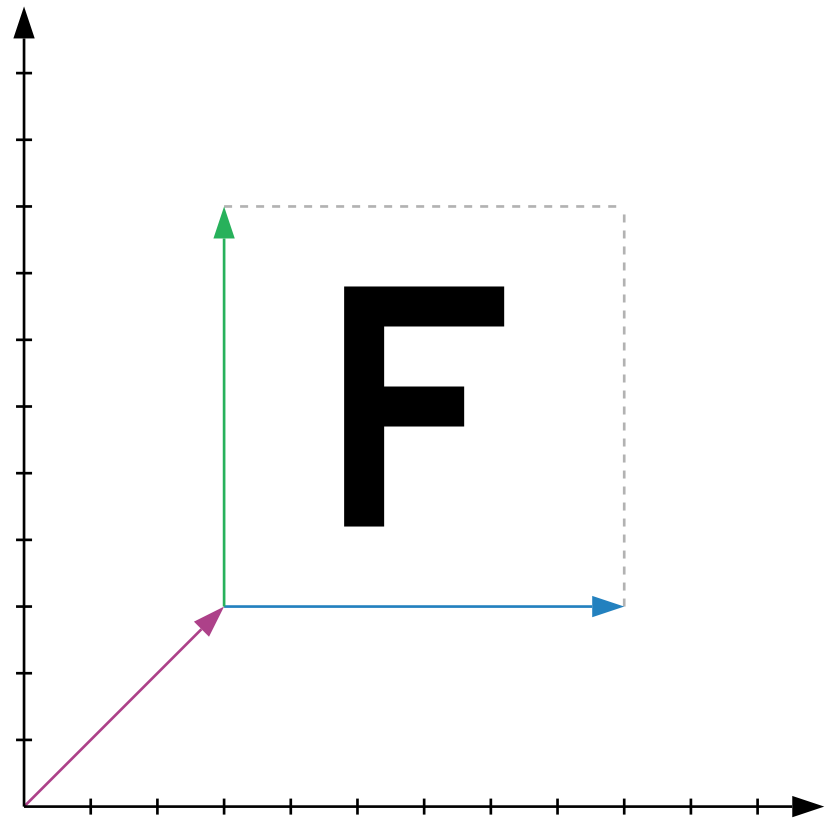
=>



turn >> turn >> turn >> turn



=>



flip

F

=>

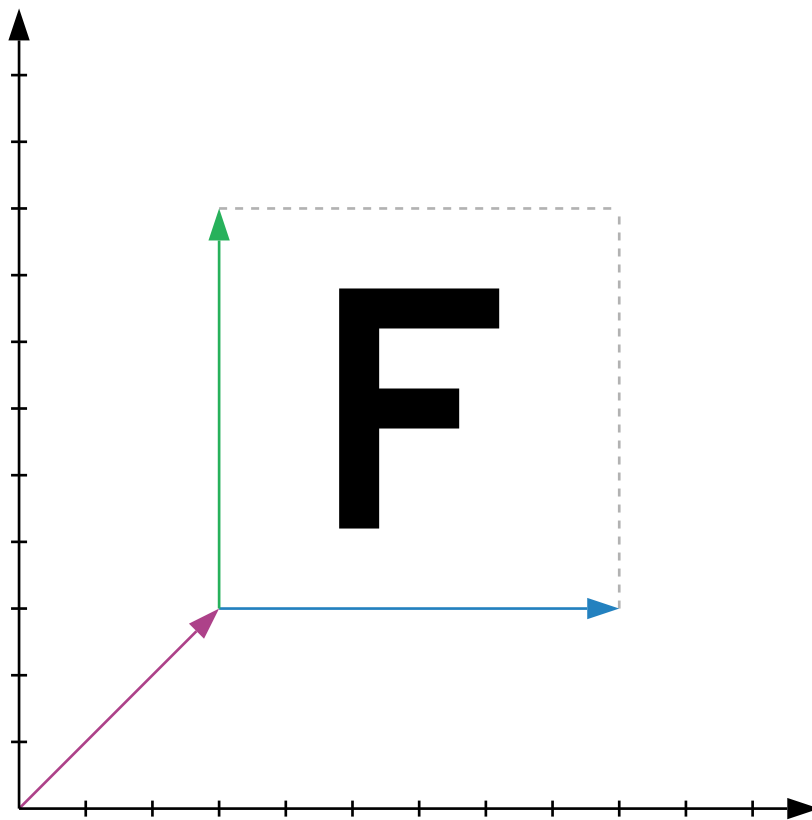
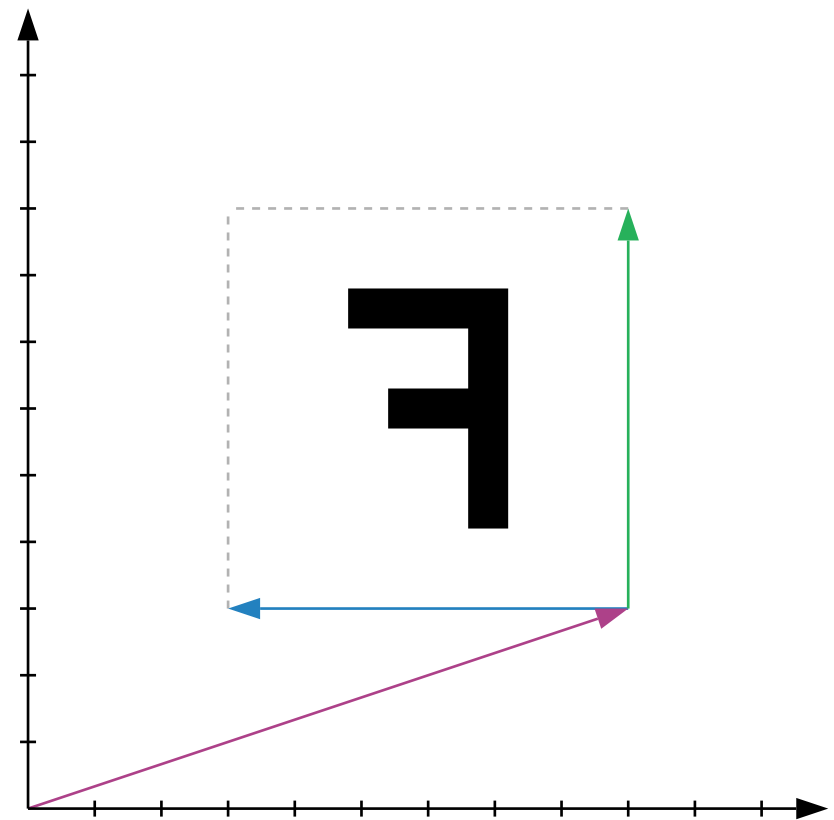
Ɔ

```
flipBox : Box -> Box
flipBox { a, b, c } = { a = add a b
                        , b = neg b
                        , c = neg c }
```

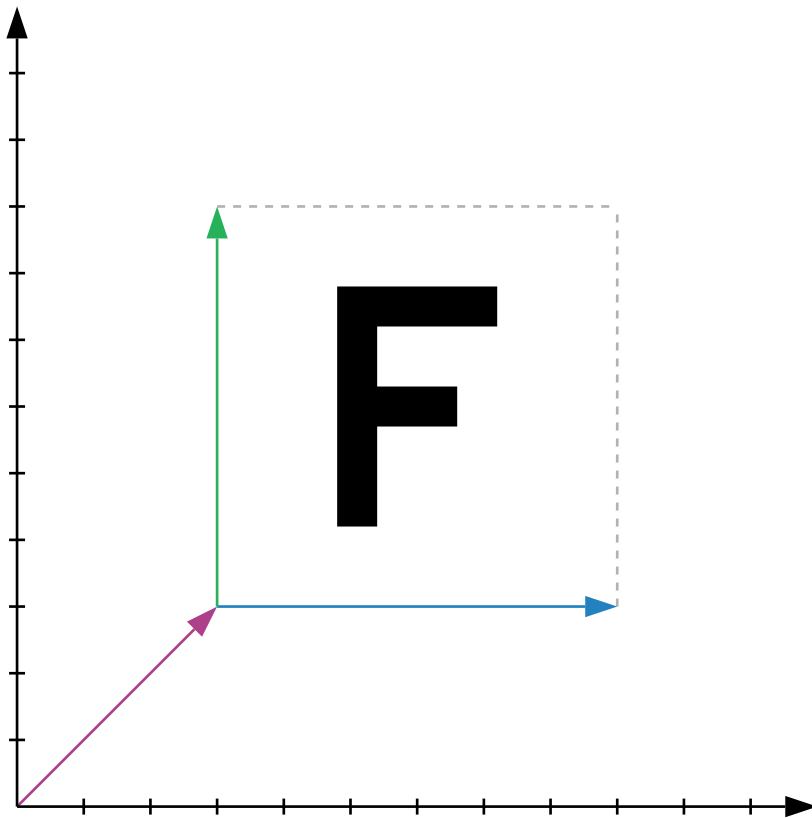
```
flip : Picture -> Picture
flip p = flipBox >> p
```



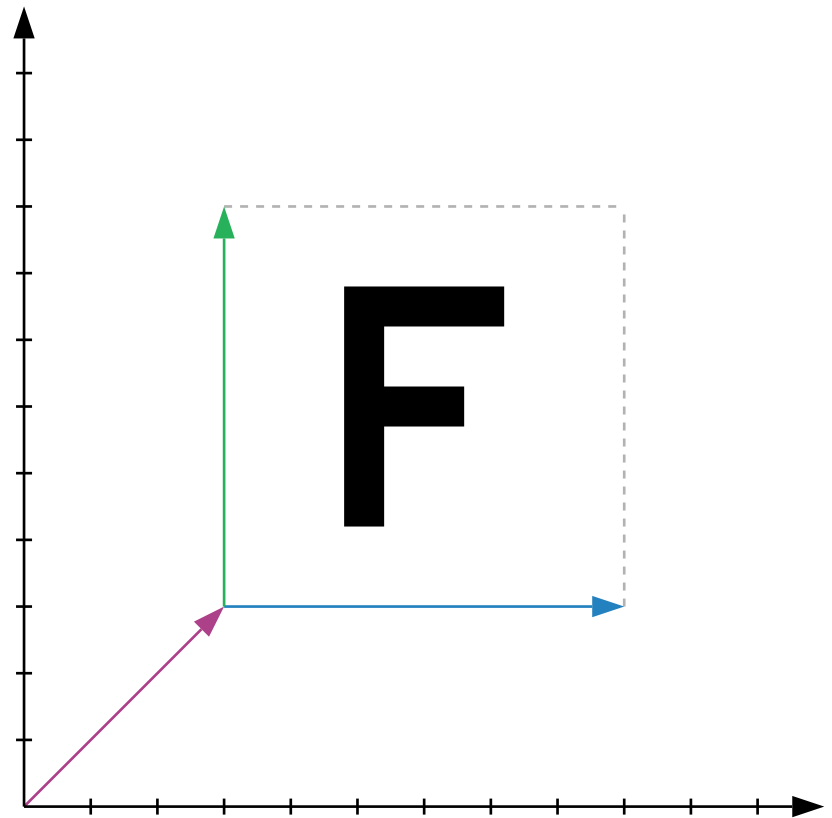
flip

 $\Rightarrow$ 

flip >> flip



=>



toss

F

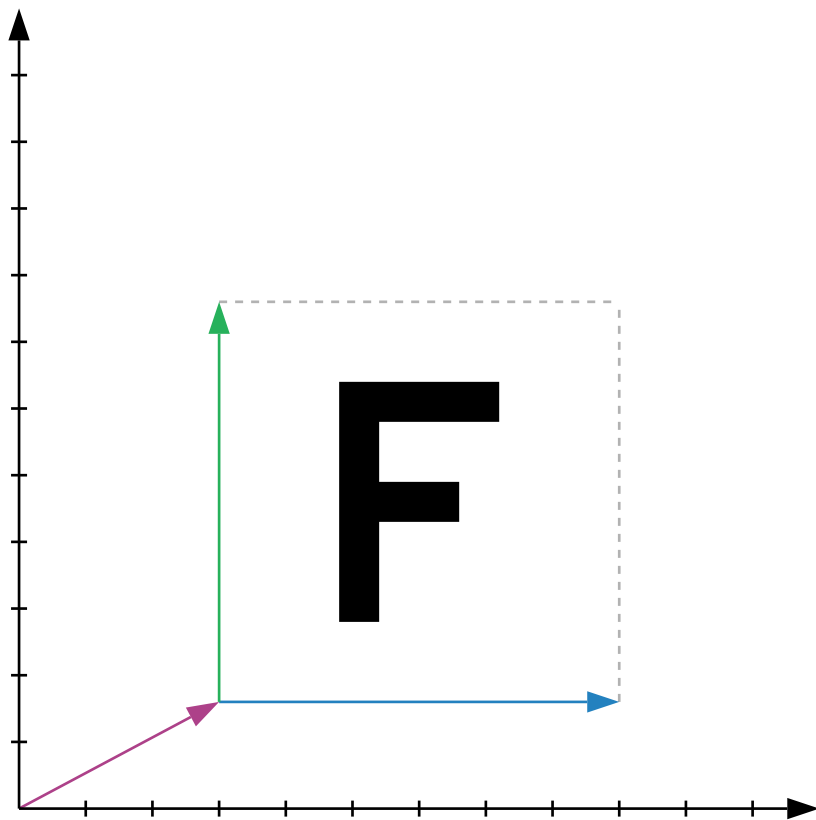
=>

F

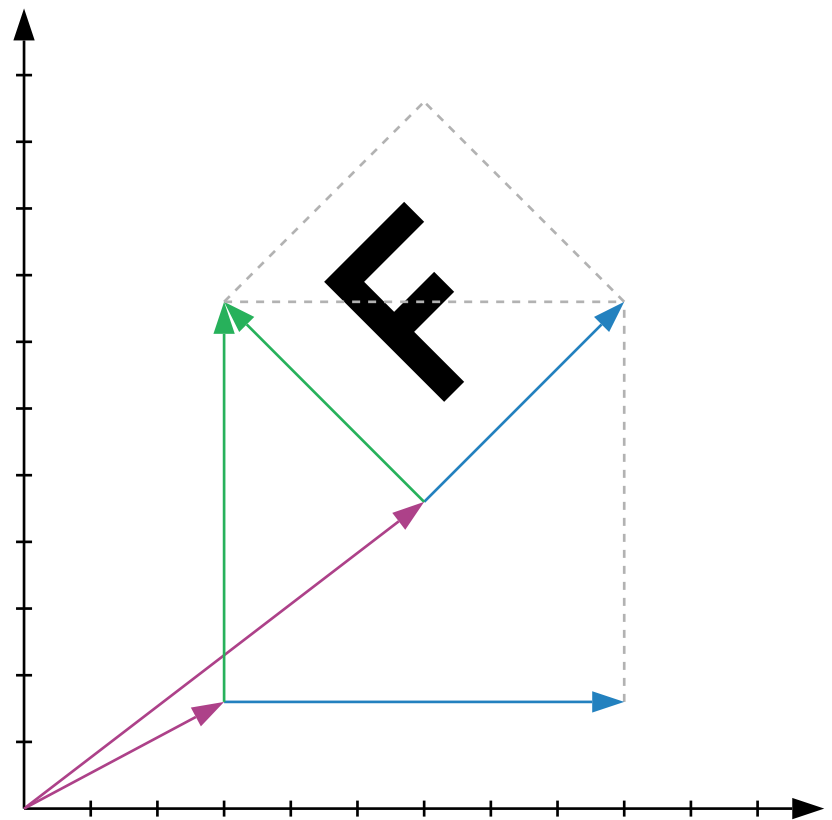
```
tossBox : Box -> Box
tossBox { a, b, c } =
  { a = add a (scale 0.5 (add b c))
    , b = scale 0.5 (add b c)
    , c = scale 0.5 (sub c b) }
```

```
toss : Picture -> Picture
toss p = tossBox >> p
```

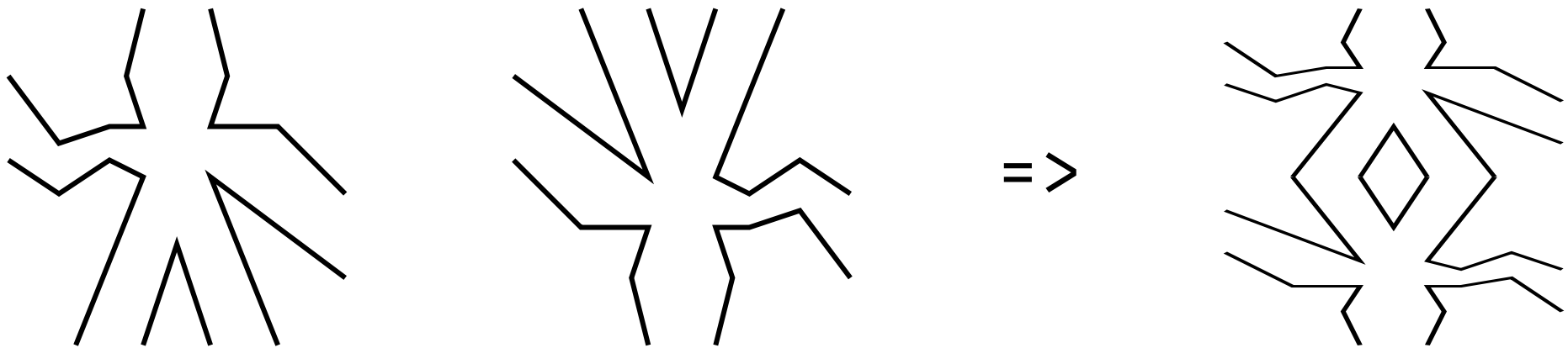
toss



$\Rightarrow$



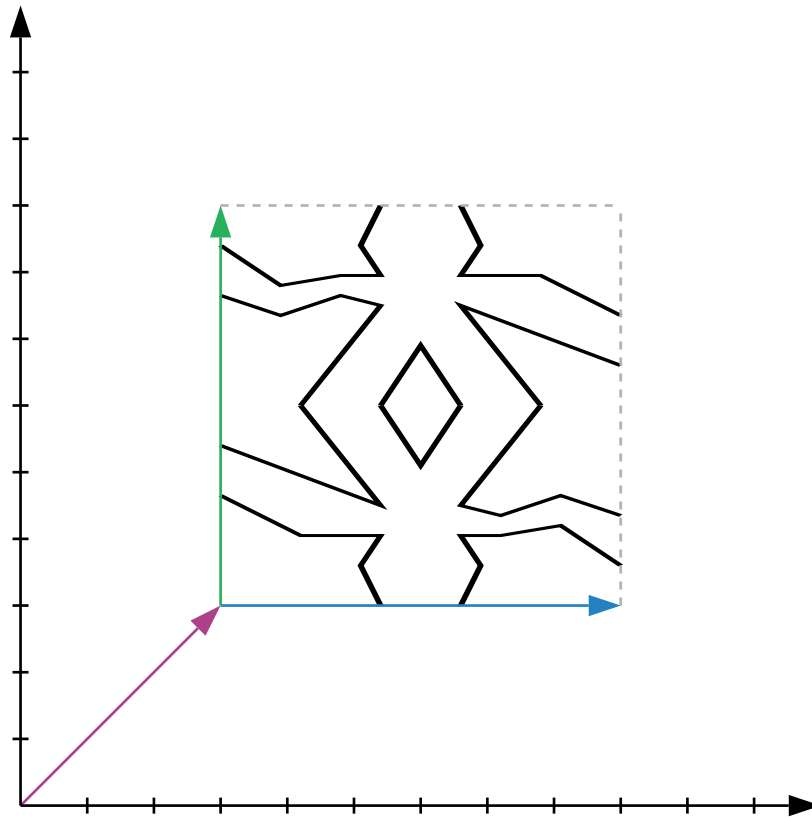
above george ((turn >> turn) george)



```
aboveRatio : Int -> Int -> Pic -> Pic -> Pic
aboveRatio m n p1 p2 =
    \box ->
        let
            f = m / (m + n)
            (b1, b2) = splitVertically f box
        in
            (p1 b1) ++ (p2 b2)

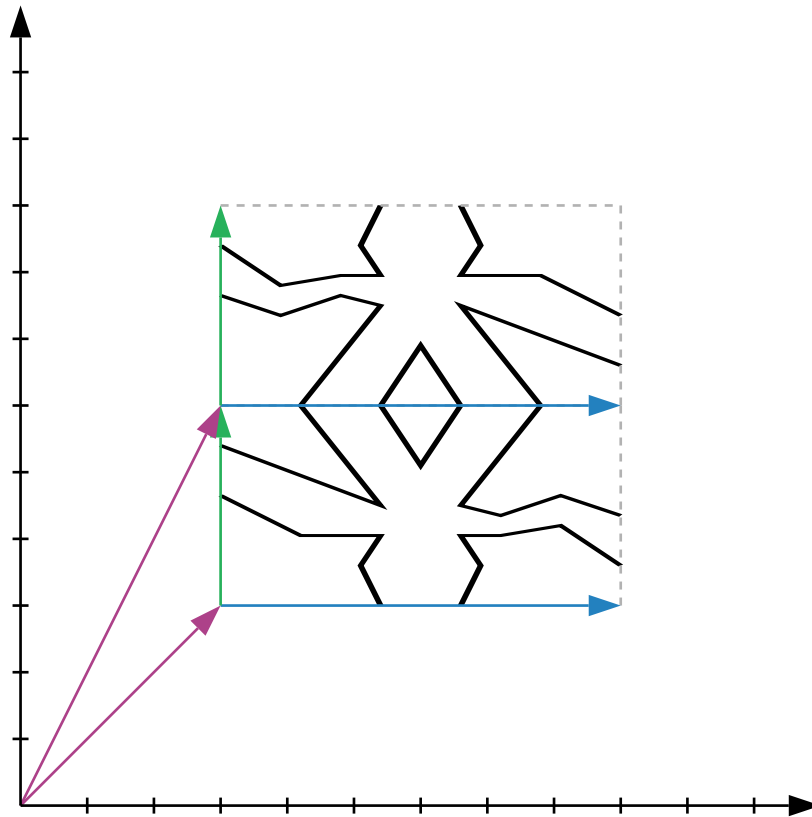
above : Pic -> Pic -> Pic
above p1 p2 = aboveRatio 1 1
```

above george ((turn >> turn) george)

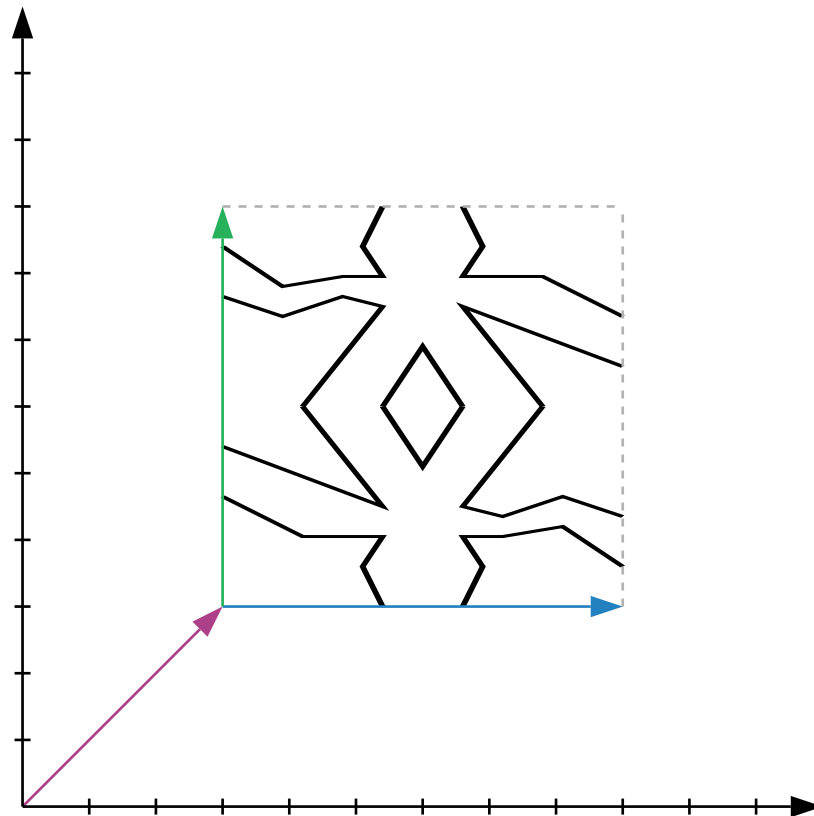




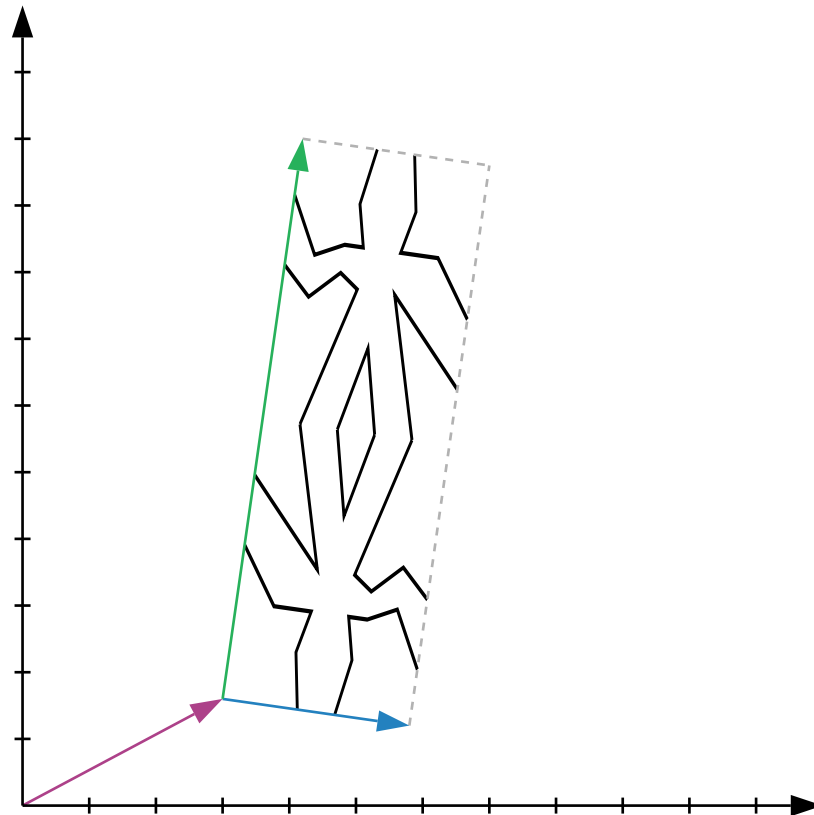
above george ((turn >> turn) george)



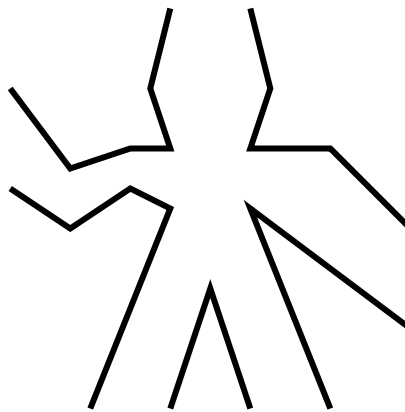
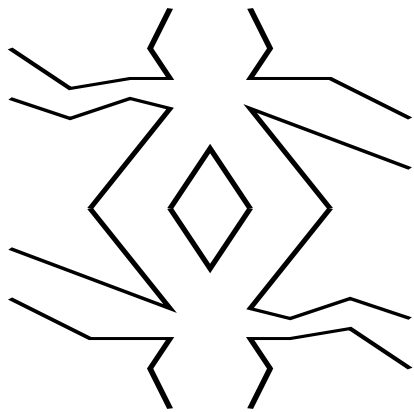
mirrorgeorge



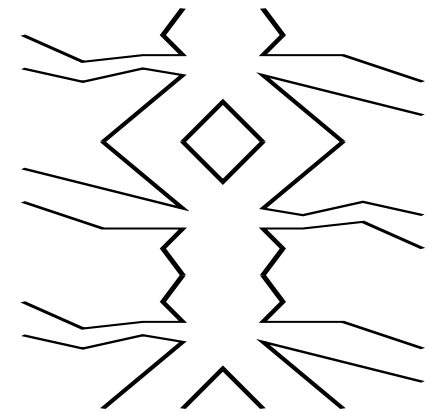
mirrorgeorge



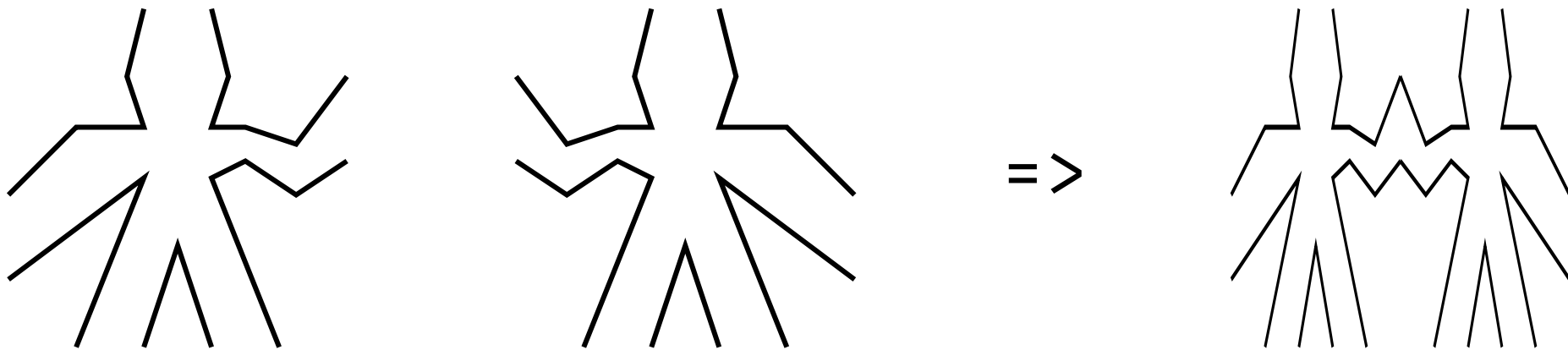
aboveRatio 2 1 mirrorgeorge george



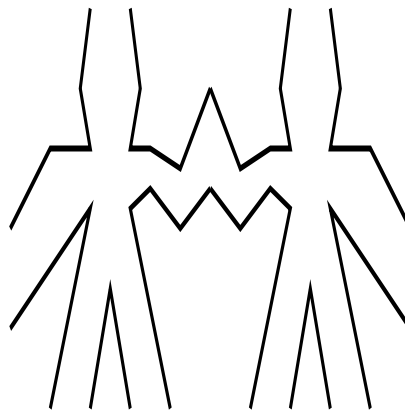
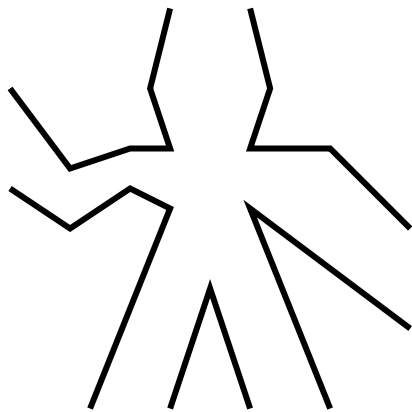
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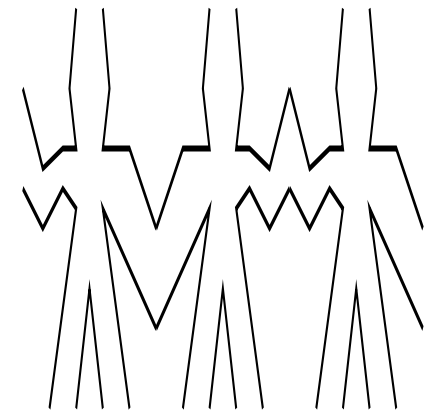
beside (flip george) george



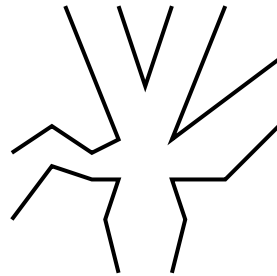
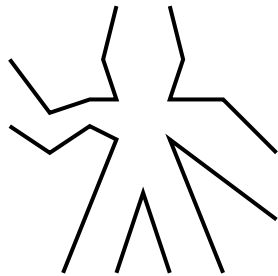
besideRatio 1 2 george twingeorge



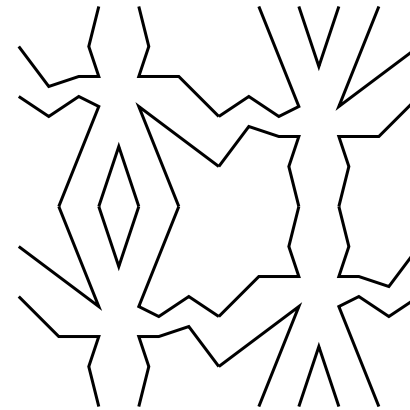
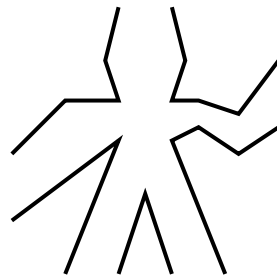
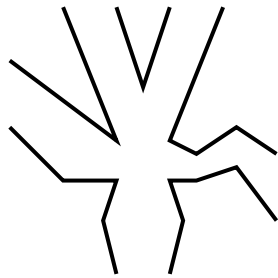
=>



quartet g1 g2 g3 g4



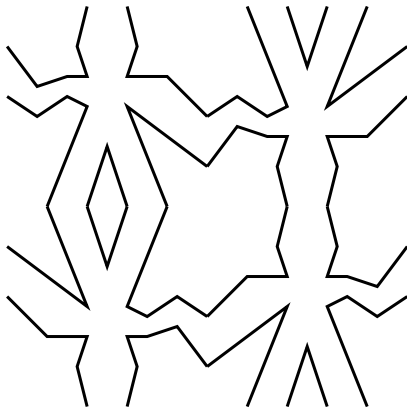
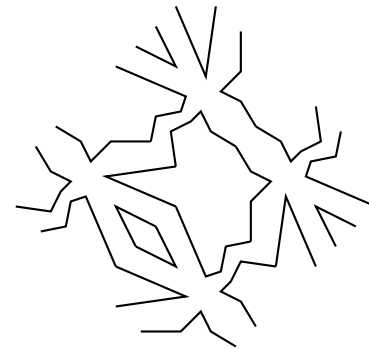
=>



```
quartet : P -> P -> P -> P -> P
quartet nw ne sw se =
    above (beside nw ne)
          (beside sw se)
```



toss

 $\Rightarrow$ 

nonet h e n d e r s o n

H E N

D E R

S O N

=>

H E N

D E R

S O N

```
nonet : P -> P -> P -> P -> P -> P -> P -> P -> P
  let
    row w m e = besideRatio 1 2 w (beside m e)
    col n m s = aboveRatio 1 2 n (above m s)
  in
    col (row nw nm ne)
        (row nw nm ne)
        (row nw nm ne)
```

nonets are just pictures



a fish picture

