Finite difference methods for vibration problems

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Problem 1

In this problemset we will be considering the following ordinary differential equation

$$u'' + \omega^2 u = f(t), \quad u(0) = I, \quad u'(0) = V, \tag{1}$$

in the interval $t \in (0, T]$.

a)

We now wish to discretize this ODE. First we create a meshgrid with N points and therafter define the following timestep as $t_n = n\Delta$ where $\Delta t = T/N$ (keep in mind that $N \in \mathbb{N}_0$). Equation (1) can be discretized as following with operators:

$$[D_t D_t u + \omega^2 u = f(t)]^n, \quad u(0) = I, \quad [D_{2t}u]^0 = V \tag{2}$$

Where we assume the differential operators $D_t D_t$ and D_{2t} are perfectly linear. Using finite difference method for these differential operators, the equation above can be written as:

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + \omega^2 u^n = f^n, \quad u^0 = I, \quad \frac{u^1 - u^{-1}}{2\Delta t} = V \tag{3}$$

The first timestep will look as following:

$$\frac{u^1 - 2u^0 + u^{-1}}{\Delta t^2} + \omega^2 u^0 = f^0, \quad u^0 = I, \quad \frac{u^1 - u^{-1}}{2\Delta t} = V \tag{4}$$

b)

For the sake of verification purposes, we will use the MMS with an exact solution looking like $u_e(t) = ct + d$ (keep in mind that boundary and intial condition above is unchanged). As we analyze this $u_e(t)$ we notice that there are no restriction on the constant b and c. The reason for this is that we can define b and c, in this case these constant are defined as $c = I$ and $d = V$. Thus giving us

$$u_e(t) = It + V$$

Using the linear function above the source term will look like when using (1) (assuming that differential operator $D_t D_t$ is perfectly linear)

$$f = \omega^2(It + V) \tag{5}$$

We will now continue with showing that $D_t D_t t]^n = 0$. Consider the following

$$D_t D_t t]^n = \frac{t^{n+1} - 2t^n + t^{n-1}}{\Delta t^2} \tag{6}$$

Where $t_{n+1} = n\Delta t + \Delta t$ and $t_{n-1} = n\Delta t - \Delta t$. Substituting this into the expression above, we will obtain the following

$$D_t D_t t]^n = \frac{n\Delta t + \Delta t - 2n\Delta t + n\Delta t - \Delta t}{\Delta t^2} \tag{7}$$

Thus $D_t D_t t]^n = 0$ is fullfilled. Using this information we will show that $u_e(t)$ is a perfect solution to equation (1).

$$[D_t D_t u_e + \omega^2 u_e = f(t)]^n = [D_t D_t u_e]^n + [\omega^2 u]^n = f^n \tag{8}$$

Inserting for $u_e(t)$ and f^n we see that u_e is a perfect solution for this system. Thus

$$\underbrace{I[D_t D_t t]^n}_{=0} + \underbrace{[D_t D_t V]^n}_{=0} + \omega^2(It + V) = \omega^2(It + V) \tag{9}$$

c)

We will now create program which uses symbolic calculations to solve the equations above. The program is named "vib_undamped_verify_mms.py".

Local trunaction error: Keep in mind when looking at the function "residual_discrete_eq_step1(u)" we are interested in the error in the first step. For some functions it does exist an small error in the begining when solving the ODE.

$$\frac{u^1 - 2u^0 + u^{-1}}{\Delta t^2} + \omega^2 u^0 = f^0 \tag{10}$$

solving this for u^1 we will obtain:

$$u^1 = f^0 \Delta t^2 - \omega^2 u^0 \Delta t^2 + 2u^0 - u^{-1} \tag{11}$$

Recall from previous that $u^{-1} = u^1 - V\Delta t$. Thus Substituting this into our expression, we will obtain:

$$u^1 = \frac{1}{2} f^0 \Delta t^2 + I \left(1 - \frac{1}{2}\omega^2 \Delta t^2\right) - V\Delta t \tag{12}$$

The residual R is then:

$$R^1 = u_e - u = \frac{1}{2} f^0 \Delta t^2 + I \left(1 - \frac{1}{2}\omega^2 \Delta t^2\right) - V\Delta t \tag{13}$$

As we see, the truncation error is heavily dependent on the source term and the order of the polynomial. For linear and quadratic we see no truncation error in the first timestep, for a cubic polynomial the residual error is:

$$R^1 = a\Delta^3 \tag{14}$$

Global trunaction error: Consider the following residual:

$$R^n = [D_t D_t u + \omega^2 u - f(t)]^n \rightarrow \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2} + \omega^2 u^n - f^n \tag{15}$$

Taylor expanding u^{n+1} and u^{n-1} around the point t_n, we will get following expansion:

$$u_{n+1} = u_n + u'_n \Delta t + \frac{u''_n}{2} \Delta t^2 + \frac{u'''_n}{6} \Delta t^3 + ... \tag{16}$$

$$u_{n-1} = u_n - u'_n \Delta t + \frac{u''_n}{2} \Delta t^2 - \frac{u'''_n}{6} \Delta t^3 + ... \tag{17}$$

Substituting this to (15), we will get:

$$R^n = u''_n + \frac{u''''_n}{12} \Delta t^2 + ... + \omega^2 u_n - f_n \tag{18}$$

Keep in mind that $u''_n + \omega^2 u_n = f_n$ thus this kan be approximated to:

$$R^n \approx \frac{u''''_n}{12} \Delta t^2 \tag{19}$$

This shows that the global truncation goes as the fourth derivative. This means up to third order polynomial the trunaction error lies in the first step. Thus it does not accumulate after $n = 2$.

d)

We will now extend the code and add new function called quadratic expressed as

$$u_e(t) = bt^2 + Vt + I \tag{20}$$

According to the analytical expression and sympy the new solution $u_e(t) = bt^2 + Vt + I$ fullfills the ODE, thus it is a solution of the ODE.

e)

When considering a cubic polynomial on the form

$$u_e(t) = at^3 + bt^2 + Vt + I \tag{21}$$

Using (19) we see that this solution does accumulate a error which to lead to global truncation error. Keep in mind, even though the global error is approximated to zero. This solution is not a perfect solution since it does have a truncation error in the first step. Which is expressed as $R^1 = a\Delta t^3$ which can be shown by using equation (13) or the sympy program.

e)

We will now extend the code and add a solver function.