FYS3150 - Project 2

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The GitHub repository for this project can be found at https://github.uio.no/emiljk/FYS3150.

PROBLEM 1

Starting from the equation

$$\gamma \frac{d^2 u(x)}{dx^2} = -Fu(x) \tag{1}$$

we introduce the dimensionless variable $\hat{x} = x/L$, and $\lambda = \frac{FL^2}{\gamma}$, by first multiplying the left hand side by $d\hat{x}^2/d\hat{x}^2$.

$$\gamma \frac{d^2}{dx^2}\frac{d\hat{x}^2}{d\hat{x}^2} u(x) = -Fu(x) \tag{2}$$

Using $d\hat{x}/dx = L$ we get

$$\gamma \frac{d^2}{L^2 d\hat{x}^2} u(\hat{x}) = -Fu(\hat{x}). \tag{3}$$

And by multiplying with L^2/γ

$$\frac{d^2 u(\hat{x})}{d\hat{x}^2} = -\frac{FL^2}{\gamma} u(\hat{x}) \tag{4}$$

$$\frac{d^2 u(\hat{x})}{d\hat{x}^2} = -\lambda u(\hat{x}) \tag{5}$$

which is what we wanted to show.

PROBLEM 2

We make a program for setting up a tridiagonal matrix, and use it to create the matrix $\mathbf{A}$.

$$\mathbf{A} = \frac{1}{h^2}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \tag{6}$$

We solve the equation $\mathbf{A}\vec{v} = \lambda\vec{v}$ using both the analytical method and the `arma::eig_sym` package which both gives the eigenvalues (using one decimal)

$$\lambda_1 = 9.7$$
$$\lambda_2 = 36.9$$
$$\lambda_3 = 76.2$$
$$\lambda_4 = 119.8$$
$$\lambda_5 = 159.1$$
$$\lambda_6 = 186.3$$

and corresponding eigenvectors as columns in the matrix (now using two decimals)

$$10^{-2}\begin{bmatrix} 23 & 42 & 52 & 52 & 42 & 23 \\ 42 & 52 & 23 & -23 & -52 & -42 \\ 52 & 23 & -42 & -42 & 23 & 52 \\ 52 & -23 & -42 & 42 & 23 & -52 \\ 42 & -52 & 23 & 23 & -52 & 42 \\ 23 & -42 & 52 & -52 & 42 & -23 \end{bmatrix}.$$

PROBLEM 3

We make a function `max_offdiag_symmetric` to find the maximum absolute off-diagonal value in a symmetric matrix, and we test it on the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0.5 \\ 0 & 1 & -0.7 & 0 \\ 0 & -0.7 & 1 & 0 \\ 0.5 & 0 & 0 & 1 \end{bmatrix}.$$

The function returns the value -0.7 in row one and column two (counting from zero) as expected.

PROBLEM 4

Using the lecture notes from 2022, we implement a code of Jacobi's rotation algorithm for solving the equation $\mathbf{A}\vec{v} = \lambda\vec{v}$. By using the same 6×6 matrix as in problem 2, we get the same eigenvalues and eigenvectors as the analytical method.

PROBLEM 5

a)

By running our program for different matrix sizes N from 10 to 20, we estimate that the number of similarity transformations needed before reaching our tolerance of $\epsilon = 10^{-14}$ will scale with cN^2, where c is some constant. From FIG 1 we see that the number of iterations indeed is growing exponentially. In this figure we have fitted a linear curve using `scipy.stats.linregress` which gives an exponential growth of 2.08, which is 0.08 away from what we expected.

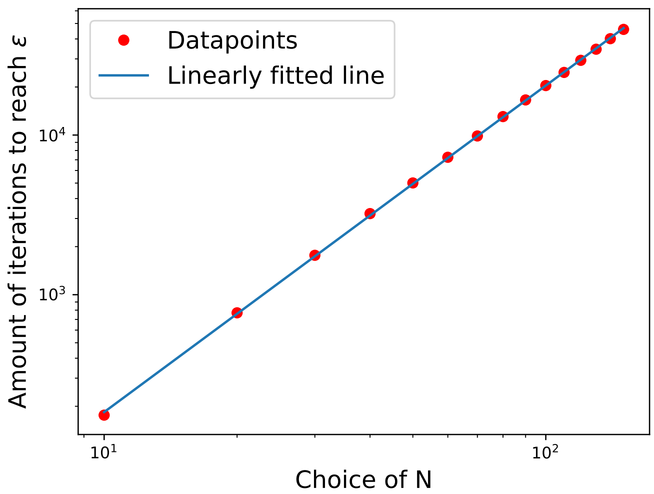

FIG. 1. A logarithmic representation of the number of similarity transformations needed before reaching the tolerance $\epsilon = 10^{-14}$ for different choices of matrix sizes N, compared to a linear fit with gradient 2.08.

b)

If **A** was a dense matrix, and not a triangular one, we would still expect the number of similarity transformations to scale with N^2. When doing Jacobi rotations, we change the elements of entire rows and columns, and therefore the entire matrix will need to be stored during the transformations, regardless of how many zero elements we start with. The scaling behavior is therefore mostly determined by the size of the matrix, and not by whether it is dense or not.

PROBLEM 6

a)

For $n = 10$ steps we make a plot of the eigenvectors corresponding to the three lowest eigenvalues found from both the Jacobi rotation method and the analytical method. This is shown in FIG 2 where we see that the solutions for the two methods are overlapping. We could therefore argue that the Jacobi method gives accurate answers compared to the discretization. The buckling of the beam for each eigenvalue seems quite extreme, so one could guess that the beam is made of a quite flexible material.

b)

We make the same plot, now for $n = 100$, as shown in FIG 3. Here we see that the curves are smoother than for $n = 10$ in FIG 2, as expected.

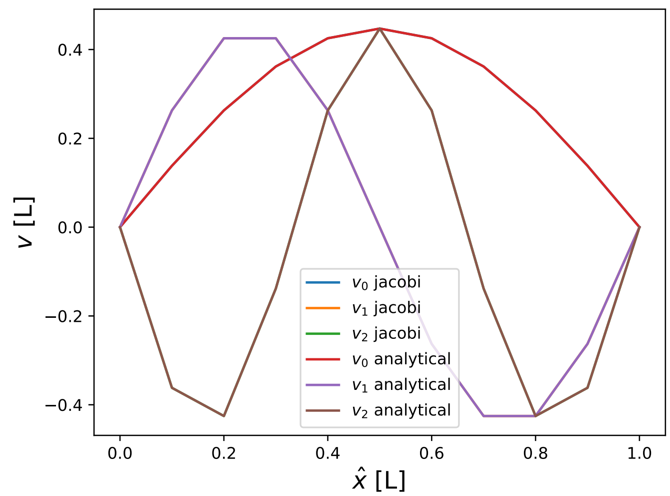

FIG. 2. Three pairs of eigenvector-solutions of the buckling beam with $n = 10$ discretization steps using both the analytical method and the Jacobi rotation method. The solutions for the different methods are overlapping.

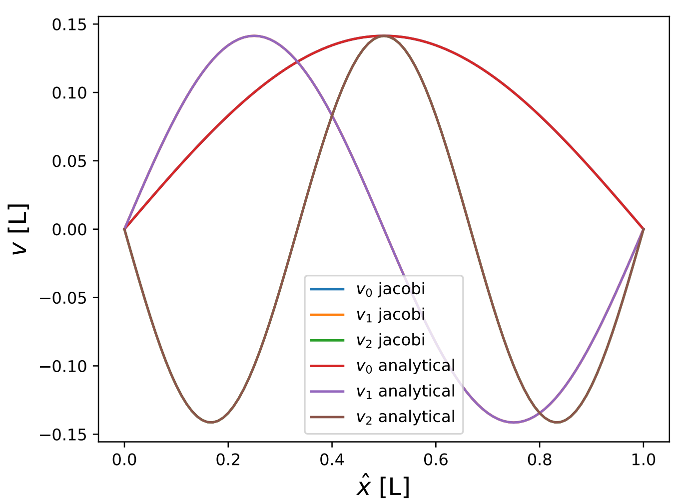

FIG. 3. Three pairs of eigenvector-solutions of the buckling beam with $n = 100$ discretization steps using both the analytical method and the Jacobi rotation method. The solutions for the different methods are overlapping.