In[4]:=
```
(* baseline = 0.202 -- radius of circle on which hydrophones lie
   SoS = 1550 -- speed of sound*)
```

In[5]:=
```
HydrophonePositions := {{baseline, 0}, {baseline, 0}.RotationMatrix[-2 Pi / 3],
   {baseline, 0}.RotationMatrix[-4 Pi / 3]}; HydrophonePositions // MatrixForm
```

Out[5]//MatrixForm=

$$\begin{pmatrix} \text{baseline} & 0 \\ -\frac{\text{baseline}}{2} & \frac{\sqrt{3}\ \text{baseline}}{2} \\ -\frac{\text{baseline}}{2} & -\frac{\sqrt{3}\ \text{baseline}}{2} \end{pmatrix}$$

In[6]:=
```
RotatedPositions := HydrophonePositions . RotationMatrix[θ]; RotatedPositions // MatrixForm
```

Out[6]//MatrixForm=

$$\begin{pmatrix} \text{baseline Cos}[\theta] & -\text{baseline Sin}[\theta] \\ -\frac{1}{2}\text{baseline Cos}[\theta]+\frac{1}{2}\sqrt{3}\ \text{baseline Sin}[\theta] & \frac{1}{2}\sqrt{3}\ \text{baseline Cos}[\theta]+\frac{1}{2}\text{baseline Sin}[\theta] \\ -\frac{1}{2}\text{baseline Cos}[\theta]-\frac{1}{2}\sqrt{3}\ \text{baseline Sin}[\theta] & -\frac{1}{2}\sqrt{3}\ \text{baseline Cos}[\theta]+\frac{1}{2}\text{baseline Sin}[\theta] \end{pmatrix}$$

In[7]:=
```
EstimatedDelays := RotatedPositions[[All, 2]] * Cos[ϕ] / SoS;
EstimatedDelays * SoS // MatrixForm
```

Out[7]//MatrixForm=

$$\begin{pmatrix} -\text{baseline Cos}[\phi]\ \text{Sin}[\theta] \\ \text{Cos}[\phi]\left(\frac{1}{2}\sqrt{3}\ \text{baseline Cos}[\theta]+\frac{1}{2}\text{baseline Sin}[\theta]\right) \\ \text{Cos}[\phi]\left(-\frac{1}{2}\sqrt{3}\ \text{baseline Cos}[\theta]+\frac{1}{2}\text{baseline Sin}[\theta]\right) \end{pmatrix}$$

In[8]:=
```
EstimatedLags :=
 FullSimplify[{EstimatedDelays[[1]] - EstimatedDelays[[2]], EstimatedDelays[[1]] -
    EstimatedDelays[[3]], EstimatedDelays[[2]] - EstimatedDelays[[3]]}];
EstimatedLags / (baseline * Cos[ϕ] / SoS) // MatrixForm (* note manual factorization *)
```

Out[8]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\left(-\sqrt{3}\ \text{Cos}[\theta]-3\ \text{Sin}[\theta]\right) \\ \frac{1}{2}\left(\sqrt{3}\ \text{Cos}[\theta]-3\ \text{Sin}[\theta]\right) \\ \sqrt{3}\ \text{Cos}[\theta] \end{pmatrix}$$

In[9]:=
```
soln := Solve[{EstimatedLags[[1]] == m1,
   EstimatedLags[[2]] == m2, EstimatedLags[[3]] == m3}, {θ, ϕ}]; soln
```

Out[9]= { }

In[10]:=
```
(* FAIL *)
```

In[11]:=
```
(* Aaannnnddd... try another approach: conjugate gradient on sum-of-squared error *)

MSE := (m1 - EstimatedLags[[1]] * 10^6)^2 +
  (m2 - EstimatedLags[[2]] * 10^6)^2 + (m3 - EstimatedLags[[3]] * 10^6)^2; MSE
```

Out[11]=

$$\left(\text{m3}-\frac{1\,000\,000\sqrt{3}\ \text{baseline Cos}[\theta]\ \text{Cos}[\phi]}{\text{SoS}}\right)^2+$$
$$\left(\text{m2}-\frac{500\,000\ \text{baseline Cos}[\phi]\left(\sqrt{3}\ \text{Cos}[\theta]-3\ \text{Sin}[\theta]\right)}{\text{SoS}}\right)^2+$$
$$\left(\text{m1}+\frac{500\,000\ \text{baseline Cos}[\phi]\left(\sqrt{3}\ \text{Cos}[\theta]+3\ \text{Sin}[\theta]\right)}{\text{SoS}}\right)^2$$

In[12]:= subst := $\left\{\sqrt{3}\ (\text{m1}-\text{m2}-2\ \text{m3})\ \text{SoS}\to \text{y},\ 3\ (\text{m1}+\text{m2})\ \text{SoS}\to \text{z}\right\}$;

In[13]:=
```
dEdθϕ := FullSimplify[D[MSE, {{θ, ϕ}}]];
tmp := FullSimplify[dEdθϕ / (baseline * 10^6)] //. subst; tmp
```

Out[13]=
$$\left\{\frac{\mathrm{Cos}[\phi]\left(3\,(\mathrm{m1}+\mathrm{m2})\,\mathrm{Cos}[\theta]+\sqrt{3}\,(-\mathrm{m1}+\mathrm{m2}+2\,\mathrm{m3})\,\mathrm{Sin}[\theta]\right)}{\mathrm{SoS}},\ -\frac{(\mathrm{y}\,\mathrm{Cos}[\theta]+9\,000\,000\ \mathrm{baseline}\,\mathrm{Cos}[\phi]+\mathrm{z}\,\mathrm{Sin}[\theta])\,\mathrm{Sin}[\phi]}{\mathrm{SoS}^2}\right\}$$

In[14]:=
```
FullSimplify[Numerator[tmp[[1]]] == ((1 / SoS * Cos[ϕ] * ( z Cos[θ] - y Sin[θ])) //.
    {y -> √3 (m1 - m2 - 2 m3) SoS, z → 3 (m1 + m2) SoS})]
```

Out[14]= True

In[15]:=
```
(* Measured Lags {m1, m2, m3} correspond to θ→6° and ϕ→26° *)
Plot3D[
  Evaluate[MSE /. {m1 → -114.58, m2 → 84.63, m3 → 199.22, baseline → 0.202, SoS → 1550}],
  {θ, 0 °, 12 °}, {ϕ, 20 °, 32 °}];
```

In[16]:=
```
(* Plot derivatives *)
Plot3D[Evaluate[dEdθϕ[[1]] /. {m1 → -114.58, m2 → 84.63, m3 → 199.22,
     baseline → 0.202, SoS → 1550}], {θ, 0 °, 360 °}, {ϕ, 0 °, 89 °}];
```

In[17]:=
```
Plot3D[Evaluate[
   dEdθϕ[[2]] /. {m1 → -114.58, m2 → 84.63, m3 → 199.22, baseline → 0.202, SoS → 1550}],
  {θ, 0 °, 360 °}, {ϕ, 0 °, 89 °}];
```

In[18]:=
```
Reduce[dEdθϕ[[1]] == 0, θ]
```

Out[18]=
$$\left(\mathrm{m2}==-\mathrm{m1}\ \&\&\ \left(\left(\frac{-\pi+\theta}{2\,\pi}\notin \mathrm{Integers}\ \&\&\ \mathrm{SoS}\neq 0\ \&\&\ \mathrm{m3}==\mathrm{m1}\right)\ ||\ (\mathrm{C}[1]\in \mathrm{Integers}\ \&\&\ \mathrm{m1}-\mathrm{m3}\neq 0\ \&\&\ \mathrm{SoS}\neq 0\ \&\&\ \theta==2\,\pi\,\mathrm{C}[1])\right)\right)\ ||$$

$$(\mathrm{SoS}\neq 0\ \&\&\ (\mathrm{baseline}==0\ ||\ \mathrm{Cos}[\phi]==0))\ ||$$

$$\left(\mathrm{m1}+\mathrm{m2}\neq 0\ \&\&\ \mathrm{C}[1]\in \mathrm{Integers}\ \&\&\ \left(\left(\left(2\,\mathrm{m1}^2+2\,\mathrm{m1}\,\mathrm{m2}+2\,\mathrm{m2}^2-2\,\mathrm{m1}\,\mathrm{m3}+2\,\mathrm{m2}\,\mathrm{m3}+2\,\mathrm{m3}^2+\mathrm{m1}\sqrt{\mathrm{m1}^2+\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-\mathrm{m1}\,\mathrm{m3}+\mathrm{m2}\,\mathrm{m3}+\mathrm{m3}^2}-\mathrm{m2}\sqrt{\mathrm{m1}^2+\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-\mathrm{m1}\,\mathrm{m3}+\mathrm{m2}\,\mathrm{m3}+\mathrm{m3}^2}-2\,\mathrm{m3}\sqrt{\mathrm{m1}^2+\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-\mathrm{m1}\,\mathrm{m3}+\mathrm{m2}\,\mathrm{m3}+\mathrm{m3}^2}\right)\mathrm{SoS}\neq 0\ \&\&\right.\right.\right.$$

$$\left.\theta==2\,\mathrm{ArcTan}\left[\frac{-\sqrt{3}\,\mathrm{m1}+\sqrt{3}\,\mathrm{m2}+2\sqrt{3}\,\mathrm{m3}-2\sqrt{3}\sqrt{\mathrm{m1}^2+\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-\mathrm{m1}\,\mathrm{m3}+\mathrm{m2}\,\mathrm{m3}+\mathrm{m3}^2}}{3\,(\mathrm{m1}+\mathrm{m2})}\right]+2\,\pi\,\mathrm{C}[1]\right)\ ||$$

$$\left(\left(-2\,\mathrm{m1}^2-2\,\mathrm{m1}\,\mathrm{m2}-2\,\mathrm{m2}^2+2\,\mathrm{m1}\,\mathrm{m3}-2\,\mathrm{m2}\,\mathrm{m3}-2\,\mathrm{m3}^2+\mathrm{m1}\sqrt{\mathrm{m1}^2+\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-\mathrm{m1}\,\mathrm{m3}+\mathrm{m2}\,\mathrm{m3}+\mathrm{m3}^2}-\mathrm{m2}\sqrt{\mathrm{m1}^2+\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-\mathrm{m1}\,\mathrm{m3}+\mathrm{m2}\,\mathrm{m3}+\mathrm{m3}^2}-2\,\mathrm{m3}\sqrt{\mathrm{m1}^2+\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-\mathrm{m1}\,\mathrm{m3}+\mathrm{m2}\,\mathrm{m3}+\mathrm{m3}^2}\right)\mathrm{SoS}\neq 0\ \&\&\right.$$

$$\left.\left.\left.\theta==2\,\mathrm{ArcTan}\left[\frac{-\sqrt{3}\,\mathrm{m1}+\sqrt{3}\,\mathrm{m2}+2\sqrt{3}\,\mathrm{m3}+2\sqrt{3}\sqrt{\mathrm{m1}^2+\mathrm{m1}\,\mathrm{m2}+\mathrm{m2}^2-\mathrm{m1}\,\mathrm{m3}+\mathrm{m2}\,\mathrm{m3}+\mathrm{m3}^2}}{3\,(\mathrm{m1}+\mathrm{m2})}\right]+2\,\pi\,\mathrm{C}[1]\right)\right)\right)\ ||$$

$$(\mathrm{C}[1]\in \mathrm{Integers}\ \&\&\ \mathrm{SoS}\neq 0\ \&\&\ \mathrm{m2}==-\mathrm{m1}\ \&\&\ \theta==\pi\,(1+2\,\mathrm{C}[1]))$$

In[19]:= **myθ = 2 ArcTan**$\left[\frac{-\sqrt{3}\ \mathrm{m1}+\sqrt{3}\ \mathrm{m2}+2\sqrt{3}\ \mathrm{m3}\pm 2\sqrt{3}\ \sqrt{\mathrm{m1}^2+\mathrm{m1\,m2}+\mathrm{m2}^2-\mathrm{m1\,m3}+\mathrm{m2\,m3}+\mathrm{m3}^2}}{3\ (\mathrm{m1}+\mathrm{m2})}\right]$

Out[19]= $2\,\mathrm{ArcTan}\left[\frac{-\sqrt{3}\ \mathrm{m1}+\sqrt{3}\ \mathrm{m2}+\left(2\sqrt{3}\ \mathrm{m3}\pm 2\sqrt{3}\ \sqrt{\mathrm{m1}^2+\mathrm{m1\,m2}+\mathrm{m2}^2-\mathrm{m1\,m3}+\mathrm{m2\,m3}+\mathrm{m3}^2}\right)}{3\ (\mathrm{m1}+\mathrm{m2})}\right]$

In[20]:= **Reduce[dEd$\theta\phi$[[2]] == 0, ϕ]**

Out[20]= $\Big(\mathrm{C}[1]\in \mathrm{Integers}\ \&\&$

$\Big(\Big(\mathrm{baseline}\neq 0\ \&\&\ \mathrm{SoS}\neq 0\ \&\&\ \Big(\phi == -\mathrm{ArcCos}\Big[\frac{1}{9\,000\,000\ \mathrm{baseline}}\Big(-\sqrt{3}\ \mathrm{m1\ SoS\ Cos}[\theta]+\sqrt{3}\ \mathrm{m2\ SoS}$

$\mathrm{Cos}[\theta]+2\sqrt{3}\ \mathrm{m3\ SoS\ Cos}[\theta]-3\ \mathrm{m1\ SoS\ Sin}[\theta]-3\ \mathrm{m2\ SoS\ Sin}[\theta]\Big)\Big]+2\,\pi\,\mathrm{C}[1]\ ||$

$\phi == \mathrm{ArcCos}\Big[\frac{1}{9\,000\,000\ \mathrm{baseline}}\Big(-\sqrt{3}\ \mathrm{m1\ SoS\ Cos}[\theta]+\sqrt{3}\ \mathrm{m2\ SoS\ Cos}[\theta]+$

$2\sqrt{3}\ \mathrm{m3\ SoS\ Cos}[\theta]-3\ \mathrm{m1\ SoS\ Sin}[\theta]-3\ \mathrm{m2\ SoS\ Sin}[\theta]\Big)\Big]+2\,\pi\,\mathrm{C}[1]\Big)\Big)\ ||$

$(\mathrm{SoS}\neq 0\ \&\&\ (\phi == 2\,\pi\,\mathrm{C}[1]\ ||\ \phi == \pi+2\,\pi\,\mathrm{C}[1]))\Big)\Big)\ ||\ \Big(\mathrm{baseline} == 0\ \&\&$

$\Big((\mathrm{m2} == -\mathrm{m1}\ \&\&\ ((\mathrm{SoS}\neq 0\ \&\&\ \mathrm{m3} == \mathrm{m1})\ ||\ (\mathrm{m1}-\mathrm{m3}\neq 0\ \&\&\ \mathrm{SoS}\neq 0\ \&\&\ \mathrm{Cos}[\theta] == 0)))\ ||$

$\Big(\mathrm{m1}+\mathrm{m2}\neq 0\ \&\&\ \mathrm{SoS}\neq 0\ \&\&\ \mathrm{Sin}[\theta] == \frac{-\sqrt{3}\ \mathrm{m1\ Cos}[\theta]+\sqrt{3}\ \mathrm{m2\ Cos}[\theta]+2\sqrt{3}\ \mathrm{m3\ Cos}[\theta]}{3\ (\mathrm{m1}+\mathrm{m2})}\Big)\Big)\Big)\ ||$

$(\mathrm{SoS}\neq 0\ \&\&\ \mathrm{baseline} == 0)$

In[21]:= **myϕ = FullSimplify**$\Big[$**ArcCos**$\Big[$**1 / (9 000 000 baseline)** $\Big(-\sqrt{3}$ **m1 SoS Cos[θ] +**
$\sqrt{3}$ **m2 SoS Cos[θ] + 2** $\sqrt{3}$ **m3 SoS Cos[θ] − 3 m1 SoS Sin[θ] − 3 m2 SoS Sin[θ]**$\Big)\Big]\Big]$

Out[21]= $\mathrm{ArcCos}\left[-\frac{\mathrm{SoS}\left(\sqrt{3}\ (\mathrm{m1}-\mathrm{m2}-2\ \mathrm{m3})\ \mathrm{Cos}[\theta]+3\ (\mathrm{m1}+\mathrm{m2})\ \mathrm{Sin}[\theta]\right)}{9\,000\,000\ \mathrm{baseline}}\right]$