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Normalized units

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SUMMARY

When experts talk of the mass of the Sun being 1·5 km, they are using *normalized units*. Normalized units are such units in terms of which a certain physical constant or constants have the value 1. The normalization of k constants involving n ($\geqslant k$) physical dimensions leads to an (n, k) system of normalized units consisting of $n-k$ arbitrarily *chosen* units and k *defined* units. Nine such systems used by experts in diverse topics are collected in Table II. The AU (astronomical unit) can be seen as the 1 defined unit in a (3, 1) system, in which the 3 dimensions are the 3 dimensions of dynamics, the 1 normalized constant is the square root of the Gravitational Constant (normalized to 1 radian) and the 2 chosen units are the solar mass and $\tau_0 = 58{\cdot}132\,440\,87$ d. For electromagnetism, 1 new dimension (the electric charge, for example) must be added, and the e.s.u. (e.m.u.) system consists of the the 3 c.g.s. units as the 3 chosen units and a new, defined unit of charge, qs (qm), resulting from the normalization of the permittivity ε_0 (permeability μ_0) of free space. In the so-called 'Maxwell's equations in Gaussian units', the symbol c stands for the numerical ratio qm/qs, and *not* for the velocity of light. These are designated 'Type-II' equations, in which symbols stand for numerical measures in some specialized (normalized or 'mixed') units. They should be distinguished from 'Type-I' equations, in which symbols stand for physical quantities, which have validity in any self-consistent system. These two types of equations for a number of well-known laws in physics and astronomy are exhibited side by side in Table IV.

1 'A MASS OF 1·5 KILOMETRES'? 'A DISTANCE OF 6000 KILOMETRES PER SECOND'?

We all know what units are: we immediately think of conversions such as 1 inch = 2·54 cm, 1 lb = 454 gms, or, as professional astronomers,

$$1 \text{ year} = 3{\cdot}1557 \times 10^7 \text{ s} = 3{\cdot}1557(+7) \text{ s}, \tag{1a}$$

$$1 \text{ parsec} = 3{\cdot}0857 \times 10^{18} \text{ cm} = 3{\cdot}0857(+18) \text{ cm}, \tag{1b}$$

$$1 \text{ AU y}^{-1} = 4{\cdot}74 \text{ km s}^{-1}. \tag{1c}$$

(The bracket notation for powers of ten, exemplified in (1 *a*) and (1 *b*), will be used generally in this essay.) The notions implicit in these conversions are: that a unit is always associated with a certain physical dimension, that units of the same dimension are related by numerical factors while it is meaningless to relate units of different dimensions. Accordingly, we are puzzled when we read in Misner, Thorne & Wheeler's (1973) text, 'The Sun has a mass of 1·477 km'. For we have always associated km with length, so how can it be used to express a mass? Again we hear Lynden-Bell (1986) talk of a distance range between 3000 and 6000 km s^{-1} and we wonder, how can that be? The specialists and experts must be using units in some different way from usual usage. In this Section, I shall begin by re-interpreting these two arcane statements in terms of ordinary statements of the type '1 inch = 2·54 cm'.

The first step towards understanding the Misner–Thorne–Wheeler (MTW) statement is to realize that the km therein is *not* a unit of length, that it is, in fact, a unit of mass. It is really an abbreviation for 'km-of-mass' and, as will be shown below,

$$1 \text{ km-of-mass} = 1{\cdot}35\,(+33) \text{ gm}. \tag{2}$$

This is a simple statement of the type '1 lb = 454 gm'. With this conversion ratio, we readily recover the familiar figure of the solar mass in grams:

$$1 \text{ Solar Mass} = 1{\cdot}477 \text{ km-of-mass} = 1{\cdot}99\,(+33) \text{ gm}.$$

Similarly, in the Lynden-Bell statement, the 'km s^{-1} is short for km s^{-1}-of-distance', and

$$1 \text{ km s}^{-1}\text{-of-distance} = (1/H_0) \text{ Mpc}, \tag{3}$$

where H_0 is Hubble's constant in km s^{-1} Mpc^{-1}. Thus, the implied distance range is between 30 and 60 Mpc, if $H_0 = 100$, or between 60 and 120 Mpc if $H_0 = 50$.

The above cm-of-mass and km s^{-1}-of-distance are examples of what I shall call 'normalized units'. Normalized units are units defined by experts in such a way that, in these units, a certain physical constant or constants will have the value 1. The motivation for using normalized units arises from the following circumstances: it often happens that, for the problem on hand, we are not interested in the numerical values of certain physical constants – all we require of them is that they be constant; in such cases, if units can be chosen such that they have the value 1, then they will no longer be explicitly present in the formulae and the reader's attention can be better directed at the quantities of interest.

Normalization is always possible if the number of constants to be normalized is not greater than the number of physical dimensions involved. In the Lynden-Bell example, there is just one such constant, the Hubble constant H_0, and there are two dimensions, the distance and the velocity. The idea is that, if we express all distances in the 'km s^{-1}-of-distance' defined at (3), and all velocities in the usual km s^{-1}, then H_0 will have the value 1, and Hubble's Law becomes simply $v = r$.

In the MTW example, there are two such constants to be normalized, the velocity of light c and the gravitational constant G, and the authors define their unconventional units of mass and time ('cm-of-mass' and 'cm-of-time') so that, when all masses and times are expressed in these units and all lengths in cm, then both c and G will have the value 1. These two requirements determine uniquely how many grams there must be in one cm-of-mass, and how many seconds in one cm-of-time.

Let us find these conversion ratios. In attempting to do so, it is helpful to employ a notation that distinguishes a physical quantity and its numerical value in some particular units. This procedure may appear pedantic at times, but my own experience is that it will clear up all our mental confusion about units. In this essay, I shall use roman characters for physical quantities and the corresponding italic characters, suffixed or primed if necessary, for their numerical values. Thus, I shall write

$$\mathrm{c} = c \text{ cm s}^{-1}, \tag{4}$$

where c stands for the physical quantity 'velocity of light' and c stands for a number which, to 4 significant figures, is

$$c = 2{\cdot}998\,(+10). \tag{5}$$

Similarly, I shall write

$$\mathrm{G} = G \text{ cm}^3 \text{ s}^{-2} \text{ gm}^{-1}, \tag{6}$$

where

$$G = 6{\cdot}67\,(-8). \tag{7}$$

Let there be t_0 seconds in one cm-of-time, and m_0 gm in one cm-of-mass:

$$1 \text{ cm-of-time} = t_0 \text{ s}, \tag{8}$$

$$1 \text{ cm-of-mass} = m_0 \text{ gm}. \tag{9}$$

The requirements are

$$\mathrm{c} = 1 \text{ cm (cm-of-time)}^{-1}, \tag{10}$$

$$\mathrm{G} = 1 \text{ cm}^3 \text{ (cm-of-time)}^{-2} \text{ (cm-of-mass)}^{-1}. \tag{11}$$

Put (8) into (10) and compare with (4), and we obtain a relation between the two numbers c and t_0,

$$c = t_0^{-1}. \tag{12}$$

Similarly, putting (8) and (9) into (11) and comparing with (6), we have a relation among the three numbers G, t_0, m_0,

$$G = t_0^{-2}\, m_0^{-1}. \tag{13}$$

Inverting (12) and (13) gives

$$t_0 = c^{-1} \tag{14}$$

$$m_0 = c^2\, G^{-1}, \tag{15}$$

that is, on recalling the definitions of t_0 and m_0 at (8) and (9) and the numerical values shown at (5) and (7),

$$1 \text{ cm-of-time} = c^{-1} \text{ s} = 3{\cdot}336\,(-11) \text{ s}, \tag{16}$$

$$1 \text{ cm-of-mass} = c^2\, G^{-1} \text{ gm} = 1{\cdot}35\,(+28) \text{ gm}. \tag{17}$$

2 L-UNITS, T-UNITS AND M-UNITS

The above system of units, cm for length, cm-of-time for time and cm-of-mass for mass, were called by MTW 'geometric units'. It is a particular case of what I shall call 'L-units'. Furthermore, the L-units are paralleled by two other types, the T-units and the M-units.

Let ul, ut, um denote any new system of units of length, time and mass, in which c and G have the value 1, and let

$$1 \text{ ul} = l_0 \text{ cm}, \quad 1 \text{ ut} = t_0 \text{ s}, \quad 1 \text{ um} = m_0 \text{ gm}. \tag{18a–c}$$

We then require

$$\mathrm{c} = 1 \text{ ul ut}^{-1}, \quad \mathrm{G} = 1 \text{ ul}^3 \text{ ut}^{-2} \text{ um}^{-1}. \tag{19a, b}$$

TABLE I

Three types of normalized units (c = G = 1)

	L-Units	T-Units	M-Units
1 ul =	l_0 cm	ct_0 cm	$(G/c^2)m_0$ cm
1 ut =	$(1/c)l_0$ s	t_0 s	$(G/c^3)m_0$ s
1 um =	$(c^2/G)l_0$ gm	$(c^3G)t_0$ gm	m_0 gm

TABLE Ia

Numerical form of Table I

	L-Units	T-Units	M-Units
1 ul =	l_0 cm	$2{\cdot}998(+10)\,t_0$ cm	$7{\cdot}42(-29)\,m_0$ cm
1 ut =	$3{\cdot}336(-11)\,l_0$ s	t_0 s	$2{\cdot}48(-39)\,m_0$ s
1 um =	$1{\cdot}35(+28)\,l_0$ gm	$4{\cdot}04(+38)\,t_0$ gm	m_0 gm

Putting (18) in (19) and comparing with (4) and (6) leads to 2 relations expressing the two known numbers c and G in terms of the three unknown numbers l_0, t_0, m_0,

$$c = l_0\, t_0^{-1}, \quad G = l_0^3\, t_0^{-2}\, m_0^{-1}. \tag{20a, b}$$

If we take l_0 as known, and solve for t_0, m_0 in terms of c, G and l_0, then we get a system of 'L-units'. The results are

$$1 \text{ ul} = l_0 \text{ cm}, \quad 1 \text{ ut} = (1/c)\, l_0 \text{ s}, \quad 1 \text{ um} = (c^2\, G^{-1})\, l_0 \text{ gm}.$$

These are repeated in the first column of Table I. Similarly, if we take t_0 to be known and solve for l_0 and m_0, we get the 'T-units' shown in the second column. Lastly, if we take a known m_0, we get the 'M-units' of the third column.

It is obvious that MTW's 'geometric units' are a particular case of L-units, particularized for $l_0 = 1$. Two particular cases of T-units and M-units now follow, the first is sometimes used in cosmology, the second, in motion around a black hole.

3 THE HUBBLE UNITS

The Hubble time τ_0 is at the very basis of modern observational cosmology. It defines a system of T-units, which can be conveniently called the 'Hubble units'. For definiteness, suppose

$$1 \text{ Hubble time} = 15 \text{ billion years} = 4{\cdot}74(+17) \text{ s}, \tag{21a}$$

then, with $t_0 = 4{\cdot}74(+17)$, the second column of Table Ia gives

$$1 \text{ Hubble Length} = 1{\cdot}42(+28) \text{ cm} = 4600 \text{ Mpc}, \tag{21b}$$

$$1 \text{ Hubble Mass} = 1{\cdot}91\,(+56) \text{ gm} = 9{\cdot}6(+22)\, M_{\odot}. \tag{21c}$$

We may note that, given a Hubble Time, we can define a Hubble Length without reference to a Hubble Mass, but not vice versa. This is because one of the two normalizable constants involves time and length only and the other one involves all three, length, time and mass.

It may be of passing interest to note that the Hubble Mass deduced here is about 10 times (and only 10 times) the value of the total content of the universe given more than half a century ago by Eddington's (1933) 'Celestial Multiplication Table':

> "A hundred thousand million stars make one Galaxy.
> A hundred thousand million Galaxies make one Universe".

4 TIME-SCALE AND DISTANCE-SCALE AROUND A BLACK HOLE

An uncharged, non-rotating black hole is uniquely characterized by its mass, M_{BH}. Even for a rotating black hole, it is also convenient sometimes to take M_{BH} as a new unit of mass. This then leads to a set of M-units, which may be called the 'standard black hole units'. For definiteness, suppose

$$1 \text{ standard black hole mass} = 1 \text{ million suns} = 1{\cdot}99\,(+39) \text{ gm.} \quad (22a)$$

Then, with $m_0 = 1{\cdot}99\ (+39)$, the third column of Table Ia gives

$$1 \text{ standard black hole length} = 1{\cdot}48\,(+11) \text{ cm} = 2{\cdot}12\ R_\odot \quad (22b)$$

$$1 \text{ standard black hole time} = 4{\cdot}93 \text{ s.} \quad (22c)$$

If the mass of the black hole is f million suns, then the units of length and time will each just be f times greater.

Note the so-called Schwarzschild (or gravitational) radius of the black hole is 2 in the above units.

5 GENERAL CHARACTERIZATION OF SYSTEMS OF NORMALIZED UNITS

A system of normalized units is characterized by two integers, the number of independent (or elementary) physical dimensions (or units), n, and the number of physical constants to be normalized (or normalizable constants), k. A system so parametrized will be referred to as an (n, k) system. The systems of L-units, T-units and M-units, described above, are all (3, 2) systems, for in each case we deal with 3 physical dimensions (length, time, mass) and 2 normalizable constants (c, G). For a (3, 2) system, one of three units must be singled out to be a 'chosen' unit, then the other two will be uniquely defined by the normalization of the two constants. In general, for an (n, k) system, there are $n-k$ 'chosen units' and k 'defined units'.

Table II is an analytic table of all the systems of normalized units discussed in this essay. The first two lines list two (2, 1) systems. The first one is in fact in common use, though perhaps not recognized as such. The two dimensions are length and time, the one normalizable constant is the velocity of light c, and the units are 'year' (chosen) and 'light-year' (defined). The second is Lynden-Bell's system of km s^{-1} (chosen) and km s^{-1}-of-distance (defined), operating in the two dimensions of velocity and length, by means of the normalization of the Hubble constant H_0. The next three lines list three (3, 2) systems, the MTW's geometric units, the Hubble units and the standard black hole units, described above. The rest of the Table refers to systems discussed in the following sections.

TABLE II

Analytic table of systems of normalized units

Line	(n, k)	n independent physical dimensions	k physical constants to be normalized	$(n-k)$ chosen units	k defined units	Equation
1	(2, 1)	L, T	c	year	light-year	—
2	(2, 1)	L, V	H_0	km s^{-1}	km s^{-1}-of-distance	(3)
3	(3, 2)	L, T, M	c, G	cm	cm-of-time, cm-of-mass	(16), (17)
4	(3, 2)	L, T, M	c, G	Hubble Time	Hubble Length, Hubble Mass	(21)
5	(3, 2)	L, T, M	c, G	$M_{\text{black hole}}$	BH Length, BH Time	(22)
6	(3, 3)	L, T, M	c, G	—	Planck Length, Time, Mass	(26)
7	(3, 1)	L, T, M	k	τ_0, $M_\odot$	AU	(33)
8	(4, 1)	L, T, M, Q	ε_0	cm, s, gm	qs	—
9	(4, 1)	L, T, M, Q	μ_0	cm, s, gm	qm	—
10	(4, 3)	L, T, M, Q	c, G, e	qs	Stoney Length, Time, Mass	(54)
11	(4, 4)	L, T, M, Q	c, G, e, ε_0	—	Stoney units, qs	—
12	(4, 4)	L, T, M, Q	c, G, e, μ_0	—	Stoney units, qm	—

6 PLANCK'S FUNDAMENTAL UNITS

Consider a (3, 3) system. The 3 dimensions are again length, time and mass, and the 3 normalizable constants are c, G, and the reduced Planck's constant ħ. Analogous to (4) and (5), I write

$$\text{ħ} = \hbar\,\text{erg}/(\text{rad s}^{-1}) = \hbar\,\text{cm}^2\,\text{s}^{-1}\,\text{gm rad}^{-1}, \tag{23}$$

where
$$\hbar = 1{\cdot}055\,(-27). \tag{24}$$

Note rad (radian) is an *angular* unit, hence its presence in the specification of the dimension of ħ at (23) leaves the number of *physical* dimensions at 3. The normalization of ħ then gives, on using (18),

$$\hbar = l_0^2\,t_0^{-1}\,m_0. \tag{25}$$

Solving the three equations (20*a*), (20*b*) and (25), we obtain a unique set of new units, called Planck's fundamental units:

$$1 \text{ Planck Length} = (G\hbar/c^3)^{\frac{1}{2}}\text{ cm} = 1{\cdot}62\,(-33)\text{ cm} \tag{26a}$$

$$1 \text{ Planck Time} = (G\hbar/c^5)^{\frac{1}{2}}\text{ s} = 5{\cdot}39\,(-44)\text{ s} \tag{26b}$$

$$1 \text{ Planck Mass} = (c\hbar/G)^{\frac{1}{2}}\text{ gm} = 2{\cdot}18\,(-5)\text{ gm}. \tag{26c}$$

In this case, since $n = k = 3$, all 3 units are 'defined' units, and none are 'chosen' units.

In Section 3, the Hubble units were regarded as belonging to a (3, 2)-system, with the Hubble Time τ_0 as a chosen unit and the other two as defined units. Of course, we can also regard τ_0 as a third constant to be normalized, then we have a (3, 3)-system, and all the three Hubble units are defined units, and then there is an exact parallel between the Hubble units and the Planck units. Throughout practically the whole history of the expanding universe, when general relativity applies, the Hubble units are appropriate, but very close to the initial singularity when the age of the universe was of the order of the Planck Time and its radius was of the order of the Planck Length, then general relativity is not valid. The significance of the Planck Mass is that, if the elementary particles were of the order of one Planck Mass, their mutual gravitation far exceeded their electromagnetic and nuclear forces. At the present, their masses are some 17–20 powers of 10 below the Planck Mass, and their mutual gravitation is altogether negligible compared to the other forces.

Some twenty years before Planck's discovery of the quantum, the Irish physicist G.J.Stoney derived a set of 'natural units' which differed from the Planck units by only a constant factor (Ray 1981, Barrow 1983). In the present terminology, the Stoney units belong to a (4, 3) system, rather than a (3, 3) system, and it is proper to defer their discussion until we have dealt with the electrostatic units (e.s.u.)

7 THE ASTRONOMICAL UNIT (AU) AND THE GAUSSIAN CONSTANT

We are so used to thinking that the astronomical unit (AU) is the mean distance between the Sun and the Earth that it is well to remind ourselves that this identification cannot be correct in principle. For, obviously, we must require the AU, as we require any unit, to be constant in time, but this immediately disqualifies the Earth–Sun mean distance, since the Earth's orbit about the Sun is varying all the time due to the gravitational attraction of the other bodies of the solar system. So, at the most, we might only identify the AU with the semi-major axis of the osculating orbit of the Earth at some particular epoch, T_0, say, but this would still be impracticable.

Let a_0 be the Earth's mean distance at T_0. It is not a directly observed quantity; it has to be calculated from n_0, the Earth's mean motion at T_0. The calculation is by means of Kepler's Third Law for the Sun–Earth system:

$$n_0^2 a_0^3 = G\,(M_\odot + M_{E+M}). \qquad (27)$$

Note, besides n_0 and a_0, which are time-dependent, equation (27) contains 3 quantities which are presumably constant in time. Thus, if we were to evaluate a_0 in some conventional (un-normalized) units, we need to know not only the numerical value of n_0, but also those of G, $M_\odot$ and M_{E+M} (the combined mass of the Earth and the Moon) in those units. A simplification results if we take $M_\odot$ as unit of mass: then we do not require the two masses separately, only their (numerical) ratio $m = M_{E+M}/M_\odot$. We still need to know G and m. Now, both G and m are only known to certain, not-very-high accuracy at any time. (Even at present, G is only known to 4 or 5 significant figures, while Gauss's adopted value of m is some 7·5 per cent below the present best estimate.) Hence, with every improvement in the determination of G or m, the value of a_0 would be changed, and this disqualifies a_0 as a unit of length.

In summary, we cannot identify the AU with the Sun–Earth mean distance in general because the latter is a varying quantity in time, and we cannot even identify it with the mean distance at a particular epoch, because the evaluation of the latter requires the values of certain physical constants and these are always subject to updating.

What, then, is the AU? Can we not find a physical interpretation for it at all?

The answer is this: the AU is the semi-major axis of the Keplerian orbit of a (hypothetical) planet having the following mean motion and mass:

$$n = 2\pi/365{\cdot}256\,383\,5 \text{ rad d}^{-1} \tag{28a}$$

$$M_p = 1/354710\, M_\odot. \tag{28b}$$

These were actually Gauss's estimates for the Earth–Moon system (Roy 1978, p. 266); but here, they are to be regarded as arbitrary, fixed constants for defining a new, time-independent unit of length called AU.

To drive home the distinction between a_0 and AU, let us evaluate a_0 in terms of AU, for 1900·0 and using the current best estimate of m. Kepler's Third Law states that a_0 varies as n_0 to the power $-\frac{2}{3}$, and as $(1+m)$ to the power $+\frac{1}{3}$. The current best estimate for m is 1/328910 (Allen 1973, p. 140). With Gauss's value at (28b), this corresponds to a fractional change in $(1+m)$ of $+2{\cdot}21\ (-7)$, hence a fractional change in a_0 of $+7{\cdot}4\ (-8)$. Again, from Allen's book, (p. 19), we find $n_0\ (1900{\cdot}0) = 2\pi/365{\cdot}256\,265\,56$. This represents a fractional change of $+4{\cdot}90\ (-8)$ over Gauss's value at (28a), hence a fractional change of $-3{\cdot}3\ (-8)$ in a_0. The sum of the two fractional changes in a_0 is $+4{\cdot}1\ (-8)$. That is,

$$a_0\ (1900{\cdot}0) = 1{\cdot}000\,000\,041 \text{ AU}.$$

This evaluation illustrates the theoretical nature of the distinction between a_0 and the AU.

A common practice in celestial mechanics is to replace the gravitational constant G by its square root, called the Gaussian Constant,

$$k = G^{\frac{1}{2}}; \tag{29}$$

thus Kepler's Third Law can now be written as

$$k = n a^{\frac{3}{2}} (M_\odot + M_p)^{-\frac{1}{2}}. \tag{30}$$

Inserting the values (28) and putting a = 1 AU, we have

$$k = k \text{ d}^{-1} \text{ AU}^{\frac{3}{2}} M_\odot^{-\frac{1}{2}}, \tag{31}$$

where
$$k = 0{\cdot}017\,202\,098\,950 \text{ rad}. \tag{32}$$

An alternative interpretation of the AU can now be given: it is the semi-major axis of the Keplerian orbit of an infinitesimal mass-point having a mean motion of 0·017 202 098 950 radians per day. Now if we define a new unit of time τ_0,

$$\tau_0 = 0{\cdot}017\,202\,098\,950^{-1} \text{ d} = 58{\cdot}132\,440\,87 \text{ d}, \tag{33}$$

then 0·017 202 098 950 radians per day is equal to 1 radian per τ_0, and in units of AU, τ_0 and $M_\odot$, k has the value 1 radian:

$$k = 1 \text{ rad } \tau_0^{-1} \text{ AU}^{\frac{3}{2}} M_\odot^{-\frac{1}{2}}. \tag{34}$$

It follows that, in the present terminology, the AU is the one defined unit in a (3, 1) system, in which the two chosen units are τ_0 and $M_\odot$, and the physical constant that is normalized is the Gaussian Constant k. Usually, constants are normalized to the *number* 1, here, the constant is normalized to the *angle* 1 radian.

In terms of this system of normalized units, Kepler's Third Law for a body of negligible mass assumes the simple numerical form,

$$n^2a^3 = 1 \tag{35a}$$

while for an object of finite mass m (in units of $M_\odot$), the Law is

$$n^2a^3 = 1+m. \tag{35b}$$

8 UNITS OF TIME IN THE PROBLEM OF THREE BODIES

Much of Celestial Mechanics deals with the restricted problem of three bodies, typified by the Sun–Jupiter–Asteroid system. Here, Jupiter is assumed to move in a fixed ellipse (or circle) of semi-major axis (radius) a_J and it is natural to take a_J as unit of length. There are then two possible new time units, depending on what we take next as unit of mass. If we still keep $M_\odot$ as unit of mass, then from (34) it is obvious that the new unit of time τ_1 is related to τ_0 and a_J by

$$\tau_0^{-1}\,AU^{\frac{3}{2}} = \tau_1^{-1}\,a_J^{\frac{3}{2}}. \tag{36}$$

To be specific, if we take (Allen 1973, p. 140),

$$a_J = 5{\cdot}2028\ AU \tag{37}$$

then,

$$\tau_1 = 11{\cdot}8674\,\tau_0 = 689{\cdot}881\ d = 1{\cdot}88883\ y. \tag{38}$$

Alternatively, if we take the combined mass of the Sun and Jupiter as unit of mass, then the new unit of time τ_2 is related to τ_1 and the masses by

$$\tau_1^{-1}\,M_\odot^{-\frac{1}{2}} = \tau_2^{-1}\,(M_\odot + M_J)^{-\frac{1}{2}}. \tag{39}$$

To be specific, if we take (Allen *ibid.*)

$$m' \equiv M_J/M_\odot = 1/1047{\cdot}39 \tag{40}$$

then

$$\tau_2 = (1+m')^{-\frac{1}{2}}\,\tau_1 = \tau_1/1{\cdot}000477,$$
$$= 689{\cdot}551\ d = 1{\cdot}88793\ y, \tag{41}$$

In units of the first set (a_J, τ_1, $M_\odot$), Kepler's Third Law for the asteroid

TABLE III

Two systems of units for the Sun–Jupiter–Asteroid System

$a_J = 5{\cdot}2028$ AU $m' = M_J/M_\odot = 1/1047{\cdot}39$

	Unit of			Kepler's Third Law	
System	length	mass	time	asteroid	Jupiter
I	a_J	$M_\odot$	$\tau_1 = 689{\cdot}881$ d	$a^3n^2 = 1$	$n' = (1+m')^{\frac{1}{2}}$
II	a_J	$M_\odot + M_J$	$\tau_2 = 689{\cdot}551$ d	$a^3n^2 = (1+m')^{-1}$	$n' = 1$

assumes the simple form of (35*a*), but the mean motion of Jupiter has the value $n' = (1+m')^{\frac{1}{2}}$. In units of the second set, $(a_J, \tau_2, M_\odot + M_J)$, the mean motion of Jupiter has value 1, but Kepler's Third Law for the asteroid becomes $n^2a^3 = (1+m')^{-1}$. These results are collected in Table III.

9 ELECTROSTATIC UNITS (e.s.u.) AND ELECTROMAGNETIC UNITS (e.m.u.)

So far we have been dealing with dynamics, and the well-known and remarkable fact is that the whole of the dynamical world is describable in terms of three and just three, physical dimensions. We usually take these to be length, time and mass, but we could equally take any set of three independent combinations of these.

When we come to deal with electric and magnetic phenomena, we must expect to be involved in some new dimensions. For those centuries when electric and magnetic phenomena were thought to be distinct happenings (as distinct as amber is from lodestone), there was a strong presumption that two new dimensions were called for. But one of the great triumphs of nineteenth-century physics is the demonstration that magnetism is just electricity in motion and that one new dimension will cover both. It does not matter whether we take electric charge or electric current as the fourth dimension. In this essay, I shall take the charge Q.

As is well-known, Coulomb's Law of static charges has the same form as Newton's Law of Gravitation. In both, it is stated that a force is *proportional* to some physical quantities raised to certain powers. The word 'proportional' is expressly emphasized here. The constant of proportionality in Newton's Law is the gravitational constant G. In the case of Coulomb's Law, the constant of proportionality depends on the medium in which the charges are located. As a definite point of reference, we take free space. Coulomb's Law for the force between two charges q_1, q_2 at a distance r apart is then

$$F = q_1\, q_2/\varepsilon_0\, r^2, \tag{42}$$

where ε_0, called the permittivity of free space, is the constant of proportionality in question (more precisely, $1/\varepsilon_0$ is the analogue of G). It is easy to see that there is practical convenience in using units in which ε_0 has the value 1.

We are concerned here with a (4, 1) system. The 4 dimensions are length, time, mass and charge; the one normalizable constant is ε_0. After we have chosen centimetre, second and gram as the units for the 3 dimensions of dynamics, the normalization of ε_0 defines a new unit of charge. This unit of charge will be denoted by qs in this essay. The system of normalized units consists, then, of the 4 units, cm, s, gram and qs. This is the precise content of what is known as the system of electrostatic units (abbreviation e.s.u.).

Common usage of the abbreviation e.s.u. can be a source of confusion for the lay reader. We may hear talk of a current of so many e.s.u., or an electric field of so many e.s.u. etc. This, to the uninitiated, can be as confusing as the talk of a mass of so many cm, or a time of so many cm. Of course, what is meant is, in the first case, so many 'e.s.u. of current', and, in the second case, so many 'e.s.u. of electric field'. In fact, in terms of the above 4 basic units,

$$1 \text{ e.s.u. of current} = 1 \text{ qs s}^{-1},$$

$$1 \text{ e.s.u. of electric field} = 1 \text{ qs}^{-1} \text{ cm s}^{-2} \text{ gm}.$$

Note, in these unit conversions, the numerical factor on the right is always 1. This follows from the fact that it is cm, s and gm themselves, and not some multiples of them, that are implicit in the system of e.s.u.

Just as Coulomb's Law describes the force between static charges, Ampère's Law describes the force between moving charges or currents. The statement of Ampère's Law involves the geometry of the currents; for our purpose, we can take the simple configuration of two short-current elements forming the opposite sides of a small square. Then the law states that the force is just proportional (again *proportional*) to the two currents I_1, I_2. Again take free space, and we have

$$F = \mu_0 I_1 I_2, \tag{43}$$

where μ_0 is known as the permeability of free space. Normalization of μ_0, together with the choice again of cm, s and gm, defines another unit of charge, which will be denoted by qm in this essay, and we have the full complement of the system of electromagnetic units (e.m.u.).

Again, the knowledgeable may talk of a current of so many e.m.u. or a magnetic flux density of so many e.m.u., when they actually mean so many 'e.m.u. of current' or so many 'e.m.u. of magnetic flux density'. And we have

$$1 \text{ e.m.u. of current} = 1 \text{ qm s}^{-1},$$

$$1 \text{ e.m.u. of magnetic flux density} = 1 \text{ qm}^{-1} \text{ s}^{-1} \text{ gm}.$$

There is, of course, no reason to suppose qs and qm to be equal. On the other hand, since both are units for the same physical dimension of electric charge, their ratio must be just a pure number. It caused no small surprise when physicists first found from experiments that the ratio qm/qs came out close to the value denoted by c at (5). Things fell into place after Maxwell had deduced from the equations now bearing his name the existence of electromagnetic waves and realized that this numerical coincidence must mean that light is just such waves. Let us now proceed the other way: start with Maxwell's physical identification and deduce the numerical equality.

According to Maxwell's equations, electromagnetic waves propagate in free space with a speed of $(\varepsilon_0 \mu_0)^{-\frac{1}{2}}$. Hence, we have the following relation among the three physical constants, the permittivity and permeability of free space and the velocity of light:

$$c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}}. \tag{44}$$

From (42), by the definition of qs, we have

$$\varepsilon_0 = 1 \text{ qs}^2 \text{ cm}^{-2} \text{ dyne}^{-1}, \tag{45}$$

where 'dyne' is retained for brevity. Similarly, from (42) and the definition of qm, we have

$$\mu_0 = 1 \text{ qm}^{-2} \text{ s}^2 \text{ dyne}. \tag{46}$$

Substituting these in (43), we obtain

$$c = (\mathrm{qm}/\mathrm{qs})\ \mathrm{cm\ s^{-1}}. \tag{47}$$

Hence, on comparing this with (4), we obtain the equality

$$\mathrm{qm}/\mathrm{qs} = c. \tag{48}$$

In other words, 1 qm is equal to c qs.

10 MAXWELL'S EQUATIONS IN GAUSSIAN UNITS

It follows from the exchange rate, 1 qm = c qs, that if a physical quantity contains charge to some power n, then its e.s.u. measure must be c^n times its e.m.u. measure. Now, magnetic flux density B contains charge to the power -1, so if we write B_s for its e.s.u. measure and B_m for its e.m.u. measure, we shall have

$$B_s = c^{-1} B_m. \tag{49}$$

Similarly, since electric displacement D and current density J both contain charge to the power $+1$, we have

$$D_s = cD_m, \quad J_s = cJ_m. \tag{50}$$

We are now ready to consider two of Maxwell's equations. They are

$$\mathrm{curl\ E} = -\frac{\partial \mathrm{B}}{\partial t}, \tag{51}$$

$$\mathrm{curl\ H} = +\frac{\partial \mathrm{D}}{\partial t} + 4\pi \mathrm{J}. \tag{52}$$

The numerical factor 4π representing the surface area of the unit sphere in (52) is necessary when ε_0 and μ_0 are defined as in (42) and (43). An alternative scheme, called 'rationalized', is to have the 4π appearing instead in the denominator of the right sides of (42) and (43). From our point of view, this alternative scheme is more awkward, for we then have to say that the definition of qs is by the normalization of ε_0 not to the number 1, but to the 'reciprocal solid angle' $(4\pi\ \mathrm{sterad})^{-1}$, and that qm is defined by the normalization of μ_0 to the solid angle 4π sterad. We shall, therefore, stick to the 'unrationalized' scheme.

The important thing to note in (51) and (52) is that the field variables E, B, H, D, J are written in roman, because they are intended to represent the physical quantities themselves. At least in the first instance – as will be more full explained in the next Section – they can also stand for numerical measures of the quantities, *provided one system of units is used for all.* Complications only arise when some quantities are expressed in one system of units, and some in another. And this is what happens when we use the so-called Gaussian units (or, very rightly, 'mixed' units). The convention with Gaussian units is that all 'electric' quantities like E, D, and J are to be in e.s.u., and all 'magnetic' quantities like B and H are to be in e.m.u.

Write (51) in e.s.u., that is, italicise E and B and add suffix s:

$$\operatorname{curl} E_s = -\frac{\partial B_s}{\partial t}. \tag{51a}$$

According to the Gaussian convention, we must replace B_s by B_m. This we do according to (49) and so we obtain

$$\operatorname{curl} E_s = -\frac{1}{c}\frac{\partial B_m}{\partial t}. \tag{51b}$$

This is what people mean when they write

$$\operatorname{curl} E = -\frac{1}{c}\frac{\partial B}{\partial t} \tag{51c}$$

and say it is the equation in Gaussian units.

Similarly, first writing (52) in e.m.u.,

$$\operatorname{curl} H_m = +\frac{\partial D_m}{\partial t} + 4\pi J_m, \tag{52a}$$

then, on using (50), we obtain

$$\operatorname{curl} H_m = \frac{1}{c}\left(\frac{\partial D_s}{\partial t} + 4\pi J_s\right), \tag{52b}$$

and this is what people mean when they write

$$\operatorname{curl} H = \frac{1}{c}\left(\frac{\partial D}{\partial t} + 4\pi J\right) \tag{52c}$$

and say it is in Gaussian units.

11 THE TWO TYPES OF EQUATIONS IN PHYSICS

I wonder how many students of electromagnetism have anguished over the difference between (51 *c*) and (51), and between (52 *c*) and (52), the presence of *c* in one and not in the other. We have been so accustomed to think that *c* is the velocity of light that we come to a rude shock when we make this identification in (51 *c*) and (52 *c*): we find the equations do not balance dimensionally. We must remind ourselves that the *c* in these equations does not stand for the velocity of light, that it stands for just a pure number, the rate of exchange between two units of charge, qm and qs, and the number is 2·998 (+10), to 4 significant digits. It is entirely analogous to the κ in the well-known equation in stellar kinematics,

$$V_t = \kappa\mu/p, \tag{53}$$

connecting the transverse velocity of a star in km s^{-1}, V_t, to its proper motion in arcsec y^{-1}, μ, and its parallax in arcsec, p: κ is simply the ratio between two units of velocity, AU y^{-1} and km s^{-1}, and is nearly equal to 4·74. We must also remember that the letters *E*, *B*, *H*, *D*, *J* in (51 *c*) and (52 *c*) do not stand for physical quantities either; they stand for the numerical measures of certain quantities in two particular systems of units (*E*, *D*, *J* in e.s.u., *B*, *H*

TABLE IV

Type-I (unit-free) and Type-II (unit specific) equations

LINE NAME	TYPE-I	TYPE-II	SPECIALIZED UNITS
1 Hubble	$\mathrm{v} = \mathrm{H}_0 \mathrm{r}$	$v = r$	v in km s^{-1} r in km s^{-1}-of-distance, eq. (3)
2 Kepler	$\mathrm{n}^2\mathrm{a}^3 = \mathrm{G}(\mathrm{M}_\odot + \mathrm{M}_\mathrm{p})$	$n^2a^3 = 1+m$	n in rad τ_0^{-1}, eq. (33) a in AU, $m = \mathrm{M}_\mathrm{p}/\mathrm{M}_\odot$
3 Coulomb	$\mathrm{F} = \mathrm{q}_1\,\mathrm{q}_2/\varepsilon_0\,\mathrm{r}^2$	$F = q_1\,q_2/r^2$	q_1, q_2 in qs, r, F in c.g.s.
4 Ampère	$\mathrm{F} = \mu_0\,\mathrm{I}_1\,\mathrm{I}_2$	$F = I_1\,I_2$	I_1, I_2 in qm s^{-1}, F in c.g.s.
5 *	$\mathrm{D} = \varepsilon_0\,\mathrm{E}$	$D = E$	D, E in e.s.u.
6 *	$\mathrm{B} = \mu_0\,\mathrm{H}$	$B = H$	B, H in e.m.u.
7 *	$\alpha = \mathrm{e}^2/\varepsilon_0\, c\hbar$	$\alpha = e^2/c\hbar$	e in qs; $\hbar$, c in c.g.s. $\alpha = 1/137{\cdot}036$
8 Maxwell	$\mathrm{curl}\ \mathrm{E} = -\dfrac{\partial \mathrm{B}}{\partial \mathrm{t}}$	$\mathrm{curl}\ E = -\dfrac{1}{c}\dfrac{\partial B}{\partial t}$	E in e.s.u., B in e.m.u. $c = 2{\cdot}998\ (+10)$
9 Maxwell	$\mathrm{curl}\ \mathrm{H} = \dfrac{\partial \mathrm{D}}{\partial \mathrm{t}} + 4\pi \mathrm{J}$	$\mathrm{curl}\ H = \dfrac{1}{c}\left(\dfrac{\partial D}{\partial t} + 4\pi J\right)$	H in e.m.u., D, J in e.s.u., $c = 2{\cdot}998\ (+10)$
10 Lorentz	$\mathrm{F} = \mathrm{q}(\mathrm{E} + \mathrm{v} \times \mathrm{B})$	$F = q\left(E + \dfrac{1}{c} v \times B\right)$	q, E in e.s.u., B in e.m.u., F, v in c.g.s., $c = 2{\cdot}998\ (+10)$
11 *	$\mathrm{v}_\mathrm{t} = \mathrm{a}_0\,\mu/\mathrm{p}$	$v_t = \kappa\mu/p$	v_t in km s^{-1}, μ in arcsec y^{-1}, p in arcsec, $\kappa = 4{\cdot}74$

* Line 5: relation between electric field and displacement for free space.
Line 6: relation between magnetic field and flux density for free space.
Line 7: the fine structure constant.
Line 11: relation between transverse velocity, proper motion and parallax.

in e.m.u.). These equations express, then, relations among pure numbers, so no wonder we came to grief when we tried to balance them dimensionally.

In contrast, equations (51) and (52) do express relations among physical quantities, and we can talk meaningfully of the physical dimensions of the various terms and ask whether all the terms within each equation have the same dimension. When such an equation balances dimensionally, then it is complete in the sense that all relevant physical quantities, including physical constants, are explicitly present. Then, the symbols can also be understood to represent the numerical measures of the quantities in some one system of units. Now both sides of (51) have the dimension $\mathrm{Q}^{-1}\,\mathrm{T}^{-2}\,\mathrm{M}$, so the equation balances dimensionally; (52) is also dimensionally correct since all its three terms have the common dimension $\mathrm{QT}^{-1}\mathrm{L}^{-2}$. The symbols E, B, H, D, J in these two equations can thus represent *either* the physical quantities, electric field, magnetic flux density, magnetic field, electric displacement and current density, *or* the numerical measures of these quantities in some *one* system of units. It does not matter what system we take; we could, if we like, express all lengths in feet, all times in minutes, all masses in pounds, all currents in amperes and all other quantities in combinations of these, and the equations will still hold. In this sense, then, (51) and (52) are 'unit-free'. In the literature, the common practice has been to refer to them as being in 'MKS or SI units' (unrationalized). This is most regrettable, for it creates the false impression that these equations are only true in that particular system of units. Worse, it leaves out their most important property: they connect physical quantities. It would be far better to describe such equations as 'physical equations' or 'unit-free equations'.

The use of mixed units thus downgrades a unit-free equation into a 'unit-specific' equation. The use of normalized units similarly downgrades: the original equation explicitly contains the normalized constants. The superficial effect is different in the two cases: with mixed units, new symbols representing conversion ratios are added; with normalized units, existing symbols representing the normalized constants are removed. But the really important effect is the same: the symbols no longer represent physical quantities; they have become numerical measures in some specialized ('mixed' or 'normalized') system of units.

I shall call the physical or unit-free equations Type-I, and the unit-specific equations Type-II. In this essay, Type-I equations are written in roman, Type-II in italic, so the distinction is clear. But I am not advocating the general adoption of this practice, for practical reasons. The ambiguity which has existed throughout the entire corpus and history of physics literature, the non-distinction of these two types of equations, is likely to go on. Given an equation, the reader should be clear in his mind as to which type it belongs, and if it is Type-II, what specialized units are implied, and what the original Type-I equation is. Equation (53), for example, is a Type-II equation, and its primitive Type-I form is $\mathrm{V_t} = \mathrm{a_0}\, \mu/p$, where $\mathrm{a_0}$ is the semi-major axis of the Earth's orbit around the Sun. I have collected together, in Table IV, a number of Type-II equations, with their specialized units used and their Type-I primitives. The only entry that requires comment is Line 7, relating to the fine-structure constant. The comment will be made in the next Section.

12 THE STONEY UNITS AND THE FINE-STRUCTURE CONSTANT

In a lecture to the British Association in 1874 entitled 'On the Physical Units of Nature', G.J.Stoney (1881) derived a set of units of length, time and mass by normalizing 3 quantities regarded as constants of nature, the velocity of light c, the gravitational constant G and the charge on the electron e. Now, these 3 constants involve 4 physical dimensions; so in the present terminology, we are dealing with a (4, 3) system, not a (3, 3) system, and there must be 1 'chosen' unit in addition to the 3 'derived' or 'defined' units. It is easy to find what the implicitly chosen unit was from the expressions for the derived units. These are

$$1 \text{ Stoney Length} = (Ge^2/c^4)^{\frac{1}{2}} \text{ cm}, \tag{54a}$$

$$1 \text{ Stoney Time} = (Ge^2/c^6)^{\frac{1}{2}} \text{ s}, \tag{54b}$$

$$1 \text{ Stoney Mass} = (e^2/G)^{\frac{1}{2}} \text{ gm}, \tag{54c}$$

where G and c are the c.g.s. measures of G and c and $e = 4{\cdot}8032\,(-10)$ is the e.s.u. measure of e. Hence Stoney implicitly chose qs as the unit of charge. The point is that it is not possible to derive a unique set of units of length, time and mass from the normalization of c, G and e, and that Stoney's particular set is dependent on his arbitrary choice of qs as the unit of charge.

Alternatively, we could say that the three Stoney units *plus* qs are uniquely defined by the normalization of the *four* constants, c, G, e and ε_0. This removes the arbitrary element and it is quite arguable that ε_0 is just as fundamental a constant of nature as e is. Obviously, we could also say that the three Stoney units plus qm are uniquely defined by the normalization of c, G, e and μ_0. These results complete the tabulation of Table II.

If we compare the Stoney units (54) with the Planck units (26), we see that they only differ by the constant factor $(e^2/c\hbar)^{\frac{1}{2}} = \alpha^{\frac{1}{2}}$, the square root of the fine-structure constant $\alpha = 1/137{\cdot}036$. The existence of the fine-structure constant is the reason why Stoney was able to derive his 'physical' units before the discovery of the quantum.

Note the commonly used expression for the fine-structure constant, $\alpha = e^2/c\hbar$ is a Type-II expression. Its Type-I primitive is $\alpha = \mathrm{e}^2/\varepsilon_0\, \mathrm{c}\hbar$. This comment completes Table IV and concludes the present essay.

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REFERENCES

Allen, C.W., 1973. *Astrophysical Quantities*, 3rd edn, pp. 19, 140, Athlone Press, London.
Barrow, J.D., 1983. *Q. Jl R. astr. Soc.*, **24**, 24.
Eddington, A., 1933. *The Expanding Universe*, p. 4, Cambridge University Press.
Lang, K.R., 1974. *Astrophysical Formulae*, pp. 1, 2, Springer, Berlin.
Lynden-Bell, D., 1986. *Presidential Address of the R.A.S. 14 February 1986. 'How Smooth is the Hubble Flow?'*, *Q. Jl R. astr. Soc.*, **27**, 319.
Misner, C.W., Thorne, K.S. & Wheeler, J.A., 1973. *Gravitation*, p. 36, Freeman, San Francisco.
Ray, T.P., 1981. *Irish astr. J.*, **15**, 152.
Roy, A.E., 1978. *Orbital Motion*, p. 266, Adam Hilger, London.
Stoney, G.J., 1881. *Phil. Mag.* (ser. 5), **11**, 381.