The Vorticity-Streamfunction Formulation

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Computational Fluid Dynamics
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2D Constant Density Navier-Stokes

A

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\nabla^2 u$$

B

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\nabla^2 v$$

C

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

but there is no equation for the pressure

The equation of state is no longer valid to obtain the pressure

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Define Vorticity as $\boxed{\boldsymbol{\omega} = \nabla \times \mathbf{u}}$

$$\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \qquad \omega_y = -\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\frac{\partial B}{\partial y} - \frac{\partial A}{\partial x} \Rightarrow \boxed{\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y} = \nu \nabla^2 \omega_z}$$

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$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0}$$

Introduce the streamfunction:

$$u=\frac{\partial \psi}{\partial y} \qquad v=-\frac{\partial \psi}{\partial x}$$

Continuity is automagically satisfied

$$\boxed{\frac{\partial}{\partial x}\frac{\partial \psi}{\partial y}-\frac{\partial}{\partial y}\frac{\partial \psi}{\partial x}=0}$$

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$$\boxed{\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y} = \nu \nabla^2 \omega_z}$$

$$\boxed{\frac{\partial \omega_z}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega_z}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega_z}{\partial y} = \nu \nabla^2 \omega_z}$$

1 equation, two unknowns - need one more

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \Rightarrow \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega_z$$

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Vorticity Streamfunction Formulation

$$\frac{\partial \omega_z}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega_z}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega_z}{\partial y} = \nu \nabla^2 \omega_z$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega_z$$

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$$\frac{\partial \omega_z}{\partial t} = -\frac{\partial \psi}{\partial y}\frac{\partial \omega_z}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \omega_z}{\partial y} + \nu \nabla^2 \omega_z$$

$$\frac{\partial \omega_z}{\partial t} \approx \frac{\omega_{i,j}^{n+1} - \omega_{i,j}^{n}}{\Delta t}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial \omega_z}{\partial x} = \left(\frac{\psi_{i,j+1}^{n} - \psi_{i,j-1}^{n}}{2\Delta y}\right)\left(\frac{\omega_{i+1,j}^{n} - \omega_{i-1,j}^{n}}{2\Delta x}\right)$$

$$\frac{\partial \psi}{\partial x}\frac{\partial \omega_z}{\partial y} = \left(\frac{\psi_{i+1,j}^{n} - \psi_{i-1,j}^{n}}{2\Delta x}\right)\left(\frac{\omega_{i,j+1}^{n} - \omega_{i,j-1}^{n}}{2\Delta y}\right)$$

$$\nabla^2 \omega_z = \frac{\omega_{i+1,j}^{n} - 2\omega_{i,j}^{n} + \omega_{i-1,j}^{n}}{\Delta x^2} + \frac{\omega_{i,j+1}^{n} - 2\omega_{i,j}^{n} + \omega_{i,j-1}^{n}}{\Delta y^2}$$

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$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega_z}$$

$$\nabla^2 \psi = \frac{\psi^n_{i+1,j} - 2\psi^n_{i,j} + \psi^n_{i-1,j}}{\Delta x^2} + \frac{\psi^n_{i,j+1} - 2\psi^n_{i,j} + \psi^n_{i,j-1}}{\Delta y^2} = -\omega^n_{i,j}$$

$$\psi^n_{i,j} = \frac{\Delta x^2 \Delta y^2}{2\left(\Delta x^2 + \Delta y^2\right)}\omega^n_{i,j} + \frac{\Delta y^2}{2\left(\Delta x^2 + \Delta y^2\right)}\left(\psi^n_{i+1,j} + \psi^n_{i-1,j}\right) + \frac{\Delta x^2}{2\left(\Delta x^2 + \Delta y^2\right)}\left(\psi^n_{i,j+1} + \psi^n_{i,j-1}\right)$$

$$\psi^{k+1}_{i,j} = \beta\frac{\Delta x^2 \Delta y^2}{2\left(\Delta x^2 + \Delta y^2\right)}\omega^n_{i,j} + \beta\frac{\Delta y^2}{2\left(\Delta x^2 + \Delta y^2\right)}\left(\psi^k_{i+1,j} + \psi^{k+1}_{i-1,j}\right) + \beta\frac{\Delta x^2}{2\left(\Delta x^2 + \Delta y^2\right)}\left(\psi^k_{i,j+1} + \psi^{k+1}_{i,j-1}\right) + (1-\beta)\psi^k_{i,j}$$

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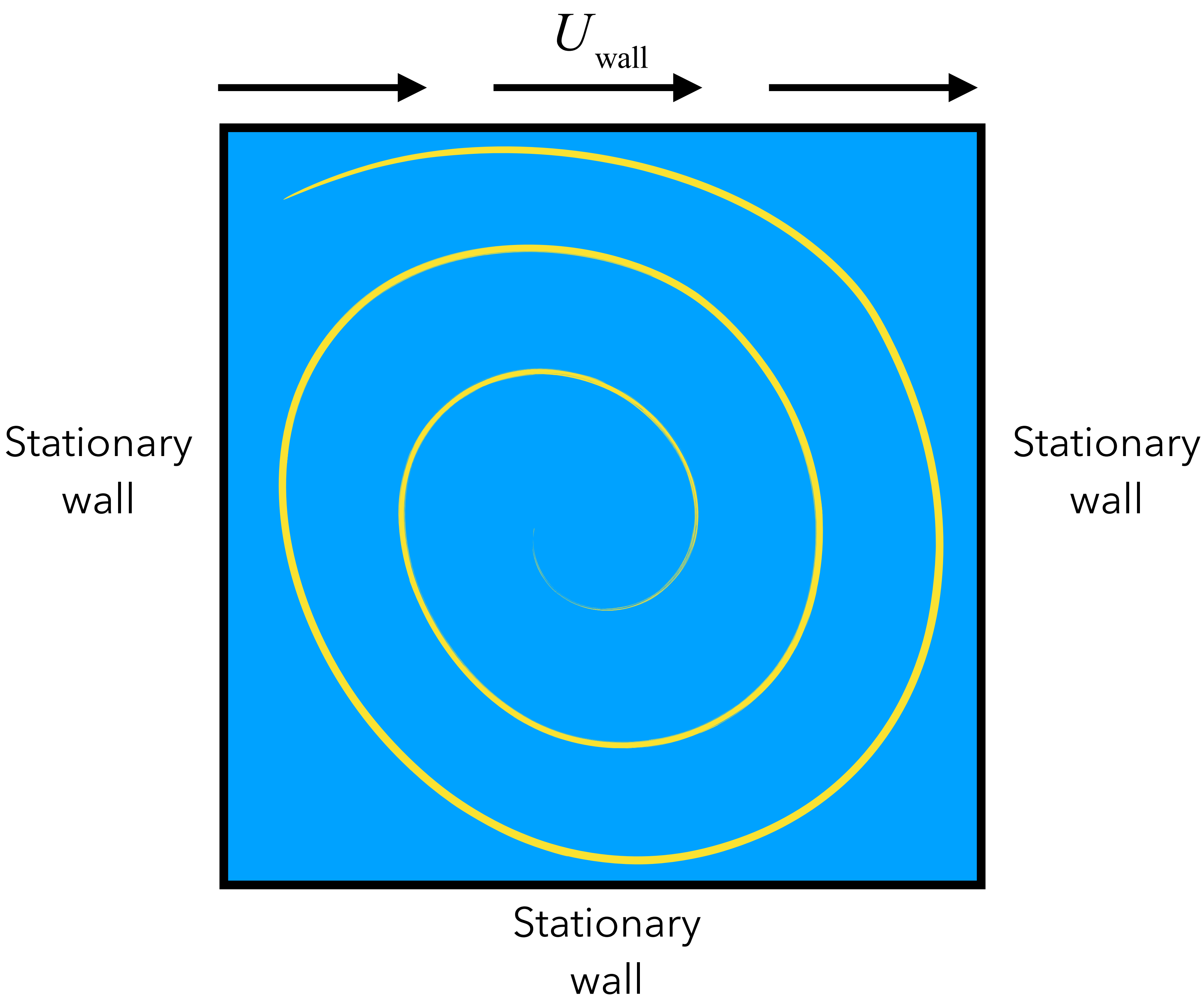
Lid-Driven Cavity



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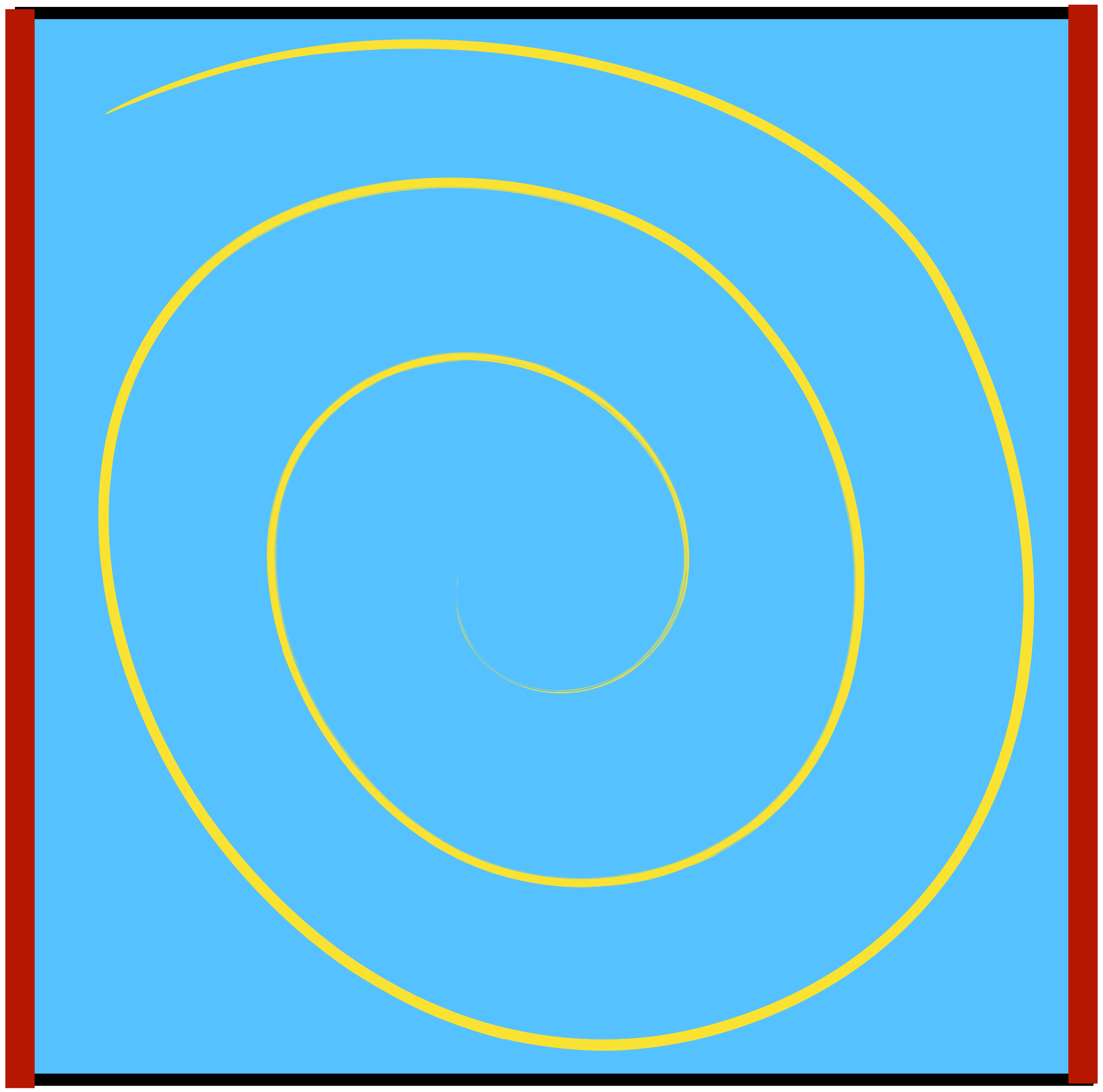
$$u(x=0)=0$$

$$\frac{\partial \psi}{\partial y}=0$$

$$\psi(x=0)=\text{Constant}$$



$$u(x=L)=0$$

$$\frac{\partial \psi}{\partial y}=0$$

$$\psi(x=L)=\text{Constant}$$

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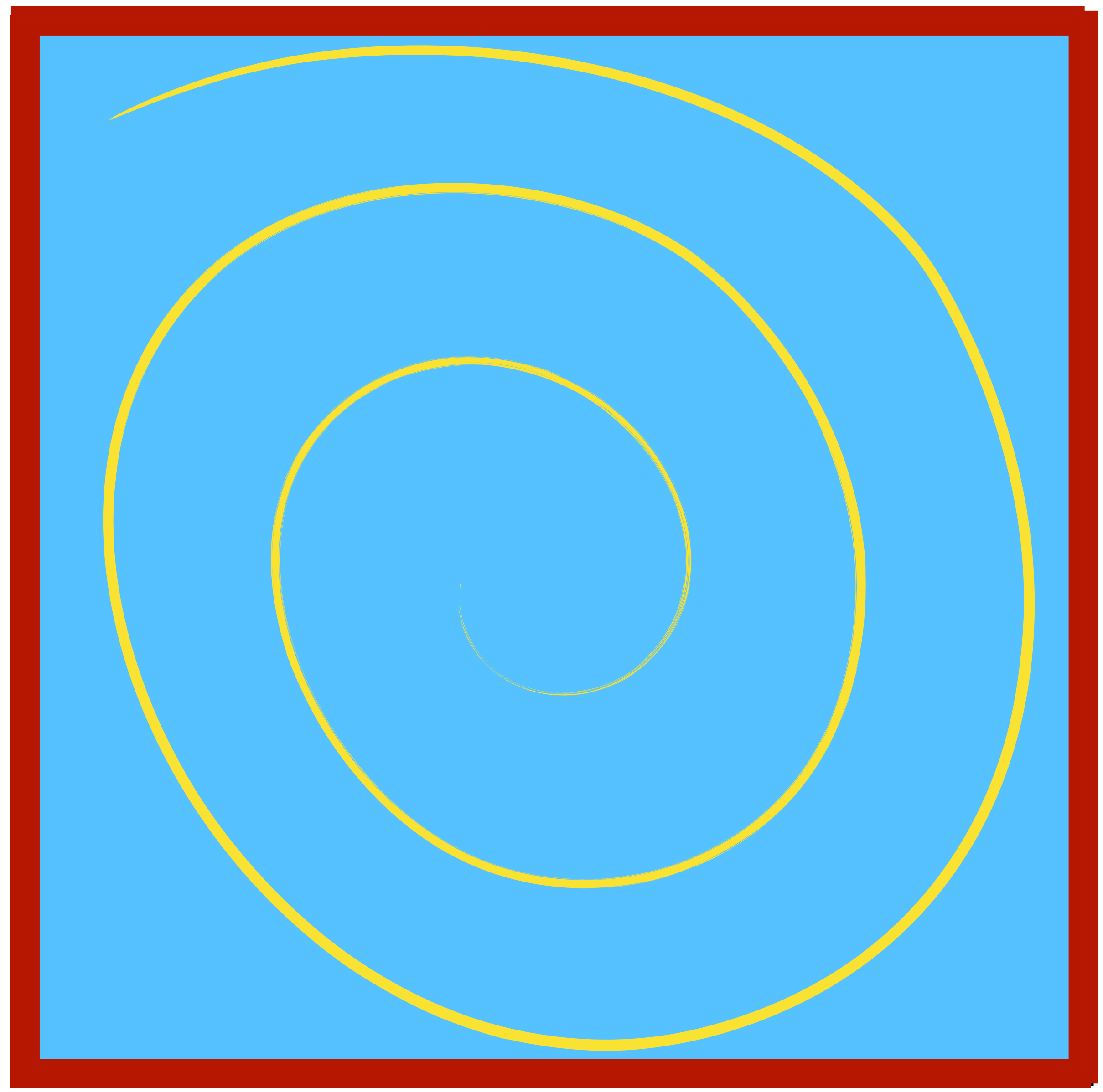
$$v(y=L)=0$$

$$\frac{\partial \psi}{\partial x}=0 \qquad \psi(y=L)=\text{Constant}$$

$$u(x=0)=0$$

$$\frac{\partial \psi}{\partial y}=0$$

$$\psi(x=0)=\text{Constant}$$



$$u(x=L)=0$$

$$\frac{\partial \psi}{\partial y}=0$$

$$\psi(x=L)=\text{Constant}$$

$$v(y=0)=0$$

$$\frac{\partial \psi}{\partial x}=0 \qquad \psi(y=0)=\text{Constant}$$

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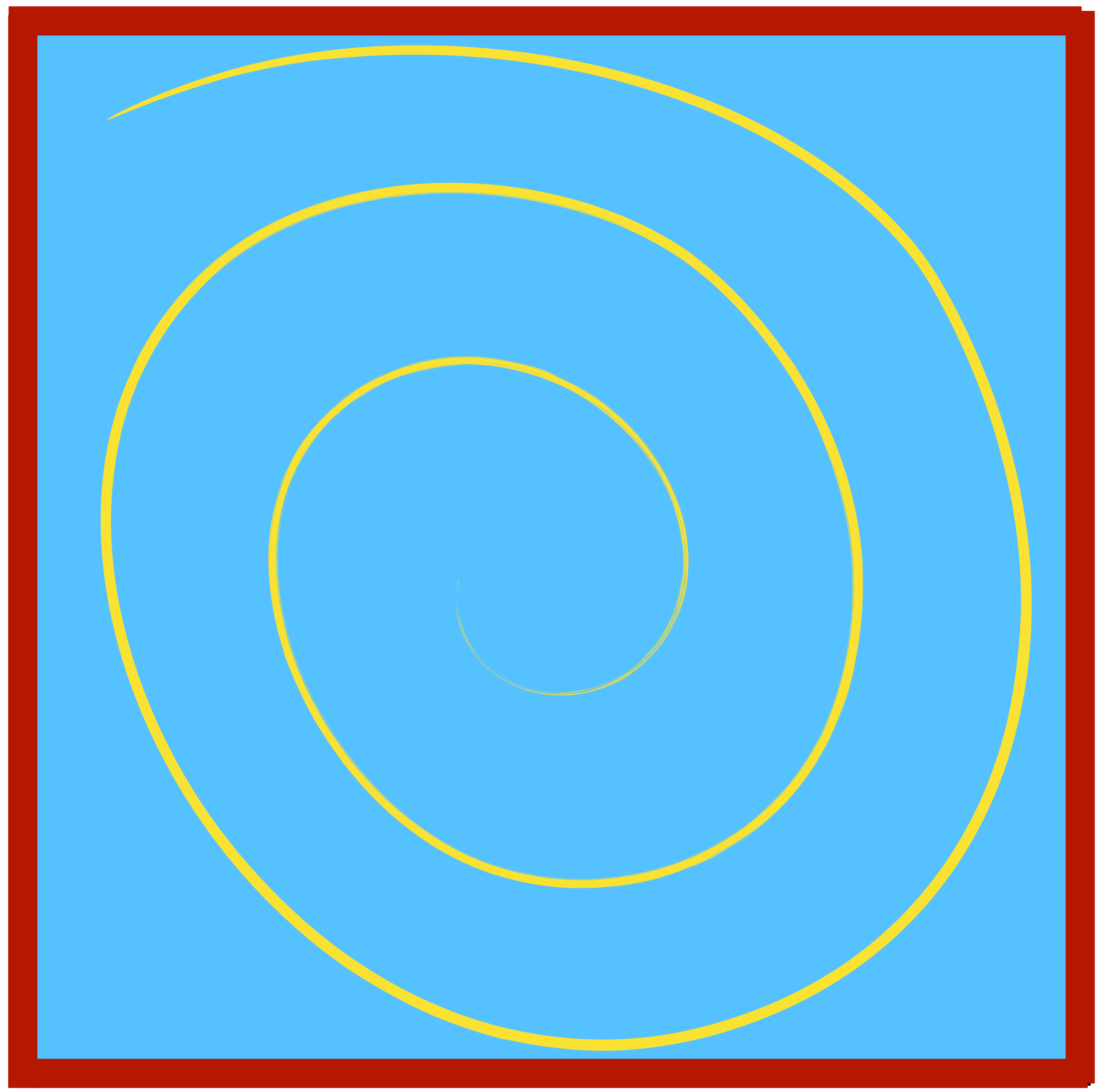
$$\psi_{\text{all boundaries}} = \text{Constant}$$



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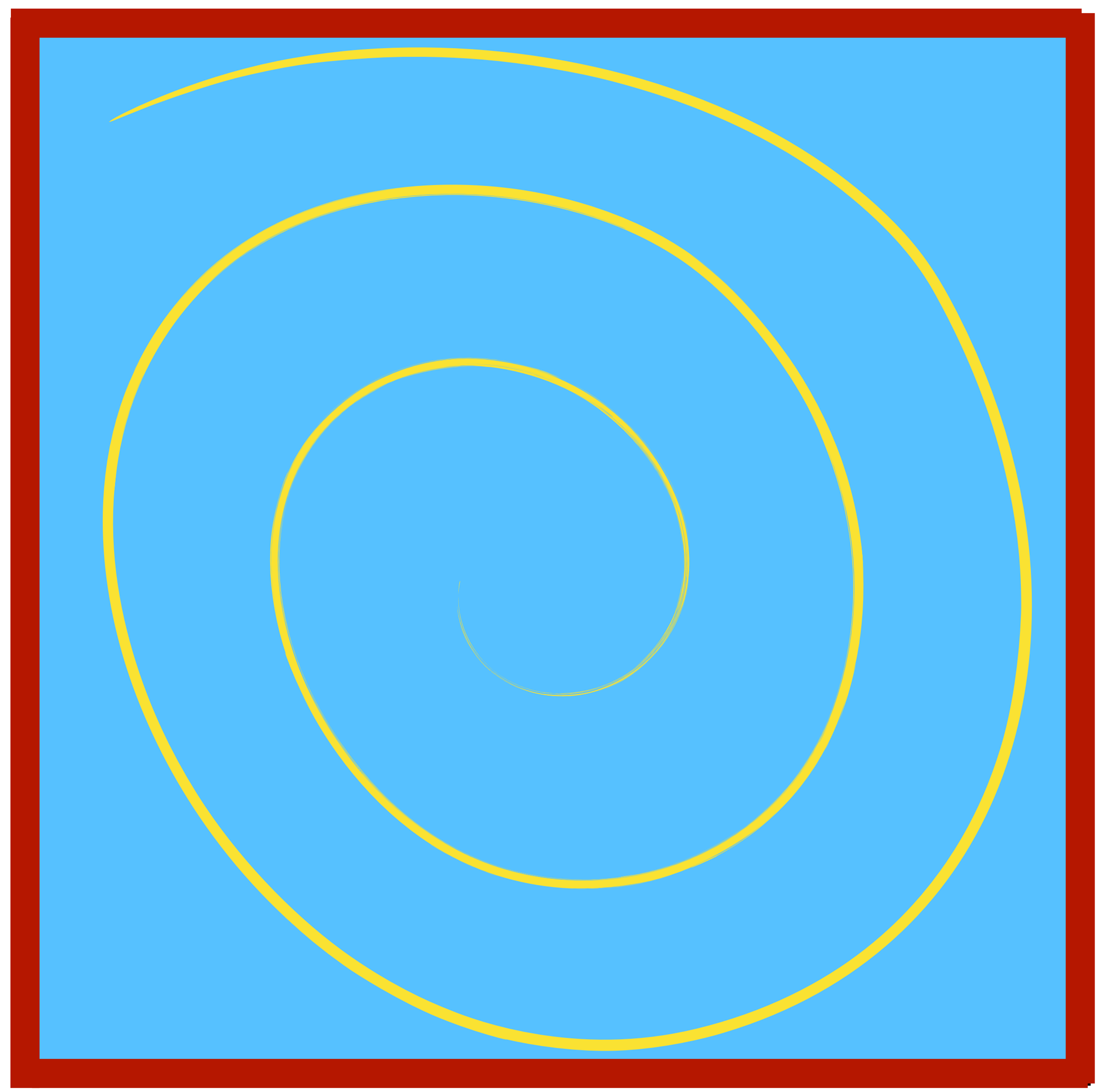
$$u = U_{\text{wall}} \qquad \frac{\partial \psi}{\partial y} = U_{\text{wall}}$$

$$v = 0$$

$$-\frac{\partial \psi}{\partial x} = 0$$



$$v = 0$$

$$-\frac{\partial \psi}{\partial x} = 0$$

$$u = 0$$

$$\frac{\partial \psi}{\partial y} = 0$$

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At left and right boundaries:

$$v = 0$$

$$-\frac{\partial\psi}{\partial x} = 0$$

$$\omega_z = -\frac{\partial^2\psi}{\partial x^2} - \cancel{\frac{\partial^2\psi}{\partial y^2}}$$

At top boundary:

$$u = U_{\text{wall}}$$

$$\frac{\partial\psi}{\partial y} = U_{\text{wall}}$$

$$\omega_z = -\cancel{\frac{\partial^2\psi}{\partial x^2}} - \frac{\partial^2\psi}{\partial y^2}$$

At bottom boundary:

$$u = 0$$

$$\frac{\partial\psi}{\partial y} = 0$$

$$\omega_z = -\cancel{\frac{\partial^2\psi}{\partial x^2}} - \frac{\partial^2\psi}{\partial y^2}$$

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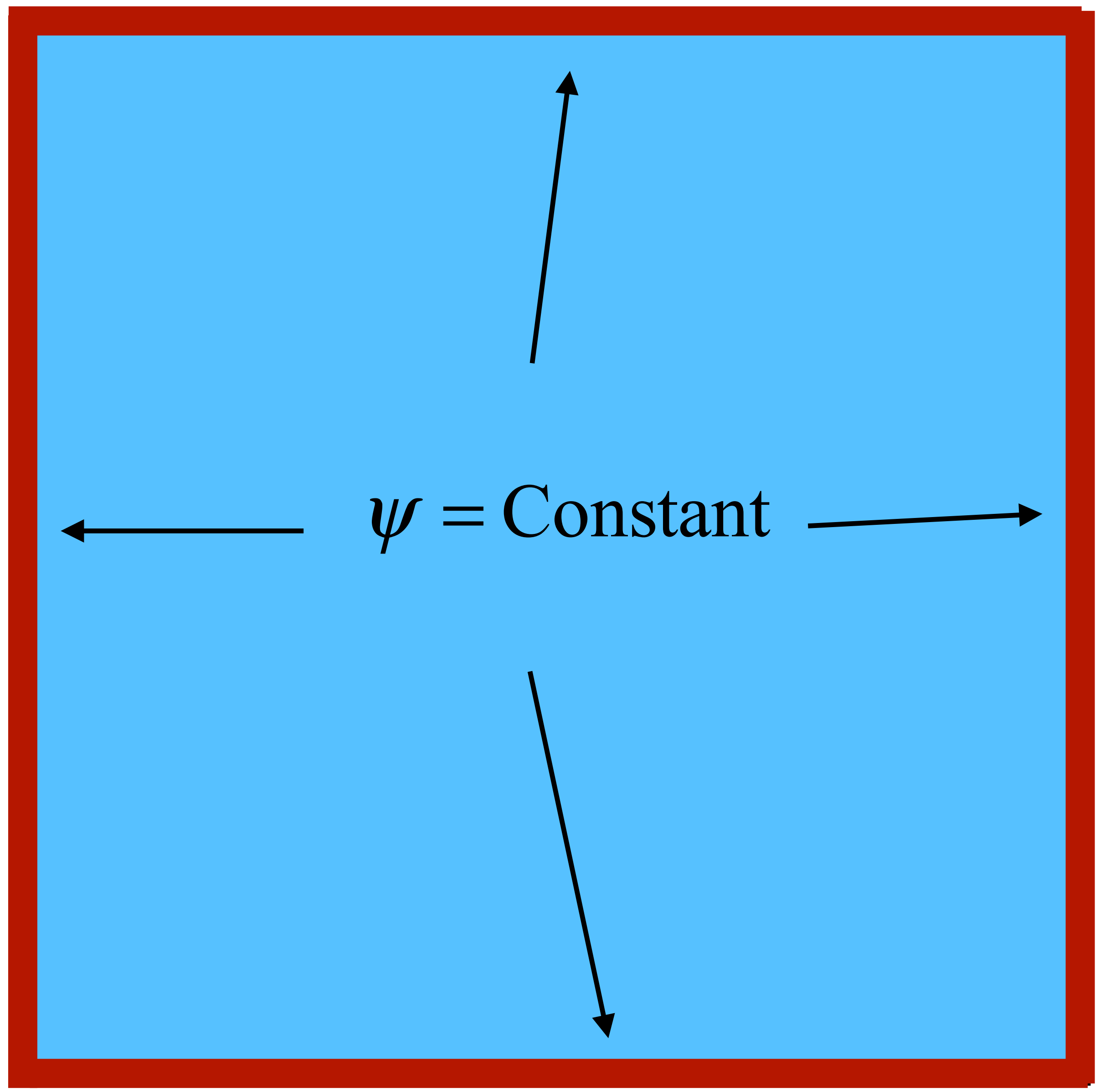
$$\frac{\partial \psi}{\partial y} = U_{\text{wall}} \qquad \omega_z = -\frac{\partial^2 \psi}{\partial y^2}$$

$$\frac{\partial \psi}{\partial x} = 0$$

$$\omega_z = -\frac{\partial^2 \psi}{\partial x^2}$$



$$\frac{\partial \psi}{\partial x} = 0$$

$$\omega_z = -\frac{\partial^2 \psi}{\partial x^2}$$

$$\frac{\partial \psi}{\partial y} = 0 \qquad \omega_z = -\frac{\partial^2 \psi}{\partial y^2}$$

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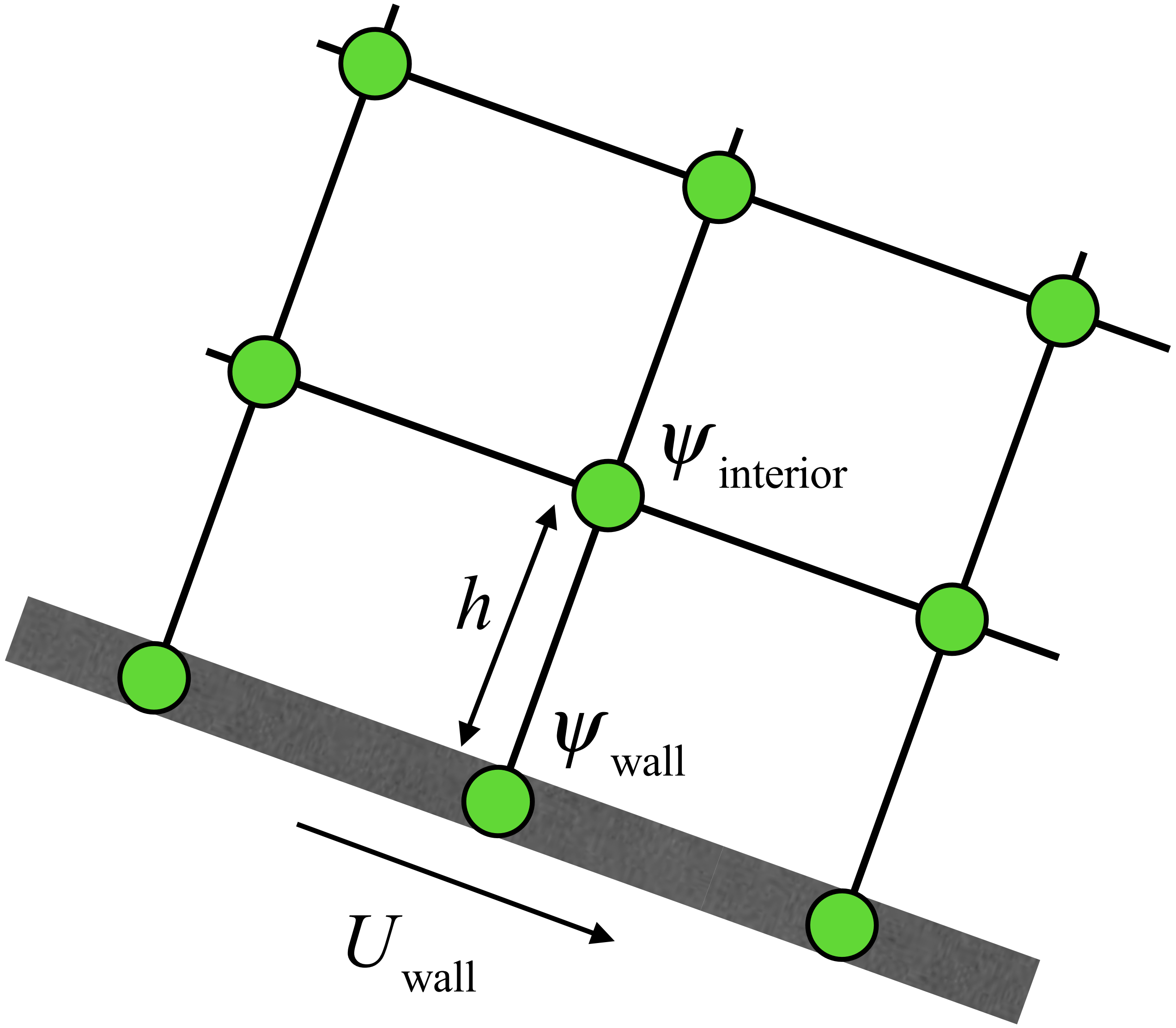

$$\psi_{\text{interior}} = \psi_{\text{wall}} + h\left.\frac{\partial \psi}{\partial n}\right|_{\text{wall}} + \frac{h^2}{2}\left.\frac{\partial^2 \psi}{\partial n^2}\right|_{\text{wall}} + \mathcal{O}(h^3)$$

$$\left.\frac{\partial \psi}{\partial n}\right|_{\text{wall}} \longrightarrow U_{\text{wall}} \qquad \left.\frac{\partial^2 \psi}{\partial n^2}\right|_{\text{wall}} \longrightarrow -\omega_{\text{wall}}$$

$$\psi_{\text{interior}} = \psi_{\text{wall}} + hU_{\text{wall}} + \frac{h^2}{2}\omega_{\text{wall}} + \mathcal{O}(h^3)$$

All quantities are known except for the wall vorticity - so we solve for it!

$$\omega_{\text{wall}} = \frac{2}{h^2}\left(\psi_{\text{wall}} - \psi_{\text{interior}}\right) + \frac{2}{h}U_{\text{wall}} + \mathcal{O}(h)$$

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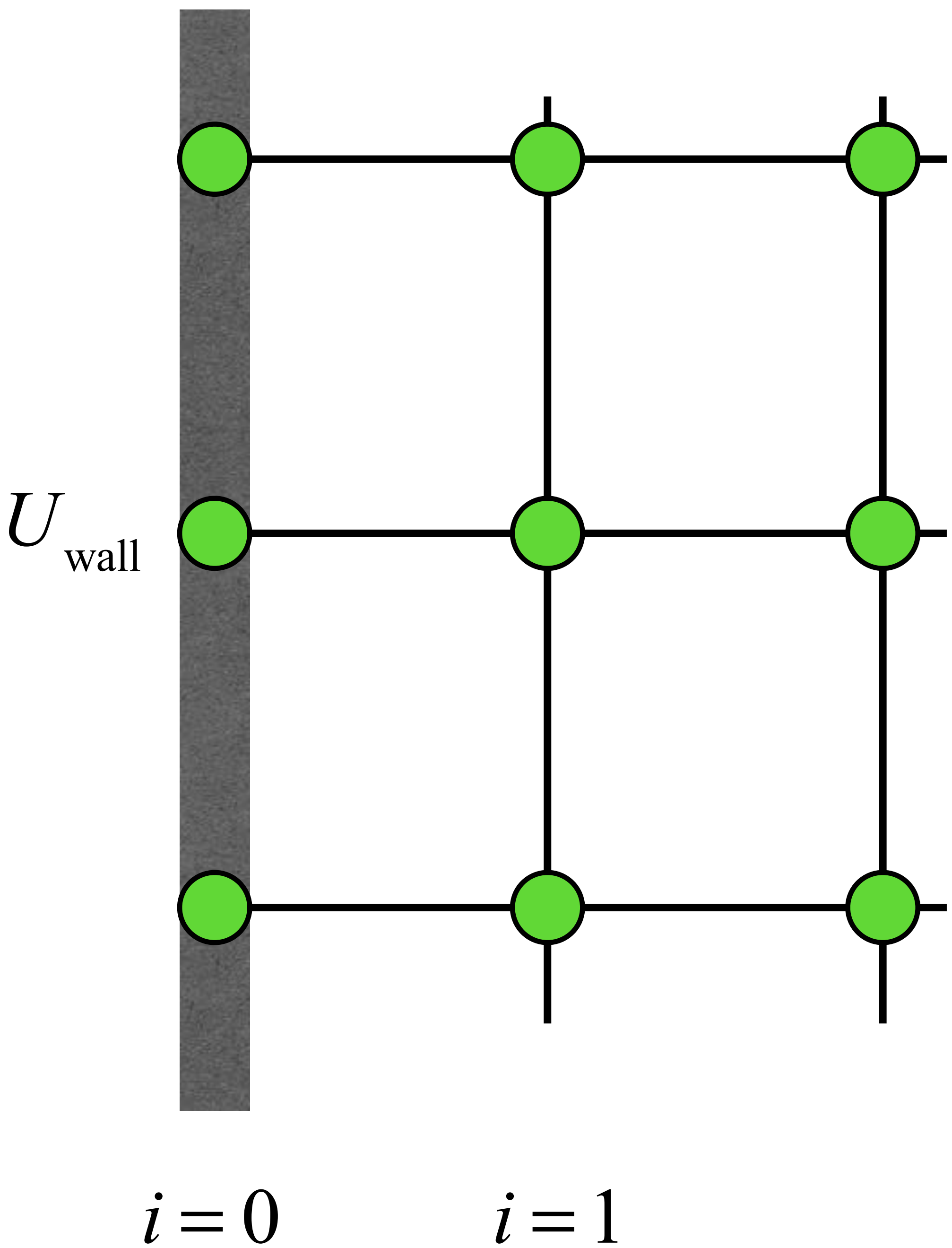

$$\psi_{1,j} = \psi_{\text{wall}} + \Delta x \left.\frac{\partial \psi}{\partial x}\right|_{\text{wall}} + \frac{\Delta x^2}{2} \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{\text{wall}} + \mathcal{O}(\Delta x^3)$$

$$\omega_{0,j} = \frac{2}{\Delta x^2}\left(\psi_{\text{wall}} - \psi_{1,j}\right) + \frac{2}{\Delta x} U_{\text{wall}} + \mathcal{O}(\Delta x)$$

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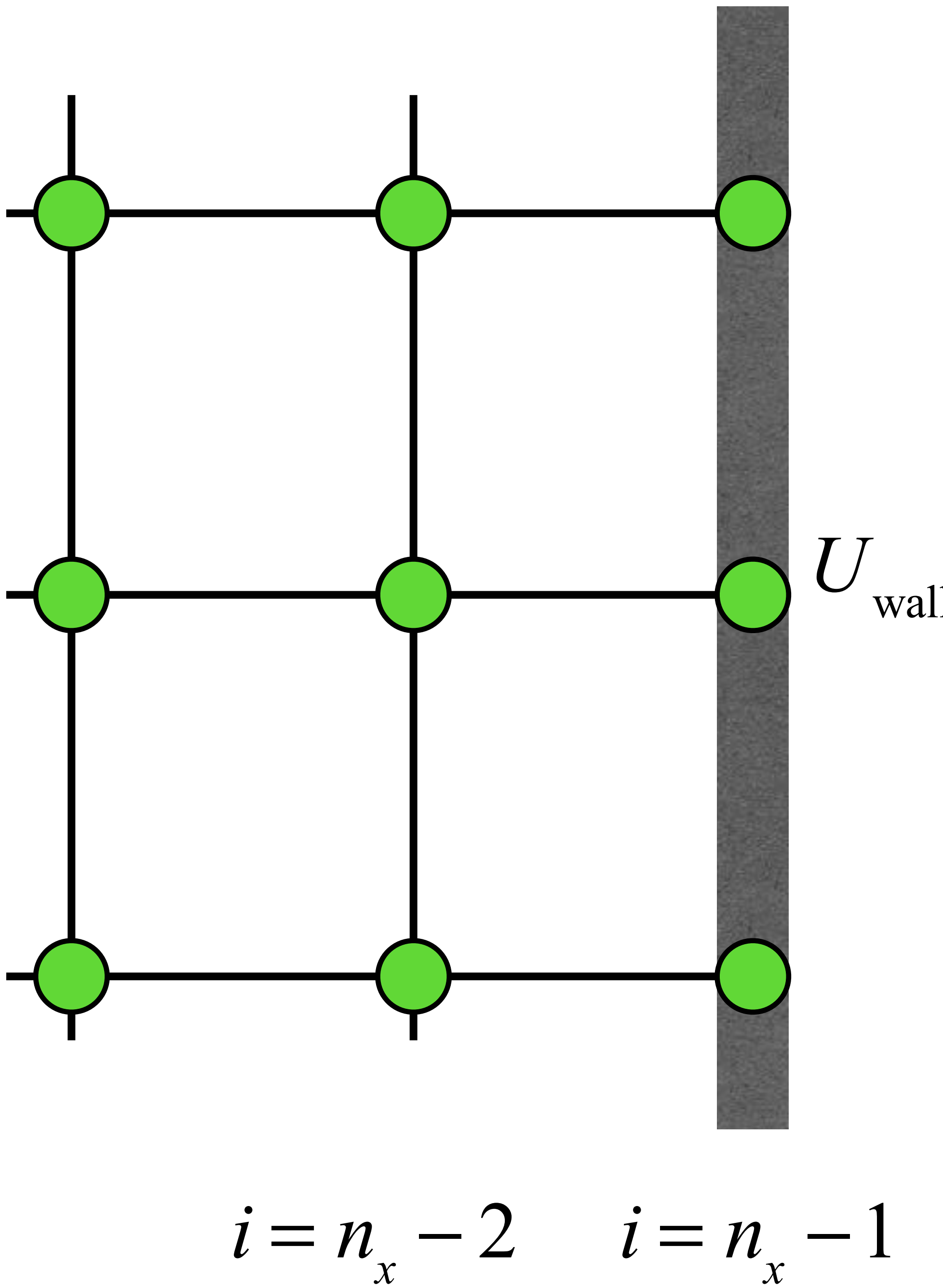
$$\psi_{n_x-2,j} = \psi_{\text{wall}} - \Delta x \left.\frac{\partial \psi}{\partial x}\right|_{\text{wall}} + \frac{\Delta x^2}{2} \left.\frac{\partial^2 \psi}{\partial x^2}\right|_{\text{wall}} + \mathcal{O}(\Delta x^3)$$

$$\omega_{n_x-1,j} = \frac{2}{\Delta x^2}\left(\psi_{\text{wall}} - \psi_{n_x-1,j}\right) - \frac{2}{h} U_{\text{wall}} + \mathcal{O}(h)$$



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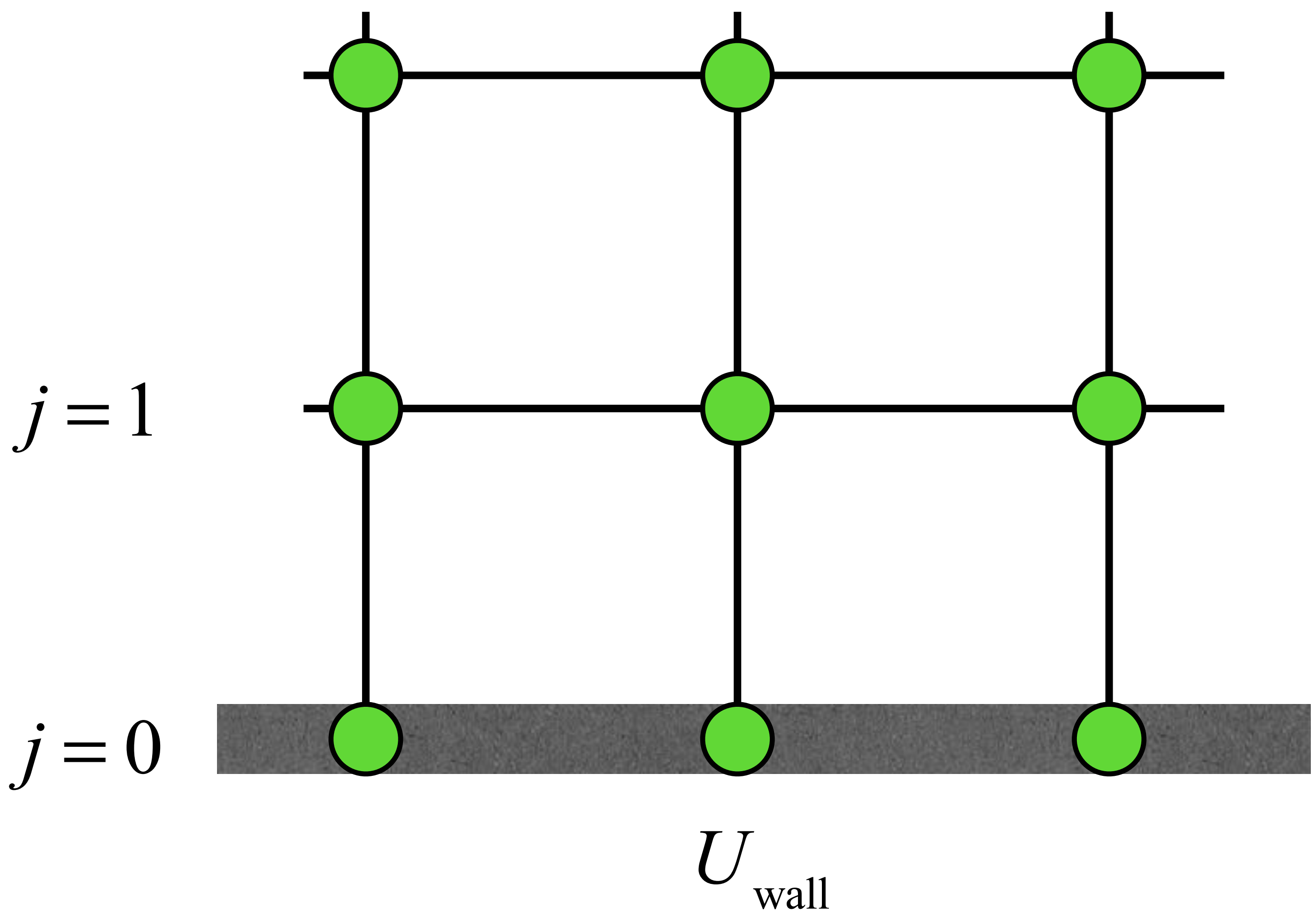
$$\psi_{i,1} = \psi_{\text{wall}} + \Delta y \left.\frac{\partial \psi}{\partial y}\right|_{\text{wall}} + \frac{\Delta y^2}{2} \left.\frac{\partial^2 \psi}{\partial y^2}\right|_{\text{wall}} + \mathcal{O}(\Delta y^3)$$

$$\omega_{i,0} = \frac{2}{\Delta y^2}\left(\psi_{\text{wall}} - \psi_{i,1}\right) + \frac{2}{\Delta y} U_{\text{wall}} + \mathcal{O}(\Delta y)$$



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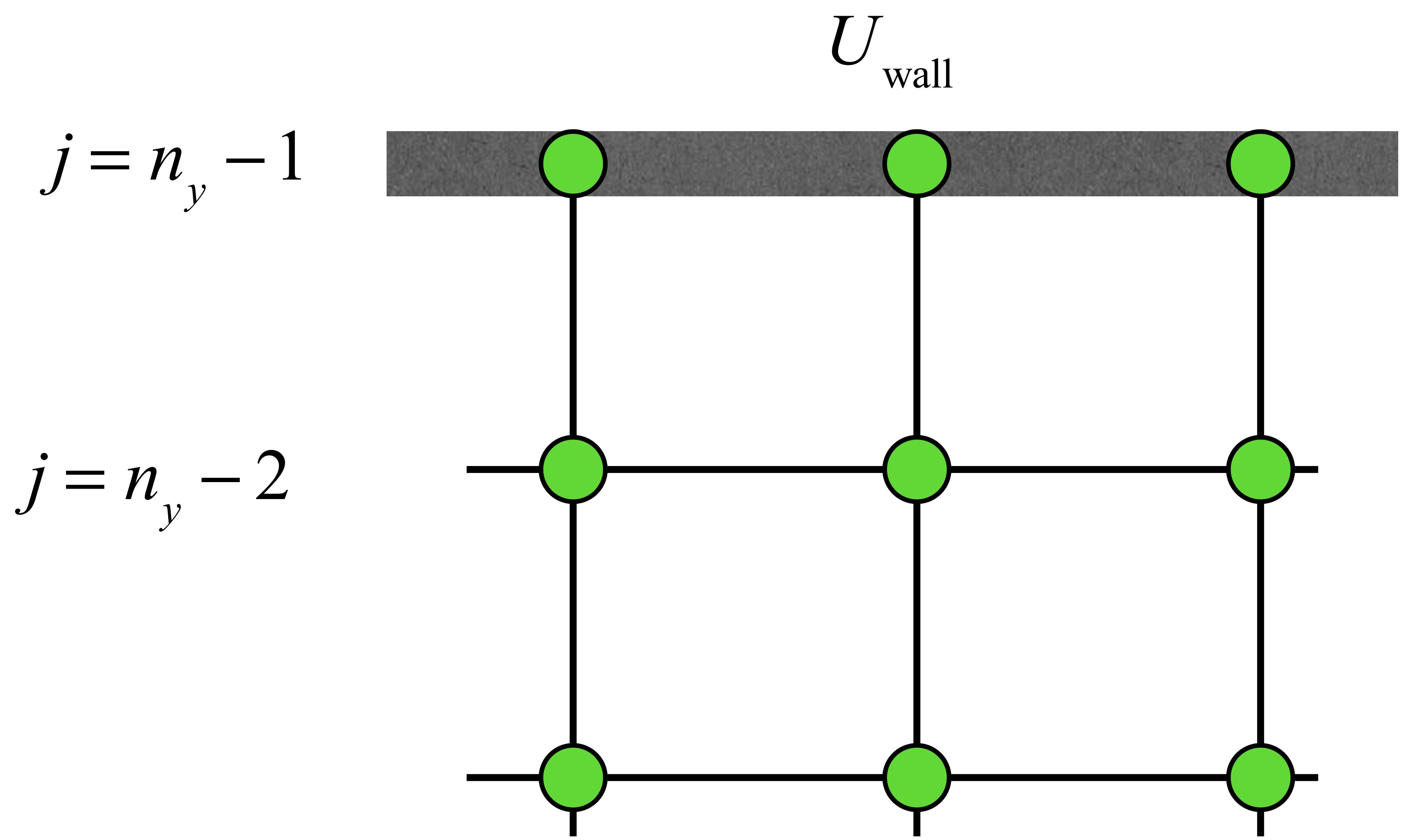

$$\psi_{i,n_y-2} = \psi_{\text{wall}} - \Delta y \left.\frac{\partial \psi}{\partial y}\right|_{\text{wall}} + \frac{h^2}{2}\left.\frac{\partial^2 \psi}{\partial y^2}\right|_{\text{wall}} + \mathcal{O}(\Delta x^3)$$

$$\omega_{i,n_y-1} = \frac{2}{\Delta y^2}\left(\psi_{\text{wall}} - \psi_{i,n_y-2}\right) - \frac{2}{\Delta y} U_{\text{wall}} + \mathcal{O}(h)$$

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