
Exercise I

1. Given $\mathcal{L}_1, \mathcal{L}_2 \in \mathsf{DTIME}(f(n))$, prove that $\mathcal{L}_1 \cup \mathcal{L}_2 \in \mathsf{DTIME}(f(n))$.
2. Given $\mathcal{L} \in \mathsf{DTIME}(f(n))$, prove that $\bar{\mathcal{L}} \in \mathsf{DTIME}(f(n))$.
3. Given $\mathcal{L}_1, \mathcal{L}_2 \in \mathsf{NTIME}(f(n))$, prove that $\mathcal{L}_1 \cup \mathcal{L}_2 \in \mathsf{NTIME}(f(n))$.
4. Prove that if $\mathsf{P} = \mathsf{NP}$, then $\forall k, \Sigma_k^{\mathsf{P}} \subseteq P$ and $\Pi_k^{\mathsf{P}} \subseteq \mathsf{P}$. [1]
5. Reminder : $\mathsf{PH} = \bigcup_{k \in \mathbb{N}} \Sigma_k^{\mathsf{P}}$. Prove that if there exists a problem $\mathcal{P}$ that is PH-complete, then $\exists k$ such that $\mathsf{PH} \subseteq \Sigma_k^{\mathsf{P}}$. [2]

Exercise II

We call HornSAT the decision problem : "Given a Horn formula φ, determine whether φ is satisfiable or not." Recall that a Horn formula is a conjunction of Horn clauses, *i.e.* clauses with at most one positive literal (either one positive literal : $x_1 \vee \neg x_2 \vee \cdots \vee \neg x_n$; or no positive literal at all : $\neg x_1 \vee \neg x_2 \vee \cdots \vee \neg x_n$).

1. Prove that HornSAT$\in \mathsf{P}$.

Exercise III

A pseudo-Boolean (PB) constraint is an (in)equality of the form

$$\textstyle\sum_i w_i l_i \quad \# \quad k$$

where w_i is a natural number, l_i is a literal (*i.e.* either a Boolean variable, or the negation of a Boolean variable), k is a natural number, and $\# \in \{<, \leqslant, =, \geqslant, >\}$. The pseudo-Boolean satisfaction problem (PB-SAT) is then

PB-SAT : “Given a set of PB constraints, is there an interpretation that satisfies all the constraints ?"

For instance,

$$\begin{array}{rcl} 3a + 2b & \geqslant & 3 \\ a + b & = & 1 \end{array}$$

is satisfied by the interpretation $\omega = \{a\}$:

$$\begin{array}{rcl} 3 \times 1 + 2 \times 0 & \geqslant & 3 \\ 1 + 0 & = & 1 \end{array}$$

1. Prove that PB-SAT is NP-complete.

Exercise IV

In propositional logic, a tautology is a formula which is always true : it means that any interpretation is a model of the formula. We also say that the formula is valid.

Complexity of Tautology We call Valid the problem “Given a propositional formula φ, is φ valid ?”.

1. Prove that Valid is coNP-complete.

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1. If this is true, we say that the polynomial hierarchy collapses to P.
2. If this is true, we say that the polynomial hierarchy collapses to its k^{th} level.