Université Paris Cité – LIPADE

Algorithmic Complexity

Space Complexity

Jean-Guy Mailly (jean-guy.mailly@u-paris.fr)

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Outline

Space Used by a Turing Machine

- If $\mathcal{M}$ is a Turing Machine and x an input word, the space used by $\mathcal{M}$ on x is the number of different squares visited by $\mathcal{M}$ during the computation.
- For $f : \mathbb{N} \rightarrow \mathbb{N}$, we say that a Turing Machine $\mathcal{M}$ works with space $\mathcal{O}(f(n))$ if
  - $\mathcal{M}$ stops for every input x
  - on every input x, $\mathcal{M}$ uses on x a space $s(n)$ with $s(n) \in \mathcal{O}(f(n))$

Space Complexity Classes

- For $f : \mathbb{N} \rightarrow \mathbb{N}$, DSPACE($f(n)$) is the set of languages which are decided by a deterministic Turing Machine $\mathcal{M}$ working with space $\mathcal{O}(f(n))$.
- For $f : \mathbb{N} \rightarrow \mathbb{N}$, NSPACE($f(n)$) is the set of languages which are decided by a non-deterministic Turing Machine $\mathcal{M}$ working with space $\mathcal{O}(f(n))$.

Deterministic vs Non-Deterministic Space

Intuition: Non-determinism is not "so important" for space complexity...

Theorem [Savitch 1970]

For all function f space-constructible,

$$\mathrm{DSPACE}(f(n)) \subseteq \mathrm{NSPACE}(f(n)) \subseteq \mathrm{DSPACE}(f(n)^2)$$

Complement Classes

Proposition

For all function $f(n)$ space-constructible,

$$\mathrm{DSPACE}(f(n)) = \mathrm{coDSPACE}(f(n))$$

Theorem [Immerman 1988, Szelepcsényi 1988]

For all function $f(n) \geq \log(n)$ space-constructible,

$$\mathrm{NSPACE}(f(n)) = \mathrm{coNSPACE}(f(n))$$

Outline

Deterministic Space Complexity

- $\mathsf{L} = \mathsf{DSPACE}(\log(n))$ is the class of decision problems that can be solved by a DTM using logarithmic space
- $\mathsf{PSPACE} = \bigcup_{k>0} \mathsf{DSPACE}(n^k)$ is the class of decision problems that can be solved by a DTM using polynomial space

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$$\mathrm{L} \subseteq \mathrm{PSPACE}$$

Non-Deterministic Space Complexity

- $\mathrm{NL} = \mathrm{NSPACE}(\log(n))$ is the class of decision problems that can be solved by a NDTM using logarithmic space
- $\mathrm{NPSPACE} = \bigcup_{k>0} \mathrm{NSPACE}(n^k)$ is the class of decision problems that can be solved by a NDTM using polynomial space

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$$\mathsf{NL} \subseteq \mathsf{NPSPACE}$$

PSPACE vs NPSPACE

Proposition

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Proof:

- For any $k > 0$,
  $\text{NSPACE}(n^k) \subseteq \text{DSPACE}((n^k)^2) = \text{DSPACE}((n^{2k}) \subseteq \text{PSPACE}$,
  so $\text{NPSPACE} \subseteq \text{PSPACE}$
- For any $k > 0$, $\text{DSPACE}(n^k) \subseteq \text{NSPACE}(n^k) \subseteq \text{NPSPACE}$, so
  $\text{PSPACE} \subseteq \text{NPSPACE}$

Relations Between Time and Space (1/3)

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For any mapping $f : \mathbb{N} \to \mathbb{N}$, NTIME$(f(n)) \subseteq$ DSPACE$(f(n))$

Corollary

$$\text{NP} \subseteq \text{PSPACE}$$

Relations Between Time and Space (2/3)

Polynomial Hierarchy vs Polynomial Space

$$\text{PH} \subseteq \text{PSPACE}$$

Intuition

For each class of the polynomial hierarchy, there is a complete problem $\forall_i$QBF or $\exists_i$QBF. All these problems are special instances of TQBF, so they are in PSPACE.

Relations Between Time and Space (3/3)

$$\mathrm{L} \subseteq \mathrm{NL} \subseteq P \subseteq \begin{matrix}\mathrm{NP} \\ \mathrm{coNP}\end{matrix} \subseteq \Delta_2^{\mathrm{P}} \subseteq \begin{matrix}\Sigma_2^{\mathrm{P}} \\ \Pi_2^{\mathrm{P}}\end{matrix} \subseteq \cdots \subseteq \mathrm{PH} \subseteq \mathrm{PSPACE} \subseteq \mathrm{EXP} \subseteq \mathrm{NEXP}$$

We also know that some inclusions are strict:

- NL $\subset$ PSPACE
- P $\subset$ EXP

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  - A problem is C-complete if it is in C and it is C-hard
  - This definition works for PSPACE-hardness and PSPACE-completeness
  - It does not work for L and NL

Logarithmic Space and Completeness

Logarithmic-Space Functional Reduction

A logarithmic-space functional reduction f is a total computable function from a problem $\mathcal{P}_1$ to a problem $\mathcal{P}_2$ such that, for any instance i of $\mathcal{P}_1$,

- $f(i)$ can be computed in logarithmic-space in the size of i
- i is a positive instance of $\mathcal{P}_1$ iff $f(i)$ is a positive instance of $\mathcal{P}_2$

Notation: $\mathcal{P}_1 \leq_f^{\mathsf{L}} \mathcal{P}_2$

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NL-Hardness and NL-Completeness

- A problem $\mathcal{P}$ is NL-hard iff $\forall \mathcal{P}' \in \mathsf{NL}$, $\mathcal{P}' \leq_f^{\mathsf{L}} \mathcal{P}$
- A problem $\mathcal{P}$ is NL-complete iff $\mathcal{P}$ is NL-hard and $\mathcal{P} \in \mathsf{NL}$

What About L-Completeness?

Theorem

Any problem in L is L-complete with respect to $\leq_f^{\mathsf{L}}$!

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Any problem in L is L-complete with respect to $\leq_f^{L}$!

To define a meaningful notion of L-completeness, the hardness must be defined with another kind of functional reductions
More details in [Garey and Johnson 1979]

Canonical PSPACE-Complete Problem

Reminder:

- A canonical QBF is a formula $\mathcal{Q}_1\mathcal{X}_1, \mathcal{Q}_2\mathcal{X}_2, \dots \mathcal{Q}_n\mathcal{X}_n, \phi$:
  - $\mathcal{Q}_i \in \{\forall, \exists\}$ and $\mathcal{Q}_i \neq \mathcal{Q}_{i+1}$
  - $\mathcal{X}_1, \dots \mathcal{X}_n$ form a partition of the Boolean variables in ϕ
  - ϕ is a propositional formula
- $\exists_n$QBF: is the QBF $\exists\mathcal{X}_1, \forall\mathcal{X}_2, \dots \mathcal{Q}_n\mathcal{X}_n, \phi$ true?
- $\forall_n$QBF: is the QBF $\forall\mathcal{X}_1, \exists\mathcal{X}_2, \dots \mathcal{Q}_n\mathcal{X}_n, \phi$ true?

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Complexity of TQBF

$$\text{TQBF} = \bigcup_{n \geq 1} (\exists_n\text{QBF} \cup \forall_n\text{QBF})$$

TQBF is PSPACE-complete

PSPACE vs Games
Tic-Tac-Toe

- Classical Tic-Tac-Toe: In a 3×3 grid, two players must draw some marks (X or O). The winner is the first who has placed three marks in a horizontal, vertical or diagonal row
- Generalized Tic-tac-toe ((m, n, k) game): In a $m \times n$ grid, a player wins when he has placed k marks in a row
- A winning strategy at time t for a player is a sequence of moves (drawing a mark in the grid) such that the player wins.

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Complexity of (m, n, k)-STRAT

$(m - n - k)$-STRAT is the decision problem: “Is there a winning strategy for player X in the (m, n, k) game?”
$(m - n - k)$-STRAT is PSPACE-complete

PSPACE vs Games

Tic-Tac-Toe: Intuition

- Determining if there is a winning strategy for player X is like asking the question:

  Is there a way to draw an X at step 1, such that for all possible moves of the O player at step 2, there is a way to draw an X at step 3, such that for all possible moves of the O player at step 4, there is ... such that X player wins?

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- This can be represented as a QBF formula

Generalized Super Mario Bros.

[Demaine *et al.* 2016]

Figure – Super Mario Bros. is PSPACE-complete

Lemmings
[Viglietta 2015]

Figure – Lemmings is PSPACE-complete

NL-Complete Problem in Graphs

Reach

Given a directed graph $G = \langle N, E \rangle$ and two nodes $n_1, n_2 \in N$, is there a path from n_1 to n_2 in G?

Theorem [Papadimitriou 1994]

Reach is NL-complete

Reach is NL-complete: Proof

We will prove that

- Reach $\in$ NL
- Reach is NL-hard

Reach $\in$ NL?

- Let us find a non-deterministic logspace algorithm that solves Reach

Reach $\in$ NL: Proof

Algorithm 1 Reach

Input: $G = \langle N, E \rangle, n_1, n_2 \in N$

$v = n_1$
$cpt = 0$
while $cpt < |N|$ **do**
  Non-deterministically pick v' some neighbour of v
  $v = v'$
  if $v == n_2$ **then**
    return YES
  end if
  $cpt{+}{+}$
end while
return NO

- Space is logarithmic: we only need to store v and cpt

Reach is NL-hard: Sketch of proof

- We need to prove that $\forall \mathcal{P} \in \mathsf{NL}$, $\mathcal{P} \leq_f^{\mathsf{L}}$ Reach

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  - $\mathcal{M}$ accepts x iff there is a path from n_1 to n_2

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  - $\mathcal{M}$ accepts x iff there is a path from n_1 to n_2
  - See [Papadimitriou 1994] for more details

NL-Complete Problem in Logics

Reminder on 2SAT

2SAT is a special case of satisfiability problem with CNF formulas with only clauses of length ≤ 2

Theorem [Papadimitriou 1994]

2SAT is NL-complete

2SAT is NL-complete: Sketch of proof

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- Reminder: From previous theorem [Immerman 1988, Szelepcsényi 1988], coNL = NL
- We define Unreach $= \{(G = \langle N, E \rangle, n_1, n_2) \mid G$ is a directed graph, $n_1, n_2 \in N$, and there is no path from n_1 to n_2 in $G\}$
- By definition, Unreach $\in$ coNL = NL

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- By definition, Unreach $\in$ coNL = NL
- We will use Reach and Unreach to prove that 2SAT is NL-complete

Implication graph of a 2CNF formula

- Given ϕ a 2CNF formula on variables X, we build $G_\phi = \langle N, E \rangle$ a directed graph such that
  - $N = X \cup \{\bar{x} \mid x \in X\}$
  - $(l_1, l_2) \in E$ iff $\neg l_1 \vee l_2$ is a clause in ϕ (*i.e.* $l_1 \Rightarrow l_2$)

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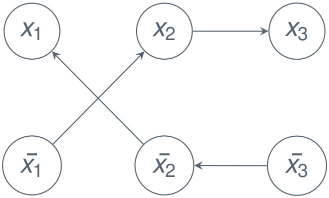
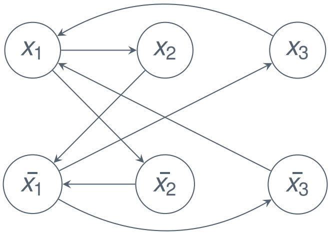
$\phi_1 = (x_1 \vee x_2) \wedge (\neg x_2 \vee x_3)$



$$\phi_2 = (\neg x_1 \vee x_2) \wedge (\neg x_2 \vee \neg x_1) \wedge (x_1 \vee x_3) \wedge (\neg x_3 \vee x_1)$$



2SAT and the implication graph

- Lemma: The formula ϕ is unsatisfiable iff $\exists x$ such that there is a path from x to $\bar{x}$ and a path from $\bar{x}$ to x in G_ϕ
- The following non-deterministic algorithm solves 2UNSAT

Algorithm 2 2UNSAT

Input: ϕ
  Build G_ϕ
  Non-deterministically pick x
  return $(G_\phi, x, \bar{x}) \in$ Reach AND $(G_\phi, \bar{x}, x) \in$ Reach

---

- So 2UNSAT$\in$ NL, and 2SAT$\in$ coNL $=$ NL

2SAT is NL-hard: Sketch of proof (1/2)

- We reduce Unreach to 2SAT
- Let $G = \langle N, E \rangle$ be a directed graph, and n_1, n_2 in N. We define $f(G, n_1, n_2) = \phi$ with
  - if $(x, y) \in E$, then $\neg x \vee y$ is a clause in ϕ
  - n_1 is a unit clause in ϕ
  - $\neg n_2$ is a unit clause in ϕ

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- if $(G, n_1, n_2) \in$ Unreach, we can make a partition of N: N_1, N_2 s. t. $n_1 \in N_1$ and $n_2 \in N_2$, and there is no edge from N_1 to N_2

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- $\forall x \in N_1$, choose $\omega(x) = 1$, $\forall x \in N_2$, choose $\omega(x) = 0$. $\omega \models \phi$

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- $\forall x \in N_1$, choose $\omega(x) = 1$, $\forall x \in N_2$, choose $\omega(x) = 0$. $\omega \models \phi$
- $(G, n_1, n_2) \in$ Unreach $\Rightarrow \phi \in$ 2SAT

2SAT is NL-hard: Sketch of proof (2/2)

- if $(G, n_1, n_2) \notin$ Unreach, then there is a path $(n_1, x_1, \ldots, x_k, n_2)$ in G. The translation of edges in clauses means that $n_1 \Rightarrow x_1 \Rightarrow \cdots \Rightarrow x_k \Rightarrow n_2$, so if n_1 is true, then n_2. But ϕ contains both unitary clauses n_1 and $\neg n_2$
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- We can conclude Unreach $\leq_f^{\mathsf{L}}$ 2SAT

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