I Turing Machines

Exercise I

1. Define a Turing Machine $\mathcal{M}_{invert}$ which reads a sequence of bits and inverts the value of each bit. We suppose that the machine starts on the first square of the number.
   Example : on the input 100110, the execution of $\mathcal{M}_{invert}$ gives on the tape 011001.

2. Give the sequence of transitions of $\mathcal{M}_{invert}$ on the input words $w_1 = 100110$, $w_2 = 0011100$.

Exercise II

1. Define a Turing Machine $\mathcal{M}_{add}$ which reads two integers written in unary notation, separated by a blank symbol B, and prints on the tape the sum of these integers. Here we suppose that the numbers are preceded and followed by (possibly infinitely) blank symbols B.
   Example : on the input $BBBB111B1111BBBB$, the execution of $\mathcal{M}_{add}$ gives on the tape $BBBB1111111BBBB$.

2. Give the sequence of transitions of $\mathcal{M}_{add}$ on the input words 1111 and 11 (*i.e.* the addition of 4 and 2 in unary notation).

Exercise III

1. Define a Turing Machine $\mathcal{M}_{\mathcal{L}}$ which decides the language $\mathcal{L} = \{a^n b^m c^p \mid n, m, p \in \mathbb{N} \setminus \{0\}\}$.

2. Give the sequence of transitions of $\mathcal{M}_{\mathcal{L}}$ on the input word $w_1 = aaabcccc$. Is w_1 accepted by $\mathcal{M}_{\mathcal{L}}$?

3. Give the sequence of transitions of $\mathcal{M}_{\mathcal{L}}$ on the input word $w_2 = aaabbbcca$. Is w_2 accepted by $\mathcal{M}_{\mathcal{L}}$?

II Recognizability and Decidability

Exercise IV

1) Given $\mathcal{L}_1, \mathcal{L}_2$ two decidable languages, prove that $\mathcal{L}_1 \cup \mathcal{L}_2$ is decidable.

2) Given $\mathcal{L}_1, \mathcal{L}_2$ two decidable languages, prove that $\mathcal{L}_1 \cap \mathcal{L}_2$ is decidable.

Exercise V

1. Given a decidable language $\mathcal{L}$ of pairs of integers (n,m), we define $\mathcal{L}'$ as :

$$\mathcal{L}' = \{n \mid \exists m \text{ such that } (n,m) \in \mathcal{L}\}$$

   Prove that $\mathcal{L}'$ is recognizable.