I Mappings and Asymptotic Bounds

Exercise I

1. Prove that for any $a_0, a_1, a_2 \in \mathbb{R}^+$, $f(n) = a_0 + a_1 n + a_2 n^2 \in \mathcal{O}(n^2)$, using the formal definition of the $\mathcal{O}$ notation.

> **Solution:** Let $c = a_2 + 1$. Let us prove that there is some n_0 such that $\forall n \geqslant n_0$, $f(n) \leqslant c \times n^2$.
>
> $$\begin{aligned} a_0 + a_1 n + a_2 n^2 &\leqslant (a_2+1)n^2 \\ a_0 + a_1 n + a_2 n^2 - (a_2+1)n^2 &\leqslant 0 \\ a_0 + a_1 n - n^2 &\leqslant 0 \end{aligned}$$
>
> So $\Delta = a_1^2 - 4 \times (-1) \times a_0 = a_1^2 + 4a_0$; the polynomial $a_0 + a_1 n - n^2$ has two roots $x_1 = \frac{-a_1+\sqrt{\Delta}}{-2}$ and $x_2 = \frac{-a_1-\sqrt{\Delta}}{-2}$. For $n_0 = \lceil \max(x_1, x_2) \rceil$, the property is satisfied.

Exercise II

1. Each expression in the following list gives the processing time $t(n)$ for some algorithm to solve a problem of size n. For each of them, give a function $f(n)$ such that $t(n) \in \mathcal{O}(f(n))$:

   (a) $8 + 4n^2 + 8n^4$

   > **Solution:** $8 + 4n^2 + 8n^4 \in \mathcal{O}(n^4)$

   (b) $5 + 0.01n^3 + 4n$

   > **Solution:** $5 + 0.01n^3 + 4n \in \mathcal{O}(n^3)$

   (c) $100n + 0.001n^2$

   > **Solution:** $100n + 0.001n^2 \in \mathcal{O}(n^2)$

   (d) $10\log(n) + 5(\log(n))^3 + 7n + 6n^3$

   > **Solution:** $10\log(n) + 5(\log(n))^3 + 7n + 6n^3 \in \mathcal{O}(n^3)$

   (e) $200n + n^2 + 50n\log_{10}(n)$

   > **Solution:** $200n + n^2 + 50n\log_{10}(n) \in \mathcal{O}(n^2)$

   (f) $3n + n\log_2(n)$

   > **Solution:** $3n + n\log_2(n) \in \mathcal{O}(n\log(n))$

(g) $n^{100} + 2^n$

Solution: $n^{100} + 2^n \in \mathcal{O}(2^n)$

(h) $100n\log_3(n) + n^3 + 100n$

Solution: $100n\log_3(n) + n^3 + 100n \in \mathcal{O}(n^3)$

Exercise III

1. Let us suppose that algorithms A and B need respectively $t_A(n) = 5n\log_{10}(n)$ and $t_B(n) = 25n$ seconds to solve a problem of size n.

(a) Which algorithm is better (with respect to the $\mathcal{O}$ notation) ?

Solution: Algorithm B is better : $t_A(n) \in \mathcal{O}(n\log(n))$ and $t_B(n) \in \mathcal{O}(n)$.

(b) For which problem size does it become better ?

Solution: B is better than A when

$$\begin{aligned} & t_B(n) \leqslant t_A(n) \\ \Leftrightarrow\ & 25n \leqslant 5n\log_{10}(n) \\ \Leftrightarrow\ & 5 \leqslant \log_{10}(n) \\ \Leftrightarrow\ & 10^5 \leqslant 10^{\log_{10}(n)} \\ \Leftrightarrow\ & 100000 \leqslant n \end{aligned}$$

II Problems

Exercise IV

1. Classify these problems : decision problem, function problem, optimization problem, enumeration problem.

(a) Given a list of integers L, determine whether L is sorted in increasing order.

Solution: Decision

(b) Given a list of integers L, sort L in increasing order.

Solution: Function

(c) Given G a graph, n_1, n_2 two nodes, is there a path from n_1 to n_2 ?

Solution: Decision

(d) Given G a graph, n_1, n_2 two nodes, find the shortest path from n_1 to n_2.

Solution: Optimization

(e) Given G a graph, n_1, n_2 two nodes and $k \in \mathbb{N}$, find if there is a path from n_1 to n_2 with length k.

Solution: Decision

(f) Given two integers a, b in binary notation, and $k \in \mathbb{N}$, what is the k^{th} bit of $a \times b$?

Solution: Decision : the answer is binary (0/1). The problem could be rephrased : "Given two integers a, b in binary notation, and $k \in \mathbb{N}$, is the k^{th} bit of $a \times b$ equal to 1 ?"

(g) Given $P(X)$ a polynomial, find all the roots of $P(X)$.

Solution: Enumeration

(h) Given $P(X)$ a polynomial, find the smallest root of $P(X)$.

Solution: Optimization

III Languages

Exercise V

Give the decision problem corresponding to each of these languages :

1. $\mathcal{L}_1 = \{k \in \mathbb{N} \mid k \text{ is a multiple of 3 or 4}\}$.

   Solution: $\mathcal{P}_1$: Given $k \in \mathbb{N}$, is k a multiple of 3 or 4 ?

2. $\mathcal{L}_2 = \{k \in \mathbb{N} \mid \exists k', k' \neq 1, k' \neq k \text{ and } k \text{ is a multiple of } k'\}$.

   Solution: $\mathcal{P}_2$: Given $k \in \mathbb{N}$, is k a non-prime number ?

3. $\mathcal{L}_3 = \{k \in \mathbb{N}, p \text{ a prime number}, i \in \mathbb{N} \mid p^i \text{ belongs to the prime decomposition of } k\}$.

   Solution: $\mathcal{P}_3$: Given $k \in \mathbb{N}$, p a prime number and $i \in \mathbb{N}$, is k a multiple of p^i but not of p^{i+1} ?

4. $\mathcal{L}_4 = \{P(X) \text{ a polynomial,} x_0 \in \mathbb{R} \mid P(x_0) = 0\}$.

   Solution: $\mathcal{P}_4$: Given $P(X)$ a polynomial and $x_0 \in \mathbb{R}$, is x_0 a root of $P(X)$?

5. $\mathcal{L}_5 = \{\varphi, \psi \text{ two propositional formulae} \mid \varphi \vdash \psi\}$.

Solution: $\mathcal{P}_5$: Given φ, ψ two propositional formulae, is ψ a logical consequence of φ ?

6. $\mathcal{L}_6 = \{\varphi$ a propositional formula$|$ φ has at least one model$\}$.

Solution: $\mathcal{P}_6$: Given φ a propositional formula, is φ consistent ?
This problem is actually the well-known SAT problem, we will see more details about it later this semester.

Exercise VI
Give the language corresponding to each of these decision problems :

1. $\mathcal{P}_1$: Given $k_1, k_2, k_3 \in \mathbb{N}$, is $k_1 + k_2$ a multiple of k_3 ?

Solution: $\mathcal{L}_1 = \{k_1, k_2, k_3 \in \mathbb{N} \mid k_1 + k_2$ is a multiple of $k_3\}$.
Remark : to be more correct, from a mathematical point of view, we should write $\{(k_1, k_2, k_3) \in \mathbb{N}^3 \mid k_1 + k_2$ is a multiple of $k_3\}$ instead. The same remark applies for the following languages, but in this course we will accept this slight simplifcation.

2. $\mathcal{P}_2$: Given $k_1, k_2 \in \mathbb{N}$, is $\sqrt{k_1} \leqslant k_2$ true ?

Solution: $\mathcal{L}_2 = \{k_1, k_2 \in \mathbb{N} \mid \sqrt{k_1} \leqslant k_2\}$.

3. $\mathcal{P}_3$: Given $P(X), Q(X), R(X)$ three polynomials, is $P(X) = Q(X) + R(X)$ true ?

Solution: $\mathcal{L}_3 = \{P(X), Q(X), R(X) \mid P(X) = Q(X) + R(X)\}$.

4. $\mathcal{P}_4$: Given $G = \langle N, E\rangle$ a directed graph, is $|E| \leqslant |N|^2$ true ?

Solution: $\mathcal{L}_4 = \{G = \langle N, E\rangle \mid |E| \leqslant |N|^2\}$.

5. $\mathcal{P}_5$: Given $G_1 = \langle N_1, E_1\rangle, G_2 = \langle N_2, E_2\rangle$ two directed graphs, is G_1 a subgraph of G_2 ? (*i.e.* $N_1 \subseteq N_2$ and $E_1 \subseteq E_2 \cap (N_1 \times N_1)$)

Solution: $\mathcal{L}_5 = \{G_1 = \langle N_1, E_1\rangle, G_2 = \langle N_2, E_2\rangle \mid N_1 \subseteq N_2$ and $E_1 \subseteq E_2 \cap (N_1 \times N_1)\}$.

6. $\mathcal{P}_6$: Given $\varphi_1, \varphi_2, \varphi_3$ three propositional formulae, is $\varphi_1 \equiv \varphi_2 \wedge \varphi_3$ true ?

Solution: $\mathcal{L}_6 = \{\varphi_1, \varphi_2, \varphi_3 \mid \varphi_1 \equiv \varphi_2 \wedge \varphi_3\}$.