Univ. de Paris — Algorithmic Complexity

Master 1 Informatique — **Examen** — 03/01/2022

- You can answer the questions in English or in French.

Exercise I

Multiple Choice Questions. For the following affirmations, there is **exactly one good answer**. You don't need to justify. For each question, a good answer is worth 0.5, no answer is worth 0, and a bad answer is worth −0.5.

(½) 1. $5n^3 + 2n^2 + 8n + 13 \in \mathcal{O}(n!)$.
☐ True ☐ False

(½) 2. The decision problem $\exists_3 QBF$ belongs to the complexity class:
☐ NP ☐ Σ_4^{P} ☐ Π_3^{P}

(½) 3. If $\mathsf{P} = \mathsf{NP}$, then $\Sigma_2^{\mathsf{P}} \subseteq \mathsf{P}$.
☐ True ☐ False

(½) 4. Among these decision problems, which one is the "easiest" to solve (w.r.t. time complexity)?
☐ SAT ☐ $\exists_3 QBF$ ☐ Horn-SAT ☐ IS

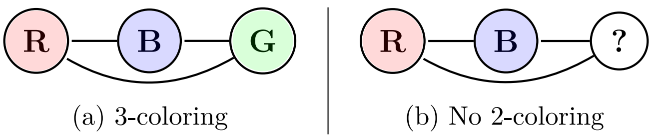
Exercise II

(2½) 1. We suppose that the Turing machine starts on the first square of the input word (there are no blank symbols before it). There are (infinitely) many blank symbols after the input word. Define a Turing Machine which reads a sequence of letters (in the set $\{a,b,c,d,e\}$), and decides whether it contains an even number of vowels. Recall that vowels are a and e (and i, o, u and y, but these ones are not used in the exercise), and even numbers are $\{0,2,4,6,\dots\}$.
Example:

- | a | b | c | a | e | B | B | ... | the machine stops on "NO" (there are 3 vowels);
- | b | a | e | d | c | B | B | ... | the machine stops on "YES" (there are 2 vowels).

Exercise III

Given a non-directed graph $G = \langle N, E \rangle$ and a set of colors $C = \{c_1, \dots, c_k\}$, a k-coloring of G is a (total) mapping $\gamma : N \to C$ such that, for all $n_i, n_j \in N$, if $\{n_i, n_j\} \in E$, then $\gamma(n_i) \neq \gamma(n_j)$. Said otherwise, it means that one color from C is assigned to each node, and two nodes that are neighbors in the graph cannot have the same color. See Figure 1 for an example of graph that can be 3-colored, but cannot be 2-colored.



(a) 3-coloring (b) No 2-coloring

Figure 1: A Graph with a 3-coloring ($C_3 = \{R, B, G\}$) but no 2-coloring ($C_2 = \{R, B\}$)

We call k-COLOR the decision problem:

"Given a non-directed graph $G = \langle N, E \rangle$ and a set $C = \{c_1, \dots, c_k\}$ of k colors, is there a k-coloring of G?"

(3) 1. Prove that k-COLOR $\in$ NP.

Exercise IV

In this exercise, we are interested in a variant the *traveling salesman problem* (TSP). The TSP consists in searching the shortest route that visits all the cities in a list of cities, such that each city is visited once, and the route ends where it started. So we suppose that we have a weighted non-directed graph $G = \langle N, E, w \rangle$ where N are the nodes, E are the (non-directed) edges, and $w : E \to \mathbb{R}$. Intuitively, the nodes represent the cities, the edges represent the existence of a way from a city to another one, and the weight of an edge is the distance between the cities. See Figure 2 for an example. This graph means that the distance between Paris and Lille is 204km (formally, $w(\{Paris, Lille\}) = 204$), the distance between Bordeaux and Marseille is 615km ($w(\{Bordeaux, Marseille\}) = 615$),...

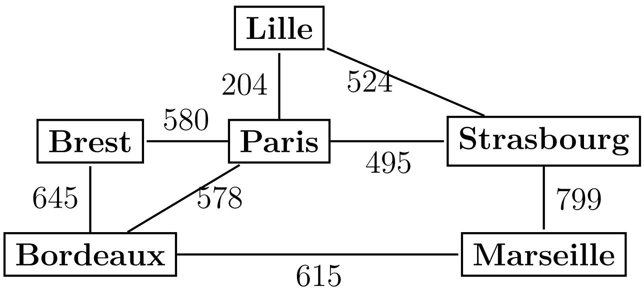


Figure 2: A (Simplified) Map of France

A TSP path is a cycle such that every city in the graph appears once in the cycle (except the starting point, which appears also as the last city). Formally, a TSP path is a vector of nodes $(n_1, n_2, \ldots, n_k)$ such that:

- for each $i \in \{1, \ldots, k-1\}$, $\{n_i, n_{i+1}\} \in E$,
- $n_1 = n_k$,
- $k - 1 = |N|$.

For instance, $P =$ (Brest, Paris, Lille, Strasbourg, Marseille, Bordeaux, Brest) is a TSP path. The length of a TSP path is the sum of the weights of the edges in the path. For instance, the length of P is $580 + 204 + 524 + 799 + 615 + 645 = 3367$.

We call k-TSP the following decision problem: "Given a weighted non-directed graph $G = \langle N, E, w \rangle$, and $k \in \mathbb{R}$, is there a TSP path P such that the length of P is $\leqslant k$?"

(3) 1. Prove that k-TSP $\in$ NP .

Exercise V

We define an extension of SAT, called MAX-SAT, as follows. Given a set of propositional variables V, a weighted formula is a set of weighted clauses $\{(cl_1, w_1), \ldots, (cl_n, w_n)\}$, where cl_i are clauses (*i.e.* disjunction of literals), and w_i are natural numbers. For instance, $\varphi = \{(a \vee b, 2), (\neg a, 1), (\neg b, 1)\}$ is a weighted formula. Then, a MAX-SAT instance is a pair (φ, k) where φ is a weighted formula, and $k \in \mathbb{N}$. Given an interpretation ω, its score is the sum of the weights of the satisfied weighted clauses. For instance,

- for $\omega_1 = \{a, b\}$, the score is 2 (because ω_1 satisfies $(a \vee b, 2)$),
- for $\omega_2 = \{a\}$, the score is 3 (because ω_2 satisfies $(a \vee b, 2)$ and $(\neg b, 1)$).

The decision problem is then MAX-SAT= "Given a weighted formula φ and a natural number k, is there an interpretation with a score $\geqslant k$?".
For instance, with φ given previously and $k_1 = 3$, the answer to the question is "YES" (ω_2 is an example of interpretation with a score $\geqslant k_1$). But with $k_2 = 4$, the answer is "NO" (there is no interpretation that satisfies all the weighted clauses at the same time).

(½) 1. Reformulate MAX-SAT as a language.

(1) 2. Given an instance of MAX-SAT, what is a positive certificate for this instance of the problem? Give a polynomial algorithm for verifying such a certificate.

(½) 3. From the previous question, can you deduce an upper bound for the complexity of MAX-SAT? Justify your answer.

(2) 4. Prove that SAT$\leqslant_f^{\mathsf{P}}$ MAX-SAT.

(½) 5. For which complexity class of the polynomial hierarchy is MAX-SAT complete? Justify your answer.

Exercise VI

We call N-VALID the problem: "Given a propositional formula φ, is φ **not** valid?". Recall that φ is valid if every possible interpretation ω is a model of φ.

(5) 1. Prove that N-VALID is NP-complete.