Correction de l'interrogation du 6/12/2022

Exercice 1 :

Soit A : " avoir au moins un accident " et soit C_i : "appartenir à la classe i " $\forall i = 1,2,3,4$.

1) $P(A) = \sum_{i=1}^{4} P(A \mid C_i) \times P(C_i)$: formule des probabilités totales

$= P(A \mid C_1) \times P(C_1) + P(A \mid C_2) \times P(C_2) + P(A \mid C_3) \times P(C_3) + P(A \mid C_4) \times P(C_4)$

$= 0{,}1 \times \frac{200\,000}{1\,000\,000} + 0{,}04 \times \frac{250\,000}{1\,000\,000} + 0{,}05 \times \frac{400\,000}{1\,000\,000} + 0{,}09 \times \frac{150\,000}{1\,000\,000}$

$= 0{,}1 \times 0{,}2 + 0{,}04 \times 0{,}25 + 0{,}05 \times 0{,}4 + 0{,}09 \times 0{,}15$

$= 0{,}0635$.

2) Soit B : "être âgé de 45 ans ou moins"

$B = C_1 \cup C_2$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap (C_1 \cup C_2))}{P(C_1 \cup C_2)} = \frac{P((A \cap C_1) \cup (A \cap C_2))}{P(C_1 \cup C_2)}$$

$$= \frac{P(A \cap C_1) + P(A \cap C_2)}{P(C_1) + P(C_2)} = \frac{P(A \mid C_1) \times P(C_1) + P(A \mid C_2) \times P(C_2)}{P(C_1) + P(C_2)}$$

car $C_1 \cap C_2 = \emptyset$
et $(A \cap C_1) \cap (A \cap C_2) = \emptyset$

$$= \frac{0{,}1 \times 0{,}2 + 0{,}04 \times 0{,}25}{0{,}2 + 0{,}25} = \frac{0{,}03}{0{,}45} = \frac{1}{15} \simeq 0{,}06667.$$

3) $P(C_1 \mid A) = \frac{P(C_1 \cap A)}{P(A)} = \frac{P(A \mid C_1) \times P(C_1)}{P(A)} = \frac{0{,}2 \times 0{,}1}{0{,}0635} \simeq 0{,}31496$.

4) Soient A^c : "ne pas avoir déclaré d'accident" et B^c : "être âgé de 46 ans ou plus". $B^c = C_3 \cup C_4$.

$$P(B^c \mid A^c) = \frac{P(B^c \cap A^c)}{P(A^c)} = \frac{P((C_3 \cup C_4) \cap A^c)}{1 - P(A)} = \frac{P((C_3 \cap A^c) \cup (C_4 \cap A^c))}{1 - P(A)}$$

$$= \frac{P(C_3 \cap A^c) + P(C_4 \cap A^c)}{1 - P(A)} = \frac{P(C_3) \times P(A^c \mid C_3) + P(C_4) \times P(A^c \mid C_4)}{1 - P(A)}$$

car $C_3 \cap C_4 = \emptyset$

$$\underset{\text{car } C_3 \cap C_4 = \emptyset}{=} \frac{0,4\times(1-0,05)+0,15\times(1-0,09)}{1-0,0635} = \frac{0,4\times 0,95+0,15\times 0,91}{0,9365} \simeq 0,55152.$$

Exercice 2 :

$$f(x) = \begin{cases} 0 & \text{si } x<0 \\ \dfrac{e^x}{e-1} & \text{si } x \in [0;a] \\ 0 & \text{si } x>a \end{cases}$$

1) $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = 1 \iff \int_0^a \frac{e^x}{e-1}\,dx = 1 \iff \frac{1}{e-1}\int_0^a e^x\,dx = 1$

$\displaystyle\iff \frac{1}{e-1}\left[e^x\right]_0^a = 1 \iff \frac{1}{e-1}(e^a - e^0) = 1 \iff \frac{e^a-1}{e-1} = 1$

$\iff e^a - 1 = e-1 \iff e^a = e \iff a\ln(e) = \ln(e) \underset{\text{car } \ln(e)=1}{\iff} a = 1.$

2) • Si $x<0$ alors $F(x) = 0$.

• Si $0 \leq x < a$ i.e. $0 \leq x < 1$ alors $F(x) = \displaystyle\int_{-\infty}^{x} f(u)\,du$

$\displaystyle = \int_{-\infty}^{0} f(u)\,du + \int_0^x f(u)\,du = F(0) + \int_0^x \frac{e^u}{e-1}\,du = 0 + \frac{1}{e-1}\left[e^u\right]_0^x$

$\displaystyle = \frac{e^x-1}{e-1}.$

• Si $x \geq a$ ie $x \geq 1$ alors $F(x) = \displaystyle\int_{-\infty}^{x} f(u)\,du = \int_{-\infty}^{1} f(u)\,du + \int_1^x f(u)\,du$

$\displaystyle = F(1) + 0 = \frac{e^1-1}{e-1} = 1.$

Ainsi, $F(x) = \begin{cases} 0 & \text{si } x<0 \\ \dfrac{e^x-1}{e-1} & \text{si } 0 \leq x < 1 \\ 1 & \text{si } x \geq 1 \end{cases}$

3) $\displaystyle E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^1 \frac{x e^x}{e-1}\,dx = \frac{1}{e-1}\int_0^1 x e^x\,dx$

Posons $\begin{cases} u(x) = x & \Rightarrow u'(x) = 1 \\ \ldots \end{cases}$

Rappel : $\displaystyle\int^b u(x)v'(x)\,dx$

Posons $\begin{cases} u(x) = x & \Rightarrow u'(x) = 1 \\ v'(x) = e^x & \Rightarrow v(x) = e^x \end{cases}$

Rappel : $\displaystyle\int_a^b u(x)v'(x)\,dx \underset{\text{I.P.P.}}{=} \left[u(x)\cdot v(x)\right]_a^b - \int_a^b u'(x)v(x)\,dx$

- $$E(X) \underset{\text{I.P.P.}}{=} \frac{1}{e-1}\left(\left[x\cdot e^x\right]_0^1 - \int_0^1 e^x\,dx\right)$$

$$= \frac{1}{e-1}\left(e - \left[e^x\right]_0^1\right) = \frac{1}{e-1}\left(e - (e-1)\right)$$

$$= \frac{1}{e-1}\left(\cancel{e} - \cancel{e} + 1\right) = \frac{1}{e-1}.$$

- $$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \int_0^1 \frac{x^2 e^x}{e-1}\,dx = \frac{1}{e-1}\int_0^1 x^2 e^x\,dx$$

I.P.P. Posons $\begin{cases} u(x) = x^2 & \Rightarrow u'(x) = 2x \\ v'(x) = e^x & \Rightarrow v(x) = e^x \end{cases}$

$$E(X^2) = \frac{1}{e-1}\left(\left[x^2 e^x\right]_0^1 - \int_0^1 2x\,e^x\,dx\right) = \frac{1}{e-1}\left(e - 2\int_0^1 x\,e^x\,dx\right)$$

$$= \frac{e}{e-1} - 2\,E(X) = \frac{e}{e-1} - \frac{2}{e-1} = \frac{e-2}{e-1}.$$

D'où $V(X) = E(X^2) - \left(E(X)\right)^2 = \dfrac{e-2}{e-1} - \left(\dfrac{1}{e-1}\right)^2 = \dfrac{(e-2)(e-1)-1}{(e-1)^2}$

$$= \frac{e^2 - e - 2e + 2 - 1}{(e-1)^2} = \frac{e^2 - 3e + 1}{(e-1)^2}.$$

Exercice 3 :

$$f(x) = \begin{cases} e^x & \text{si } x < 0 \\ 0 & \text{si } x \geq 0 \end{cases}$$

1) • Si $x < 0$ alors $F_X(x) = \displaystyle\int_{-\infty}^x f(u)\,du = \int_{-\infty}^x e^u\,du = \left[e^u\right]_{-\infty}^x = e^x.$

• Si $x \geq 0$ alors $F_X(x) = \displaystyle\int_{-\infty}^x f(u)\,du = \int_{-\infty}^0 f(u)\,du + \int_0^x f(u)\,du$

$$= F_X(0) + 0 = e^0 = 1.$$

Ainsi, $F_X(x) = \begin{cases} e^x & \text{si } x < 0 \\ 1 & \text{si } x \geq 0 \end{cases}$

2) $E(X) = \displaystyle\int_{-\infty}^{+\infty} x\,f(x)\,dx = \int_{-\infty}^0 x\,e^x\,dx.$

$\quad 1 \quad$ si $x \geq 0$

2) $E(X) = \int_{-\infty}^{+\infty} x\, f(x)\, dx = \int_{-\infty}^{0} x\, e^{x}\, dx$.

I.P.P. Posons $\begin{cases} u(x) = x & \Rightarrow u'(x) = 1 \\ v'(x) = e^{x} & \Rightarrow v(x) = e^{x} \end{cases}$

$E(X) = \left[x e^{x} \right]_{-\infty}^{0} - \int_{-\infty}^{0} e^{x}\, dx = 0 - \left[e^{x} \right]_{-\infty}^{0} = -(1-0)$

$= -1$.

3) $Y := 2X + 1$. $F_Y(y) := P(Y \leq y) = P(2X + 1 \leq y) = P(2X \leq y - 1)$

$= P\left(X \leq \frac{y-1}{2}\right) = F_X\left(\frac{y-1}{2}\right)$.

- Si $\frac{y-1}{2} < 0$ i.e $y < 1$ alors $F_Y(y) = F_X\left(\frac{y-1}{2}\right) = e^{\frac{y-1}{2}}$.

- Si $\frac{y-1}{2} \geq 0$ i.e. $y \geq 1$ alors $F_Y(y) = F_X\left(\frac{y-1}{2}\right) = 1$.

Ainsi, $F_Y(y) = \begin{cases} e^{\frac{y-1}{2}} & \text{si } y < 1 \\ 1 & \text{si } y \geq 1 \end{cases}$

4) $f_Y(y) = F_Y'(y)$

$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{2} e^{\frac{y-1}{2}} & \text{si } y < 1 \\ 0 & \text{si } y > 1 \end{cases}$