Linear Algebra HW3
Exercises 262, 269, 277, 320, 321, 379, 385, 387, 421, 431

Neal D. Nesbitt

June 28, 2016

Problem 6.262. Let V and W be vector spaces over a field F. Let $\alpha \in \text{Hom}(V, W)$ and $\beta \in \text{Hom}(W, V)$ satisfy the condition that $\alpha\beta\alpha = \alpha$. If $w \in \text{im}(\alpha)$, show that $\alpha^{-1}(w) = \{\beta(w) + v - \beta\alpha(v) | v \in V\}$

Solution 6.262. Let our notation be as above, $w \in \text{im}(\alpha)$, and recall that $\alpha^{-1}(w) = \{v \in V | \alpha(v) = w\}$.

Assume $x \in \{\beta(w) + v - \beta\alpha(v) | v \in V\}$ and notice that $\{\beta(w) + v - \beta\alpha(v) | v \in V\} \subset V$. Then there is some $v \in V$ such that $x = \beta(w) + v - \beta\alpha(v)$. Since $w \in \text{im}(\alpha)$ there is also some $v\prime \in V$ such that $\alpha(v\prime) = w$. Then by applying α, using its linearity, and its listed property we find

$$\begin{aligned}
\alpha(x) &= \alpha(\beta(w) + v - \beta\alpha(v)) \\
\alpha(x) &= \alpha(\beta\alpha(v\prime) + v - \beta\alpha(v)) \\
\alpha(x) &= \alpha\beta\alpha(v\prime) + \alpha(v) - \alpha\beta\alpha(v) \\
\alpha(x) &= \alpha(v\prime) + \alpha(v) - \alpha(v) \\
\alpha(x) &= \alpha(v\prime) \\
\alpha(x) - \alpha(v\prime) &= 0 \\
\alpha(x - v\prime) &= 0
\end{aligned}$$

showing that $x - v\prime$ is in the kernel of α. Then by Proposition 6.6 on pg 96, and the fact that $v\prime \in \alpha^{-1}(w)$, we have that $(x - v\prime) + v\prime = x \in \alpha^{-1}(w)$

Now assume instead that $y \in \alpha^{-1}(w) = \{v \in V | \alpha(v) = w\}$. Then for some $w \in W$, $\alpha(y) = w$.