# Receding Horizon Control for MJS

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October 20, 2016

Given the control system

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1}$$

where $u$ is dependent on $x$, we can recursively calculate the highest order term in terms of all the lower ones: $\forall k \in \mathbb{N}$

$$\begin{aligned}
x_1 &= A_0x_0 + B_0u_0 \\
x_2 &= A_1\left(A_0x_0 + B_0u_0\right) + B_1u_1 \\
&= A_1A_0x_0 + A_1B_0u_0 + B_1u_1 \\
x_3 &= A_2x_2 + B_2u_2 \\
&= A_2\left(A_1A_0x_0 + A_1B_0u_0 + B_1u_1\right) + B_2u_2 \\
&= A_2A_1A_0x_0 + A_2A_1B_0u_0 + A_2B_1u_1 + B_2u_2 \\
&\ \ \vdots \\
x_k &= \left(A_{k-1}\dots A_1A_0\right)x_0 + \left(A_{k-2}\dots A_2A_1\right)B_0u_0 + \left(A_{k-3}\dots A_3A_2\right)B_1u_1 + \cdots \\
&\qquad + \left(A_{k-2}A_{k-1}\right)B_{k-3}u_{k-3} + A_{k-1}B_{k-2}u_{k-2} + B_{k-1}u_{k-1}
\end{aligned}$$

$$x_{k+1} = \left[\prod_{n=k}^{0} A_n\right]x_0 + \sum_{j=1}^{k}\left[\prod_{n=k}^{j} A_n\right]B_{j-1}u_{j-1} + B_ku_k$$

So consider the following cost function:

$$\begin{aligned}
J(x_0) &= \sum_{k=1}^{L} x_k^{\mathrm{T}}R_kx_k + \sum_{k=0}^{L-1} u_k^{\mathrm{T}}Q_ku_k \\
J(x_0) &= \sum_{k=1}^{L} x_k^{\mathrm{T}}R_kx_k + u_{k-1}^{\mathrm{T}}Q_{k-1}u_{k-1}
\end{aligned}$$

Notice that there is no well defined $R_0$ or $Q_L$, and $x_0$ and $u_L$ are not taken into account.

To minimize the cost function with respect to the control vector, $u$, we take it's derivative with respect to the highest order term, since it has no other variables that depend on it. So then, because

$$\frac{\partial}{\partial u_{k-1}}\left[x_k\right] = \frac{\partial}{\partial u_{k-1}}\left[A_{k-1}x_{k-1} + B_{k-1}u_{k-1}\right] = B_{k-1}$$

the chain rule gives

$$\frac{\partial}{\partial u_{L-1}}\left[J(x_0)\right] = 2\left(x_L^{\mathrm{T}}R_LB_{L-1} + u_{L-1}^{\mathrm{T}}Q_{L-1}\right)$$

Set this equal to zero to find critical points.

$$\begin{aligned}
2\left(x_L^{\mathrm{T}}R_LB_{L-1} + u_{L-1}^{\mathrm{T}}Q_{L-1}\right) &= 0\\
x_L^{\mathrm{T}}R_LB_{L-1} + u_{L-1}^{\mathrm{T}}Q_{L-1} &= 0\\
u_{L-1}^{\mathrm{T}}Q_{L-1} &= -x_L^{\mathrm{T}}R_LB_{L-1}\\
u_{L-1}^{\mathrm{T}} &= -x_L^{\mathrm{T}}R_LB_{L-1}Q_{L-1}^{-1}\\
u_{L-1} &= -Q_{L-1}^{-1}B_{L-1}^{\mathrm{T}}R_Lx_L\\
u_{L-1} &= -Q_{L-1}^{-1}B_{L-1}^{\mathrm{T}}R_L\left(A_{L-1}x_{L-1} + B_{L-1}u_{L-1}\right)\\
Q_{L-1}u_{L-1} &= -B_{L-1}^{\mathrm{T}}R_LA_{L-1}x_{L-1} - B_{L-1}^{\mathrm{T}}R_LB_{L-1}u_{L-1}\\
B_{L-1}^{\mathrm{T}}R_LB_{L-1}u_{L-1} + Q_{L-1}u_{L-1} &= -B_{L-1}^{\mathrm{T}}R_LA_{L-1}x_{L-1}\\
\left(B_{L-1}^{\mathrm{T}}R_LB_{L-1} + Q_{L-1}\right)u_{L-1} &= -B_{L-1}^{\mathrm{T}}R_LA_{L-1}x_{L-1}
\end{aligned}$$

So assuming that $\left(B_{L-1}^{\mathrm{T}}R_LB_{L-1} + Q_{L-1}\right)$ is invertible for all $L \in \mathbb{N}$, then

$$\boxed{u_{L-1} = -\left(B_{L-1}^{\mathrm{T}}R_LB_{L-1} + Q_{L-1}\right)^{-1}B_{L-1}^{\mathrm{T}}R_LA_{L-1}x_{L-1}}$$

Now equivalently, for all $k \in \mathbb{Z}^+$, we have assumed that $\left(B_k^{\mathrm{T}}R_{k+1}B_k + Q_k\right)$ is invertible. So call

$$\boxed{G_k = -\left(B_k^{\mathrm{T}}R_{k+1}B_k + Q_k\right)^{-1}B_k^{\mathrm{T}}R_{k+1}A_k}$$

so that

$$\boxed{u_{L-1} = G_{L-1}x_{L-1}}$$

Starting with $L = 1$ gives

$$\boxed{u_0 = G_0x_0}$$

and finding the next predicted position:

$$\begin{aligned}
x_1 &= A_0x_0 + B_0u_0\\
x_1 &= A_0x_0 + B_0G_0x_0\\
x_1 &= \left(A_0 + B_0G_0\right)x_0
\end{aligned}$$

Increase the prediction distance to $L = 2$ and reuse the prediction for $x_1$ to solve for the next control:

$$u_1 = G_1 x_1$$
$$u_1 = G_1 (A_0 + B_0 G_0) x_0$$

Predict forward one more time to $L = 3$ and the pattern will emerge:

$$x_2 = A_1 x_1 + B_1 u_1$$
$$x_2 = A_1 (A_0 + B_0 G_0) x_0 + B_1 G_1 (A_0 + B_0 G_0) x_0$$
$$x_2 = (A_1 + B_1 G_1)(A_0 + B_0 G_0) x_0$$

solving for $u_2$ just adds a $G_2$ term on the front.

$$u_2 = G_2 x_2$$
$$u_2 = G_2 (A_1 + B_1 G_1)(A_0 + B_0 G_0) x_0$$

**Theorem 0.1.** For any $k \in \mathbb{N}$, $k \leq L$, if the predictions for $x_1, \ldots, x_{k-1}$ are used recursively, and the control vector $u_k$ is found by minimizing the cost function $J(x_0)$ at each step, then

$$\boxed{x_k = \left[ \prod_{n=k-1}^{0} A_n + B_n G_n \right] x_0}$$

*Proof.* Notice that for $k \in \mathbb{N}$, if $k = 1$, then as noted above:

$$\boxed{x_1 = (A_0 + B_0 G_0) x_0}$$

Assume that for $k \geq 1$,

$$x_k = \left[ \prod_{n=k-1}^{0} A_n + B_n G_n \right] x_0$$

Then by the derived control formula above: $\forall k \in \mathbb{N}$

$$\boxed{u_{k-1} = G_{k-1} x_{k-1}}$$

and using the original system equation, it follows that,

$$x_{k+1} = A_k x_k + B_k u_k$$
$$x_{k+1} = A_k x_k + B_k G_k x_k$$
$$x_{k+1} = (A_k + B_k G_k) x_k$$
$$x_{k+1} = (A_k + B_k G_k) \left[ \prod_{n=k-1}^{0} A_n + B_n G_n \right] x_0$$
$$x_{k+1} = \left[ \prod_{n=k}^{0} A_n + B_n G_n \right] x_0$$

So by induction, the formula must be valid for all $k \in \mathbb{N}$ $\square$

**Corollary 0.1.** For any $k, L \in \mathbb{N}$, $L > 2$, and $k < L$; if the predictions for $x_1, \ldots, x_{k-1}$ are used recursively, and the control vector $u_k$ is found by minimizing the cost function $J(x_0)$ at each step, then

$$\boxed{u_k = G_k \left[ \prod_{n=k-1}^{0} A_n + B_n G_n \right] x_0}$$

*Proof.* This follows directly from the theorem, and the derived control formula above (after shifting up one index): $\forall k \in \mathbb{N}$

$$\begin{aligned} u_k &= G_k x_k \\ u_k &= G_k \left[ \prod_{n=k-1}^{0} A_n + B_n G_n \right] x_0 \end{aligned}$$

$\square$