# Linear Algebra HW4
# Exercises 476, 516, 542, 585, 622, 670, 683, 760, 880

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**Problem 9.476.** Let $F$ be a field. Find a matrix $A \in \mathcal{M}_{4\times 4}(F)$ satisfying $A^4 = I \neq A^3$.

**Solution 9.476.** Let the notation be as given in the problem.

$$\text{Examine } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}:$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}^4 = I$$

and thus this matrix fulfills the required conditions.

**Problem 9.516.** Let $n$ be a positive integer, let $F$ be a field, and let $A \in \mathcal{M}_{n\times n}(F)$ satisfy the condition $A = AA^T$. Show that $A^2 = A$.

**Solution 9.516.** Let our notation be as given in the problem. Then $A = AA^T$ implies $A^T = (AA^T)^T = (A^T)^T A^T = AA^T = A$. Thus $A = AA^T = A^2$.

**Problem 9.542.** Let $n$ and $p$ be positive integers and let $F$ be a field. Let $A \in \mathcal{M}_{n\times n}(F)$ and let $B, C \in \mathcal{M}_{n\times p}(F)$ be matrices satisfying the condition that $A$ and $(I+C^TA^{-1}B)$ are nonsingular. Show that $A+BC^T$ is non-singular, and that

$$\left(A+BC^T\right)^{-1} = A^{-1} - A^{-1}B\left(I+C^TA^{-1}B\right)^{-1}C^TA^{-1}$$

**Solution 9.542.** Let our notation be as given in the problem. Then since $(I+C^TA^{-1}B)$ is non-singular, by Proposition 9.1 on pg 154, $(I+BC^TA^{-1})$ is also nonsingular. So since $A$ is also non-singular, then their product will be as well by the same proposition:

$$(I+BC^TA^{-1})A = A+BC^T$$

Also by the same proposition

$$\begin{aligned}
[A+BC^T]^{-1} &= [(I+BC^TA^{-1})A]^{-1} \\
&= A^{-1}(I+BC^TA^{-1})^{-1} \\
&= A^{-1}(I-B(I+C^TA^{-1}B)^{-1}C^TA^{-1}) \\
&= A^{-1} - A^{-1}B(I+C^TA^{-1}B)^{-1}C^TA^{-1}
\end{aligned}$$

**Problem 10.585.** Find all solutions to the system

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \\ -5 \end{bmatrix}$$

**Solution 10.585.** Using Gaussian elimination reduces the augmented matrix as follows:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 2 & 1 \\ 4 & 3 & 2 & 1 & -5 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & -3 & -4 & -5 & -9 \\ 0 & -4 & -8 & -10 & -14 \\ 0 & -5 & -10 & -15 & -25 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & -3 & -4 & -5 & -9 \\ 0 & 0 & -8/3 & -10/3 & -2 \\ 0 & 0 & -10/3 & -20/3 & -10 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & -3 & -4 & -5 & -9 \\ 0 & 0 & -8/3 & -10/3 & -2 \\ 0 & 0 & -10/3 & -20/3 & -10 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & -3 & -4 & -5 & -9 \\ 0 & 0 & -8/3 & -10/3 & -2 \\ 0 & 0 & 0 & -5/2 & -15/2 \end{array}\right]$$

Then by back substitution

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -3 \\ 3 \end{bmatrix}$$

**Problem 10.622.** Let $k$ and $n$ be positive integers and let $F$ be a field. For matrices $A, B \in \mathcal{M}_{k\times n}(F)$, show that the rank of $A+B$ is no more than the sum of the ranks of $A$ and of $B$.

**Solution 10.622.** Let our notation be as given in the problem, and let us call $a \leq n$ the rank of $A$, and $b \leq n$ the rank of $B$.

Then the column space of each matrix has a respective basis, $\mathcal{B}(A) = \{u_1, \cdots, u_a\}$ and $\mathcal{B}(B) = \{v_1, \cdots, v_b\}$. The column space of $A+B$ can have at most $a+b$ vectors, where this is only if all of the basis vectors for $A \cup B$ are linearly independent: $\mathcal{B}(A) \cup \mathcal{B}(B) = \{u_1, \cdots, u_a, v_1, \cdots, v_b\}$.

When they are independent like this, the rank of $A+B$ is its maximum: the number of basis vectors, $a+b$, which is no more than the sum of the ranks of $A$ and $B$.

**Problem 11.670.** Let $n$ be a positive integer and let $A = [a_{ij}] \in \mathcal{M}_{n\times n}(\mathbb{R})$ be the matrix defined by

$$a_{ij} = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{otherwise} \end{cases}$$

Calculate $|A|$.

**Solution 11.670.** Let our notation be that same as given in the problem.

Then since for every $j \in \mathbb{N}$ such that $j \leq n$, we know $a_{jj} = 0$, then by Proposition 11.11 on pg 231,

$$|A| = \sum_{j=1}^{n} (-1)^{j+j} a_{jj} \cdot |A_{ij}| = \sum_{j=1}^{n} 0 \cdot |A_{ij}| = 0$$

**Problem 11.683.** Let $F$ be a field, let $n$ be a positive integer, and let $A = [a_{ij}] \in \mathcal{M}_{n\times n}(F)$ be nonsingular. Show that $\mathrm{adj}(\mathrm{adj}(A)) = |A|^{n-2}A$.

**Solution 11.683.** Let our notation be the same as given in the problem. Then notice that $A$ non-singular means that $A^{-1}$ is well defined.

Now look at Proposition 11.13 on pg 235. Since $A$ is nonsingular, then

$$\begin{aligned} A[\mathrm{adj}(A)] &= |A| I \\ \mathrm{adj}(A) &= |A| A^{-1} \\ \mathrm{adj}(\mathrm{adj}(A)) &= \left| |A| A^{-1} \right| (|A| A^{-1})^{-1} \end{aligned}$$

Now we take note that multiplying a whole matrix by a scalar $c \in F$ is equivalent to multiplying each column by the same scalar:

$$cA = \left( \prod_{i=1}^{n} E_{i;c} \right) A$$

and thus by the properties of determinant functions and elementary matrices:

$$
\begin{aligned}
\operatorname{adj}(\operatorname{adj}(A)) &= \left||A|A^{-1}\right| (|A|A^{-1})^{-1} \\
&= \left|\left(\prod_{i=1}^{n} E_{i;|A|}\right) A^{-1}\right| (\left(\prod_{i=1}^{n} E_{i;|A|}\right) A^{-1})^{-1} \\
&= |A|^n |A^{-1}| A \left(\prod_{i=1}^{n} E_{i;|A|}\right)^{-1} \\
&= |A|^{n-1} A \prod_{i=n}^{1} E_{i;|A|}^{-1} \\
&= |A|^{n-1} A \prod_{i=n}^{1} E_{i;|A|^{-1}} \\
&= |A|^{n-1} A |A|^{-1} \\
\operatorname{adj}(\operatorname{adj}(A)) &= |A|^{n-2} A
\end{aligned}
$$

**Problem 12.760.** Fine the eigenvalues of the matrix $\begin{bmatrix} 5 & 6 & -3 \\ -1 & 0 & 1 \\ 2 & 2 & -1 \end{bmatrix} \in \mathcal{M}_{3\times 3}(\mathbb{R})$ and, for each such eigenvalue, find the associated eigenspace.

**Solution 12.760.** If we call the matrix above $A$, we can compute the characteristic polynomial by letting $\lambda \in \mathbb{C}$ be an eigenvalue, and setting the determinant of $|A - \lambda I|$ to zero:

$$
\begin{aligned}
|A - \lambda I| = \begin{vmatrix} 5-\lambda & 6 & -3 \\ -1 & -\lambda & 1 \\ 2 & 2 & -1-\lambda \end{vmatrix}& \\
&= (5-\lambda)(1+\lambda)\lambda + 12 + 6 - (6\lambda + 6(1+\lambda) + 2(5-\lambda)) \\
&= (5 + 4\lambda - \lambda^2)\lambda + 18 - (6\lambda + 6 + 6\lambda + 10 - 2\lambda) \\
&= (5 + 4\lambda - \lambda^2)\lambda + 18 - (10\lambda + 16) \\
&= (5 + 4\lambda - \lambda^2)\lambda - 10\lambda + 2 \\
&= 5\lambda + 4\lambda^2 - \lambda^3 - 10\lambda + 2 \\
&= 2 - 5\lambda + 4\lambda^2 - \lambda^3 \\
&= (2-\lambda)(1 - 2\lambda + \lambda^2) \\
&= (2-\lambda)(1-\lambda)^2 = 0
\end{aligned}
$$

and thus $A$ has eigenvalues $\lambda = 1, 2$. We then consider these separately and find their respective eigenspaces.

So begin with $\lambda = 1$, and to find the eigenspace, we find the equivalent nullspace of $A - \lambda I$:

$$\left[\begin{array}{ccc|c} 4 & 6 & -3 & 0 \\ -1 & -1 & 1 & 0 \\ 2 & 2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 4 & 6 & -3 & 0 \\ 0 & 1/2 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We then let $v \in \mathbb{R}^3$ be in the nullspace of $A - \lambda I$, and take $v_3 \in \mathbb{R}$ to be a free variable. This allows us to see:

$$\begin{aligned} \frac{v_2}{2} + \frac{v_3}{4} &= 0 \\ \frac{v_2}{2} &= \frac{-v_3}{4} \\ v_2 &= \frac{-v_3}{2} \end{aligned}$$

and

$$\begin{aligned} 4v_1 + 6v_2 - 3v_3 &= 0 \\ 4v_1 &= -6v_2 + 3v_3 \\ 4v_1 &= 3v_3 + 3v_3 \\ 4v_1 &= 6v_3 \\ v_1 &= \frac{3v_3}{2} \end{aligned}$$

Then we have that vectors in the eigenspace for $\lambda = 1$ are of the form

$$v_3 \begin{bmatrix} 3/2 \\ -1/2 \\ 1 \end{bmatrix} = \frac{v_3}{2} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

So now let us consider $\lambda = 2$. The associated $A - \lambda I$ reduces as follows:

$$\left[\begin{array}{ccc|c} 3 & 6 & -3 & 0 \\ -1 & -2 & 1 & 0 \\ 2 & 2 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 6 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -1 & 0 \end{array}\right]$$

and for $v \in \mathbb{R}^3$ in the nullspace of $A - \lambda I$, we take $v_2$ to be a free variable this time so that:

$$\begin{aligned} -2v_2 - v_3 &= 0 \\ v_3 &= -2v_2 \end{aligned}$$

and then

$$\begin{aligned} 3v_1 + 6v_2 - 3v_3 &= 0 \\ 3v_1 &= -6v_2 + 3v_3 \\ 3v_1 &= -6v_2 - 6v_2 \\ 3v_1 &= -12v_2 \\ v_1 &= -4v_2 \end{aligned}$$

Showing that vectors in the eigenspace for $\lambda = 2$ are of the form:

$$v_2 \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix} = -v_2 \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

**Problem 13.880.** Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$ differ from $I$ and $O$. If $A$ is idempotent, show that its Jordan canonical form is a diagonal matrix.

**Solution 13.880.** Let $A$ be as given in the problem.

Then by Proposition 13.7, there is some matrix $B \in \mathcal{M}_{n\times n}(\mathbb{R})$ expressed in canonical form that is similar to $A$. So then if we take the resulting $P \in \mathcal{M}_{n\times n}(\mathbb{R})$ where $A = P^{-1}BP$, we can then use the idepotence of $A$ to see

$$\begin{aligned} A^2 &= P^{-1}BPP^{-1}BP = A = P^{-1}BP \\ &= P^{-1}B^2P = P^{-1}BP \end{aligned}$$

which implies that $B^2 = B$, showing $B$ is also idepotent.

Since $B$ is in canonical form, then there is an $m \in \mathbb{N}$ where for any $j \in \mathbb{N}$ such that $j \leq m$, we have $c_j \in \mathbb{R}$ where blocks of the form

$$B_j = \begin{bmatrix} c_j & 0 & 0 & \cdots & 0 \\ 1 & c_j & 0 & \cdots & 0 \\ 0 & 1 & c_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & c_j \end{bmatrix}$$

make up the matrix $B$ in the following way:

$$B = \begin{bmatrix} B_1 & O & \cdots & O \\ 0 & B_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_m \end{bmatrix}$$

So now we notice that

$$B^2 = \begin{bmatrix} B_1^2 & O & \cdots & O \\ 0 & B_2^2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & B_m^2 \end{bmatrix}$$

where for each $j$, the idepotence of $B$ implies that

$$B_j^2 = B_j$$

$$\begin{bmatrix} c_j & 0 & 0 & \cdots & 0 \\ 1 & c_j & 0 & \cdots & 0 \\ 0 & 1 & c_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & c_j \end{bmatrix} \begin{bmatrix} c_j & 0 & 0 & \cdots & 0 \\ 1 & c_j & 0 & \cdots & 0 \\ 0 & 1 & c_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & c_j \end{bmatrix} = \begin{bmatrix} c_j & 0 & 0 & \cdots & 0 \\ 1 & c_j & 0 & \cdots & 0 \\ 0 & 1 & c_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & c_j \end{bmatrix}$$

$$\begin{bmatrix} c_j^2 & 0 & 0 & 0 & \cdots & 0 \\ 2c_j & c_j^2 & 0 & 0 & \cdots & 0 \\ 1 & 2c_j & c_j^2 & 0 & \cdots & 0 \\ 0 & 1 & 2c_j & c_j^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 2c_j & c_j^2 \end{bmatrix} = \begin{bmatrix} c_j & 0 & 0 & \cdots & 0 \\ 1 & c_j & 0 & \cdots & 0 \\ 0 & 1 & c_j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & c_j \end{bmatrix}$$

Since this implies that each $c_j^2 = c_j$ and $2c_j = 1$, then $c_j$ must be simultaneously $1/2$ and either 1 or 0. This is a contradiction unless our blocks are only one row by one column:

$$B_j = [c_j]$$

showing that $B$ is a diagonal matrix.