Linear Algebra HW2
Exercises 169, 185, 187, 202, 205, 214, 232, 234

Neal D. Nesbitt

June 22, 2016

Problem 5.169. Let F be a field of characteristic different from 2 and let V be a vector space over F containing a linearly independent subset $\{v_1, v_2, v_3\}$. Show that the set $\{v_1 + v_2, v_2 + v_3, v_1 + v_3\}$ is also linearly independent.

Solution 5.169. *Proof.* Let our notation be as above, and assume by way of contradiction that $\{v_1 + v_2, v_2 + v_3, v_1 + v_3\}$ is not linearly independent. Then it is linearly dependent, and there are some $a_1, a_2, a_3 \in F$ such that $a_1 \neq 0, a_2 \neq 0, a_3 \neq 0$ and:

$$a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_1 + v_3) = 0$$

Now distribute the scalars over the vectors and use commutativity to reorder them. If we recollect the vectors as instead distributed over the scalars, we find:

$$\begin{aligned} a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_1 + v_3) &= 0 \\ a_1v_1 + a_1v_2 + a_2v_2 + a_2v_3 + a_3v_1 + a_3v_3 &= 0 \\ a_1v_1 + a_3v_1 + a_1v_2 + a_2v_2 + a_2v_3 + a_3v_3 &= 0 \\ (a_1 + a_3)v_1 + (a_1 + a_2)v_2 + (a_2 + a_3)v_3 &= 0 \end{aligned}$$

Since we know F is not of characteristic two, and each a is non-zero, then $a_1 + a_3 \neq 0, a_1 + a_2 \neq 0, a_2 + a_3 \neq 0$ also, showing $\{v_1, v_2, v_3\}$ must be linearly dependent.

This contradicts our assumption that $\{v_1, v_2, v_3\}$ is linearly independent. So our assumption that $\{v_1 + v_2, v_2 + v_3, v_1 + v_3\}$ is linearly dependent must be wrong, and it must instead be linearly independent as desired. □

Problem 5.185. Let $\{v_1, \cdots, v_n\}$ be a basis for a vector space V over a field F. Is the set $\{v_1 + v_2, v_2 + v_3, \cdots, v_{n-1} + v_n, v_n + v_1\}$ necessarily also a basis for V over F?

Solution 5.185. No, this is not always the case.

Proof. Take F to be a field of characteristic 2, and let $\{v_1, \cdots, v_n\}$ be a basis for a vector space V over F. Then take $a \in F$ such that $a \neq 0$. Since every basis is linearly independent,

$$\sum_{i=1}^{n} av_i \neq 0$$

Now notice how this vector added to itself is a linear combination of $\{v_1 + v_2, v_2 + v_3, \cdots, v_{n-1} + v_n, v_n + v_1\}$.

$$\begin{aligned}
\sum_{i=1}^{n} av_i + \sum_{i=1}^{n} av_i &= a\left(\sum_{i=1}^{n} v_i + \sum_{i=1}^{n} v_i\right) \\
&= a\left(v_1 + \sum_{i=2}^{n} v_i + \sum_{i=1}^{n-1} v_i + v_n\right) \\
&= a\left(v_1 + \sum_{i=1}^{n-1} v_{i+1} + \sum_{i=1}^{n-1} v_i + v_n\right) \\
&= a\left(v_1 + \sum_{i=1}^{n-1} (v_{i+1} + v_i) + v_n\right) \\
&= a\left(\sum_{i=1}^{n-1} (v_i + v_{i+1}) + (v_1 + v_n)\right) \\
\sum_{i=1}^{n} av_i + \sum_{i=1}^{n} av_i &= \sum_{i=1}^{n-1} a(v_i + v_{i+1}) + a(v_1 + v_n)
\end{aligned}$$

But since F is of characteristic 2,

$$\sum_{i=1}^{n} av_i + \sum_{i=1}^{n} av_i = (a + a)\sum_{i=1}^{n} v_i = a(1 + 1)\sum_{i=1}^{n} v_i = 0\sum_{i=1}^{n} v_i = 0$$

Then we must also have that

$$\sum_{i=1}^{n-1} a(v_i + v_{i+1}) + a(v_1 + v_n) = 0$$

showing $\{v_1 + v_2, v_2 + v_3, \cdots, v_{n-1} + v_n, v_n + v_1\}$ is not linearly independent, and therefore cannot be a basis. This shows that not every set of the given type constitutes a basis of the original vector space. □

Problem 5.187. For which values of $a \in \mathbb{R}$ is the set

$$\left\{ \begin{bmatrix} a & 2a \\ 2 & 3a \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2a & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2a \\ a+1 & a+2 \end{bmatrix}, \begin{bmatrix} 1 & a+1 \\ 2 & 2a+1 \end{bmatrix} \right\}$$

a basis for $\mathcal{M}_{2\times 2}(\mathbb{R})$ as a vector space over $\mathbb{R}$?

Solution 5.187. Let the notation be as above.

A basis is a linearly independent set that spans the space. Since $\mathcal{M}_{2\times 2}(\mathbb{R})$ as a vector space over $\mathbb{R}$ has dimension 4, then if the given set is linearly independent it will be a subspace with the same dimension (since the set has 4 elements), and therefore must span the whole space.

So let us consider the conditions on a for which the set is linearly dependent and take the complement.

If the set is linearly dependent, there exist $c_1, c_2, c_3, c_4 \in \mathbb{R}$ such that $c_1 \neq 0, c_2 \neq 0, c_3 \neq 0, c_4 \neq 0$. Then

$$c_1 \begin{bmatrix} a & 2a \\ 2 & 3a \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 \\ 2a & 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 2a \\ a+1 & a+2 \end{bmatrix} + c_4 \begin{bmatrix} 1 & a+1 \\ 2 & 2a+1 \end{bmatrix} = 0$$

$$\begin{aligned} & c_1 \left(a \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \right) \\ & + c_2 \left(a \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \right) \\ & + c_3 \left(a \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \right) \\ & + c_4 \left(a \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \right) = 0 \end{aligned}$$

$$\begin{aligned} & a \left(c_1 \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \right) = \\ & \qquad - \left(c_1 \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \right) \end{aligned}$$

We can then consider each constant's equation separately to cancel them out, representing this using row vectors in $\{\mathcal{M}_{2\times 2}(\mathbb{R})\}^4$ as a vector space over $\mathbb{R}$ gives a compact way of showing this:

$$a \left(\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = - \left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \right) \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$a \left(\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \right) = - \left(\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \right)$$

but since

$$\begin{bmatrix} a & 2a \\ 0 & 3a \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 0 \\ 2a & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & -3 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 2a \\ a & a \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -1 & -2 \end{bmatrix}$$
$$\begin{bmatrix} 0 & a \\ 0 & 2a \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix}$$

each have no solution for $a \in \mathbb{R}$, (since $0 \neq -2, 0 \neq -1$, etc ...) then the row vectors will never be equal, and the set must always be linearly independent. So since it also spans the entire space, it is thus a basis for $\mathcal{M}_{2\times 2}(\mathbb{R})$ as a vector space over $\mathbb{R}$, for any $a \in \mathbb{R}$.

Problem 5.202. Let V be a vector space of finite dimension n over a field F, and let W be a subspace of V of dimension $n-1$. If U is a subspace not contained in W, show that $\dim(W \cap U) = \dim(U) - 1$.

Solution 5.202. *Proof.* Let our notation be as above. Since we know that

$$\begin{aligned} \dim(W \cap U) + \dim(W \cup U) &= \dim(W) + \dim(U) \\ \dim(V) &= n \\ \dim(W) &= n - 1 \end{aligned}$$

And since $U \not\subset W \subset V$ and $U \subset V$ implies $\dim(V) \geq \dim(U), \dim(V) \geq \dim(W)$, then $\dim(V) = n \geq \dim(W \cup U) > \dim(W) = n - 1$, implying that $\dim(V) = n = \dim(W \cup U)$ since dimensions only take on integer values.

Thus we can substitute our findings into our initial equation to find:

$$\begin{aligned} \dim(W \cap U) &= \dim(W) + \dim(U) - \dim(W \cup U) \\ \dim(W \cap U) &= (n-1) + \dim(U) - n = \dim(U) - 1 \end{aligned}$$

□

Problem 5.205. Let

$$W = \mathbb{R}\left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}$$

Determine the dimension of W and find a basis for it.

Solution 5.205. Let our notation be as above. Since

$$\begin{bmatrix} -1 \\ 1 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

and for any $a \in \mathbb{R}$

$$a \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

then the set

$$W \supset V = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$$

is linearly independent, but W is not. Thus $\dim(W) = \dim(V) = 2$ and V serves as a basis for the space.

Problem 5.214. Let V be a vector space of finite dimension n over a field F and let W and Y be distinct subspaces of V, each of dimension $n-1$. What is $\dim(W \cap Y)$?

Solution 5.214. Let our notation be as above. Since W and Y are distinct, $W \neq Y$, then $\dim(W \cup Y) = n$ since $\dim(V) = n \geq \dim(W \cup Y) > \dim(W) = n-1$, and dimensions only take on integer values.

Then since

$$\begin{aligned} \dim(W \cap Y) + \dim(W \cup Y) &= \dim(W) + \dim(Y) \\ \dim(W \cap Y) + n &= 2n - 2 \\ \dim(W \cap Y) &= n - 2 \end{aligned}$$

we have found our solution.

Problem 5.232. Let V be a vector space over a field F and let D be a finite minimal linearly dependent subset of V. Find $\dim(FD)$.

Solution 5.232. Let our notation be as above. Then since D minimal linearly dependent, that means that removing any one element makes it linearly independent, and thus the dimension of D is one less than the number of elements it contains.

So since linear combinations of a field over some subset of vectors in a space produces a subspace, $\dim(V) \geq \dim(FD)$ and $\dim(FD) = |D| - 1$.

Problem 6.234. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a linear transformation satisfying $\alpha\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\3\\4\end{bmatrix}$, $\alpha\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\alpha\left(\begin{bmatrix}1\\2\\-1\end{bmatrix}\right) = \begin{bmatrix}3\\1\\4\end{bmatrix}$. What is $\alpha\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$?

Solution 6.234. Since

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}1\\2\\-1\end{bmatrix} + 2\begin{bmatrix}1\\-1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix}\right)$$

then

$$\begin{aligned}
\alpha\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) &= \frac{1}{2}\alpha\left(\begin{bmatrix}1\\2\\-1\end{bmatrix} + 2\begin{bmatrix}1\\-1\\1\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix}\right)\\
&= \frac{1}{2}\alpha\left(\begin{bmatrix}1\\2\\-1\end{bmatrix}\right) + \alpha\left(\begin{bmatrix}1\\-1\\1\end{bmatrix}\right) - \frac{1}{2}\alpha\left(\begin{bmatrix}1\\0\\1\end{bmatrix}\right)\\
&= \frac{1}{2}\begin{bmatrix}3\\1\\4\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\3\\4\end{bmatrix}\\
&= \begin{bmatrix}2\\0\\0\end{bmatrix}
\end{aligned}$$