Electrodynamics HW3
Ch2 - 7,11,23 (pg85)

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Problem 7. Consider a potential problem in the half-space defined by $z \geq 0$, with Derichlet boundary conditions on the plane $z = 0$ (and at infinity)

- Write down the appropriate Green function $G(\mathbf{x}, \mathbf{x}')$
- If the potential on the plane $z = 0$ is specified to be $\Phi = V$ inside a circle of radius a centered at the origin, and $\Phi = 0$ outside that circle, find an integral expression for the potential at the point P specified in terms of cylindrical coordinates (ρ, ϕ, z)
- Show that, along the axis of the circle ($\rho = 0$), the potential is given by

$$\Phi = V\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right)$$

- Show that at large distances ($\rho^2 + z^2 \gg a^2$) the potential can be expanded in a power series in $(\rho^2 + z^2)^{-1}$, and that the leading terms are

$$\Phi = \frac{Va^2}{2}\frac{z}{(\rho^2 + z^2)^{3/2}}\left[1 - \frac{3a^2}{4(\rho^2 + z^2)} + \frac{5(3\rho^2 a^2 + a^4)}{8(\rho^2 + z^2)^2} + \cdots\right]$$

  Verify the consistence of the solution with your previous results.

We know from the book's derivations using Green's theorem that

$$\Phi = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G(\mathbf{x}, \mathbf{x}')d^3x' + \frac{1}{4\pi}\oint_S \left[G(\mathbf{x}, \mathbf{x}')\frac{\partial \Phi}{\partial n'}(\mathbf{x}') - \Phi(\mathbf{x}')\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'})\right] da'$$

Where for Dirichlet boundary conditions, $G_D(\mathbf{x}, \mathbf{x}') = 0$ for $\mathbf{x}'$ on S ($\mathbf{x}' = (x, y, 0)$), and this reduces to

$$\Phi = \frac{1}{4\pi\epsilon_0}\int_V \rho(\mathbf{x}')G_D(\mathbf{x}, \mathbf{x}')d^3x' + \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\frac{\partial G_D(\mathbf{x}, \mathbf{x}')}{\partial n'})da'$$

We also know that a Green function is defined by

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} + F(\mathbf{x}, \mathbf{x}')$$
$$\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}')$$

and that inside the specified volume (in this case $z \geq 0$)

$$\nabla'^2 F(\mathbf{x}, \mathbf{x}') = 0$$

We are then looking for the function $F_D(\mathbf{x}, \mathbf{x}')$ that makes the Green function match the boundary conditions, but still satisfies the restrictions on their Laplacians.

To use the method of images, we notice that the first term of the Green function $1/|\mathbf{x}-\mathbf{x}'|$ is the potential of a point charge $q = 4\pi\epsilon_0$. We then search for a corresponding distribution of imaginary charges (outside of the given volume) whose potential will satisfy the boundary conditions when added to $1/|\mathbf{x} - \mathbf{x}'|$.

This will replace the problem with the given boundary conditions to an equivalent problem where we only need to compute the potential of the charge distribution. The imaginary charges "stand in" for the boundary because they produce the same effect, and by including them in the charge distribution they allow us to remove the boundary from the problem. We then set F as the potential of these imaginary charges, and this will properly complete the Green function.

Because this imaginary charge distribution would be outside of the volume, we are guaranteed its potential, given by F, will satisfy Laplace's equation inside the volume ($\nabla'^2 F(\mathbf{x}, \mathbf{x}') = 0$). We also know by definition of the Green function that $\nabla'^2 G(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}')$, so if we can find such a setup, the resulting Green function G will provide a unique solution to the original potential problem.

In cases of symmetrical boundaries, such as this flat sheet, this scheme may work out, and the imaginary charges needed to produce a usable F will hopefully be simple to find. But in general cases with potentially wild boundaries they may not be. It is important to note that the boundaries are what matter. Since we are only finding the Green function, the charge distribution for the associated problem plays no role. We only have the first term of this function $1/|\mathbf{x} - \mathbf{x}'|$, which represents the potential of one charge, to account for with image potentials from F. So it is how our placement of the image charges (giving image potential F) account for the boundary conditions alone that form the bulk of the problem for finding the full Green function.

In this case we imagine a plane with the single charge above it, and then, given the symmetry of a plane, it does not seem a large step to guess that another charge opposite the plane from the first may produce the desired result.

We write the position of the charge above the plane in Cartesian coordinates ($\mathbf{x} = (x, y, z)$), and if for each coordinate i, we call $\Delta x_i = (x_i - x'_i)$, our definition for G becomes

$$G_D(\mathbf{x}, \mathbf{x}') = \frac{1}{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}} + F_D(\mathbf{x}, \mathbf{x}')$$

We then use our guess and take F to represent the potential of a charge equal and opposite to the first, and therefore of charge $-4\pi\epsilon_0$ located at $(x, y, -z)$ such that

$$\begin{aligned} F_D(\mathbf{x}, \mathbf{x}') &= \frac{-1}{\sqrt{\Delta x^2 + \Delta y^2 + (z+z')^2}} \\ G_D(\mathbf{x}, \mathbf{x}') &= \boxed{\frac{1}{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}} - \frac{1}{\sqrt{\Delta x^2 + \Delta y^2 + (z+z')^2}}} \end{aligned}$$

Notice that by construction this Green function satisfies the boundary conditions

$$G_D(\mathbf{x} \ni (z=0), \mathbf{x}') = \frac{1}{\sqrt{\Delta x^2 + \Delta y^2 + (z')^2}} - \frac{1}{\sqrt{\Delta x^2 + \Delta y^2 + (z')^2}} = 0$$

It also satisfies the restrictions on the Laplacians as previously mentioned, and therefore represents the unique function needed for any charge distribution problem with this given boundary.

Now we turn our attention to finding the potential at some point in space $P = (\rho, \phi, z)$ given that inside a circle of radius a centered at the origin on the plane $z' = 0$ there is a fixed potential $\Phi = V$, and everywhere else on the same plane the potential is $\Phi = 0$.

Let us begin by noticing that we are still using Dirichlet boundaries by specifying the potential on the plane. There is also no specified charge in the volume, only a fixed potential on the surface to account for. Thus our original motivating equation reduces as before, and more so: $G_D(\mathbf{x}, \mathbf{x}') = 0$ for $\mathbf{x}'$ on S, and for $\mathbf{x}'$ in V, $\rho(\mathbf{x}') = 0$

$$\begin{aligned} \Phi &= \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') d^3x' + \frac{1}{4\pi} \oint_S \left[G(\mathbf{x}, \mathbf{x}') \frac{\partial \Phi}{\partial n'}(\mathbf{x}') - \Phi(\mathbf{x}') \frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n'}) \right] da' \\ \Phi &= \frac{1}{4\pi} \oint_S \Phi(\mathbf{x}') \frac{\partial G_D(\mathbf{x}, \mathbf{x}')}{\partial n'}) da' \end{aligned}$$

Then, since we are using the same planar boundary, even with the new potential specifications, the restrictions on the Green function itself have not changed, allowing us to reuse the function found previously:

$$\begin{aligned} G_D(\mathbf{x}, \mathbf{x}') &= \frac{1}{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}} - \frac{1}{\sqrt{\Delta x^2 + \Delta y^2 + (z+z')^2}} \\ \frac{\partial G_D}{\partial n'}(\mathbf{x}, \mathbf{x}') = \frac{\partial G_D}{\partial z'}(\mathbf{x}, \mathbf{x}') &= \frac{z - z'}{[\Delta x^2 + \Delta y^2 + \Delta z^2]^{3/2}} + \frac{z + z'}{[\Delta x^2 + \Delta y^2 + (z+z')^2]^{3/2}} \end{aligned}$$

(notice the chain rule causes a change in sign)

If we then switch to cylindrical coordinates

$$\begin{aligned}\Delta x^2 = (x-x')^2 = (\rho\cos\phi - \rho'\cos\phi')^2 &= \rho^2\cos^2\phi - 2\rho\rho'\cos\phi\cos\phi' + (\rho')^2\cos^2\phi' \\ \Delta y^2 = (y-y')^2 = (\rho\sin\phi - \rho'\sin\phi')^2 &= \rho^2\sin^2\phi - 2\rho\rho'\sin\phi\sin\phi' + (\rho')^2\sin^2\phi' \\ \Delta x^2 + \Delta y^2 &= \rho^2 + (\rho')^2 - 2\rho\rho'(\cos\phi\cos\phi' + \sin\phi\sin\phi') \\ &= \rho^2 + (\rho')^2 - 2\rho\rho'\cos(\phi-\phi')\end{aligned}$$

and thus

$$\begin{aligned}\frac{\partial G_D}{\partial z'}(\mathbf{x},\mathbf{x}') = &\frac{z-z'}{[\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2]^{3/2}} \\ &+ \frac{z+z'}{[\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')+(z+z')^2]^{3/2}}\end{aligned}$$

$$\begin{aligned}\Phi = \frac{1}{4\pi}\oint_S \Phi(\mathbf{x}')\Bigg(&\frac{z-z'}{[\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')+(z-z')^2]^{3/2}} \\ &+ \frac{z+z'}{[\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')+(z+z')^2]^{3/2}}\Bigg)da'\end{aligned}$$

Now, we are integrating over a surface that should contain the upper half space. So imagine a pillbox resting on the x-y plane that grows to infinity on the sides and upwards on the z-axis. The side components cancel as in any pillbox setup, the roof falls out because we specified $\Phi(\mathbf{x}' \ni (z \to \infty)) = 0$, and we are left with the surface on the plane $z' = 0$. Also, $\Phi(\mathbf{x}') = 0$ outside our circle of radius a, so all together our integral reduces to

$$\begin{aligned}\Phi &= \frac{1}{4\pi}\int_{(z'=0)} \frac{2z\Phi(\mathbf{x}')}{[\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')+z^2]^{3/2}}da' \\ \Phi &= \boxed{\frac{Vz}{2\pi}\int_0^{2\pi}\int_0^a \frac{\rho' d\rho' d\phi'}{[\rho^2+(\rho')^2-2\rho\rho'\cos(\phi-\phi')+z^2]^{3/2}}}\end{aligned}$$

If we then set $\rho = 0$ we can see what the potential would be along the z-axis

$$\begin{aligned}\Phi(\rho=0) &= \frac{Vz}{2\pi}\int_0^{2\pi}\int_0^a \frac{\rho' d\rho' d\phi'}{[(\rho')^2+z^2]^{3/2}} \\ &= Vz\int_0^a \frac{\rho' d\rho'}{[(\rho')^2+z^2]^{3/2}} \\ &= -Vz\left.\frac{1}{\sqrt{(\rho')^2+z^2}}\right|_{\rho'=0}^{a} \\ \Phi(\rho=0) &= \frac{-Vz}{\sqrt{a^2+z^2}} - \frac{-Vz}{\sqrt{z^2}} = \boxed{V\left(1-\frac{z}{\sqrt{a^2+z^2}}\right)}\end{aligned}$$

Although we really need to be careful here. $z/\sqrt{z^2}$ is actually $z/|z|$ which breaks us into two cases. Since we are above the x-y plane, $(z \geq 0) \rightarrow z/|z| = 1$ and this equation is fine. If on the other hand we wished to include points below the plane, we would have to adjust our Green function for the new boundary, and the given reduction would change: $(z < 0) \rightarrow z/|z| = -1$. But symmetry allows us to find the full solution with relative ease if we note that the potential at both $\pm z$ should be the same.

$$\Phi(\rho = 0) = \boxed{V\left(1 - \frac{|z|}{\sqrt{a^2 + z^2}}\right)}$$

Now, going back to the general potential in the upper half space, we consider the case where $(\rho^2 + z^2 \gg a^2)$. So let us look again at the formula we calculated

$$\begin{aligned}
\Phi &= \frac{Vz}{2\pi}\int_0^{2\pi}\int_0^a \frac{\rho' d\rho' d\phi'}{[\rho^2 + (\rho')^2 - 2\rho\rho'\cos(\phi - \phi') + z^2]^{3/2}} \\
&= \frac{Vz}{2\pi}\int_0^{2\pi}\int_0^a \left[\rho^2 + z^2 + (\rho')^2 - 2\rho\rho'\cos(\phi - \phi')\right]^{-3/2} \rho' d\rho' d\phi' \\
&= \frac{Vz}{2\pi}\int_0^{2\pi}\int_0^a \left[(\rho^2 + z^2)\left[1 + \frac{(\rho')^2 - 2\rho\rho'\cos(\phi - \phi')}{(\rho^2 + z^2)}\right]\right]^{-3/2} \rho' d\rho' d\phi' \\
\Phi &= \frac{Vz}{2\pi(\rho^2 + z^2)^{3/2}}\int_0^{2\pi}\int_0^a \left[1 + \frac{(\rho')^2 - 2\rho\rho'\cos(\phi - \phi')}{(\rho^2 + z^2)}\right]^{-3/2} \rho' d\rho' d\phi'
\end{aligned}$$

We then expand the integrand using the generalized binomial theorem.

$$(1 + x)^z = \sum_{k=0}^{\infty}\binom{z}{k}x^k = 1 + zx + \frac{z(z-1)}{2!}x^2 + \cdots$$

$$\left[1 + \frac{(\rho')^2 - 2\rho\rho'\cos(\phi - \phi')}{(\rho^2 + z^2)}\right]^{-3/2} = \sum_{k=0}^{\infty}\binom{-3/2}{k}\left(\frac{(\rho')^2 - 2\rho\rho'\cos(\phi - \phi')}{(\rho^2 + z^2)}\right)^k$$

This only converges properly if

$$\left|\frac{(\rho')^2 - 2\rho\rho'\cos(\phi - \phi')}{(\rho^2 + z^2)}\right| < 1$$

or equivalently

$$\left|(\rho')^2 - 2\rho\rho'\cos(\phi - \phi')\right| < (\rho^2 + z^2)$$

But since by the triangle inequality, and the fact that $\rho' \leq a$

$$\left|(\rho')^2 - 2\rho\rho'\cos(\phi - \phi')\right| \leq \left|(\rho')^2\right| + |2\rho\rho'\cos(\phi - \phi')| \leq a^2 + 2\rho a$$

Then $a^2 \ll \rho^2 + z^2$ implies we can assume the smaller $2\rho a$ term drops out, and

$$\left|(\rho')^2 - 2\rho\rho'\cos(\phi - \phi')\right| \leq a^2 + 2\rho a < (\rho^2 + z^2)$$

which verifies that the expansion is valid.

So expanding using generalized binomials:

$$\left[1+\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}\right]^{-3/2}=1-\frac{3}{2}\left(\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}\right)$$
$$+\frac{(-\frac{3}{2})(-\frac{3}{2}-1)}{2!}\left(\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}\right)^2-\cdots$$

We then integrate this term by term

$$\int_0^{2\pi}\int_0^a\left[1+\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}\right]^{-3/2}\rho' d\rho' d\phi'$$
$$=\int_0^{2\pi}\int_0^a \rho' d\rho' d\phi'$$
$$-\int_0^{2\pi}\int_0^a \frac{3}{2}\left(\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}\right)\rho' d\rho' d\phi'$$
$$+\int_0^{2\pi}\int_0^a \frac{(-\frac{3}{2})(-\frac{3}{2}-1)}{2!}\left(\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}\right)^2\rho' d\rho' d\phi'$$
$$-\cdots$$

Separating each term to make the integration easier to follow:

$$\int_0^{2\pi}\int_0^a \rho' d\rho' d\phi' = \boxed{\pi a^2}$$

$$\int_0^{2\pi}\int_0^a \frac{-3}{2}\left(\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}\right)\rho' d\rho' d\phi'$$

$$=\frac{-3}{2(\rho^2+z^2)}\int_0^{2\pi}\int_0^a\left[(\rho')^3-2\rho(\rho')^2\cos(\phi-\phi')\right]d\rho' d\phi'$$
$$=\frac{-3}{2(\rho^2+z^2)}\left[\frac{\pi a^4}{2}-2\rho\int_0^{2\pi}\int_0^a\left[(\rho')^2\cos(\phi-\phi')\right]d\rho' d\phi'\right]$$
$$=\frac{-3}{2(\rho^2+z^2)}\left[\frac{\pi a^4}{2}-\frac{2a\rho}{3}\int_0^{2\pi}\cos(\phi-\phi')d\phi'\right]$$
$$=\frac{-3}{2(\rho^2+z^2)}\left[\frac{\pi a^4}{2}+\frac{2a\rho}{3}\left[\sin(\phi-2\pi)-\sin(\phi)\right]\right]$$
$$=\frac{-3}{2(\rho^2+z^2)}\left[\frac{\pi a^4}{2}\right]=\boxed{\frac{-3\pi a^4}{4(\rho^2+z^2)}}$$

$$\int_0^{2\pi}\int_0^a \frac{(-\frac{3}{2})(-\frac{3}{2}-1)}{2!}\left(\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}\right)^2\rho' d\rho' d\phi'$$

$$= \frac{15}{8(\rho^2+z^2)^2}\int_0^{2\pi}\int_0^a \left[(\rho')^2-2\rho\rho'\cos(\phi-\phi')\right]^2\rho' d\rho' d\phi'$$
$$= \frac{15}{8(\rho^2+z^2)^2}\int_0^{2\pi}\int_0^a \left[(\rho')^4-4\rho(\rho')^3\cos(\phi-\phi')+4\rho^2(\rho')^2\cos^2(\phi-\phi')\right]\rho' d\rho' d\phi'$$
$$= \frac{15}{8(\rho^2+z^2)^2}\int_0^{2\pi}\int_0^a \left[(\rho')^5-4\rho(\rho')^4\cos(\phi-\phi')+4\rho^2(\rho')^3\cos^2(\phi-\phi')\right] d\rho' d\phi'$$
$$= \frac{3}{8(\rho^2+z^2)^2}\left[\frac{\pi a^6}{3}-4\rho\int_0^{2\pi}\int_0^a \left[(\rho')^4\cos(\phi-\phi')-\rho(\rho')^3\cos^2(\phi-\phi')\right] d\rho' d\phi'\right]$$
$$= \frac{15}{8(\rho^2+z^2)^2}\left[\frac{\pi a^6}{3}-4\rho\left(\frac{a^5}{5}\int_0^{2\pi}\cos(\phi-\phi')d\phi'-\rho\int_0^{2\pi}\int_0^a(\rho')^3\cos^2(\phi-\phi')d\rho' d\phi'\right)\right]$$
$$= \frac{15}{8(\rho^2+z^2)^2}\left[\frac{\pi a^6}{3}+4\rho^2\int_0^{2\pi}\int_0^a(\rho')^3\cos^2(\phi-\phi')d\rho' d\phi'\right]$$
$$= \frac{15}{8(\rho^2+z^2)^2}\left[\frac{\pi a^6}{3}+\rho^2 a^4\int_0^{2\pi}\cos^2(\phi-\phi')d\phi'\right] = \frac{15}{8(\rho^2+z^2)^2}\left[\frac{\pi a^6}{3}-\pi\rho^2 a^4\right]$$
$$= \frac{15}{8(\rho^2+z^2)^2}\left[\frac{\pi a^6-3\pi\rho^2 a^4}{3}\right] = \boxed{\frac{5\pi a^4(a^2-3\rho^2)}{8(\rho^2+z^2)^2}}$$

Adding these back together shows

$$\int_0^{2\pi}\int_0^a\left[1+\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}\right]^{-3/2}\rho' d\rho' d\phi'$$
$$= \pi a^2-\frac{3\pi a^4}{4(\rho^2+z^2)}+\frac{5\pi a^4(a^2-3\rho^2)}{8(\rho^2+z^2)^2}-\cdots = \boxed{\pi a^2\left[1-\frac{3a^2}{4(\rho^2+z^2)}+\frac{5a^2(a^2-3\rho^2)}{8(\rho^2+z^2)^2}-\cdots\right]}$$

And thus

$$\Phi = \frac{Vz}{2\pi(\rho^2+z^2)^{3/2}}\int_0^{2\pi}\int_0^a\left[1+\frac{(\rho')^2-2\rho\rho'\cos(\phi-\phi')}{(\rho^2+z^2)}\right]^{-3/2}\rho' d\rho' d\phi'$$
$$\Phi = \frac{Vz}{2\pi(\rho^2+z^2)^{3/2}}\left[\pi a^2\left[1-\frac{3a^2}{4(\rho^2+z^2)}+\frac{5a^2(a^2-3\rho^2)}{8(\rho^2+z^2)^2}-\cdots\right]\right]$$
$$\Phi = \boxed{\frac{Vza^2}{2(\rho^2+z^2)^{3/2}}\left[1-\frac{3a^2}{4(\rho^2+z^2)}+\frac{5a^2(a^2-3\rho^2)}{8(\rho^2+z^2)^2}-\cdots\right]}$$

which is what we were looking for.

Finally we check that this is consistent with what we got previously by setting $\rho = 0$:

$$
\begin{aligned}
\Phi(\rho=0) &= \frac{Vza^2}{2(z^2)^{3/2}}\left[1 - \frac{3a^2}{4(z^2)} + \frac{5a^2(a^2)}{8(z^2)^2} - \cdots\right] \\
&= \frac{Va^2}{2z^2}\left[1 - \frac{3a^2}{4z^2} + \frac{5a^4}{8z^4} - \cdots\right] \\
&= V\left[\frac{1}{2}\left(\frac{a^2}{z^2}\right) - \frac{3}{8}\left(\frac{a^2}{z^2}\right)^2 + \frac{5}{16}\left(\frac{a^2}{z^2}\right)^3 - \cdots\right] \\
&= V\left[1 - \left(1 - \frac{1}{2}\left(\frac{a^2}{z^2}\right) + \frac{3}{4\cdot 2}\left(\frac{a^2}{z^2}\right)^2 - \frac{5\cdot 3}{8\cdot 6}\left(\frac{a^2}{z^2}\right)^3 + \cdots\right)\right] \\
&= V\left[1 - \left(1 - \frac{1}{2}\left(\frac{a^2}{z^2}\right) + \frac{\frac{3}{4}}{2!}\left(\frac{a^2}{z^2}\right)^2 + \frac{\frac{-15}{8}}{3!}\left(\frac{a^2}{z^2}\right)^3 + \cdots\right)\right] \\
&= V\left[1 - \left(1 - \frac{1}{2}\left(\frac{a^2}{z^2}\right) + \frac{\frac{-1}{2}(\frac{-3}{2})}{2!}\left(\frac{a^2}{z^2}\right)^2 + \frac{\frac{-1}{2}(\frac{-3}{2})\frac{-5}{2}}{3!}\left(\frac{a^2}{z^2}\right)^3 + \cdots\right)\right] \\
\Phi(\rho=0) &= V\left[1 - \left(1 - \frac{1}{2}\left(\frac{a^2}{z^2}\right) + \frac{\frac{-1}{2}(\frac{-1}{2}-1)}{2!}\left(\frac{a^2}{z^2}\right)^2 + \frac{\frac{-1}{2}(\frac{-1}{2}-1)(\frac{-1}{2}-2)}{3!}\left(\frac{a^2}{z^2}\right)^3 + \cdots\right)\right]
\end{aligned}
$$

So if we recognize that this is a binomial expansion that converges when

$$
\left|\frac{a^2}{z^2}\right| < 1 \implies a^2 < z^2 \implies a < |z|
$$

We can substitute and find

$$
\begin{aligned}
\Phi(\rho=0) &= V\left[1 - \left(1 + \frac{a^2}{z^2}\right)^{-1/2}\right] \\
&= V\left[1 - \left(\frac{z^2+a^2}{z^2}\right)^{-1/2}\right] \\
\Phi(\rho=0) &= \boxed{V\left[1 - \frac{|z|}{\sqrt{a^2+z^2}}\right]}
\end{aligned}
$$

which matches up with our previous answer under the given conditions.

Problem 11. A line charge with linear charge density τ is placed parallel to, and a distance R away from, the axis of a conducting cylinder of radius b held at a fixed voltage such that the potential vanishes at infinity. Find

- The magnitude and position of the image charge(s)
- The potential at any point (expressed in polar coordinates with the origin at the axis of the cylinder and the direction from the origin to the line charge as the x axis), including the asymptotic form far from the cylinder

- The induced surface-charge density, and plot it as a function of angle for $R/b = 2, 4$ in units of $\tau/2\pi b$
- The force per unit length on the line charge

So let us first imagine the scenario. Put the axis of a cylinder of radius b along the z-axis, and regulate a constant voltage v on its surface. Add a line charge with charge density τ parallel to the cylinder and a distance R away from its axis. If we keep the surface of the cylinder fixed at this constant potential, and say that the potential drops off as you move away from the axis, we can imagine this as a boundary value problem as before.

So it seems easiest to work in cylindrical coordinates given the geometry of the problem. So as usual, call $\mathbf{x} = (\rho, \phi, z)$ the observation point, and $\mathbf{x}' = (\rho', \phi', z')$ the points from which our integration components are calculated.

It also seems reasonable to imagine a parallel imaginary line charge on the same radial vector as the given one, but inside the cylinder. This we will say is a distance R' away from the axis of the cylinder, and carries a charge density of τ'.

So we start by finding the potential of each of the line charges, real and imaginary, and then find the position and magnitude of the image line such that the boundary conditions are satisfied.

Construct a Gaussian cylinder of radius α and length L around our line charge. This makes the radial field the line produces constant along the curved surface of the Gaussian cylinder, and assures the field also points normal to it at every point. The top and bottom cancel for sufficiently large L, and thus Gauss' law gives us the magnitude of e the field at he radius α away from the line:

$$\begin{aligned}
\oint_S \mathbf{E} \cdot \mathbf{n} da &= \frac{1}{\epsilon_0} \int_V \rho(\mathbf{x}') d^3x \\
|\mathbf{E}| \oint_S da &= \frac{\tau L}{\epsilon_0} \\
|\mathbf{E}|\, 2\pi\alpha L &= \frac{\tau L}{\epsilon_0} \\
|\mathbf{E}| &= \frac{\tau}{2\pi\alpha\epsilon_0}
\end{aligned}$$

Where this was assumed to point radially away from the line charge.

Thus we use our definition of the potential, and if we call $\hat{\alpha}$ the radial vector pointing away from the line, we can dot our gradient to show that the potential at the same radius is

$$\begin{aligned}
\mathbf{E} &= -\nabla\Phi \\
\mathbf{E}\cdot\hat{\alpha} &= \frac{-\partial\Phi}{\partial\alpha} \\
\frac{\partial\Phi}{\partial\alpha} &= \frac{-\tau}{2\pi\alpha\epsilon_0} \\
\Phi &= \frac{-\tau}{2\pi\epsilon_0}\int\frac{d\alpha}{\alpha} = \frac{-\tau}{2\pi\epsilon_0}\ln\alpha
\end{aligned}$$

where we can have no integration constant because potential is arbitrarily set against some constant value. So imagine taking the integral from some arbitrarily defined point, say the surface of the cylinder, to infinity. This removes the constant of integration, and makes life a little easier.

If we construct a Gaussian cylinder of radius β around the image line in a similar fashion, we will come up with an equivalent formula for its potential a radius β away from it:

$$\Phi' = \frac{-\tau'}{2\pi\epsilon_0}\ln\beta$$

We then recall the distance formula in cylindrical coordinates as computed in the previous problem

$$\begin{aligned}
\Delta x^2 = (x-x')^2 = (\rho\cos\phi-\rho'\cos\phi')^2 &= \rho^2\cos^2\phi - 2\rho\rho'\cos\phi\cos\phi' + (\rho')^2\cos^2\phi' \\
\Delta y^2 = (y-y')^2 = (\rho\sin\phi-\rho'\sin\phi')^2 &= \rho^2\sin^2\phi - 2\rho\rho'\sin\phi\sin\phi' + (\rho')^2\sin^2\phi' \\
\Delta x^2 + \Delta y^2 &= \rho^2 + (\rho')^2 - 2\rho\rho'(\cos\phi\cos\phi' + \sin\phi\sin\phi') \\
&= \rho^2 + (\rho')^2 - 2\rho\rho'\cos(\phi-\phi')
\end{aligned}$$

$$\Delta\mathbf{x} = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} = \sqrt{\rho^2 + (\rho')^2 - 2\rho\rho'\cos(\phi-\phi') + (z-z')^2}$$

Since we imagined the line charge and image line charge to be on the same radial vector, their angles are the same, and we can imagine them on the x-axis so $\phi' = 0$. Since they are considered infinitely long, we can ignore the z components of their distances to the observation point by just picking the same one as the observation point itself ($[z' = z] \to [(z-z')^2 = 0]$). If we can find the charge density of the image line, we then have only the radial position of the line and image lines and to account for, which we called R and R' respectively.

So substitute in the appropriate values for α and β and add together the potentials to get

$$\begin{aligned}
\Phi_{Total} &= \frac{-\tau}{2\pi\epsilon_0}\ln\alpha + \frac{-\tau'}{2\pi\epsilon_0}\ln\beta \\
&= \frac{-1}{2\pi\epsilon_0}\left[\tau\ln\alpha + \tau'\ln\beta\right] \\
&= \frac{-1}{2\pi\epsilon_0}\ln(\alpha^{\tau}\beta^{\tau'}) \\
\Phi_{Total} &= \frac{-1}{2\pi\epsilon_0}\ln([\rho^2 + R^2 - 2R\rho\cos\phi]^{\tau/2}[\rho^2 + (R')^2 - 2R'\rho\cos\phi]^{\tau'/2})
\end{aligned}$$

So to make this a little more palatable let us take $\rho \to \infty$ where then $\Phi \to 0$. We will leave the ρ^2 terms that dominate in this case to isolate τ and τ':

$$\begin{aligned}
0 &= \frac{-1}{2\pi\epsilon_0}\ln((\rho^2)^{\tau/2}(\rho^2)^{\tau'/2}) \\
0 &= \ln(\rho^{\tau+\tau'}) \\
\tau + \tau' &= 0 \\
\tau' &= \boxed{-\tau}
\end{aligned}$$

Substituting this in gives

$$\begin{aligned}
\Phi_{Total} &= \frac{-1}{2\pi\epsilon_0}\ln(\alpha^{\tau}\beta^{\tau'}) \\
&= \frac{-\tau}{2\pi\epsilon_0}\ln\left(\frac{\alpha}{\beta}\right) \\
&= \frac{-\tau}{2\pi\epsilon_0}\ln\left(\frac{\sqrt{\rho^2 + R^2 - 2R\rho\cos\phi}}{\sqrt{\rho^2 + (R')^2 - 2R'\rho\cos\phi}}\right) \\
\Phi_{Total} &= \frac{-\tau}{4\pi\epsilon_0}\ln\left(\frac{\rho^2 + R^2 - 2R\rho\cos\phi}{\rho^2 + (R')^2 - 2R'\rho\cos\phi}\right)
\end{aligned}$$

and if we then use our main boundary condition $\rho = b \to \Phi = V$, we can isolate

$\cos\phi$ and take advantage of symmetry

$$
\begin{aligned}
V &= \frac{-\tau}{4\pi\epsilon_0}\ln\left(\frac{b^2+R^2-2Rb\cos\phi}{b^2+(R')^2-2R'b\cos\phi}\right)\\
\frac{-4\pi\epsilon_0 V}{\tau} &= \ln\left(\frac{b^2+R^2-2Rb\cos\phi}{b^2+(R')^2-2R'b\cos\phi}\right)\\
\exp\left(\frac{-4\pi\epsilon_0 V}{\tau}\right) &= \frac{b^2+R^2-2Rb\cos\phi}{b^2+(R')^2-2R'b\cos\phi}\\
\exp\left(\frac{-4\pi\epsilon_0 V}{\tau}\right)(b^2+(R')^2-2R'b\cos\phi) &= b^2+R^2-2Rb\cos\phi\\
\exp\left(\frac{-4\pi\epsilon_0 V}{\tau}\right)(b^2+(R')^2)-(b^2+R^2) &= \exp\left(\frac{-4\pi\epsilon_0 V}{\tau}\right)(2R'b\cos\phi)-2Rb\cos\phi\\
\exp\left(\frac{-4\pi\epsilon_0 V}{\tau}\right)(b^2+(R')^2)-(b^2+R^2) &= \left[\exp\left(\frac{-4\pi\epsilon_0 V}{\tau}\right)(2R')-2R\right]b\cos\phi\\
b\cos\phi &= \frac{\exp\left(\frac{-4\pi\epsilon_0 V}{\tau}\right)(b^2+(R')^2)-(b^2+R^2)}{\exp\left(\frac{-4\pi\epsilon_0 V}{\tau}\right)(2R')-2R}
\end{aligned}
$$

Then since the potential on the cylinder is the same at every angle we view the surface, this equation must be true for any angle ϕ we put into it.

Thus the numerator must be zero to account for when we look down the y-axis $(\phi = \pi/2) \to (\cos\phi = 0)$, and the numerator and denominator must be proportional to account for looking down the x-axis $(\phi = 0) \to (\cos\phi = 1)$. This means the denominator must be zero as well since both these values of ϕ must give the same result.

Note that this is only the case where we are looking at the potential on the surface of the cylinder alone. If we were accounting for the potential anywhere in space, there would not be the symmetry we are making use of, and we would have to account for the line and image charges separately.

But given our situation currently we proceed by setting the numerator and denominator to zero to find R'

$$\exp\left(\frac{-4\pi\epsilon_0 V}{\tau}\right)(b^2+(R')^2)-(b^2+R^2)=0 \qquad \exp\left(\frac{-4\pi\epsilon_0 V}{\tau}\right)(2R')-2R=0$$

$$\exp\left(\frac{-4\pi\epsilon_0 V}{\tau}\right)(b^2+(R')^2)=b^2+R^2 \qquad \exp\left(\frac{-4\pi\epsilon_0 V}{\tau}\right)=\frac{R}{R'}$$

$$\begin{aligned}
\frac{R}{R'}(b^2+(R')^2) &= b^2+R^2\\
\frac{b^2}{R'}+R' &= \frac{b^2}{R}+R\\
b^2\left(\frac{1}{R'}-\frac{1}{R}\right) &= R-R'\\
b^2\left(\frac{R-R'}{RR'}\right) &= R-R'\\
b^2 &= RR'\\
R' &= \boxed{\frac{b^2}{R}}
\end{aligned}$$

We then have the full solution to the potential by substituting in the new found values. I now drop the "Total" subscript and flip the numerator and denominator in the log with the negative constant, as I should have done previously.

$$\Phi=\boxed{\frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{\rho^2+(\frac{b^2}{R})^2-\frac{2b^2}{R}\rho\cos\phi}{\rho^2+R^2-2R\rho\cos\phi}\right)}$$

If we then set this up to expand ln in a power series

$$\begin{aligned}
\Phi &= \frac{\tau}{4\pi\epsilon_0}\ln\left(1-1+\frac{\rho^2+(\frac{b^2}{R})^2-\frac{2b^2}{R}\rho\cos\phi}{\rho^2+R^2-2R\rho\cos\phi}\right)\\
&= \frac{\tau}{4\pi\epsilon_0}\ln\left(1+\frac{\rho^2+(\frac{b^2}{R})^2-\frac{2b^2}{R}\rho\cos\phi}{\rho^2+R^2-2R\rho\cos\phi}-\frac{\rho^2+R^2-2R\rho\cos\phi}{\rho^2+R^2-2R\rho\cos\phi}\right)\\
&= \frac{\tau}{4\pi\epsilon_0}\ln\left(1+\frac{(\frac{b^2}{R})^2-R^2-\left(\frac{b^2}{R}-R\right)2\rho\cos\phi}{\rho^2+R^2-2R\rho\cos\phi}\right)\\
\Phi &= \frac{\tau}{4\pi\epsilon_0}\ln\left(1+\frac{(b^4-R^4)\frac{1}{R^2}+(R^2-b^2)\frac{2\rho}{R}\cos\phi}{\rho^2+R^2-2R\rho\cos\phi}\right)
\end{aligned}$$

and taking $\rho \gg R$, and $\rho \gg b$, we can approximate the potential very far away from the system and drop all but the largest terms

$$\Phi = \frac{\tau}{4\pi\epsilon_0}\ln\left(1+\frac{(b^4-R^4)\frac{1}{R^2}+(R^2-b^2)\frac{2\rho}{R}\cos\phi}{\rho^2+R^2-2R\rho\cos\phi}\right)$$

$$\Phi = \frac{\tau}{4\pi\epsilon_0}\ln\left(1+\frac{(R^2-b^2)\frac{2\rho}{R}\cos\phi}{\rho^2}\right)$$

$$\Phi = \frac{\tau}{4\pi\epsilon_0}\ln\left(1+\frac{(R^2-b^2)2\cos\phi}{R\rho}\right)$$

We then use the McLauren series expansion of $\ln(1+x)$

$$\ln(1+x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \cdots$$

to show that for $\ln(1+x) \approx x$ we can use the simpler approximation

$$\Phi = \boxed{\frac{\tau}{2\pi\epsilon_0}\left(\frac{(R^2-b^2)\cos\phi}{R\rho}\right)}$$

To find the surface charge density induced on the cylinder we again make use of Gauss' law.

$$\oint_S \mathbf{E}\cdot\hat{\mathbf{n}}da = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{x}')d^3x'$$

where with a cylindrical Gaussian surface of length L that we shrink down to the radius of the cylinder b, this becomes

$$\oint_S \mathbf{E}\cdot\hat{\mathbf{r}}da = \frac{1}{\epsilon_0}\int_V \sigma(\mathbf{x}')d^3x'$$

$$|\mathbf{E}|\oint_S da = \frac{\sigma 2\pi bL}{\epsilon_0}$$

$$|\mathbf{E}| = \frac{\sigma}{\epsilon_0}$$

$$\sigma = |\mathbf{E}|\,\epsilon_0$$

Then since the electrical field induced will point in a radial direction, we can use the fact that $\mathbf{E} = -\nabla\Phi$ and dot both sides with $\hat{\mathbf{r}}$ to get

$$\mathbf{E}\cdot\hat{\mathbf{r}} = |\mathbf{E}| = -\frac{\partial\Phi}{\partial\hat{\mathbf{r}}} = -\frac{\partial\Phi}{\partial\rho}$$

So then I substitute in our previous formula where

$$\Phi = \boxed{\frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{\rho^2+(\frac{b^2}{R})^2-\frac{2b^2}{R}\rho\cos\phi}{\rho^2+R^2-2R\rho\cos\phi}\right)}$$

to compute

$$
\begin{aligned}
\sigma = |\mathbf{E}|\,\epsilon_0 &= -\epsilon_0 \frac{\partial \Phi}{\partial \rho} \\
&= -\epsilon_0 \frac{\partial}{\partial \rho}\left[\frac{\tau}{4\pi\epsilon_0}\ln\left(\frac{\rho^2 + (\frac{b^2}{R})^2 - \frac{2b^2}{R}\rho\cos\phi}{\rho^2 + R^2 - 2R\rho\cos\phi}\right)\right] \\
&= \frac{\tau}{4\pi}\frac{\partial}{\partial \rho}\left[\ln\left(\frac{\rho^2 + R^2 - 2R\rho\cos\phi}{\rho^2 + (\frac{b^2}{R})^2 - \frac{2b^2}{R}\rho\cos\phi}\right)\right] \\
&= \frac{\tau}{4\pi}\frac{\partial}{\partial \rho}\left[\ln\left(\rho^2 + R^2 - 2R\rho\cos\phi\right) - \ln\left(\rho^2 + \left(\frac{b^2}{R}\right)^2 - \frac{2b^2}{R}\rho\cos\phi\right)\right] \\
\sigma &= \frac{\tau}{4\pi}\left[\frac{2\rho - 2R\cos\phi}{\rho^2 + R^2 - 2R\rho\cos\phi} - \frac{2\rho - \frac{2b^2}{R}\cos\phi}{\rho^2 + \left(\frac{b^2}{R}\right)^2 - \frac{2b^2}{R}\rho\cos\phi}\right]
\end{aligned}
$$

Then since we are evaluating on the surface of the cylinder with radius b, we can take $\rho = b$ to find

$$
\begin{aligned}
\sigma &= \frac{\tau}{4\pi}\left[\frac{2b - 2R\cos\phi}{b^2 + R^2 - 2Rb\cos\phi} - \frac{2b - 2\frac{b^2}{R}\cos\phi}{b^2 + \left(\frac{b^2}{R}\right)^2 - 2\frac{b^3}{R}\cos\phi}\right] \\
&= \frac{\tau}{4\pi}\left[\frac{R^2(2\frac{b}{R^2} - \frac{2}{R}\cos\phi)}{R^2(\left(\frac{b}{R}\right)^2 + 1 - 2\frac{b}{R}\cos\phi)} - \frac{b^2(\frac{2}{b} - \frac{2}{R}\cos\phi)}{b^2(1 + \left(\frac{b}{R}\right)^2 - 2\frac{b}{R}\cos\phi)}\right] \\
&= \frac{\tau}{4\pi}\left[\frac{(2\frac{b}{R^2} - \frac{2}{R}\cos\phi) - (\frac{2}{b} - \frac{2}{R}\cos\phi)}{1 + \left(\frac{b}{R}\right)^2 - 2\frac{b}{R}\cos\phi}\right] \\
&= \frac{\tau}{4\pi}\left[\frac{2\frac{b}{R^2} - \frac{2}{b}}{1 + \left(\frac{b}{R}\right)^2 - 2\frac{b}{R}\cos\phi}\right] \\
&= \frac{\tau}{2\pi}\left[\frac{(b^2 - R^2)/R^2 b}{1 + \left(\frac{b}{R}\right)^2 - 2\frac{b}{R}\cos\phi}\right] \\
\sigma &= \boxed{\frac{\tau}{2\pi b}\left[\frac{b^2 - R^2}{b^2 + R^2 - 2bR\cos\phi}\right]}
\end{aligned}
$$

To investigate ratios of R/b we can reformulate this as

$$
\sigma = \frac{\tau}{2\pi b}\left[\frac{1 - (R/b)^2}{1 + (R/b)^2 - 2(R/b)\cos\phi}\right]
$$

And then for $R/b = 2$

$$
\sigma = \frac{\tau}{2\pi b}\left[\frac{1 - 4}{1 + 4 - 4\cos\phi}\right] = \boxed{\frac{\tau}{2\pi b}\left[\frac{-3}{5 - 4\cos\phi}\right]}
$$

And for $R/b = 4$

$$\sigma = \frac{\tau}{2\pi b}\left[\frac{1-16}{1+16-8\cos\phi}\right] = \boxed{\frac{\tau}{2\pi b}\left[\frac{-15}{17-8\cos\phi}\right]}$$

So now we seek to find the force per unit length on the line charge from the induced charge on the cylinder. We do this by again utilizing Gauss' law (with a cylindrical Gaussian surface of radius d and length L) to find the field a distance d away from the image line we are using to represent the charge on the cylinder, noting the field is again normal to this surface, and the charge density on the image is $\tau' = -\tau$ making the field point towards it.

$$\begin{aligned}
\oint_S \mathbf{E}\cdot\hat{\mathbf{n}}da &= \frac{1}{\epsilon_0}\int_V \rho(\mathbf{x}')d^3x' \\
|\mathbf{E}|\oint_S da &= \frac{\tau'}{\epsilon_0} \\
|\mathbf{E}|\,2\pi dL &= \frac{\tau L}{\epsilon_0} \\
|\mathbf{E}| &= \frac{\tau}{2\pi d\epsilon_0} \\
\mathbf{E} &= \frac{-\tau}{2\pi d\epsilon_0}\hat{\rho}
\end{aligned}$$

Thus since we are trying to find the force on the line charge which is a distance $R - R'$ away from the image charge, we can use the fact that we computed $R' = b^2/R$ to substitute $d = R - (b^2/R)$ and find

$$\mathbf{E} = \frac{-\tau}{2\pi\epsilon_0(R-(b^2/R))}\hat{\rho} = \frac{-\tau R}{2\pi\epsilon_0(R^2-b^2)}\hat{\rho}$$

Then since the force is $\mathbf{F} = q\mathbf{E}$ and the charge of the line is $q = \tau L$

$$\mathbf{F} = \frac{-\tau^2 RL}{2\pi\epsilon_0(R^2-b^2)}\hat{\rho}$$

and for charge per length, we just divide by the length of our calculation to find

$$\frac{\mathbf{F}}{L} = \boxed{\frac{-\tau^2 R}{2\pi\epsilon_0(R^2-b^2)}\hat{\rho}}$$

Problem 23. A hollow cube has conducting walls defined by six planes $x = 0, y = 0, z = 0$ and $x = a, y = a, z = a$. The walls $z = 0$ and $z = a$ are held at a constant potential V. The other four sides are at zero potential.

- Find the potential $\Phi(x, y, z)$ at any point in the cube

- Evaluate the potential at the center of the cube numerically, accurate to three significant figures. How many terms in the series is it necessary to keep in order to attain this accuracy? Compare your numerical result with the average value of the potential on the walls. See Problem 2.28
- Find the surface-charge density on the wall $z = a$

So imagining the box with a corner at the origin and sides of length a, to find the potential we begin with the Poisson equation. This reduces to Laplace because there is no charge inside the box, and working in Cartesian we can expand

$$\nabla^2\Phi(x,y,z) = -\rho/\epsilon_0 = 0$$
$$\frac{\partial\Phi}{\partial x^2} + \frac{\partial\Phi}{\partial y^2} + \frac{\partial\Phi}{\partial z^2} = 0$$

So using the method of separation of variables, we assume that we can split Φ into a product of three functions, $X(x)$,$Y(y)$, and $Z(z)$, each depending respectively on one of the coordinates used in Φ. Thus we can break up Poisson's equation using $\Phi = X(x)Y(y)Z(z)$ and divide through again by the same $\Phi = X(x)Y(y)Z(z)$

$$\frac{\partial}{\partial x^2}\left[X(x)Y(y)Z(z)\right] + \frac{\partial}{\partial y^2}\left[X(x)Y(y)Z(z)\right] + \frac{\partial}{dz^2}\left[X(x)Y(y)Z(z)\right] = 0$$
$$Y(y)Z(z)\frac{\partial}{\partial x^2}X(x) + X(x)Z(z)\frac{\partial}{\partial x^2}Y(y) + X(x)Y(y)\frac{\partial}{\partial x^2}Z(z) = 0$$
$$\frac{1}{X(x)}\frac{\partial}{\partial x^2}X(x) + \frac{1}{Y(y)}\frac{\partial}{\partial x^2}Y(y) + \frac{1}{Z(z)}\frac{\partial}{\partial x^2}Z(z) = 0$$

We are now left with three equations that depend on only one coordinate, and we can collapse our partials to full fledged derivatives. Then the terms in the sum cannot effect each other and therefore each must be a constant. Moreover when all are added together, they must cancel each other out.

Imagine one term was variable depending on that coordinate. For example $X(x)$. Then I could evaluate it at different coordinates, its value would change, and therefore sum would change.

Eg. If $\exists x_0 \neq x_1 \ni X(x_0) \neq X(x_1)$ then

$$\begin{aligned}0 = \frac{1}{X(x_0)}\frac{\partial}{\partial x^2}X(x_0) + \frac{1}{Y(y_0)}\frac{\partial}{\partial x^2}Y(y_0) + \frac{1}{Z(z_0)}\frac{\partial}{\partial x^2}Z(z_0)\\ \neq \frac{1}{X(x_1)}\frac{\partial}{\partial x^2}X(x_1) + \frac{1}{Y(y_0)}\frac{\partial}{\partial x^2}Y(y_0) + \frac{1}{Z(z_0)}\frac{\partial}{\partial x^2}Z(z_0) = 0\end{aligned}$$

Implying $0 \neq 0$

But this makes no sense. So each term in the sum must have the same value irregardless of where we evaluate it. (Eg. $\forall x_0, x_1,\ X(x_0) = X(x_1)$)

So with our new found constant terms, let us give the following names to their constant values:

$$\frac{1}{X(x)}\frac{\partial}{\partial x^2}X(x) = j$$
$$\frac{1}{Y(y)}\frac{\partial}{\partial y^2}Y(y) = k$$
$$\frac{1}{Z(z)}\frac{\partial}{\partial z^2}Z(z) = -(j+k)$$

Our equation is then reduced to ordinary differential equations

$$\frac{\partial}{\partial x^2}X(x) = jX(x)$$
$$\frac{\partial}{\partial y^2}Y(y) = kY(y)$$
$$\frac{\partial}{\partial z^2}Z(z) = -(j+k)Z(z)$$

where for some other constants $\alpha_n, \beta_n, \gamma_n$ for $n \in \mathbb{N}$ we have solutions of the form

$$\begin{aligned}
\frac{\partial}{\partial x^2}X(x) &= jX(x) \\
X(x) &= \sum_l \alpha_l e^{\pm x\sqrt{j}} = \alpha_{(+)}e^{x\sqrt{j}} + \alpha_{(-)}e^{-x\sqrt{j}} \\
\frac{\partial}{\partial y^2}Y(y) &= kY(y) \\
Y(y) &= \sum_l \beta_l e^{\pm y\sqrt{k}} = \beta_{(+)}e^{y\sqrt{k}} + \beta_{(-)}e^{-y\sqrt{k}} \\
\frac{\partial}{\partial z^2}Z(z) &= -(j+k)Z(z) \\
Z(z) &= \sum_l \gamma_l e^{\pm iz\sqrt{j+k}} = \gamma_{(+)}e^{iz\sqrt{j+k}} + \gamma_{(-)}e^{-iz\sqrt{j+k}}
\end{aligned}$$

We then try to find the constants where the solution satisfies the given boundary conditions for the resulting product $\Phi = XYZ$.

So let us begin by looking at

$$X(x) = \alpha_{(+)}e^{x\sqrt{j}} + \alpha_{(-)}e^{-x\sqrt{j}}$$

and recall that we are given $\Phi(x=0) = 0$ and $\Phi(x=a) = 0$. Then we also know that $X(0) = 0$ and $X(a) = 0$.

$X(0) = 0$ implies:

$$X(0) = \alpha_{(+)} + \alpha_{(-)} = 0$$
$$\alpha_{(-)} = -\alpha_{(-)}$$

so that we can simplify calling $\alpha = \alpha_+ = -\alpha_-$, and thus

$$X(x) = \alpha \left(e^{x\sqrt{j}} - e^{-x\sqrt{j}} \right)$$

Then $X(a) = 0$ implies

$$X(a) = \alpha \left(e^{a\sqrt{j}} - e^{-a\sqrt{j}} \right) = 0$$
$$e^{a\sqrt{j}} = e^{-a\sqrt{j}}$$
$$e^{2a\sqrt{j}} = 1$$

It is at this point we must be careful. Naively using the real logarithm would let us simply remove the exponential to find either a or $\sqrt{j}$ must be 0. But a is not zero by supposition, and j could be less than zero making $\sqrt{j}$ complex. In this case we cannot use the real logarithm, but find instead that $\sqrt{j} = im\pi/a$ for any $m \in \mathbb{Z}$ would also work (given Euler's ubiquitous formula: $e^{i\theta} = \cos\theta + i\sin\theta$)

$$e^{2a(im\pi/a)} = e^{i2m\pi} = \cos(2m\pi) + i\sin(2m/\pi) = 1$$

Since this is a more general solution, (ie. it includes the case where $j \geq 0$) we proceed by using it rather than assuming the degenerate case where j is non-negative.

Thus $j = -(m\pi/a)^2$ and all together we have

$$X(x) = \alpha \sum_{m \text{ odd}} \left(e^{im\pi x/a} - e^{-im\pi x/a} \right) = i2\alpha \sum_{m \text{ odd}} \sin\left(\frac{m\pi x}{a}\right)$$

and a similar process will show $\beta = \beta_{(+)} = -\beta_{(-)}$ and $\sqrt{k} = in\pi/a$ for any $n \in \mathbb{Z}$ giving

$$Y(y) = \beta \sum_{n \text{ odd}} \left(e^{in\pi y/a} - e^{-in\pi y/a} \right) = i2\beta \sum_{n \text{ odd}} \sin\left(\frac{n\pi y}{a}\right)$$

This leaves us to find $Z(z)$ which has the more interesting boundary conditions.

We first replace $i\sqrt{j+k} = i\sqrt{-(m\pi/a)^2 - (n\pi/a)^2} = (\pi/a)\sqrt{m^2+n^2}$. Then since $\Phi(z = 0) = V$ does not necessarily imply $Z(0) = V$, we must instead use our newfound X and Y along with our previously agreed upon formula $\Phi = XYZ$ to work out the rest of the constants, collapsing our current ones into $\gamma_{(+)}$ and $\gamma_{(-)}$, which I should rename, but I will abuse notation for fear of having to pick another letter to keep track of, or having to use more tildes.

$$\Phi(x,y,z) = \sum_{m,n \text{ odd}} \left[i2\alpha \sin\left(\frac{m\pi x}{a}\right)\right]\left[i2\beta \sin\left(\frac{n\pi y}{a}\right)\right]\left[\gamma_{(+)}e^{z(\pi/a)\sqrt{m^2+n^2}} + \gamma_{(-)}e^{-z(\pi/a)\sqrt{m^2+n^2}}\right]$$
$$\Phi(x,y,z) = \sum_{m,n \text{ odd}} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\left[\gamma_{(+)}e^{z(\pi/a)\sqrt{m^2+n^2}} + \gamma_{(-)}e^{-z(\pi/a)\sqrt{m^2+n^2}}\right]$$

Then we use $\Phi(x,y,0) = V$ to find

$$V = \sum_{m,n \text{ odd}} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\left[\gamma_{(+)} + \gamma_{(-)}\right]$$

This sets us up to use Fourier's trick, where we let $\tilde{m}, \tilde{n} \in \mathbb{Z}$, multiply through by $\sin(\tilde{m}\pi x/a)$ and $\sin(\tilde{n}\pi y/a)$, and integrate with respect to x and y. This takes advantage of the orthogonality condition

$$\int_0^a \sin(p\theta)\sin(\tilde{p}\theta)d\theta = \begin{cases} 0 & \text{if } p \neq \tilde{p}, \\ \frac{1}{4p}(2pa - \sin(2pa)) & \text{if } p = \tilde{p} \end{cases}$$

So starting with multiplying through we get

$$V = \sum_{m,n \text{ odd}} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\left[\gamma_{(+)} + \gamma_{(-)}\right]$$
$$V\sin\left(\frac{\tilde{m}\pi x}{a}\right)\sin\left(\frac{\tilde{n}\pi y}{a}\right) = \sum_{m,n \text{ odd}} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{\tilde{m}\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{\tilde{n}\pi y}{a}\right)\left[\gamma_{(+)} + \gamma_{(-)}\right]$$

We then integrate and first evaluate the left side

$$\begin{aligned} V\int_0^a\int_0^a \sin\left(\frac{\tilde{m}\pi x}{a}\right)\sin\left(\frac{\tilde{n}\pi y}{a}\right)dxdy &= V\int_0^a \sin\left(\frac{\tilde{m}\pi x}{a}\right)dx\int_0^a \sin\left(\frac{\tilde{n}\pi y}{a}\right)dy \\ &= \frac{Va^2}{\pi^2\tilde{m}\tilde{n}}[\cos(\tilde{m}\pi) - 1][\cos(\tilde{n}\pi) - 1] \\ &= \frac{Va^2}{\pi^2\tilde{m}\tilde{n}}[(-1)^{\tilde{m}} - 1][(-1)^{\tilde{n}} - 1] \end{aligned}$$

followed by matching and evaluating the integral of the right side

$$= \sum_{m,n \text{ odd}} \left[\gamma_{(+)} + \gamma_{(-)}\right] \int_0^a \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{\tilde{m}\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{\tilde{n}\pi y}{a}\right) dxdy$$

$$= \sum_{m,n \text{ odd}} \left[\gamma_{(+)} + \gamma_{(-)}\right] \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{\tilde{m}\pi x}{a}\right) dx \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{\tilde{n}\pi y}{a}\right) dy$$

$$= \sum_{m,n \text{ odd}} \frac{a^2}{16\pi^2 mn} \left[\gamma_{(+)} + \gamma_{(-)}\right] [(2\pi m) - \sin(2\pi m)][(2\pi n) - \sin(2\pi n)]$$

$$= \sum_{m,n \text{ odd}} \frac{a^2}{16\pi^2 mn} \left[\gamma_{(+)} + \gamma_{(-)}\right] (2\pi m)(2\pi n)$$

$$= \frac{a^2}{4} \left[\gamma_{(+)} + \gamma_{(-)}\right]$$

Where together we get

$$\frac{Va^2}{\pi^2 \tilde{m}\tilde{n}} [(-1)^{\tilde{m}} - 1][(-1)^{\tilde{n}} - 1] = \frac{a^2}{4} \left[\gamma_{(+)} + \gamma_{(-)}\right]$$

This has $\left[\gamma_{(+)} + \gamma_{(-)}\right]$, and thus Φ, collapsing for either m or n even, so to again avoid a degenerate case we take both to be odd. We can also remove the tildes because of our orthogonality condition that required us to use the case where $\tilde{m} = m$ and $\tilde{n} = n$.

$$\frac{16V}{\pi^2 mn} = \gamma_{(+)} + \gamma_{(-)}$$

We can also use the same Fourier trick with our other boundary condition $\Phi(x, y, a) = V$ to get a similar answer:

$$V = \sum_{m,n \text{ odd}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{a}\right) \left[\gamma_{(+)} e^{a(\pi/a)\sqrt{m^2+n^2}} + \gamma_{(-)} e^{-a(\pi/a)\sqrt{m^2+n^2}}\right]$$

$$\frac{16V}{\pi^2 mn} = \gamma_{(+)} e^{\pi\sqrt{m^2+n^2}} + \gamma_{(-)} e^{-\pi\sqrt{m^2+n^2}}$$

and with these two equations we have a linear system that we can solve:

$$\begin{aligned} \gamma_{(+)} + \gamma_{(-)} &= \frac{16V}{\pi^2 mn} \\ \gamma_{(+)} e^{\pi\sqrt{m^2+n^2}} + \gamma_{(-)} e^{-\pi\sqrt{m^2+n^2}} &= \frac{16V}{\pi^2 mn} \end{aligned}$$

Multiply the first equation by $\exp\left(\pi\sqrt{m^2+n^2}\right)$ and subtract it from the second

$$
\begin{aligned}
\gamma_{(-)}\left(e^{-\pi\sqrt{m^2+n^2}} - e^{\pi\sqrt{m^2+n^2}}\right) &= \frac{16V}{\pi^2 mn}\left(1 - e^{\pi\sqrt{m^2+n^2}}\right) \\
\gamma_{(-)}\left(e^{-\pi\sqrt{m^2+n^2}} - e^{\pi\sqrt{m^2+n^2}}\right) &= \frac{16V}{\pi^2 mn}\left(e^{\pi\sqrt{m^2+n^2}-\pi\sqrt{m^2+n^2}} - e^{\pi\sqrt{m^2+n^2}}\right) \\
\gamma_{(-)}\left(e^{-\pi\sqrt{m^2+n^2}} - e^{\pi\sqrt{m^2+n^2}}\right) &= \frac{16V}{\pi^2 mn}\left(e^{-\pi\sqrt{m^2+n^2}} - 1\right)e^{\pi\sqrt{m^2+n^2}} \\
\gamma_{(-)}\left(e^{-\pi\sqrt{m^2+n^2}} - e^{\pi\sqrt{m^2+n^2}}\right)e^{-\pi\sqrt{m^2+n^2}} &= \frac{16V}{\pi^2 mn}\left(e^{-\pi\sqrt{m^2+n^2}} - 1\right) \\
\gamma_{(-)}\left(e^{-2\pi\sqrt{m^2+n^2}} - 1\right) &= \frac{16V}{\pi^2 mn}\left(e^{-\pi\sqrt{m^2+n^2}} - 1\right) \\
\gamma_{(-)}\left(e^{-\pi\sqrt{m^2+n^2}} + 1\right)\left(e^{-\pi\sqrt{m^2+n^2}} - 1\right) &= \frac{16V}{\pi^2 mn}\left(e^{-\pi\sqrt{m^2+n^2}} - 1\right) \\
\gamma_{(-)} &= \frac{16V}{\pi^2 mn\left(1 + e^{-\pi\sqrt{m^2+n^2}}\right)}
\end{aligned}
$$

Then multiply it by $\exp\left(-\pi\sqrt{m^2+n^2}\right)$ instead and subtract it from the second, reusing the method from the last equation to simplify the result

$$
\begin{aligned}
\gamma_{(+)}\left(e^{\pi\sqrt{m^2+n^2}} - e^{-\pi\sqrt{m^2+n^2}}\right) &= \frac{16V}{\pi^2 mn}\left(1 - e^{-\pi\sqrt{m^2+n^2}}\right) \\
\gamma_{(+)}\left(e^{\pi\sqrt{m^2+n^2}} - e^{-\pi\sqrt{m^2+n^2}}\right) &= \frac{16V}{\pi^2 mn}\left(e^{\pi\sqrt{m^2+n^2}} - 1\right)e^{-\pi\sqrt{m^2+n^2}} \\
\gamma_{(+)}\left(e^{2\pi\sqrt{m^2+n^2}} - 1\right) &= \frac{16V}{\pi^2 mn}\left(e^{\pi\sqrt{m^2+n^2}} - 1\right) \\
\gamma_{(+)} &= \frac{16V}{\pi^2 mn\left(1 + e^{\pi\sqrt{m^2+n^2}}\right)}
\end{aligned}
$$

This allows us to fill in the last constants

$$
\begin{aligned}
\Phi(x,y,z) &= \sum_{m,n\text{ odd}} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\left[\gamma_{(+)}e^{(z\pi/a)\sqrt{m^2+n^2}} + \gamma_{(-)}e^{-(z\pi/a)\sqrt{m^2+n^2}}\right] \\
&= \sum_{m,n\text{ odd}} \left(\frac{16V}{\pi^2 mn}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\left[\frac{e^{(z\pi/a)\sqrt{m^2+n^2}}}{1+e^{\pi\sqrt{m^2+n^2}}} + \frac{e^{-(z\pi/a)\sqrt{m^2+n^2}}}{1+e^{-\pi\sqrt{m^2+n^2}}}\right] \\
&= \sum_{m,n\text{ odd}} \left(\frac{16V}{\pi^2 mn}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\left[\frac{e^{(z\pi/a)\sqrt{m^2+n^2}}}{1+e^{\pi\sqrt{m^2+n^2}}} + \frac{e^{-(z\pi/a)\sqrt{m^2+n^2}}e^{\pi\sqrt{m^2+n^2}}}{1+e^{\pi\sqrt{m^2+n^2}}}\right] \\
\Phi(x,y,z) &= \boxed{\sum_{m,n\text{ odd}} \left(\frac{16V}{\pi^2 mn}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\left[\frac{e^{(z/a)\pi\sqrt{m^2+n^2}} + e^{(1-(z/a))\pi\sqrt{m^2+n^2}}}{1+e^{\pi\sqrt{m^2+n^2}}}\right]}
\end{aligned}
$$

Now we can evaluate this at the center of the cube:

$$
\begin{aligned}
\Phi(a/2,a/2,a/2) &= \sum_{m,n \text{ odd}} \left(\frac{16V}{\pi^2 mn}\right) \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right) \left[\frac{e^{(\pi/2)\sqrt{m^2+n^2}} + e^{(\pi/2)\sqrt{m^2+n^2}}}{1+e^{\pi\sqrt{m^2+n^2}}}\right] \\
&= \sum_{m,n \text{ odd}} \left(\frac{32V}{\pi^2 mn}\right) \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right) \left[\frac{e^{(\pi/2)\sqrt{m^2+n^2}}}{1+e^{\pi\sqrt{m^2+n^2}}}\right] \\
&= \sum_{m,n \text{ odd}} \left(\frac{32V}{\pi^2 mn}\right) \left[\frac{\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)}{e^{-(\pi/2)\sqrt{m^2+n^2}} + e^{(\pi/2)\sqrt{m^2+n^2}}}\right] \\
&= \sum_{m,n \text{ odd}} \left(\frac{16V}{\pi^2 mn}\right) \left[\frac{\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)}{\cosh\left((\pi/2)\sqrt{m^2+n^2}\right)}\right] \\
&= \left(\frac{16V}{\pi^2}\right)\left[\frac{\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)}{\cosh\left((\pi/2)\sqrt{1^2+1^2}\right)}\right] + \left(\frac{16V}{3\pi^2}\right)\left[\frac{\sin\left(\frac{3\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)}{\cosh\left((\pi/2)\sqrt{3^2+1^2}\right)}\right] \\
&+ \left(\frac{16V}{3\pi^2}\right)\left[\frac{\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)}{\cosh\left((\pi/2)\sqrt{1^2+3^2}\right)}\right] + \left(\frac{16V}{5\pi^2}\right)\left[\frac{\sin\left(\frac{5\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)}{\cosh\left((\pi/2)\sqrt{5^2+1^2}\right)}\right] \\
&+ \left(\frac{16V}{9\pi^2}\right)\left[\frac{\sin\left(\frac{3\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)}{\cosh\left((\pi/2)\sqrt{3^2+3^2}\right)}\right] + \left(\frac{16V}{5\pi^2}\right)\left[\frac{\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{5\pi}{2}\right)}{\cosh\left((\pi/2)\sqrt{1^2+5^2}\right)}\right] + \cdots \\
\Phi(a/2,a/2,a/2) &= V\left[\frac{16}{\pi^2\cosh\left((\pi/2)\sqrt{2}\right)} + \frac{-16}{3\pi^2\cosh\left((\pi/2)\sqrt{10}\right)} + \frac{-16}{3\pi^2\cosh\left((\pi/2)\sqrt{10}\right)}\right. \\
&\left. + \frac{16}{5\pi^2\cosh\left((\pi/2)\sqrt{26}\right)} + \frac{16}{9\pi^2\cosh\left((\pi/2)\sqrt{18}\right)} + \frac{16}{5\pi^2\cosh\left((\pi/2)\sqrt{26}\right)} + \cdots\right] \\
\Phi(a/2,a/2,a/2) &= V\left[0.3475 - 0.0075 - 0.0075 + 0.0002 + 0.0005 + 0.0002 + \cdots\right]
\end{aligned}
$$

Then our partial sums are:

$$
\begin{aligned}
\Phi(a/2,a/2,a/2)_1 &= V\left[0.3475\right] \\
\Phi(a/2,a/2,a/2)_2 &= V\left[0.3475 - 0.0075\right] = V\left[0.3400\right] \\
\Phi(a/2,a/2,a/2)_3 &= V\left[0.3400 - 0.0075\right] = V\left[0.3325\right] \\
\Phi(a/2,a/2,a/2)_4 &= V\left[0.3325 + 0.0002\right] = V\left[0.3327\right] \\
\Phi(a/2,a/2,a/2)_5 &= V\left[0.3325 + 0.0005\right] = V\left[0.3332\right] \\
\Phi(a/2,a/2,a/2)_6 &= V\left[0.3325 + 0.0002\right] = V\left[0.3334\right]
\end{aligned}
$$

Which shows we only need 3 terms in the sum before we can round to our numerical factor to 0.333 times the voltage which is accurate to three decimal places.

This makes it look as if the potential at the center of the cube approaches the average value of the walls = 1/3, since 2 out of the 6 walls are fixed at potential V while the rest are grounded at potential 0.

We can then find the charge density on the top surface. Start with Gauss's law and remember how we can use the definition of the potential:

$$\begin{aligned}
\oint \mathbf{E}\cdot\hat{\mathbf{n}}da &= \frac{1}{\epsilon_0}\oint \rho(\mathbf{x})da = \frac{1}{\epsilon_0}\oint \sigma(\mathbf{x})da \\
-\oint \nabla\Phi\cdot\hat{\mathbf{n}}da &= \frac{1}{\epsilon_0}\oint \sigma(\mathbf{x})da \\
-\oint \frac{\partial\Phi}{\partial\hat{\mathbf{n}}}da &= \frac{1}{\epsilon_0}\oint \sigma(\mathbf{x})da \\
0 &= \oint \frac{\sigma(\mathbf{x})}{\epsilon_0} + \frac{\partial\Phi}{\partial\hat{\mathbf{n}}}da \\
0 &= \frac{\sigma(\mathbf{x})}{\epsilon_0} + \frac{\partial\Phi}{\partial\hat{\mathbf{n}}} \\
\sigma(\mathbf{x}) &= -\epsilon_0\frac{\partial\Phi}{\partial\hat{\mathbf{n}}}
\end{aligned}$$

Then we must notice that the inward facing normal for the top of the cube is $-\hat{\mathbf{z}}$, and then use the formula we found for the potential in the box to get that on the top face

$$\begin{aligned}
\sigma(\mathbf{x})_{\text{Top}} &= -\epsilon_0\frac{\partial\Phi}{\partial\hat{\mathbf{n}}} = \epsilon_0\frac{\partial\Phi}{\partial z} \\
&= \epsilon_0\frac{\partial}{\partial z}\left[\sum_{m,n\text{ odd}}\left(\frac{16V}{\pi^2 mn}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\left[\frac{e^{(z/a)\pi\sqrt{m^2+n^2}}+e^{(1-(z/a))\pi\sqrt{m^2+n^2}}}{1+e^{\pi\sqrt{m^2+n^2}}}\right]\right] \\
&= \frac{16V\epsilon_0}{\pi^2}\sum_{m,n\text{ odd}}\left(\frac{\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}{mn\left(1+e^{\pi\sqrt{m^2+n^2}}\right)}\right)\frac{\partial}{\partial z}\left[e^{(z/a)\pi\sqrt{m^2+n^2}}+e^{(1-(z/a))\pi\sqrt{m^2+n^2}}\right] \\
&= \frac{16V\epsilon_0}{\pi^2}\sum_{m,n\text{ odd}}\left(\frac{\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}{mn\left(1+e^{\pi\sqrt{m^2+n^2}}\right)}\right)\left(\frac{\pi\sqrt{m^2+n^2}}{a}e^{(z/a)\pi\sqrt{m^2+n^2}}-\frac{\pi\sqrt{m^2+n^2}}{a}e^{(1-(z/a))\pi\sqrt{m^2+n}}\right. \\
\sigma(\mathbf{x})_{\text{Top}} &= \frac{16V\epsilon_0}{\pi^2}\sum_{m,n\text{ odd}}\left(\frac{\sqrt{m^2+n^2}}{mn}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\left[\frac{e^{(z/a)\pi\sqrt{m^2+n^2}}-e^{(1-(z/a))\pi\sqrt{m^2+n^2}}}{1+e^{\pi\sqrt{m^2+n^2}}}\right]
\end{aligned}$$

Since this is only valid on the top face where $z = a$, we find

$$\begin{aligned}
\sigma(x,y,a)_{\text{Top}} &= \frac{16V\epsilon_0}{\pi a}\sum_{m,n\text{ odd}}\left(\frac{\sqrt{m^2+n^2}}{mn}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\left[\frac{e^{\pi\sqrt{m^2+n^2}}}{1+e^{\pi\sqrt{m^2+n^2}}}\right] \\
&= \frac{16V\epsilon_0}{\pi a}\sum_{m,n\text{ odd}}\left(\frac{\sqrt{m^2+n^2}}{mn}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\left[\frac{e^{\pi\sqrt{m^2+n^2}}}{\left(e^{-\pi\sqrt{m^2+n^2}}+1\right)e^{\pi\sqrt{m^2+n^2}}}\right] \\
\sigma(x,y,a)_{\text{Top}} &= \boxed{\frac{16V\epsilon_0}{\pi a}\sum_{m,n\text{ odd}}\left(\frac{\sqrt{m^2+n^2}}{mn\left(e^{-\pi\sqrt{m^2+n^2}}+1\right)}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}
\end{aligned}$$