Linear Algebra HW3
Exercises 262, 269, 277, 320, 321, 379, 385, 387, 421, 431

Neal D. Nesbitt

July 7, 2016

Problem 6.262. Let V and W be vector spaces over a field F. Let $\alpha \in \mathrm{Hom}(V, W)$ and $\beta \in \mathrm{Hom}(W, V)$ satisfy the condition that $\alpha\beta\alpha = \alpha$. If $w \in \mathrm{im}(\alpha)$, show that $\alpha^{-1}(w) = \{\beta(w) + v - \beta\alpha(v) | v \in V\}$

Solution 6.262. Let our notation be as above, $B = \{\beta(w) + v - \beta\alpha(v) | v \in V\}$, $w \in \mathrm{im}(\alpha)$, and recall that $\alpha^{-1}(w) = \{v \in V | \alpha(v) = w\}$.

Notice that $B \subseteq V$, and assume $x \in B$. Then there is some $v \in V$ such that $x = \beta(w) + v - \beta\alpha(v)$. Since $w \in \mathrm{im}(\alpha)$ there is also some $v' \in V$ such that $\alpha(v') = w$. Then by applying α, using its linearity, and its listed property we find

$$\begin{aligned}
\alpha(x) &= \alpha(\beta(w) + v - \beta\alpha(v)) \\
\alpha(x) &= \alpha(\beta\alpha(v') + v - \beta\alpha(v)) \\
\alpha(x) &= \alpha\beta\alpha(v') + \alpha(v) - \alpha\beta\alpha(v) \\
\alpha(x) &= \alpha(v') + \alpha(v) - \alpha(v) \\
\alpha(x) &= \alpha(v') \\
\alpha(x) - \alpha(v') &= 0_W \\
\alpha(x - v') &= 0_W
\end{aligned}$$

showing that $x - v'$ is in the kernel of α. Then by Proposition 6.6 on pg 96, and the fact that $v' \in \alpha^{-1}(w)$, we have that $(x - v') + v' = x \in \alpha^{-1}(w)$. Then $B \subseteq \alpha^{-1}(w)$.

Now assume instead that $y \in \alpha^{-1}(w)$. Then $\alpha(y) = w$, and since $w \in \mathrm{im}(\alpha)$, there is some $u \in V$ such that $\alpha(u) = w = \alpha(y)$. Now pick some $u' \in V$. Using the linearity of α, and its listed property, we find

$$
\begin{aligned}
\alpha(y) &= \alpha(u)\\
\alpha(y) &= \alpha(u) + \alpha(u') - \alpha(u')\\
\alpha(y) &= \alpha\beta\alpha(u) + \alpha(u') - \alpha\beta\alpha(u')\\
0 &= \alpha\beta\alpha(u) + \alpha(u') - \alpha\beta\alpha(u') - \alpha(y)\\
0 &= \alpha(\beta\alpha(u) + u' - \beta\alpha(u') - y)\\
0 &= \alpha(\beta(w) + u' - \beta\alpha(u') - y)
\end{aligned}
$$

which implies that $\beta(w) + u' - \beta\alpha(u') - y \in \ker(\alpha)$.

Then since $y \in \alpha^{-1}(w)$, by Proposition 9.6 on pg 96,

$$\beta(w) + u' - \beta\alpha(u') - y + y = \beta(w) + u' - \beta\alpha(u') \in \alpha^{-1}(w)$$

and since $u' \in V$ was arbitrary, $\beta(w) + u' - \beta\alpha(u')$ is an arbitrary element of B. Thus $\alpha^{-1}(w) \subseteq B$, showing $\alpha^{-1}(w) = B = \{\beta(w) + v - \beta\alpha(v) | v \in V\}$ as desired.

Problem 6.269. Let $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$\alpha : \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mapsto \begin{bmatrix} a+b+c \\ -a-c \\ b \end{bmatrix}$$

Find $\ker(\alpha)$ and $\operatorname{im}(\alpha)$.

Solution 6.269.a. Let α be as above. Then for $a, b, c \in \mathbb{R}$, $E = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \ker(\alpha)$ iff

$$\alpha\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a+b+c \\ -a-c \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies that $b = 0$, and thus $a + c = 0$ and $-(a + c) = 0$ consistently agree that $c = -a$.

So then

$$\ker(\alpha) = \left\{ \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix} \middle| a \in \mathbb{R} \right\}$$

Solution 6.269.b. Let α be as above. Then for $a, b, c \in \mathbb{R}$, $A = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \operatorname{im}(\alpha)$ iff $\exists x, y, z \in \mathbb{R}$ where

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \alpha\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y+z \\ -x-z \\ y \end{bmatrix}$$

Then $y = c$, implying c could be any real number. Also $-(x+z) = b$ and $x+y+z = a$ imply $c - b = a$.

Since $\forall r, s \in \mathbb{R}$, $r = -((r-s)+s)$, every number is the negative of the sum of many binary combinations, and since $-(x+z) = b$, then b could be any real number. Since we know c and b could be any real number, and we know $c - b = a$, then we can see that any vector in the image of α must have the form:

$$\begin{bmatrix} c-b \\ b \\ c \end{bmatrix} \in \mathrm{im}(\alpha)$$

Problem 6.277. Let V be a finite dimensional vector space over a field F and let $\alpha, \beta \in \mathrm{Hom}(V, V)$ be linear transformations satisfying $\mathrm{im}(\alpha) + \mathrm{im}(\beta) = V = \ker(\alpha) + \ker(\beta)$. Show that $\mathrm{im}(\alpha) \cap \mathrm{im}(\beta) = \{0_V\} = \ker(\alpha) \cap \ker(\beta)$.

Solution 6.277. Let our notation be as above.

Then since V is finite dimensional, then by Proposition 6.10 on pg 98,

$$\dim(V) = \dim(\mathrm{im}(\alpha)) + \dim(\ker(\alpha)) = \dim(\mathrm{im}(\beta)) + \dim(\ker(\beta))$$

and by Grassmann's Theorem on pg 77,

$$\dim(V) = \dim(\mathrm{im}(\alpha)+\mathrm{im}(\beta)) = \dim(\mathrm{im}(\alpha))+\dim(\mathrm{im}(\beta))-\dim(\mathrm{im}(\alpha)\cap\mathrm{im}(\beta))$$

where similarly

$$\dim(V) = \dim(\ker(\alpha)+\ker(\beta)) = \dim(\ker(\alpha))+\dim(\ker(\beta))-\dim(\ker(\alpha)\cap\ker(\beta))$$

We then collect these to find:

$$\dim(\mathrm{im}(\alpha)) + \dim(\ker(\alpha)) = \dim(\mathrm{im}(\alpha)) + \dim(\mathrm{im}(\beta)) - \dim(\mathrm{im}(\alpha) \cap \mathrm{im}(\beta))$$
$$\dim(\mathrm{im}(\beta)) - \dim(\ker(\alpha)) = \dim(\mathrm{im}(\alpha) \cap \mathrm{im}(\beta))$$

$$\dim(\mathrm{im}(\beta)) + \dim(\ker(\beta)) = \dim(\mathrm{im}(\alpha)) + \dim(\mathrm{im}(\beta)) - \dim(\mathrm{im}(\alpha) \cap \mathrm{im}(\beta))$$
$$\dim(\mathrm{im}(\alpha)) - \dim(\ker(\beta)) = \dim(\mathrm{im}(\alpha) \cap \mathrm{im}(\beta))$$

Similarly we find with the other equation:

$$\dim(\mathrm{im}(\alpha)) + \dim(\ker(\alpha)) = \dim(\ker(\alpha)) + \dim(\ker(\beta)) - \dim(\ker(\alpha) \cap \ker(\beta))$$
$$\dim(\ker(\beta)) - \dim(\mathrm{im}(\alpha)) = \dim(\ker(\alpha) \cap \ker(\beta))$$

$$\dim(\mathrm{im}(\beta)) + \dim(\ker(\beta)) = \dim(\ker(\alpha)) + \dim(\ker(\beta)) - \dim(\ker(\alpha) \cap \ker(\beta))$$
$$\dim(\ker(\alpha)) - \dim(\mathrm{im}(\beta)) = \dim(\ker(\alpha) \cap \ker(\beta))$$

Thus

$$\dim(\mathrm{im}(\alpha) \cap \mathrm{im}(\beta)) + \dim(\ker(\alpha) \cap \ker(\beta)) = 0$$

But since dimension is always non-negative, this can only be true if

$$\dim(\operatorname{im}(\alpha) \cap \operatorname{im}(\beta)) = \dim(\ker(\alpha) \cap \ker(\beta)) = 0$$

implying $\operatorname{im}(\alpha) \cap \operatorname{im}(\beta)$ and $\ker(\alpha) \cap \ker(\beta)$ are both the trivial subspace $\{0_V\}$, which is what we desired.

Problem 7.320. Let V be a finitely-generated vector space over a field F and let $\alpha \in \operatorname{End}(V)$. Show that α is not monic if and only if there exists an endomorphism $\beta \neq \sigma_0$ of V satisfying $\alpha\beta = \sigma_0$.

Solution 7.320. Let our notation be as used for the problem, and let us start by assuming there is some endomorphism $\beta \neq \sigma_0$ of V where $\alpha\beta = \sigma_0$.

Since $\beta \neq \sigma_0$, then there are some $v, w \in V$ where $\beta(v) = w \neq 0$, but $\alpha\beta(v) = \alpha(w) = 0$, where we notice that $0 = \alpha(0)$ because α is an endomorphism. So since $w \neq 0$, but $\alpha(w) = \alpha(0)$, then α is not monic.

Now instead assume that α is not monic. Then there are some $a, b \in V$ such that $a \neq b$, but $\alpha(a) = \alpha(b)$. So since V is finitely generated, there is some $n \in \mathbb{Z}$ where $B = \{v_1, \cdots, v_n\}$ is a basis for V, and there are corresponding $\{a_1, \cdots, a_n\}, \{b_1, \cdots, b_n\} \subset \mathbb{R}$ such that $a = a_1v_1 + \cdots + a_nv_n$, and $b = b_1v_1 + \cdots + b_nv_n$. Then we know:

$$\begin{aligned}
\alpha(a) &= \alpha(b) \\
\alpha(a_1v_1 + \cdots + a_nv_n) &= \alpha(b_1v_1 + \cdots + b_nv_n) \\
\alpha((a_1v_1 - b_1v_1) + \cdots + (a_nv_n - b_nv_n)) &= 0 \\
\alpha((a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n) &= 0
\end{aligned}$$

So let $\beta \in \operatorname{End}(V)$ such that $\forall x \in V$, $\beta(x) = (a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n$.

Since $a \neq b$, there is at least one $i \in \mathbb{N}$ where $a_i - b_i \neq 0$, the function is a non-zero constant function, and thus an endomorphism where $\beta \neq \sigma_0$. Then based on our observation above

$$\alpha\beta(x) = \alpha((a_1 - b_1)v_1 + \cdots + (a_n - b_n)v_n) = 0 = \sigma_0(x)$$

which is the property we desired.

Problem 7.321. Let V be a vector space over a field F and let $\alpha \in \operatorname{End}(V)$. Show that $\ker(\alpha) = \ker(\alpha^2)$ if and only if $\ker(\alpha)$ and $\operatorname{im}(\alpha)$ are disjoint.

Solution 7.321. Let our notation be the same as in the problem, and let us start by assuming that $\ker(\alpha) = \ker(\alpha^2)$. Then for any $e \in V$, $\alpha^2(e) = \alpha(\alpha(e)) = 0$ iff $\alpha(e) = 0$.

Now let $e \in \ker(\alpha) \cap \operatorname{im}(\alpha)$. Then there is some $v \in V$ such that $\alpha(v) = e$ and $\alpha(e) = \alpha(\alpha(v)) = \alpha^2(v) = 0$. Thus $v \in \ker(\alpha^2)$, and by our property, $v \in \ker(\alpha)$ implying $\alpha(v) = e = 0$. Thus $\ker(\alpha)$ and $\operatorname{im}(\alpha)$ are disjoint.

Now instead assume that $\ker(\alpha)$ and $\operatorname{im}(\alpha)$ are disjoint to start.

Since for any $e \in \ker(\alpha)$, $\alpha(e) = 0$ implies that $\alpha^2(v) = \alpha(\alpha(v)) = \alpha(0) = 0$, since α is an endomorphism, showing $e \in \ker(\alpha^2)$, and thus $\ker(\alpha) \subseteq \ker(\alpha^2)$.

So let $e \in \ker(\alpha^2)$, so that $\alpha^2(e) = 0$. Then there is some $v \in \operatorname{im}(\alpha)$ where $\alpha(e) = v$ and $\alpha(v) = 0$. Thus $v \in \ker(\alpha)$, but since $\ker(\alpha)$ and $\operatorname{im}(\alpha)$ are disjoint, then we must have $v = 0$, showing that $\alpha(e) = 0$, and thus $e \in \ker(\alpha)$. Then we know $\ker(\alpha^2) \subseteq \ker(\alpha)$, and together with our previous result, $\ker(\alpha^2) = \ker(\alpha)$.

Problem 7.379. Let α and β be endomorphisms of a vector space V over a field F satisfying $\alpha\beta = \beta\alpha$. Is $\ker(\alpha)$ invariant under β?

Solution 7.379. Let our notation be as used in the problem. Then $\ker(\alpha)$ is invariant under β iff $\beta(\ker(\alpha)) \subseteq \ker(\alpha)$. So let $w \in \beta(\ker(\alpha))$. Then there is some $e \in \ker(\alpha)$ such that $\beta(e) = w$. So since $\alpha(e) = 0$, and $\alpha\beta = \beta\alpha$, then

$$\begin{aligned} w &= \alpha(\beta(e)) \\ &= \beta(\alpha(e)) \\ &= \beta(0) \\ w &= 0 \end{aligned}$$

Thus w must be in every kernel, implying $w \in \ker(\alpha)$, and $\beta(\ker(\alpha)) \subseteq \ker(\alpha)$, which shows $\ker(\alpha)$ is invariant under β.

Problem 7.385. Let V be a vector space over a field F and let W and Y be subspaces of V satisfying $W + Y = V$. Let Y' be a complement of Y in V and let Y'' be a complement of $W \cap Y$ in W. Show that $Y' \cong Y''$.

Solution 7.385. Let the notation be as used in the problem. Then since Y' is a complement of Y in V, we have $Y' \cap Y = \{0\}$, and $Y' + Y = V$.

And since Y'' is a complement of $W \cap Y$ in W, then $Y'' \cap W \cap Y = \{0\}$, and $Y'' + (W \cap Y) = W$, also showing that $Y'' \subseteq W$.

Then since $\{0\} \subseteq Y'' \cap Y \subseteq Y'' \cap W \cap Y = \{0\}$, we know $Y'' \cap Y = \{0\}$.

So $W + Y = V$ and $Y + (W \cap Y) = Y$, then imply

$$V = W + Y = Y'' + (W \cap Y) + Y = Y'' + Y$$

showing that Y'' is a complement of Y in V.

Now since Y' and Y'' are both complements of Y in V, then by Proposition 7.8 on pg 119 they must be isomorphic.

Problem 7.387. Let V be a vector space over F and let $\alpha, \beta \in \text{End}(V)$. Show that α and β are projections satisfying $\ker(\alpha) = \ker(\beta)$ if and only if $\alpha\beta = \alpha$ and $\beta\alpha = \beta$.

Solution 7.387. Let our notation be as above, and begin by assuming $\alpha\beta = \alpha$ and $\beta\alpha = \beta$.

Then $\alpha = \alpha\beta$, implies $\beta\alpha = \beta\alpha\beta$, and then $\beta\alpha = \beta$ imples $\beta = \beta^2$. By similar logic, $\alpha = \alpha^2$ and thus is also a projection.

So let $e \in \ker(\alpha)$. Then $\alpha(e) = 0$, and $\beta\alpha(e) = \beta(0) = 0$. But since $\beta\alpha = \beta$, then $\beta(e) = 0$, showing $e \in \ker(\beta)$. Again, similar logic will show that $e \in \ker(\beta)$ implies $e \in \ker(\alpha)$, and thus α and β are projections satisfying $\ker(\alpha) = \ker(\beta)$.

So instead begin by assuming α and β are projections satisfying $\ker(\alpha) = \ker(\beta)$. Then

$$\begin{aligned} \alpha^2 &= \alpha \\ \alpha^2 - \alpha &= \sigma_0 \\ \alpha(\alpha - \sigma_1) &= \sigma_0 \end{aligned}$$

Thus $\alpha - \sigma_1 \in \ker(\alpha)$. Then since $\ker(\alpha) = \ker(\beta)$, we know $\beta(\alpha - \sigma_1) = \sigma_0$. So distributing β gives $\beta\alpha - \beta = \sigma_0$, and thus $\beta\alpha = \beta$. Similar logic beginning with $\beta^2 = \beta$ will show $\alpha\beta = \alpha$, as we desired.

Problem 8.421. Let $B = \{1+i, 2+i\}$, which is a basis for $\mathbb{C}$ as a vector space over $\mathbb{R}$. Let α be the endomorphism of this space defined by $\alpha : z \mapsto \bar{z}$. Find $\Phi_{BB}(\alpha)$.

Solution 8.421. Let our notation be as given in the problem, call $b_1 = 1 + i$, $b_2 = 2 + i$, and let $z \in \mathbb{C}$. Then there are $x, y \in \mathbb{R}$ such that $z = xb_1 + yb_2$, and $u, v \in \mathbb{R}$ such that $\bar{z} = ub_1 + vb_2$.

Thus

$$\begin{aligned} \alpha(z) &= \bar{z} \\ \alpha(x(1+i) + y(2+i)) &= u(1+i) + v(2+i) \\ x(1-i) + y(2-i) &= u(1+i) + v(2+i) \\ x - ix + 2y - iy &= u + iu + 2v + iv \\ (x+2y) - i(x+y) &= (u+2v) + i(u+v) \end{aligned}$$

$$[(x+2y) - (u+2v)] - i[(x+y) + (u+v)] = 0$$

This can only happen if

$$\begin{aligned} x + 2y &= u + 2v \\ x + y &= -u - v \\ \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned}$$

So since

$$\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$\begin{aligned} \begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} -3 & -4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

Which is then how we define $\Phi_{BB}(\alpha)$.

Problem 8.431. Find a nonzero matrix A in $\mathcal{M}_{2\times 2}(\mathbb{R})$ satisfying $v \odot Av = 0$ for all $v \in \mathbb{R}^2$.

Solution 8.431. Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$, and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{R})$.

Then

$$\begin{aligned} v \odot Av &= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ &= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \odot \begin{bmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{bmatrix} \\ &= v_1(av_1 + bv_2) + v_2(cv_1 + dv_2) \\ &= av_1^2 + bv_1v_2 + cv_1v_2 + dv_2^2 \\ &= av_1^2 + dv_2^2 + (b + c)v_1v_2 \end{aligned}$$

So for the product to be zero for any v, we must have $a = d = 0$, and $c = -b$.

$$A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$