

Linear Algebra HW1

Exercises 1, 9, 12, 18, 61, 70, 87, 104, 117

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June 14, 2016

Problem 1. Let F be a field and let $G = F \times F$. Define operations of addition and multiplication on G by setting $(a, b) + (c, d) = (a + b, c + d)$ and $(a, b) \cdot (c, d) = (ac, bd)$. Do these operations define the structure of a field on G ?

Solution 1. These operations do not define a field on G .

Proof. Let F and $(G, +, \cdot)$ be defined as above. Then for any $[a \neq 0] \in F$, $[(a, 0) \neq (0, 0)] \in G$ has no inverse in G , since 0 has no inverse in F , and multiplication in G is done component-wise.

Thus $(G, +, \cdot)$ does not have inverses for every non-identity element, and is not a field. \square

Problem 9. Let F be a field and define a new operation \star on F by setting $a \star b = a + b + ab$. When is $(F, +, \star)$ a field?

Solution 9. Let $(F, +, \star)$ be defined as above, and notice the following:

Inverses Check the form of the inverse of any $a \in F$

$$\begin{aligned} a \star a^{-1} &= u \\ a + a^{-1} + aa^{-1} &= 0 \\ (1 + a)a^{-1} &= -a \\ a^{-1} &= \frac{-a}{1 + a} \end{aligned}$$

This inverse solution is only valid for $a \neq -1$, and thus F is cannot be a field since there is a non-identity element (namely -1) without an inverse.

Distributivity Moreover we can see that \star does not distribute across addition for any $a, b, c \in F$:

$$\begin{aligned} a \star (b + c) &= \boxed{a + b + c + ab + ac} \\ (a \star b) + (a \star c) &= (a + b + ab) + (a + c + ac) \\ &= \boxed{2a + b + c + ab + ac} \end{aligned}$$

meaning that $a = 1$ is the only element that distributes.

Problem 12. Let F be a field. Show that the function $a \mapsto a^{-1}$ is a permutation of $F \setminus \{0_F\}$

Solution 12. *Proof.* Let the notation be as above. A permutation is a function that is bijective and closed. Since every non-identity element has an inverse, then $F \setminus \{0_F\}$ will be closed under the inverse function.

So it remains to verify bijectivity. Since every non-identity element in F has an inverse, then the function must be surjective (onto) in $F \setminus \{0_F\}$, and since no two elements have the same inverse (inverses are unique), then the function is injective (one-to-one) in $F \setminus \{0_F\}$ as well. This shows together that the inverse function is bijective and, with closure, a permutation on $F \setminus \{0_F\}$. \square

Problem 18. Show that for all $z \in \mathbb{C}$, $|z + 1| \leq |z + 1|^2 + |z|$.

Solution 18. *Proof.* Let $z \in \mathbb{C}$. Then by the triangle inequality and the fact that $0 \leq |z|$:

$$\begin{aligned} 1 &= |1 + z - z| \leq |z + 1| + |z| \\ (|z + 1| - |z|) &\leq (|z + 1| + |z|) (|z + 1| - |z|) \\ |z + 1| - |z| &\leq |z + 1|^2 - |z|^2 \\ |z + 1| + |z|^2 &\leq |z + 1|^2 + |z| \\ |z + 1| &\leq |z + 1|^2 + |z| \end{aligned}$$

\square

Problem 61. Let V be a non-trivial vector space over \mathbb{R} . For each $v \in V$ and each complex number $a + bi$, let us define $(a + bi)v = av$. Does V , together with this new scalar multiplication, form a vector space over \mathbb{C} .

Solution 61. The construction above does not form a vector space over \mathbb{C} .

Proof. Let the notation be as above, and let $(a + bi), (c + di) \in \mathbb{C}$ and $v \in V$. Then if we check the associativity of vector multiplication we find:

$$\begin{aligned} [(a + bi)(c + di)]v &= ((ac - bd) + (ad + bc)i)v = (ac - bd)v \\ (a + bi)[(c + di)v] &= (a + bi)[cv] = acv \end{aligned}$$

which means that only real numbers associate with scalar multiplication, and this construction does not form a vector space over \mathbb{C} . \square

Problem 70. Show that \mathbb{Z} is not a vector space over any field.

Solution 70. *Proof.* Let F be a field, and assume by way of contradiction that \mathbb{Z} forms a vector space over F .

Then if we pick $a \in F$ such that $a1 > 1$ (without loss of generality since $[a1 < 1] \rightarrow [1 < a^{-1}1]$). Since $\forall a \in F, 1_F = a^{-1}a, 0 = a0$, and $1 = 1_F 1$, then

$$\begin{aligned} a1 &> 1_F 1 > 0 \\ a1 &> a^{-1}a1 > a0 \\ 1 &> a^{-1}1 > 0 \end{aligned}$$

but there is no integer between zero and one. Therefore $a^{-1}1 \notin \mathbb{Z}$ and is thus not in the vector space, contradicting the need for closure under scalar multiplication. Since F was arbitrary, \mathbb{Z} cannot form a vector space over any F . \square

Problem 87. Let F be a field, and let $V = F^F$, which is a vector space over F .

Let W be the set of all functions $f \in V$ such that $f(1) = f(-1)$. Is W a subspace of V ?

Solution 87. Let the notation be as above. W is a subspace of V . To check its validity we verify its closure under addition and scalar multiplication for all $v, w \in W$ and $a \in F$.

Addition Since $v(1) = v(-1)$ and $w(1) = w(-1)$, then $(v + w)(1) = v(1) + w(1) = v(-1) + w(-1) = (v + w)(-1)$, showing W is closed under vector addition.

Scalar Multiplication Since $v(1) = v(-1)$, then $av(1) = av(-1)$, showing W is closed under scalar multiplication, and is thus a vector space as constructed above.

Problem 104. Find subspaces W and Y of \mathbb{R}^3 having the property that $W \cup Y$ is not a subspace of \mathbb{R}^3 .

Solution 104. Let $W \subset \mathbb{R}^3$ be the x-y plane, and let $Y \subset \mathbb{R}^3$ be the x-z plane. Then both are subspaces individually since they are equivalent to \mathbb{R}^2 , but if we take a non-identity vector from each, their sum will not be in either one of the planes.

Problem 117. Define \cdot on \mathbb{R}^2 such that for any $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{R}^2$

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2ac - bd \\ ad + bc \end{bmatrix}$$

Show this definition of vector multiplication makes \mathbb{R}^2 an \mathbb{R} -algebra.

Solution 117. Let the notation be as given. Then to verify the \mathbb{R} -algebra we must check that the vector multiplication distributes across vector addition, and that it associates with scalar multiplication.

So let $u = (u_1, u_2), v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{R}^2$ and $a \in \mathbb{R}$.

Distributivity

$$\begin{aligned} u \cdot (v + w) &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} \\ &= \begin{bmatrix} 2u_1(v_1 + w_1) - u_2(v_2 + w_2) \\ u_1(v_2 + w_2) + u_2(v_1 + w_1) \end{bmatrix} \\ &= \begin{bmatrix} 2u_1v_1 - u_2v_2 \\ u_1v_2 + u_2v_1 \end{bmatrix} + \begin{bmatrix} 2u_1w_1 - u_2w_2 \\ u_1w_2 + u_2w_1 \end{bmatrix} \\ &= \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) + \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) \\ &= (u \cdot v) + (u \cdot w) \end{aligned}$$

And since

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2u_1v_1 - u_2v_2 \\ u_1v_2 + u_2v_1 \end{bmatrix} = \begin{bmatrix} 2v_1u_1 - v_2u_2 \\ v_1u_2 + v_2u_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Then vector multiplication is commutative, and vector multiplication distributes over vector addition from the left or the right:

$$\begin{aligned} u \cdot (v + w) &= (u \cdot v) + (u \cdot w) \\ (v + w) \cdot u &= (v \cdot u) + (w \cdot u) \end{aligned}$$

Associativity

$$\begin{aligned} a(u \cdot v) &= a \begin{bmatrix} 2u_1v_1 - u_2v_2 \\ u_1v_2 + u_2v_1 \end{bmatrix} \\ &= \begin{bmatrix} a(2u_1v_1 - u_2v_2) \\ a(u_1v_2 + u_2v_1) \end{bmatrix} \\ &= a \begin{bmatrix} 2au_1v_1 - au_2v_2 \\ au_1v_2 + au_2v_1 \end{bmatrix} \\ &= (au) \cdot v \end{aligned}$$

And thus vector multiplication associates properly, and with the distributivity property above this verifies \mathbb{R}^2 as an \mathbb{R} -algebra.