Numerical Analysis HW7
Ch10 - 1,3,4,13 (pg267)
Ch11 - 1,6,8 (pg280)

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March 29, 2016

Problem 1. Determine the number of flops as a function of the number of equations n for

(a) factorization

(b) forward substitution

(c) back substitution

of LU factorization

Solution 1.1.

```
%---------------------------------------------------------------------------
% Forward Elimination:

% for each pivot column
for k = 1:(n-1)

    %***********************************************************************
    % Partial Pivoting:

    % find the maximum leading term in the current pivot column
    [~,i] = max(abs(A(k:n,k)));
    % if it is not the current row
    if i ~= 1
        % switch the current row and the row with the largest leading term
        A([k,i+k-1],:)=A([i+k-1,k],:);
        P([k,i+k-1],:)=P([i+k-1,k],:);

        display([P,A]);    % display the result of the pivot
    end
    %***********************************************************************
```

```matlab
    % for each row past the diagonal
    for i = k+1:n

        % eliminate the leading term using row operations:

        % find the factor that changes the column's leading term
        % to match the leading term of the current elimination row
        factor = A(i,k)/A(k,k);

        % set the leading term (that normally is eliminated)
        % instead to its corresponding factor to store until needed
        A(i,k)=factor;

        % multiply the factor through the pivot row
        % and subtract the result from the current elimination row
        A(i,(k+1):n) = A(i,(k+1):n) - factor*A(k,(k+1):n);
    end
    % (n-k) cycles in i,
    %   1 flop per for finding elimination row factor,
    %   2(n-k) flops per with:
    %       n-k for multiplying factor through pivot row and
    %       n-k for subtracting the resulting row from the elimination row
    % Total Flops:(n-k)[2(n-k)+1]

    display(A);   % display after each pivot column elimination
end
% (n-1) cycles in k
% Total Flops: sum_{k=1}^{n-1} (n-k)[2(n-k)+1] =
% = n(n-1)(4n-1)/6 = (2/3)n^3 -(5/6)n^2 +(1/6)n


%------------------------------------------------------------------------
% Seperation:

% set up the place to store L
L = eye(n);

% copy each column of A below the diagonal into L
% and set the value in A to zero
for i = 1:n-1
    % copy each column of A below the diagonal into L
    L((i+1):n,i) = A((i+1):n,i);
    A((i+1):n,i) = 0; % and set original the value in A to zero
end
```

```
U = A;
```

So for Forward Elimination, or factoring an input matrix $A_{m,n}$ into the lower triangular factor matrix L, and upper triangular matrix U:

For each pivot column there are $(n-1)$ cycles in k:
For each row elimination there are $(n-k)$ cycles in i, each with

- 1 flop per for finding elimination row factor,
- $2(n-k)$ flops per with:

  $n-k$ for multiplying factor through pivot row and

  $n-k$ for subtracting the resulting row from the elimination row

Giving a total of $(n-k)[2(n-k)+1]$ flops per row of elimination.

Then for the $(n-1)$ cycles in k, the total number of flops is:

$$
\begin{aligned}
&\sum_{k=1}^{n-1}(n-k)[2(n-k)+1] \\
&= \sum_{k=1}^{n-1}(2(n-k)^2+n-k) = \sum_{k=1}^{n-1}(2n^2-4nk+2k^2+n-k) \\
&= \sum_{k=1}^{n-1}(2n^2+n) + \sum_{k=1}^{n-1}2k^2 + \sum_{k=1}^{n-1}(-4nk-k) \\
&= (n-1)(2n^2+n) + 2\sum_{k=1}^{n-1}k^2 - (4n+1)\sum_{k=1}^{n-1}k \\
&= (n-1)(2n^2+n) + 2\left(\frac{(n-1)n(2(n-1)+1)}{6}\right) - (4n+1)\left(\frac{(n-1)n}{2}\right) \\
&= (n-1)n\left[(2n+1)+\frac{(2n-1)}{3}-\frac{(4n+1)}{2}\right] \\
&= (n-1)n\left[(12n+6)+(4n-4)-(12n+3)\right]/6 \\
&= (n-1)n(4n-1)/6 = \boxed{\frac{2}{3}n^3-\frac{5}{6}n^2+\frac{1}{6}n}
\end{aligned}
$$

Solution 1.2.
```
%====================================================================
% Forward Substitution:

D = zeros(size(B));      % Set up our intermediary solution
D(1,:) = P(1,:)*B;       % Find the initial values to start substitution
% (Accounting for row changes in the decomposition)
```

```
% (nB)n flops for picking the correct row of B

for i=2:n
    D(i,:) = (P(i,:)*B)-(L(i,1:(i-1))*D(1:(i-1),:));
end
% (n-1) cycles in i,
%   (nb)n flops for picking the correct row of B
%   (nb)(i-1) flops for producing the variables to subtract
%   (nb) flops for subtracting the row to find the next iteration's values
% Total Flops: (n-1)((nB)n+(nB))+(nB)sum_{i=2}^{n}(i-1)
% = (nB)(n-1)((3/2)n+1) = (nB)((3/2)n^{2}-(1/2)n-1)

display(D);
```

For forward substitution we have n flops per vector on the right hand side. If we set up the column vectors on the right hand side into a matrix: $B = [\mathbf{b}_1 \cdots \mathbf{b}_1]$

Problem 3. Use naive Gauss elimination to factor the following system according to the description in Section 10.2:

$$\begin{aligned} 7x_1 + 2x_2 - 3x_3 &= -12 \\ 2x_1 + 5x_2 - 3x_3 &= -20 \\ x_1 - x_2 - 6x_3 &= -26 \end{aligned}$$

Then, multiply the resulting $[L]$ and $[U]$ matrices to determine that $[A]$ is produced.

Problem 4. Use LU factorization to solve the system of equations in Problem 3. Show all the steps of the computation. Also solve the system for the alternate right-vand-side vector

$$b = (12, 18, -6)$$

Problem 13. Use Cholesky factorization to determine $[U]$ so that

$$[A] = [U]^T[U] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Problem 1. Determine the matrix inverse for the following system:

$$\begin{aligned} 10x_1 + 2x_2 - x_3 &= 27 \\ -3x_1 - 6x_2 + 2x_3 &= -61.5 \\ x_1 + x_2 + 5x_3 &= -21.5 \end{aligned}$$

Check your results by verifying $[A][A]^{-1} = [I]$

Problem 6. Determine $\|A\|_f$, $\|A\|_1$, and $\|A\|_\infty$ for

$$[A] = \begin{bmatrix} 8 & 2 & -10 \\ -9 & 1 & 3 \\ 15 & -1 & 6 \end{bmatrix}$$

Before determining the norms, scale the matrix by making the maximum element in each row equal to one.

Problem 8. Use MATLAB to determine the spectral condition number for the following system. Do not normalize the system:

$$\begin{bmatrix} 1 & 4 & 9 & 16 & 25 \\ 4 & 9 & 16 & 25 & 36 \\ 9 & 16 & 25 & 36 & 49 \\ 16 & 25 & 36 & 49 & 64 \\ 25 & 36 & 49 & 64 & 81 \end{bmatrix}$$

Compute the condition number based on the row-sum norm.