Numerical Analysis HW11
Ch23 - 2a,7 (pg613)
Ch24 - 1,4 (pg635)

Neal D. Nesbitt

April 27, 2016

23

Problem 23.3. The following ODEs have been proposed as a model of an epidemic:

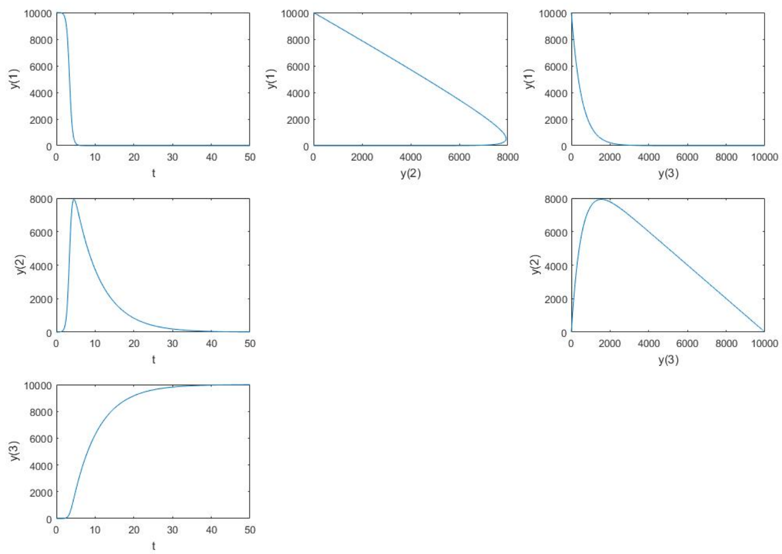
$$\frac{dS}{dt} = -aSI$$
$$\frac{dI}{dt} = aSI - rI$$
$$\frac{dR}{dt} = rI$$

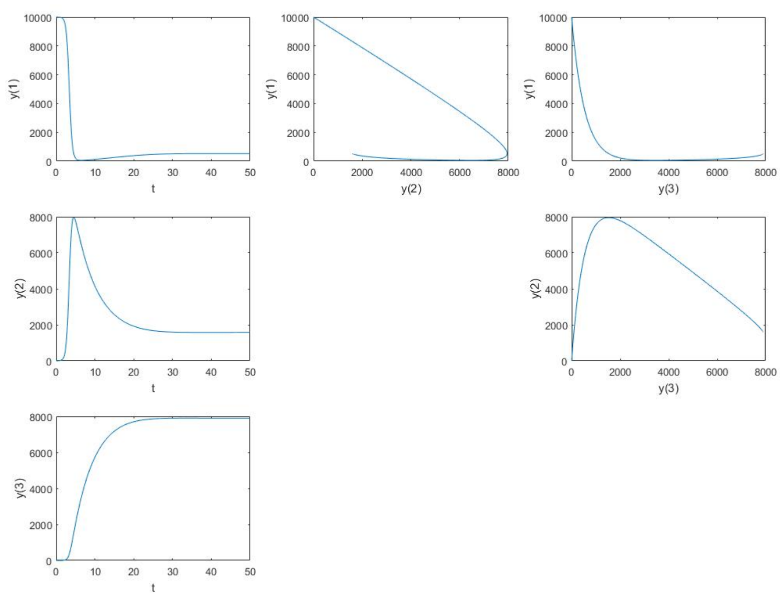
where S is the number of susceptible individuals, I is the number of infected, R is the number of the recovered, a is the infection rate, and r is the recovery rate. A city has 10,000 people, all of whom are susceptible.

(a) If a single infectious individual enters the city at $t = 0$, compute the progression of the epidemic until the number of infected individuals falls below 10. Use the following parameters: $a = 0.0002/(\text{person} \cdot \text{week})$ and $r = 0.15/\text{d}$. Develop time series plots of all the state variables. Also generate a phase-plane plot of S versus I versus R

(b) Suppose that after recovery, there is a loss of immunity that causes recovered individuals to become susceptible. This reinfection mechanism can be computed as ρR, where ρ is the reinfection rate. Modify the model to include this mechanism and repeat the computations in (a) using $\rho = 0.03/\text{d}$.

Let y(1) = S, y(2) = I, y(3) = R, and count t in days. Below are the results for the full and partial immunity models respectively.


y(1)
t
y(2)
y(3)


y(1)
t
y(2)
y(3)

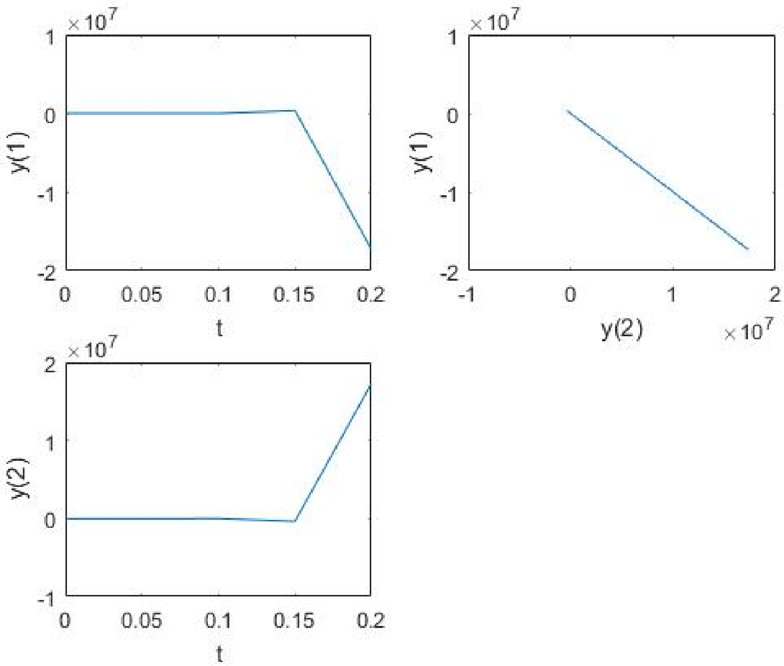
Problem 23.7. Given

$$\frac{dx_1}{dt} = 999x_1 + 1999x_2$$
$$\frac{dx_2}{dt} = -1000x_1 - 2000x_2$$

If $x_1(0) = x_2(0) = 1$, obtain a solution from $t = 0$ to 0.2 using a step size of 0.05 with

(a) the explicit Euler method

(b) and the implicit Euler method



Above is the Explicit version where $y = x$ is the independent variable we solved for.

24

Problem 24.1. A steady-state heat balance for a rod can be represented as

$$\frac{d^2T}{dx^2} - 0.15T = 0$$

Obtain a solution for a 10-m rod with $T(0) = 240$ and $T(10) = 150$

(a) analytically

(b) with the shooting method

(c) using the finite difference method with $\Delta x = 1$

Solution 24.1.a.

$$\begin{aligned}
T(x) &= Ae^{x\sqrt{0.15}} + Be^{-x\sqrt{0.15}} \\
T(0) &= A + B = 240 \\
B &= 240 - A \\
T(10) &= Ae^{\sqrt{15}} + Be^{-\sqrt{15}} = 150 \\
150 &= Ae^{\sqrt{15}} + (240 - A)e^{-\sqrt{15}} \\
150 &= 240e^{-\sqrt{15}} + Ae^{\sqrt{15}} - Ae^{-\sqrt{15}} \\
150 &= 240e^{-\sqrt{15}} + 2Ai\sin(\sqrt{15}) \\
A &= \frac{150 - 240e^{-\sqrt{15}}}{2i\sin(\sqrt{15})} \\
A &\approx -1073.43i \\
B &\approx -1313.43i
\end{aligned}$$

$$T(x) = \frac{1073.43e^{x\sqrt{0.15}} + 1313.43e^{-x\sqrt{0.15}}}{i}$$

Solution 24.1.b. content...

Solution 24.1.c. Using the finite difference approximation of the second derivative we find

$$\begin{aligned}
\frac{d^2T}{dx^2} \approx \frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2} &= 0.15T_i \\
T_{i-1} - 2T_i + T_{i+1} &= 0.15T_i\Delta x^2 \\
T_{i-1} - T_i(2 + 0.15\Delta x^2) + T_{i+1} &= 0
\end{aligned}$$

So extending this to each step we turn this into a tridiagonal system:

$$\begin{bmatrix} T_0 & -(2+0.15\Delta x^2)T_1 & T_2 & 0 & \cdots & 0 \\ 0 & T_1 & -(2+0.15\Delta x^2)T_2 & T_3 & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & T_{n-3} & -(2+0.15\Delta x^2)T_{n-2} & T_{n-1} & 0 \\ 0 & \cdots & 0 & T_{n-2} & -(2+0.15\Delta x^2)T_{n-1} & T_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -(2+0.15\Delta x^2)T_1 & T_2 & 0 & \cdots \\ T_1 & -(2+0.15\Delta x^2)T_2 & T_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & T_{n-3} & -(2+0.15\Delta x^2)T_{n-2} & T_{n-1} \\ \cdots & 0 & T_{n-2} & -(2+0.15\Delta x^2)T_{n-1} \end{bmatrix} = \begin{bmatrix} -T_0 \\ 0 \\ \cdots \\ 0 \\ -T_n \end{bmatrix}$$

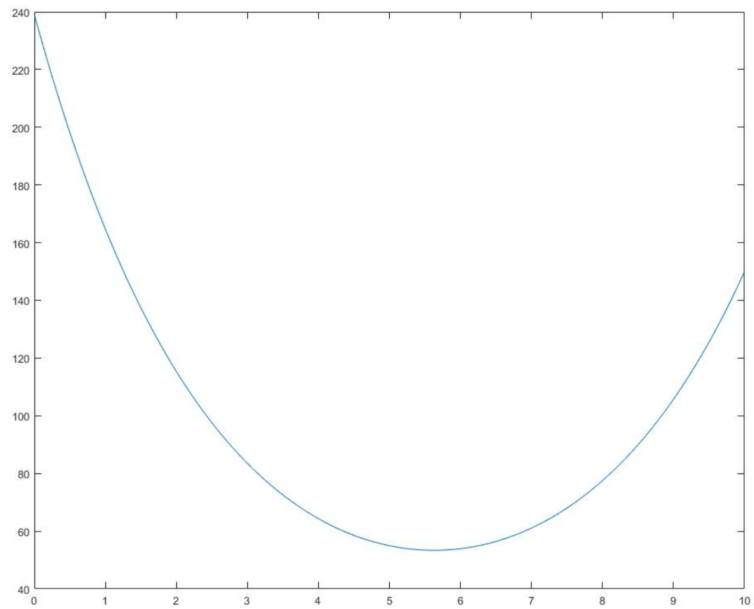
$$\begin{bmatrix} -(2+0.15\Delta x^2) & 1 & 0 & \cdots \\ 1 & -(2+0.15\Delta x^2) & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & -(2+0.15\Delta x^2) & 1 \\ \cdots & 0 & 1 & -(2+0.15\Delta x^2) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \cdots \\ T_{n-2} \\ T_{n-1} \end{bmatrix} = \begin{bmatrix} -T_0 \\ 0 \\ \cdots \\ 0 \\ -T_n \end{bmatrix}$$

Then with the given boundary conditions

$$\begin{bmatrix} -(2+0.15(10/n)^2) & 1 & 0 & \cdots \\ 1 & -(2+0.15(10/n)^2) & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & -(2+0.15(10/n)^2) & 1 \\ \cdots & 0 & 1 & -(2+0.15(10/n)^2) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \cdots \\ T_{n-2} \\ T_{n-1} \end{bmatrix} = \begin{bmatrix} -240 \\ 0 \\ \cdots \\ 0 \\ -150 \end{bmatrix}$$

$$\begin{bmatrix} -(2+15/n^2) & 1 & 0 & \cdots \\ 1 & -(2+15/n^2) & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & -(2+15/n^2) & 1 \\ \cdots & 0 & 1 & -(2+15/n^2) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \cdots \\ T_{n-2} \\ T_{n-1} \end{bmatrix} = \begin{bmatrix} -240 \\ 0 \\ \cdots \\ 0 \\ -150 \end{bmatrix}$$

We then solve this system using MATLAB to give



Problem 24.4. Use the finite difference method to solve

$$7\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y + x = 0$$

with the boundary conditions $y(0) = 5$ and $y(20) = 8$, and $\Delta x = 2$.