Electrodynamics HW1
Ch1 - 5 (pg51)

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Problem 1. The time-averaged potential of a hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right)$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

Given the potential Φ, we make use of the Poisson equation in spherical coordinates with only an r dependence to find

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\Phi}{\partial r}\right] = -\rho/\epsilon_0$$

which implies

$$\rho = -\epsilon_0\nabla^2\Phi = \frac{-\epsilon_0}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\Phi}{\partial r}\right]$$

So if we begin by assuming that $r \neq 0$ (to account for the singularity in our potential) we can take the Laplacian directly:

$$\begin{aligned}\rho &= \frac{-\epsilon_0}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left[\frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1 + \frac{\alpha r}{2}\right)\right]\right] \\ &= \frac{-q}{4\pi r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left[e^{-\alpha r}\left(\frac{1}{r} + \frac{\alpha}{2}\right)\right]\right] \\ &= \frac{-q}{4\pi r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left[e^{-\alpha r}\left(\frac{1}{r} + \frac{\alpha}{2}\right)\right]\right]\end{aligned}$$

At his stage I would like to bring the r^2 term into the inner partial, and so

I add and subtract the necessary component to account for the product rule.

$$
\begin{aligned}
\rho &= \frac{-q}{4\pi r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left[e^{-\alpha r}\left(\frac{1}{r}+\frac{\alpha}{2}\right)\right]+\frac{\partial}{\partial r}\left[r^2\right]e^{-\alpha r}\left(\frac{1}{r}+\frac{\alpha}{2}\right)-\frac{\partial}{\partial r}\left[r^2\right]e^{-\alpha r}\left(\frac{1}{r}+\frac{\alpha}{2}\right)\right] \\
&= \frac{-q}{4\pi r^2}\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\left[r^2 e^{-\alpha r}\left(\frac{1}{r}+\frac{\alpha}{2}\right)\right]-\frac{\partial}{\partial r}\left[r^2\right]e^{-\alpha r}\left(\frac{1}{r}+\frac{\alpha}{2}\right)\right] \\
&= \frac{-q}{4\pi r^2}\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\left[r^2 e^{-\alpha r}\left(\frac{1}{r}+\frac{\alpha}{2}\right)\right]-2re^{-\alpha r}\left(\frac{1}{r}+\frac{\alpha}{2}\right)\right] \\
&= \frac{-q}{4\pi r^2}\frac{\partial}{\partial r}\left[\frac{\partial}{\partial r}\left[e^{-\alpha r}\left(r+\frac{\alpha r^2}{2}\right)\right]-e^{-\alpha r}\left(2+\alpha r\right)\right]
\end{aligned}
$$

We then proceed with the computation.

$$
\begin{aligned}
\rho &= \frac{-q}{4\pi r^2}\left(\frac{\partial^2}{\partial r^2}\left[e^{-\alpha r}\left(r+\frac{\alpha r^2}{2}\right)\right]-\frac{\partial}{\partial r}\left[e^{-\alpha r}\left(2+\alpha r\right)\right]\right) \\
&= \frac{-q}{4\pi r^2}\left(\frac{\partial}{\partial r}\left[-\alpha e^{-\alpha r}\left(r+\frac{\alpha r^2}{2}\right)+e^{-\alpha r}\left(1+\alpha r\right)\right]+\alpha e^{-\alpha r}\left(2+\alpha r\right)-\alpha e^{-\alpha r}\right) \\
&= \frac{-q}{4\pi r^2}\left(\frac{\partial}{\partial r}\left[e^{-\alpha r}\left(1-\frac{\alpha^2 r^2}{2}\right)\right]+\alpha e^{-\alpha r}\left(1+\alpha r\right)\right) \\
&= \frac{-q}{4\pi r^2}\left(-\alpha e^{-\alpha r}\left(1-\frac{\alpha^2 r^2}{2}\right)-\alpha^2 r e^{-\alpha r}+\alpha e^{-\alpha r}\left(1+\alpha r\right)\right) \\
&= \frac{q\alpha e^{-\alpha r}}{4\pi r^2}\left(1-\frac{\alpha^2 r^2}{2}+\alpha r-\left(1+\alpha r\right)\right) \\
&= \frac{q\alpha e^{-\alpha r}}{4\pi r^2}\frac{\alpha^2 r^2}{2} = \boxed{\frac{q\alpha^3 e^{-\alpha r}}{8\pi}}
\end{aligned}
$$

So having computed the charge density for $r \neq 0$ we now proceed to account for the case where $r = 0$ by taking the limit as $r \to 0$ of the potential, leaving the singularity, and then computing ρ.

$$
\lim_{r\to 0}\Phi = \lim_{r\to 0}\left(\frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)\right) = \frac{q}{4\pi\epsilon_0}\lim_{r\to 0}\left(\frac{1}{r}\right)
$$

And since $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r)$ we know that

$$
\rho(r\to 0) = -\epsilon_0\nabla^2\left[\lim_{r\to 0}\Phi\right] = \boxed{q\delta(r)}
$$

implying there is a point charge of magnitude q at $r = 0$.