Separation of Variables In Multiple Coordinates

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Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a scalar field, and recall that the Laplacian of Φ is

$$\nabla^2 \Phi = \nabla \cdot \nabla \Phi = \nabla \cdot \left[\sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left[\Phi(x_1, \ldots, x_n) \right] \hat{\mathbf{x}}_{\mathbf{k}} \right] = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} \left[\Phi(x_1, \ldots, x_n) \right]$$

The crux of the separation of variables technique relies on the assumption that Φ can be written as the product of n independent scalar functions: $\exists X_1(x_1), \ldots, X_n(x_n) : \mathbb{R} \to \mathbb{R}$ st. $\Phi(x_1, \ldots, x_n) = X_1(x_1) \ldots X_n(x_n)$.

Then if we substitute this into the Laplacian we can find

$$\begin{aligned}
\nabla^2 \Phi &= \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} \left[\Phi(x_1, \ldots, x_n) \right] = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} X_1(x_1) \ldots X_n(x_n) \\
&= \sum_{k=1}^{n} X_1(x_1) \ldots X_{k-1}(x_{k-1}) X_{k+1}(x_{k+1}) \ldots X_n(x_n) \frac{d^2}{dx_k^2} X_k(x_k) \\
\frac{\nabla^2 \Phi}{\Phi} &= \sum_{k=1}^{n} \frac{1}{X_k(x_k)} \frac{d^2}{dx_k^2} X_k(x_k)
\end{aligned}$$