Numerical Analysis HW9
Ch19 - 2 (pg492)
Ch20 - 2 (pg517)

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Problem 19.2. Evaluate:

$$\int_0^4 \left(1 - e^{-x}\right) dx$$

(a) analytically

(b) single application of the trapezoidal rule

(c) composite trapezoidal rule with $n = 2, 4$

(d) single application of Simpson's 1/3 rule

(e) composite Simpson's 1/3 rule with $n = 4$

(f) Simpson's 3/8 rule

(g) composite Simpson's rule, with $n = 5$

Compute the percent relative error for each.

Solution 19.2.a. Analytically:

$$\int_0^4 \left(1 - e^{-x}\right) dx = \left[x + e^{-x}\right]_0^4 = 3 + e^{-4} \approx \boxed{3.01831}$$

Solution 19.2.b. Trapezoid:

$$\begin{aligned} \int_0^4 (1 - e^{-x}) dx &\approx (b - a) \left[(1 - e^{-a}) + (1 - e^{-b})\right] / 2 \\ &= 4 \left(2 - (e^{-0} + e^{-4}\right) / 2 \\ &= 2 \left(1 - e^{-4}\right) \approx \boxed{1.96336} \end{aligned}$$

Where the relative error is

$$|\varepsilon_r| = \left|\frac{3.01831 - 1.96336}{3.01831}\right| = \boxed{34.951\%}$$

Solution 19.2.c. Composite Trapezoid $n = 2, 4$:

$$h = (a - b)/n$$

Then for $k \in 1, \cdots, n$

$$x_k = a + kh = b - (n - k)h$$

Plugging in $a = 0, b = 4, n = 2$ gives

$$\begin{aligned}
\int_0^4 (1 - e^{-x})dx &\approx h\left[(1 - e^{-x_0}) + 2(1 - e^{-x_1}) + (1 - e^{-x_2})\right]/2 \\
&= 2\left[(1 - e^0) + 2(1 - e^{-2}) + (1 - e^{-4})\right]/2 \\
&= 4 - (e^0 + 2e^{-2} + e^{-4}) \\
&= 3 - (2e^{-2} + e^{-4}) \approx \boxed{2.71101}
\end{aligned}$$

Where the relative error is

$$|\varepsilon_r| = \left|\frac{3.01831 - 2.71101}{3.01831}\right| = \boxed{10.181\%}$$

While plugging in $n = 4$ gives

$$\begin{aligned}
\int_0^4 (1 - e^{-x})dx &\approx h\left[(1 - e^{-x_0}) + 2(1 - e^{-x_1}) + 2(1 - e^{-x_2}) + 2(1 - e^{-x_0}) + (1 - e^{-x_2})\right]/2 \\
&= \left[(1 - e^0) + 2(1 - e^{-1}) + 2(1 - e^{-2}) + 2(1 - e^{-3}) + (1 - e^{-4})\right]/2 \\
&= \left[7 - e^{-1} - 2e^{-2} - 2e^{-3} - e^{-4}\right]/2 \approx \boxed{3.12178}
\end{aligned}$$

Where the relative error is

$$|\varepsilon_r| = \left|\frac{3.01831 - 3.12178}{3.01831}\right| = \boxed{3.428\%}$$

Solution 19.2.d. Simpson 1/3:

$$\begin{aligned}
\int_0^4 (1 - e^{-x})dx &\approx h\left[(1 - e^{-x_0}) + 4(1 - e^{-x_1}) + (1 - e^{-x_2})\right]/3 \\
&= \frac{2}{3}\left[(1 - e^0) + 4(1 - e^{-2}) + (1 - e^{-4})\right] \\
&= \frac{2}{3}\left[6 - (e^0 + 4e^{-2} + e^{-4})\right] \\
&= \frac{2}{3}\left[5 - (4e^{-2} + e^{-4})\right] \approx \boxed{2.96023}
\end{aligned}$$

Where the relative error is

$$|\varepsilon_r| = \left|\frac{3.01831 - 2.96023}{3.01831}\right| = \boxed{1.924\%}$$

Solution 19.2.e. Composite Simpson 1/3 $n = 4$:

$$h = (a - b)/n$$

Then for $k \in 1, \cdots, n$

$$x_k = a + kh = b - (n - k)h$$

Then

$$\begin{aligned}
\int_0^4 (1 - e^{-x})dx &\approx \frac{h}{3}\left[(1 - e^{-x_0}) + 4(1 - e^{-x_1}) + 2(1 - e^{-x_2}) + 4(1 - e^{-x_3}) + (1 - e^{-x_4})\right] \\
&= \frac{1}{3}\left[(1 - e^0) + 4(1 - e^{-1}) + 2(1 - e^{-2}) + 4(1 - e^{-3}) + (1 - e^{-4})\right] \\
&= \frac{1}{3}\left[11 - (4e^{-1} + 2e^{-2} + 4e^{-3} + e^{-4})\right] \\
&= \frac{1}{3}\left[11 - 4(e^{-1} + e^{-3}) - 2e^{-2} - e^{-4}\right] \approx \boxed{3.01345}
\end{aligned}$$

Where the relative error is

$$|\varepsilon_r| = \left|\frac{3.01831 - 3.01345}{3.01831}\right| = \boxed{0.161\%}$$

Solution 19.2.f. Simpson 3/8:

$$\begin{aligned}
\int_0^4 (1 - e^{-x})dx &\approx \frac{3h}{8}\left[(1 - e^{-x_0}) + 3(1 - e^{-x_1}) + 3(1 - e^{-x_2}) + (1 - e^{-x_3})\right] \\
&= \frac{1}{2}\left[(1 - e^0) + 3(1 - e^{-4/3}) + 3(1 - e^{-8/3}) + (1 - e^{-4})\right] \\
&= \frac{1}{2}\left[7 - 3e^{-4/3} - 3e^{-8/3} - e^{-4}\right] \\
&= \frac{1}{2}\left[7 - 3(e^{-4/3} + e^{-8/3}) - e^{-4}\right] \\
&\approx \boxed{2.99122}
\end{aligned}$$

Where the relative error is

$$|\varepsilon_r| = \left|\frac{3.01831 - 2.99122}{3.01831}\right| = \boxed{0.898\%}$$

Solution 19.2.g. Composite Simpson $n = 5$:

$$h = (a - b)/n$$

Then for $k \in 1, \cdots, n$

$$x_k = a + kh = b - (n-k)h$$

$$\begin{aligned}
&\int_0^4 (1 - e^{-x})dx \\
&\approx \frac{3h}{8}\left[(1-e^{-x_0}) + 3(1-e^{-x_1}) + 3(1-e^{-x_2}) + (1-e^{-x_3})\right] + \frac{h}{3}\left[(1-e^{-x_3}) + 4(1-e^{-x_4}) + (1-e^{-x_5})\right] \\
&= \frac{3h}{8}\left[(1-e^{-0}) + 3(1-e^{-4/5}) + 3(1-e^{-8/5}) + (1-e^{-12/5})\right] + \frac{h}{3}\left[(1-e^{-12/5}) + 4(1-e^{-16/5}) + (1-e^{-2})\right] \\
&= \frac{3}{10}\left[7 - 3(e^{-4/5} + e^{-8/5}) - e^{-12/5}\right] + \frac{4}{15}\left[6 - e^{-12/5} - 4e^{-16/5} - e^{-2}\right] \\
&\approx \frac{3}{10}[4.95560] + \frac{4}{15}[5.88157] \\
&= 1.48668 + 1.56842 = \boxed{3.0551}
\end{aligned}$$

Where the relative error is

$$|\varepsilon_r| = \left|\frac{3.01831 - 3.05510}{3.01831}\right| = \boxed{1.219\%}$$

Problem 20.2. Evaluate:

$$I = \int_0^8 -0.055x^4 + 0.86x^3 - 4.2x^2 + 6.3x + 2dx$$

(a) analytically

(b) Romberg integration ($\varepsilon_s = 0.5\%$)

(c) Three point Gauss quadrature

(d) MATLAB `quad` function

Solution 20.2.a. Analytically:

$$\begin{aligned}
I &= \int_0^8 -0.055x^4 + 0.86x^3 - 4.2x^2 + 6.3x + 2dx \\
&= \left[-\frac{0.055}{5}x^5 + \frac{0.86}{4}x^4 - \frac{4.2}{3}x^3 + \frac{6.3}{2}x^2 + 2x\right]_0^8 \\
&= -\frac{0.055}{5}8^5 + \frac{0.86}{4}8^4 - \frac{4.2}{3}8^3 + \frac{6.3}{2}8^2 + 16 \\
&= -(0.011)32768 + (0.215)4096 - (1.4)512 + (3.15)64 + 16 \\
&= -360.448 + 880.64 - 716.8 + 201.6 + 16 = \boxed{20.992}
\end{aligned}$$

Solution 20.2.b. The matrix containing the steps for Romberg Integration (as in Fig 20.1), where we stop at a relative error of $\varepsilon_s = 0.5\%$, is given by:

2.5600	5.9733	20.9920	20.9920
5.1200	20.0533	20.9920	
16.3200	20.9333		
19.7800			

It takes three steps (of doubling the segments), each with full use of Richardson extrapolation to refine the approximation to produce these results.

Solution 20.2.c. Three point Gauss quadrature

Solution 20.2.d. Using the `quad` function in MATLAB gives the same result of 20.9920.