Complex Analysis HW1

Neal D. Nesbitt

August 31, 2016

Problem 1. Show that

(a)
$$\mathrm{Re}(iz) = -\mathrm{Im}(z)$$

(b)
$$\mathrm{Im}(iz) = -\mathrm{Re}(z)$$

Solution 1. (a) *Proof.* Let $z \in \mathbb{C}$ such that $z = x + iy$ for some $x, y \in \mathbb{R}$.

Then since $i^2 = -1$,
$$\mathrm{Re}(iz) = \mathrm{Re}(i(x+iy)) = \mathrm{Re}(-y+ix) = -y$$
and
$$-\mathrm{Im}(z) = -\mathrm{Im}(x+iy) = -y$$
implying
$$\mathrm{Re}(iz) = -\mathrm{Im}(z)$$

□

(b) *Proof.* Let $z \in \mathbb{C}$ such that $z = x + iy$ for some $x, y \in \mathbb{R}$.

Then since $i^2 = -1$,
$$\mathrm{Im}(iz) = \mathrm{Im}(i(x+iy)) = \mathrm{Im}(-y+ix) = x$$
and
$$\mathrm{Re}(z) = \mathrm{Re}(x+iy) = x$$
implying
$$\mathrm{Im}(iz) = \mathrm{Re}(z)$$

□

Problem 2. Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing
$$(x,y)(x,y) + (x,y) + (1,0) = (0,0)$$

Solution 2. Let our notation be as above, and then work out z^2, and match the real and imaginary parts to find:

$$\begin{aligned}
(x,y)(x,y)+(x,y)+(1,0)&=(0,0)\\
(x^2-y^2,2xy)+(x,y)+(1,0)&=(0,0)\\
x^2-y^2+x+1=0 &\qquad 2xy+y=0\\
x^2+x=y^2-1 &\qquad (2x+1)y=0
\end{aligned}$$

The imaginary component's equation implies that potential solutions have components $x=-1/2$ and $y=0$.

So beginning with the first possibility we plug $x=-1/2$ back into the real component's equation to see

$$\begin{aligned}
x^2+x&=y^2-1\\
\frac{1}{4}-\frac{1}{2}&=y^2-1\\
\frac{-1}{4}&=y^2-1\\
\frac{3}{4}&=y^2\\
y&=\pm\sqrt{\frac{3}{4}}=\pm\sqrt{3}/2
\end{aligned}$$

giving the pair of complex solutions $z=\left(-1\pm i\sqrt{3}\right)/2$.

Similarly, using $y=0$ in the same equation would show

$$\begin{aligned}
x^2+x&=-1\\
x^2+x+1&=0\\
x&=\left(-1\pm\sqrt{1-4}\right)/2\\
x&=\left(-1\pm\sqrt{-3}\right)/2\\
x&=\left(-1\pm i\sqrt{3}\right)/2
\end{aligned}$$

giving the same pair of solutions, but requiring the quadratic formula.

Problem 3. Reduce each of these quantities to a real number:

(a)

$$\frac{1+i2}{3-i4}+\frac{2-i}{5i}$$

(b) $(1-i)^4$

Solution 3. (a) Start by multiplying the first fraction by th complex conjugate of the denominator over itself. Then simplify:

$$\begin{aligned}
\frac{1+i2}{3-i4}+\frac{2-i}{5i} &= \frac{(1+i2)(3-i4)}{(9+16)}+\frac{1+i2}{5}\\
&= \frac{3+8+i(6-4)}{9+16}+\frac{1+i2}{5}\\
&= \frac{11+i2}{25}+\frac{1+i2}{5}\\
&= \frac{11+i2}{25}+\frac{5+i10}{25}\\
&= \frac{11+i2}{25}+\frac{5+i10}{25}\\
&= \boxed{\frac{16+i12}{25}}
\end{aligned}$$

(b) Using $i^2=-1$, we can find

$$\begin{aligned}
(1-i)^4 &= \left((1-i)^2\right)^2\\
&= (-2i)^2 = \boxed{-4}
\end{aligned}$$

Problem 4. Verify that $\sqrt{2}|z| \geq |\mathrm{Re}(z)|+|\mathrm{Im}(z)|$.

Solution 4. I we call $z=x+iy$, then we can see

$$\begin{aligned}
(|x|-|y|)^2 &\geq 0\\
|x|^2+|y|^2-2|x||y| &\geq 0\\
|x|^2+|y|^2 &\geq 2|x||y|\\
|z|^2 &\geq 2|x||y|\\
2|z|^2 &\geq 2|x||y|+|x|^2+|y|^2\\
2|z|^2 &\geq (|x|+|y|)^2\\
\sqrt{2}|z| &\geq |x|+|y| = |\mathrm{Re}(z)|+|\mathrm{Im}(z)|
\end{aligned}$$

Problem 5. In each case, sketch the set of points determined by the given condition:

(a) $|z-1+i|=1$

(b) $|z+i|\leq 3$

(c) $|z-4i|\geq 4$

Solution 5. (a)

Problem 6. Use properties of conjugates and moduli to show that

(a) $\overline{z+3i} = z - 3i$

(b) $\overline{iz} = -i\overline{z}$

Solution 6. (a) $\overline{z+3i} = \overline{z} - 3i$

If we call $z = x + iy$, then

$$\overline{z+3i} = \overline{x+iy+3i} = \overline{x+i(y+3)} = x - i(y+3) = x - iy - 3i = \overline{z} - 3i$$

(b) Similarly

$$\overline{iz} = \overline{i(x+iy)} = \overline{ix-y} = -ix - y = -i(x+iy) - i\overline{z}$$

Problem 7. Sketch the set of points determined by the condition $\text{Re}(\overline{z}-i) = 2$.

Solution 7. If $z = x + iy$, then

$$\text{Re}(\overline{z}-i) = \text{Re}(x-iy-i) = \text{Re}(x-i(y+1)) = x = 2$$

Showing that this is the vertical line in the complex plane of all points with real part 2.

Problem 8. By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using inequality (8), Section 4, show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \leq \frac{1}{3}$$

Solution 8. Take z such that $|z| = 2$. Then $|z| = |x + iy| = \sqrt{x^2+y^2} = 2$. So let us examine the following:

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| = \frac{1}{|z^2-3||z^2-1|}$$

Then by our given inequality, $|z^2-3| \geq \big||z|^2 - 3\big|$ and $|z^2-1| \geq \big||z|^2-1\big|$ which implies

$$\frac{1}{|z^2-3||z^2-1|} \leq \frac{1}{||z|^2-3|\,||z|^2-1|} = \frac{1}{|4-3|\,|4-1|} = \frac{1}{3}$$