Linear Algebra HW1
Exercises 1, 9, 12, 18, 61, 70, 87, 104, 117

Neal D. Nesbitt

June 16, 2016

Problem 1. Let F be a field and let $G = F \times F$. Define operations of addition and multiplication on G by setting $(a,b)+(c,d) = (a+b, c+d)$ and $(a,b)\cdot(c,d) = (ac, bd)$. Do these operations define the structure of a field on G?

Solution 1. These operations do not define a field on G.

Proof. Let F and $(G, +, \cdot)$ be defined as above. Then for any $[a \neq 0] \in F$, $[(a,0) \neq (0,0)] \in G$ has no inverse in G, since 0 has no inverse in F, and multiplication in G is done component-wise.

Thus $(G, +, \cdot)$ does not have inverses for every non-identity element, and is not a field. □

Problem 9. Let F be a field and define a new operation $\star$ on F by setting $a \star b = a + b + ab$. When is $(F, +, \star)$ a field?

Solution 9. Let $(F, +, \star)$ be defined as above, and notice the following:

Distributivity $\star$ does not distribute across addition for any $a, b, c \in F$.

$$\begin{aligned} a \star (b+c) &= \boxed{a+b+c+ab+ac} \\ (a \star b) + (a \star c) &= (a+b+ab) + (a+c+ac) \\ &= \boxed{2a+b+c+ab+ac} \end{aligned}$$

Meaning that $a = 0$ is the only element that distributes:

$$\begin{aligned} a \star (b+c) &= (a \star b) + (a \star c) \\ a+b+c+ab+ac &= 2a+b+c+ab+ac \\ a &= 2a \\ 0 &= a \end{aligned}$$

Thus F is only a field in the trivial case where $F = \{0\}$.

Problem 12. Let F be a field. Show that the function $a \mapsto a^{-1}$ is a permutation of $F\backslash\{0_F\}$

Solution 12. *Proof.* Let the notation be as above. A permutation is a function that is bijective and closed. Since every non-identity element has an inverse, then $F\backslash\{0_F\}$ will be closed under the inverse function.

So it remains to verify bijectivity. Since every non-identity element in F has an inverse, then the function must be surjective (onto) in $F\backslash\{0_F\}$, and since no two elements have the same inverse (inverses are unique), then the function is injective (one-to-one) in $F\backslash\{0_F\}$ as well. This shows together that the inverse function is bijective and, with closure, a permutation on $F\backslash\{0_F\}$. $\square$

Problem 18. Show that for all $z \in \mathbb{C}$, $|z+1| \leq |z+1|^2 + |z|$.

Solution 18. *Proof.* Let $z \in \mathbb{C}$. Note that for any $w \in \mathbb{C}$, $0 \leq |w|$. Consider the following cases:

If $|z+1| \leq |z|$, then

$$\begin{aligned} |z+1| - |z| &\leq 0 \leq |z+1|^2 \\ |z+1| &\leq |z+1|^2 + |z| \end{aligned}$$

and we are finished.

So instead assume that $|z+1| > |z|$, so that $|z+1| - |z| > 0$. Then by the triangle inequality,

$$\begin{aligned} 1 = |1 + z - z| &\leq |z+1| + |z| \\ (|z+1| - |z|) &\leq (|z+1| + |z|)\,(|z+1| - |z|) \\ |z+1| - |z| &\leq |z+1|^2 - |z|^2 \\ |z+1| + |z|^2 &\leq |z+1|^2 + |z| \\ |z+1| &\leq |z+1|^2 + |z| \end{aligned}$$

and again we have reached our desired result. $\square$

Problem 61. Let V be a non-trivial vector space over $\mathbb{R}$. For each $v \in V$ and each complex number $a+bi$, let us define $(a+bi)v = av$. Does V, together with this new scalar multiplication, form a vector space over $\mathbb{C}$?

Solution 61. The construction above does not form a vector space over $\mathbb{C}$.

Proof. Let the notation be as above, and let $(a+bi), (c+di) \in \mathbb{C}$ and $v \in V$. Then if we check the associativity of vector multiplication we find:

$$\begin{aligned} [(a+bi)(c+di)]\,v &= ((ac-bd) + (ad+bc)i)\,v = (ac-bd)v \\ (a+bi)\,[(c+di)]\,v &= (a+bi)[cv] = acv \end{aligned}$$

which means that only real numbers associate with scalar multiplication, and this construction does not form a vector space over $\mathbb{C}$. $\square$

Problem 70. Show that $\mathbb{Z}$ is not a vector space over any field.

Solution 70. *Proof.* Let F be a field, and assume by way of contradiction that $\mathbb{Z}$ forms a vector space over F.

Then if we pick $a \in F$ such that $a1 > 1$ (without loss of generality since $[a1 < 1] \to [1 < a^{-1}1]$). Since $\forall a \in F$, $1_F = a^{-1}a$, $0 = a0$, and $1 = 1_F 1$, then

$$\begin{aligned} a1 &> 1_F 1 > 0 \\ a1 &> a^{-1}a1 > a0 \\ 1 &> a^{-1}1 > 0 \end{aligned}$$

but there is no integer between zero and one. Therefore $a^{-1}1 \notin \mathbb{Z}$ and is thus not in the vector space, contradicting the need for closure under scalar multiplication. Since F was arbitrary, $\mathbb{Z}$ cannot form a vector space over any F. $\square$

Problem 87. Let F be a field, and let $V = F^F$, which is a vector space over F.

Let W be the set of all functions $f \in V$ such that $f(1) = f(-1)$. Is W a subspace of V?

Solution 87. Let the notation be as above. W is a subspace of V. To check its validity we verify its closure under addition and scalar multiplication for all $v, w \in W$ and $a \in F$.

Addition Since $v(1) = v(-1)$ and $w(1) = w(-1)$, then $(v+w)(1) = v(1) + w(1) = v(-1) + w(-1) = (v+w)(-1)$, showing W is closed under vector addition.

Scalar Multiplication Since $v(1) = v(-1)$, then $av(1) = av(-1)$, showing W is closed under scalar multiplication, and is thus a vector space as constructed above.

Problem 104. Find subspaces W and Y of $\mathbb{R}^3$ having the property that $W \cup Y$ is not a subspace of $\mathbb{R}^3$.

Solution 104. Let $W \subset \mathbb{R}^3$ be the x-y plane, and let $W \subset \mathbb{R}^3$ be the x-z plane. Then both are subspaces individually since they are equivalent to $\mathbb{R}^2$, but if we take a non-identity vector from each, their sum will not be in either one of the planes. For example $(1,1,0) + (0,1,1) = (1,2,1)$ which is not on either of the planes.

Problem 117. Define $\cdot$ on $\mathbb{R}^2$ such that for any $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{R}^2$

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2ac - bd \\ ad + bc \end{bmatrix}$$

Show this definition of vector multiplication makes $\mathbb{R}^2$ an $\mathbb{R}$-algebra.

Solution 117. Let the notation be as given. Then to verify the $\mathbb{R}$-algebra we must check that the vector multiplication distributes across vector addition, and that it associates with scalar multiplication.

So let $u = (u_1, u_2), v = (v_1, v_2), w = (w_1, w_2) \in \mathbb{R}^2$ and $a \in \mathbb{R}$.

Distributivity

$$\begin{aligned}
u \cdot (v + w) &= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \end{bmatrix} \\
&= \begin{bmatrix} 2u_1(v_1 + w_1) - u_2(v_2 + w_2) \\ u_1(v_2 + w_2) + u_2(v_1 + w_1) \end{bmatrix} \\
&= \begin{bmatrix} 2u_1v_1 - u_2v_2 \\ u_1v_2 + u_2v_2 \end{bmatrix} + \begin{bmatrix} 2u_1w_1 - u_2w_2 \\ u_1w_2 + u_2w_1 \end{bmatrix} \\
&= \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right) + \left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right) \\
&= (u \cdot v) + (u \cdot w)
\end{aligned}$$

And since

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2u_1v_1 - u_2v_2 \\ u_1v_2 + u_2v_1 \end{bmatrix} = \begin{bmatrix} 2v_1u_1 - v_2u_2 \\ v_1u_2 + v_2u_1 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Then vector multiplication is commutative, and vector multiplication distributes over vector addition from the left or the right:

$$\begin{aligned}
u \cdot (v + w) &= (u \cdot v) + (u \cdot w) \\
(v + w) \cdot u &= (v \cdot u) + (w \cdot u)
\end{aligned}$$

Associativity

$$\begin{aligned}
a(u \cdot v) &= a \begin{bmatrix} 2u_1v_1 - u_2v_2 \\ u_1v_2 + u_2v_1 \end{bmatrix} \\
&= \begin{bmatrix} a(2u_1v_1 - u_2v_2) \\ a(u_1v_2 + u_2v_1) \end{bmatrix} \\
&= a \begin{bmatrix} 2au_1v_1 - au_2v_2 \\ au_1v_2 + au_2v_1 \end{bmatrix} \\
&= (au) \cdot v
\end{aligned}$$

And thus vector multiplication associates properly, and with the distribution property above this verifies $\mathbb{R}^2$ as an $\mathbb{R}$-algebra.