Numerical Analysis HW3
Ch4 - 2,4,10,16 (pg120)

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Problem 2. Convert the following numbers from binary into decimal:

- `1011001`

  $2^0 + 2^3 + 2^4 + 2^6 = 1 + 8 + 16 + 64 = \boxed{89}$

- `0.01011`

  $2^{-2} + 2^{-4} + 2^{-5} = 0.25 + 0.0625 + 0.03125 = \boxed{0.34375}$

- `110.01001`

  $2^2 + 2^1 + 2^{-2} + 2^{-5} = 4 + 2 + 0.25 + 0.03125 = \boxed{6.28125}$

Problem 4. The machine epsilon is the smallest number that can be added to 1 and register as greater than 1 by the computer. Write a MATLAB program (based on the given algorithm in the book) to compute this number, and validate the script by comparing to the built in function `eps`.

```
% Clear our memory and working space
clear;
clc;

% Begin by displaying the value we wish to achieve
check = eps;
display(check);

% Calculate and display the manually computed value
ep = 1;
while ep+1>1
        ep = ep/2;
end
ep = ep*2;
```

```
display(ep);
```

Problem 10. The following infinite series can be used to approximate e^x

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$

- Prove that this Maclaurin series expansion is a special case of the Taylor series expansion (Eq. 4.13) with $x_i = 0$ and $h = x$.

  Proof. By equation 4.13 we know that a complete Taylor series expansion of f about x_i is given by:

  $$f(x_{i+1}) = \sum_{k=0}^{n} \frac{f^{(k)}(x_i)}{k!} h^k + R_n$$

  where $h = x_{i+1} - x_i$.

  Then if we set $x_i = 0$ and $h = x$ as proposed, then $x_{i+1} = x_i + h = 0 + x = x$, and we have that

  $$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n$$

  Thus if we take $f(x) = e^x$ noting that $e^0 = 1$, we arrive at the given formula as desired.

  $\square$

- Use the Taylor series to estimate $f(x) = e^{-x}$ at $x_{i+1} = 1$ for $x_i = 0.25$. Employ the zero through third order versions and compute the $|\epsilon_t|$ in each case.

  Again, using the given formula, we substitute in the appropriate values, finding that $h = x_{i+1} - x_i = 1 - 0.25 = 0.75$.

  $$f(1) = \sum_{k=0}^{n} \frac{f^{(k)}(0.25)}{k!} (0.75)^k + R_n$$

  Implying that for $f(x) = e^{-x}$, where $\forall m \in \mathbb{N}$,

  $$\frac{d^m}{dx^m}\left[e^{-x}\right] = (-1)^n e^{-x}$$

we have that

$$e^{-1} = \sum_{k=0}^{n} \frac{(-1)^k e^{-0.25}}{k!}(0.75)^k + R_n$$

Thus is remains only to employ each given order and compute their respective errors (when the true value is $e^{-1} \approx 0.3679$):

Zero Order:

$$\begin{aligned} e^{-1} &\approx e^{-0.25} \approx \boxed{0.7788} \\ |\epsilon_t| &= \left| \frac{e^{-1} - e^{-0.25}}{e^{-1}} \right| \approx \left| \frac{0.3679 - 0.7788}{0.3679} \right| \approx 111.70\% \end{aligned}$$

First Order:

$$\begin{aligned} e^{-1} &\approx e^{-0.25} - e^{-0.25}(0.75) \approx \boxed{0.1947} \\ |\epsilon_t| &= \left| \frac{e^{-1} - e^{-0.25}}{e^{-1}} \right| \approx \left| \frac{0.3679 - 0.1947}{0.3679} \right| \approx 47.07\% \end{aligned}$$

Second Order:

$$\begin{aligned} e^{-1} &\approx e^{-0.25} - e^{-0.25}(0.75) + \frac{e^{-0.25}}{2}(0.75)^2 \approx \boxed{0.4137} \\ |\epsilon_t| &= \left| \frac{e^{-1} - e^{-0.25}}{e^{-1}} \right| \approx \left| \frac{0.3679 - 0.4137}{0.3679} \right| \approx 12.47\% \end{aligned}$$

Third Order:

$$\begin{aligned} e^{-1} &\approx e^{-0.25} - e^{-0.25}(0.75) + \frac{e^{-0.25}}{2}(0.75)^2 - \frac{e^{-0.25}}{6}(0.75)^3 \approx \boxed{0.3590} \\ |\epsilon_t| &= \left| \frac{e^{-1} - e^{-0.25}}{e^{-1}} \right| \approx \left| \frac{0.3679 - 0.3590}{0.3679} \right| \approx 2.42\% \end{aligned}$$

Problem 14. Prove that Eq.4.11 is exact for all value of x if $f(x) = ax^2 + bx + c$.

Proof. Note that $f'(x) = 2ax + b$, $f''(x) = 2a$, and $\forall n \in \mathbb{N}, n > 2$

$$\frac{d^n}{dx^n}[f(x)] = 0$$

We then use equation 4.13 as in the previous problems, and notice that with f defined as above,

$$
\begin{aligned}
f(x_{i+1}) &= \sum_{k=0}^{n} \frac{f^{(k)}(x_i)}{k!} h^k + R_n \\
ax_{i+1}^2 + bx_{i+1} + c &= (ax_i^2 + bx_i + c) + (2ax_i + b)h + \frac{2a}{2} h^2 + R_n \\
&= ax_i^2 + bx_i + c + 2ax_i h + bh + ah^2 + R_n
\end{aligned}
$$

Which implies

$$
\begin{aligned}
R_n &= (ax_{i+1}^2 - ax_i^2 - 2ax_i h - ah^2) + (bx_{i+1} - bx_i - bh) + (c - c) \\
&= a(x_{i+1}^2 - x_i^2 - 2x_i h - h^2) + b(x_{i+1} - x_i - h) + 0 \\
&= a(x_{i+1}^2 - x_i^2 - 2x_i(x_{i+1} - x_i) - (x_{i+1} - x_i)^2) + b(0) \\
&= a(x_{i+1}^2 - x_i^2 - 2x_i x_{i+1} + 2x_i^2 - (x_{i+1}^2 - 2x_i x_{i+1} + x_i^2)) \\
&= a(x_{i+1}^2 - 2x_i x_{i+1} + x_i^2 - (x_{i+1}^2 - 2x_i x_{i+1} + x_i^2)) \\
&= a(0) \\
R_n &= 0
\end{aligned}
$$

But this is true for all $n \geq 2$, so then for any order Taylor expansion greater than or equal to 2,

$$
ax_{i+1}^2 + bx_{i+1} + c = f(0) + f'(0) + \frac{f''(x_i)}{2} h^2
$$

exactly and without approximation.

$\square$