Receding Horizon Control for MJS

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Given the control system

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1}$$

where u is dependent on x, we can recursively calculate the highest order term in terms of all the lower ones:

$$x_L = \left[\prod_{n=L-1}^{0} A_n\right] x_0 + \sum_{l=1}^{L-1}\left[\prod_{n=L-1}^{l} A_n\right] B_{l-1}u_{l-1} + B_{L-1}u_{L-1}$$

Use the following cost function:

$$J(x_0) = \sum_{k=1}^{L} x_k^{\mathrm{T}} R_k x_k + u_{k-1}^{\mathrm{T}} Q_{k-1} u_{k-1}$$

To minimize the control vector u with respect to the cost function, we take it's derivative with respect to the highest order term, since it is calculated with respect to the others.

$$\frac{\partial}{\partial u_{L-1}}\left[J(x_0)\right] = 2\left(x_L^{\mathrm{T}} R_L B_{L-1} + u_{L-1}^{\mathrm{T}} Q_{L-1}\right)$$

Then set this equal to zero to find critical points.

$$\begin{aligned}
2\left(x_L^{\mathrm{T}} R_L B_{L-1} + u_{L-1}^{\mathrm{T}} Q_{L-1}\right) &= 0\\
x_L^{\mathrm{T}} R_L B_{L-1} + u_{L-1}^{\mathrm{T}} Q_{L-1} &= 0\\
u_{L-1}^{\mathrm{T}} Q_{L-1} &= -x_L^{\mathrm{T}} R_L B_{L-1}\\
u_{L-1}^{\mathrm{T}} &= -x_L^{\mathrm{T}} R_L B_{L-1} Q_{L-1}^{-1}\\
u_{L-1} &= -Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L x_L
\end{aligned}$$

Replace x_L with the recursive formula, and solve for u_{L-1}

$$u_{L-1} = -Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L \left(\left[\prod_{n=L-1}^{0} A_n\right] x_0 + \sum_{l=1}^{L-1}\left[\prod_{n=L-1}^{l} A_n\right] B_{l-1}u_{l-1} + B_{L-1}u_{L-1}\right)$$

$$u_{L-1} = -Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L \left(\left[\prod_{n=L-1}^{0} A_n\right] x_0 + \sum_{l=1}^{L-1}\left[\prod_{n=L-1}^{l} A_n\right] B_{l-1}u_{l-1}\right) - Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L B_{L-1} u_{L-1}$$

$$u_{L-1} + Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L B_{L-1} u_{L-1} = -Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L \left(\left[\prod_{n=L-1}^{0} A_n \right] x_0 + \sum_{l=1}^{L-1} \left[\prod_{n=L-1}^{l} A_n \right] B_{l-1} u_{l-1} \right)$$

$$\left(I + Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L B_{L-1}\right) u_{L-1} = -Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L \left(\left[\prod_{n=L-1}^{0} A_n \right] x_0 + \sum_{l=1}^{L-1} \left[\prod_{n=L-1}^{l} A_n \right] B_{l-1} u_{l-1} \right)$$

$$u_{L-1} = -\left(I + Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L B_{L-1}\right)^{-1} Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L \left(\left[\prod_{n=L-1}^{0} A_n \right] x_0 + \sum_{l=1}^{L-1} \left[\prod_{n=L-1}^{l} A_n \right] B_{l-1} u_{l-1} \right)$$

$$u_{L-1} = -\left[\left(Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L\right)^{-1} \left(I + Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L B_{L-1}\right) \right]^{-1} \left(\left[\prod_{n=L-1}^{0} A_n \right] x_0 + \sum_{l=1}^{L-1} \left[\prod_{n=L-1}^{l} A_n \right] B_{l-1} u_{l-1} \right)$$

$$u_{L-1} = -\left[\left(Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L\right)^{-1} + B_{L-1} \right]^{-1} \left(\left[\prod_{n=L-1}^{0} A_n \right] x_0 + \sum_{l=1}^{L-1} \left[\prod_{n=L-1}^{l} A_n \right] B_{l-1} u_{l-1} \right)$$

$$u_{L-1} = -B_{L-1}^{-1} \left[\left(B_{L-1} Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L\right)^{-1} + I \right]^{-1} \left(\left[\prod_{n=L-1}^{0} A_n \right] x_0 + \sum_{l=1}^{L-1} \left[\prod_{n=L-1}^{l} A_n \right] B_{l-1} u_{l-1} \right) u_{L-1} \qquad =$$

Now call $\boxed{G_k = B_k Q_k^{-1} B_k^{\mathrm{T}} R_{k+1}}$, call $\boxed{H_k = [G_k + I]^{-1}}$, and notice both the commutativity of the matrices, the following identity,

$$\begin{aligned}
G_k H_k &= (G_k^{-1})^{-1} [G_k + I]^{-1} \\
G_k H_k &= [(G_k + I) G_k^{-1}]^{-1} \\
G_k H_k &= \boxed{[I + G_k^{-1}]^{-1}} \\
G_k H_k &= [G_k^{-1} (G_k + I)]^{-1} \\
G_k H_k &= [G_k + I]^{-1} G_k = H_k G_k
\end{aligned}$$

and the combination that produces G_k^{-1}:

$$\begin{aligned}
G_k H_k &= [I + G_k^{-1}]^{-1} \\
(G_k H_k)^{-1} &= I + G_k^{-1} \\
(G_k H_k)^{-1} - I &= G_k^{-1}
\end{aligned}$$

Then starting from the smallest term possible, solve upwards in terms of the original using the result above: $\forall k \geq 1$

$$u_k = -B_k^{-1} G_k H_k \left(\left[\prod_{n=k}^{0} A_n \right] x_0 + \sum_{l=1}^{k} \left[\prod_{n=k}^{l} A_n \right] B_{l-1} u_{l-1} \right)$$

For $k = 0$, begin the process with:

$$\boxed{u_0 = -B_0^{-1} G_0 H_0 A_0 x_0}$$

$k=1$ substitutes in the result,

$$
\begin{aligned}
u_1 &= -B_1^{-1} G_1 H_1 \left(A_1 A_0 x_0 + A_1 B_0 u_0 \right) \\
u_1 &= -B_1^{-1} G_1 H_1 A_1 \left(A_0 x_0 + B_0 u_0 \right) \\
u_1 &= -B_1^{-1} G_1 H_1 A_1 \left(A_0 x_0 - G_0 H_0 A_0 x_0 \right) \\
u_1 &= -B_1^{-1} G_1 H_1 A_1 \left(I - G_0 H_0 \right) A_0 x_0 \\
u_1 &= -B_1^{-1} G_1 H_1 A_1 G_0 H_0 \left((G_0 H_0)^{-1} - I \right) A_0 x_0 \\
u_1 &= -B_1^{-1} G_1 H_1 A_1 G_0 H_0 G_0^{-1} A_0 x_0 \\
u_1 &= -B_1^{-1} G_1 H_1 A_1 H_0 A_0 x_0
\end{aligned}
$$

$$\boxed{u_1 = -B_1^{-1} G_1 H_1 A_1 H_0 A_0 x_0}$$

And with $k=2$, the pattern should start to emerge:

$$
\begin{aligned}
u_2 &= -B_2^{-1} G_2 H_2 \left(A_2 A_1 A_0 x_0 + A_2 A_1 B_0 u_0 + A_2 B_1 u_1 \right) \\
u_2 &= -B_2^{-1} G_2 H_2 A_2 \left(A_1 A_0 x_0 + A_1 B_0 u_0 + B_1 u_1 \right) \\
u_2 &= -B_2^{-1} G_2 H_2 A_2 \left(A_1 A_0 x_0 - A_1 G_0 H_0 A_0 x_0 - G_1 H_1 A_1 H_0 A_0 x_0 \right) \\
u_2 &= -B_2^{-1} G_2 H_2 A_2 \left(A_1 - A_1 G_0 H_0 - G_1 H_1 A_1 H_0 \right) A_0 x_0 \\
u_2 &= -B_2^{-1} G_2 H_2 A_2 \left(A_1 G_0 H_0 ([G_0 H_0]^{-1} - I) - G_1 H_1 A_1 H_0 \right) A_0 x_0 \\
u_2 &= -B_2^{-1} G_2 H_2 A_2 \left(A_1 H_0 - G_1 H_1 A_1 H_0 \right) A_0 x_0 \\
u_2 &= -B_2^{-1} G_2 H_2 A_2 \left(I - G_1 H_1 \right) A_1 H_0 A_0 x_0 \\
u_2 &= -B_2^{-1} G_2 H_2 A_2 G_1 H_1 \left([G_1 H_1]^{-1} - I \right) A_1 H_0 A_0 x_0 \\
u_2 &= -B_2^{-1} G_2 H_2 A_2 H_1 A_1 H_0 A_0 x_0
\end{aligned}
$$

$$\boxed{u_2 = -B_2^{-1} G_2 H_2 A_2 H_1 A_1 H_0 A_0 x_0}$$

With these results, it seems that

$$\boxed{u_k = -B_k^{-1} G_k \left(\prod_{i=k}^{0} H_i A_i \right) x_0}$$

where

$$
\begin{aligned}
G_k &= B_k Q_k^{-1} B_k^{\mathrm{T}} R_{k+1} \\
H_k &= (G_k + I)^{-1}
\end{aligned}
$$

This should be enough to start simulations, though I have to complete the formal proof for the general case.