While we have to separate R into parts where

$$S(r) = \frac{\partial R}{\partial r}(r) \implies \frac{\partial S}{\partial r}(r) = \frac{\partial^2 R}{\partial r^2}(r)$$

$$\frac{\partial S}{\partial r}(r) = \frac{-1}{r} S(r) + \frac{\alpha}{r^2} R(r) \qquad\qquad \frac{\partial R}{\partial r}(r) = S(r)$$

$$\boxed{\begin{bmatrix} (\partial S/\partial r) \\ (\partial R/\partial r) \end{bmatrix} (r) = \frac{1}{r^2} \begin{bmatrix} -r & \alpha \\ r^2 & 0 \end{bmatrix} \begin{bmatrix} S(r) \\ R(r) \end{bmatrix}}$$

We can find the eigenvalues λ with the characteristic polynomial

$$|A - \lambda I| = \begin{vmatrix} -r - \lambda & \alpha \\ r^2 & -\lambda \end{vmatrix} = 0$$

$$\lambda(r + \lambda) - \alpha r^2 = 0$$
$$\lambda^2 + \lambda r - \alpha r^2 = 0$$

$$\lambda = \frac{1}{2}\left(-r \pm \sqrt{r^2 + 4\alpha r^2}\right)$$
$$\lambda = \boxed{\frac{-r}{2}\left(1 \mp \sqrt{1 + 4\alpha}\right)}$$

Then substituting this back into our original equation we find

$$\begin{bmatrix} -r + \frac{r}{2}\left(1 \mp \sqrt{1+4\alpha}\right) & \alpha \\ r^2 & \frac{r}{2}\left(1 \mp \sqrt{1+4\alpha}\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-r}{2}\left(1 \pm \sqrt{1+4\alpha}\right) & \alpha \\ r^2 & \frac{r}{2}\left(1 \mp \sqrt{1+4\alpha}\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-r}{2}\left(1 \pm \sqrt{1+4\alpha}\right) & \alpha \\ 0 & \frac{r}{2}\left(1 \mp \sqrt{1+4\alpha}\right) - \frac{2\alpha r}{(1 \pm \sqrt{1+4\alpha})} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-r}{2}\left(1 \pm \sqrt{1+4\alpha}\right) & \alpha \\ 0 & \frac{r}{2}\left(1 \mp \sqrt{1+4\alpha}\right) + \frac{2\alpha r\left(1 \mp \sqrt{1+4\alpha}\right)}{(1-(1+4\alpha))} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-r}{2}\left(1 \pm \sqrt{1+4\alpha}\right) & \alpha \\ 0 & \frac{r}{2}\left(1 \mp \sqrt{1+4\alpha}\right) - \frac{r}{2}\left(1 \mp \sqrt{1+4\alpha}\right) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{-r}{2}\left(1 \pm \sqrt{1+4\alpha}\right) & \alpha \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Showing the system is singular as expected, and we can trust our eigenvalue computations a little more.

We then use one of the equations to find our eigenvector lines. Using the second one with a little algebra gives

$$-2rv_1 = \left(1 \mp \sqrt{1+4\alpha}\right) v_2$$

where if we pick an arbitrary values of v_2 to simplify the math, $v_2 = -2r$ will make $v_1 = \left(1 \mp \sqrt{1+4\alpha}\right)$ the matrix of eigenvectors:

$$P = \boxed{\begin{bmatrix} \left(1-\sqrt{1+4\alpha}\right) & \left(1+\sqrt{1+4\alpha}\right) \\ -2r & -2r \end{bmatrix}}$$

Then

$$P^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{|P|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
$$P^{-1} = \frac{1}{|P|}\begin{bmatrix} -2r & -\left(1+\sqrt{1+4\alpha}\right) \\ 2r & \left(1-\sqrt{1+4\alpha}\right) \end{bmatrix}$$
$$P^{-1} = \frac{1}{\left|-2r\left(1-\sqrt{1+4\alpha}\right)+2r\left(1+\sqrt{1+4\alpha}\right)\right|}\begin{bmatrix} -2r & -\left(1+\sqrt{1+4\alpha}\right) \\ 2r & \left(1-\sqrt{1+4\alpha}\right) \end{bmatrix}$$
$$P^{-1} = \frac{1}{\left|4r\sqrt{1+4\alpha}\right|}\begin{bmatrix} -2r & -\left(1+\sqrt{1+4\alpha}\right) \\ 2r & \left(1-\sqrt{1+4\alpha}\right) \end{bmatrix}$$

and since r is always positive, and we always take the positive root unless specified otherwise, we can just write

$$P^{-1} = \boxed{\frac{1}{4r\sqrt{1+4\alpha}}\begin{bmatrix} -2r & -\left(1+\sqrt{1+4\alpha}\right) \\ 2r & \left(1-\sqrt{1+4\alpha}\right) \end{bmatrix}}$$

So double check

$$
\begin{aligned}
P^{-1}AP &= \frac{1}{4r\sqrt{1+4\alpha}} \begin{bmatrix} -2r & -\left(1+\sqrt{1+4\alpha}\right) \\ 2r & \left(1-\sqrt{1+4\alpha}\right) \end{bmatrix} \begin{bmatrix} -r & \alpha \\ r^2 & 0 \end{bmatrix} \begin{bmatrix} \left(1-\sqrt{1+4\alpha}\right) & \left(1+\sqrt{1+4\alpha}\right) \\ -2r & -2r \end{bmatrix} \\
&= \frac{1}{4r\sqrt{1+4\alpha}} \begin{bmatrix} 2r^2 - r^2\left(1+\sqrt{1+4\alpha}\right) & -2\alpha r \\ -2r^2 + r^2\left(1-\sqrt{1+4\alpha}\right) & 2\alpha r \end{bmatrix} \begin{bmatrix} \left(1-\sqrt{1+4\alpha}\right) & \left(1+\sqrt{1+4\alpha}\right) \\ -2r & -2r \end{bmatrix} \\
&= \frac{1}{4r\sqrt{1+4\alpha}} \begin{bmatrix} r^2\left(1-\sqrt{1+4\alpha}\right) & -2\alpha r \\ -r^2\left(1+\sqrt{1+4\alpha}\right) & 2\alpha r \end{bmatrix} \begin{bmatrix} \left(1-\sqrt{1+4\alpha}\right) & \left(1+\sqrt{1+4\alpha}\right) \\ -2r & -2r \end{bmatrix} \\
&= \frac{1}{4r\sqrt{1+4\alpha}} \begin{bmatrix} r^2\left(1-\sqrt{1+4\alpha}\right)^2 + 4\alpha r^2 & -4\alpha r + 4\alpha r \\ 4\alpha r^2 - 4\alpha r^2 & -r^2\left(1+\sqrt{1+4\alpha}\right)^2 - 4\alpha r^2 \end{bmatrix} \\
&= \frac{r}{4\sqrt{1+4\alpha}} \begin{bmatrix} 1 - 2\sqrt{1+4\alpha} + (1+4\alpha) + 4\alpha & 0 \\ 0 & -\left(1 + 2\sqrt{1+4\alpha} + (1+4\alpha)\right) - 4\alpha \end{bmatrix} \\
&= \frac{r}{4\sqrt{1+4\alpha}} \begin{bmatrix} 2 + 8\alpha - 2\sqrt{1+4\alpha} & 0 \\ 0 & -\left(2 + 8\alpha + 2\sqrt{1+4\alpha}\right) \end{bmatrix} \\
&= \frac{r}{2\sqrt{1+4\alpha}} \begin{bmatrix} (1+4\alpha) - \sqrt{1+4\alpha} & 0 \\ 0 & -\left((1+4\alpha) + \sqrt{1+4\alpha}\right) \end{bmatrix} \\
&= \frac{r}{2} \begin{bmatrix} \sqrt{1+4\alpha} - 1 & 0 \\ 0 & -\left(\sqrt{1+4\alpha} + 1\right) \end{bmatrix} \\
P^{-1}AP &= \begin{bmatrix} \frac{r}{2}(-1+\sqrt{1+4\alpha}) & 0 \\ 0 & \frac{r}{2}(-1-\sqrt{1+4\alpha}) \end{bmatrix} = \operatorname{diag}(\lambda_1, \lambda_2)
\end{aligned}
$$

So our P matrices can be used as expected to uncouple the system, and we can trust more in our eigenvector computations.

Then if we start with the system $\dot{\mathbf{x}} = cA\mathbf{x}$ and use the change of variables $\mathbf{y} = P\mathbf{x}$ such that $\mathbf{x} = P^{-1}\mathbf{y}$

$$
\dot{\mathbf{y}} = P\dot{\mathbf{x}} = PcA\mathbf{x} = cPAP^{-1}\mathbf{y} = c\ \operatorname{diag}\{\lambda_1, \lambda_2\}\mathbf{y}
$$

which gives a diagonal matrix with the solution

$$
\mathbf{y}(r) = e^{cPAP^{-1}(r)}\mathbf{y}(0) = \operatorname{diag}\{e^{c\lambda_1(r)}, e^{c\lambda_2(r)}\}\mathbf{y}(0)
$$

and then since $\mathbf{y} = P\mathbf{x}$ and

$$
P = \begin{bmatrix} \left(1-\sqrt{1+4\alpha}\right) & \left(1+\sqrt{1+4\alpha}\right) \\ -2r & -2r \end{bmatrix}, P^{-1} = \frac{1}{4r\sqrt{1+4\alpha}} \begin{bmatrix} -2r & -\left(1+\sqrt{1+4\alpha}\right) \\ 2r & \left(1-\sqrt{1+4\alpha}\right) \end{bmatrix}
$$

$$
\begin{aligned}
P\mathbf{x}(r) &= \operatorname{diag}\{e^{c\lambda_1(r)}, e^{c\lambda_2(r)}\}P\mathbf{x}(0) \\
\mathbf{x}(r) &= e^c P^{-1}\operatorname{diag}\{e^{\lambda_1(r)}, e^{c\lambda_2(r)}\}P\mathbf{x}(0)
\end{aligned}
$$

$$\begin{bmatrix} S(r) \\ R(r) \end{bmatrix} = \frac{e^{1/r^2}}{4r\sqrt{1+4\alpha}}$$

$$\begin{bmatrix} -2r & -\left(1+\sqrt{1+4\alpha}\right) \\ 2r & \left(1-\sqrt{1+4\alpha}\right) \end{bmatrix} \begin{bmatrix} e^{r^2(-1+\sqrt{1+4\alpha})/2} & 0 \\ 0 & e^{-r^2(1+\sqrt{1+4\alpha})/2} \end{bmatrix} \begin{bmatrix} \left(1-\sqrt{1+4\alpha}\right) & \left(1+\sqrt{1+4\alpha}\right) \\ -2r & -2r \end{bmatrix} \begin{bmatrix} S(0) \\ R(0) \end{bmatrix}$$

$$\begin{bmatrix} S(r) \\ R(r) \end{bmatrix} = \frac{e^{1/r^2}}{4r\sqrt{1+4\alpha}}$$

$$\begin{bmatrix} -2r & -\left(1+\sqrt{1+4\alpha}\right) \\ 2r & \left(1-\sqrt{1+4\alpha}\right) \end{bmatrix} \begin{bmatrix} e^{-r^2/2}e^{r^2\sqrt{1+4\alpha}/2} & 0 \\ 0 & e^{-r^2/2}e^{-r^2\sqrt{1+4\alpha}/2} \end{bmatrix} \begin{bmatrix} \left(1-\sqrt{1+4\alpha}\right) & \left(1+\sqrt{1+4\alpha}\right) \\ -2r & -2r \end{bmatrix} \begin{bmatrix} S(0) \\ R(0) \end{bmatrix}$$

$$\begin{bmatrix} S(r) \\ R(r) \end{bmatrix} = \frac{e^{1/r^2}e^{-r^2/2}}{4r\sqrt{1+4\alpha}}$$

$$\begin{bmatrix} -2r & -\left(1+\sqrt{1+4\alpha}\right) \\ 2r & \left(1-\sqrt{1+4\alpha}\right) \end{bmatrix} \begin{bmatrix} e^{r^2\sqrt{1+4\alpha}/2} & 0 \\ 0 & e^{-r^2\sqrt{1+4\alpha}/2} \end{bmatrix} \begin{bmatrix} \left(1-\sqrt{1+4\alpha}\right) & \left(1+\sqrt{1+4\alpha}\right) \\ -2r & -2r \end{bmatrix} \begin{bmatrix} S(0) \\ R(0) \end{bmatrix}$$

$$\begin{bmatrix} S(r) \\ R(r) \end{bmatrix} = \frac{e^{r^{-2}-r^2/2}}{4r\sqrt{1+4\alpha}}$$

$$\begin{bmatrix} -2re^{r^2\sqrt{1+4\alpha}/2} & -\left(1+\sqrt{1+4\alpha}\right)e^{-r^2\sqrt{1+4\alpha}/2} \\ 2re^{r^2\sqrt{1+4\alpha}/2} & \left(1-\sqrt{1+4\alpha}\right)e^{-r^2\sqrt{1+4\alpha}/2} \end{bmatrix} \begin{bmatrix} \left(1-\sqrt{1+4\alpha}\right) & \left(1+\sqrt{1+4\alpha}\right) \\ -2r & -2r \end{bmatrix} \begin{bmatrix} S(0) \\ R(0) \end{bmatrix}$$

$$\begin{bmatrix} S(r) \\ R(r) \end{bmatrix} = \frac{e^{r^{-2}-r^2/2}}{4r\sqrt{1+4\alpha}}$$

$$\begin{bmatrix} -2r\left(1-\sqrt{1+4\alpha}\right)e^{r^2\sqrt{1+4\alpha}/2} + 2r\left(1+\sqrt{1+4\alpha}\right)e^{-r^2\sqrt{1+4\alpha}/2} & -2r\left(1+\sqrt{1+4\alpha}\right)e^{r^2\sqrt{1+4\alpha}/2} + 2r\left(1+\sqrt{1+4\alpha}\right)e^{-r^2\sqrt{1+4\alpha}/2} \\ 2r\left(1-\sqrt{1+4\alpha}\right)e^{r^2\sqrt{1+4\alpha}/2} - 2r\left(1-\sqrt{1+4\alpha}\right)e^{-r^2\sqrt{1+4\alpha}/2} & 2r\left(1+\sqrt{1+4\alpha}\right)e^{r^2\sqrt{1+4\alpha}/2} - 2r\left(1-\sqrt{1+4\alpha}\right)e^{-r^2\sqrt{1+4\alpha}/2} \end{bmatrix} \begin{bmatrix} S(0) \\ R(0) \end{bmatrix}$$

$$\begin{bmatrix} S(r) \\ R(r) \end{bmatrix} = \frac{e^{r^{-2}-r^2/2}}{2\sqrt{1+4\alpha}}$$

$$\begin{bmatrix} -\left(1-\sqrt{1+4\alpha}\right)e^{r^2\sqrt{1+4\alpha}/2} + \left(1+\sqrt{1+4\alpha}\right)e^{-r^2\sqrt{1+4\alpha}/2} & -\left(1+\sqrt{1+4\alpha}\right)e^{r^2\sqrt{1+4\alpha}/2} + \left(1+\sqrt{1+4\alpha}\right)e^{-r^2\sqrt{1+4\alpha}/2} \\ \left(1-\sqrt{1+4\alpha}\right)e^{r^2\sqrt{1+4\alpha}/2} - \left(1-\sqrt{1+4\alpha}\right)e^{-r^2\sqrt{1+4\alpha}/2} & \left(1+\sqrt{1+4\alpha}\right)e^{r^2\sqrt{1+4\alpha}/2} - \left(1-\sqrt{1+4\alpha}\right)e^{-r^2\sqrt{1+4\alpha}/2} \end{bmatrix} \begin{bmatrix} S(0) \\ R(0) \end{bmatrix}$$

Now we attempt to solve this system of equations using Gaussian elimination. Take the first pivot coefficient to be

$$\frac{\left(1-\sqrt{1+4\alpha}\right)e^{r^2\sqrt{1+4\alpha}/2}-\left(1-\sqrt{1+4\alpha}\right)e^{-r^2\sqrt{1+4\alpha}/2}}{-\left(1-\sqrt{1+4\alpha}\right)e^{r^2\sqrt{1+4\alpha}/2}+\left(1+\sqrt{1+4\alpha}\right)e^{-r^2\sqrt{1+4\alpha}/2}}$$

such that the second column of the first row is scaled to

$$\frac{4\alpha e^{r^2\sqrt{1+4\alpha}}-8\alpha+4\alpha e^{-r^2\sqrt{1+4\alpha}}}{-\left(1-\sqrt{1+4\alpha}\right)e^{r^2\sqrt{1+4\alpha}/2}+\left(1+\sqrt{1+4\alpha}\right)e^{-r^2\sqrt{1+4\alpha}/2}}$$