Receding Horizon Control for MJS

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Given the control system

$$x_k = A_{k-1}x_{k-1} + B_{k-1}u_{k-1}$$

where u is dependent on x, we can recursively calculate the highest order term in terms of all the lower ones: $\forall k \in \mathbb{N}$

$$\begin{aligned}
x_1 &= A_0x_0 + B_0u_0 \\
x_2 &= A_1\left(A_0x_0 + B_0u_0\right) + B_1u_1 \\
&= A_1A_0x_0 + A_1B_0u_0 + B_1u_1 \\
x_3 &= A_2x_2 + B_2u_2 \\
&= A_2\left(A_1A_0x_0 + A_1B_0u_0 + B_1u_1\right) + B_2u_2 \\
&= A_2A_1A_0x_0 + A_2A_1B_0u_0 + A_2B_1u_1 + B_2u_2 \\
&\;\;\vdots \\
x_k &= \left(A_{k-1}\dots A_1A_0\right)x_0 + \left(A_{k-2}\dots A_2A_1\right)B_0u_0 + \left(A_{k-3}\dots A_3A_2\right)B_1u_1 + \cdots \\
&\qquad + \left(A_{k-2}A_{k-1}\right)B_{k-3}u_{k-3} + A_{k-1}B_{k-2}u_{k-2} + B_{k-1}u_{k-1}
\end{aligned}$$

$$x_{k+1} = \left[\prod_{n=k}^{0} A_n\right]x_0 + \sum_{j=1}^{k}\left[\prod_{n=k}^{j} A_n\right]B_{j-1}u_{j-1} + B_ku_k$$

So consider the following cost function:

$$\begin{aligned}
J(x_0) &= \sum_{k=1}^{L} x_k^{\mathrm{T}}R_kx_k + \sum_{k=0}^{L-1} u_k^{\mathrm{T}}Q_ku_k \\
J(x_0) &= \sum_{k=1}^{L} x_k^{\mathrm{T}}R_kx_k + u_{k-1}^{\mathrm{T}}Q_{k-1}u_{k-1}
\end{aligned}$$

Notice that there is no well defined R_0 or Q_L, and x_0 and u_L are not taken into account.

To begin minimizing the cost function, J, with respect to the control vector, u, we take it's derivative with respect to the highest order term of u, since it has no other variables that depend on it. Then, because

$$\frac{\partial}{\partial u_{k-1}}[x_k] = \frac{\partial}{\partial u_{k-1}}[A_{k-1}x_{k-1} + B_{k-1}u_{k-1}] = B_{k-1}$$

the chain rule gives

$$\frac{\partial}{\partial u_{L-1}}[J(x_0)] = 2\left(x_L^{\mathrm{T}} R_L B_{L-1} + u_{L-1}^{\mathrm{T}} Q_{L-1}\right)$$

Set this equal to zero to find critical points.

$$\begin{aligned}
2\left(x_L^{\mathrm{T}} R_L B_{L-1} + u_{L-1}^{\mathrm{T}} Q_{L-1}\right) &= 0 \\
x_L^{\mathrm{T}} R_L B_{L-1} + u_{L-1}^{\mathrm{T}} Q_{L-1} &= 0 \\
u_{L-1}^{\mathrm{T}} Q_{L-1} &= -x_L^{\mathrm{T}} R_L B_{L-1} \\
u_{L-1}^{\mathrm{T}} &= -x_L^{\mathrm{T}} R_L B_{L-1} Q_{L-1}^{-1} \\
u_{L-1} &= -Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L x_L \\
u_{L-1} &= -Q_{L-1}^{-1} B_{L-1}^{\mathrm{T}} R_L \left(A_{L-1}x_{L-1} + B_{L-1}u_{L-1}\right) \\
Q_{L-1}u_{L-1} &= -B_{L-1}^{\mathrm{T}} R_L A_{L-1} x_{L-1} - B_{L-1}^{\mathrm{T}} R_L B_{L-1} u_{L-1} \\
B_{L-1}^{\mathrm{T}} R_L B_{L-1} u_{L-1} + Q_{L-1}u_{L-1} &= -B_{L-1}^{\mathrm{T}} R_L A_{L-1} x_{L-1} \\
\left(B_{L-1}^{\mathrm{T}} R_L B_{L-1} + Q_{L-1}\right) u_{L-1} &= -B_{L-1}^{\mathrm{T}} R_L A_{L-1} x_{L-1}
\end{aligned}$$

So assuming that $\left(B_{L-1}^{\mathrm{T}} R_L B_{L-1} + Q_{L-1}\right)$ is invertible for all $L \in \mathbb{N}$, then

$$\boxed{u_{L-1} = -\left(B_{L-1}^{\mathrm{T}} R_L B_{L-1} + Q_{L-1}\right)^{-1} B_{L-1}^{\mathrm{T}} R_L A_{L-1} x_{L-1}}$$

So substitute this into the original system to find x_L:

$$\begin{aligned}
x_L &= A_{L-1}x_{L-1} + B_{L-1}u_{L-1} \\
x_L &= A_{L-1}x_{L-1} - B_{L-1}\left(B_{L-1}^{\mathrm{T}} R_L B_{L-1} + Q_{L-1}\right)^{-1} B_{L-1}^{\mathrm{T}} R_L A_{L-1} x_{L-1} \\
x_L &= \left(A_{L-1} - B_{L-1}\left(B_{L-1}^{\mathrm{T}} R_L B_{L-1} + Q_{L-1}\right)^{-1} B_{L-1}^{\mathrm{T}} R_L A_{L-1}\right) x_{L-1}
\end{aligned}$$

and let

$$\boxed{G_{L-1} = \left(B_{L-1}^{\mathrm{T}} R_L B_{L-1} + Q_{L-1}\right)^{-1} B_{L-1}^{\mathrm{T}} R_L A_{L-1}}$$

so that

$$\boxed{u_{L-1} = -G_{L-1}x_{L-1}}$$

and

$$x_L = \left(A_{L-1} - B_{L-1}G_{L-1}\right) x_{L-1}$$

And call

$$\boxed{H_{L-1} = (A_{L-1} - B_{L-1}G_{L-1})}$$

so that

$$\boxed{x_L = H_{L-1}x_{L-1}}$$

Now reevaluate the cost function with the resulting derivations:

$$\begin{aligned}
x_L^{\mathrm{T}} R_L x_L &= (H_{L-1}x_{L-1})^{\mathrm{T}} R_L H_{L-1} x_{L-1} \\
&= x_{L-1}^{\mathrm{T}} H_{L-1}^{\mathrm{T}} R_L H_{L-1} x_{L-1} \\
u_{L-1}^{\mathrm{T}} Q_{L-1} u_{L-1} &= (G_{L-1}x_{L-1})^{\mathrm{T}} R_L G_{L-1} x_{L-1} \\
&= x_{L-1}^{\mathrm{T}} G_{L-1}^{\mathrm{T}} R_L G_{L-1} x_{L-1}
\end{aligned}$$

Then

$$\boxed{x_L^{\mathrm{T}} R_L x_L + u_{L-1}^{\mathrm{T}} Q_{L-1} u_{L-1} = x_{L-1}^{\mathrm{T}} \left(H_{L-1}^{\mathrm{T}} R_L H_{L-1} + G_{L-1}^{\mathrm{T}} R_L G_{L-1} \right) x_{L-1}}$$

So consider $L \geq 2$. Substitute the result into the original system equation to reduce the number of terms by one.

$$J(x_0) = \left(\sum_{k=1}^{L-2} x_k^{\mathrm{T}} R_k x_k + u_{k-1}^{\mathrm{T}} Q_{k-1} u_{k-1} \right) + x_{L-1}^{\mathrm{T}} R_{L-1} x_{L-1} + u_{L-2}^{\mathrm{T}} Q_{L-2} u_{L-2} + x_L^{\mathrm{T}} R_L x_L + u_{L-1}^{\mathrm{T}} Q_{L-1} u_{L-1}$$

The trailing terms become

$$\begin{aligned}
& x_{L-1}^{\mathrm{T}} R_{L-1} x_{L-1} + x_L^{\mathrm{T}} R_L x_L + u_{L-1}^{\mathrm{T}} Q_{L-1} u_{L-1} \\
&= x_{L-1}^{\mathrm{T}} R_{L-1} x_{L-1} + x_{L-1}^{\mathrm{T}} \left(H_{L-1}^{\mathrm{T}} R_L H_{L-1} + G_{L-1}^{\mathrm{T}} R_L G_{L-1} \right) x_{L-1} \\
&= x_{L-1}^{\mathrm{T}} \left[R_{L-1} + H_{L-1}^{\mathrm{T}} R_L H_{L-1} + G_{L-1}^{\mathrm{T}} R_L G_{L-1} \right] x_{L-1}
\end{aligned}$$

Then, to simplify the math, call

$$\boxed{R_{L-1}^{\star} = R_{L-1} + H_{L-1}^{\mathrm{T}} R_L H_{L-1} + G_{L-1}^{\mathrm{T}} R_L G_{L-1}}$$

so that the last terms of the cost function reduce to

$$\boxed{x_{L-1}^{\mathrm{T}} R_{L-1} x_{L-1} + u_{L-2}^{\mathrm{T}} Q_{L-2} u_{L-2} + x_L^{\mathrm{T}} R_L x_L + u_{L-1}^{\mathrm{T}} Q_{L-1} u_{L-1} = x_{L-1}^{\mathrm{T}} R_{L-1}^{\star} x_{L-1} + u_{L-2}^{\mathrm{T}} Q_{L-2} u_{L-2}}$$

Now again since the last terms do not have anything else dependent on them, and since

$$\frac{\partial}{\partial u_{k-1}} [x_k] = \frac{\partial}{\partial u_{k-1}} [A_{k-1}x_{k-1} + B_{k-1}u_{k-1}] = B_{k-1}$$

the chain rule with the new derivation gives

$$\frac{\partial}{\partial u_{L-2}} [J(x_0)] = 2 \left(x_{L-1}^{\mathrm{T}} R_{L-1}^{\star} B_{L-2} + u_{L-1}^{\mathrm{T}} Q_{L-1} \right)$$

and minimizing with respect to this term is equivalent to the process above, producing the following for $L \geq 2$:

$$\boxed{u_{L-2} = -\left(B_{L-2}^{\mathrm{T}} R_{L-1}^{\star} B_{L-2} + Q_{L-2}\right)^{-1} B_{L-2}^{\mathrm{T}} R_{L-1}^{\star} A_{L-2} x_{L-2}}$$

$$\boxed{G_{L-2} = \left(B_{L-2}^{\mathrm{T}} R_{L-1}^{\star} B_{L-2} + Q_{L-2}\right)^{-1} B_{L-2}^{\mathrm{T}} R_{L-1}^{\star} A_{L-2}}$$

$$\boxed{u_{L-2} = -G_{L-2} x_{L-2}}$$

$$x_{L-1} = \left(A_{L-2} - B_{L-2} G_{L-2}\right) x_{L-2}$$

$$\boxed{H_{L-2} = \left(A_{L-2} - B_{L-2} G_{L-2}\right)}$$

$$\boxed{x_{L-1} = H_{L-2} x_{L-2}}$$

and for any $j \in \mathbb{N}$, where $L > j$, this can be repeated where

$$\boxed{u_{L-(j+1)} = -\left(B_{L-(j+1)}^{\mathrm{T}} R_{L-j}^{\star} B_{L-(j+1)} + Q_{L-(j+1)}\right)^{-1} B_{L-(j+1)}^{\mathrm{T}} R_{L-j}^{\star} A_{L-(j+1)} x_{L-(j+1)}}$$

$$\boxed{G_{L-(j+1)} = \left(B_{L-(j+1)}^{\mathrm{T}} R_{L-j}^{\star} B_{L-(j+1)} + Q_{L-(j+1)}\right)^{-1} B_{L-(j+1)}^{\mathrm{T}} R_{L-j}^{\star} A_{L-(j+1)}}$$

$$\boxed{u_{L-(j+1)} = -G_{L-(j+1)} x_{L-(j+1)}}$$

$$x_{L-j} = \left(A_{L-(j+1)} - B_{L-(j+1)} G_{L-j}\right) x_{L-(j+1)}$$

$$\boxed{H_{L-(j+1)} = \left(A_{L-(j+1)} - B_{L-(j+1)} G_{L-(j+1)}\right)}$$

$$\boxed{x_{L-j} = H_{L-(j+1)} x_{L-(j+1)}}$$

$$\boxed{R_{L-(j+1)}^{\star} = R_{L-(j+1)} + H_{L-(j+1)}^{\mathrm{T}} R_{L-j}^{\star} H_{L-(j+1)} + G_{L-(j+1)}^{\mathrm{T}} R_{L-j}^{\star} G_{L-(j+1)}}$$

all the way to the last terms in the cost function where:

$$x_1^{\mathrm{T}} R_1 x_1 + u_0^{\mathrm{T}} Q_0 u_0 + x_2^{\mathrm{T}} R_2 x_2 + u_1^{\mathrm{T}} Q_1 u_1 = x_1^{\mathrm{T}} R_1^{\star} x_1 + u_0^{\mathrm{T}} Q_0 u_0$$

$$\boxed{u_0 = -\left(B_0^{\mathrm{T}} R_1^{\star} B_0 + Q_0\right)^{-1} B_0^{\mathrm{T}} R_1^{\star} A_0 x_0}$$

$$\boxed{G_0 = \left(B_0^{\mathrm{T}} R_1^{\star} B_0 + Q_0\right)^{-1} B_0^{\mathrm{T}} R_1^{\star} A_0}$$

$$\boxed{u_0 = -G_0 x_0}$$

$$\boxed{x_1 = \left(A_0 - B_0 G_0\right) x_0}$$

Then if $L > 2$, for any $j \in \mathbb{Z}^+$ such that $j < L$, an explicit formula for $R_j^\star$ is needed.

Begin by recalling the initial case,

$$\boxed{R_{L-1}^\star = R_{L-1} + G_{L-1}^{\mathrm{T}} R_L G_{L-1} + H_{L-1}^{\mathrm{T}} R_L H_{L-1}}$$

and noticing that we can write this as

$$R_{L-1}^\star = R_{L-1} + \begin{bmatrix} G_{L-1}^{\mathrm{T}} & H_{L-1}^{\mathrm{T}} \end{bmatrix} R_L \begin{bmatrix} G_{L-1} \\ H_{L-1} \end{bmatrix}$$

Substituting this into the next iteration:

$$\begin{aligned} R_{L-2}^\star = R_{L-2} & \\ & + \begin{bmatrix} G_{L-2}^{\mathrm{T}} & H_{L-2}^{\mathrm{T}} \end{bmatrix} R_{L-1} \begin{bmatrix} G_{L-2} \\ H_{L-2} \end{bmatrix} + \begin{bmatrix} G_{L-2}^{\mathrm{T}} & H_{L-2}^{\mathrm{T}} \end{bmatrix} \left(\begin{bmatrix} G_{L-1}^{\mathrm{T}} & H_{L-1}^{\mathrm{T}} \end{bmatrix} R_L \begin{bmatrix} G_{L-1} \\ H_{L-1} \end{bmatrix} \right) \begin{bmatrix} G_{L-2} \\ H_{L-2} \end{bmatrix} \end{aligned}$$

and if we expand algebraically instead,

$$\begin{aligned} R_{L-2}^\star = R_{L-2} & \\ & + G_{L-2}^{\mathrm{T}} \left(R_{L-1} + G_{L-1}^{\mathrm{T}} R_L G_{L-1} + H_{L-1}^{\mathrm{T}} R_L H_{L-1} \right) G_{L-2} \\ & + H_{L-2}^{\mathrm{T}} \left(R_{L-1} + G_{L-1}^{\mathrm{T}} R_L G_{L-1} + H_{L-1}^{\mathrm{T}} R_L H_{L-1} \right) H_{L-2} \\ R_{L-2}^\star = R_{L-2} & \\ & + G_{L-2}^{\mathrm{T}} R_{L-1} G_{L-2} + H_{L-2}^{\mathrm{T}} R_{L-1} H_{L-2} \\ & + G_{L-2}^{\mathrm{T}} G_{L-1}^{\mathrm{T}} R_L G_{L-1} G_{L-2} + H_{L-2}^{\mathrm{T}} G_{L-1}^{\mathrm{T}} R_L G_{L-1} H_{L-2} \\ & + G_{L-2}^{\mathrm{T}} H_{L-1}^{\mathrm{T}} R_L H_{L-1} G_{L-2} + H_{L-2}^{\mathrm{T}} H_{L-1}^{\mathrm{T}} R_L H_{L-1} H_{L-2} \end{aligned}$$

Where in vector notation this becomes

$$\begin{aligned} R_{L-2}^\star = R_{L-2} & \\ & + \begin{bmatrix} G_{L-2}^{\mathrm{T}} & H_{L-2}^{\mathrm{T}} \end{bmatrix} R_{L-1} \begin{bmatrix} G_{L-2} \\ H_{L-2} \end{bmatrix} \\ & + \begin{bmatrix} G_{L-1}^{\mathrm{T}} G_{L-2}^{\mathrm{T}} & H_{L-2}^{\mathrm{T}} G_{L-1}^{\mathrm{T}} & G_{L-2}^{\mathrm{T}} H_{L-1}^{\mathrm{T}} & H_{L-1}^{\mathrm{T}} H_{L-2}^{\mathrm{T}} \end{bmatrix} R_{L-1} \begin{bmatrix} G_{L-1} G_{L-2} \\ G_{L-1} H_{L-2} \\ H_{L-1} G_{L-2} \\ H_{L-1} H_{L-2} \end{bmatrix} \end{aligned}$$

So consider following the improper notation:

$$\boxed{\begin{bmatrix} A \\ B \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} AC \\ AD \\ AE \\ BC \\ BD \\ BE \end{bmatrix}}$$

so that

$$\begin{bmatrix} A \\ B \end{bmatrix}^2 = \begin{bmatrix} AA \\ AB \\ BA \\ BB \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix}^3 = \begin{bmatrix} AAA \\ AAB \\ ABA \\ ABB \\ BAA \\ BAB \\ BBA \\ BBB \end{bmatrix}$$

and so on.

Additionally, call

$$\boxed{\begin{bmatrix} A \\ B \end{bmatrix}^{\mathrm{T}\star} = \begin{bmatrix} A^{\mathrm{T}} & B^{\mathrm{T}} \end{bmatrix}}$$

so that we can write

$$\boxed{R^{\star}_{L-j} = R_{L-j} + \sum_{k=0}^{j-1} \left(\left(\prod_{h=k+1}^{j} \begin{bmatrix} G_{L-h} \\ H_{L-h} \end{bmatrix} \right)^{\mathrm{T}\star} R_{L-k} \left(\prod_{h=k+1}^{j} \begin{bmatrix} G_{L-h} \\ H_{L-h} \end{bmatrix} \right) \right)}$$

We must then use this to find an explicit formula for G, and consequently H.

So again, consider the initial cases for G and H,

$$\boxed{G_{L-1} = \left(B^{\mathrm{T}}_{L-1} R_L B_{L-1} + Q_{L-1}\right)^{-1} B^{\mathrm{T}}_{L-1} R_L A_{L-1}}$$

$$\boxed{H_{L-1} = \left(A_{L-1} - B_{L-1} G_{L-1}\right)}$$

and for the first case including an $R^{\star}$ term,

$$G_{L-2} = \left(B^{\mathrm{T}}_{L-2} R^{\star}_{L-1} B_{L-2} + Q_{L-2}\right)^{-1} B^{\mathrm{T}}_{L-2} R^{\star}_{L-1} A_{L-2}$$

replace in the formula for $R^{\star}_{L-1}$.

$$\begin{aligned} G_{L-2} &= \left(B^{\mathrm{T}}_{L-2} R^{\star}_{L-1} B_{L-2} + Q_{L-2}\right)^{-1} B^{\mathrm{T}}_{L-2} R^{\star}_{L-1} A_{L-2} \\ G_{L-2} &= \left(B^{\mathrm{T}}_{L-2} \left(R_{L-1} + G^{\mathrm{T}}_{L-1} R_L G_{L-1} + H^{\mathrm{T}}_{L-1} R_L H_{L-1}\right) B_{L-2} + Q_{L-2}\right)^{-1} \\ &\quad B^{\mathrm{T}}_{L-2} \left(R_{L-1} + G^{\mathrm{T}}_{L-1} R_L G_{L-1} + H^{\mathrm{T}}_{L-1} R_L H_{L-1}\right) A_{L-2} \end{aligned}$$

Then compute separately

$$
\begin{aligned}
H_{L-1}^{\mathrm{T}} R_L H_{L-1} &= \left(A_{L-1} - B_{L-1} G_{L-1}\right)^{\mathrm{T}} R_L \left(A_{L-1} - B_{L-1} G_{L-1}\right) \\
&= \left(A_{L-1}^{\mathrm{T}} - G_{L-1}^{\mathrm{T}} B_{L-1}^{\mathrm{T}}\right) R_L \left(A_{L-1} - B_{L-1} G_{L-1}\right) \\
&= \left(A_{L-1}^{\mathrm{T}} R_L - G_{L-1}^{\mathrm{T}} B_{L-1}^{\mathrm{T}} R_L\right) \left(A_{L-1} - B_{L-1} G_{L-1}\right) \\
&= A_{L-1}^{\mathrm{T}} R_L A_{L-1} + G_{L-1}^{\mathrm{T}} B_{L-1}^{\mathrm{T}} R_L B_{L-1} G_{L-1} - A_{L-1}^{\mathrm{T}} R_L B_{L-1} G_{L-1} - G_{L-1}^{\mathrm{T}} B_{L-1}^{\mathrm{T}} R_L A_{L-1} \\
&= \begin{bmatrix} A_{L-1}^{\mathrm{T}} & G_{L-1}^{\mathrm{T}} B_{L-1}^{\mathrm{T}} \end{bmatrix} R_L \begin{bmatrix} A_{L-1} \\ B_{L-1} G_{L-1} \end{bmatrix} - \begin{bmatrix} A_{L-1}^{\mathrm{T}} & G_{L-1}^{\mathrm{T}} B_{L-1}^{\mathrm{T}} \end{bmatrix} R_L \begin{bmatrix} B_{L-1} G_{L-1} \\ A_{L-1} \end{bmatrix} \\
&= \begin{bmatrix} A_{L-1}^{\mathrm{T}} & G_{L-1}^{\mathrm{T}} B_{L-1}^{\mathrm{T}} \end{bmatrix} R_L \left(\begin{bmatrix} A_{L-1} \\ B_{L-1} G_{L-1} \end{bmatrix} - \begin{bmatrix} B_{L-1} G_{L-1} \\ A_{L-1} \end{bmatrix} \right)
\end{aligned}
$$