Lecture 3

Changing variables & Jacobians

3.1 Introduction

The slope of the $f = f(x, y)$ surface depends on the direction in which one moves from the point (x, y), and in Lecture 1 we learnt that the partial derivative must be defined in terms of the change along a particular direction or axis.

For a function $f(x, y)$ we decided that the obvious directions to choose are along the x and y axes, keeping y and x fixed, respectively.

The question now is are these *really* the obvious directions?

For example, consider the function

$$f(x, y) = e^{-(x^2+y^2)} \cos(4(x^2 + y^2))$$

shown in Figure 3.1. Does it really make sense to impose a square "x-constant, y-constant" cake-rack mesh onto this function? Probably not! It would be in more sympathy with the function to use a mesh with radial symmetry as in Figure 3.2.

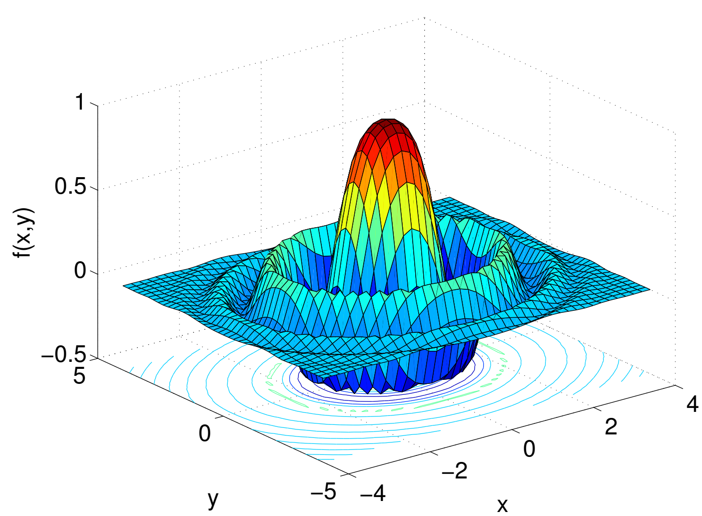

Figure 3.1: The function $\exp -(x^2 + y^2) \cos(4(x^2 + y^2))$.

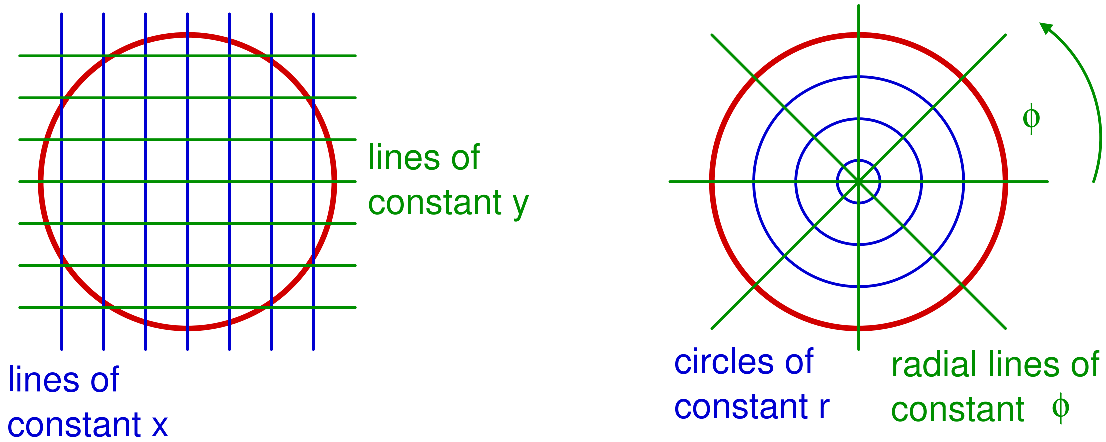

Figure 3.2: A radial "dartboard" mesh reflects the symmetry rather better than the x, y constant cake rack.

To achieve this we must make an appropriate **transformation** to a **new set of variables**. This raises the questions of

- How to choose the new variables and thus describe the transformation.
- How to describe the function in the new variables.
- How to find the partial derivatives with respect to these new variables.

3.1.1 The general problem

The more general problem is illustrated in Figure 3.3. Subfigure (a) shows a surface $f(x, y)$ with lines of constant x and y underneath. The partial derivatives wrt x and y are found by slicing or moving along these directions.

Figure 3.3(b) shows the exactly the same surface with lines of constant u and v underneath. The partial derivatives wrt u and v are found by slicing along these new directions.

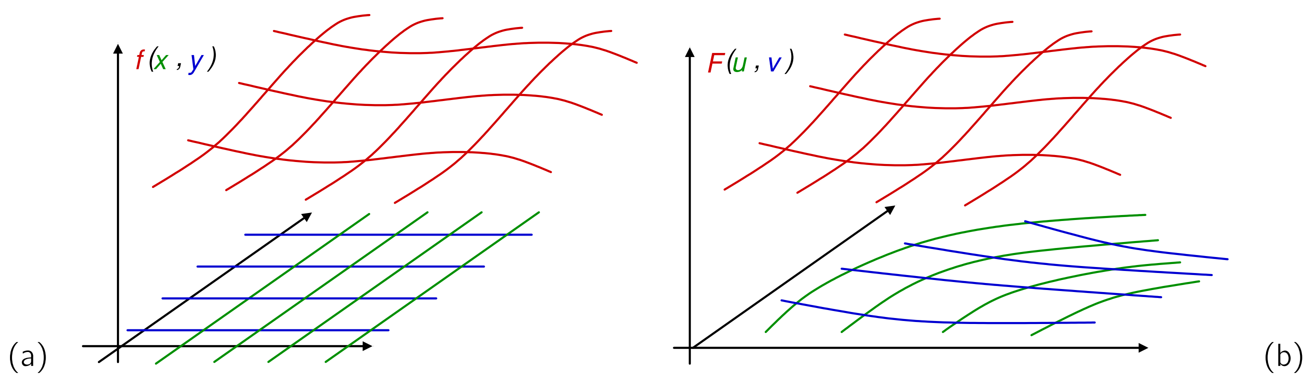

Figure 3.3: (a) Surface $f(x, y)$ above a set of x-constant and y-constant lines. (b) Surface $F(u, v)$ above a set of u-constant and v-constant curves but plotted in the x, y frame. The surface has exactly the same shape.

3.1.2 Choosing the new variables

There is no routine method for doing this, but usually the problem symmetry drops very large hints as to the best transformation.

Later we will consider some of the most standard transformations, viz between Cartesian and

- Plane Polar Coordinates (2D): for radial symmetry
- Spherical Polar Coordinates (3D): for spherical symmetry, and
- Cylindrical Polars Coordinates (3D): for cylindrical symmetry.

We also consider what would make a bad transformation.

3.2 Rewriting the function and finding the derivatives

First though we should consider the general problems of rewriting the function and finding the partial derivatives with respect to arbitrary new variables (u, v).

There are two main cases to consider:

1. The straightforward case. **OLD variables in terms of NEW**.

$$x = x(u, v) \quad \text{and} \quad y = y(u, v) \quad . \tag{3.1}$$

2. The more complicated case. **NEW variables in terms of OLD**.

$$u = u(x, y) \quad \text{and} \quad v = v(x, y) \quad . \tag{3.2}$$

3.2.1 CASES 1A and 1B: OLD variables in terms of NEW

We are transforming from (x, y) to (u, v) coordinates, and the transformation is given as "old in terms of new" variables. That is:

$$x = x(u, v) \quad \text{and} \quad y = y(u, v) \quad . \tag{3.3}$$

Case 1A:

If we know what the function $f(x, y)$ is explicitly, we can replace x and y in the function to get the new function as

$$f = f(x(u, v), y(u, v)) = F(u, v) \quad . \tag{3.4}$$

Then one can work out the partial derivatives with respect to u and v directly from the new function $F(u, v)$.

♣ Example of Case 1A.

Let $f(x, y) = x/y$, and let the transformation be $x = u$ and $y = u/v$.

$$\Rightarrow f = F(u, v) = \frac{u}{(u/v)} = v \quad \text{and} \quad \Rightarrow \frac{\partial f}{\partial u} = 0, \quad \frac{\partial f}{\partial v} = 1 \ . \tag{3.5}$$

Case 1B: In this case we don't have an explicit form for $f(x, y)$, and, therefore, cannot find an explicit form for function $F(u, v)$.

But we can still write expressions for the partial derivatives using the Chain Rule for partials from Lecture 2:

Case 1B: Use the Chain Rule for partials to obtain the derivatives

$$\left(\frac{\partial f}{\partial u}\right) = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial u}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial u}\right) \tag{3.6}$$

$$\left(\frac{\partial f}{\partial v}\right) = \left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial x}{\partial v}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial y}{\partial v}\right) \tag{3.7}$$

♣ Example of Case 1B.

We are told $f = f(x, y)$, and $x = u$ and $y = u/v$.

Given this transformation, the best we can do is to write for **any** function $f(x, y)$

$$\left(\frac{\partial f}{\partial u}\right) = \left(\frac{\partial f}{\partial x}\right)(1) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{1}{v}\right) = \left(\frac{\partial f}{\partial x}\right) + \left(\frac{\partial f}{\partial y}\right)\frac{1}{v} \tag{3.8}$$

$$\left(\frac{\partial f}{\partial v}\right) = \left(\frac{\partial f}{\partial x}\right)(0) + \left(\frac{\partial f}{\partial y}\right)\left(-\frac{u}{v^2}\right) = -\left(\frac{\partial f}{\partial y}\right)\frac{u}{v^2} \ . \tag{3.9}$$

To check this result, let's use (as in the example of Case 1A) $f(x, y) = x/y$. Then

$$\left(\frac{\partial f}{\partial u}\right) = \frac{1}{y}(1) + \left(-\frac{x}{y^2}\right)\left(\frac{1}{v}\right) = \left(\frac{v}{u}\right) - \left(\frac{u}{v}\right)\left(\frac{v^2}{u^2}\right) = \left(\frac{v}{u}\right) - \left(\frac{v}{u}\right) = 0 \tag{3.10}$$

$$\left(\frac{\partial f}{\partial v}\right) = \frac{1}{y}(0) + \left(-\frac{x}{y^2}\right)\left(-\frac{u}{v^2}\right) = \frac{u^2 v}{u^2 v} = 1 \tag{3.11}$$

This is exactly as found earlier at Equation 3.5.

3.2.2 A source of confusion introduced, and cleared up

Before continuing, it is worth checking that you are happy that the functions $f()$ and $F()$ have different names. They work differently on their parameters, so should be named differently.

But notice that the chain rule for partials seems to deliver an expression for $\partial f/\partial u$ and $\partial f/\partial v$.

But you might complain that f is a function of (x, y) NOT (u, v), and so we should have written $\partial F/\partial u$ and $\partial F/\partial v$. You are correct. So how are we getting away with it?

We are using f as a "go-between" value $f = f(x, y)$ and $f = F(u, v)$. This is correct because at corresponding values of (x, y) and (u, v) the functions do return exactly the same value. (Look back at Figure 3.3.)

Some books introduce another symbol (z, say) as the "go-between". They write $z = f(x, y)$ and $z = F(u, v)$ and then write

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} , \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} , \tag{3.12}$$

and so on — but this seems over-fussy.

3.2.3 CASES 2A and 2B: NEW variables in terms of OLD

We still wish to transform the function $f = f(x, y)$ to (u, v) coordinates. But now the transformation is given as "new in terms of old" variables, that is:

$$u = u(x, y) \quad \text{and} \quad v = v(x, y) \ . \tag{3.13}$$

One's first thought is that this should be the better way of defining a transformation, but it turns out to be really inconvenient!

New-ito-Old prevents us from finding either $F(u, v)$ or using the Chain Rule for Partials. We cannot immediately work out $\partial x/\partial u$, etc.

We proceed in one of two ways.

Case 2A: Try to invert the transformation.
Use $u = u(x, y)$ and $v = v(x, y)$ as simultaneous equations from which to find $x = x(u, v)$ and $y = y(u, v)$.
If you can invert, the problem becomes a Case 1 problem.

♣ Example of Case 2A

Suppose you are told that

$$u = x \quad \text{and} \quad v = x/y \ . \tag{3.14}$$

These are new in terms of old. But it is obvious that

$$x = u, \quad \text{and} \quad \Rightarrow v = u/y, \quad \Rightarrow y = u/v \ . \tag{3.15}$$

We now have Old-in-terms-of-New, and can move to Case 1 above.

Case 2B: When the transformation is not invertible Write down the Chain Rule for Partials the "wrong" way round! That is, we want $\partial f/\partial u$, so start with $\partial f/\partial x$.

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial x} \tag{3.16}$$

$$\frac{\partial f}{\partial y} = \left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial y} \tag{3.17}$$

Treat these as simultaneous equations in the unknown $\partial f/\partial u$ and $\partial f/\partial v$ and re-arrange. Ie:

$$\frac{\partial f}{\partial u} = \left(\frac{\partial f}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial v}{\partial x}\right) / \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{3.18}$$

$$\frac{\partial f}{\partial v} = \left(\frac{\partial f}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial u}{\partial y}\right) / \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{3.19}$$

So we can recover the partial derivatives with respect to the new variables without an explicit expression for the function in the new variables.

3.2.4 SUMMARY

To summarize:

1: If you have the transformation as **OLD variables in terms of NEW** either

- **1A:** find the function explicity and find the partials from it directly, or
- **1B:** find just the partials using the Chain-rule-for-partials directly.

2: If you have the transformation as **NEW variables in terms of OLD** either

- **2A:** invert the given transformation to get OLD variables in terms of NEW, and then go to CASE 1, or
- **2B:** use the Chain-rule-for-partials for the OLD variables to give simultaneous equations for the partial derivatives with respect to to the NEW variables.

♣ Example #1

Q: The elevation of a hill is described by

$$z = z(x, y) = \exp\left\{-(x^2 + y^2)\right\} \ . \tag{3.20}$$

By deriving the function in plane polars, find $\partial z/\partial r$ and $\partial z/\partial \phi$ and comment on their values. Determine the value of r where the hill is steepest.

A: Plane polars are (r, ϕ) where $x = r\cos\phi$, $y = r\sin\phi$.

This is old in terms of new, and we know the function, ⇒CASE 1A.

The hill is therefore

$$z = \exp\left\{-r^2\right\} \ . \tag{3.21}$$

So

$$\frac{\partial z}{\partial r} = -2r\exp\left\{-r^2\right\} \quad \text{and} \quad \frac{\partial z}{\partial \phi} = 0 \ . \tag{3.22}$$

The change of height is all in the radial direction. $\partial z/\partial \phi = 0$ means that if you move at constant r (that is, "round" the hill) you will not change height at all.

The gradient is a function of r alone, so we can find the total derivative

$$\frac{\mathrm{d}}{\mathrm{d}r}\text{Slope} = \frac{\mathrm{d}}{\mathrm{d}r}(-2r\exp\left\{-r^2\right\}) = (-2 + 4r^2)\exp\left\{-r^2\right\} \tag{3.23}$$

This is zero when $r = 1/\sqrt{2}$ and Slope $= -\sqrt{2}\exp\{-1/2\}$.

♣ Example #2.

Here is one that can be solved using all approaches.

Q: Find the partial derivatives $\partial f/\partial u$ and $\partial f/\partial v$ when

$$f = f(x, y) = x^2 - y^2 \quad \text{and} \quad u = (x + y) \ \ v = (x - y) \ . \tag{3.24}$$

A: Transformation is new in terms of old. ⇒CASE 2.

Try CASE 2A: Can we invert the transformation? Yes! Adding then substracting we find

$$x = \frac{1}{2}(u + v) \quad y = \frac{1}{2}(u - v) \ . \tag{3.25}$$

Now goto CASE 1. We know the function, **⇒CASE 1A.**

$$f = F(u, v) = x^2 - y^2 = \frac{1}{4}\left((u + v)^2 - (u - v)^2\right) = uv \tag{3.26}$$

So the derivatives are

$$\frac{\partial f}{\partial u} = v \quad \text{and} \quad \frac{\partial f}{\partial v} = u \tag{3.27}$$

That's the problem solved. But now notice that if **CASE *A** works, we can always find **just the derivatives** using **CASE *B**.

For example, having inverted the transformation to get

$$x = \frac{1}{2}(u+v) \quad y = \frac{1}{2}(u-v) \quad .$$

we could use CASE 1B to find just the derivatives:

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = (2x).\frac{1}{2} + (-2y).\frac{1}{2} = x - y = v \tag{3.28}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = (2x).\frac{1}{2} + (-2y).\frac{-1}{2} = x + y = u \tag{3.29}$$

Now pretend that we did not (or could not) invert the transformation. We could still find the derivative using Case 2B.

Write the chain rule for partials the other way around:

$$\frac{\partial f}{\partial x} = \left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial x} + \left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial x} \tag{3.30}$$

$$\frac{\partial f}{\partial y} = \left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial y} + \left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial y} \tag{3.31}$$

and for simplicity fill in the values you know

$$2x = \left(\frac{\partial f}{\partial u}\right)(1) + \left(\frac{\partial f}{\partial v}\right)(1) \tag{3.32}$$

$$-2y = \left(\frac{\partial f}{\partial u}\right)(1) + \left(\frac{\partial f}{\partial v}\right)(-1) \tag{3.33}$$

Now use as simultaneous equations for $\partial f/\partial u$ and $\partial f/\partial v$:

$$2x - 2y = 2\left(\frac{\partial f}{\partial u}\right) \quad \Rightarrow \quad \left(\frac{\partial f}{\partial u}\right) = (x-y) = v \quad , \tag{3.34}$$

and subtracting gives, once again, the same results:

$$2x + 2y = 2\left(\frac{\partial f}{\partial v}\right) \quad \Rightarrow \quad \left(\frac{\partial f}{\partial v}\right) = (x+y) = u \quad . \tag{3.35}$$

♣ **Example #3.** Harder!
Q: A quantity $V = V(x, y)$ satisfies Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \ . \tag{3.36}$$

Express this equation in plane polar coordinates (r, ϕ) where $x = r\cos\phi$, $y = r\sin\phi$.

A: We want first to write down expressions for $\partial V/\partial x$ and $\partial V/\partial y$ which will involve $\partial r/\partial x$, $\partial r/\partial y$, $\partial\phi/\partial x$, and $\partial\phi/\partial y$. This is the wrong way round for the transformation. So

- Either invert the transformation (Case 2A), or
- Write the CRfP the "other way round" (Case 2B) and use sim eqs.

Both will work, but let's use 2B ...

$$\begin{aligned}
\frac{\partial V}{\partial r} &= \left(\frac{\partial V}{\partial x}\right)\frac{\partial x}{\partial r} + \left(\frac{\partial V}{\partial y}\right)\frac{\partial y}{\partial r} = \left(\frac{\partial V}{\partial x}\right)\cos\phi \qquad + \left(\frac{\partial V}{\partial y}\right)\sin\phi && (3.37)\\
\frac{\partial V}{\partial \phi} &= \left(\frac{\partial V}{\partial x}\right)\frac{\partial x}{\partial \phi} + \left(\frac{\partial V}{\partial y}\right)\frac{\partial y}{\partial \phi} = \left(\frac{\partial V}{\partial x}\right)(-r\sin\phi) + \left(\frac{\partial V}{\partial y}\right)(r\cos\phi) && (3.38)
\end{aligned}$$

Sorting out the simultaneous equations (do check!) gives

$$\begin{aligned}
\frac{\partial V}{\partial x} &= \cos\phi\frac{\partial V}{\partial r} - \frac{\sin\phi}{r}\frac{\partial V}{\partial \phi} && (3.39)\\
\frac{\partial V}{\partial y} &= \sin\phi\frac{\partial V}{\partial r} + \frac{\cos\phi}{r}\frac{\partial V}{\partial \phi} && (3.40)
\end{aligned}$$

We can drop the V and consider $\frac{\partial}{\partial x}$ as an *operator*

$$\begin{aligned}
\frac{\partial}{\partial x} &= \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi} && (3.41)\\
\frac{\partial}{\partial y} &= \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi} && (3.42)
\end{aligned}$$

So the operator

$$\frac{\partial^2}{\partial x^2} = \left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right) \tag{3.43}$$

$$= \cos^2\phi\frac{\partial^2}{\partial r^2} - \cos\phi\sin\phi\left[-\frac{1}{r^2}\frac{\partial}{\partial\phi} + \frac{1}{r}\frac{\partial^2}{\partial\phi\partial r}\right] \tag{3.44}$$

$$-\frac{1}{r}\sin\phi\left[-\sin\phi\frac{\partial}{\partial r} + \cos\phi\frac{\partial^2}{\partial\phi\partial r}\right] + \frac{-\sin\phi}{r}\left[-\frac{\cos\phi}{r}\frac{\partial}{\partial\phi} - \frac{\sin\phi}{r}\frac{\partial^2}{\partial\phi^2}\right]$$

which can be tidied up a little. Note how the operators move through to the right, operating as they go.

Let's follow through a couple of terms. The following is straightforward:

$$\left(\cos\phi\frac{\partial}{\partial r}\right)\left(\cos\phi\frac{\partial}{\partial r}\right) = \cos^2\phi\frac{\partial^2}{\partial r^2} \tag{3.45}$$

but this requires the product rule

$$\left(\cos\phi\frac{\partial}{\partial r}\right)\left(-\frac{\sin\phi}{r}\frac{\partial}{\partial\phi}\right) = -\cos\phi\sin\phi\left(-\frac{1}{r^2}\right)\frac{\partial}{\partial\phi} - \cos\phi\sin\phi\left(\frac{1}{r}\right)\frac{\partial^2}{\partial r\partial\phi} \tag{3.46}$$

The tidied expression for y is (you should check!)

$$\frac{\partial^2}{\partial y^2} = \sin\phi^2\frac{\partial^2}{\partial r^2} - \cos\phi\sin\phi\left[\frac{2}{r^2}\frac{\partial}{\partial\phi} - \frac{1}{r}\frac{\partial^2}{\partial\phi\partial r}\right] \tag{3.47}$$

$$+\frac{1}{r}\cos\phi\left[\cos\phi\frac{\partial}{\partial r} + \sin\phi\frac{\partial^2}{\partial\phi\partial r}\right] + \frac{1}{r^2}\cos\phi^2\frac{\partial^2}{\partial\phi^2} \tag{3.48}$$

Adding them up:

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right]. \tag{3.49}$$

So Laplace's equation in plane polar coordinates is

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\phi^2} = 0. \tag{3.50}$$

3.3 Jacobians (Jacobian determinants)

Recall the equations that appeared from solving the simultaneous equations in case 2B:

$$\frac{\partial f}{\partial u} = \left(\frac{\partial f}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial v}{\partial x}\right) \Big/ \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{3.51}$$

$$\frac{\partial f}{\partial v} = \left(\frac{\partial f}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial u}{\partial y}\right) \Big/ \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) \tag{3.52}$$

Notice that the denominator is the same in both expressions. It has special significance, and is called the Jacobian.

It is common to write the Jacobian as a determinant, but there is also another useful notation.

$$J = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial(u,v)}{\partial(x,y)} \tag{3.53}$$

As we shall see later, the Jacobian is especially useful when changing variables from (x, y) to (u, v) in multiple integrals.

3.3.1 Multiple definitions of the Jacobian determinant?

In some books one reads that

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} \tag{3.54}$$

while in others

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad . \tag{3.55}$$

This is puzzling, until you realize that they are **identical**! However, perhaps the first way is to be preferred for reasons we see now.

3.3.2 Jacobians and their inverses

In examples given later you will see that

$$\frac{\partial(x,y)}{\partial(u,v)} = 1 \Bigg/ \frac{\partial(u,v)}{\partial(x,y)} \tag{3.56}$$

We shall gain an intuitive understanding of why this must be so when we come to use Jacobians in multiple integration. However it is quite easy to prove using the **Jacobian matrix**. This is not on the syllabus.

The Chain rule for partials says

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \qquad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \tag{3.57}$$

or, switching the order of the products,

$$\frac{\partial f}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial f}{\partial y} \qquad \frac{\partial f}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial f}{\partial y} \tag{3.58}$$

We can treat these as operator equations: in steps ...

$$\frac{\partial}{\partial u}f = \frac{\partial x}{\partial u}\frac{\partial}{\partial x}f + \frac{\partial y}{\partial u}\frac{\partial}{\partial y}f \qquad \frac{\partial}{\partial v}f = \frac{\partial x}{\partial v}\frac{\partial}{\partial x}f + \frac{\partial y}{\partial v}\frac{\partial}{\partial y}f \tag{3.59}$$

$$\Rightarrow \frac{\partial}{\partial u} \equiv \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} \qquad \frac{\partial}{\partial v} \equiv \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} \tag{3.60}$$

In matrix notation, this operator is

$$\begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} = \begin{bmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{bmatrix} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \boldsymbol{J}_{\frac{x,y}{u,v}} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} \tag{3.61}$$

The matrix $\boldsymbol{J}_{\frac{x,y}{u,v}}$ is a **Jacobian matrix**.

Now repeat with the Chain Rule around the other way. You will arrive at the following

$$\begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \begin{bmatrix} \partial u/\partial x & \partial v/\partial x \\ \partial u/\partial y & \partial v/\partial y \end{bmatrix} \begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} = \boldsymbol{J}_{\frac{u,v}{x,y}} \begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} \tag{3.62}$$

But it follows that

$$\begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} = \boldsymbol{J}_{\frac{x,y}{u,v}} \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = \boldsymbol{J}_{\frac{x,y}{u,v}} \boldsymbol{J}_{\frac{u,v}{x,y}} \begin{bmatrix} \partial/\partial u \\ \partial/\partial v \end{bmatrix} \tag{3.63}$$

so that

$$\boldsymbol{J}_{\frac{x,y}{u,v}} \boldsymbol{J}_{\frac{u,v}{x,y}} = \boldsymbol{I} \quad \Rightarrow \quad \boldsymbol{J}_{\frac{x,y}{u,v}} = \left[\boldsymbol{J}_{\frac{u,v}{x,y}}\right]^{-1} . \tag{3.64}$$

But a result from linear algebra tells us that, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are square, $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$, and hence $|\boldsymbol{A}||\boldsymbol{A}^{-1}| = 1$. The result for Jacobian determinants follows immediately.

3.4 Three standard transformations

There are certain transformations which occur very frequently, viz:
(1) Cartesian to plane polar coordinates;
(2) Cartesian to spherical polar coordinates; and
(3) Cartesian to cylindrical polar coordinates.

3.4.1 Cartesian to plane polars

This is useful for problems with radial symmetry in the plane.

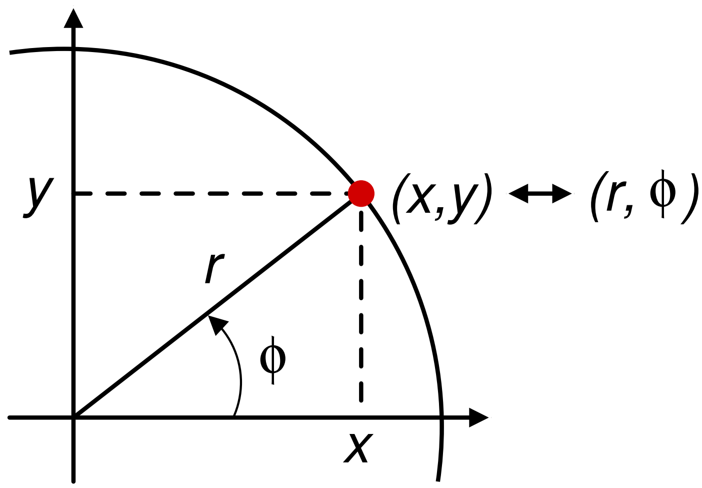

Figure 3.4: Cartesian to plane polars. Many texts use θ rather than ϕ.

The transformation is

$$x = r\cos\phi,\ \ y = r\sin\phi \tag{3.65}$$

$$\Rightarrow \frac{\partial x}{\partial r} = \cos\phi;\ \ \frac{\partial x}{\partial \phi} = -r\sin\phi;\ \ \frac{\partial y}{\partial r} = \sin\phi;\ \ \frac{\partial y}{\partial \phi} = r\cos\phi \tag{3.66}$$

Writing the CRfP in operator form gives

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y} \tag{3.67}$$

$$\frac{\partial}{\partial \phi} = \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} = -r\sin\phi\frac{\partial}{\partial x} + r\cos\phi\frac{\partial}{\partial y}\ . \tag{3.68}$$

Using these are simutaneous equations gives

$$\frac{\partial}{\partial x} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi} \quad \text{and} \quad \frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi}\ . \tag{3.69}$$

(You could also find these by inverting the transformation.)

The Jacobian

$$\frac{\partial(x,y)}{\partial(r,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{vmatrix} = r\cos^2\phi + r\sin^2\phi = r\ . \tag{3.70}$$

and

$$\frac{\partial(r,\phi)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} \end{vmatrix} = (\cos\phi)(\frac{\cos\phi}{r}) - (\sin\phi)(-\frac{\sin\phi}{r}) = \frac{1}{r}\ . \tag{3.71}$$

3.4.2 3D Cartesian to cylindrical polars

Cylindrical polars add a z-axis, perpendicular to the plane, which is identical to the Cartesian z-axis.

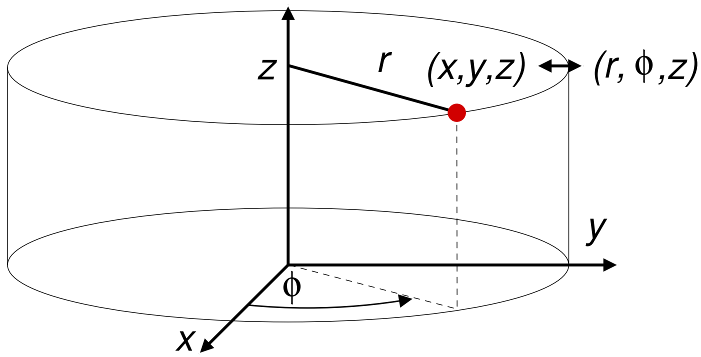

Figure 3.5: Cartesian x, y, z to cylindrical polars r, ϕ, z

The transformation is

$$x = r\cos\phi; \quad y = r\sin\phi; \quad z = z. \tag{3.72}$$

So

$$\frac{\partial}{\partial r} = \cos\phi\frac{\partial}{\partial x} + \sin\phi\frac{\partial}{\partial y} \tag{3.73}$$

$$\frac{\partial}{\partial \phi} = -r\sin\phi\frac{\partial}{\partial x} + r\cos\phi\frac{\partial}{\partial y} \qquad \text{and} \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \tag{3.74}$$

and using these as simultaneous equations (or inverting the transformation):

$$\frac{\partial}{\partial x} = \cos\phi\frac{\partial}{\partial r} - \frac{\sin\phi}{r}\frac{\partial}{\partial \phi} \tag{3.75}$$

$$\frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial r} + \frac{\cos\phi}{r}\frac{\partial}{\partial \phi} \qquad \text{and} \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} \tag{3.76}$$

The Jacobian is now a 3×3 determinant

$$\frac{\partial(x,y,z)}{\partial(r,\phi,z)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & \sin\phi & 0 \\ -r\sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r, \tag{3.77}$$

and (do check!)

$$\frac{\partial(r,\phi,z)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial \phi}{\partial x} & \frac{\partial z}{\partial x} \\ \frac{\partial r}{\partial y} & \frac{\partial \phi}{\partial y} & \frac{\partial z}{\partial y} \\ \frac{\partial r}{\partial z} & \frac{\partial \phi}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \frac{1}{r}\ . \tag{3.78}$$

Most entries can be copied over from the plane polar case.

3.4.3 Cartesian to spherical polars

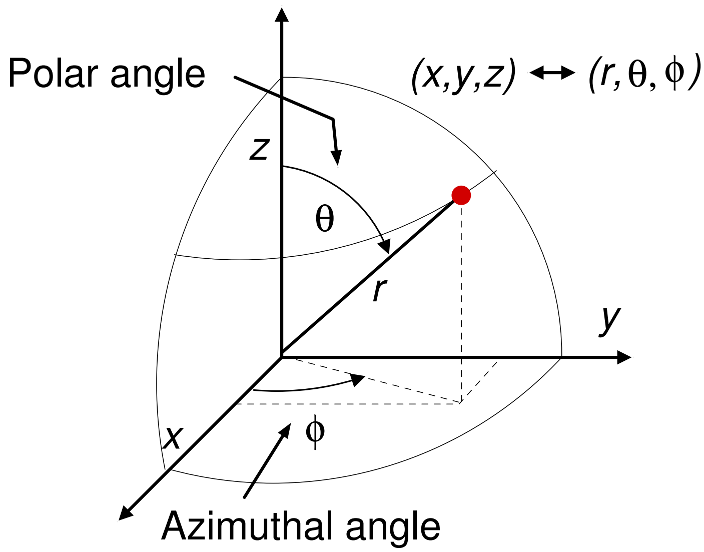

Figure 3.6: Cartesian to spherical polars

The transformation involves the radius r of a sphere $r^2 = x^2 + y^2 + z^2$, the polar angle $0 \le \theta \le \pi$ and the azimuthal angle $0 \le \phi \le 2\pi$

$$x = r\sin\theta\cos\phi; \quad y = r\sin\theta\sin\phi; \quad z = r\cos\theta. \tag{3.79}$$

Hence

$$\frac{\partial}{\partial r} = \sin\theta\cos\phi\frac{\partial}{\partial x} + \sin\theta\sin\phi\frac{\partial}{\partial y} + \cos\theta\frac{\partial}{\partial z} \tag{3.80}$$

$$\frac{\partial}{\partial \theta} = r\cos\theta\cos\phi\frac{\partial}{\partial x} + r\cos\theta\sin\phi\frac{\partial}{\partial y} - r\sin\theta\frac{\partial}{\partial z} \tag{3.81}$$

$$\frac{\partial}{\partial \phi} = -r\sin\theta\sin\phi\frac{\partial}{\partial x} + r\sin\theta\cos\phi\frac{\partial}{\partial y} \tag{3.82}$$

and

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{3.83}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{3.84}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \tag{3.85}$$

The Jacobian

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\cos\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta,$$

(3.86)

and as an exercise you should show that

$$\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \frac{1}{r^2 \sin\theta} \ . \tag{3.87}$$

3.5 A "good" transformation

Cylindrical and spherical polars are particularly useful when solving problems involving cylindrical and spherical regions. But what in general makes a "good" transformation.

If you were asked what is the nicest region shape to fit into Cartesian coordinates Oxy you would answer a rectangle.

Similarly, if you drew your Ouv coordinate system at right angles and asked the same question, you would again answer a square.

So given some arbitrary region shape in the xy plane, the best sort of transformation is one which maps it onto a rectangle in the uv plane, as illustrated in Figure 3.7(a). A specific example is given in Figure 3.7(b) for plane polars. The quadrant's extent is awkard to describe in Cartesians, but easy in describe in plane polars.

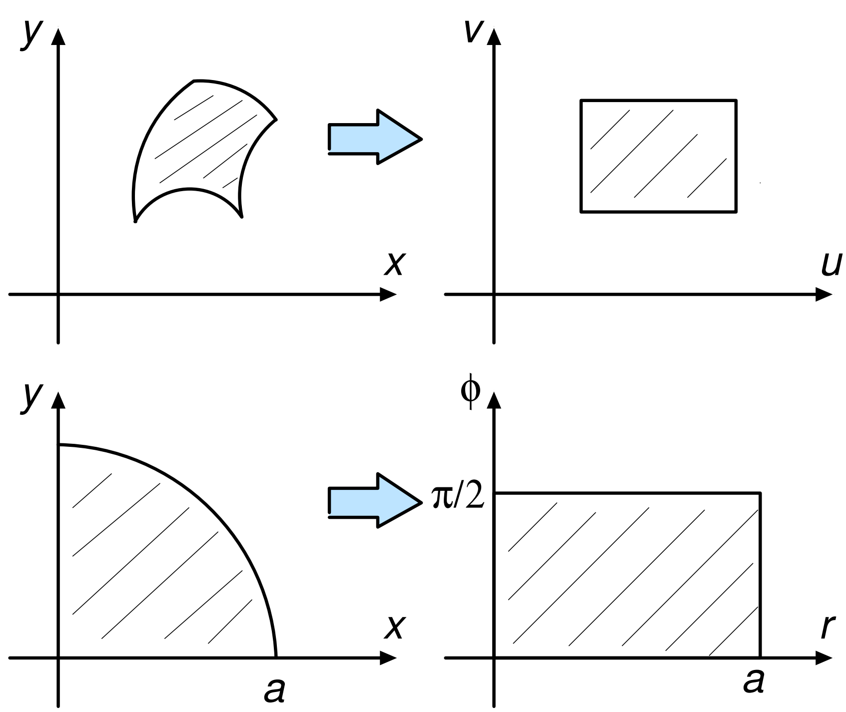

Figure 3.7: Useful transformations tend to take an awkward region shape in Oxy into a rectangle in Ouv.

3.5.1 Will any transformation work? When is one bad?

Another question is "will any transformation work?" The answer is no. If we start with two (or n) independent variables (x, y), we must ensure that the new two (or n) are also independent.

The test for independence is rather simple.

If $u(x, y)$ and $v(x, y)$ are functionally INDEPENDENT then

$$\frac{\partial(u, v)}{\partial(x, y)} \neq 0. \tag{3.88}$$

Recall that the Jacobian is the denominator in a relationship relating partial derivatives with respect to different sets of variables. Trouble is expected if this is zero.

Proof: For reading only

The proof is straightforward. If $u(x, y)$ and $v(x, y)$ are functionally dependent, then $u = u(v)$, which in turn means that a function $z = z(u, v) = 0$ exists. Differentiating z with respect to x and y gives:

$$\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 0 \tag{3.89}$$

$$\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = 0 \tag{3.90}$$

For consistency between these two equations we must have $u_x = \beta u_y$ and $v_x = \beta v_y$, where β is some constant. Thus

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = (u_x v_y - v_x u_y) = (\beta u_y v_y - \beta v_y u_y) = 0 \quad . \tag{3.91}$$

♣ **Example**

Q: Test whether there is a functional dependence between the variables (u, v) if

$$u = x^2 + y + 1 \quad \text{and} \quad v = x^4 + 2x^2 y + y^2 + x^2 - y. \tag{3.92}$$

A:

$$\begin{aligned} \frac{\partial(u, v)}{\partial(x, y)} &= (u_x v_y - v_x u_y) \qquad (3.93) \\ &= (2x)(2x^2 + 2y - 1) - (4x^3 + 4xy + 2x)(1) \\ &= (4x^3 + 4xy - 2x) - (4x^3 + 4xy + 2x) \\ &= 0 \end{aligned}$$

So, yes the variables are not independent. With a little work you'll spot that the dependence is $v = u^2 - 3u + 2$.