Complex Analysis HW1

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Problem 1. Show that

(a)

$$\mathrm{Re}(iz) = -\mathrm{Im}(z)$$

(b)

$$\mathrm{Im}(iz) = -\mathrm{Re}(z)$$

Solution 1. (a) *Proof.* Let $z \in \mathbb{C}$ such that $z = x + iy$ for some $x, y \in \mathbb{R}$.

Then since $i^2 = -1$,

$$\mathrm{Re}(iz) = \mathrm{Re}(i(x + iy)) = \mathrm{Re}(-y + ix) = -y$$

and

$$-\mathrm{Im}(z) = -\mathrm{Im}(x + iy) = -y$$

implying

$$\mathrm{Re}(iz) = -\mathrm{Im}(z)$$

□

(b) *Proof.* Let $z \in \mathbb{C}$ such that $z = x + iy$ for some $x, y \in \mathbb{R}$.

Then since $i^2 = -1$,

$$\mathrm{Im}(iz) = \mathrm{Im}(i(x + iy)) = \mathrm{Im}(-y + ix) = x$$

and

$$\mathrm{Re}(z) = \mathrm{Re}(x + iy) = x$$

implying

$$\mathrm{Im}(iz) = \mathrm{Re}(z)$$

□

Problem 2. Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0)$$

Solution 2. Let our notation be as above, and then work out z^2, and match the real and imaginary parts to find:

$$\begin{aligned}(x,y)(x,y)+(x,y)+(1,0)&=(0,0)\\(x^2-y^2,2xy)+(x,y)+(1,0)&=(0,0)\\x^2-y^2+x+1=0 &\qquad\qquad 2xy+y=0\\x^2+x=y^2-1 &\qquad\qquad (2x+1)y=0\end{aligned}$$

The imaginary component's equation implies that potential solutions have components $x=-1/2$ and $y=0$.

So beginning with the first possibility we plug $x=-1/2$ back into the real component's equation to see

$$\begin{aligned}x^2+x&=y^2-1\\\frac{1}{4}-\frac{1}{2}&=y^2-1\\\frac{-1}{4}&=y^2-1\\\frac{3}{4}&=y^2\\y&=\pm\sqrt{\frac{3}{4}}=\pm\sqrt{3}/2\end{aligned}$$

giving the pair of complex solutions $z=\left(-1\pm i\sqrt{3}\right)/2$.

Similarly, using $y=0$ in the same equation would show

$$\begin{aligned}x^2+x&=-1\\x^2+x+1&=0\\x&=\left(-1\pm\sqrt{1-4}\right)/2\\x&=\left(-1\pm\sqrt{-3}\right)/2\\x&=\left(-1\pm i\sqrt{3}\right)/2\end{aligned}$$

giving the same pair of solutions, but requiring the quadratic formula.

Problem 3. Reduce each of these quantities to a real number:

(a)

$$\frac{1+i2}{3-i4}+\frac{2-i}{5i}$$

(b) $(1-i)^4$

Solution 3. (a) Start by multiplying the first fraction by th complex conjugate of the denominator over itself. Then simplify:

$$\begin{aligned}
\frac{1+i2}{3-i4} + \frac{2-i}{5i} &= \frac{(1+i2)(3-i4)}{(9+16)} + \frac{1+i2}{5} \\
&= \frac{3+8+i(6-4)}{9+16} + \frac{1+i2}{5} \\
&= \frac{11+i2}{25} + \frac{1+i2}{5} \\
&= \frac{11+i2}{25} + \frac{5+i10}{25} \\
&= \frac{11+i2}{25} + \frac{5+i10}{25} \\
&= \boxed{\frac{16+i12}{25}}
\end{aligned}$$

(b) Using $i^2 = -1$, we can find

$$\begin{aligned}
(1-i)^4 &= \left((1-i)^2\right)^2 \\
&= (-2i)^2 = \boxed{-4}
\end{aligned}$$

Problem 4. Verify that $\sqrt{2}|z| \geq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$.

Solution 4. I we call $z = x + iy$, then we can see

$$\begin{aligned}
(|x| - |y|)^2 &\geq 0 \\
|x|^2 + |y|^2 - 2|x||y| &\geq 0 \\
|x|^2 + |y|^2 &\geq 2|x||y| \\
|z|^2 &\geq 2|x||y| \\
2|z|^2 &\geq 2|x||y| + |x|^2 + |y|^2 \\
2|z|^2 &\geq (|x| + |y|)^2 \\
\sqrt{2}|z| &\geq |x| + |y| = |\mathrm{Re}(z)| + |\mathrm{Im}(z)|
\end{aligned}$$

Problem 5. In each case, sketch the set of points determined by the given condition:

(a) $|z - 1 + 1| = 1$

(b) $|z + i| \leq 3$

(c) $|z - 4i| \geq 4$

Solution 5. (a)