University of West Florida

STA 6856 Mid-Term Exam | Professor Dr.Tharindu De Alwis

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Solution 1a.

$$Y_t = t + e_3$$

Mean function

$$\mu Y(t) = \mathbb{E}(Y_t) = \mathbb{E}(t + e_3) = t + \mathbb{E}(e_3) = t + 0 = t$$

μ is dependent on t

Autocovariance

For s, t

$$\gamma Y(s,t) = Cov(Y_s, Y_t) = Cov(s + e_3, t + e_3) = Cov(e_3, e_3) = Var(e_3) = 1$$

therefore,

$Var(Y_t) = 1$ for every t

$Cov(Y_t, Y_{t+h}) = 1$ for any lag h

Stationary

$\mu Y(t) = t$, mean and autocovarience dependent on the lag, process is not weakly stationary.

Fails to be Strict Stationary.

Process is *nonstationary*

Solution 1b.

Mean function

$$Y_t = e_t e_{t-2}$$
$$\mu Y(t) = \mathbb{E}(Y_t) = \mathbb{E}(e_t)\mathbb{E}(e_{t-2}) = 0 \cdot 0 = 0$$

e_t and e_{t-2} are independent

Autocovarience function

$$\mathbb{E}(Y_t) = \mathbb{E}(e_t \cdot e_{t-2}) = \mathbb{E}(e_t) \cdot \mathbb{E}(e_{t-2}) = 0 \times 0 = 0$$
$$\mu Y(t) = 0$$

Independence of e_t

$h = 0$

$$\gamma Y(0) = \mathbb{E}(e_t e_{t-2} e_t e_{t-2}) = \mathbb{E}(e_t^2 e_{t-2}^2)$$
$$\mathbb{E}(e_t^2 e_{t-2}^2) = \mathbb{E}(e_t^2)\mathbb{E}(e_{t-2}^2) = 1 \cdot 1 = 1$$

thus,

$$\gamma Y(0) = 1$$

$h = 2$

$$\gamma Y(2) = \mathbb{E}(e_t e_{t-2} e_{t+2} e_t) = \mathbb{E}(e_t^2 e_{t-2} e_{t+2})$$
$$\mathbb{E}(e_{t-2})\,\mathbb{E}(e_{t+2}) = 0$$

thus,

$$\gamma_Y(2) = 0$$

$h = -2$

$$\gamma_Y(-2) = \mathbb{E}[e_t e_{t-2} e_{t-2} e_{t-4}] = \mathbb{E}[e_{t-2}^2 e_t e_{t-4}]$$

thus,

$$\gamma_Y(-2) = 0$$

For any other h,

$$\mathbb{E}[e_t e_{t-2} e_{t+h} e_{t+h-2}]$$

contains at least one factor that appears only once, and because each e_i has mean 0, the entire expectation is 0:

$$\gamma_Y(h) = 0 \quad \text{for all } h \neq 0$$

Weak stationary,

$\mathbb{E}(Y_t)$ is constant, 0 for all $t

$\gamma Y(h)$ dependent on lag h

thus, Y_t is strictly and weakly stationary

Solution 2.

Standard ARMA(p,q) form

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_q e_{t-q}$$

$$\text{AR part: } Y_t = 0.5Y_{t-1} - 0.5Y_{t-2} + \cdots \quad \text{then, } p = 2,\ \phi_1 = 0.5, \phi_2 = -0.5$$

$$\text{MA part: } e_t - 0.5e_{t-1} + 0.25e_{t-2} \quad \text{then, } q = 2,\ \theta_1 = -0.5,\ \theta_2 = 0.25$$

Form check,

$$(1 - 0.5B + 0.5B^2)Y_t = (1 - 0.5B + 0.25B^2)e_t$$

Therefore,

$$\text{ARMA}(2,2): \quad \phi_1 = 0.5,\ \phi_2 = -0.5,\ \theta_1 = -0.5,\ \theta_2 = 0.25$$

Solution 3.

$E(X) = 2, \quad \mathrm{Var}(X) = 9, \quad E(Y) = 0, \quad \mathrm{Var}(Y) = 4, \quad \mathrm{Corr}(X, Y) = 0.25$

$$\mathrm{Corr}(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

$\sigma(X) = \sqrt{9} = 3, \sigma(Y) = \sqrt{4} = 2$

$\mathrm{Cov}(X, Y) = 0.25 \times 3 \times 2 = 1.5$

Solution 3a

$Var(X + Y)$

$\mathrm{Var}(X) + \mathrm{Var}(Y) + 2\ \mathrm{Cov}(X, Y) = 9 + 4 + 2(1.5) = 16$

Solution 3b.

$\mathrm{Cov}(X, X + Y)$

$$= \mathrm{Cov}(X, X) + \mathrm{Cov}(X, Y) = \mathrm{Var}(X) + \mathrm{Cov}(X, Y) = 9 + 1.5 = 10.5$$

Solution 3c.

$\mathrm{Corr}(X + Y, X - Y)$

$$= \mathrm{Cov}(X, X) - \mathrm{Cov}(X, Y) + \mathrm{Cov}(Y, X) - \mathrm{Cov}(Y, Y) = \mathrm{Var}(X) - \mathrm{Var}(Y) = 9 - 4 = 5$$

$$\mathrm{Var}(X + Y) = 16$$

$$\mathrm{Var}(X - Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\ \mathrm{Cov}(X, Y) = 9 + 4 - 3 = 10$$

$$\mathrm{Corr}(X + Y, X - Y) = \frac{\mathrm{Cov}(X + Y, X - Y)}{\sqrt{\mathrm{Var}(X + Y)\ \mathrm{Var}(X - Y)}} = \frac{5}{\sqrt{16 \times 10}} = \frac{5}{4\sqrt{10}} = \frac{\sqrt{10}}{8} \approx 0.395$$

$\mathrm{Corr}(X + Y, X - Y) = \dfrac{\sqrt{10}}{8} \approx 0.395.$

Solution 4a.

AR lags = 2 MA lags = 1

$$\text{ARMA(2,1)} \quad (\phi_1, \phi_2, \theta_1) = (1, -0.25, -0.5)$$
$$(1 - \phi_1 B - \phi_2 B^2) Y_t = (1 + \theta_1 B) e_t$$

So we have,

$$(1 - B + 0.25B^2) Y_t = (1 - 0.5B) e_t$$

Solution 4b.

Casuality and Stationary

The roots of the AR polynomial lie outside the unit circle. $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$

$$\phi(z) = 1 - z + 0.25z^2$$

$\phi(z) = 0$

$$0.25z^2 - z + 1 = 0 \quad \Longrightarrow \quad z^2 - 4z + 4 = 0$$

so,

$$(z - 2)^2 = 0 \quad \Rightarrow \quad z = 2$$

Since $|z| = 2 > 1$, the AR part satisfies the stationarity condition.

The MA polynomial

Solution 5a.

$\phi(B) = 1 - 1.1B + 0.18B^2$

Examine the roots of:

$$\phi(z) = 1 - 1.1z + 0.18z^2 = 0$$
$$0.18z^2 - 1.1z + 1 = 0$$
$$z = \frac{1.1 \pm \sqrt{(-1.1)^2 - 4(0.18)(1)}}{2(0.18)} = \frac{1.1 \pm \sqrt{1.21 - 0.72}}{0.36} = \frac{1.1 \pm 0.7}{0.36}$$

$$z_1 = \frac{1.8}{0.36} = 5, z_2 = \frac{0.4}{0.36} \approx 1.111$$

Since both roots satisfy $|z_1| > 1$ and $|z_2| > 1$, both are greater than 1 in absolute value, the series is stationary

Solution 5b.

For an AR(2) process:

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

the theoretical autocorrelation function satisfies the Yule–Walker equations:

$$\rho(h) = \phi_1 \rho(h-1) + \phi_2 \rho(h-2), \quad h \geq 2$$
$$\rho(0) = 1, \qquad \rho(1) = \frac{\phi_1}{1 - \phi_2}$$

$\phi_1 = 1.1,\ \phi_2 = -0.18$ we have:

$$\rho(1) = \frac{1.1}{1 - (-0.18)} = \frac{1.1}{1.18} \approx 0.932$$

Then:

$$\rho(2) = \phi_1\rho(1) + \phi_2\rho(0) = 1.1(0.932) - 0.18 = 0.845$$

$$\rho(h) - 1.1\rho(h-1) + 0.18\rho(h-2) = 0$$
$$r^2 - 1.1r + 0.18 = 0,$$

whose roots are:

$$r_1 = 0.9, \qquad r_2 = 0.2.$$

Therefore, the general form of the autocorrelation function is:

$$\rho(h) = C_1(0.9)^h + C_2(0.2)^h, \qquad h \geq 0.$$

Using the conditions $\rho(0) = 1$ and $(\rho(1) = 0.932,)$ we solve for (C_1) and (C_2) :

$$C_1 + C_2 = 1$$

$$0.9C_1 + 0.2C_2 = 0.932$$

which gives

$$C_1 = 1.046,\ C_2 = -0.046$$

Thus:

$$\rho(h) = 1.046(0.9)^h - 0.046(0.2)^h$$

Solution 6a.

ARMA(1,0)

-Time Series is Stationary ARMA possible

-ACF Small lag-1 correlation, then none AR(1)

-PACF Sharp cutoff at lag 1 AR(1)

-EACF Zero region starts at (1,0) AR(1)

$$Y_t = e_t - \phi_1 Y_{t-1} + e_t \text{ where } e_t\ WN(0, \sigma^2)$$

Solution 6b.

ARMA(0,1)

-Time Series is Stationary, ARMA possible

-ACF Cuts off after lag 1 MA(1)

-PACF Tails off gradually MA(1)

-EACF Triangle starts at (0,1) MA(1)

$$Y_t = e_t - \phi_1 e_{t-1} \text{where } e_t \ WN(0, \sigma^2)$$

Solution 7a.

$$Y_t = 0.7324Y_{t-1} + e_t - 0.5817e_{t-1}, e_t\ WN(0, 1.204)$$

Solution 7b.

The quarterly economic time series is stationary

ARIMA Order (1, 0, 1): The second parameter in the ARIMA model, d=0, indicates that no differencing was required to make the series stationary.

AR Coefficient $\phi_1 = 0.7324$ Within Stationary Range: For an AR(1) process to be stationary, we require, 0.7324<1, satisfy the condition for stationarity.

No Trend or Seasonality: The problem statement explicitly says the series "appears to exhibit no significant trend or cyclical behavior", supports that the mean and variance are stable over time.

Solution 7c.

The time series is invertible, because the MA(1) coefficient $= -0.5817$ satisfies $\phi_1 < 1$, the model has a valid and unique invertible representation.